Università della Calabria

Dipartimento di Matematica

Dottorato di Ricerca in Matematica ed Informatica

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Settore Disciplinare MAT/05

Tesi di Dottorato

Solutions of problems in analytic and extremal combinatorial set theory

Caterina Nardi

Supervisori
Prof. Giuseppe Marino
Dott. Giampiero Chiaselotti

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A.A. 2011 – 2012

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Abstract

The research of this thesis is part of the study of some problems in extremal combinatorial set theory, which are related to some aspects of the analytic number theory. We placed these problems in the lattice theory and using classic tools we e have obtained some results, which are related to the well-known Manickham-Miklós-Singhi-conjecture (1988). In particular, we found the maximum and the minimum values related to a class of multi-sets of real numbers with non-negative sums. Moreover we builded a Boolean map for each intermediate value between the maximum and the minimum. This Boolean map represents the abstract form of each multi-sets.

However, the lattice theory provided a partial solution. Afterwards, with different approach, we solved a generalization of a problem of Manickam e Miklós.

The last goal was to adopt analytical methods to find some estimates, which come from property of graded Posets.

Any ranked poset whose largest antichain is not bigger than its maximum level is said to possess the Sperner property, hence the Sperner property is equivalently the property that some rank level is a maximum antichain. Many tools have been developed by researchers to determine whether a poset has the property of Sperner, instead there are considerably fewer techniques for establishing if a poset has not the Sperner property. In particular, we studied a quotient of lattice of partition of the set $[n] = \{1, 2, \ldots, n\}$ and we estimate how much it is 'close' to be a Sperner poset.

Sommario

L'attività di ricerca di questa tesi rientra nell'ambito dello studio di alcuni problemi di combinatoria estremale che risultano essere connessi ad alcuni aspetti della teoria analitica dei numeri. Abbiamo collocato tali problemi nell'ambito della teoria combinatoria dei reticoli e utilizzando gli strumenti classici di tale teoria abbiamo ottenuto alcuni risultati, i quali sono connessi ad una congettura del 1988 di Manckam, Miklös e Singhi. Più in particolare, siamo riusciti a determinare il massimo e il minimo valore relativi ad una certa classe di multi-insiemi di numeri reali aventi somma non negativa. Inoltre abbiamo costruito una particolare mappa Booleana per ogni valore intermedio tra il minimo e il massimo. Questa mappa rappresenta la forma astratta di ciascun multi-insieme.

Tuttavia, la tecnica di collocare questi problemi nella teoria dei reticoli ha fornito solo una soluzione parziale. Successivamente, con un approccio diverso, abbiamo risolto una generalizzazione di un problema di Manickam e Miklós.

L'ultimo obiettivo è stato quello di applicare alcuni metodi analitici per trovare delle stime che provengono da alcune proprietà relative ai Posets graduati.

Ogni Poset graduato possiede la proprietà di Sperner se l'anticatena più larga non è più grande del livello massimo. Molti interessanti Posets nascono in Combinatorica fornendo ai ricercatori una ricchezza di problemi. La principale verità di questa ricerca è stata sviluppare metodi per stabilire la proprietà di Sperner. Al contrario ci sono molte meno tecniche per stimare la grandezza di un'anticatena massima in un Poset che non possiede la proprietà di Sperner. In particolare, abbiamo studiato un quoziente del reticolo delle Partizioni di un insieme stimando quanto questo si 'avvicina' ad essere un Sperner Poset.

Chapter 1

Introduction

In this thesis we study some problems in extremal combinatorial set theory, which is a branch of extremal combinatorics. The general problem in extremal combinatorial set theory is to start with all collections of subsets of an underlying ground set, apply restrictions, and then ask how large or small some property can be under those restrictions. In general extremal combinatorial set theory the main problem is related into finding (or estimating) the maximum or minimum number of sets satisfying given set-theoretic or combinatorial conditions on such sets.

In [35] the authors asked the following question:

> Let n be an integer strictly greater than 1 and $a_1, \dots, a_n$ be real numbers satisfying the property $\sum_{1=1}^{n} a_i \geq 0$. We may ask: how many of the subsets of the set $\{a_1, \cdots, a_n\}$ will have a non-negative sum?

Following the notations of [35], the authors denote with $A(n)$ the minimum number of the non-negative partial sums of a sum $\sum_{i=1}^{n} a_i \geq 0$, not counting the empty sum, if we take all the possible choices of the a_i's. They proved (see Theorem 1 in [35]) that $A(n) = 2^{n-1}$ and they explained as Erdös, Ko and Rado investigated a question with an answer similar to this one:

> What is the maximum number of pairwise intersecting subsets of an $n-$elements set? As in their case, here also the question becomes more difficult if we restrict ourselves to the $d-$subsets.

More details about this remark can be find in the famous theorem of Erdös-Ko-Rado [28] (see also [31] for an easy proof of it).
Formally, with the introduction of the positive integer d, the problem is the following. Let $I_n = \{1, 2 \dots, n\}$ and $1 \leq d < n$ be integers a function $f : I_n \to \mathbb{R}$ is called an $n-$weight function if $\sum_{x \in I_n} f(x) \geq 0$.
Denote with $W_n(\mathbb{R})$ the set of all the $n-$weight functions and set

$$\phi(f, d) = |\{Y \subseteq I_n : |Y| = d, \sum_{y \in Y} f(y) \geq 0\}|,$$

$$\psi(n, d) = \min\{\phi(f, d) : f \in W_n(\mathbb{R})\}.$$

If f is such that $f(1) = n-1$, $f(2) = \cdots = f(n) = -1$, it follows that $\psi(n,d) \le \binom{n-1}{d-1}$.
In [8], Bier and Manickam proved that $\psi(n,d) = \binom{n-1}{d-1}$ if $n \ge d(d-1)^d(d-2)^d + d^4$ and $\psi(n,d) = \binom{n-1}{d-1}$ if $d|n$.
Both the proofs use the Baranyai theorem on the factorization of complete hypergraphs [4](see also [50] for a modern exposition of the theorem).
In [35] and [36] it was conjectured that $\psi(n,d) \ge \binom{n-1}{d-1}$ if $n \ge 4d$.
In [36] this conjecture (the Manickam-Miklös-Singhi Conjecture) is set in the more general context of the association schemes (see [3] for general references on the subject). This conjecture is connected with the first invariant distribution of the Johnson association scheme (see [8], [36], [33], [34]). The distribution invariants were introduced by Bier [7], and later investigated in [9], [32], [33], [36]. In [36] the authors claim that this conjecture is, in some sense, dual to the theorem of Erdös-Ko-Rado [28]. Also, as pointed out in [43], this conjecture settles some cases of another conjecture on multiplicative functions by Alladi, Erdös and Vaaler, [1]. Partial results related to the Manickam-Miklös-Singhi conjecture also have been obtained in [5], [6], [18], [19], [20] and more recently in [2] and [49]. Bisi and Chiaselotti in [11] and [12] have discovered an interesting lattice structure hidden behind this conjecture. They defined a partial order $\sqsubseteq$ on the power set $\mathcal{P}(I_n)$ having the following property : if X, Y are two subsets of I_n such that $X \sqsubseteq Y$, then $\sum_{i\in X} a_i \le \sum_{i\in Y} a_i$, for each n-multiset $\{a_1,\dots,a_n\}$ of real numbers such that $a_1 \ge \cdots \ge a_r \ge 0 > a_{r+1} \ge \cdots \ge a_n$. This order defines a lattice structure on $\mathcal{P}(I_n)$ the lattice $(S(n,r),\sqsubseteq)$. They established the connection between the lattice $S(n,r)$ and some combinatorial extremal sum problems related to a conjecture of Manickam, Miklös and Singhi and they have given an interpretation of these problems in terms of a particular class of boolean maps defined on $S(n,r)$.

In the same way Chiaselotti, Marino and Nardi studied two extremal combinatorial sum problems.
Let $a_1,\dots,a_r$ be r non-negative real numbers and let $a_{r+1},\dots,a_n$ be $n-r$ negative real numbers with non-negative sum, i.e.,

$$\sum_{i=1}^{r} a_i + \sum_{j=r+1}^{n} a_j \ge 0.$$

Let $\gamma(n,r)$ $[\eta(n,r)]$ be the minimum [maximum] number of subsets of $\{a_1,\dots,a_n\}$ whose element-sum is non-negative. In [21], the authors have studied the following two problems:

($P1$) Which are the values of $\gamma(n,r)$ and $\eta(n,r)$ for each n and r, $0 < r \le n$?

($P2$) If q is an integer such that $\gamma(n,r) \le q \le \eta(n,r)$, can we find r non-negative real numbers $a_1,\dots,a_r$ and $n-r$ negative real numbers $a_{r+1},\dots,a_n$ with $\sum_{i=1}^{r} a_i + \sum_{j=r+1}^{n} a_j \ge 0$ such that the number of subsets of $\{a_1,\dots,a_n\}$ whose element-sum is non-negative is exactly q?

Using a poset-formulation of the previous problems, the authors solved the problem ($P1$) and they provided a partial result for ($P2$) showing that the answer is affirmative for particular class of boolean maps defined on $S(n,r)$. Afterwards Engel and Nardi [25], using a different method, solved completely the problem ($P2$).

The lattice $S(n,r)$ is distributive, graded, involutive, moreover the largest antichain is not bigger than its maximum level, i.e. $S(n,r)$ has the Sperner property. Many tools have been developed by researchers to determine whether a poset has the property of Sperner, instead there are considerably fewer techniques for establishing if a poset has not the Sperner property. In the last part of the thesis we consider a new poset and we estimate how much it is 'close' to be a Sperner poset.

Chapter 2

Notation and terminology

The problems considered in this thesis are most conveniently formulated in terms of partially ordered sets, or posets for short. Thus we start with discussing some basic notions concerning posets, which are assumed to be *finite* throughout this thesis. We adopt the classical terminology and notations usually used in the context of the partially ordered sets (see [22], [23] and [44] for the general aspects on this subject).

2.1 Posets

Definition 2.1.1. Let P be a set. An *order* (or *partial order*) on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

- $x \leq x$,
- $x \leq y$ and $y \leq x$ imply $x = y$,
- $x \leq y$ and $y \leq z$ imply $x \leq z$.

These conditions are referred to, respectively, as *reflexivity*, *antisymmetry* and *transitivity*. A set P equipped with an order relation $\leq$ is said to be an *ordered set* (or *partially ordered set*). Some authors use the shorthand *poset*. We use $x \leq y$ and $y \geq x$ interchangeably, and write $x \nleq y$ to mean '$x \leq y$ is false', and so on. Less familiar is the symbol $\parallel$ used to denote non-comparability: we write $x \parallel y$ if $x \nleq y$ and $y \nleq x$.

We later deal with the construction of new ordered sets from existing ones. There is one such construction which it is convenient to have available immediately. Let P be an ordered set and let Q be a subset of P. Then Q inherits an ordered relation from P; given $x, y \in Q$, $x \leq y$ in Q if only if $x \leq y$ in P. We say in these circumstances that Q has the *induced order*, or, when we wish to be a more explicit, the order *inherited from* P.

There is a simple way to represent small posets pictorially, for describing it we need the following definition.

Definition 2.1.2. Let P be an ordered set and let $x, y \in P$. We say x is *covered* by y (or y *covers* x), and write $x \lessdot y$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter conditions is demanding that there be no element z of P with $x < z < y$.

Let P be an ordered set. We can represent P by a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation). The construction goes as follows.

(1) To each point $x \in P$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$.

(2) For each covering pair $x \lessdot y$ in P, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.

(3) Carry out(1) and (2) in such a way that

   1. if $x \lessdot y$, then $p(x)$ is 'lower' than $p(y)$ (that is, in standard Cartesian coordinates, has a strictly smaller second coordinate),
   2. the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

A configuration satisfying (1)-(2)-(3) is called a *diagram* (or *Hasse diagram*) of P.
The following figure shows two alternative diagrams for the ordered set $P = \{a, b, c, d\}$ in which $a < c$; $a < d$; $b < c$ and $b < d$. (When we specify an ordered set by a set of inequalities in this way, it is to be understood that no other pairs of distinct elements are comparable.)

Definition 2.1.3. For a poset P with cardinality $|P| = n$. The *incidence matrix* of P is a $n \times n$ matrix M such that

$$M(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}.$$

We need to be able to recognize when two ordered sets, P and Q, are 'essentially the same'.

Definition 2.1.4. We say that P and Q are (*order*) *isomorphic*, and write $P \cong Q$, if there exists a map φ from P onto Q such that $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q. Then φ is called an *order-isomorphism.*

Such a map φ faithfully mirrors the order structure. It is necessarily bijective. On the other hand, not every bijective map between ordered sets is an order-isomorphism: consider, for example, $P = Q = \mathit{2}$ and define φ by $\varphi(0) = 1, \varphi(1) = 0$. Thus one can think that two posets are isomorphic if they differ only in the names of their elements. This is exactly analogous to the notion of isomorphism of groups, rings, etc. It is an instructive exercise to draw Hasse diagrams of the one poset of order (number of elements) one (up to isomorphism), the two posets of order two, the five posets of order three, and the sixteen posets of order four.
Structure-preserving maps are considered more generally.

Definition 2.1.5. Let P and Q be ordered sets. A map $\phi : P \to Q$ is said to be

(i) *order-preserving* (or, alternatively, *monotone*) if $x \leq y$ in P implies $\phi(x) \leq \phi(y)$ in Q;

(ii) *order-embedding* (and we write $\phi : P \hookrightarrow Q$) if $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q;

(iii) *order-isomorphism* if it is order-embedding which maps P onto Q

There are some ways of constructing new ordered sets from existing ones. For example, given any ordered set P we can form a new ordered set P^∂ (the *dual* of P) by defining $x \leq y$ to hold in P^∂ if and only if $y \leq x$ holds in P. For P finite, we obtain a diagram for P^∂ simply by 'turning upside down' a diagram for P.

Now, we introduce some important special elements.

Definition 2.1.6. We say P has a bottom element if there exists $\bot \in P$ (called *bottom*) with the property that $\bot \leq x$ for all $x \in P$. Dually, P has top element if there exists $\top \in P$ such that $x \leq \top$ for all $x \in P$.

Definition 2.1.7. Let P be an ordered set and $Q \subseteq P$.Then $a \in P$ is a *maximal element of* Q if $a \leq x$ and $x \in Q$ imply $a = x$. We denote the set of maximal elements of Q by Max$Q$. If Q (with the order inherited from P) has top element, $\top_Q$, then Max$Q = \{\top_Q\}$; in this case $\top_Q$ is called *greatest* (or *maximum*) element of Q, and we write $\top_Q = \max Q$. A *minimal* element of $Q \subseteq P$ and $\min Q$, the *least* (or *minimun*) element of Q (when these exist) are defined dually, that is by reversing the order.

There are several different ways to join two ordered sets together.

Definition 2.1.8. Let $P_1, \ldots, P_n$ be ordered sets. The Cartesian product $P1 \times \ldots \times P_n$ can be made into an ordered set by imposing the coordinatewise order defined by

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \Longleftrightarrow (\forall i)\ x_i \leq y_i \ in\ P_i$$

Given an ordered set P, the notation P^n is used as shorthand for the n-fold product $P \times \ldots P$.

Associated with any ordered set are two important families of sets. They play a central role in our results.

Definition 2.1.9. Let P be an ordered set and $Q \subseteq P$.

(i) Q is a *down set* (alternative terms include *decreasing set* and *order ideal*) if, whenever $x \in Q$, $y \in P$ and $y \leq x$, we have $y \in Q$.

(ii) Dually,Q is a *up set* (alternative terms include *increasing set* and *order filter*) if, whenever $x \in Q$, $y \in P$ and $x \leq y$, we have $y \in Q$.

Given an arbitrary subset Q of P and $x \in P$, we define

$$\downarrow Q := \{y \in P | (\exists x \in Q) y \leq x\} \ and\ \uparrow Q := \{y \in P | (\exists x \in Q) y \geq x\},$$

$$\downarrow x := \{y \in P | y \leq x\} \ and\ \uparrow x := \{y \in P | y \geq x\}.$$

These are read 'down Q', etc. It is easily checked that $\downarrow Q$ is the smallest down-set containing Q. and that Q is a down-set if and only if $Q =\downarrow Q$, and dually for $\uparrow Q$. Clearly $\downarrow \{x\} = \downarrow x$, and dually. Down-sets(upsets) of the form $\downarrow x$ ($\uparrow x$) are called *principal*.

A *chain* C in a poset is a totally ordered subset of P, i.e., for all $x, y \in P$, either $x \leq y$ or $y \leq x$. Let P be an ordered set.

(i) If $C = \{c_0, c_1, \ldots c_n\}$ is a finite chain in P with $|C| = n + 1$, then we say that the *length* of C is n.

(ii) P is said to have *length* n, written $l(P) = n$, if the length of the longest chain in P is n.

(iii) P is of *finite length* if it is length n for some $n \in \mathbb{N}_0$.

(iv) P has *no infinite chain* if every chain in P is finite.

(v) A chain is called *saturated* if it has the form $C = (c_0 \lessdot \ldots \lessdot c_h)$, and it is called *maximal* if, in addition, c_o and c_h are minimal and maximal elements of P, respectively.

At opposite extreme from a chain is an antichain. The ordered set P is a *antichain* for $x, y \in P$ if $x \leq y$ only if $x = y$. Subsets of a poset will often be called *families* too (motivated by families od subsets of a set). Antichains are also called *Sperner families*. A *k-family* is a family in P containing no chain of $k+1$ elements in P, thus a 1-family is an antichain.

A *rank function* of a poset P is a function r from P into the set $\mathbb{N}$ of all natural numbers such $r(p) = 0$ for a minimal element p of P and $p \lessdot q$ implies $r(q) = r(p) + 1$. A *ranked poset* is a poset with a rank function. If in a ranked poset every minimal element has rank 0 and every maximal element has the same rank, we speak of a *graded poset*. The number $r(P) := max\{r(p) : p \in P\}$ is called the *rank of* P. If P is a finite poset, equivalently we say that the poset P is graded of rank n if every maximal chain has length n. If $r(P) < \infty$ the *dual of a ranked poset* P is ranked.

The *product of two ranked posets* P, Q is defined to be the poset $P \times Q$ together with the rank function $r_{P\times Q}$ given by $r_{P\times Q}(p,q) := r_{P(p)} + r_{Q(q)}$. For a ranked poset P, we define the *i th level* by $N_i(P) := \{p \in P \quad r(p) = i\}$; its size $W_i(P) := |N_i(P)|$ is called the *i th Whitney number*, $i = 0, \ldots, r(P)$ (when there is no danger of ambiguity, we write briefly N_i and W_i). It is useful to define $N_i := \varnothing$ and $W_i := 0$ if $i \notin \{0, \ldots, r(P)\}$. Obviously, each level of a ranked poset is an antichain, and the union of k levels is a *k-family*.

Given a group G of automorphisms of a poset P, a nonempty subset A of P is called an *orbit* if for all $p, q \in A$ there is some $\phi \in G$ such that $\phi(p) = q$ and if A is maximal with respect to this property. It is easy to see that union of all orbits is a partition of P. Now the *quotient of* P *under* G (denoted by P/G) is the poset of all orbits ordered in the following way: $A \leq B$ in P/G if there exist $a \in A$, $b \in B$ such that $a \leq b$ in P.

The *rank-generating function* $F(P;x)$ of a ranked poset is defined by $F(P;x) := \sum_{p\in P} x^{r(p)} = \sum_{i=0}^{r(P)} W_i x^i$. It is easy to see that $F(P \times Q; x) = F(P;x)F(Q,x)$ if P and Q are ranked.

2.2 Lattices

Many important properties of an ordered set P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. Two of the most important classes of ordered set defined in this way are lattices and complete lattices.

Definition 2.2.1. Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. A *lower bound* is defined dually. The set of all upper bounds of S is denoted by S^u:

$$S^u := \{x \in P | (\forall s \in S) s \leq x\} \textit{ and } S^l := \{x \in P | (\forall s \in S) s \geq x\}$$

Since $\leq$ is transitive, S^u is always an up-set and S^l a down-set. If S^u has a least element x, then x is called the *least upper bound* of S. Equivalently, x is the least upper bound of S if

(i) x is an upper bound of S, and

(ii) $x \leq y$ for all upper bounds y of S.

The least upper bound of S exists if and only if there exists $x \in P$ such that

$$(\forall y \in P)[((\forall s \in S) s \leq y) \iff x \leq y]$$

and this characterizes the least upper bound of S. This way of presenting the Definition is slicker, but is less transparent until the two-step version has been fully mastered.

Dually, if S^l has a greatest element, x, then x is called the *greatest lower bound* of S. Since least elements and greatest elements are unique, least upper bounds and greatest lower bounds are unique when they exist. The last upper bound of S is also called *supremum* of S and is denoted by sup$S$; the greatest lower bound of S is also called the *infimum* of S and is denoted by inf$S$. It is easily seen that if P has a top element $\top$, then $P^u = \{\top\}$ in which case sup$P = \top$. By duality, inf$P = \bot$ whenever P has a bottom element.

Looking ahead, we shall adopt the following notation: we write $x \vee y$ (read as '*x join y*') in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ (read as '*x meet y*') in place of $\inf\{x, y\}$ when it exists. Similarly, we write $\bigvee S$ (the '*join of S*') and $\bigwedge S$ (the '*meet of S*') instead of $\sup S$ and $\inf S$ when these exist. It is sometimes necessary to indicate that the join or meet is being found in a particular ordered set P, which case we write $\bigvee_P S$ or $\bigwedge_P S$.

We shall be particularly interested in ordered sets in which $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$.

Definition 2.2.2. Let P be a non-empty ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a *lattice.*

(ii) if $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a *complete lattice.*

Given a lattice L, we may define binary operations *join* and *meet* on the non-empty set L by

$$a \vee b := \sup\{a, b\} \textit{ and } a \wedge b := \inf\{a, b\} \ (a, b \in L).$$

We view a lattice as an algebraic structure $\langle L; \vee, \wedge \rangle$ and explore the properties of these binary operations. We amplify the connection between $\vee, \wedge$, and $\leq$ using the following results (see [22] for a proof).

Lemma 2.2.3. *Let L be a lattice and let $a, b \in L$. Then the following are equivalent:*

(i) $a \leq b$;

(ii) $a \vee b = b$;

(iii) $a \wedge b = a$.

Theorem 2.2.4. *Let L be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c \in L$,*

$(L1)$	$(a \vee b) \vee c = a \vee (b \vee c)$	(*associative laws*)
$(L1)^{\partial}$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	
$(L2)$	$a \vee b = b \vee a$	(*commutative laws*)
$(L2)^{\partial}$	$a \wedge b = b \wedge a$	
$(L3)$	$a \vee a = a$	(*idempotency laws*)
$(L3)^{\partial}$	$a \wedge a = a$	
$(L4)$	$a \vee (a \wedge b) = a$	(*absorption laws*)
$(L4)^{\partial}$	$a \wedge (a \vee b) = a$	

We can derive new lattice from existing ones.

Definition 2.2.5. Let L be a lattice and $\varnothing \neq M \subseteq L$. Then M is a *sublattice* of L if

$$a, b \in M \text{ implies } a \vee b \in M \text{ and } a \wedge b \in M.$$

Definition 2.2.6. Let L and K be lattices. Define $\vee$ and $\wedge$ coordinate-wise on $L \times K$, as follows:

$$(l_1, k_1) \vee (l_2, k_2) = (l_1 \vee l_2, k_1 \vee k_2),$$

$$(l_1, k_1) \wedge (l_2, k_2) = (l_1 \wedge l_2, k_1 \wedge k_2).$$

It is routine to check that $L \times K$ satisfies the identities (L1)-(L4)$^{\partial}$ and therefore is a lattice.

There are important lattices which satisfy additional identities.

Definition 2.2.7. Let L be a lattice.

(i) L is said to be a *distributive* if it satisfies the *distributive law*,

$$(\forall a, b, c \in L) \ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

(ii) L is said to be a *modular* if it satisfies the *modular law*,

$$(\forall a, b, c \in L) \ a \geq c \Longrightarrow a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Proposition 2.2.8. *Every distributive lattice is modular.*

Proof. Let L be a distributive lattice, then $(\forall a, b, c \in L) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. If $a \geq c$ we have $a \wedge c = c$, thus $a \wedge (b \vee c) = (a \wedge b) \vee c$. □

2.3 Sequences

A sequence of nonnegative real numbers $\{a_n\}$ is called *unimodal* if there is a number h such that $a_i \le a_{i+1}$ for $i < h$ and $a_i \ge a_{i+1}$ if $i \ge h$. It is called *logarithmically concave* (or log concave) if $a_i^2 \ge a_{i-1}a_{i+1}$ for all i. For a finite sequence $(a_0, \ldots, a_n)$, we say that it is *symmetric* if $a_i = a_{n-i}$ for all i. If the Whitney numbers of P are unimodal (resp. symmetric), then P is said to be a *rank unimodal* (resp. rank symmetric).

If $\{a_n\}$ and $\{b_n\}$ are two infinite sequences of real numbers, the following notations for $n \to \infty$ are well known: $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$; $a_n = O(b_n)$ if there exists some $c \in (R)$ such that $|a_n| \le c|b_n|$ for all n; $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $a_n \lesssim b_n$ if $a_n \le b_n(1 + o(1))$.

Chapter 3

The Lattice $S(n,r)$

Bisi and Chiaselotti [11] have built a family of lattices related to some combinatorial extremal sum problems and we studied the main properties of this family of lattices.

Let n, r be two fixed integers such that $0 \le r \le n$ and let $I_n = \{1, 2 \dots, n\}$. In the first part of this chapter (section 3.1) we define a partial order $\sqsubseteq$ on the power set $\mathcal{P}(I_n)$ having the following property : if X, Y are two subsets of I_n such that $X \sqsubseteq Y$, then $\sum_{i \in X} a_i \le \sum_{i \in Y} a_i$, for each n-multiset $\{a_1, \dots, a_n\}$ of real numbers such that $a_1 \ge \cdots \ge a_r \ge 0 > a_{r+1} \ge \cdots \ge a_n$. This order defines a lattice structure on $\mathcal{P}(I_n)$ that we will denote by $(S(n,r), \sqsubseteq)$. We show as this lattice is distributive, graded (section 3.2), involutive (section 3.3), i.e. $X \sqsubseteq Y$ implies $Y^c \sqsubseteq X^c$ and a recursive formula to count the number of its elements having fixed rank (section 3.4).

3.1 A new lattice

Let n and r be two fixed integers such that $0 \le r \le n$. We denote by $A(n,r)$ an alphabet composed by the following $(n+1)$ formal symbols: $\tilde{1}, \dots, \tilde{r}$, $0^{\S}$, $\overline{1}, \dots, \overline{n-r}$. We introduce on $A(n,r)$ the following total order:

$$\overline{n-r} \prec \cdots \prec \overline{2} \prec \overline{1} \prec 0^{\S} \prec \tilde{1} \prec \tilde{2} \prec \cdots \prec \tilde{r} \tag{3.1}$$

where $\overline{n-r}$ is the minimal element and $\tilde{r}$ is the maximal element in this chain. If $i, j \in A(n,r)$, then we shall write : $i \preceq j$ for $i = j$ or $i \prec j$; $i \wedge j$ for the minimum and $i \vee j$ for the maximum between i and j with respect to $\preceq$; $i \lessdot j$ if j covers i with respect to $\preceq$; $j \succ i$ for $i \prec j$; $j \succeq i$ for $i \preceq j$. We shall denote by $(\mathcal{C}(n,r), \sqsubseteq)$ the n-fold cartesian product poset $A(n,r)^n$. An arbitrary element of $\mathcal{C}(n,r)$ can be identified with an n-string $t_1 \cdots t_n$ where $t_i \in A(n,r)$ for all $i = 1, \dots, n$. Therefore, if $t_1 \cdots t_n$ and $s_1 \cdots s_n$ are two strings of $\mathcal{C}(n,r)$, we have

$$t_1 \cdots t_n \sqsubseteq s_1 \cdots s_n \Longleftrightarrow t_1 \preceq s_1, \dots, t_n \preceq s_n.$$

We introduce now a particular subset $S(n,r)$ of $\mathcal{C}(n,r)$. We denote by $S(n,r)$ the set of all the formal expressions $i_1 \cdots i_r | j_1 \cdots j_{n-r}$ (hereafter called *strings*) that satisfy the following properties:

i) $i_1,\dots,i_r \in \{\tilde{1},\dots,\tilde{r},0^{\S}\}$,

ii) $j_1,\dots,j_{n-r} \in \{\overline{1},\dots,\overline{n-r},0^{\S}\}$,

iii) $i_1 \geq \cdots \geq i_r \geq 0^{\S} \geq j_1 \geq \cdots \geq j_{n-r}$,

iv) the unique element which can be repeated is $0^{\S}$.

In the sequel we often use the lowercase letters $u,w,z,...$ to denote a generic string in $S(n,r)$. Moreover to make smoother reading, in the numerical examples the formal symbols which appear in a string will be written without $\tilde{}$ $\bar{}$ and § ; in such way the vertical bar $|$ will indicate that the symbols on the left of $|$ are in $\{\tilde{1},\dots,\tilde{r},0^{\S}\}$ and the symbols on the right of $|$ are elements in $\{0^{\S},\overline{1},\dots,\overline{n-r}\}$.

Example 3.1.1. a) If $n=3$ and $r=2$, then $A(3,2)=\{\tilde{2}>\tilde{1}>0^{\S}>\overline{1}\}$ and $S(3,2)=\{21|0,\ 21|1,\ 10|0,\ 20|0,\ 10|1,\ 20|1,\ 00|1,\ 00|0\}$.

b) If $n=3$ and $r=0$, then $A(3,0)=\{0^{\S}>\overline{1}>\overline{2}>\overline{3}\}$ and $S(3,0)=\{|123,\ |023,\ |013,\ |012,\ |003,\ |002,\ |001,\ |000\}$.

c) If $n=3$ and $r=3$, then $A(3,3)=\{\tilde{3}>\tilde{2}>\tilde{1}>0^{\S}\}$ and $S(3,3)=\{000|,\ 300|,\ 200|,\ 100|,\ 320|,\ 310|,\ 210|,\ 321|\}$.

d) If $n=3$ and $r=1$, then $A(3,1)=\{\tilde{1}>0^{\S}>\overline{1}>\overline{2}\}$ and $S(3,1)=\{0|12,\ 0|02,\ 0|01,\ 0|00,\ 1|02,\ 1|01,\ 1|12,\ 1|00\}$.

e) If $n=0$ and $r=0$, then $S(0,0)$ will be identified with a singleton Γ corresponding to $|$ without symbols.

f) $S(1,0)=\{1|,\ 0|\}$, $S(0,1)=\{|1,\ |0\}$.

□

In the following, $S(n,r)$ will be considered as sub-poset of $\mathcal{C}(n,r)$ with the induced order from $\sqsubseteq$ after the restriction to $S(n,r)$. Therefore, if $w = i_1\cdots i_r|j_1\cdots j_{n-r}$ and $w' = i_1'\cdots i_r'|j_1'\cdots j_{n-r}'$ are two strings in $S(n,r)$, by definition of induced order we have

$$w \sqsubseteq w' \Longleftrightarrow i_1 \preceq i_1',\dots,i_r \preceq i_r',\ j_1 \preceq j_1',\dots,j_{n-r} \preceq j_{n-r}'.$$

As it is well known, $(\mathcal{C}(n,r),\sqsubseteq)$ is a distributive lattice whose binary operations of *inf* and *sup* are given respectively by $(t_1\cdots t_n)\wedge(s_1\cdots s_n)=(t_1\wedge s_1)\cdots(t_n\wedge s_n)$, and $(t_1\cdots t_n)\vee(s_1\cdots s_n)=(t_1\vee s_1)\cdots(t_n\vee s_n)$.

Example 3.1.2. If $n=7$ and $r=4$, and if $w_1=4310|023$, and $w_2=2100|012$, are two elements of $S(7,4)$, then $w_1\wedge w_2=2100|023$, and $w_1\vee w_2=4310|012$. If $n=8$ and $r=5$, and if $w_1=00000|001$, and $w_2=10000|003$, are two elements of $S(8,5)$, then $w_3\wedge w_4=00000|003$, and $w_3\vee w_4=10000|001$. □

In general, if $w_1,w_2\in S(n,r)$, then it is trivial to verify that $w_1\wedge w_2\in S(n,r)$ and $w_1\vee w_2\in S(n,r)$. Therefore the following Proposition holds:

Proposition 3.1.3. *$(S(n,r),\sqsubseteq)$ is a distributive lattice.*

Proof. $(\mathcal{C}(n,r),\sqsubseteq)$ is a distributive lattice and $S(n,r)$ is closed respect to $\wedge$ and $\vee$. Hence $S(n,r)$ is a distributive sublattice of $\mathcal{C}(n,r)$. □

Definition 3.1.4. If $w_1, w_2 \in S(n,r)$, then

i) $w_1 \sqsubset w_2$ if $w_1 \sqsubseteq w_2$ and $w_1 \neq w_2$;

ii) $w_1 \lessdot w_2$ if w_2 covers w_1 with respect to the order $\sqsubseteq$ in $S(n,r)$ (i.e. if $w_1 \sqsubset w_2$ and there does not exist $w \in S(n,r)$ such that $w_1 \sqsubset w \sqsubset w_2$);

iii) $w_1 \not\lessdot w_2$ if w_2 does not cover w_1 with respect to the order $\sqsubseteq$ in $S(n,r)$.

The minimal element of $S(n,r)$ is the string $0 \cdots 0|12 \cdots (n-r)$ and the maximal element is $r(r-1) \cdots 1|0 \cdots 0$. Sometimes they are denoted respectively with $\hat{0}$ and $\hat{1}$. Note that there is a natural bijective set-correspondence $* : w \in S(n,r) \mapsto w^* \in \mathcal{P}(I(n,r))$ between $S(n,r)$ and $\mathcal{P}(I(n,r))$ defined as follows: if $w = i_1 \cdots i_r | j_1 \cdots j_{n-r} \in S(n,r)$ then w^* is the subset of $I(n,r)$ made with the elements i_k and j_l such that $i_k \neq 0^{\S}$ and $j_l \neq 0^{\S}$. For example, if $w = 4310|013 \in S(7,4)$, then $w^* = \{\tilde{1}, \tilde{3}, \tilde{4}, \overline{1}, \overline{3}\}$. In particular, if $w = 0 \cdots 0|0 \cdots 0$ then $w^* = \varnothing$.

The lattice $(S(n,r), \sqsubseteq)$ has the following unary complementary operation c: $(p_1 \cdots p_k 0 \cdots 0|0 \cdots 0 q_1 \cdots q_l)^c = p'_1 \cdots p'_{r-k} 0 \cdots 0|0 \cdots 0 q'_1 \cdots q'_{n-r-l}$, where $\{p'_1, \ldots, p'_{r-k}\}$ is the complement of $\{p_1, \ldots, p_k\}$ in $\{\tilde{1}, \ldots, \tilde{r}\}$, and $\{q'_1, \ldots, q'_{n-r-l}\}$ is the complement of $\{q_1, \ldots, q_l\}$ in $\{\overline{1}, \ldots, \overline{n-r}\}$ (for example, in $S(7,4)$, we have that $(4310|001)^c = 2000|023)$.

3.2 Fundamental Properties of the Lattice $S(n,r)$

The Hasse diagrams of the lattices $S(n,r)$ for the first values of n and r are reported in Figure 3.1.

Proposition 3.2.1. *If $0 \le r \le n$, then $S(n,r) \cong S(r,r) \times S(n-r,0)$.*

Proof. Let $(w_P, w_N) \in S(r,r) \times S(n-r,0)$, with $w_P = i_1 \cdots i_r|$ and $w_N = |j_1 \cdots j_{n-r}$, where $i_1, \ldots, i_r \in \{\tilde{1}, \tilde{2}, \ldots, \tilde{r}, 0^{\S}\}$ and $j_1, \ldots, j_{n-r} \in \{0^{\S}, \overline{1}, \overline{2}, \ldots, \overline{n-r}\}$. We set $\varphi(w_P, w_N) = i_1 \cdots i_r | j_1 \cdots j_{n-r}$. It is easy to verify that φ is an isomorphism between $S(r,r) \times S(n-r,0)$ and $S(n,r)$. □

Proposition 3.2.2. *Let $w = l_1 \cdots l_n$, $w' = l'_1 \cdots l'_n$ be two strings in $S(n,r)$. Then: $w \lessdot w'$ if and only if w and w' differ between them only in one place k, where $l_k \lessdot l'_k$.*

Proof. $(\Rightarrow)$ By contradiction, we distinguish two cases:

Case 1. there exists a place k such that $l_k \neq l'_k$ and $l_k \not\lessdot l'_k$. Since by hypothesis $w \lessdot w'$, we have $w \sqsubset w'$; therefore it must be $l_k \prec l'_k$. This implies that

$$l_k \prec l \prec l'_k$$

for some $l \in A(n,r)$. Hence the string $w_l = l_1 \cdots l_{k-1} l_k l_{k+1} \cdots l_n$ is such that $w \sqsubset w_l \sqsubset w'$; against the hypothesis.

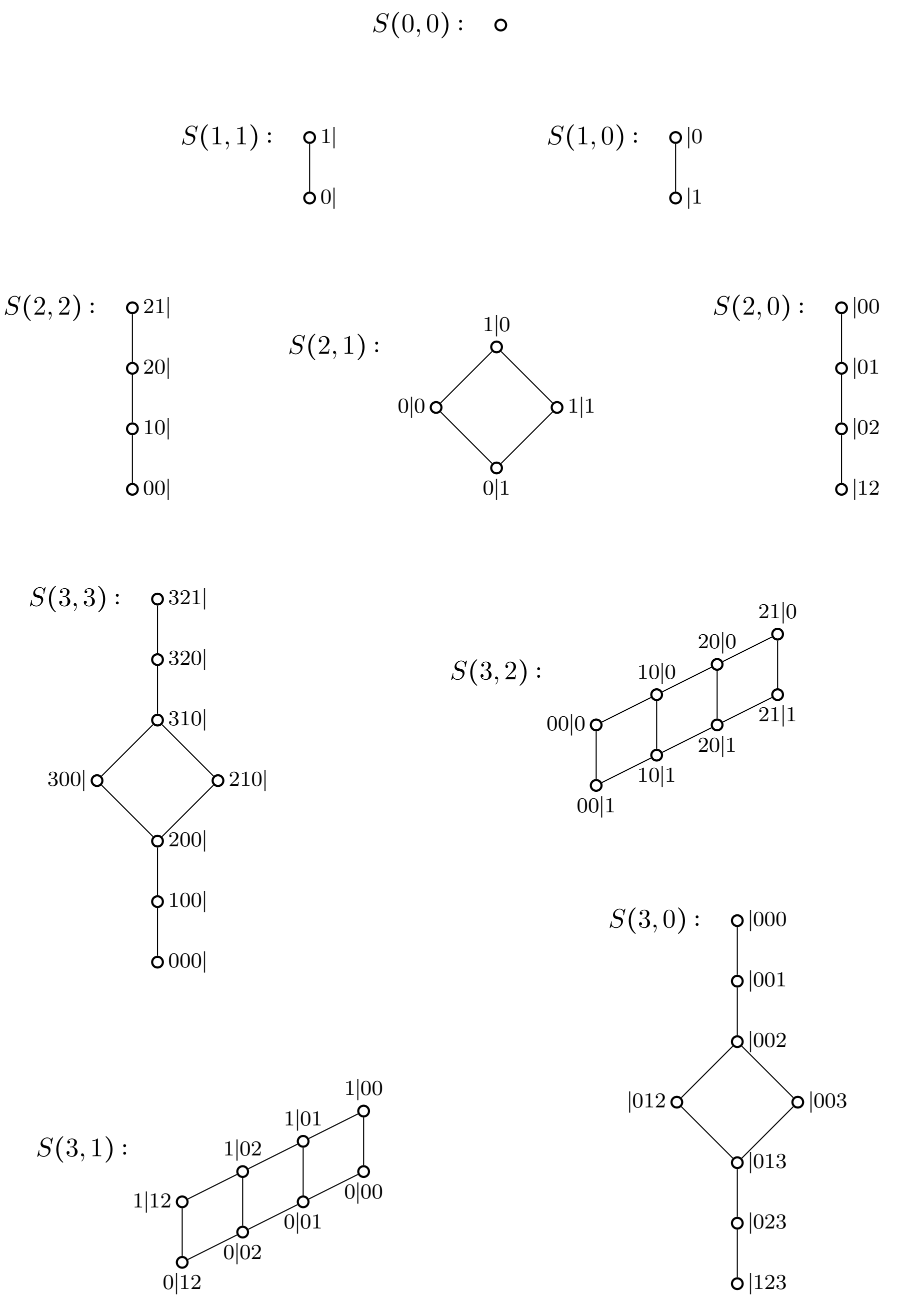


Figure 3.1: Hasse diagrams of the lattices $S(n,r)$ for the first values of n and r.

Case 2. there exist at least two places k and s, with $s > k$, such that $l_k \lessdot l'_k$ and $l_s \lessdot l'_s$. Then, if we consider the string:

$$u = l_1 \cdots l_{k-1} l_k l_{k+1} \cdots l_{s-1} l'_s l_{s+1} \cdots l_n,$$

it follows that $w \sqsubset u \sqsubset w'$; against the hypothesis.
($\Leftarrow$) From the hypothesis we have that $w \sqsubset w'$. Suppose that the thesis is false, then there exists an element $w'' \in S(n,r)$ such that $w \sqsubset w'' \sqsubset w'$. Let $w'' = l''_1 \cdots l''_n$. Then

$$l_i = l''_i = l'_i$$

if $i \neq k$, and $l_k \prec l''_k \prec l'_k$, and hence $l_k \nless l'_k$; against the hypothesis. □

We define now the function $\rho : S(n,r) \to \mathbb{N}_0$ as follows: If $w \in S(n,r)$ and we consider the symbols $i_1, \ldots, i_r, j_1, \ldots, j_{n-r}$ as non-negative integers (without $\tilde{}$ and $\bar{}$), then we set

$$\begin{aligned} \rho(w) &= i_1 + \cdots + i_r + |j_1 - 1| + \cdots + |j_{n-r} - (n-r)| \\ &= i_1 + \cdots + i_r + (1 - j_1) + \cdots + ((n-r) - j_{n-r}). \end{aligned}$$

Proposition 3.2.3. *The function ρ satisfies the following two properties:*

i) $\rho(\hat{0}) = 0$;

ii) *if $w, w' \in S(n,r)$ and $w \lessdot w'$, then $\rho(w') = \rho(w) + 1$.*

Proof. *i)* Since $\hat{0} = 0 \cdots 0 | 12 \cdots (n-r)$, the thesis follows by the definition of ρ.

ii) If $(w = l_1 \cdots l_n) \lessdot (w' = l'_1 \cdots l'_n)$, by Proposition 3.2.2 we have that $l_t \lessdot l'_t$ for some $t \in \{1, \ldots, n\}$ and $l_k = l'_k$ in all the other places $k \neq t$.

We distinguish different cases:

Case 1. Suppose that $1 \leq t \leq r$ and $l_t \succ 0^{\S}$. In this case we have that $w = l_1 \cdots l_{t-1} l_t l_{t+1} \cdots l_r | l_{r+1} \cdots l_n$ and $w' = l_1 \cdots l_{t-1} (l_t + 1) l_{t+1} \cdots l_r |\ l_{r+1} \cdots l_n$. Hence $\rho(w') = l_1 + \cdots + l_{t-1} + (l_t + 1) + l_{t+1} + \cdots + l_r + \sum_{k=1}^{n-r} (k - l_{r+k}) = \sum_{k=1}^{r} l_k + \sum_{k=1}^{n-r} (k - l_{r+k}) + 1 = \rho(w) + 1$.

Case 2. Suppose that $1 \leq t \leq r$ and $l_t = 0^{\S}$. In this case we have that $l'_t = 1$, hence $w = l_1 \cdots l_{t-1} 00 \cdots 0 | l_{r+1} \cdots l_n$ and $w' = l_1 \cdots l_{t-1} 10 \cdots 0 | l_{r+1} \cdots l_n$, from which it holds that $\rho(w') = \rho(w) + 1$.

Case 3. Suppose that $(r+1) \leq t \leq n$ and that $l_t = 0^{\S}$. In this case we have a contradiction because there does not exist an element l'_t in $\{0^{\S}, \overline{1}, \ldots, \overline{n-r}\}$ which covers $0^{\S}$.

Case 4. Suppose that $(r+1) \leq t \leq n$ and that $l_t \prec 0^{\S}$; since we consider l_t as an integer, it means that $1 \leq l_t \leq (n-r)$. In this case we have that $w = l_1 \cdots l_r | l_{r+1} \cdots l_{t-1} l_t l_{t+1} \cdots l_n$ and $w' = l_1 \cdots l_r | l_{r+1} \cdots l_{t-1} (l_t - 1) l_{t+1} \cdots l_n$. Therefore $\rho(w') = \sum_{i=1}^{r} l_i + \sum_{k=1, k \neq t-r}^{n-r} (k - l_{r+k}) + [(t-r) - (l_t - 1)] = \sum_{i=1}^{r} l_i + \sum_{k=1}^{n-r} (k - l_{r+k}) + 1 = \rho(w) + 1$.

□

Proposition 3.2.4. *$S(n,r)$ is a graded lattice having rank $R(n,r) = \binom{r+1}{2} + \binom{n-r+1}{2}$ and its rank function coincides with ρ.*

Proof. A finite distributive lattice is also graded, therefore $S(n,r)$ is also graded. In order to calculate the rank of $S(n,r)$ we need to determine a maximal chain and its length. We consider the following chain C in $S(n,r)$:

$\hat{1} = r(r-1)\cdots 21\vert 00\cdots 0$	1 string with r elements different from 0 on the left of |
$\vdots$	
$\left\{\begin{array}{l} r(r-1)(r-2)0\cdots 0\vert 00\cdots 0 \\ \vdots \\ r(r-1)10\cdots 0\vert 00\cdots 0 \end{array}\right.$	(r-2) strings with 3 elements different from 0 on the left of |
$\left\{\begin{array}{l} r(r-1)0\cdots 0\vert 00\cdots 0 \\ \vdots \\ r10\cdots 0\vert 00\cdots 0 \end{array}\right.$	(r-1) strings with 2 elements different from 0 on the left of |
$\left\{\begin{array}{l} r0\cdots 0\vert 00\cdots 0 \\ \vdots \\ 10\cdots 0\vert 00\cdots 0 \end{array}\right.$	r strings with 1 element different from 0 on the left of |
$00\cdots 0\vert 00\cdots 0$	
$\left\{\begin{array}{l} 00\cdots 0\vert 00\cdots 1 \\ \vdots \\ 00\cdots 0\vert 00\cdots (n-r) \end{array}\right.$	(n-r) strings with 1 element different from 0 on the right of |
$\vdots$	
$\left\{\begin{array}{l} 00\cdots 0\vert 001(4\cdots(n-r)) \\ 00\cdots 0\vert 002(4\cdots(n-r)) \\ 00\cdots 0\vert 003(4\cdots(n-r)) \end{array}\right.$	3 strings with (n-r-2) elements different from 0 on the right of |
$\left\{\begin{array}{l} 00\cdots 0\vert 01(34\cdots(n-r)) \\ 00\cdots 0\vert 02(34\cdots(n-r)) \end{array}\right.$	2 strings with (n-r-1) elements different from 0 on the right of |
$\hat{0} = 00\cdots 00\vert (12\cdots(n-r))$	1 string with (n-r) elements different from 0 on the right of |

Therefore C has exactly $(1+2+\cdots+(r-1)+r)+1+(1+2+\cdots+(n-r)) = \frac{r(r+1)}{2} + \frac{(n-r+1)(n-r)}{2} + 1 = \binom{r+1}{2} + \binom{n-r+1}{2} + 1$ elements and hence the length of C is $\binom{r+1}{2} + \binom{n-r+1}{2} + 1$. By Proposition 3.2.2, each element of the chain covers the previous one with respect to the order $\sqsubseteq$ in $S(n,r)$. Furthermore, C has minimal element $\hat{0}$ (the minimum of $S(n,r)$) and maximal element $\hat{1}$ (the maximum of $S(n,r)$), hence C is a maximal chain in $S(n,r)$. Finally, since $S(n,r)$ is a graded lattice of rank $R(n,r)$ and it has $\hat{0}$ as minimal element, its rank function has to be the unique function defined on $S(n,r)$ and with values in $\{0,1,\ldots R(n,r)\}$ which satisfies the $i)$ and $ii)$ of Proposition 3.2.3. Hence such a function coincides with ρ, by the uniqueness property. □

The following Proposition shows that w and w^c are symmetric in the Hasse diagram of $S(n,r)$.

Proposition 3.2.5. *If $w \in S(n,r)$, then $\rho(w) + \rho(w^c) = R(n,r)$.*

Proof. Let $w = i_1 \cdots i_r | j_1 \cdots j_{n-r}$ and $w^c = i_1^{'} \cdots i_r^{'} | j_1^{'} \cdots j_{n-r}^{'}$. Then $\rho(w) + \rho(w^c) = \sum_{k=1}^{r} i_k + \sum_{k=1}^{n-r} (k - j_k) + \sum_{k=1}^{r} i'_k + \sum_{k=1}^{n-r} (k - j_k^{'})$. By definition of w^c in $S(n,r)$, it follows that $\sum_{k=1}^{r} i_k + \sum_{k=1}^{r} i_k^{'} = \sum_{k=1}^{r} k$ and $\sum_{k=1}^{n-r} j_k + \sum_{k=1}^{n-r} j_k^{'} = \sum_{k=1}^{n-r} k$. Hence $\rho(w) + \rho(w^c) = \sum_{k=1}^{r} k + 2\sum_{k=1}^{n-r} k - \sum_{k=1}^{n-r} k = \sum_{k=1}^{r} k + \sum_{k=1}^{n-r} k = \binom{r+1}{2} + \binom{n-r+1}{2} = R(n,r)$. □

3.3 Involutive property of $S(n,r)$

In general, $(S(n,r), \sqsubseteq)$ is not a Boolean lattice. For example, if we take $w = 54210|012 \in S(8,5)$, it is easy to verify that there does not exist an element $w' \in S(8,5)$ such that $w \wedge w' = \hat{0}$ and $w \vee w' = \hat{1}$. In this section we will prove that the function $w \in S(n,r) \mapsto w^c \in S(n,r)$ is *involutive* with respect to the order $\sqsubseteq$, in the sense that if $w_1 \sqsubseteq w_2$, then $w_2^c \sqsubseteq w_1^c$.

Proposition 3.3.1. *Let $w, w' \in S(n,r)$ be such that $w' \lessdot w$ then $w^c \lessdot (w')^c$*

Proof. The proof is structured in four distinct cases.

Case 1. Let w and w' be distinct in the following way:

$$w = i_1 \cdots i_{r-s-1} 10 \cdots 0 | \cdots$$

$$w' = i_1 \cdots i_{r-s-1} 00 \cdots 0 | \cdots,$$

where $i_1 > \cdots > i_{r-s-1} > \tilde{1}$.
Consider now $(w^*)^\pi$ and $((w')^*)^\pi$: they are two elements of $\mathcal{P}(A(n,r) \setminus \{0^\S\})$. In $(w^*)^\pi$ there are (s) elements of $A(n,r) > 0^\S$ and in $((w')^*)^\pi$ there are $(s+1)$ elements of $A(n,r) > 0^\S$. Furthermore, $\tilde{1} \in ((w')^*)^\pi$ and $\tilde{1} \notin (w^*)^\pi$, hence the symmetric difference between $(w^*)^\pi$ and $((w')^*)^\pi$ is equal to $\{\tilde{1}\}$. From this, it follows that:

$$(w')^c = \overline{((w')^*)^\pi} = t_1 \cdots t_s 10 \cdots 0 | \cdots$$

$$w^c = \overline{(w^*)^\pi} = t_1 \cdots t_s 00 \cdots 0 | \cdots,$$

where $\{t_1, \ldots, t_s, \tilde{1}\} = \{i_1, \ldots, i_{r-s-1}, \overline{1}, \ldots, \overline{n-r}\}^c$ in $A(n,r) \setminus \{0^\S\}$ and $t_1 > \cdots > t_s > \tilde{1}$. By Proposition 3.2.2, it follows that $(w')^c$ covers w^c.

Case 2. Adaptation of Case 1 to the elements on the right of |.

Case 3. Let k be the index in which w and w' are distinct,$1 \le k \le r$, then:

$$w = i_1 \cdots i_{k-1} (i_k + 1) i_{k+1} \cdots i_p 0 \cdots 0 | \cdots$$

$$w' = i_1 \cdots i_{k-1} i_k i_{k+1} \cdots i_p 0 \cdots 0 | \cdots,$$

with $i_1 > \cdots > i_{k-1} > i_k + 1 > i_k > i_{k+1} > \cdots > i_p > 0^\S$.
Then, in $(w^*)^\pi$ there are exactly $q = (r-p)$ elements $> 0^\S$ and in $((w')^*)^\pi$ there are also $q = (r-p)$ elements $> 0^\S$. Moreover, it follows that $i_k \in (w^*)^\pi \setminus ((w')^*)^\pi$ and $i_{k+1} \in ((w')^*)^\pi \setminus (w^*)^\pi$, hence the symmetric difference between $(w^*)^\pi$ and $((w')^*)^\pi$ is equal to $\{i_k, i_{k+1}\}$. From this, it follows that:

$$(w')^c = \overline{((w')^*)^\pi} : t_1 \cdots t_{l-1} (i_k + 1) t_{l+1} \cdots t_q 0 \cdots 0 | \cdots$$

$$w^c = \overline{(w^*)^\pi} : t_1 \cdots t_{m-1} i_k t_{m+1} \cdots t_q 0 \cdots 0 | \cdots,$$

where $i_k + 1$ appears in the l–th place $(1 \le l \le r)$ in $(w')^c$ and i_k appears in the m–th place $(1 \le m \le r)$ in w^c, with

$$\{t_1, \dots t_{l-1}, t_{l+1}, \dots, t_q\} = \{t_1, \dots t_{m-1}, t_{m+1}, \dots, t_q\}, \tag{3.2}$$

where $\{t_1, \dots, t_q\} = \{i_1, \dots i_k, i_k + 1, i_{k+1}, \dots, i_p, \overline{1}, \dots \overline{n-r}\}^\pi$ in $A(n, r) \smallsetminus \{0^\S\}$. We prove now that the place l coincides with the place m. Let $t \in \{t_{l+1}, \dots, t_q\}$ and suppose by contradiction that $t \notin \{t_{m+1}, \dots, t_q\}$. By (3.2) it follows that $t \in \{t_1, \dots, t_{m-1}\}$, hence we will have $i_k + 1 > t$ and $t > i_k$, and hence $i_k + 1 > t > i_k$ in $A(n, r)$ and this contradicts $i_k \lessdot (i_k + 1)$.
Let now $t \in \{t_1, \dots, t_{m-1}\}$. Suppose by contradiction that $t \notin \{t_1, \dots, t_{l-1}\}$. By (3.2) it follows that $t \in \{t_{l+1}, \dots, t_q\}$, hence we will have $t > i_k$ and $i_k + 1 > t$, by which $i_k + 1 > t > i_k$, and this contradicts $i_k \vdash i_k + 1$. By (3.2) hence follows that $m = l$, and this proves that $w^c \lessdot (w')^c$.

Case 4. Analogously to Case 3 with k such that $r + 1 \le k \le n$. □

Proposition 3.3.2. *If $w, w' \in S(n, r)$ are such that $w' \sqsubseteq w$, then $w^c \sqsubseteq (w')^c$.*

Proof. It is enough to consider a sequence of elements $w_0, w_1, \dots, w_n$ such that $w' = w_0 \sqsubseteq w_1 \sqsubseteq \cdots \sqsubseteq w_{n-1} \sqsubseteq w_n = w$ where w_i covers w_{i-1} for $i = 1, \dots, n$ and apply Proposition 3.3.1 to $w_{i-1} \lessdot w_i$. □

In general, a poset $\mathbb{P} = (P, \le)$ is called an *involution poset* if there exists a map $' : P \to P$ such that (i) $(x')' = x$ and (ii) $x \le y$, then $y' \le x'$ for all $x, y \in P$. Hence by Proposition 3.3.1 and 3.3.2, $(S(n, r), \sqsubseteq, ^c, \hat{0}, \hat{1})$ is an involutive and distributive bounded lattice. If $w = i_1 \cdots i_r | j_1 \cdots j_{n-r}$ is an element of $S(n, r)$, with $0 \le r \le n$, we can also consider the symbols $i_1, \dots i_r, j_1, \dots, j_{n-r}$ as elements in the alphabet $A(n, n-r)$, where $j_1, \dots, j_{n-r} \in \{\widetilde{n-r} > \cdots > \tilde{1} > 0^\S\}$ and $i_1, \dots, i_r \in \{0^\S > \overline{1} > \cdots > \overline{r}\}$; in such case we will set $w^t = j_{n-r} \cdots j_1 | i_r \cdots i_1$. Then it holds that the map $w \in S(n, r) \mapsto w^t \in S(n, n-r)$ is bijective and is such that

$$w \sqsubseteq w' \text{ in } S(n, r) \iff (w')^t \sqsubseteq w^t \text{ in } S(n, n-r), \tag{3.3}$$

since $(w^t)^t = w$, for each $w \in S(n, r)$. Also the map $w \in S(n, r) \mapsto w^c \in S(n, r)$ is bijective, and since $(w^c)^c = w$, by Proposition 3.3.2, it follows that

$$w \sqsubseteq w' \text{ in } S(n, r) \iff (w')^c \sqsubseteq w^c \text{ in } S(n, r). \tag{3.4}$$

Therefore it holds the following isomorphism of lattices:

Proposition 3.3.3. *If $0 \le r \le n$, then $S(n, r) \cong S(n, n-r)$.*

Proof. It is enough to consider the map $\varphi : S(n, r) \to S(n, n-r)$ defined by $\varphi(w) = (w^t)^c$. Since the map φ is the composition of the map $w \in S(n, r) \mapsto w^t \in S(n, n-r)$ with the map $u \in S(n, n-r) \mapsto u^c \in S(n, n-r)$, it follows that that φ is bijective. Furthermore, by (3.3) and (3.4), it holds that

$$w \sqsubseteq w' \text{ in } S(n, r) \iff \varphi(w) \sqsubseteq \varphi(w') \text{ in } S(n, n-r).$$

Hence φ is an isomorphism of lattices. □

Example 3.3.4. $S(3,1) \cong S(3,2)$ and for example $\varphi(0|01) = ((0|01)^t)^c = (10|0)^c = (20|1)$, or $\varphi(1|02) = ((1|02)^t)^c = (20|1)^c = (10|0)$, see the Hasse diagrams in Figure 3.1.. □

3.4 A Recursive Formula for the Number

In this section we give a recursive formula which counts the number of elements in $S(n,r)$ having fixed rank. At first we show that $S(n,r)$ can be seen as a translate union of two copies of $S(n-1,r)$ if $0 \le r < n$ and of $S(n-1,n-1)$ if $r = n$.

Proposition 3.4.1. *Let $n \ge 1$ and $r \in \mathbb{N}$ such that $0 \le r \le n$. Then there exist two disjoint sublattices $S_1(n,r)$, $S_2(n,r)$ of $S(n,r)$ such that $S(n,r) = S_1(n,r) \cup S_2(n,r)$, where:*

- *i) $S_i(n,r) \cong S(n-1,r)$ for $i = 1,2$, if $0 \le r < n$;*
- *ii) $S_i(n,n) \cong S(n-1,n-1)$ for $i = 1,2$, if $r = n$.*

Proof. We distinguish two cases:

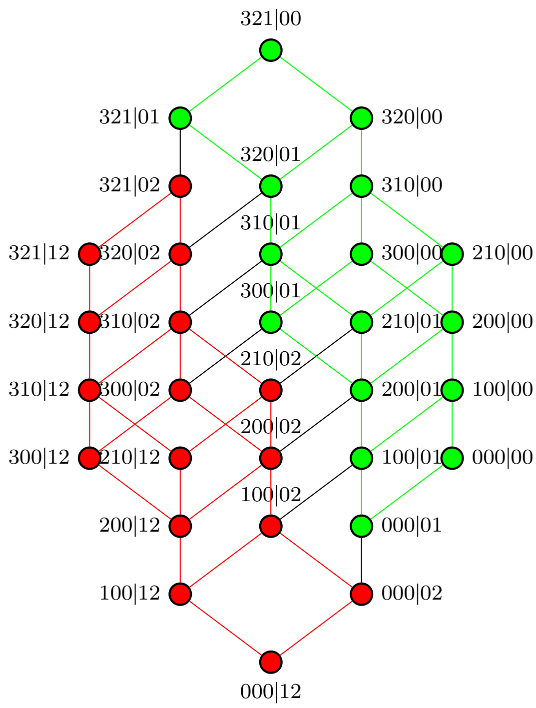
i) $0 \le r < n$; we denote by $S_1(n,r)$ the subset of $S(n,r)$ of all the strings w of the form $w = i_1 \cdots i_r | j_1 \cdots j_{n-1-r}(n-r)$, with $j_1 \cdots j_{n-1-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$; moreover, we denote by $S_2(n,r)$ the subset of $S(n,r)$ of all the strings w of the form $w = i_1 \cdots i_r | 0 j_2 \cdots j_{n-r}$, with $j_2 \cdots j_{n-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$.

It is clear that $S(n,r)$ is a disjoint union of $S_1(n,r)$ and $S_2(n,r)$. We prove now that $S_i(n,r) \cong S(n-1,r)$ for $i = 1,2$. Let $i = 1$ (the case $i = 2$ is analogous). It is obvious that there exists a bijective correspondence between $S_1(n,r)$ and $S(n-1,r)$. Furthermore, if $w, w' \in S_1(n,r)$ are such that $w = i_1 \cdots i_r | j_1, \cdots j_{n-r-1}(n-r)$, $w' = i_1' \cdots i_r' | j_1' \cdots j_{n-r-1}'(n-r)$, it follows that $w \sqsubseteq w'$ (with respect to the order on $S(n,r)$) if and only if $i_1 \cdots i_r | j_1 \cdots j_{n-r-1} \sqsubseteq i_1' \cdots i_r' | j_1' \cdots j_{n-r-1}'$ (with respect to the order in $S(n-1,r)$). Hence $S_1(n,r)$ is isomorphic to $S((n-1),r)$.
Finally, since the order on $S(n,r)$ is component by component, it follows that each $S_i(n,r)$ (for $i = 1,2$) is a sublattice of $S(n,r)$.

ii) $r = n$; by *i*), there exist two disjoint sublattices $S_1(n,0)$, $S_2(n,0)$, of $S(n,0)$ such that $S(n,0) = S_1(n,0) \cup S_2(n,0)$, with $S_i(n,0) \cong S_i(n-1,0)$, for $i = 1,2$. By Proposition 3.3.3, it follows that $S(n,n) \cong S(n,0)$, therefore there also exist two disjoint sublattices $S_1(n,n)$, $S_2(n,n)$, of $S(n,n)$ such that $S(n,n) = S_1(n,n) \cup S_2(n,n)$, where $S_i(n,n) \cong S_i(n,0) \cong S(n-1,0) \cong S(n-1,n-1)$, for $i = 1,2$, again by Proposition 3.3.3. □

If $n \ge 1$, the element of minimal rank of the sublattice $S_2(n,r)$ is obviuosly $\hat{w} = 0 \cdots 0 | 012 \cdots (n-r-1)$. This element has rank 0 as element of $(S_2(n,r), \sqsubseteq)$, but in $S(n,r)$ has rank given by $\rho(\hat{w}) = (1-0)+(2-1)+(3-2)+\cdots+((n-r)-(n-r-1)) = n-r$. Therefore we can visualize $S_2(n,r)$ (in the Hasse diagram of $S(n,r)$) as an upper-translation of the sublattice $S_1(n,r)$, of height $(n-r)$.

Example 3.4.2. For example, this is the Hasse diagram of $S(5,3)$ as a translate union of $S_1(5,3) \cong S(4,3)$ (red lattice) and of $S_2(5,3) \cong S(4,3)$ (green lattice).



□

Given the lattice $S(n,r)$, for each k such that $0 \leq k \leq R(n,r)$, we denote with $s(n,r,k)$ the number of elements of $S(n,r)$ with rank k. It holds the following ricorsive formula for $s(n,r,k)$:

Proposition 3.4.3. *Let $n \geq 1$. If $r \in \mathbb{N}$ is such that $0 \leq r < n$, then*

$$s(n,r,k) = \begin{cases} s(n-1,r,k) & \text{if} \quad 0 \leq k < (n-r) \\ s(n-1,r,k) + & \\ \quad s(n-1,r,k-(n-r)) & \text{if} \quad (n-r) \leq k \leq R(n-1,r) \\ s(n-1,r,k-(n-r)) & \text{if} \quad R(n-1,r) < k \leq R(n,r) \end{cases}$$

If $r = n$, then $s(n,n,k) = s(n,0,k)$.

Proof. We distinguish three cases:

Case 1. Let k be such that $0 \leq k < (n-r)$. By what we have asserted before, the element $\hat{w}$ (i.e. the minimum of $S_2(n,r)$) has rank $(n-r)$ in $S(n,r)$, hence by Proposition 3.4.1, it follows that $s(n,r,k)$ coincides with the number of elements of rank k in $S_1(n,r)$ and since $S_1(n,r) \cong S(n-1,r)$, it follows that $s(n,r,k) = s(n-1,r,k)$.

Case 2. Let k be such that $(n-r) \leq k \leq R(n-1,r)$. In this case, the number of elements of rank k in $S(n,r)$ coincides with the sum of the number of elements of rank k in $S_1(n,r)$ and of the number of elements of rank $[k-(n-r)]$ in $S_2(n,r)$. Since $S_1(n,r) \cong S_2(n,r) \cong S(n-1,r)$, it follows that $s(n,r,k) = s(n-1,r,k) + s(n-1,r,k-(n-r))$.

Case 3. Let k be such that $R(n-1,r) < k \leq R(n,r)$. In this case $s(n,r,k)$ coincides with the number of elements of rank $[k-(n-r)]$ in $S_2(n,r)$, and since $S_2(n,r) \cong S(n-1,r)$ it follows that $s(n,r,k) = s(n-1,r,k-(n-r))$.

Finally, if $r = n$, the last equality follows from the isomorphism $S(n,n) \cong S(n,0)$. □

By the recursive formula stated in Proposition 3.4.3, the first values of $s(n,r,k)$ are given by:

$$
\begin{array}{rcl}
s(0,0,0) & = & 1 \\
s(1,0,0) & = & s(0,0,0) = 1 \\
s(1,0,1) & = & s(0,0,0) = 1 \\
s(1,1,0) & = & s(1,0,0) = 1 \\
s(1,1,1) & = & s(1,0,1) = 1 \\
s(2,0,0) & = & 1 \\
s(2,0,1) & = & s(1,0,1) = 1 \\
s(2,0,2) & = & s(1,0,0) = 1 \\
s(2,0,3) & = & s(1,0,1) = 1 \\
s(2,1,0) & = & s(1,1,0) = 1 \\
s(2,1,1) & = & s(1,1,1) + s(1,1,0) = 2 \\
s(2,1,2) & = & s(1,1,1) = 1 \\
s(2,2,0) & = & s(2,0,0) = 1 \\
s(2,2,1) & = & s(2,0,1) = 1 \\
s(2,2,2) & = & s(2,0,2) = 1 \\
s(2,2,3) & = & s(2,0,3) = 1
\end{array}
$$

The rank generating function of $P \times Q$ leads to the following Cauchy-type formula for $s(n,r,k)$.

Proposition 3.4.4. *If* $0 \leq r \leq n$ *and* $0 \leq k \leq R(n,r)$ *then* $s(n,r,k) = \sum_{i=0}^{k} s(r,r,i) \cdot s(n-r,n-r,k-i)$.

Proof. The rank generating function of $S(n,r)$ is

$$F(S(n,r),t) = \sum_{k=0}^{R(n,r)} s(n,r,k)t^k.$$

By Proposition 3.3.3 it follows that

$$S(n,r) \cong S(r,r) \times S(n-r,0) \cong S(r,r) \times S(n-r,n-r).$$

Hence

$$F(S(n,r),t) = F(S(r,r) \times S(n-r,n-r),t) = F(S(r,r,t)) \cdot F(S(n-r,n-r,t)).$$

Then the thesis follows by the following equality:

$$
\begin{array}{rcl}
F(S(n,r),t) & = & \left(\sum_{l=0}^{R(r,r)} s(r,r,l)t^l\right) \cdot \left(\sum_{j=0}^{R(n-r,n-r)} s(n-r,n-r,j)t^j\right) \\
& = & \sum_{k=0}^{R(r,r)+R(n-r,n-r)} \sum_{i=0}^{k} s(r,r,i)s(r,r,k-i)t^k \\
& = & \sum_{k=0}^{R(n,r)} \sum_{i=0}^{k} s(r,r,i)s(n-r,n-r,k-i)t^k.
\end{array}
$$

□

The following Proposition shows a symmetric property of $S(n.r)$.

Proposition 3.4.5. *If* $0 \le r \le n$ *and* $k = R(n,r)$*, then* $s(n,r,i) = s(n,r,k-i)$ *for* $0 \le i \le k$.

Proof. We recall that $S_l(n,r)$ is the set of elements of $S(n,r)$ with rank l for each $0 \le l \le k$. It is enough to consider the map $f : S_i(n,r) \to S_{k-i}(n,r)$ defined by $f(w) = w^c$.
At first we observe that f is well defined, because if $w \in S_i(n,r)$ then $\rho(w) = i$ and by Proposition 3.2.5 $\rho(w^c) = k - i$, therefore $w^c \in S_{k-i}(n,r)$. The map f is injective, because by $(w^c)^c = w$ it follows that $w_1^c = w_2^c \Rightarrow w_1 = w_2$. To show that f is also onto, we take $v \in S_{k-i}(n,r)$ and $w = v^c$. Since $\rho(v) = k - i$, by Proposition 3.2.5 we have that $k = \rho(v^c) + \rho(v) = \rho(w) + \rho(v) = \rho(w) + (k - i)$, hence $\rho(v^c) = i$, i.e. $w \in S_i(n,r)$ and $f(w) = w^c = (v^c)^c = v$, so f is onto and hence f is bijective. □

Chapter 4

Solutions of problems on non-negative subset sums

Using results of [11] and [12], Chiaselotti, Marino and Nardi solved some combinatorial extremal sum problems (section 4.3). For one problem they provided a partial result, showing that the answer is affirmative for a related poset-formulation (section 4.4). Afterwards Engel and Nardi solved completely the problem (section 4.5).

4.1 Relation between weight functions and maps on $S(n,r)$

Our aim in this section is to connect the study of particular weight functions and of the related extremal problems to some Boolean maps on $S(n,r)$.

In the sequel n and r will denote two fixed integers such that $1 \le r \le n$. We set $I(n,r) = \{\tilde{r}, \dots, \tilde{1}, \overline{1}, \dots, \overline{n-r}\}$ and we consider $I(n,r)$ as a n-set in which we simply have marked the difference between the first r formal symbols $\tilde{r}, \dots, \tilde{1}$ and the remaining $(n-r)$ formal symbols $\overline{1}, \dots, \overline{n-r}$, so we can write $A(n,r) = I(n,r) \cup \{0^{\S}\}$.

Definition 4.1.1. The function $f : A(n,r) \to \mathbb{R}$ is a (n,r)–weight function such that:

$$f(\tilde{r}) \ge \dots \ge f(\tilde{1}) \ge f(0^{\S}) = 0 > f(\overline{1}) \ge \dots \ge f(\overline{n-r}), \tag{4.1}$$

$$f(\tilde{1}) + \dots + f(\tilde{r}) + f(\overline{1}) + \dots + f(\overline{n-r}) \ge 0. \tag{4.2}$$

We call $WF(n,r)$ the set of the (n,r)–weight functions and if $f \in WF(n,r)$ we set $\alpha(f) := |\{Y \subseteq I(n,r) : \sum_{y \in Y} f(y) \ge 0\}|$. If $f \in WF(n,r)$, the *sum function* $\Sigma_f : S(n,r) \to \mathbb{R}$ induced by f on $S(n,r)$ is the function that associates to $w = i_r \cdots i_1 | j_1 \cdots j_{n-r} \in S(n,r)$ the real number $\Sigma_f(w) = f(i_1) + \dots + f(i_r) + f(j_1) + \dots + f(j_{n-r})$.

Proposition 4.1.2. *If f is a (n,r)–weight function and if $w, w' \in S(n,r)$ are such that $w \sqsubseteq w'$, then $\Sigma_f(w) \le \Sigma_f(w')$.*

Proof. If $w = i_1 \cdots i_r \mid j_1 \cdots j_{n-r} \sqsubseteq w' = i_1^{'} \cdots i_r^{'} | j_1^{'} \cdots j_{n-r}^{'}$, then we have that $i_1 \preceq i_1^{'}$, $\cdots$, $i_r \preceq i_r^{'}$, $j_1 \preceq j_1^{'}$, $\cdots$ $j_{n-r} \preceq j_{n-r}^{'}$; hence, since f is increasing on $A(n,r)$, the assertion follows immediately by Definition of sum function Σ_f. □

Proposition 4.1.3. *If f is a (n,r)–weight function and if $w \in S(n,r)$ is such that $\Sigma_f(w) < 0$, then $\Sigma_f(w^c) > 0$.*

Proof. By Definition of the two binary operations $\sqcap$ and $\sqcup$ and of the complement operation ^{c} on $S(n,r)$, we have that

$$w \sqcup w^c = 12 \cdots r | 1 \cdots (n-r) \quad \text{and} \quad w \sqcap w^c = 00 \cdots 0 | 0 \cdots 0.$$

Hence, by Definition of Σ_f and since $f(0^{\S}) = 0$, we have that

$$\Sigma_f(w) + \Sigma_f(w^c) = \Sigma_f(w \sqcup w^c) = f(\tilde{1}) + \cdots + f(\tilde{r}) + f(\overline{1}) + \cdots + f(\overline{n-r}) \geq 0,$$

by (4.2). Hence, if $\Sigma_f(w) < 0$, we will have that $\Sigma_f(w^c) > 0$. □

We denote by **2** the Boolean lattice composed of a chain with two elements that we denote N (the minimum element) and P (the maximum element). A *Boolean map* (briefly BM) *on* $S(n,r)$ is a map $A : dom(A) \subseteq S(n,r) \to \mathbf{2}$, in particular if $dom(A) = S(n,r)$ we also say that A is a *Boolean total map* (briefly BTM) *on* $S(n,r)$. If A is BM on $S(n,r)$, we set $S_A^+(n,r) = \{w \in dom(A) : A(w) = P\}$.

We can associate to $f \in WF(n,r)$ the map $A_f : S(n,r) \to \mathbf{2}$ setting

$$A_f(w) = \begin{cases} P & \text{if} \quad \Sigma_f(w) \geq 0 \\ N & \text{if} \quad \Sigma_f(w) < 0 \end{cases} \tag{4.3}$$

Let us note that $|S^+_{A_f}(n,r)| = |\{w \in S(n,r) : A_f(w) = P\}| = \alpha(f)$.
It is easy to observe that the map A_f has the following properties:

(i) A_f is order-preserving

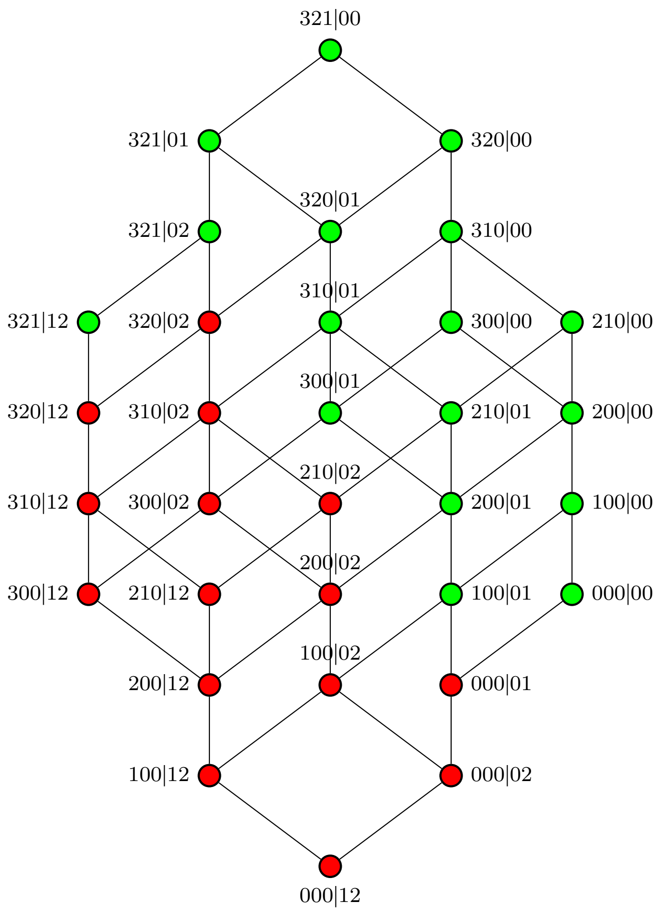
(ii) If $w \in S(n,r)$ is such that $A_f(w) = N$, then $A_f(w^c) = P$

(iii) $A_f(0 \cdots 0 | 0 \cdots 0) = P$, $A(0 \cdots 0 | 0 \cdots 01) = N$ and $A_f(r \cdots 21 | 12 \cdots (n-r)) = P$

Example 4.1.4. Let f be the following $(5,3)$-weight function :

$$f : \begin{array}{ccccc} \tilde{3} & \tilde{2} & \tilde{1} & \overline{1} & \overline{2} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0.9 & -0.8 & -2.1 \end{array}$$

We represent the map A_f defined on $S(5,3)$ by using the Hasse diagram of this lattice, on which we color green the nodes where the map A_f assumes value P and red the nodes where it assumes value N:



□

Note that if we have a generic Boolean total map $A : S(n, r) \to \mathbf{2}$ which has the properties (i), (ii) and (iii) of A_f, i.e. the following:

(BM1) A is order-preserving

(BM2) If $w \in S(n, r)$ is such that $A(w) = N$, then $A(w^c) = P$

(BM3) $A(0 \cdots 0|0 \cdots 0) = P$, $A(0 \cdots 0|0 \cdots 01) = N$ and $A(r \cdots 21|12 \cdots (n - r)) = P$

In general there does not exist a function $f \in WF(n, r)$ such that $A_f = A$ (see Example 4.1.5 [12]),

We denote by $\mathcal{W}_+(S(n, r), \mathbf{2})$ the set of all the maps $A : S(n, r) \to \mathbf{2}$ which satisfy (BM1) and (BM2) and by $\mathcal{W}_+(n, r)$ the subset of all the maps in $\mathcal{W}_+(S(n, r), \mathbf{2})$ which satisfy also (BM3).

Example 4.1.5. Let us consider the map $A : S(6, 3) \to \mathbf{2}$ reported in Figure 4.1, where green nodes are P and red nodes are N.

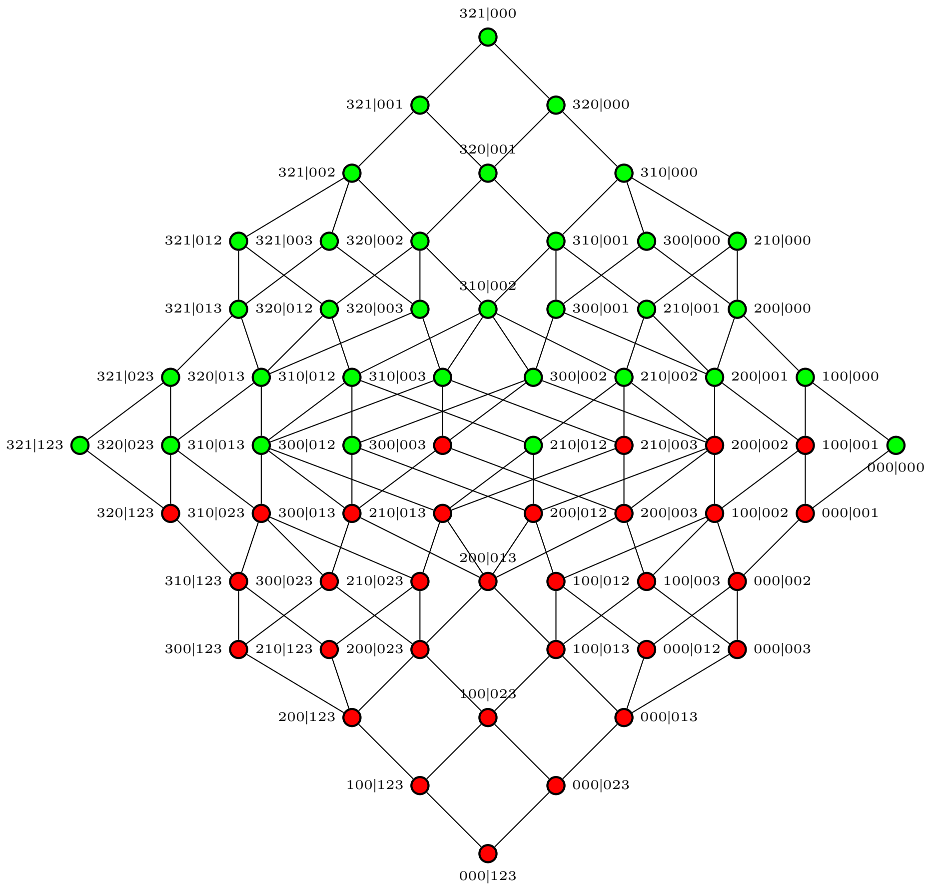


Figure 4.1: Map $A : S(6,3) \to \mathbf{2}$.

It is easy to observe that $A \in \mathcal{W}_+(6,3)$, but for this Boolean total map there does not exist an $f \in WF(n,r)$ such that $A = A_f$. To show that for previous map does not exist an $f \in WF(n,r)$ such that $A = A_f$ we need to introduce new Definitions and results. □

4.2 Relation between weight functions and (n,r)-systems

The purpose of this section is to show that for the Boolean map in Example 4.1.5 there does not exist an $f \in WF(n,r)$ such that $A = A_f$ [12].

Let us suppose that we have r real variables $x_{\tilde{r}}, \dots, x_{\tilde{1}}$ and other $(n-r)$ real variables $y_{\overline{1}}, \dots, y_{\overline{n-r}}$. In the sequel, to simplify the notations, we write simply x_i instead of $x_{\tilde{i}}$ and y_j instead of $y_{\overline{j}}$. However, it is important to mark that the index i in x_i corresponds to the symbol $\tilde{i}$, while the index i in y_i corresponds to the symbol $\overline{i}$ and that $\tilde{i} \neq \overline{i}$. We call (n,r)-*system* of *size* p a system $\mathcal{S}$ of linear inequalities having the following form:

$$\mathcal{S} : \begin{cases} x_r \geq \cdots \geq x_1 \geq 0 > y_1 \geq \cdots \geq y_{n-r} \\ \sum_{i \in A_1} t_i \geq 0 \ \text{(or } < 0) \\ \sum_{i \in A_2} t_i \geq 0 \ \text{(or } < 0) \\ \cdots \\ \cdots \\ \sum_{i \in A_p} t_i \geq 0 \ \text{(or } < 0) \end{cases} \tag{4.4}$$

where $A_1, \dots, A_p$ are non-empty and different subsets of $I(n, r)$, all different from the singletons of $I(n, r)$; moreover $t_i = x_i$ if $i \in \{\tilde{r}, \dots, \tilde{1}\}$ and $t_i = y_i$ if $i \in \{\overline{1}, \dots, \overline{n-r}\}$. Formally we set $\sum_{i \in \varnothing} t_i = 0$. When the subsets $A_1, \dots, A_p$ coincide with all the possible subsets of $I(n, r)$ different from the singletons and from the empty set, we say that the (n, r)-system (4.4) is *total*. Furthermore, when in (4.4) appears the inequality

$$x_r + \cdots + x_1 + y_1 + \cdots + y_{n-r} \geq 0 \tag{4.5}$$

we say that it is a (n, r)-*weighted* system. A (n, r)-total weighted system of type (4.4) can be identified with a Boolean total map A defined on the lattice $(S(n, r), \sqsubseteq)$. A subsystem of (4.4) which is also equivalent to it, can be identified with a particular restriction of the map A which represents (4.4): such a restriction of A is called *core* of A.

Let $\mathcal{S}$, $\mathcal{S}'$ be two (n, r)-systems: we say that they are *equals* (in symbols $\mathcal{S} = \mathcal{S}'$) if they have exactly the same inequalities, otherwise we say that they are *different* (in symbols $\mathcal{S} \neq \mathcal{S}'$). If they are both compatibles (i.e. they have solutions) and equivalents (i.e. they have the same solutions) we shall write $\mathcal{S} \equiv \mathcal{S}'$. We denote by $Syst(n, r)$ $[TSyst(n, r)]$ the set of all the (n, r)-systems [(n, r)-total systems] and by $WCSyst(n, r)$ $[WCTSyst(n, r)]$ the (n, r)-compatible [total] weighted systems. Let us consider a (n, r)-system $\mathcal{S}$ as in (4.4). Since there is an obvious bijection between the power set $\mathcal{P}(I(n, r))$ and $S(n, r)$, all the subsets $A_1, \dots, A_p$ in (4.4) can be identified with strings of $S(n, r)$, that we denote by $w_1, \dots, w_p$.

We denote by $(S(n, r) \rightsquigarrow \mathbf{2})$ the poset of the Boolean partial maps on $S(n, r)$ ([22]). We set now $\xi_r = r0 \cdots 0|0 \cdots 0, \dots, \xi_1 = 10 \cdots 0|0 \cdots 0, \xi_0 = 00 \cdots 0|0 \cdots 0 \eta_1 = 0 \cdots 0|0 \cdots 01, \dots, \eta_{n-r} = 0 \cdots 0|0 \cdots 0(n-r)$, $\Omega_{\mathcal{S}} = \{w_1, \dots, w_p, \xi_r, \dots, \xi_1, \xi_0, \eta_1, \dots, \eta_{n-r}\}$.

Definition 4.2.1. Let $\mathcal{S} \in Syst(n, r)$. A $\mathcal{S}$-Boolean partial map ($\mathcal{S}$–BPM) $A_{\mathcal{S}} : \Omega_{\mathcal{S}} \subseteq S(n, r) \to \mathbf{2}$ is defined as follows: for $j \in \{1, \dots, p\}$

$$A_{\mathcal{S}}(w_j) = \begin{cases} P & \text{if} \quad \sum_{i \in A_j} t_i \geq 0 \\ N & \text{if} \quad \sum_{i \in A_j} t_i < 0 \end{cases}$$

$A_{\mathcal{S}}(\xi_0) = A_{\mathcal{S}}(\xi_1) = \cdots = A_{\mathcal{S}}(\xi_r) = P$ and $A_{\mathcal{S}}(\eta_1) = \cdots = A_{\mathcal{S}}(\eta_{n-r}) = N$.

Definition 4.2.2. If $\mathcal{S}, \mathcal{S}' \in Syst(n, r)$, we set $\mathcal{S} \lesssim \mathcal{S}'$ if $\mathcal{S}$ is a subsystem of $\mathcal{S}'$.

This obviously defines a partial order $\lesssim$ on $Syst(n, r)$. We denote with $\mathcal{B}(n, r)$ the sub-poset of all the Boolean partial maps $A \in (S(n, r) \rightsquigarrow \mathbf{2})$ such that $\xi_r, \dots, \xi_1, \xi_0, \eta_1, \dots, \eta_{n-r} \in dom(A)$ and $A(\xi_0) = A(\xi_1) = \cdots = A(\xi_r) = P$, $A(\eta_1) = \cdots = A(\eta_{n-r}) = N$ and with $\mathcal{BT}(n, r)$ the subset of all the total maps of $\mathcal{B}(n, r)$. Then the map $\chi : Syst(n, r) \to \mathcal{B}(n, r)$ such that $\chi(\mathcal{S}) = A_{\mathcal{S}}$, for each $\mathcal{S} \in Syst(n, r)$, is an isomorphism of posets. We denote by $\tau : \mathcal{B}(n, r) \to Syst(n, r)$ the inverse of χ and we set $\tau(A) = \mathcal{S}_A$ if $A \in \mathcal{B}(n, r)$. Obviously the restriction of

χ to $\mathcal{BT}(n,r)$ defines an isomorphism between $\mathcal{BT}(n,r)$ and $TSyst(n,r)$ and we continue respectively to denote with χ and τ this isomorphism and its inverse.

We say that a (n,r)-weight function f is a *solution* of the system (4.4) if the assignment

$$x_r = f(\tilde{r}),\ldots,x_1 = f(\tilde{1}),\ldots y_1 = f(\overline{1}),\ldots,y_{n-r} = f(\overline{n-r}) \tag{4.6}$$

provides a solution of (4.4).

If $f \in WF(n,r)$, we denote by $\mathcal{S}_f$ the (n,r)-compatible total system having f as one of its solutions. If $f \in WF(n,r)$ the map $A_f : S(n,r) \to \mathbf{2}$ defined as (4.3) then it is obvious that $A_{\mathcal{S}_f} = A_f$.

Definition 4.2.3. Let $\mathcal{H}$ be a family of maps of $\mathcal{BT}(n,r)$ and let $A \in \mathcal{H}$; we say that a Boolean partial map $B \in \mathcal{B}(n,r)$ is a core for A if $A_{|W} = B$ (where $W = dom(B)$) and if $A' \in \mathcal{H}$ is such that $A'_{|W} = B$, then $A = A'$. We simply say that B is a core if it is a core for some $A \in \mathcal{H}$.

The main result of Bisi-Chiaselotti [12] is the **local criterion**:
criterion *Let $\mathcal{H}$ be a family of maps of $\mathcal{BT}(n,r)$ such that $\chi(WCTSyst(n,r)) \subseteq \mathcal{H}$ and let $A \in \mathcal{H}$. Let B denote a core of A. Then, $\mathcal{S}_A$ is compatible if and only if $\mathcal{S}_B \in WCSyst(n,r)$ and, in this case, if $f \in WF(n,r)$ is a solution of $\mathcal{S}_B$, it is also a solution of $\mathcal{S}_A$.*

The authors use this criterion 'from global to local', to decide if a map A that we choose in a special family $\mathcal{H}$ of Boolean total maps of $\mathcal{BT}(n,r)$ determines a (n,r)-compatible total system. In this case the previous criterion are useful if we know, for each given map $A \in \mathcal{H}$ a core that is 'sufficiently' small. The main results of the paper [12] is building some appropriate families $\mathcal{H}$ of Boolean total maps that satisfy the previous local criterion and such that for each $A \in \mathcal{H}$ there exists a unique core of A with minimal cardinality (the *fundamental* core of A).

Definition 4.2.4. If $Z \subset X$ we denote with $Min(Z)$ the set of minimal elements of Z and with $Max(Z)$ the set of maximal elements of Z.

If A is a Boolean partial map on X and if Z is a subset of X, we set:

$$Z_P^A = A^{-1}(P) \cap Z = \{x \in Z \cap dom(A) : A(x) = P\},$$

$$Z_N^A = A^{-1}(N) \cap Z = \{x \in Z \cap dom(A) : A(x) = N\}.$$

Definition 4.2.5. Let X be an arbitrary poset and let $\mathcal{H}$ a family of Boolean partial maps on X. Let $W \subset X$ and let $\varphi : W \to \mathbf{2}$. A couple (W,φ) is a core on X if:

N1) φ is a BPM on X such that $dom(\varphi) = W$;

N2) there exists a unique $A \in \mathcal{H}$ such that $A_{|W} = \varphi$.

Definition 4.2.6. We say that W is fundamental core for A if it is a core for A and if, for each core V for A, $W \subseteq V$.

Obviously, if there exists fundamental core of A, then it is unique, therefore we can speak of *the* fundamental core for A. Let $Core(X)$ be the family of all the cores (W,φ) on X.

Proposition 4.2.7. *Let* $A \in \mathcal{W}_+(X, \mathbf{2})$. *Then, setting*

$$N(A) = Min(X_P^A) \cup Max(X_N^A)$$

it follows that $N(A)$ *is a core for* A *on* X.

Theorem 4.2.8. *Let* $A \in \mathcal{W}_+(X, \mathbf{2})$. *Then, setting*

$$Core(A) = N(A) \setminus [Max(A^{-1}(N))]^c$$

it results that $Core(A)$ *is the fundamental core for* A.

Continuing the Example 4.1.5, it is easy to observe that the map $A \in \mathcal{W}_+(6,3)$ and that the fundamental core of A is the following partial map:
$B = \{321|123P, 300|003N, 210|003N, 200|002N, 100|001N, 000|000P\}$.
Hence $\mathcal{S}_B$ is the following $(6,3)$-weighted system :

$$\begin{cases} x_3 \geq x_2 \geq x_1 \geq 0 > y_1 \geq y_2 \geq y_3 \\ x_1 + x_2 + x_3 + y_1 + y_2 + y_3 \geq 0 \\ x_3 + y_3 < 0 \\ x_2 + y_2 < 0 \\ x_1 + y_1 < 0 \\ x_2 + x_1 + y_3 < 0. \end{cases}$$

Obviously the previous system $\mathcal{S}_B$ is not compatible, therefore also $\mathcal{S}_A$ is not compatible, hence $A \in \mathcal{W}_+(6,3)$, but $A \notin \chi(W_+CTSyst(6,3))$ and so does not exist an $f \in WF(n,r)$ such that $A = A_f$.

4.3 An application of the Boolean maps

Let n and r be two integers such that $0 < r \leq n$; we denote by $\gamma(n,r)$ $[\eta(n,r)]$ the minimum [maximum] number of the non-negative partial sums of a sum $\sum_{1=1}^n a_i \geq 0$, when $a_1, \ldots, a_n$ are n real numbers arbitrarily chosen in such a way that r of them are non-negative and the remaining $n-r$ are negative. Inspired by some interesting extremal combinatorial sum problems raised by Manickam, Miklös and Singhi in 1987 and 1988 Chiaselotti, Marino and Nardi [21] studied and solved the following problem $(P1)$:

> *Which are the values of* $\gamma(n,r)$ *and* $\eta(n,r)$ *for each* n *and* r, $0 < r \leq n$?

If we use notations of weight function we have that $\gamma(n,r) = \min\{\alpha(f) : f \in WF(n,r)\}$, $\eta(n,r) = \max\{\alpha(f) : f \in WF(n,r)\}$. We also set $\gamma^*(n,r) := \min\{|S_A^+(n,r)| : A \in \mathcal{W}_+(n,r)\}$ and $\eta^*(n,r) := \max\{|S_A^+(n,r)| : A \in \mathcal{W}_+(n,r)\}$. Let us observe that $\gamma^*(n,r) \leq \gamma(n,r) \leq \eta(n,r) \leq \eta^*(n,r)$.

Theorem 4.3.1. *If* n *and* r *are two integers such that* $0 < r < n$, *then:* $\gamma(n,r) = \gamma^*(n,r) = 2^{n-1} + 1$ *and* $\eta(n,r) = \eta^*(n,r) = 2^n - 2^{n-r} + 1$.

Proof. Assume that $0 < r < n$. We denote by $S_1(n,r)$ the sublattice of $S(n,r)$ of all the strings w of the form $w = i_1 \cdots i_r | j_1 \cdots j_{n-r-1}(n-r)$, with $j_1 \cdots j_{n-r-1} \in \{0^\S, \overline{1}, \ldots, \overline{n-r-1}\}$ and by $S_2(n,r)$ the sublattice of $S(n,r)$ of all the strings w of the form $w = i_1 \cdots i_r | 0 j_2 \cdots j_{n-r}$, with $j_2 \cdots j_{n-r} \in \{0^\S, \overline{1}, \ldots, \overline{n-r-1}\}$. It is clear that $S(n,r) = S_1(n,r)\dot{\bigcup}S_2(n,r)$ and $S_1(n,r) \cong S_2(n,r) \cong S(n-1,r)$. We consider now the following further sublattices of $S(n,r)$:

$$\begin{array}{lcl}
S_1^+(n,r) & := & \{w \in S_1(n,r) : w = r(r-1)\cdots 21|j_1\cdots j_{n-r-1}(n-r)\} \\
S_1^\pm(n,r) & := & \left\{w \in S_1(n,r) : w = i_1\cdots i_{r-1}0|j_1\cdots j_{n-r-1}(n-r), i_1 > 0^\S\right\} \\
S_1^-(n,r) & := & \{w \in S_1(n,r) : w = 0\cdots 0|j_1\cdots j_{n-r-1}(n-r)\} \\
S_2^+(n,r) & := & \{w \in S_2(n,r) : w = r(r-1)\cdots 21|0j_2\cdots j_{n-r}\} \\
S_2^\pm(n,r) & := & \left\{w \in S_2(n,r) : w = i_1\cdots i_{r-1}0|0j_2\cdots j_{n-r}, i_1 > 0^\S\right\} \\
S_2^-(n,r) & := & \{w \in S_2(n,r) : w = 0\cdots 0|0j_2\cdots j_{n-r}\}
\end{array}$$

It occurs immediately that: $S_i(n,r) = S_i^+(n,r)\dot{\bigcup}S_i^\pm(n,r)\dot{\bigcup}S_i^-(n,r)$, for $i = 1,2$ and $S_i^\pm(n,r)$ is a distributive sublattice of $S_i(n,r)$ with $2^{n-1} - 2\cdot 2^{n-r-1} = 2^{n-1} - 2^{n-r}$ elements, for $i = 1,2$.
Now we consider the following (n,r)-weight function $f : A(n,r) \to \mathbb{R}$

$$f: \begin{array}{cccccccc}
\widetilde{r} & \cdots & \widetilde{1} & 0^\S & \overline{1} & \cdots & \overline{(n-r-1)} & \overline{(n-r)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(n-r) & \cdots & (n-r) & 0 & -1 & \cdots & -1 & (n-r)(1-r)-1
\end{array}$$

Then it follows that $\Sigma_f : S(n,r) \to \mathbb{R}$ is such that

$$\Sigma_f(w)\begin{cases} \geq 0 & \text{if } w \in S_2^\pm(n,r)\dot{\bigcup}\{0\cdots 0|0\cdots 0\} \\ < 0 & \text{if } w \in S_1^\pm(n,r) \end{cases}$$

It means that the Boolean map $A_f \in \mathcal{W}_+(n,r)$ is such that:

$$A_f(w) = \begin{cases} P & \text{if } w \in S_2^\pm(n,r)\dot{\bigcup}\{0\cdots 0|0\cdots 0\} \\ N & \text{if } w \in S_1^\pm(n,r) \end{cases}$$

This shows that: $\left|S^+_{A_f}(n,r)\right| = \left|S_1^+(n,r)\dot{\bigcup}S_2^+(n,r)\dot{\bigcup}S_2^\pm(n,r)\dot{\bigcup}\{0\cdots 0|0\cdots 0\}\right| = 2^{n-r-1} + 2^{n-r-1} + 2^{n-1} - 2\cdot 2^{n-r-1} + 1 = 2^{n-1} + 1$.
In [35] (Theorem 1) it has been proved that $\gamma(n,1) = 2^{n-1}+1$ and $\gamma(n,r) \geq 2^{n-1}+1$. Since $\gamma(n,r) \geq \gamma^*(n,r)$, using a technique similar to that used in the proof of the Theorem 1 of [35], it easily follows that $\gamma^*(n,r) \geq 2^{n-1}+1$. As shown above, it results that $|S^+_{A_f}(n,r)| = 2^{n-1}+1$. Hence $2^{n-1}+1 \geq \gamma(n,r) \geq \gamma^*(n,r) \geq 2^{n-1}+1$, i.e. $\gamma(n,r) = \gamma^*(n,r) = 2^{n-1}+1$. This prove the first part of Theorem, it remains to prove the latter part.

We consider now the following (n,r)–weight function $g : A(n,r) \to \mathbb{R}$:

$$g: \begin{array}{cccccccc}
\widetilde{r} & \cdots & \widetilde{1} & 0^\S & \overline{1} & \cdots & \overline{(n-r-1)} & \overline{(n-r)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \cdots & 1 & 0 & \frac{-1}{n-r} & \cdots & \frac{-1}{n-r} & \frac{-1}{n-r}
\end{array}$$

It results then that the sum $\Sigma_g : S(n,r) \to \mathbb{R}$ is such that $\Sigma_g(w) \geq 0$ if $w \in S_1^\pm(n,r)\dot{\bigcup}S_2^\pm(n,r)\dot{\bigcup}\{0\cdots 0|0\cdots 0\}$, i.e. the Boolean map $A_g \in \mathcal{W}_+(n,r)$ is such that $A_g(w) = P$ if $w \in S_1^\pm(n,r)\dot{\bigcup}S_2^\pm(n,r)\dot{\bigcup}\{0\cdots 0|0\cdots 0\}$. This shows that $\left|S^+_{A_g}(n,r)\right| = \left|S_1^\pm(n,r)\dot{\bigcup}S_2^\pm(n,r)\dot{\bigcup}S_1^+(n,r)\dot{\bigcup}S_2^+(n,r)\dot{\bigcup}\{0\cdots 0|0\cdots 0\}\right| = (2^{n-1} - 2^{n-r}) + (2^{n-1} -$

$2^{n-r}) + 2^{n-r-1} + 2^{n-r-1} + 1 = 2^n - 2^{n-r} + 1$.
On other hand, it is clear that for any $A \in \mathcal{W}_+(n,r)$ it results $A(w) = P$ for each $w \in S_1^+(n,r) \dot{\cup} S_2^+(n,r) \dot{\cup} \{0\cdots0|0\cdots0\}$ and $A(w) = N$ for each $w \in S_1^-(n,r) \dot{\cup} \{S_2^-(n,r) \setminus \{0\cdots0|0\cdots0\}\}$. Moreover, $2^n - 2^{n-r} + 1$ is the biggest number of values P that a Boolean map $A \in \mathcal{W}_+(n,r)$ can assume. Hence, since $\eta(n,r) \le \eta^*(n,r)$, we have $2^n - 2^{n-r} + 1 = \left|S^+_{A_g}(n,r)\right| \le \eta(n,r) \le \eta^*(n,r) \le 2^n - 2^{n-r} + 1$, i.e. $\eta(n,r) = \eta^*(n,r) = 2^n - 2^{n-r} + 1$. This conclude the proof of the Theorem 4.3.1. □

To better visualize the previous result, we give a numerical example on a specific Hasse diagram. Let $n = 6$ and $r = 2$ and let f as given in previous Theorem, i.e.

$$f: \begin{array}{ccccccc} \widetilde{2} & \widetilde{1} & 0^{\S} & \overline{1} & \overline{2} & \overline{3} & \overline{4} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 4 & 0 & -1 & -1 & -1 & -5 \end{array}$$

In the Figure 4.2 we have shown the Hasse diagram of the lattice S(6,2), where $S_1^+(n,r)$ is black; $S_1^{\pm}(n,r)$ is violet; $S_1^-(n,r)$ are is red; $S_2^+(n,r)$ is blue; $S_2^{\pm}(n,r)$ is brown; $S_2^-(n,r)$ is green. Therefore A_f assume the following values:

- the blue, black and brown nodes correspond to values P of A_f
- the violet, red and green nodes correspond to values N of A_f, except for $w = 0\cdots0|0\cdots0$, where the map assume value P.

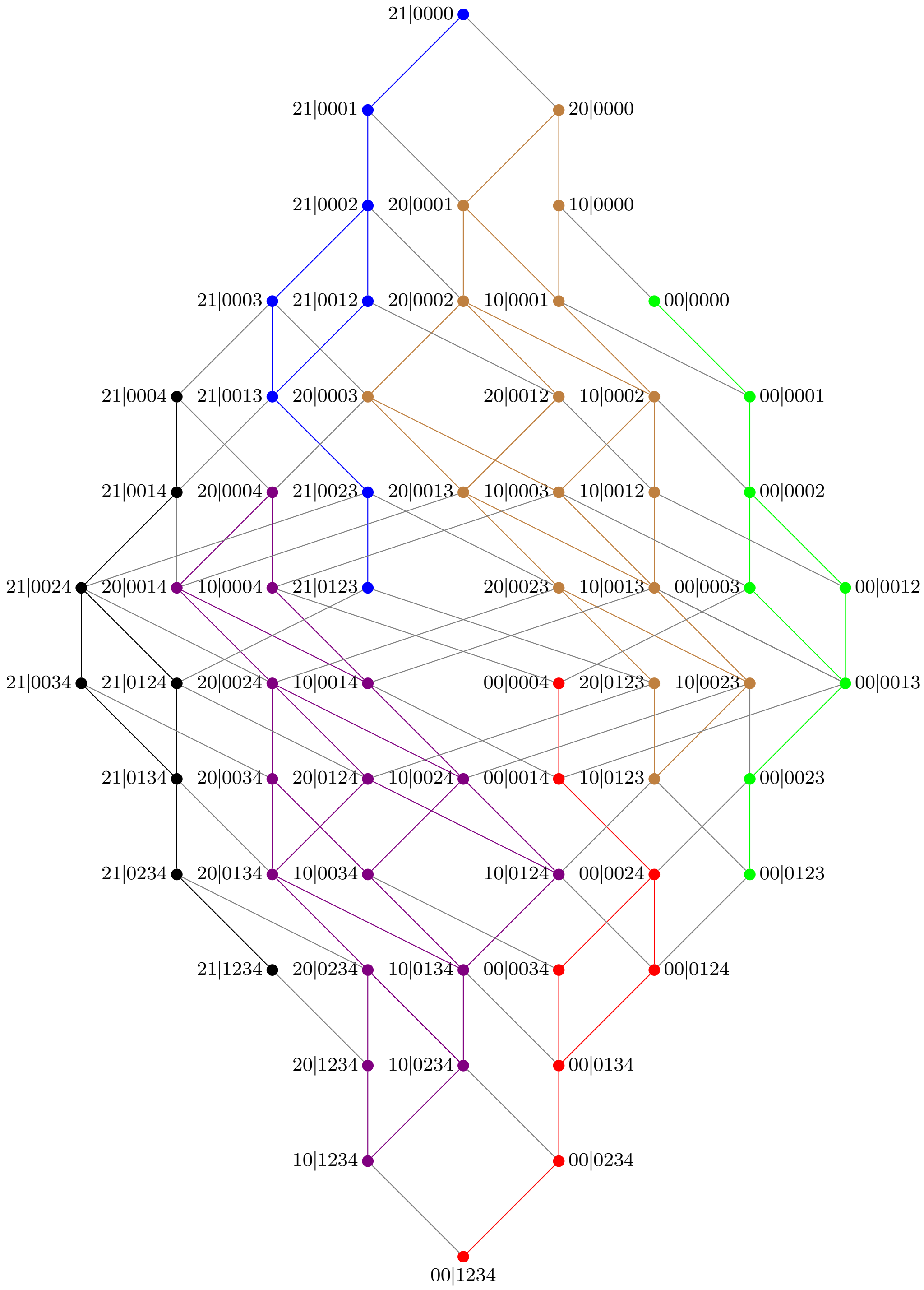


Figure 4.2: Hasse Diagram $S(6,2)$.

4.4 An application of concept of basis for $S(n,r)$

In this section we consider the following problem (Q):

> *If q is an integer such that $\gamma^*(n,r) \le q \le \eta^*(n,r)$, does there exist a map $A \in \mathcal{W}_+(n,r)$ with the property that $|S_A^+(n,r)| = q$?*

Chiaselotti, Marino and Nardi [21] gave an affirmative answer to the question (Q) building the map A.

First to give the proof of the Theorem 4.4.4 we need to introduce some useful results and the concept of basis in $S(n,r)$.
In the following first Lemma we show some properties of the sublattices of $S(n,r)$.

Lemma 4.4.1. *Here hold the following properties, where $\theta = 0\cdots0|0\cdots0$ and $\Theta = r\cdots21|12\cdots(n-r)$*

i) $\uparrow\Theta = S_1^+(n,r)\dot{\bigcup}S_2^+(n,r)$

ii) $\downarrow\theta = S_1^-(n,r)\dot{\bigcup}S_2^-(n,r)$

iii) $\uparrow S_2^\pm(n,r) \subseteq S_2^\pm(n,r)\dot{\bigcup}S_2^+(n,r)$

iv) $\downarrow S_1^\pm(n,r) \subseteq S_1^\pm(n,r)\dot{\bigcup}S_1^-(n,r)$

v) $((S_1^\pm(n,r))^c = S_2^\pm(n,r)$

Proof. We use the properties of Up-sets and Down-sets.

i) If $w \in (\uparrow\Theta)$ then $\Theta \sqsubseteq w$, i.e. it has the form $w = r\cdots1|j_1\cdots j_{n-r}$, where $j_1\cdots j_{n-r} \in \{0^\S,\overline{1},\ldots,\overline{n-r}\}$; therefore $w \in S_1^+(n,r)\dot{\bigcup}S_2^+(n,r)$. If $w \in S_1^+(n,r)$, it has the form $w = r\cdots1|j_1\cdots j_{n-r-1}(n-r)$, where $j_1\cdots j_{n-r-1} \in \{0^\S,\overline{1},\ldots,\overline{n-r-1}\}$; instead if $w \in S_2^+(n,r)$, it has the form $w = r\cdots1|$ $0j_2\cdots j_{n-r}$, where $j_2\cdots j_{n-r} \in \{0^\S,\overline{1},\ldots,\overline{n-r-1}\}$. In both cases it results that $\Theta \sqsubseteq w$, i.e. $w \in (\uparrow\Theta)$.

ii) It is analogue to i).

iii) The minimum of the sublattice $S_2^\pm(n,r)$ is $\alpha = 10\cdots0|01\cdots(n-r-1)$; since $\uparrow S_2^\pm(n,r) \subseteq\uparrow\alpha$, it is sufficient to show that $\uparrow\alpha = S_2^\pm(n,r)\bigcup S_2^+(n,r)$. The inclusion $S_2^\pm(n,r)\bigcup S_2^+(n,r) \subseteq\uparrow\alpha$ follows by the Definition of $S_2^\pm(n,r)$ and of $S_2^+(n,r)$. On the other hand, if $w \in\uparrow\alpha$, it follows that $\alpha \sqsubseteq w$, i.e. $w = i_1\cdots i_r|0j_2\cdots j_{n-r}$, with $i_1 > 0^\S$ and $j_2\cdots j_{n-r} \in \{0^\S,\overline{1},\ldots,\overline{n-r-1}\}$. Therefore $w \in S_2^\pm(n,r)\bigcup S_2^+(n,r)$, and this proves the other inclusion.

iv) Let us consider the maximum of the sublattice $S_1^\pm(n,r)$, that is $t_1 = r(r-1)\cdots20|0\cdots0(n-r)$. Since $\downarrow S_1^\pm(n,r) \subseteq\downarrow\beta$, it is sufficient to show that $\downarrow t_1 = S_1^\pm(n,r)\bigcup S_1^-(n,r)$; this proof is similar to iii).

v) If $w \in S_1^\pm(n,r)$, it has the form $w = i_1\cdots i_{r-1}0|j_1\cdots j_{n-r-1}(n-r)$, with $i_1 > 0^\S$ and $j_1\cdots j_{n-r-1} \in \{0^\S,\overline{1},\ldots,\overline{n-r-1}\}$, therefore his complement has the form $w^c = i'_1\cdots i'_{r-1}0|0j'_2\cdots j'_{n-r}$, with $i'_1 > 0^\S$ and $j'_2\cdots j'_{n-r} \in \{0^\S,\overline{1},\ldots,\overline{n-r-1}\}$; hence $w^c \in S_2^\pm(n,r)$. This shows that $(S_1^\pm(n,r))^c \subseteq S_2^\pm(n,r)$. Now, if $w \in S_2^\pm(n,r)$, we write w in the form $(w^c)^c$; then $w^c \in (S_2^\pm(n,r))^c \subseteq S_1^\pm(n,r)$, therefore $w \in (S_1^\pm(n,r))^c$. This shows that $S_2^\pm(n,r) \subseteq (S_1^\pm(n,r))^c$.

□

At this point let us recall the Definition of basis for $S(n,r)$, as given in [12] in a more general context. In the same way as an anti-chain uniquely determines a Boolean order-preserving map, a basis uniquely determines a Boolean map that has the properties (BM1) and (BM2) (see [12] for details). Hence the concept of basis will be fundamental in the sequel of this proof.

Definition 4.4.2. A basis for $S(n,r)$ is an ordered couple $\langle Y_+|Y_-\rangle$, where Y_+ and Y_- are two disjoint anti-chains of $S(n,r)$ such that :

B1) $(\downarrow Y_+)\bigcap(Y_-^c) = \varnothing$;

B2) $((\uparrow Y_+)\bigcup\uparrow(Y_-^c))\bigcap\downarrow Y_- = \varnothing$;

B3) $S(n,r) = ((\uparrow Y_+)\bigcup\uparrow(Y_-^c))\bigcup\downarrow Y_-$.

In the proof of Theorem 4.4.4 we will construct explicitly a such basis. We will also use the following result that was proved in [12].

Lemma 4.4.3. *Let $\langle W_+|W_-\rangle$ be a basis for $S(n,r)$. Then the map*

$$A(x) = \begin{cases} P & \text{if} \quad x \in\uparrow(W_+)\bigcup\uparrow(W_-^c) \\ N & \text{if} \quad x \in\downarrow W_- \end{cases}$$

is such that $A \in \mathcal{W}_+(S(n,r),\mathbf{2})$.

Theorem 4.4.4. *If q is a fixed integer with $2^{n-1}+1 \le q \le 2^n - 2^{n-r}+1$ then there is a Boolean map $A_q \in \mathcal{W}_+(n,r)$ such that $\left|S^+_{A_q}(n,r)\right| = q$.*

Proof. Let q be a fixed integer such that $2^{n-1}+1 \le q \le 2^n-2^{n-r}+1$. We determine a specific Boolean total map $A \in \mathcal{W}_+(n,r)$ such that $|S^+_A(n,r)| = q$. We proceed as follows.
The case $r=1$ is proved in the previous Theorem 4.3.1. Let us assume $r>1$.
Since $S_1^{\pm}(n,r)$ is a finite distributive sublattice of graded lattice $S(n,r)$, also $S_1^{\pm}(n,r)$ is a graded lattice. We denote by R the rank of $S_1^{\pm}(n,r)$ and with ρ_1 its rank function. The bottom of $S_1^{\pm}(n,r)$ is $b_1 = 10\cdots0|1\cdots(n-r-1)(n-r)$ and the top is $t_1 = r(r-1)\cdots20|0\cdots0(n-r)$.
We write q in the form $q = 2^{n-1}+1+p$ with $0 \le p \le 2^{n-1}-2^{n-r} = |S_1^{\pm}(n,r)|$. We will build a map $A \in W_+(n,r)$ such that $|S^+_A(n,r)| = 2^{n-1}+1+p$.
If $p=0$ we take $A = A_f$, with f as in Theorem 4.3.1 and hence we have $|S^+_A(n,r)| = 2^{n-1}$, instead if $p = 2^{n-1}-2^{n-r}$ we take $A = A_g$, with g as in Theorem 4.3.1, and we have $|S^+_A(n,r)| = 2^n-2^{n-r}+1$.
Therefore we can assume that $0 < p < 2^{n-1}-2^{n-r}$.
If $0 \le i \le R$, we denote by $\mathfrak{R}_i$ the set of elements $w \in S_1^{\pm}(n,r)$ such that $\rho_1(w) = R-i$ and we also set $\beta_i := |\mathfrak{R}_i|$. We write each $\mathfrak{R}_i$ in the following form $\mathfrak{R}_i = \{v_{i1},\ldots,v_{i\beta_i}\}$. If $0 \le l \le R$ we set $\mathfrak{U}_l := \dot{\bigcup}_{i=0,\ldots,l} \mathfrak{R}_i$. If $0 \le l \le R-2$ we set $\mathfrak{B}_l := \dot{\bigcup}_{i=l+2,\ldots,R} \mathfrak{R}_i$ and $\mathfrak{B}_{R-1} := \mathfrak{B}_R := \varnothing$. We can then write p in the form $p = \sum_{i=0}^k |\mathfrak{R}_i|+s = |\mathfrak{U}_k|+s$, for some $0 \le s < |\mathfrak{R}_{k+1}|$ and some $0 \le k \le R-1$. Depending of the previous number s we partition $\mathfrak{R}_{k+1}$ into the following two disjoint subsets : $\mathfrak{R}_{k+1} = \{v_{(k+1)1},\ldots,v_{(k+1)s}\}\dot{\bigcup}\{v_{(k+1)(s+1)},\ldots,v_{(k+1)\beta_{k+1}}\}$, where the first subset is considered empty if $s=0$. Let us note that $S_1^{\pm}(n,r) = \mathfrak{U}_k\dot{\bigcup}\mathfrak{R}_{k+1}\dot{\bigcup}\mathfrak{B}_k$.

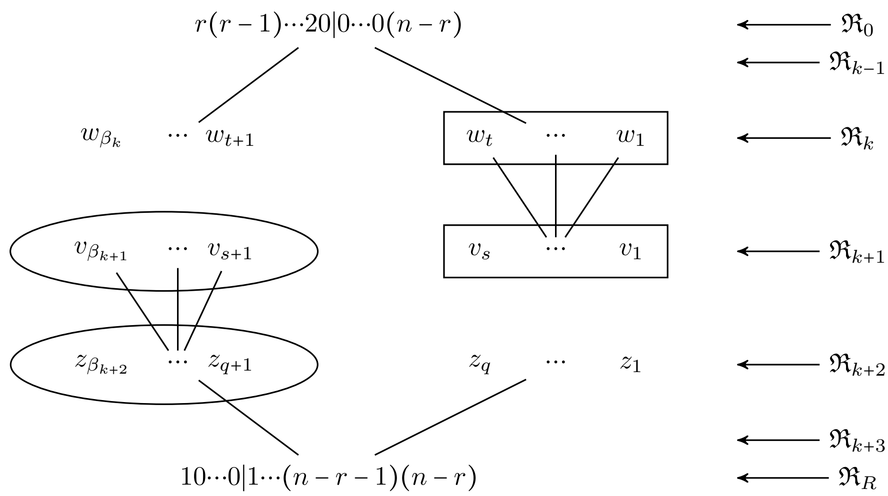
In the sequel, to avoid an overload of notations, we shall write simply v_i instead of $v_{(k+1)i}$, for $i = 1, \dots, \beta_{k+1}$.

We define now the map $A : S(n,r) \to \mathbf{2}$

$$A(w) = \begin{cases} P & \text{if } w \in S_2^{\pm}(n,r) \dot{\bigcup} S_1^{+}(n,r) \dot{\bigcup} S_2^{+}(n,r) \dot{\bigcup} \{0\cdots 0|0\cdots 0\} \\ P & \text{if } w \in \mathfrak{U}_k \dot{\bigcup} \{v_1, \dots, v_s\} \\ N & \text{if } w \in \mathfrak{B}_k \dot{\bigcup} \{v_{s+1}, \dots, v_{\beta_{k+1}}\} \\ N & \text{if } w \in S_1^{-}(n,r) \dot{\bigcup} \{S_2^{-}(n,r) \smallsetminus \{0\cdots 0|0\cdots 0\}\} \end{cases} \tag{4.7}$$

Let us observe that $|S_A^{+}(n,r)| = |S_2^{\pm}(n,r) \dot{\bigcup} S_1^{+}(n,r) \dot{\bigcup} S_2^{+}(n,r) \dot{\bigcup} \{0\cdots 0|0\cdots 0\}| + |\mathfrak{U}_k \dot{\bigcup} \{v_1, \dots, v_s\}| = (2^{n-1} - 2^{n-r}) + 2^{n-r-1} + 2^{n-r-1} + 1 + |\mathfrak{U}_k| + s = 2^{n-1} + 1 + p = q$. Therefore, if we show that $A \in \mathcal{W}_+(n,r)$, the Theorem is proved.
We write $\mathfrak{R}_k$ in the following way: $\mathfrak{R}_k = \{w_1, \dots, w_t\} \dot{\bigcup} \{w_{t+1}, \dots, w_{\beta_k}\}$, where $\{w_1, \dots, w_t\} = \mathfrak{R}_k \bigcap \uparrow \{v_1, \dots, v_s\}$ and $\{w_{t+1}, \dots, w_{\beta_k}\} = \mathfrak{R}_k \smallsetminus \{w_1, \dots, w_t\}$. Analogously $\mathfrak{R}_{k+2} = \{z_1, \dots, z_q\} \dot{\bigcup} \{z_{q+1}, \dots, z_{\beta_{k+2}}\}$, where $\{z_{q+1}, \dots, z_{\beta_{k+2}}\} = \mathfrak{R}_{k+2} \bigcap \downarrow \{v_{s+1}, \dots, v_{\beta_{k+1}}\}$ and $\{z_1, \dots, z_q\} = \mathfrak{R}_{k+2} \smallsetminus \{z_{q+1}, \dots, z_{\beta_{k+2}}\}$.

We can see a picture of this partition of the sublattice $S_1^{\pm}(n,r)$ in the following Figure:



Depending of s and k, we build now a particular basis for $S(n,r)$. For this we consider the minimum $\alpha = 10\cdots 0|01\cdots(n-r-1)$ of $S_2^{\pm}(n,r)$ and the subsets $T_+ := \{v_1, \dots, v_s, w_{t+1}, \dots, w_{\beta_k}\}$ and $T_- := \{v_{s+1}, \dots, v_{\beta_{k+1}}, z_1, \dots, z_q\}$. Let us distinguish two cases:

(a1) $\alpha \in \uparrow T_+$

(a2) $\alpha \notin \uparrow T_+$.

Let $\eta_1 := \{0\cdots 0|00\cdots 1\}$. We define two different couples of subsets as follows:
in the case $(a1)$ we set $Y_+ := T_+ \dot{\bigcup} \{\theta\}$ and $Y_- := T_- \dot{\bigcup} \{\eta_1\}$; in the case $(a2)$ we set $Y_+ := T_+ \dot{\bigcup} \{\theta\} \dot{\bigcup} \{\alpha\}$ and $Y_- := T_- \dot{\bigcup} \{\eta_1\}$.

Step 1. *$\langle Y_+|Y_-\rangle$ is a couple of two disjoint anti-chains of $S(n,r)$*

In both cases (a1) and (a2) it is obvious that $Y_+ \cap Y_- = \varnothing$.

Case (a1): The elements $\{v_1,\dots,v_s\}$ are not comparable between them because they have all rank $R-(k+1)$ and analogously for the elements $\{w_{t+1},\dots,w_{\beta_k}\}$ that have all rank $R-k$. Let now $v \in \{v_1,\dots,v_s\}$ and $w \in \{w_{t+1},\dots,w_{\beta_k}\}$; then $w \notin\downarrow v$ because $\rho_1(v) < \rho_1(w)$ and $w \notin\uparrow v$, because $\{w_{t+1},\dots,w_{\beta_k}\} \cap \uparrow\{v_1,\dots,v_s\} = \varnothing$ by construction. For the elements in Y_- different from η_1, we can proceed as for Y_+. On the other hand, we can observe that η_1 is not comparable with none of the elements $v_{s+1},\dots,v_{\beta_{k+1}},z_1,\dots,z_q$ since these are all in $S_1^{\pm}(n,r)$ while $\eta_1 \in S_2^{-}(n,r)$. Thus Y_+ is an anti-chain.

Case (a2): In this case we only must show that α is not comparable with none of the elements $v_1,\dots,v_s,w_{t+1},\dots,w_{\beta_k}$. At first from the fact that $\alpha \notin\uparrow T_+$ it follows $v_i \notin\downarrow \alpha$ for each $i=1,\dots,s$ and $w_j \notin\downarrow \alpha$ for each $j=t+1,\dots,\beta_k$. Moreover, the elements $v_1,\dots,v_s$ and $w_{t+1},\dots,w_{\beta_k}$ are all in $S_1^{\pm}(n,r)$, hence they have the form $i_1\cdots i_{r-1}0|j_1\cdots j_{n-r-1}(n-r)$, with $i_1 > 0^{\S}$, while $\alpha = 10\cdots0|01\cdots(n-r-1)$; so $\alpha \notin\downarrow v_i$ and $\alpha \notin\downarrow w_j$ for each $i=1,\dots,s$ and each $j=t+1,\dots,\beta_k$.

Step 2 *$\langle Y_+|Y_-\rangle$ is a basis for $S(n,r)$*

Case (a1): We must see that B1), B2) and B3) hold in this case.

B1) Let us begin to observe that $T_-^c = \left\{v_{s+1}^c,\dots,v_{\beta_{k+1}}^c,z_1^c,\dots,z_q^c\right\} \subseteq (S_1^{\pm}(n,r))^c = S_2^{\pm}(n,r)$. Then we have that $Y_-^c \subseteq S_2^{\pm}(n,r)\dot{\cup}S_1^{+}(n,r)$. On the other hand, by Lemma 4.4.1 iv) we have also that $\downarrow Y_+ \subseteq S_1^{\pm}(n,r)\dot{\cup}S_1^{-}(n,r)\dot{\cup}\{\theta\}$. Hence $(\downarrow Y_+)\cap(Y_-^c) = \varnothing$. This proves B1).

B2) We show at first that $\uparrow Y_-^c \cap \downarrow Y_- = \varnothing$.
Since $\uparrow Y_-^c \subseteq\uparrow (S_2^{\pm}(n,r)) \cup \uparrow \Theta$, by Lemma 4.4.1 i) and iii) we have then $\uparrow Y_-^c \subseteq S_2^{\pm}(n,r)\dot{\cup}S_1^{+}(n,r)\dot{\cup}S_2^{+}(n,r)$.
On the other hand, since $\downarrow T_- \subseteq\downarrow S_1^{\pm}(n,r))$, by Lemma 4.4.1 iv) it follows that $\downarrow T_- \subseteq S_1^{\pm}(n,r)\dot{\cup}S_1^{-}(n,r)$. By Proposition 4.4.1 ii), we have $\downarrow \eta_1 \subseteq S_1^{-}(n,r)\dot{\cup}S_2^{-}(n,r)$. Hence we have $\downarrow Y_- \subseteq S_1^{\pm}(n,r)\dot{\cup}S_1^{-}(n,r)\dot{\cup}$ $S_2^{-}(n,r)$. This proves that $\uparrow Y_-^c \cap \downarrow Y_- = \varnothing$. We show now that also $\uparrow Y_+ \cap \downarrow Y_- = \varnothing$; we proceed by contradiction. Let us suppose that there exists an element $z \in\uparrow Y_+ \cap \downarrow Y_-$, then there are two elements $w_+ \in Y_+$ and $w_- \in Y_-$ such that $w_+ \sqsubseteq z \sqsubseteq w_-$,hence $w_+ \sqsubseteq w_-$. We will distinguish the following six cases, and in each of them we will find a contradiction.

1. $w_+ \in \{v_1,\dots,v_s\}$ and $w_- \in \{v_{s+1},\dots,v_{\beta_{k+1}}\}$. In this case w_+ and w_- are two distinct elements having both rank $R-(k+1)$ and such that $w_+ \sqsubseteq w_-$; it is not possible.
2. $w_+ \in \{v_1,\dots,v_s\}$ and $w_- \in \{z_1,\dots,z_q\}$. In this case w_+ has rank $R-(k+1)$ while $w-$ has rank $R-(k+2) < R-(k+1)$, and this contradicts the condition $w_+ \sqsubseteq w_-$.
3. $w_+ \in \{w_{t+1},\dots,w_{\beta_k}\}$ and $w_- \in \{v_{s+1},\dots,v_{\beta_{k+1}}\}$. This case is similar to the previous because w_+ has rank $R-k$ while w_- has rank $R-(k+1)$.

4. $w_+ \in \{w_{t+1},\ldots,w_{\beta_k}\}$ and $w_- \in \{z_1,\ldots,z_q\}$. Similar to the previous because w_+ has rank $R-k$ while w_- has rank $R-(k+2)$.
5. $w_+ \in \{v_1,\ldots,v_s,w_{t+1},\ldots,w_{\beta_k}\}$ and $w_- = \eta_1$. In this case $w_+ \sqsubseteq w_-$ implies that $w_+ \in\downarrow \eta_1$; since $\downarrow \eta_1 \subseteq S_1^-(n,r)\dot{\bigcup} S_2^-(n,r)$ this is not possible since $w_+ \in S_1^{\pm}(n,r)$.
6. $w_+ = \theta$ and $w_- \in Y_-$ it is obvious that does not exist z such that $w_+ \sqsubseteq w_-$.

B3) Since α is the minimum of $S_2^{\pm}(n,r)$ we have that $S_2^{\pm}(n,r) \subseteq\uparrow \alpha$, moreover $\alpha \in\uparrow Y_+$; therefore $S_2^{\pm}(n,r) \subseteq\uparrow \alpha \subseteq\uparrow Y_+$. Since $\uparrow \Theta \subseteq\uparrow Y_-^c$, by Lemma 4.4.1 i) we have then that $(S_1^+(n,r)\dot{\bigcup} S_2^+(n,r)) =\uparrow \Theta \subseteq\uparrow Y_-^c$. By Lemma 4.4.1 ii) we have also that $(S_1^-(n,r)\dot{\bigcup}\{S_2^-(n,r) \smallsetminus \{0\cdots0|0\cdots0\}\}) =\downarrow \eta_1 \subseteq\downarrow Y_-$. To complete the proof let us observe that

$$S_1^{\pm}(n,r) = (\bigcup_{i=0}^{k}\mathfrak{R}_i)\dot{\bigcup}\mathfrak{R}_{k+1}\dot{\bigcup}(\bigcup_{j=k+2}^{R}\mathfrak{R}_j)$$

where

$$(\bigcup_{i=0}^{k}\mathfrak{R}_i) \subseteq\uparrow \mathfrak{R}_k \subseteq\uparrow Y_+ (\bigcup_{j=k+2}^{R}\mathfrak{R}_j) \subseteq\downarrow \mathfrak{R}_{k+2} \subseteq\downarrow Y_-.$$

Moreover, since $\mathfrak{R}_{k+1} = \{v_1,\ldots,v_s\}\dot{\bigcup}\{v_{s+1},\ldots,v_{\beta_{k+1}}\}$, and $\{v_1,\ldots,v_s\} \subseteq Y_+ \subseteq\uparrow Y_+$ and $\{v_{s+1},\ldots,v_{\beta_{k+1}}\} \subseteq Y_- \subseteq\downarrow Y_-$, then $\mathfrak{R}_{k+1} \subseteq (\uparrow Y_+ \cup \downarrow Y_-)$. This shows that $S_1^{\pm}(n,r) \subseteq (\uparrow Y_+ \cup \downarrow Y_-)$. Since the six sublattices $S_i^+(n,r)$, $S_i^{\pm}(n,r)$ and $S_i^-(n,r)$ for $i=1,2$ are a partition of $S(n,r)$, the property B3) is proved.

Case (a2): We must see that B1), B2) and B3) hold also in this case.

B1) We begin to show that $\alpha \notin Y_-^c$. In fact, $\alpha = (r(r-1)\ldots 20|0\ldots 0(n-r))^c$; suppose that $\alpha \in Y_-^c$ and show that we obtain a contradiction. By $\alpha \in Y_-^c$ it follows $\alpha^c = t_1 \in Y_-$. But t_1 is the top of $S_1^{\pm}(n,r)$, so $Y_- = t_1$, since Y_- is anti-chain. This means that $\mathfrak{R}_{k+1} = \mathfrak{R}_0$ and this is not possible. Since $Y_+ = T_+\dot{\bigcup}\{\alpha\}$, we have $(\downarrow Y_+)\bigcap Y_-^c = ((\downarrow T_+)\bigcap Y_-^c)\bigcup((\downarrow \alpha)\bigcap Y_-^c)$ and, moreover, as in the proof of B1) in the case (a1), we also have that $(\downarrow T_+)\bigcap Y_-^c = \varnothing$; therefore, to prove B1) it is sufficient to show that $(\downarrow \alpha)\bigcap Y_-^c = \varnothing$. As in the proof of B1) We have that $Y_-^c \subseteq S_2^{\pm}(n,r)\dot{\bigcup}\{r\cdots1|02\cdots n-r\}$. Since α is the minimum of $S_2^{\pm}(n,r)$ and $\{r\cdots1|02\cdots n-r\} \notin\downarrow \alpha$, it follows that the unique element of $\downarrow \alpha$ that can belong to Y_-^c is α, but we have before shown that this is not true.

B2) As in the previous case we have $(\uparrow Y_-^c)\bigcap \downarrow Y_- = \varnothing$, moreover $(\uparrow Y_+)\bigcap(\downarrow Y_-) = ((\uparrow T_+)\bigcup(\uparrow \alpha))\bigcap(\downarrow Y_-)((\uparrow T_+)\bigcap(\downarrow Y_-))\bigcup((\uparrow \alpha)\bigcap(\downarrow Y_-))$. As in the case (a1) we have $((\uparrow T_+)\bigcap(\downarrow Y_-)) = \varnothing$, therefore, to prove B2) also in the case (a2), it is sufficient to show that $((\uparrow \alpha)\bigcap(\downarrow Y_-)) = \varnothing$. As in the proof of B2) in the case (a1) it results that $\downarrow Y_- \subseteq S_2^{\pm}(n,r)\dot{\bigcup} S_1^-(n,r)\dot{\bigcup}\{S_2^-(n,r)\smallsetminus\{\theta\}\}$ and, by Definition of α, it is easy to observe that $(\uparrow \alpha)\bigcap(S_1^{\pm}(n,r)\dot{\bigcup} S_1^-(n,r)\dot{\bigcup}\{S_2^-(n,r)\smallsetminus\{\theta\}\}) = \varnothing$. Hence $((\uparrow \alpha)\bigcap(\downarrow Y_-)) = \varnothing$.

B3) Identical to that of Case (a1).

Step 3 *The map A defined in (4.7) is such that $A \in \mathcal{W}_+(n,r)$*

Since we have proved that $\langle Y_+|Y_-\rangle$ is a basis for $S(n,r)$, by Lemma 4.4.3 it follows that the map $A \in \mathcal{W}_+(S(n,r),\mathbf{2})$ if the two following identities hold:

$$\begin{aligned}(\uparrow Y_+)\cup(\uparrow Y_-^c) &= S_2^{\pm}(n,r)\ \dot\cup\ S_1^{+}(n,r)\ \dot\cup\ S_2^{+}(n,r)\ \dot\cup \\ &\quad \mathfrak{U}_k\ \dot\cup\ \{v_1,\dots,v_s\}\ \dot\cup\ \{\theta\}\end{aligned} \tag{4.8}$$

and

$$\downarrow Y_- = \mathfrak{B}_k \ \dot\bigcup\ \{v_{s+1},\dots,v_{\beta_{k+1}}\}\ \dot\bigcup\ S_1^-(n,r)\ \dot\bigcup\ \{S_2^-(n,r)\smallsetminus\{\theta\}\} \tag{4.9}$$

We prove at first (4.9). By Definition of $\mathfrak{B}_k$ and T_- it easy to observe that $\mathfrak{B}_k\dot\cup\{v_{s+1},\dots,v_{\beta_{k+1}}\}\subseteq\downarrow T_-$ and moreover by Lemma 4.4.1 ii) we also have that $\downarrow\eta_1\subseteq S_1^-(n,r)\cup S_2^-(n,r)$, hence, since $\downarrow Y_-=\downarrow\eta_1\cup\downarrow T_-$, it results that $\mathfrak{B}_k\dot\cup\{v_{s+1},\dots,v_{\beta_{k+1}}\}\cup S_1^-(n,r)\dot\cup\{S_2^-(n,r)\smallsetminus\{\theta\}\}\subseteq\downarrow Y_-$. On the other hand, by Lemma 4.4.1 iv) we have that $\downarrow T_-\subseteq S_1^{\pm}(n,r)\dot\cup S_1^-(n,r)$, because T_- is a subset of $S_1^{\pm}(n,r)$. At this point let us note that the elements of $\downarrow T_-$ that are also in $S_1^{\pm}(n,r)$ must belong necessarily to the subset $\mathfrak{B}_k\dot\cup\{v_{s+1},\dots,v_{\beta_{k+1}}\}$. This proves the other inclusion and hence (4.9).
To prove now (4.8) we must distinguish the cases (a1) and (a2). We set $\Delta := S_2^{\pm}(n,r)\dot\cup S_1^+(n,r)\dot\cup S_2^+(n,r)\cup\mathfrak{U}_k\dot\cup\{v_1,\dots,v_s\}\dot\cup\{\theta\}$. Let us at first to examine the case (a1). Since α is the minimum of $S_2^{\pm}(n,r)$ we have $S_2^{\pm}(n,r)\subseteq\uparrow\alpha\subseteq\uparrow Y_+$. Moreover, since $Y_-=T_-\cup\{\eta_1\}$, it follows that $\uparrow Y_-^c\supseteq\uparrow(\Theta)=S_1^+(n,r)\dot\cup S_2^+(n,r)$ by Lemma 4.4.1 i). Finally, since $\mathfrak{U}_k\dot\cup\{v_1,\dots,v_s\}\subseteq\uparrow T_+=\uparrow Y_+$ the inclusion $\supseteq$ in (4.8) is proved. To prove the other inclusion $\subseteq$ we begin to observe that $\uparrow Y_+\cap(S_1^-(n,r)\cup S_1^-(n,r))=\varnothing$, therefore the elements of $\uparrow Y_+$ that are not in $S_2^{\pm}(n,r)\dot\cup S_1^+(n,r)\dot\cup S_2^+(n,r)$ must be necessarily in $S_1^{\pm}(n,r)$, and such elements, by Definition of T_+ must be necessarily in $\mathfrak{U}_k\dot\cup\{v_1,\dots,v_s\}$. This proves that $\uparrow Y_+\subseteq\Delta$. For $\uparrow(Y_-^c)$, we have $\uparrow(Y_-^c)=\uparrow(T_-^c\cup\{\eta_1\}^c)$ and since $T_-\subseteq S_1^{\pm}(n,r)$, also $T_-^c\subseteq(S_1^{\pm}(n,r))^c=S_2^{\pm}(n,r)$, by Lemma 4.4.1 v). Therefore $\uparrow T_-^c\subseteq\uparrow S_2^{\pm}(n,r)\subseteq S_2^{\pm}(n,r)\dot\cup S_2^+(n,r)$ by Lemma 4.4.1 iii). This shows that $\uparrow(Y_-^c)\subseteq S_2^{\pm}(n,r)\dot\cup S_2^+(n,r)\dot\cup S_1^+(n,r)\subseteq\Delta$, hence the inclusion $\subseteq$. The proof of (4.8) in the case (a1) is therefore complete.
Finally, to prove (4.8) in the case (a2), it easy to observe that the only difference with to respect case (a1) is when we must show that $\uparrow Y_+\subseteq\Delta$. In fact, in the case (a2) it results that $\alpha\notin\uparrow T_+$ and $Y_+=T_+\cup\{\alpha\}$, while Y_- is the same in both cases (a1) and (a2). Therefore, in the case (a2), the elements of $\uparrow Y_+$ that are not in $S_2^{\pm}(n,r)\dot\cup S_1^+(n,r)\dot\cup S_2^+(n,r)$ must be in $(\uparrow T_+)\cap S_1^{\pm}(n,r)$ or in $(\uparrow\alpha)\cap S_1^{\pm}(n,r)$. As in the case (a1) we have $(\uparrow T_+)\cap S_1^{\pm}(n,r)=\mathfrak{U}_k\dot\cup\{v_1,\dots,v_s\}$ and, since α is the minimum of $S_2^{\pm}(n,r)$, it results that $\uparrow\alpha=\uparrow S_2^{\pm}(n,r)\subseteq S_2^{\pm}(n,r)\dot\cup S_2^+(n,r)$ by Lemma 4.4.1 iii), hence $(\uparrow\alpha)\cap S_1^{\pm}(n,r)=\varnothing$. Therefore, also in the case (a2), the elements of $\uparrow Y_+$ that are not in $S_2^{\pm}(n,r)\dot\cup S_1^+(n,r)\dot\cup S_2^+(n,r)$ must be necessarily in $\mathfrak{U}_k\dot\cup\{v_1,\dots,v_s\}$. The other parts of the proof are the same as in the case (a1). Hence we have proved the identities (4.8) and (4.9). By Lemma 4.4.3 it follows then that the map $A\in\mathcal{W}_+(S(n,r),\mathbf{2})$. Finally, by Definition of A, we have obviously $A(\eta_1)=N$, $A(\theta)=P$ and $A(\Theta)=P$. This shows that $A\in\mathcal{W}_+(n,r)$. The proof is complete. □

To conclude we emphasize the elegant symmetry of the induced partitions on $S(n,r)$ from the Boolean total maps A_q's constructed in the proof.

4.5 Solution of a problem extremal sums

Question (Q) can be seen as a refinement the following problem $(P2)$:

> If q is an integer such that $\gamma(n,r) \le q \le \eta(n,r)$, can we find n real numbers $a_1, \dots, a_n$, which r of them are non-negative and the remaining $n - r$ are negative with $\sum_{1=1}^{n} a_i \ge 0$, such that the number of the non-negative sums formed from these numbers is exactly q?

If we use the notations of weight function, problem $(P2)$ can be written as follows $(P2)$:

> If q is an integer such that $\gamma(n,r) \le q \le \eta(n,r)$, does there exist a function $f \in W(n,r)$ with the property that $\alpha(f) = q$?

A solution to problem $(P2)$ has been provided by Engel and Nardi [25], which will be presented in detail in this section.

For $f \in WF(n,r)$ and $X \in A(n,r)$, let $\Sigma(X) := \sum_{x \in X} f(x)$. We define:

$$\begin{aligned} \mathcal{F}^+(f) &= \{X \subseteq A(n,r) : \Sigma(X) \ge 0\} \qquad \text{and} \\ \mathcal{F}^-(f) &= \{X \subseteq A(n,r) : \Sigma(X) < 0\}. \end{aligned}$$

Since f is always clear from the context, we will usually omit it in the following. Note that $\varnothing \in \mathcal{F}^+$, $|\mathcal{F}^+| = \alpha(f)$ and $|\mathcal{F}^+| + |\mathcal{F}^-| = 2^n$.

Theorem 4.5.1. *Let n and r be positive integers with $1 \le r \le n-1$ and let q be an integer with $2^{n-1} + 1 \le q \le 2^n - 2^{n-r} + 1$. Then there exists an $f \in WF(n,r)$ such that $|\mathcal{F}^+| = q$.*

Proof. For $X \subseteq A(n,r)$ we set $X^+ = X \cap = \{\widetilde{r}, \dots, \widetilde{1}\}$ and $X^- = X \cap = \{\overline{1}, \dots, \overline{n-r}\}$.

Case 1. $2^{n-1} + 1 \le q \le 2^n - 2^{n-1} + 2^{r-1}$. Then we define f as follows:

$$f: \begin{array}{cccccccccc} \widetilde{r} & \widetilde{r-1} & \widetilde{r-2} & \cdots & \widetilde{1} & 0^{\S} & \overline{1} & \overline{2} & \cdots & \overline{(n-r)} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ q-2^r & 2^{r-2} & 2^{r-3} & \cdots & 2^0 & 0 & -2^{r-1} & -2^r & \cdots & -2^{n-2} \end{array}$$

Then $f(\tilde{r}) \le 0$ and $f(\tilde{1}) + \dots + f(\tilde{r}) + f(\overline{1}) + \dots + f(\overline{n-r}) = q - 1 - 2^{n-1} \ge 0$, hence $f \in WF(n,r)$. Let

$$\begin{aligned} \mathcal{F}_1 &= \{X \subseteq A(n,r) : X^- = \varnothing\}, \\ \mathcal{F}_2 &= \{X \subseteq A(n,r) : X^- \ne \varnothing, \widetilde{r} \notin X\}, \\ \mathcal{F}_3 &= \{X \subseteq A(n,r) : X^- \ne \varnothing, \widetilde{r} \in X, X^- \ne \{\overline{1}, \dots, \overline{n-r}\}\}, \\ \mathcal{F}_4 &= \{X \subseteq A(n,r) : \tilde{r} \in X, X^- = \{\overline{1}, \dots, \overline{n-r}\}\}. \end{aligned}$$

Obviously,

$$\{X : X \subseteq A(n,r)\} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2 \dot{\cup} \mathcal{F}_3 \dot{\cup} \mathcal{F}_4.$$

It is clear that $\mathcal{F}_1 \subseteq \mathcal{F}^+$.

Now let $X \in \mathcal{F}_2$. Then $\Sigma(X) = \Sigma(X^+) + \Sigma(X^-) \le 2^{r-1} - 1 - 2^{r-1} < 0$, hence $\mathcal{F}_2 \subseteq \mathcal{F}^-$. If $X \in \mathcal{F}_3$ then $\Sigma(X) = \Sigma(X^+) + \Sigma(X^-) \ge q - 2^r - 2^{r-1}(2^{n-r} - 2) \ge 2^{n-1} + 1 - 2^r - 2^{r-1}(2^{n-r} - 2) = 1 \ge 0$ hence $\mathcal{F}_3 \subseteq \mathcal{F}^+$. Finally let $X \in \mathcal{F}_4$. We put $Y = X^+ \setminus \{\tilde{r}\}$. Then $\Sigma(X) = q - 2^r + \Sigma(Y) - (2^{n-1} - 2^{r-1})$. Hence $\Sigma(X) \ge 0$ iff $\Sigma(Y) \ge 2^{n-1} + 2^{r-1} - q$.

In the actual Case 1 we have $0 \le 2^{n-1} + 2^{r-1} - q \le 2^{r-1} - 1$. Using binary expansions it is easy to see that $\Sigma(Y)$ takes on all integer values from 0 to $2^{r-1} - 1$ exactly once if Y runs through all subsets of $[2, r]$. Hence,

$$|\mathcal{F}_4 \cap \mathcal{F}^+| = 2^{r-1} - 1 - (2^{n-1} + 2^{r-1} - q) + 1 = q - 2^{n-1}.$$

Consequently, we have

$$|\mathcal{F}^+| = |\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_4 \cap \mathcal{F}^+| = 2^r + 2^{r-1}(2^{n-r} - 2) + (q - 2^{n-1}) = q.$$

Case 2. $2^n - 2^{n-1} + 2^{r-1} < q \le 2^n - 2^{n-r} + 1$.

Then there exists a unique integer $j \in [2, r]$ such that

$$2^n - 2^{n-j+1} + 2^{r-j+1} < q \le 2^n - 2^{n-j} + 2^{r-j}. \tag{4.10}$$

Let

$$d = q - 2^n + 2^{n-j+1} - 2^{r-j+1}.$$

By (4.10),

$$0 < d \le 2^{n-j} - 2^{r-j}. \tag{4.11}$$

We write d in the form

$$d = k \cdot 2^{r-j} + \ell,$$

where k is a non-negative integer and $1 \le \ell \le 2^{r-j}$, i.e.

$$k = \left\lceil \frac{d}{2^{r-j}} \right\rceil - 1, \quad \ell = d - k \cdot 2^{r-j}.$$

Note that by (4.11)

$$0 \le k \le 2^{n-r} - 2.$$

We define f as follows:

$$f: \begin{array}{ccccccc} \widetilde{r} & \cdots & \widetilde{r-j+2} & \widetilde{r-j+1} & \widetilde{r-j} & \cdots & \widetilde{1} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2^{n-1} - 2^{r-1} & \cdots & 2^{n-1} - 2^{r-1} & ((k+1) \cdot 2^{r-1} - 2^{r-j} + \ell) & 2^{r-j+1} & \cdots & 2^{r-r} \end{array}$$

$$\begin{array}{cccc} 0^{\S} & \overline{1} & \cdots & \overline{(n-r)} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & -2^{r-1} & \cdots & -2^{n-2} \end{array}$$

Then $f(i) \ge 0$ for $i \in \{\tilde{r}, \ldots, \widetilde{r-j+1}\}$ and $f(\tilde{1}) + \cdots + f(\tilde{r}) + f(\overline{1}) + \cdots + f(\overline{n-r}) =$

$q - 1 - 2^{n-1} \geq 2^{n-1} - 2^{r-1} - 2^{r-1}(2^{n-r} - 1) = 0$, hence indeed $f \in WF(n,r)$. Let

$$\mathcal{F}_1 = \{X \subseteq A(n,r) : X^+ \cap \{\widetilde{r}, \dots, \widetilde{r-j+2}\} \neq \emptyset\},$$
$$\mathcal{F}_2 = \{X \subseteq A(n,r) : X^+ \cap \{\widetilde{r}, \dots, \widetilde{r-j+2}\} = \emptyset, j \notin X^+, X^- \neq \emptyset\},$$
$$\mathcal{F}_3 = \{X \subseteq A(n,r) : X^+ \cap \{\widetilde{r}, \dots, \widetilde{r-j+2}\} = \emptyset, j \notin X^+, X^- = \emptyset\},$$
$$\mathcal{F}_4 = \{X \subseteq A(n,r) : X^+ \cap \{\widetilde{r}, \dots, \widetilde{r-j+2}\} = \emptyset, j \in X^+\}.$$

Obviously,

$$\{X : X \subseteq A(n,r)\} = \mathcal{F}_1 \dot\cup \mathcal{F}_2 \dot\cup \mathcal{F}_3 \dot\cup \mathcal{F}_4.$$

If $X \in \mathcal{F}_1$ then $\Sigma(X) \geq (2^{n-1} - 2^{r-1}) - 2^{r-1}(2^{n-r} - 1) = 0$, hence $\mathcal{F}_1 \subseteq \mathcal{F}^+$.

Now let $X \in \mathcal{F}_2$. Then $\Sigma(X) = \Sigma(X^+) + \Sigma(X^-) \leq (2^{r-j} - 1) - 2^{r-1} < 0$, hence $\mathcal{F}_2 \subseteq \mathcal{F}^-$.

Clearly, $\mathcal{F}_3 \subseteq \mathcal{F}^+$.

Now we study $\mathcal{F}_4$. Let

$$\mathcal{F}_{4,1} = \{X \in \mathcal{F}_4 : \Sigma(X^-) \geq -k \cdot 2^{r-1}\},$$
$$\mathcal{F}_{4,2} = \{X \in \mathcal{F}_4 : \Sigma(X^-) \leq -(k+2) \cdot 2^{r-1}\},$$
$$\mathcal{F}_{4,3} = \{X \in \mathcal{F}_4 : \Sigma(X^-) = -(k+1) \cdot 2^{r-1}\}.$$

Then

$$\mathcal{F}_4 = \mathcal{F}_{4,1} \dot\cup \mathcal{F}_{4,2} \dot\cup \mathcal{F}_{4,3}.$$

If $X \in \mathcal{F}_{4,1}$ then $\Sigma(X) > ((k+1)2^{r-1} - 2^{r-j}) - k \cdot 2^{r-1} = 2^{r-1} - 2^{r-j} \geq 0$, hence $\mathcal{F}_{4,1} \subseteq \mathcal{F}^+$. Obviously, $\Sigma(X^-)$ takes on all values from $\{0, -2^{r-1}, -2 \cdot 2^{r-1}, \dots, -(2^{n-r} - 1) \cdot 2^{r-1}\}$ exactly once if X^- runs through all subsets of $\{\overline{1}, \dots, \overline{n-r}\}$. Hence

$$|\mathcal{F}_{4,1}| = (k+1) \cdot 2^{r-j}.$$

If $X \in \mathcal{F}_{4,2}$ then $\Sigma(X) \leq ((k+1) \cdot 2^{r-1} - 2^{r-j} + l) + (2^{r-j} - 1) - (k+2) \cdot 2^{r-1} = l - 1 - 2^{r-1} \leq 2^{r-j} - 1 - 2^{r-1} < 0$, hence $\mathcal{F}_{4,2} \subseteq \mathcal{F}^-$.

Finally, let $X \in \mathcal{F}_{4,3}$. We put $Y = X^+ \setminus \{\widetilde{r-j+1}\}$. Then $\Sigma(X) = ((k+1) \cdot 2^{r-1} - 2^{r-j} + \ell) + \Sigma(Y) - (k+1) \cdot 2^{r-1}$, hence $\Sigma(X) \geq 0$ iff $\Sigma(Y) \geq 2^{r-j} - \ell$. Obviously, $\Sigma(Y)$ takes on all integer values from 0 to $2^{r-j} - 1$ exactly once if Y runs through all subsets of $\{\widetilde{r-j}, \dots, \widetilde{1}\}$. Accordingly,

$$|\mathcal{F}_{4,3} \cap \mathcal{F}^+| = \ell.$$

Consequently,

$$\begin{aligned} |\mathcal{F}^+| &= |\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_{4,1}| + |\mathcal{F}_{4,3} \cap \mathcal{F}^+| \\ &= (2^n - 2^{n-j+1}) + 2^{r-j} + (k+1) \cdot 2^{r-j} + \ell \\ &= (2^n - 2^{n-j+1}) + 2^{r-j+1} + d \\ &= q. \end{aligned}$$

□

Chapter 5

Sperner and Non-Sperner Posets

Let $b(P)$ be the largest size of a level of the graded poset P, i.e. $b(P) := max\{W_i, i = 0, \dots, r(P)\}$. Let $d(P)$ be the largest size of an antichain in P, the parameter $d(P)$ is called the *the width* of poset P. Obviously, for each graded poset P, $\frac{d(P)}{b(P)} \geq 1$.

After Sperner [42], it was proven for many interesting classes of graded posets that the inequality is in fact an equality, the posets for which $d(P) = b(P)$ are called Sperner posets [23]. In other words, any ranked poset whose largest antichain is not bigger than its maximum level is said to possess the Sperner property, hence the Sperner property is equivalently the property that some rank level is a maximum antichain.

Many interesting posets arise in combinatorics, supplying researchers with a wealth of problems. The main thrust of this research has been to develop tools for establishing the Sperner property; there are considerably fewer techniques for bounding $d(P)$ in non-Sperner posets.

In the first part of this chapter we show that the lattice $S(n, r)$ is a Sperner poset, instead in the latter part we consider a new poset and we estimate how much it is 'close' to be a Sperner poset.

5.1 A Sperner poset

5.1.1 The full rank property and Jordan functions

Let P be a ranked poset with rank function r of rank $n := r(P)$. We remember that for a ranked poset P we define the *i th level* by $N_i(P) := \{p \in P : r(p) = i\}$; its size $W_i(P) := |N_i(P)|$ is called the *i th Whitney number*, $i = 0, \dots, r(P)$ (when there is no danger of ambiguity we write briefly N_i and W_i). With P we associate the *poset space* $\tilde{P}$, which is the vector space of all functions $\varphi : P \to \mathbb{R}$ with the usual vector space operations. With the element $p \in P$ we associate its *characteristic function* $\varphi_p \in \tilde{P}$ defined by

$$\varphi_p(q) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{otherwise} \end{cases}$$

For the sake of brevity, we write $\tilde{p}$ instead of φ_p, but $\tilde{0}$ denotes the zero vector. Obviously, $\{\tilde{p} : p \in P\}$ is a basis of $\tilde{P}$. Thus every element φ of $\tilde{P}$ has the form $\varphi = \sum_{p \in P} \mu_p \tilde{p}$, where μ_p is real number, and $\tilde{P}$ can be considered as the vector space of all formal linear combinations of elements of P with real coefficients. With the standard scalar product

$$\left\langle \sum_{p \in P} \mu_p \tilde{p}, \sum_{p \in P} \nu_p \tilde{p} \right\rangle = \sum_{p \in P} \mu_p \nu_p$$

the space $\tilde{P}$ becomes a Euclidean space. With a subset F of P we associate the subspace $\tilde{F}$ of $\tilde{P}$, which is generated by $\{\tilde{p} : p \in F\}$. Obviously, the dimension of $\tilde{F}$, denoted by $\dim \tilde{F}$, equals $|F|$. Note that $\tilde{P} = \tilde{N}_0 \oplus \ldots \oplus \tilde{N}_n$, where $\oplus$ denotes the direct sum. Any basis B of $\tilde{P}$ with the property $B = \bigcup_{i=0}^{n} B \cap \tilde{N}_i$, that is, all basis elements belong to some $\tilde{N}_i$, is said to be a *ranked basis*. In particular, $\{\tilde{p} : p \in \tilde{P}\}$ is a ranked basis. In the following we consider linear operators $\tilde{P} \to \tilde{P}$. We define $\phi\psi$ to be the operator for which $\phi\psi(\varphi) = \phi(\psi(\varphi))$ for all $\phi \in \tilde{P}$. Further let $\phi^i := \phi \ldots \phi$ (i times) if $i \geq 1$. It is useful to define ϕ^j to be the *identity operator* $\tilde{I}$ if $j \leq 0$. For a subspace E if $\tilde{P}$, let $\phi(E) := \{\phi(\varphi) : \phi \in E\}$ and $\phi|_E$ be the restriction of ϕ to E. Let ϕ^* be the *the adjoint operator* of ϕ, that is, $\phi^* : \tilde{P} \to \tilde{P}$ with $\langle \phi(\varphi), \psi \rangle = \langle \varphi, \phi^*(\psi) \rangle$ for all $\varphi, \psi \in \tilde{P}$.

Definition 5.1.1. If $j \notin \{0, \ldots, n\}$, let $\tilde{N}_j = \{\tilde{0}\}$. A linear operator $\tilde{\nabla} : \tilde{P} \to \tilde{P}$ is called *raising operator* if $\tilde{\nabla}(\tilde{N}_i) \subseteq \tilde{N}_{i+1}$

Definition 5.1.2. A raising operator $\tilde{\nabla}$ defined on the basis $\{\tilde{p} : p \in P\}$ by

$$\tilde{\nabla}(\tilde{p}) = \sum_{q : p \lessdot q} c(p, q) \tilde{q}$$

with $c(p, q) \in \mathbb{R}$, is called an *order-raising* operator

Definition 5.1.3. If $j \notin \{0, \ldots, n\}$, let $\tilde{N}_j = \{\tilde{0}\}$. A linear operator $\tilde{\Delta} : \tilde{P} \to \tilde{P}$ is called *lowering operator* if $\tilde{\Delta}(\tilde{N}_i) \subseteq \tilde{N}_{i-1}$

Definition 5.1.4. A lowering operator $\tilde{\Delta}$ defined on the basis $\{\tilde{p} : p \in P\}$ by

$$\tilde{\Delta}(\tilde{q}) = \sum_{p : p \lessdot q} d(p, q) \tilde{p}$$

with $d(p, q) \in \mathbb{R}$, is called an *order-lowering* operator

Thus order-raising (resp. lowering) operators are defined by functions $E(P) \to \mathbb{R}$, where $E(P)$ is the arc set of the Hasse diagram of P. If $c(p, q) = 1$ (resp. $d(p, q) = 1$) for all $p \lessdot q$ we speak of the *Lefschetz raising* (resp. lowering). We denote by $\tilde{\nabla}$ (resp. $\tilde{\Delta}$) a raising (resp. lowering) operator and by $\tilde{\nabla}_L = \tilde{\nabla}_L(P)$ (resp. $\tilde{\Delta}_L = \tilde{\Delta}_L(P)$) the Lefschetz raising (resp. lowering) operator in P. Intuitively speaking the Lefschetz raising operator $\tilde{\nabla}_L$, when acting on a poset element, produces the sum of poset elements covering that element.

Let $\tilde{\nabla}_{ij} := \tilde{\nabla}^{j-i}|_{\tilde{N}_i}$ (resp. $\tilde{\Delta}_{ji} := \tilde{\Delta}^{j-i}|_{\tilde{N}_j}$). Then $\tilde{\nabla}_{ij} \quad \tilde{N}_i \to \tilde{N}_j$ (resp. $\tilde{\Delta}_{ji} \quad \tilde{N}_j \to \tilde{N}_i$). If we fix the bases $\{\tilde{p} : \; p \in N_i\}$ and $\{\tilde{q} : \; q \in N_j\}$, we can associate whit the linear operator $\tilde{\nabla}_{ij}$ (resp. $\tilde{\Delta}_{ji}$) a matrix R_{ij} (resp. L_{ji}) whose columns (resp. rows) and rows (resp. columns) are indexed by rank i and rank j elements and whose entry in p's column (resp. row) and q's row (resp. column) equals $\langle \tilde{p}, \tilde{\Delta}_{ji}(\tilde{q}), \tilde{q} \rangle$ (resp.$\langle \tilde{\nabla}_{ij}(\tilde{p}) \rangle$), $p \in N_i$, $q \in N_j$. If $\tilde{\Delta} = \tilde{\nabla}^*$, then

$$L_{ji} = R_{ji}^{\mathbf{T}} \text{ for all } i, j.$$

For a fixed raising (resp. lowering) operator $\tilde{\nabla}$ (resp. $\tilde{\Delta}$) and for $0 \le i \le j \le k \le n$, we have

$$\tilde{\nabla}_{jk}\tilde{\nabla}_{ij} = \tilde{\nabla}_{ik} \qquad \text{(resp. } \tilde{\Delta}_{ji}\tilde{\Delta}_{kj} = \tilde{\Delta}_{ki}\text{)},$$

which implies

$$R_{jk}R_{ij} = R_{ik} \qquad \text{(resp. } L_{ji}L_{kj} = L_{ki}\text{)}.$$

If $\tilde{\nabla}$ (resp. $\tilde{\Delta}$) is the Lefschetz raising (resp. lowering) operator, we use the notation LR_{ij}(resp. LL_{ji}). In particular, $LR_{i,i+1}$ is the *incidence matrix* of rank $i+1$ versus rank i elements, and the entry in p's column (resp. row) and q's row (resp. column) of LR_{ij} (resp. LL_{ji}) equals the number of saturated chains from p to q, $p \in N_i$, $q \in N_j$. Note that, as above, $LL_{ji} = LR_{ij}^{\mathbf{T}}$. We call the matrices LR_{ij} the *Lefschetz matrices of P.*

Let $\ker_{ij}(\tilde{\nabla})$ be the kernel of $\tilde{\nabla}_{ij}$, that is, $\ker_{ij}(\tilde{\nabla}) := \{\varphi \in \tilde{N}_i \;\; \tilde{\nabla}_{ij}(\varphi) = \tilde{0}\}$, and let $\operatorname{rank}_{ij}(\tilde{\nabla})$ be the rank of $\tilde{\nabla}_{ij}$, that is, the dimension of $\tilde{\nabla}_{ij}(\tilde{(N_i)})$. From linear algebra we know that $\ker_{ij}$ is a subspace of $\tilde{N}_i$ and that rank_{ij} equals the matrix theoretical rank of the matrix R_{ij} and, moreover,

$$\operatorname{rank}_{ij} + \dim\ker_{ij} = \dim \tilde{N}_i = W_i, \quad \operatorname{rank}_{ij} = \operatorname{rank} R_{ij} \le \min\{W_i, W_j\}.$$

If $\operatorname{rank}_{ij} = \min\{W_i, W_j\}$ we say that R_{ij} and $\tilde{\nabla}_{ij}$ have *full rank*. Moreover, we say that $\tilde{\nabla}$ has the *full rank property* if all $\tilde{\nabla}_{ij}$, $0 \le i \le j \le n$, have full rank. The definitions for lowering operators are analogous.

The full rank property is a criterion for the strong Sperner property. But let us look first at a weaker condition that already implies the Sperner property.

Theorem 5.1.5. *Let P be a rank unimodal. If there exists an order-raising operator $\tilde{\nabla}$ in $\tilde{P}$ such that $\tilde{\nabla}_{i,i+1}$ is of full rank for all $0 \le i \le n-1$, then P has the Sperner property.*

Let $\tilde{\nabla}$ be a raising operator in $\tilde{P}$. A set $S = \{b^i, b^{i+1}, \ldots, b^j\}$ of elements of $\tilde{P}$ is called a *ranked Jordan string from* $\tilde{N}_i$ to $\tilde{N}_j$ with respect to $\tilde{\nabla}$ if $b^k \in \tilde{N}_k$, $k = i, \ldots, j$, and $\tilde{\nabla}(b^k) = b^{k+1}$,$k = i, \ldots, j-1$,$\tilde{\nabla}(b^j) = 0$. Note that a ranked Jordan string generates an invariant subspace of $\tilde{P}$ with respect to $\tilde{\nabla}$, denoted in the following by V_{ijh} (h enumerates the ranked Jordan strings from $\tilde{N}_i$ to $\tilde{N}_j$). A basis of $\tilde{P}$ that is a union of ranked Jordan strings is said to be a *ranked Jordan basis of $\tilde{P}$ with respect to* $\tilde{\nabla}$. We define:

$$s_{ij}(\tilde{\nabla}) := \begin{cases} rank_{ij} + rank_{i-1,j+1} - rank_{i-1,j} - rank_{i,j+1} & \text{if } 0 \le i \le j \le n, \\ 0 & \text{otherwise} \end{cases}$$

(If the operator $\tilde{\nabla}$ is clear from the context we will write briefly s_{ij}).

With a raising operator $\tilde{\nabla}$ in $\tilde{P}$ we associate its *Jordan function* $J(\tilde{\nabla}, \tilde{P}; x, y)$ defined by

$$J(\tilde{\nabla}, \tilde{P}; x, y) = \sum_{0 \le i \le j \le n} s_{ij}(x^i + x^{i+1} + \ldots + x^j)y^{i+j},$$

that is, a polynomial in the variables x and y with nonnegative integer coefficients.

We say that the ranked poset P of rank $n := r(P)$ has *property* T if for all $0 \le i \le j \le n$ there exist $\min\{W_i, W_j\}$ pairwise disjoint saturated chains starting at some point in the ith level and ending at some point in the jth level. The following theorem is mainly due to Stanley [45] and Griggs [29] ((i)↔(ii)); see also [39].

Theorem 5.1.6. *Let P be a ranked poset. The following conditions are equivalent:*

(i) P is rank unimodal and has the strong Sperner property

(ii) P has property T

(iii) There exist an order-raising operator $\tilde{\nabla}$ on $\tilde{P}$ with the full rank property

(iv) There exists an order-lowering operator $\tilde{\triangle}$ on $\tilde{P}$ with the full rank property

5.1.2 Peck poset

Let $d_k(P)$ be the largest size of k-family (union of k antichains) in P. A ranked poset P is said to have the *k-Sperner property* if the maximum k-family in P equals the largest sum of k Whitney numbers in P, that is, if

$$d_k(P) = \max\{W_{i_1} + \ldots + W_{i_k} \ 0 \le i_1 < \ldots < i_k \le r(P)\}$$

in other words, P is k-Sperner if no union of k antichains is larger than the union of the k-largest levels N_i. Further, P has the *strong Sperner property* (P is *strongly Sperner*) if P has the k-Sperner property for all $k = 1, 2, \ldots$.

In honor of the 'dummy' mathematician G.W. Peck and his best friends Graham, West, Purdy, Erdős, Chung and Kleitman, a ranked poset P is called a *Peck poset* if it is rank symmetric, rank unimodal and if it has the strong Sperner property. This means that if a poset is Peck is also Sperner.
One finds equivalent conditions in Theorem 5.1.6, where one has always to add rank symmetry. In particular, a poset P is Peck, iff it is rank symmetric and there exists and order-raising operator with the full rank property. If, in particular, the Lefschetz raising operator $\tilde{\nabla}$ has the full rank property, then we speak (in the case of rank symmetry) of a *unitary Peck poset.*

Lemma 5.1.7. *Let P be a ranked poset of rank n. It is Peck iff there exists an order raising operator $\tilde{\nabla}$ such that for the Jordan function there holds $J(\tilde{\nabla}, P; x, y) = F(P; x)y^m$, where $F(P; x)$ is the rank-generating function of P. The poset P is unitary Peck iff $J(\tilde{\nabla}_L, \tilde{P}; x, y) = F(P; x)y^n$.*

The following product theorem was first proved by Canfield [13], and then in different ways by Proctor, Saks, Sturtevant [40], by Saks in a more general version [41], and by Proctor [38].

Theorem 5.1.8. (**Peck Product Theorem**. *If P_1 and P_2 are (unitary) Peck posets, then $P_1 \times P_2$ is a (unitary) Peck poset, too.*

Let P be a ranked poset of rank n and let $\tilde{\nabla}$ and $\tilde{\triangle}$ be a raising and lowering operator on $\tilde{P}$, respectively. We say that the pair $(\tilde{\nabla}, \tilde{\triangle})$ has the *commutation property*, briefly *property C* (see [24]), if there are real numbers $\mu_0, \dots, \mu_n$ such that

$$(\tilde{\nabla}\tilde{\triangle} - \tilde{\triangle}\tilde{\nabla})(\tilde{p}) = \mu_{r(p)}\tilde{p} \qquad \text{for all } p \in P.$$

If in particular $(\tilde{\nabla}_L, \tilde{\triangle})$ has property C and if P has unique minimal element, then P is called by Stanley [46] *μ–differential*. If in particular $\mu_i = 2i - n$, $i = 0 \dots n$, then we say that $(\tilde{\nabla}, \tilde{\triangle})$ has the *Lie property*.

The following theorem is mainly due to Proctor [37], who proved the equivalence (i)↔(ii).

Theorem 5.1.9. *The following conditions are equivalent for a ranked poset P of rank n.*

(i) P is Peck poset

(ii) There exist an order-raising operator $\tilde{\nabla}$ and a lowering operator $\tilde{\triangle}$ such that $(\tilde{\nabla}, \tilde{\triangle})$ has the Lie-property

(iii) There exist an order-raising operator $\tilde{\nabla}$ and a lowering operator $\tilde{\triangle}$ such that $(\tilde{\nabla}, \tilde{\triangle})$ has property C with a regular sequence $(\mu_0, \dots, \mu_n)$

Remark 5.1.10. *P is unitary Peck iff conditions (ii)(resp. (iii)) in Theorem 5.1.9 hold, with $\tilde{\nabla} = \tilde{\nabla}_L$.*

One has still much freedom in constructing $(\tilde{\nabla}, \tilde{\triangle})$ satisfying conditions (ii)(resp. (iii)) of Theorem 5.1.9. But one has to solve a nonlinear (quadratic) system of equations with a very number of variables. So let us restrict ourselves to the special case when $\tilde{\nabla} = \tilde{\nabla}_L$ and when $\tilde{\triangle}$ is an order-lowering operator. Moreover, we will consider only modular lattices though the results can be given in a more general form (see Proctor [37]). Note that whenever p and q are two elements that cover (resp. which are covered by) a third element then necessarily this third element is $p \wedge q$ (resp. $p \vee q$), and $r(p) = r(q) = r(p \wedge q) + 1$ (resp. $r(p \vee q) - 1$). Since $r(p) + r(q) = r(p \wedge q) + r(p \vee q)$, p and q are covered by (resp. cover) a unique fourth element, namely $p \vee q$ (resp. $p \wedge q$).

We say that the modular lattice P of rank n is *edge-labelable* if there is a function $f: E(P) \to \mathbb{R}$ such that

$$f(p, p \vee q) = f(p \wedge q, q) \qquad \text{for all } p, q \in P \text{ with } p, q \lessdot p \vee q \tag{5.1}$$

and

$$\sum_{e^+ = p} f(e) - \sum_{e^- = p} f(e) = 2r(p) - n \qquad \text{for all } p \in P. \tag{5.2}$$

Corollary 5.1.11. *Every edge-labelable modular lattice is unitary Peck.*

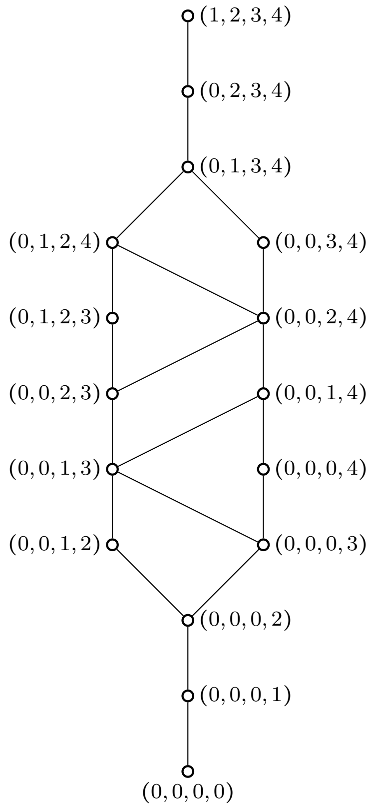
Proof. For $p \in P$, we define

$$\tilde{\triangle}(\tilde{p}) = \sum_{q : q \lessdot p} f(qp)\tilde{q}.$$

It is easy to check that $(\tilde{\nabla}_L, \tilde{\triangle})$ has the Lie property. □

5.1.3 The poset $M(n)$

The poset $M(n)$ is the poset of all n-tuples $a = (a_1, \dots, a_n)$ with $0 = a_1 = \cdots = a_h < a_{h+1} < \cdots < a_n \le n$, $h \in \{0, \dots, n\}$ ($h = 0$ means $a_1 > 0$), with the order given by $a \le b$ iff $a_i \le b_i$ for all i. For the rank of an element a, we have $r(a) = a_1 + \cdots + a_n$; thus $r(M(n)) = \frac{n(n+1)}{2}$. In the following we show $M(4)$.



In the following Theorem we prove that the poset $M(n)$ is an edge-labelable modular lattice, which as we have seen above implies Peck property (Corollary 5.1.11).

Theorem 5.1.12. *The poset $M(n)$ is an edge-labelable modular lattice.*

Proof. Observe at first that, if $\mathbf{b} = (b_1, \dots, b_n)$ covers $\mathbf{a} = (a_1, \dots, a_n)$ in $M(n)$, then there is some index i such that $a_i = b_i - 1$ and $a_j = b_j$ for $j \ne i$. Put for $M(n)$:

$$f(\mathbf{a}, \mathbf{b}) := \begin{cases} \frac{n(n+1)}{2} & \text{if } a_i = 0, \\ n(n+1) - a_i(a_i + 1) & \text{otherwise} \end{cases}.$$

We have to verify (5.1) and (5.2). Let $\mathbf{a}, \mathbf{b} \lessdot \mathbf{c} := \mathbf{a} \vee \mathbf{b}$. We pass from $\mathbf{c}$ to $\mathbf{a}$ by decreasing some component c_i by 1 (i.e., $a_i = c_i - 1$) and to $\mathbf{b}$ by decreasing some other component c_j ($i \ne j$)by 1. Moreover, we pass from $\mathbf{b}$ to $\mathbf{d} := \mathbf{a} \wedge \mathbf{b}$

by decreasing the component $b_i = c_i$ by 1 (i.e., $d_i = c_i - 1 = a_i$). Consequently,

$$f(\mathbf{a},\mathbf{c}) = f(\mathbf{b},\mathbf{d}) := \begin{cases} \frac{n(n+1)}{2} & \text{if } a_i = 0, \\ n(n+1) - a_i(a_i+1) & \text{otherwise} \end{cases}$$

thus (5.1) is satisfied. To verify (5.2) consider any $\mathbf{a} = (a_1, \ldots, a_n) \in N_k$ where $0 = a_1 = \ldots = a_h < a_{h+1} < a_n \leq n$, $h \in \{0, \ldots, n\}$ and define $a_0 := 0, a_{n+1} := n+1$. We have

$$\sum_{e^+=a} f(e) = \sum_{\substack{h+1 \leq i \leq n: \\ a_{i-1} < a_i - 1}} n(n+1) - (a_i - 1)a_i \qquad \left(+\frac{n(n+1)}{2} \text{ if } a_{h+1} = 1\right)$$

$$\sum_{e^-=a} f(e) = \sum_{\substack{h+1 \leq i \leq n: \\ a_i + 1 < a_{i+1}}} n(n+1) - a_i(a_i+1) \qquad \left(+\frac{n(n+1)}{2} \text{ if } a_{h+1} > 1\right)$$

Now it is not difficult to see that in $\sum_{e^+=a} f(e) - \sum_{e^-=a} f(e)$ we may drop the extra conditions on the summation (any new term appearing will cancel each other), and we obtain

$$\sum_{e^+=a} f(e) - \sum_{e^-=a} f(e) = \sum_{h+1 \leq i \leq n} -(a_i - 1)a_i + a_i(a_i+1) - \frac{n(n+1)}{2}$$

$$\sum_{i=h+1}^{n} 2a_i - \frac{n(n+1)}{2} = 2r(\mathbf{a}) - r(M(n)).$$

□

There is a nice application of the Peck property of $M(n)$ to a number-theoretic problem. Let $R = \{r_1, \ldots, r_n\}$ be a set of n distinct positive real numbers and let α be real number. Moreover, let $m(R, \alpha)$ denote the number of subsets of R adding up to α. Thus

$$m(R, \alpha) := \left| \left\{ X \subseteq [n] : \sum_{i \in X} r_i = \alpha \right\} \right|$$

In 1963 Erdős-Moser [27] posed the problem of maximizing $m(R, \alpha)$. First observe that there is a natural bijection $\varphi : 2^{[n]} \to M(n)$ diven by $\varphi(X) := (0, \ldots, 0, i_1, \ldots, i_l)$ for any set $X = \{i_1, \ldots, i_l\} \in 2^{[n]}$ with $i_1 < \cdots < i_l$.

Lemma 5.1.13. *If $|R| = n$, then for all $\alpha \in \mathbb{R}$ the set $\{\varphi(X) : X \subseteq [n]$ and $\sum_{i \in X} r_i = \alpha\}$ is an antichain in $M(n)$.*

Proof. Let, w.l.o.g., $r_1 < \cdots < r_n$. Assume, in the contrary, that there are $X = \{i_1, \ldots, i_l\} \neq Y = \{j_1, \ldots, j_m\} \in 2^{[n]}, i_1 < \cdots < i_l, j_1 < \cdots < j_m$ such that

$$\sum_{k=1}^{l} r_{i_k} = \sum_{k=1}^{m} r_{j_k} = \alpha$$

but $\varphi(X) \leq_{M(n)} \varphi(Y)$. Then it is easy to see that $l \leq m$ and $i_l \leq j_m, i_{l-1} \leq j_{m-1}, \dots, i_1 \leq j_{m-l+1}$. If there was some strict inequality under these inequalities, then we would have $\alpha = \sum_{k=1}^{l} r_{i_k} \leq \sum_{k=1}^{l} r_{j_{m-l+k}} \leq \sum_{k=1}^{m} r_{j_k} = \alpha$ with at least one strict inequality, a contradiction. Consequently, $l = m$ and $i_l = j_l, \dots, i_1 = j_1$, that is, $\varphi(X) = \varphi(Y)$, implying $X = Y$, a contradiction. □

Theorem 5.1.14. *if* $|R| = n$, *then for all* $\alpha \in \mathbb{R}$,

$$m(R, \alpha) \leq m\left([n], \left\lfloor \frac{n(n+1)}{4} \right\rfloor\right) = d(M(n)).$$

Proof. From Lemma 5.1.13 we derive directly

$$m(R, \alpha) \leq d(M(n)).$$

But $M(n)$ is Peck and of rank $\frac{n(n+1)}{2}$. Consequently,

$$d(M(n)) = W_{\lfloor n(n+1)/4 \rfloor}(M(n)) = \left| \left\{ X \subseteq [n] : \sum_{i \in X} i = \left\lfloor \frac{n(n+1)}{4} \right\rfloor \right\} \right|$$

$$= m\left([n], \left\lfloor \frac{n(n+1)}{4} \right\rfloor\right),$$

and the proof is complete. □

5.1.4 $S(n, r)$ is a Sperner poset

It is easy to show that the dual poset $M(n)^{\partial}$ is isomorphic to $M(n)$, thus $M(n)^{\partial}$ is also a Peck poset. It turns out that the lattice $S(n, r)$ is isomorphic to the poset $M(r) \times M(n-r)^{\partial}$ and by Peck Product Theorem 5.1.8 we can conclude that $S(n, r)$ is a Sperner poset.

The poset $M(r) \times M(n-r)^{\partial}$ was studied by Stanley [45], who noticed that the Peck property of $M(n)$ together with Lemma5.1.13 solves the modification of the Erdős-Moser problem. The original Erdős-Moser problem concerns integer sums including negative numbers. Let $S = \{s_1, \dots, s_n\}$ be a set of n distinct real numbers and α any real number. Let

$$em(S, \alpha) := \left| \left\{ X \subseteq [n] : \sum_{i \in X} s_i = \alpha \right\} \right|.$$

Theorem 5.1.15. *For all* $\alpha \in \mathbb{R}$,

$$em(S, \alpha) \leq em(\{-n, -n+1, \dots, 0, \dots, n-1, n\}, 0) \qquad \text{if } |S| = 2n+1,$$

$$em(S, \alpha) \leq em(\{-n+1, \dots, 0, \dots, n-1, n\}, \left\lfloor \frac{n}{2} \right\rfloor) \qquad \text{if } |S| = 2n.$$

Proof. Let us first suppose $0 \notin S$. Moreover, let

$$s_1 < \dots < s_r < 0 < s_{r+1} < \dots < s_n.$$

A subset of S is uniquely characterized by a pair $(X_1, X_2)$with $X_1 \subseteq [r]$, $X_2 \subseteq [r+1, \dots, n]$ and, as previously, by a pair (a_1, a_2) with $a_1 \in M(r)$, $a_2 \in M(n-r)$.

A family of subset of S that all have element sum α there corresponds an antichain in $M(r) \times M(n-r)^*$, which is Peck and so we have

$$em(S,\alpha) \leq d(M(r) \times M(n-r)^*) = h_n(r),$$

where

$$h_n(r) := W_{\lfloor ((r+1)r+(n-r+1)(n-r))/4 \rfloor}(M(r) \times M(n-r)^*)$$

Now consider the maximum of $h_n(r)$. W.l.o.g., we assume that $r \leq n-r$, that is, $r \leq \lfloor \frac{n}{2} \rfloor$, and we will show that the maximum of $h_n(r)$ is attained at $r = \lfloor \frac{n}{2} \rfloor$. Otherwise we would have some $r < \lfloor \frac{n}{2} \rfloor$ with $h_n(r) > h_n(r+1)$. It is easy to see the rank generating function of $M(n)$ is given by

$$F(M(n);x) = (1+x)(1+x^2)\ldots(1+x^n),$$

and consequently,

$$F(M(r) \times M(n-r)^*;x) = (1+x)\ldots(1+x^r)(1+x)\ldots(1+x^{n-r}).$$

Let

$$p(x) := F(M(r) \times M(n-r)^*;x) = \beta_0 + \beta_1 x + \ldots + \beta_d x^d$$

with $d := \frac{(r+1)r+(n-r)(n-r-1)}{2}$. By rank symmetry and rank unimodality, $\beta_0 = \beta_d \leq \beta_1 = \beta_{d-1} \leq \ldots$, and $h_n(r)$ (resp. $h_n(r+1)$) is the largest (i.e. middle) coefficient in $p(x)(1+x^{n-r})$ (resp. $p(x)(1+x^{r+1})$). We put $\beta_i := 0$ if $i \notin \{0,\ldots,d\}$. We have

$$\begin{aligned} h_n(r) &= \beta_{\lfloor (d+n-r)/2 \rfloor} + \beta_{\lfloor (d+n-r)/2 \rfloor-(n-r)} \\ &\leq \beta_{\lfloor (d+r+1)/2 \rfloor} + \beta_{\lfloor (d+r+1)/2 \rfloor-(r+1)} = h_n(r+1) \end{aligned}$$

since

$$\beta_{\lfloor (d+n-r)/2 \rfloor} \leq \beta_{\lfloor (d+r+1)/2 \rfloor}$$

in view of $\frac{d+n-r}{2} \geq \frac{d+r+1}{2} \geq \frac{d}{2}$ and $\beta_{\lfloor (d+n-r)/2 \rfloor-(n-r)} \leq \beta_{\lfloor (d+r+1)/2 \rfloor-(r+1)}$ because of $\frac{d+n-r-2(n-r)}{2} = \frac{d+r-n}{2} \leq \frac{d-r-1}{2} = \frac{d+r+1-2(r+1)}{2} \leq \frac{d}{2}$, a contradiction. Consequently, indeed $h_n(\lfloor \frac{n}{2} \rfloor) = \max\{h_n(r) : 0 \leq r \leq \lfloor \frac{n}{2} \rfloor\}$.

Thus under the supposition $0 \notin S$, for $|S| = 2n$, $em(S,\alpha) \leq h_{2n}(n) = em(\{-n,-n+1,\ldots,-1,1,\ldots,n\},0)$, and for $|S| = 2n+1$, $em(S,\alpha) \leq h_{2n+1}(n) = em(\{-n,\ldots,-1,1,\ldots,n+1\},\lfloor \frac{n+1}{2} \rfloor)$.

Finally we allow 0 to be an element of S. Clearly for $S = S' \cup \{0\}$, $0 \notin S'$, $em(S,\alpha) = 2em(S',\alpha)$(we can take the subsets with and without zero). For the sake of brevity, we consider only the case $|S| = 2n+1$. From the preceding we know that the best we can achieve when 0 is included is $2h_{2n}(n) = em(\{-n,\ldots,-1,0,1,\ldots,n\},0)$ and, without including 0, the optimum is $h_{2n+1}(n) = em(\{-n,\ldots,-1,1,\ldots,n+1\},\lfloor \frac{n+1}{2} \rfloor))$. But $2h_{2n}(n)$ is twice the largest (i.e., middle) coefficient in $(1+x)\ldots(1+x^n)(1+x)\ldots(1+x^n)$, which is not less than largest coefficient $h_{2n+1}(n)$ in $(1+x)\ldots(1+x^n)(1+x)\ldots(1+x^n)(1+x^{n+1})$ (which is the sum of two coefficient from the preceding polynomial). Thus result follows.

□

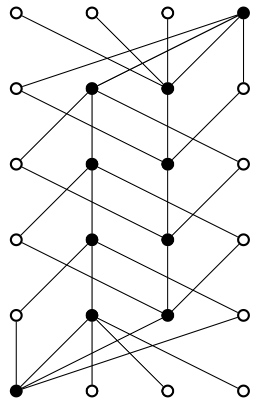


Figure 5.1: class of graded posets which for $r(P) = 5$.

5.2 The size largest antichain in the partition lattice

For many interesting classes of graded poset holds the equality $d(P) = b(P)$. But there exist graded posets where the ratio is arbitrarily large. For the class of graded posets which is illustrated in Figure 5.1 for $r(P) = 5$ we have

$$\frac{d(P)}{b(P)} = \frac{|P|}{8} + \frac{1}{2}$$

In [17] was proven an upper bound for each graded poset.

Theorem 5.2.1. *Let P be a graded poset. Then*

$$\frac{d(P)}{b(P)} \leq max\left\{\frac{|P|}{8} + \frac{1}{2}, 2\right\}$$

Proof. We proceed by induction on $r(P)$. The case $r(P) = 0$ is trivial. Let A be a maximum antichain in P.

Case 1. There is some $k \in \{0, \dots, r(P)\}$ such that $|A \cap N_k| = |N_k|$. Since P is graded, we have $A = N_k$ and thus $d(P)/b(P) = 1$.

Case 2. There is some $k \in \{0, \dots, r(P) - 1\}$ such that $|A \cap N_k| = |N_k| - 1$. Let

$$A_l := \bigcup_{i=0}^{k-1} (A \cap N_i) \quad \text{and} \quad A_u := \bigcup_{i=k+1}^{r(P)} (A \cap N_i).$$

Let p the unique element of $N_k \setminus A$. Since P is graded, all elements of A_l and A_u are comparable with p (for absurd exists $c \in A \cap N_i \quad i \neq k$ such that c is incomparable with p, hence $|A \cap N_k| = |N_k|$). Hence $A_l = \varnothing$ or $A_u = \varnothing$ (since all elements of A_l and A_u are comparable with p, all elements of A_l are comparable with elements of A_u), let w.o.l.g. $A_u = \varnothing$. Let

$$P' := \bigcup_{i=0}^{k-1} N_i.$$

Clearly, P' is also graded and

$$d(P) = |A| \le d(P') \le d(P) \qquad\qquad b(P') \le b(P)$$

Consequently, by the induction hypotesis

$$\frac{d(P)}{b(P)} \le \frac{d(P')}{b(P')} \le max\left\{\frac{|P'|}{8} + \frac{1}{2}, 2\right\} \le max\left\{\frac{|P|}{8} + \frac{1}{2}, 2\right\}.$$

Case 3. Not Case 1 and not Case 2. Then $|A \cap N_k| \le |N_k| - 2$

$$d(P) = |A| = \left|\bigcup_{i=0}^{r(P)} A \cap N_i\right| \le \left(\bigcup_{i=0}^{r(P)} N_i\right) - 2(r(P)) = |P| - 2(r(P) + 1) + 2.$$

Obviously,

$$|P| = b(P)(r(P) + 1), \text{ i.e. } r(P) + 1 \ge \frac{|P|}{b(P)}.$$

Hence,

$$d(P) \le |P| - 2\frac{|P|}{b(P)} + 2 = |P|\frac{b(P) - 2}{b(P)} + 2$$

and consequently (since $(b(P) - 2)/(b(P))^2$ attains its maximum at $b = 4$)

$$\frac{d(P)}{b(P)} \le \frac{b(P) - 2}{(b(P))^2}|P| + \frac{2}{b(P)} \le max\left\{\frac{|P|}{8} + \frac{1}{2}, 2\right\}$$

□

Some similar results have been obtained in [26].

5.2.1 The partition lattice Π_n

A *partition* of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty, pairwise disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. Let Π_n denote the set of partitions of $[n]$. In other words, if $\pi \in \Pi_n$, then we can write $\pi = A_1|A_2|\ldots|A_k$ and there is an application $z : \pi \to z(\pi) := n_1 + \ldots + n_k = n$, with $n_i := |A_i|$ for $1 \le i \le k$. We denote, for example, the partition $\{4\} \cup \{2, 5, 6\} \cup \{1, 3\}$ of $[6]$ briefly by 4|256|13 (or 526|13|4,. . .), and for this partition $k = 3$ and $n_1 = 1$,$n_2 = 3$ and $n_3 = 2$ (or $n_1 = 3$, $n_2 = 2$,$n_3 = 1$,. . .). Given two partitions $\pi, \sigma \in \Pi_n$, we say that π *refines* σ if π can be obtained from σ by further partitioning one or more blocks of y. In other words, If $\pi, \sigma \in \Pi_n$ $\pi \le \sigma$ if only if for a block $A_i \in \pi$ exists some block $B_j \in \sigma$ such that $A_i \subseteq B_j$. The reflexive closure of this relation, denoted $\pi \le \sigma$, is reflexive, transitive, and antisymmetric; hence, the pair $(\Pi_n, \le)$ is a finite partially ordered set. For example the previous partition 4|256|13 covers 4|2|56|13, 4|25|6|13, 4|5|26|13, 4|256|1|3. Let $b(\pi)$ stand for the number of blocks in π. Then the rank function is given by $r(\pi) = n - b(\pi)$. In the following, Figure 5.2 we show Π_4.

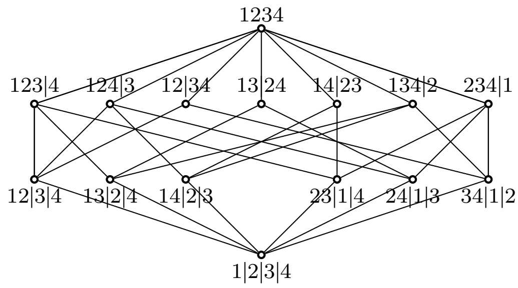


Figure 5.2: Hasse Diagram Π_4.

For the Whitney numbers, we have $W_i(\Pi_n) = S_{n,n-i}$, where $S_{n,k}$ is the corresponding *Stirling number of the second kind*, which is defined by the number of partitions of $[n]$ into exactly k blocks. The following basic recurrence holds for the Stirling numbers:

$$S_{n,k} = kS_{n-1,k} + S_{n-1,k-1} \qquad n,k \geq 1.$$

The *Bell numbers* $B_n := |\Pi_n|$ can be determined by the formula of Dobinski

$$B_n = \frac{1}{e}\sum_{i=0}^{\infty}\frac{i^n}{i!}.$$

For fixed n the sequence $S(n,k)$ is unimodal in k, and it has been of interest of investigate the relationship between $d(\Pi_n)$ and the largest Stirling number $S(n,K_n)$. From [15] we have:

Theorem 5.2.2. *Let $a := (2-e\log 2)/4$. Then for suitable constants c_1, c_2, and $n > 1$*

$$c_1 n^a (\log n)^{-a-1/4} \leq \frac{d(\Pi_n)}{b(\Pi_n)} \leq c_2 n^a (\log n)^{-a-1/4}$$

The lower bound in this theorem is proven in [14]. Moreover, corresponding limit theorems in [23] imply the following result:

Theorem 5.2.3. *We have*

$$b(\Pi_n) \sim \frac{\sqrt{\log n}}{\sqrt{2\pi}}\frac{|\Pi_n|}{\sqrt{n}} \qquad n \to \infty.$$

5.2.2 A quotient of the partition lattice Π_n

W.o.l.g hereafter we consider only patitions $\pi \in \Pi_n$, $\pi = A_1|A_2|\ldots|A_k$ such that $n_1 \geq n_2 \geq \ldots \geq n_k$. Let $\pi = A_1|A_2|\ldots|A_k$, $\sigma = B_1|B_2|\ldots|B_h$ be two partitions in Π_n. We define on Π_n the equivalence relation $\pi \sim \sigma$ iff $|A_i| = |B_j|$ $\forall\ 1 \leq i \leq k,\ 1 \leq j \leq h$. For example, $\pi = 123456|7$ and $\pi = 712345|6$ are equivalent, $\pi,\sigma \in \Pi_7$,. We denote by $[\pi]$ the equivalence class of partition $\pi \in \Pi_n$ and we denote by $\widetilde{\Pi}_n$ the quotient set of Π_n respect to the relation $\sim$. The partition with elements sorted in ascending order is chosen as representative, so $\pi = 123456|7$ is the representative of elements in $\widetilde{\Pi}_7$ which have the form $A_1|A_2$ such that $|A_1| = 6$ and $|A_2| = 1$. In the following Figure 5.3 we show $\widetilde{\Pi}_7$.

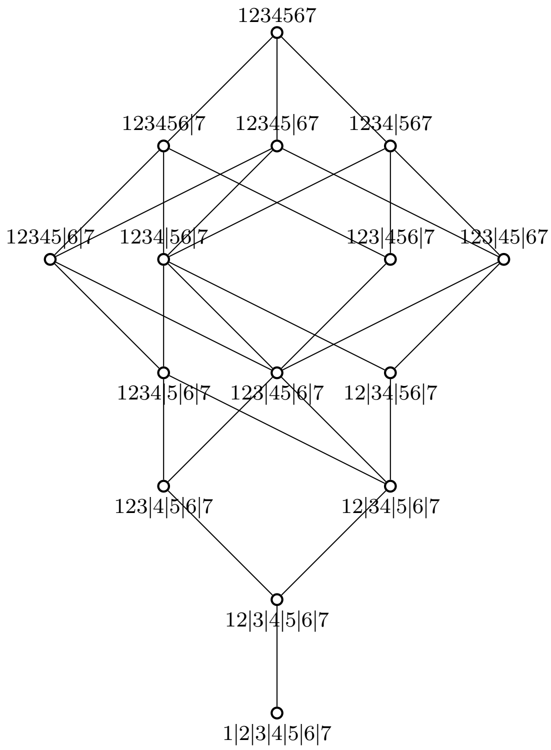


Figure 5.3: Hasse Diagram $\widetilde{\Pi}_7$.

Theorem 5.2.4. *We have*

$$b(\widetilde{\Pi}_n) \sim \frac{\pi}{e\sqrt{6}} \frac{|\widetilde{\Pi}_n|}{\sqrt{n}} \qquad \textit{as} \quad n \to \infty.$$

5.2.3 Estimation of size of the largest antichain in $\widetilde{\Pi}_n$

Let $\widetilde{\Pi}_{2,n}$ be the set of all partitions in $\widetilde{\Pi}_n$ which have blocks with cardinality greater than 1, in other words $\pi = A_1|A_2|\dots|A_k \in \widetilde{\Pi}_n$ if $|A_i| > 1$ for all $1 \le i \le k$.

Theorem 5.2.5. *We have*

$$d(\widetilde{\Pi}_n) \le |\widetilde{\Pi}_{2,n}|.$$

Proof. Let $\varphi : \widetilde{\Pi}_n \setminus \widetilde{\Pi}_{2,n} \to \widetilde{\Pi}_n$ be the mapping that assigns to the partition p (having at least a block with cardinality 1) the partition p' that can be obtained from p by combining a block with cardinality 1 and a block with the largest cardinality of p, for example $p = 1234|5|6|7 \in \widetilde{\Pi}_7 \setminus \widetilde{\Pi}_{2,7}$ we assign to the partition $p' = 12345|6|7$. Clearly, for all $p \in \widetilde{\Pi}_n \setminus \widetilde{\Pi}_{2,n}$ $p \lessdot \varphi(p)$.

The mapping φ is injective since p can be recovered from $\varphi(p)$ (partition the block with the largest cardinality s of $\varphi(p)$ into two blocks with cardinality $s-1$ and 1). Let $l(p)$ be the first natural number for which $\varphi^{l(p)}(p) \in \widetilde{\Pi}_{2,n}$. In addition, let for $p \in \widetilde{\Pi}_{2,n}$, $\varphi^0(p) := p$. If p and q are incomparable elements in $\widetilde{\Pi}_n$, then

$$\varphi^{l(p)}(p) \neq \varphi^{l(q)}(q)$$

since otherwise (say for $l(p) \geq l(q)$) by injectivity of φ

$$\varphi^{l(p)-l(q)}(p) = q,$$

i.e, $p \leq q$. Hence, for any antichain A in $\widetilde{\Pi}_n$,

$$|A| = |\{\varphi^{l(p)}(p) : p \in A\}| \leq |\widetilde{\Pi}_{2,n}|.$$

□

Theorem 5.2.6. *We have*

$$|\widetilde{\Pi}_{2,n}| \sim \frac{\pi}{\sqrt{6}} \frac{\widetilde{\Pi}_n}{\sqrt{n}} \qquad as \quad n \to \infty.$$

Note that from theorems 5.2.4, 5.2.5 and 5.2.6 follows this result:

Theorem 5.2.7.

$$1 \leq \frac{d(\widetilde{\Pi}_n)}{b(\widetilde{\Pi}_n)} \leq e + o(1) \quad as \quad n \to \infty$$

Let $P(n,k)$ (resp. $p(n,k)$) be the number of partitions of n into k or fewer (resp into exactly k) parts and let $p(n) := P(n,n)$. We note that the elements of $\widetilde{\Pi}_n$ are uniquely determined by their cardinality, then there exists an isomorphism between elements of $\widetilde{\Pi}_n$ and partitions of integer. In particular, $p(n) := P(n,n) = |\widetilde{\Pi}_n|$ and the Whitney numbers of $\widetilde{\Pi}_n$ coincide with he number of partitions of n into exactly k parts. Therefore there exists an isomorphism between the poset $\widetilde{\Pi}_n$ and the poset P_n (unordered) of partition of an integer n, ordered by refinement, as defined by G. Birkhoff [[10], pp.16, 104].

We write partitions as $x = (a_1, a_2, \ldots, a_k)$, $y = (b_1, b_2, \ldots, b_l)$ etc., where we assume that $a_1 \geq a_2 \geq \cdots \geq a_k > 0$, $a_1 + a_2 + \cdots + a_k = n$ and similarly for y. Then $x \leq y$ is defined to mean that there is a partition $\{1, \ldots, k\} = J_1 \cup J_2 \cup \cdots \cup J_l$ of the index set of x into l disjoint, nonempty subsets, such that $b_i = \sum_{j \in J_i} a_j$ for all $1 \leq j \leq l$.

Fix $n \geq 1$ then P_n is a graded modular poset of rank $n-1$, with maximal element $\hat{1} = (n)$ and minimal element $\hat{0} = (1, \ldots, 1)$. ts rank function is given by $r(a_1, a_2, \ldots, a_k) = n - k$. The Whitney numbers of the second kind for P_n are $W_k = p(n, n-k)$ number of partitions of n into $n-k$ parts. Then the estimate of the theorem 5.2.7 also holds to the poset P_n, as shown in [17].

To prove 5.2.4 and 5.2.6 we use a result of Szekeres (see [47] and [48]) which was reproved in with a new recursion method in [16] a more or less elementary way:

Theorem 5.2.8. *Let $\epsilon > 0$ be given. Then, uniformly for $k \le n^{1/6}$*

$$P(n,k) = \frac{f(u)}{n} e^{\sqrt{n}g(u)+O(n^{-1/6+\epsilon})}.$$

Here, $u = k/\sqrt{(n)}$, and the functions $f(u), g(u)$ are:

$$f(u) = \frac{v}{\sqrt{8\pi u}} \left(1 - e^{-v} - \frac{1}{2}u^2 e^{-v}\right)^{-1/2} \tag{5.3}$$

$$g(u) = \frac{2v}{u} - u\log\left(1 - e^{-v}\right), \tag{5.4}$$

where $v(= v(u))$ is determined implicitly by

$$u^2 = \frac{v^2}{\int_o^v \frac{t}{e^t - 1} dt}. \tag{5.5}$$

The function

$$F(v) := \frac{v^2}{\int_o^v \frac{t}{e^t-1} dt}$$

is an increasing (continous) function of v, because its derived function is:

$$F'(v) = \frac{v(\int_o^v \frac{2t}{e^t-1}dt - \frac{v^2}{e^v-1})}{(\int_o^v \frac{t}{e^t-1}dt)^2}$$

is positive, because

$$\int_o^v \frac{2t}{e^t-1}dt > \int_o^v \frac{t}{e^t-1}dt \ge \int_o^v \min_{s\in[0,v]} \frac{s}{e^s-1}dt = \int_o^v \frac{v}{e^v-1}dt = v\frac{v}{e^v-1} = \frac{v^2}{e^v-1}.$$

Since the RHS of 5.5 is an increasing function, also u is an increasing function of v, and hence the inverse function of u does exist.

Proposition 5.2.9. *Let C:=$\frac{\pi}{\sqrt{6}}$ with u also v tends to infinity (and vice versa)*

$$\lim_{u\to\infty} \frac{v}{u} = C$$

Proof. Since

$$\frac{1}{e^t-1} = \frac{1}{e^t(1-\frac{1}{e^t})} = e^{-t}\sum_{n=0}^{\infty} e^{-nt} = \sum_{n=1}^{\infty} e^{-nt} \qquad \forall\ t > 0,$$

we have

$$\int_o^{+\infty} \frac{t}{e^t-1}dt = \int_o^{+\infty} t\sum_{n=1}^{\infty} e^{-nt}dt = \int_o^{+\infty} \sum_{n=1}^{\infty} te^{-nt}dt = \sum_{n=1}^{\infty} \int_o^{+\infty} te^{-nt}dt = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Now we apply the identity of Parseval to the function $f(x) = x$ on the interval $[-\pi,\pi]$ to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the function $f(x) = x$

is odd the Fourier coefficients $a_n = 0 \ \forall \ n \geq 0$, instead the Fourier coefficients $b_n = \frac{(-1)^{n+1}2}{n} \ \forall n \geq 1$. By identity of Parseval we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = 4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

i.e.

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

hence $\sum_{n=1}^{\infty}\frac{1}{n^2} = C^2$. The result $\int_o^{+\infty}\frac{t}{e^t-1}dt = C^2$ implies that with u also v tends to infinity (and vice versa) and the thesis follows. □

Lemma 5.2.10. *We have for $u \to \infty$ (or $v \to \infty$)*

$$\frac{v}{u} = C - \frac{v+1}{2C}e^v + O(v^2 e^{-2v}).$$

Proof. For $t \geq 1$ we have:

$$\frac{t}{e^t - 1} = \frac{t}{e^t(1-\frac{1}{e^t})} = \frac{t}{e^t}\sum_{n=0}^{\infty}e^{-nt} \leq \frac{t}{e^t}(1+\frac{2}{e^t}).$$

Then, it results

$$\frac{t}{e^t} \leq \frac{t}{e^t-1} \leq \frac{t}{e^t} + \frac{2t}{e^{2t}}.$$

Using $\int_o^{+\infty}\frac{t}{e^t-1}dt = C^2$ and taking the integral from $v \geq 1$ yields

$$\int_v^{+\infty}\frac{t}{e^t} \leq \int_v^{+\infty}\frac{t}{e^t-1}dt = \int_o^{+\infty}\frac{t}{e^t-1}dt - \int_o^{v}\frac{t}{e^t-1}dt \leq \int_v^{+\infty}\frac{t}{e^t} + \frac{2t}{e^{2t}}$$

$$(v+1)e^{-v} \leq C^2 - \frac{v^2}{u^2} \leq (v+1)e^{-v} + \frac{e^{-2v}(2v+1)}{2},$$

and hence

$$\left(\frac{v}{u}\right)^2 = C^2 - (v+1)e^{-v} + O(ve^{-2v}).$$

Consequently,

$$\frac{v}{u} = C\left(1 - \frac{v+1}{C^2}e^{-v} + O(ve^{-2v})\right)^{1/2}.$$

Using $(1-x)^{1/2} = 1 - \frac{1}{2}x + O(x^2) \quad x \to 0$, with $x = \frac{v+1}{C^2}e^{-v} + O(ve^{-2v})$ we have

$$\frac{v}{u} = C\left(1 - \frac{v+1}{2C^2}e^{-v} + O(v^2e^{-2v})\right) = C - \frac{v+1}{2C}e^{-v} + O(v^2e^{-2v}).$$

□

Lemma 5.2.11. *We have for $u \to \infty$ (or $v \to \infty$)*

$$g(u) = 2C - \frac{1}{C}e^{-v} + O(v^2e^{-2v}).$$

Proof. Using $\log(1-x) = -x + O(x^2) \quad x \to 0$ with $x = e^{-v}$ we have

$$-u\log(1-e^{-v}) = ue^{-v} + O(ue^{-2v})$$

and consequently by 5.4

$$g(u) = \frac{2v}{u} + ue^{-v} + O(ue^{-2v}),$$

and by Lemma 5.2.10

$$g(u) = 2\left(C - \frac{v+1}{2C}e^{v} + O(v^2e^{-2v})\right) + ue^{-v} + O(ue^{-2v})$$

$$g(u) = 2C - \frac{v}{C}e^{-v} - \frac{1}{C}e^{-v} + O(v^2e^{-2v}) + ue^{-v} + O(ue^{-2v}). \tag{5.6}$$

Moreover, by Lemma 5.2.10 we have

$$\frac{v}{u} = C + O(ve^{-v})$$

i.e.

$$v = Cu + O(uve^{-v}).$$

Using Proposition 5.2.9

$$\frac{v}{C} = u + O(v^2e^{-v}),$$

hence

$$\frac{v}{C}e^{-v} = ue^{-v} + O(v^2e^{-2v})$$

i.e.

$$ue^{-v} = \frac{v}{C}e^{-v} + O(v^2e^{-2v})$$

and finally substituting the previous expression in equation 5.6 we have

$$g(u) = 2C - \frac{v}{C}e^{-v} - \frac{1}{C}e^{-v} + \frac{v}{C}e^{-v} + O(v^2e^{-2v}),$$

i.e.

$$g(u) = 2C - \frac{1}{C}e^{-v} + O(v^2e^{-2v}).$$

□

Proposition 5.2.12. *If $k \geq n \geq 1$, then $P(n,k) = p(n)$ for $n \geq k \geq 2$:*

$$P(n.k) = P(n,k-1) + P(n-k,k) \qquad P(n,1) := 1 \quad P(0,k) := 1.$$

Proof. P(n,k) is the number of solution of $x_1 + 2x_2 + \ldots + nx_n = n$ that satisfy $x_1 + x_2 + \ldots \leq k$ also. So we divide the set of solutions into two parts: first the solutions of $x_1 + 2x_2 + \ldots + nx_n = n$ that also satisfy $x_1 + x_2 + \ldots \leq k-1$: there are $P(n,k-1)$ of these; then the solutions of $x_1 + 2x_2 + \ldots + nx_n = n$ which also satisfy $x_1 + x_2 + \ldots = k$; these are just the solutions of $x_2 + 2x_3 + \ldots = n-k$ and $x_2 + x_3 + \ldots \leq k$ (since $x_1 \geq 0$); hence there are $P(n-k,k)$ of these. □

Lemma 5.2.13. *Let* $0 < \delta < \frac{1}{4C}$ *and* $I = \left[\left(\frac{1}{2C} - \delta\right)\sqrt{n}\log n, \left(\frac{1}{2C} + \delta\right)\sqrt{n}\log n\right]$. *Then, uniformly for* $k \in I$ *as* $n \to \infty$

$$P(n,k) \sim p(n) e^{-\frac{\sqrt{n}}{C} e^{-Cu}},$$

$$p(n,k) \sim p(n) e^{-Cu - \frac{\sqrt{n}}{C} e^{-Cu}},$$

$$p(n-k,k) \sim p(n) e^{-2Cu - \frac{\sqrt{n}}{C} e^{-Cu}}.$$

Here $u := k/\sqrt{n}$.

Proof. Obviously $p(n,k) = P(n,k) - P(n,k-1)$, by Proposition 5.2.12 we have

$$p(n,k) = P(n,k-1) + P(n-k,k) - P(n,k-1),$$

i.e.

$$p(n,k) = P(n-k,k) \tag{5.7}$$

and subtracting k from each part

$$p(n-k,k) = P(n-2k,k) \tag{5.8}$$

All the following estimates are uniform for $k \in I$ and taken for $n \to \infty$. Let $i \in \{0,1,2\}$. Let $u_i := \frac{k}{\sqrt{n-ik}}$ and $v_i := v(u_i)$. By equation (5.3)

$$f(u_i) = \frac{v_i}{\sqrt{8\pi} u_i}\left(1 - e^{-v_i} - \frac{1}{2}u_i^2 e^{-v_i}\right)^{-1/2},$$

for $u_i \to \infty$ by Proposition 5.2.9 we have $\lim_{u_i \to \infty} \frac{v_i}{u_i} = C$ and $v_i \sim C u_i$, so

$$f(u_i) \sim \frac{C}{\sqrt{8\pi}}.$$

Moreover, by Theorem 5.2.8

$$P(n-ik,k) = \frac{f(u_i)}{n-ik} e^{\sqrt{n-ik}\, g(u_i) + O((n-ik)^{-1/6+\epsilon})} \sim \frac{C}{\sqrt{8}\pi(n-ik)} e^{\sqrt{n-ik}\, g(u_i)}$$

i.e.

$$P(n-ik,k) \sim \frac{C}{\sqrt{8}\pi n} e^{\sqrt{n-ik}\, g(u_i)}. \tag{5.9}$$

Using $(1-x)^{1/2} = 1 - \frac{1}{2}x + o(x)$ we have

$$\sqrt{n-ik} = \sqrt{n}\left(1 - \frac{ik}{n}\right)^{1/2} = \sqrt{n}\left(1 - \frac{1}{2}\frac{ik}{n} + o\left(\frac{ik}{n}\right)\right),$$

hence

$$\sqrt{n-ik} = \sqrt{n} - \frac{iu}{2} + o(1). \tag{5.10}$$

Now,using $(1-x)^{-1/2} = 1 + O(x)$ we have

$$u_i = \frac{k}{\sqrt{n-ik}} = \frac{k}{\sqrt{n}\left(1-\frac{ik}{n}\right)^{1/2}} = u\left(1-\frac{ik}{n}\right)^{-1/2} = \frac{k}{\sqrt{n}}\left(1+O\left(\frac{ik}{n}\right)\right)$$

hence

$$u_i = u + O\left(\frac{1}{n^{3/2}}\right) = u + O\left(\frac{\log^2 n}{\sqrt{n}}\right). \tag{5.11}$$

Let $\delta < \delta_1 < \delta_2 < \frac{1}{4C}$, since $k \in I$

$$\left(\frac{1}{2C} - \delta_1\right)\sqrt{n}\log n < k < \left(\frac{1}{2C} + \delta_1\right)\sqrt{n}\log n$$

$$\left(\frac{1}{2C} - \delta_1\right)\frac{\sqrt{n}\log n}{\sqrt{n-ik}} < \frac{k}{\sqrt{n-ik}} < \left(\frac{1}{2C} + \delta_1\right)\frac{\sqrt{n}\log n}{\sqrt{n-ik}}.$$

Then for large n,

$$\left(\frac{1}{2C} - \delta_1\right)\log n < u_i < \left(\frac{1}{2C} + \delta_1\right)\log n.$$

By Proposition 5.2.9 it follows

$$\left(\frac{1}{2} - C\delta_2\right)\log n < v_i < \left(\frac{1}{2} + C\delta_2\right)\log n.$$

Consequently,

$$v_i > \log n^{\left(\frac{1}{2} - C\delta_2\right)}$$

$$-v_i < \log \frac{1}{n^{\left(\frac{1}{2} - C\delta_2\right)}}$$

and hence

$$e^{-v_i} < \frac{1}{n^{\left(\frac{1}{2} - C\delta_2\right)}}.$$

From Lemma 5.2.10

$$\frac{v}{u} = C + O(ve^{-v})$$

hence

$$v_i = Cu_i + O(u_i v_i e^{-v_i}) = Cu_i + O(u_i^2 e^{-v_i}) = Cu_i + O\left(\frac{u_i^2}{n^{\left(\frac{1}{2} - C\delta_2\right)}}\right)$$

since $u_i^2 = \frac{k^2}{n^2} = O(\log^2 n)$ we have

$$v_i = Cu_i + O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2} - C\delta_2\right)}}\right)$$

noting (5.11)

$$v_i = Cu + O\left(\frac{\log^2 n}{\sqrt{n}}\right) + O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2} - C\delta_2\right)}}\right) = Cu + O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2} - C\delta_2\right)}}\right),$$

hence,

$$e^{-v_i} = e^{-Cu+O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2}-C\delta_2\right)}}\right)} = e^{-Cu}e^{O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2}-C\delta_2\right)}}\right)},$$

using $e^x = 1 + 0(x)$

$$e^{-v_i} = e^{-Cu}\left(1 + O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2}-C\delta_2\right)}}\right)\right) = e^{-Cu} + e^{-Cu}O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2}-C\delta_2\right)}}\right).$$

Obviously,

$$e^{-Cu} = O\left(\frac{1}{n^{\left(\frac{1}{2}-C\delta\right)}}\right)$$

and thus

$$e^{-v_i} = e^{-Cu} + O\left(\frac{1}{n^{\left(\frac{1}{2}-C\delta\right)}}\right)O\left(\frac{\log^2 n}{n^{\left(\frac{1}{2}-C\delta_2\right)}}\right) = e^{-Cu} + O\left(\frac{\log^2 n}{n^{1-C(\delta+\delta_2)}}\right)$$

i.e.

$$e^{-v_i} = e^{-Cu} + o\left(\frac{1}{\sqrt{n}}\right).$$

By Lemma 5.2.11

$$g(u_i) = 2C - \frac{1}{C}e^{-v_i} + O(v_i^2 e^{-2v_i}) = 2C - \frac{1}{C}\left(e^{-Cu} + o\left(\frac{1}{\sqrt{n}}\right)\right) + O(v_i^2 e^{-2v_i})$$

and this yields

$$g(u_i) = 2C - \frac{1}{C}e^{-Cu} + o\left(\frac{1}{\sqrt{n}}\right),$$

and from (5.10) we derive

$$\sqrt{n-ik}\,g(u_i) = (\sqrt{n} - \frac{iu}{2} + o(1))\left(2C - \frac{1}{C}e^{-Cu} + o\left(\frac{1}{\sqrt{n}}\right)\right)$$

i.e.

$$\sqrt{n-ik}\,g(u_i) = (\sqrt{n} - \frac{iu}{2})\left(2C - \frac{1}{C}e^{-Cu}\right) + o(1) \tag{5.12}$$

Note that by the Hrady-Ramanujan formula [30] (put in Theorem 5.2.8$u := \sqrt{n}$)

$$p(n) := P(n,n) \sim \frac{C}{\sqrt{8}\pi n}e^{\sqrt{n}2C}. \tag{5.13}$$

From (5.9) and (5.12) we obtain

$$P(n-ik,k) \sim \frac{C}{\sqrt{8}\pi n}e^{(\sqrt{n}-\frac{iu}{2})(2C-\frac{1}{C}e^{-Cu})+o(1)}$$

i.e.

$$P(n-ik,k) \sim \frac{C}{\sqrt{8}\pi n}e^{\sqrt{n}2C}e^{-iCu-\frac{\sqrt{n}}{C}e^{-Cu}+o(1)},$$

using (5.13)

$$P(n-ik,k) \sim p(n)e^{-iCu-\frac{\sqrt{n}}{C}e^{-Cu}+o(1)}.$$

Now for $i=0$

$$P(n,k) \sim p(n)e^{-\frac{\sqrt{n}}{C}}e^{-Cu}.$$

The assertion follows from (5.7) and (5.8), for $i=1,2$ we have

$$p(n,k) = P(n-k,k) \sim p(n)e^{-Cu-\frac{\sqrt{n}}{C}e^{-Cu}},$$

$$p(n-k,k) = P(n-2k,k) \sim p(n)e^{-2Cu-\frac{\sqrt{n}}{C}e^{-Cu}}.$$

□

In the following let only $i \in \{1,2\}$. Note that

$$U_i := \frac{1}{2C}\log n - \frac{1}{C}\log iC = \frac{1}{C}\log\frac{\sqrt{n}}{iC}$$

is the unique point at which the function

$$h_i(u) := -iCu - \frac{\sqrt{n}}{C}e^{-Cu}$$

achieves its maximum. Now

$$h_i(U_i+t) = -iC(U_i+t) - \frac{\sqrt{n}}{C}e^{-C(U_i+t)}$$

$$h_i(U_i+t) = -iCU_i - iCt - \frac{\sqrt{n}}{C}e^{-CU_i}e^{-Ct}$$

$$h_i(U_i+t) = -iC\left(\frac{1}{C}\log\frac{\sqrt{n}}{iC}\right) - iCt - \frac{\sqrt{n}}{C}e^{-C\left(\frac{1}{C}\log\frac{\sqrt{n}}{iC}\right)}e^{-Ct}$$

$$h_i(U_i+t) = \log\left(\frac{iC}{\sqrt{n}}\right)^i - iCt - ie^{-Ct}$$

hence

$$e^{h_i(U_i+t)} = e^{\log\left(\frac{iC}{\sqrt{n}}\right)^i - iCt - ie^{-Ct}}$$

$$e^{h_i(U_i+t)} = \frac{(iC)^i}{n^{i/2}}e^{-iCt-ie^{-Ct}}. \tag{5.14}$$

let $0<\delta<\frac{1}{4C}$ and let $\underline{U_i} := U_i - \delta\log n$, $\overline{U_i} := U_i + \delta\log n$. Further let $\underline{k_i} := \lfloor \underline{U_i}\sqrt{n}\rfloor$, $\overline{k_i} := \lfloor \overline{U_i}\sqrt{n}\rfloor$, $k_i^* := \lfloor U_i\sqrt{n}\rfloor$ and $\underline{u_i} := \underline{k_i}/\sqrt{n}$, $\underline{v_i} := v(\underline{u_i})$.

Lemma 5.2.14. *We have for* $i \in \{1,2\}$

$$P(n,\underline{k_i}) = o\left(\frac{p(n)}{\sqrt{n}}\right).$$

Proof. Since

$$\underline{u_i} := \frac{\underline{k_i}}{\sqrt{n}} = \frac{\lfloor \underline{U_i}\sqrt{n}\rfloor}{\sqrt{n}} = \frac{\lfloor (U_i - \delta\log n)\sqrt{n}\rfloor}{\sqrt{n}}$$

$$\underline{u_i} = \frac{\lfloor \left(\frac{1}{2C}\log n - \frac{1}{C}\log iC - \delta\log n\right)\sqrt{n}\rfloor}{\sqrt{n}} = \left(\frac{1}{2C} - \delta\right)\log n + O(1),$$

we have

$$e^{-C\underline{u_i}} = e^{\log n^{C\delta - \frac{1}{2}} + O(1)} = n^{C\delta - \frac{1}{2}} e^{O(1)}$$

hence

$$-\frac{\sqrt{n}}{C} e^{-C\underline{u_i}} = -\frac{\sqrt{n}}{C}\left(n^{C\delta - \frac{1}{2}} e^{O(1)}\right) = \frac{-n^{C\delta} e^{O(1)}}{C}$$

and thus

$$e^{-\frac{\sqrt{n}}{C} e^{-C\underline{u_i}}} = e^{\frac{-n^{C\delta} e^{O(1)}}{C}} = o\left(\frac{1}{\sqrt{n}}\right).$$

From Lemma 5.2.13

$$P(n, \underline{k_i}) \sim p(n) e^{-\frac{\sqrt{n}}{C} e^{-C\underline{u_i}}}$$

the assertion follows

$$P(n, \underline{k_i}) \sim p(n) o\left(\frac{1}{\sqrt{n}}\right) = o\left(\frac{p(n)}{\sqrt{n}}\right).$$

□

Lemma 5.2.15. *We have for $i \in \{1, 2\}$*

$$p(n - \overline{k_i}) = o\left(\frac{p(n)}{\sqrt{n}}\right).$$

Proof. Let $0 < \delta_1 < \delta$. Since

$$\overline{k_i} = \lfloor \overline{U_i}\sqrt{n}\rfloor = \lfloor (U_i + \delta\log n)\sqrt{n}\rfloor = \left\lfloor \left(\left(\frac{1}{2C} + \delta\right)\log n - \frac{1}{C}\log iC\right)\sqrt{n}\right\rfloor.$$

Then, for large n,

$$\overline{k_i} \le \left(\frac{1}{2C} + \delta_1\right)\sqrt{n}\log n,$$

so

$$n - \overline{k_i} \le n - \left(\frac{1}{2C} + \delta_1\right)\sqrt{n}\log n,$$

$$\sqrt{n - \overline{k_i}} \le \left(n\left(1 - \left(\frac{1}{2C} + \delta_1\right)\frac{\sqrt{n}\log n}{n}\right)\right)^{1/2} = \sqrt{n}\left(1 - \left(\frac{1}{2C} + \delta_1\right)\frac{\log n}{\sqrt{n}}\right)^{1/2},$$

using $(1 - x)^{1/2} = 1 - \frac{1}{2}x + o(x)$

$$\sqrt{n - \overline{k_i}} \le \sqrt{n}\left(1 - \left(\frac{1}{4C} + \frac{\delta_1}{2}\right)\frac{\log n}{\sqrt{n}}\right) + o(1),$$

i.e.

$$\sqrt{n - \overline{k_i}} \le \sqrt{n} - \left(\frac{1}{4C} + \frac{\delta_1}{2}\right)\log n + o(1).$$

Form (5.13) we derive

$$p(n-\overline{k_i}) \sim \frac{C}{\sqrt{8}\pi(n-\overline{k_i})}e^{\sqrt{n-\overline{k_i}}2C} \leq \frac{C}{\sqrt{8}\pi(n-\overline{k_i})}e^{\left(\sqrt{n}-\left(\frac{1}{4C}+\frac{\delta_1}{2}\right)\log n+o(1)\right)2C}$$

but the RHS of the previous equation is equal to:

$$\frac{C}{\sqrt{8}\pi(n-\overline{k_i})}e^{\sqrt{n}2C}e^{\left(-\left(\frac{1}{4C}+\frac{\delta_1}{2}\right)\log n+o(1)\right)2C} \sim p(n)e^{\left(-\left(\frac{1}{4C}+\frac{\delta_1}{2}\right)\log n\right)2C}$$

$$= p(n)e^{\log n\left(-\frac{1}{2}+C\delta_1\right)} = \frac{p(n)}{\sqrt{n}}n^{-C\delta_1} = \frac{p(n)}{\sqrt{n}}o(1) = o\left(\frac{p(n)}{\sqrt{n}}\right).$$

□

Proof of Theorem 5.2.4. By Lemma 5.2.13

$$p(n,k_1^*) \sim p(n)e^{-C\frac{k_1^*}{\sqrt{n}}-\frac{\sqrt{n}}{C}e^{-C\frac{k_1^*}{\sqrt{n}}}} = p(n)e^{h_1\left(\frac{k_1^*}{\sqrt{n}}\right)},$$

by equation (5.14) (note $t = o(1)$)

$$p(n)e^{h_1\left(\frac{k_1^*}{\sqrt{n}}\right)} \sim \frac{C}{e\sqrt{n}}p(n).$$

Because $h_1(U_1)$ is the maximum of $h_1(u)$ we have

$$e^{h_1(u)} = -Cu - \frac{\sqrt{n}}{C}e^{-Cu} \leq e^{h_1(U_1)}$$

and again in view of Lemma 5.2.13 we have for $k \in [\overline{k_1}+1, \underline{k_1}-1]$

$$p(n,k) \lesssim p(n,k_1^*).$$

For $k \leq \underline{k_1}$, for large n

$$p(n,k) \leq P(n,\underline{k_1})$$

and by Lemma 5.2.14

$$P(n,\underline{k_1}) = o\left(\frac{p(n)}{\sqrt{n}}\right) < p(n,k_1^*)$$

thus

$$p(n,k) < p(n,k_1^*) \sim \frac{C}{e\sqrt{n}}p(n) = \frac{\pi}{e\sqrt{6}}\frac{p(n)}{\sqrt{n}}.$$

For $k \leq \overline{k_1}$, for large n, using (5.7)

$$p(n,k) = P(n-k,k)) \leq p(n-\overline{k_1})$$

and by Lemma 5.2.15

$$p(n-\overline{k_1}) = o\left(\frac{p(n)}{\sqrt{n}}\right) < p(n,k_1^*)$$

thus

$$p(n,k) < p(n,k_1^*) \sim \frac{C}{e\sqrt{n}}p(n) = \frac{\pi}{e\sqrt{6}}\frac{p(n)}{\sqrt{n}}.$$

□

Proof of Theorem 5.2.6. Obviously (substract from each part of a memeber of $\widetilde{\Pi}_{2,n}$, a one)

$$|\widetilde{\Pi}_{2,n}| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k).$$

We divide the sum into 3 parts:

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k) = \sum_{k=1}^{\underline{k}_2} p(n-k,k) + \sum_{\underline{k}_2+1}^{\overline{k}_2-1} p(n-k,k) + \sum_{\overline{k}_2}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k).$$

By lemma 5.2.13,uniformly for $k \in I$, where $I = \left[\left(\frac{1}{2C} - \delta\right)\sqrt{n}\log n, \left(\frac{1}{2C} + \delta\right)\sqrt{n}\log n\right]$

$$p(n-k,k) \sim p(n)e^{-2C\frac{k}{\sqrt{n}} - \frac{\sqrt{n}}{C}e^{-C\frac{k}{\sqrt{n}}}}.$$

and from (5.14) with $i = 2$

$$e^{U_2} = \frac{4C^2}{n}e^{-2CU_2 - 2e^{-CU_2}}.$$

Thus

$$\sum_{\underline{k}_2+1}^{\overline{k}_2-1} p(n-k,k) \sim \frac{4C^2}{n}p(n)\sum_{\underline{k}_2+1}^{\overline{k}_2-1} e^{-2C\left(\frac{k}{\sqrt{n}}-U_2\right)-2e^{-C\left(\frac{k}{\sqrt{n}}-U_2\right)}}$$

The sum can be considered as an integral approximation with step size $n^{1/2}$. Since $\underline{k}_2 \to -\infty$ and $\overline{k}_2 \to +\infty$ this sum multiplied by $\sqrt{n}$ converges for $n \to \infty$ to

$$\int_{-\infty}^{+\infty} e^{-2Ct-2e^{-Ct}}dt = \int_{-\infty}^{+\infty} e^{-2Ct}e^{-2e^{-Ct}}dt$$

putting $-2e^{-Ct} = u$ and using integration by parts we have

$$\int_{-\infty}^{+\infty} e^{-2Ct-2e^{-Ct}}dt = \frac{1}{4C}.$$

Consequently,

$$\sum_{\underline{k}_2+1}^{\overline{k}_2-1} p(n-k,k) \sim \frac{4C^2}{\sqrt{n}}p(n)\sqrt{n}\sum_{\underline{k}_2+1}^{\overline{k}_2-1} e^{-2C\left(\frac{k}{\sqrt{n}}-U_2\right)-2e^{-C\left(\frac{k}{\sqrt{n}}-U_2\right)}} \sim \frac{4C^2}{\sqrt{n}}p(n)\frac{1}{4C}$$

so

$$\sum_{\underline{k}_2+1}^{\overline{k}_2-1} p(n-k,k) \sim \frac{C}{\sqrt{n}}p(n).$$

Moreover, by Lemma 5.2.14

$$\sum_{k=1}^{\underline{k}_2} p(n-k,k) \leq P(n,\underline{k}_2) = o\left(\frac{p(n)}{\sqrt{n}}\right).$$

Finally,by Lemma 5.2.15

$$\sum_{\overline{k}_2}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k) \leq p(n-\overline{k}_2) = o\left(\frac{p(n)}{\sqrt{n}}\right).$$

Hence

$$|\widetilde{\Pi}_{2,n}| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k) \sim \frac{C}{\sqrt{n}} p(n)$$

and the assertion is proved. □

Chapter 6

Conclusion

In this thesis the main innovation was to introduce the lattice theory in some combinatorics problems related to the conjecture of Manckam, Miklös e Singhi.

First, we built the lattice $S(n,r)$ and studied the main properties. In the future, we would like to continue to study the lattice $S(n,r)$ in order to obtain new results in this context.

Afterwards, we connected the study of particular weight functions with the study of some Boolean maps on $S(n,r)$. In particular, we showed that

> Let $A \in \mathcal{W}_+(n,r)$ be a Boolean map on $S(n,r)$ exists a weight function $f \in WF(n,r)$, but the reverse is not true.

Now we are interested to find the restrictions in order to:*A Boolean map exists iff exists a weight function.*

Moreover, we have already started to think interesting asymptotic variants of the conjecture of Manckam, Miklös e Singhi. For example, let $A(n,k)$ be the minimum possible number of non-negative k-sums over all possible choices of n numbers with non-negative sum. Suppose that k is asymptotically equal to rn , where r is a real number, $0 < r < 1$. The conjecture of Manckam, Miklös e Singhi states that $A(n,k)$ equals $(n-1$ choose $k-1)$ if $r < 0.25$ and n is sufficiently large. We would be interested in the solution of the following problem:

> Determine $A(n,rn)$ asymptotically for fixed r and for $n \to \infty$, in particular also for $0.25 < r < 0.5$.

The last goal is continue to study some asymptotic techniques for bounding the largest size of an antichain in non-Sperner posets (or techniques about the rapport between $d(P)$ and $b(P)$, the largest size of an antichain and the largest size of a level of a graded poset).

The most interesting problem would be an improvement of the bound for the poset of Partitions of a set.

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