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On the Computational Complexity of Solution Concepts in Compact Coalitional Games

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alla mia famiglia

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Sommario

La teoria dei giochi cooperativi, tramite il concetto di gioco di coalizione, mette a disposizione un elegante modello per i sistemi multi-agente, nei quali gli agenti possono coalizzarsi al fine di potersi garantire il raggiungimento di alcuni risultati. I giochi di coalizione vengono classificati in due tipologie, distinte in base alla modalità che i giocatori possono adottare per suddividersi i frutti della propria collaborazione: i giochi con guadagni trasferibili (giochi TU), in cui i giocatori possono decidere a loro piacimento come suddividere fra loro la ricchezza che hanno prodotto, e i giochi senza guadagni trasferibili (giochi NTU), nei quali i giocatori tramite la loro cooperazione possono ottenere risultati specifici, sui quali hanno delle preferenze, ma dei quali non possono ridistribuirsi a piacimento la ricchezza complessiva.

Un problema rilevante da affrontare nei giochi di coalizione è che, in linea di principio, sono molti i risultati che possono essere associati ad un gioco, e diventa importante riuscire ad individuare quali di questi catturino meglio il comportamento razionale dei giocatori. Ed, infatti, in letteratura, si trovano molte proposte di solution concept, che sono proprio una caratterizzazione formale di quali proprietà debba soddisfare un risultato di un gioco per essere considerato stabile e razionale dal punto di vista dei singoli giocatori e dei gruppi di giocatori. Tra i solution concept più noti ed ampiamente utilizzati citiamo lo Shapley value, il core, il bargaining set, il kernel ed il nucleolo.

Recentemente, Deng e Papadimitriou (1994) hanno riconsiderato la definizione dei solution concept da un punto di vista informatico, sostenendo che in realtà le decisioni prese dagli agenti non possano comportare l'utilizzo di risorse di calcolo illimitate e che, quindi, questo concetto di razionalità limitata possa essere efficacemente modellato in modo formale tramite le nozioni di complessità computazionale del calcolo dei solution cocnept.

Al fine di analizzare la complessità computazionale dei solution concept, diventa rilevante scegliere un modo per rappresentare i giochi di coalizione. In letteratura sono stati proposti molti modi compatti per rappresentare questo tipo di giochi, dove con il termine compatto si intende che lo spazio necessario a rappresentarli è polinomiale nel numero dei giocatori che vi prendono parte. I

graph game (Deng e Papadimitriou, 1994), e le marginal contribution network (Ieong e Shoham, 2005) sono due modi compatti per rappresentare tali giochi. In questo lavoro noi proponiamo una rappresentazione compatta generale, cioè una rappresentazione che è in grado di generalizzare il concetto di compattezza della rappresentazione, fornendo, in tal modo, la possibilità di racchiudere nella definizione le varie rappresentazioni particolari che sono state proposte in letteratura.

L'obiettivo di questa tesi è proprio quello di condurre un'analisi di complessità dei solution concept per questa rappresentazione compatta generica, sia per i giochi TU che per quelli NTU, e sia per i constrained game (giochi vincolati) che sono particolari giochi di tipo NTU che proponiamo.

I nostri risultati di ricerca risolvono i problemi lasciati aperti da Deng e Papadimitriou (1994) sui graph game, e da Ieong e Shoham (2005) sulle marginal contribution network. Più nello specifico, noi dimostriamo che le congetture ipotizzate da Deng e Papadimitriou riguardo il bargaining set ed il kernel dei graph game sono, in effetti, valide, mostrando fra l'altro che, per il kernel, vale un risultato più stringente di quello ipotizzato. Inoltre, per quanto riguarda il problema lasciato aperto da Ieong e Shoham circa l'ammissibilità del core su marginal contribution network, dimostriamo il rispettivo risultato di membership. Per il nucleolo, proponiamo un algoritmo generale per calcolarlo in tempo $\mathbf{F}\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$, mostrando che tale upper bound è in realtà stretto, cioè che in generale non si può far di meglio considerando la nostra rappresentazione compatta generica.

Al fine di ottenere i risultati di membership sui problemi relativi ai solution concept dei giochi TU, sviluppiamo alcuni strumenti per poter operare su sistemi lineari succinti, cioè sistemi lineari il cui numero di vincoli è esponenziale nel numero di variabili, ma la cui rappresentazione ha comunque taglia polinomiale nel numero di variabili. Sebbene il fine ultimo di questi strumenti sia quello di permetterci di dimostrare la membership di problemi su giochi di coalizione, in realtà hanno un interesse rilevante di per sé, in quanto in letteratura sono noti solo i rispettivi risultati per sistemi lineari in cui la rappresentazione dei vincoli sia esplicita.

Noi ci concentriamo, inoltre, sulla complessità computazionale dei giochi NTU, mostrando che la complessità dei problemi aumenti in questo modello rispetto ai giochi TU. Abbiamo notato, in particolare, come in molti scenari reali i giochi TU non siano il miglior modello da adottare, ma che spesso i giochi di NTU offrano una flessibilità maggiore tale da renderli un scelta più adeguata per l'analisi del contesto in esame. Noi proponiamo i constrained game, quale meccanismo maneggevole per poter rappresentare giochi NTU tramite l'utilizzo di un gioco TU standard al quale vengono aggiunti dei vincoli (*constraint*, da cui il nome constrained game) nella forma di disequazioni lineari a variabili miste intere e reali. Per questo tipo di giochi, svolgiamo un'analisi strutturale e di complessità al fine di poter evidenziare se l'aggiunta di vincoli alteri le relazione fra i solution concept analizzati o la loro complessità computazionale.

Abstract

Cooperative game theory provides—under the concept of *coalitional games*—an elegant framework for modeling multi-agent systems where agents might collaborate with other agents, by forming *coalitions* in order to guarantee certain outcomes. Coalitional games are very often classified in different species based on the mechanisms underlying the payoff distribution and come in two guises (Osborne and Rubinstein, 1994): Coalitional Games *with transferable utility* (or TU Games), where there is no constraints whatsoever over the way coalitional worths can be distributed among coalition members, and Coalitional Games *with non-transferable utility* (or NTU Games), where a coalition guarantees a specific set of consequences (payoff distributions) on which its players have specific preferences.

A fundamental problem for coalitional games is that several outcomes might be associated with a given game and, hence, the relevant question comes into play about the outcomes to be looked at as those most properly capturing the rational behavior of the players. And, in fact, various *solution concepts* have been proposed in the literature (see e.g., Shubik, 1981) to individuate outcomes that embody some rational concept of stability. Well-known and widely-accepted solution concepts are the *Shapely value*, the *stable sets*, the *core*, the *kernel*, the *bargaining set*, and the *nucleolus*.

Recently, Deng and Papadimitriou (1994) reconsidered the definition of such solution concepts from a computer science perspective, by arguing that decisions taken by realistic agents cannot involve unbounded resources to support reasoning (Deng and Papadimitriou, 1994; Papadimitriou and Yannakakis, 1994; Simon, 1972, 1996), and by suggesting to formally capture such a bounded rationality principle by assessing the amount of resources needed to compute solution concepts in terms of their computational complexity (Deng and Papadimitriou, 1994; Kalai and Stanford, 1988; Simon, 1996).

In order to deal with the computational complexity of solution concept related problems, the issue of how to represent a coalitional game comes into play. Several compact representation schemes for coalitional games have been proposed in the literature, that are representation schemes whose size is polynomial in the number of players. Among them we cite graph games (Deng and Papadimitriou, 1994), and marginal contribution networks (Ieong and Shoham, 2005). We propose a general compact representation scheme that encompasses all these representation frameworks.

Within the context of such compact representation schemes of games, the very aim of this thesis is to study the computational complexity of solution concepts in TU, NTU, and constrained coalitional games, where constrained games is a particular NTU framework that we propose.

In our research we close the problems left open by Deng and Papadimitriou (1994) on graph games, and by Ieong and Shoham (2005) on marginal contribution networks. In particular, we show that the complexity results on the bargaining set and the kernel only conjectured by Deng and Papadimitriou (1994) actually holds, giving for the kernel a even tighter result. As for the problem left open by Ieong and Shoham (2005) about the non-emptiness problem of the core, we are able to prove the sought membership result. For the nucleolus we devise a general algorithm for its computation, giving a tight upper bound, $\mathbf{F}\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$, to the complexity of this computation.

In order to obtain the membership results of our problems, we develop some tools to deal with succinct linear programs, that are linear programs with exponentially many constraints represented in polynomial space. This results are important in their own.

We also investigate the computational complexity associated with NTU games, showing an increase in the complexity with respect to TU games. We noted that, in many real world scenarios, TU games are not the best model to adopt, but often NTU games are more suitable for the analyzed context. We propose constrained games as an handy way to represent NTU game through standard TU games on which constraints are imposed in the form of mixed-integer linear (in)equalities. On this type of games, we conduct a structural and computational analysis in order to asses whether the issuing of constraints affects relationships among solution concepts or their computational complexity.

Contents

1

Introduction

Game theory is that branch of mathematics studying the way in which strategic interactions amongst rational players produce outcomes with respect to the utilities of those players (Ross, 2006). Even if an extensive mathematical formulation of this theory is only recent, as its formal rendering was proposed by John von Neumann and Oskar Morgenstern in 1944 (von Neumann and Morgenstern, 1953, 3rd edition), game-theoretic insights can be found among philosophers and political commentators going back to ancient times (Ross, 2006).

1.1 The aim of Game Theory

One may ask wether the aim of game theory is concretely to build models and provide tools to "predict" the outcomes of a strategic interaction amongst selfish players. Even if this view of the theory, quite common among non-insiders, is somewhat linked to the aim of the theory, it does not characterize the main aspects of game theory since this has not, in fact, that "predictive" flavor of other sciences like physics. Rather, the three aspects of the theory are the *descriptive*, the *normative*, and the *classifying* one (Aumann, 1983).

Through its interaction's models, game theory aims at describing what will be the agents' behavior, assuming that players taking part in the game are perfect rational agents. Since perfect rational agents model only in some respects human beings that are the real subject of interest and that often cannot, or simply do not want to, act as perfect rational agents maximizing their utility function, studying interactions between perfect rational agents cannot have strictly "predictive" capability with respect to the outcome of humans' interactions. Nonetheless game theory can give us some insight into the behavior of real humans, and it is really on this feature that game theory is descriptive, that is, it is descriptive because sheds some lights on human behavior and not because it predicts the results of their interactions.

From the descriptive aspects descend the normative one, the one prescribing optimal behaviors to be adopted in real interactions. This connection between the two aspect is due to the fact that game theory can individuate in a game what are the behaviors conducted by its perfect rational players and hence these specific behaviors can be the subject of a suggestion to humans interacting in a situation modeled by the analyzed game. An example of prescription that can be drawn from analyzing games through the theory is that in negotiating an international treaty, where each side should be (quite) sure that the proposed treaty indeed represents an equilibrium point, i.e. that it is written in such a way that it is a priori common knowledge that it is not worthwhile for either side to violate it.

The theory is characterized by another aspect, that differs from the descriptive one by the fact that, instead of studying players' behavior, it analyzes the interactions themselves in order to classify them, and this is precisely the classifying aspect of game theory. Classifying games is not a pretended issues purely connected to the developing of a nomenclature of questionable usefulness. On the contrary, it is a need deriving from the deep differences existing between game situations related to their specific and intrinsic peculiarities. In fact, highlighting the peculiar features of a class of games is the first step to let the researcher of the field to develop suitable models and tools to analyze games of the new defined class.

1.2 Strategic vs. Cooperative Game Theory

Since its origin, game theory has been classifying games in two families, that of *strategic games* and that of *coalitional games*, and models and tools tailored for them have been developed. The former type are those games in which their modeling does not account for players to cooperate, instead the latter type are games for which their modeling consider the possibility for players to form coalitions given the availability of a mechanism that can enforce agreements. Despite the fact that works linking these two type of games are present in the literature (see, e.g., Aumann, 1961, 1967; Peleg and Sudhölter, 2007), research on these two areas has mostly been carried out separately.

1.2.1 Strategic games

Well known games like chess or checkers are strategic games since their players play one against the other and do not cooperate. But the strategic theory does not study only games that are in some form similar to these familiar games. Rather, it has been used to model every interaction between self-interested agents having a set of actions they can choose from. Look at the famous *The Prisoner's Dilemma* example (Osborne and Rubinstein, 1994) reported in Figure 1.1.

	Confess	Don't Confess
Confess	$-3, -3$	$0, -4$
Don't Confess	$-4, \ 0$	$-1, -1$

Fig. 1.1. The Prisoner's Dilemma

Figure 1.1 represents the utility functions of the two player taking part in the game, one plays "on the rows", let us call her *player* A, and one plays "on the columns", say *player* B. These two players are two prisoners charged with a crime. They are separated (they cannot communicate and hence cooperate) and they have to choose individually whether confess the crime or not. The absolute value of the utility functions is the year in prison they spend depending on their combined actions.

Let us carry out now an argumentation based on the players' rationality. At first we focus our attention on what player A thinks. She knows that player B is rational and knows the utility functions and hence player B will think "if I don't Confess then player A will prefer to Confess and my utility will be -4, while if Confess also in this case player A will prefer to Confess but my utility will be -3, hence it is better for me to Confess". Since player A is confident of B's rationality she knows that B will Confess and then it is better for her to Confess. A similar reasoning is carried out by B regarding A's choice, and hence also for her it is better to Confess.

By the previous argumentation we know that for both players it is better to Confess and hence we expect to observe, in a context modeled by this game, that both players do Confess. The confession of the two players is characterized by two aspects: it is rational and it is an equilibrium. We have just analyzed its rational aspect, let us concentrate now on the other one. This profile is an equilibrium, more specifically a Nash equilibrium (Nash, 1950), since none of the player prefers to deviate (choose not to confess) given that the other player confesses.

In this case, the *solution concept* "Nash equilibrium", has the two aspects to be descriptive and normative in the sense explained above. A solution concept is a mathematical characterization of the features that a game outcome has to satisfy in order to be observable in the real context modeled through the analyzed game. For this reason, solution concepts—yes there are *many* solution concepts—embody in their definition notions of stability or rationality depending on which particular aspect of the interaction the modeler wants to highlight.

1.2.2 The cooperative perspective

Now, we focus back our attention on the Prisoner's Dilemma. We can note that (Confess,Confess) is not the best outcome that players can obtain, but it is the best they can achieve if they are put aside and they must decide individually at the same time without any kind of prior communications or agreements. Given these specific circumstances, the interaction is classified as a strategic game, and hence strategic models and solution concepts can be adopted to study it. On the contrary, if we assume that the two prisoners are in the same room, or, simply they can communicate, and there is a mechanism to enforce agreements available to them—here the reader can imagine what she prefers, good or evil, from mutual loyalty and support to extreme means of retaliation and revenge—they can agree not to confess in order to receive both only one year of prison instead of three. In this last scenario, the game would be classified as a coalitional game, and hence models of cooperative game theory and cooperative solution concepts should be used to model and analyze the game in this new perspective. In this thesis, we will focus our attention on cooperative game theory and related solution concepts.

1.3 Cooperative Game Theory

In the framework of cooperative game theory, it is assumed—and hence models account for this—that players can form coalitions to achieve their goals. Although one may think that cooperative game theory can be built upon the strategic theory using its models with the only additional feature to consider the possibility of cooperation, the cooperative game theory has its own models and methods. Rarely, in the literature, it is found a cooperative analysis tracking the causality between players' actions and outcomes of the game. Much more common to see is the analysis of cooperative scenarios in which a set of possible payoffs granted to the coalition's members is associated with each coalition. This particular model of coalitional games, in which only this function is given that assigns to each coalition the set of outcomes it can guarantee to itself, is called game in *characteristic function form* (von Neumann and Morgenstern, 1953; Luce and Raiffa, 1957), where the "characteristic function" is exactly the function associating coalitions with payoffs for which that coalition is *effective*.

And it is really on this concept of effectiveness that is built the not much investigated link between strategic and cooperative theory. Before devoting completely our attention to games in characteristic function form, let at first have a glimpse to the "normal form" of coalitional games (Aumann, 1961, 1967; Peleg and Sudhölter, 2007). In this form, coalitional games model the set of possible actions of players, like in the strategic case, but the model accounts for possible cooperation. Relating actions of coalitions' members to the payoff vectors for which coalitions are effective, and hence building the characteristic function, asks for seeking which payoff players can guarantee to themselves when acting jointly with other ones in this or that coalition.

Finding this relationship is not obvious, and the concepts of α- and β-effectiveness were introduced by Aumann and Peleg (1960) to this purpose (see also Aumann, 1961, 1967; Peleg and Sudhölter, 2007). Let S be a coalition of players, subset of the set of all players N, called also *grand-coalition*,

	s_1^3	s_2^3
s_1^T	$(1,-1)$	$(\ \ 0,0)$
s_2^T	$(0,\ \ 0)$	$(-1,1)$

Fig. 1.2. Aumann's (1961) example

taking part in the game. The coalition S is α-effective for the payoff vector $x = (x_i)_{i \in S}$ if there is a joint strategy for players in S such that for every joint strategy of players in $N \setminus S$ every player $i \in S$ gains a payoff of at least x_i. The coalition S is β-effective for x if for every joint strategy of players in $N \setminus S$ there is a joint strategy for players in S guaranteeing to each player $i \in S$ a payoff x_i.

Intuitively, α-effectiveness means that coalition S can guarantee itself, independently of the actions of $N \setminus S$, that each of its members i receives, as a payoff, at least the component x_i of vector x. Instead, β-effectiveness means that S can always act in such a way that each of its members i receives as payoff at least the component x_i of vector x, but the strategy S must use to achieve this result may depend on the actions of $N \setminus S$; in other words, $N \setminus S$ cannot effectively prevent S from obtaining at least the vector x.

Even if there are cases in which these two concepts of effectiveness coincide (Aumann, 1961), this is not always true. Consider, for example, the game in Figure 1.2 proposed by Aumann (1961).

There are three players in this game, that is, $N = \{1, 2, 3\}$; player 1 has only one action available $A^1 = \{s_1^1\}$, while players 2 and 3 have two possible actions $A^2 = \{s_1^2, s_2^2\}$ and $A^3 = \{s_1^3, s_2^3\}$. Assume that a coalition T is formed by players 1 and 2, $T = \{1, 2\}$, and since player 1 has only one action available the set of joint strategies of T contains only two joint strategies $A^T = A^1 \times A^2 = \{s_1^T, s_2^T\}$. To conclude, the utility function $h^T = (h^1, h^2)$ of the coalition T is represented by the matrix in Figure 1.2.

As we can note, coalition T is β-effective for the payoff vector $x = (0, 0)$ but it is not α-effective for it. Indeed, given the action performed by player 3, coalition T can act jointly and guarantee at least a payoff 0 for all its members—if player 3 plays s_1^3 then the coalition T can play s_2^T; otherwise it plays s_1^T, which gives the payoff vector $(0, 0)$. Instead, if the role are inverted and coalition T plays before player 3, there is no joint strategy for T to guarantee in all cases the payoff vector $(0, 0)$—if T plays s_1^T and player 3 plays s_1^3 then player 2 receives less than 0, as required by vector x, while if T plays s_2^T then player 3 can play s_2^3, and in this case player 1 gets less than 0.

To build a game in characteristic function form from a game in cooperative normal form, it suffices to evaluate coalition by coalition for which vector they are effective for, choosing only one concept between α- and β-effectiveness and using it for all coalitions. The example shown above illustrates that the two effectiveness concepts are not interchangeable in the general case, meaning that from a cooperative game in normal form we can obtain two different games in characteristic function form depending on which notion of effectiveness has been adopted.

Almost always, when we deal with a cooperative game in characteristic function form we ignore whether the characteristic function comes from a translation that used this or that notion of effectiveness. In fact, in the literature, games in characteristic function form are not required to be specified along with the underlying effectiveness concept, but it is always assumed that what the characteristic function returns for a coalition is the set of payoff vector for which the coalition is effective without any relation of the joint strategy carried out by that coalition nor the actions performed by players outside it.

Even if this model does not account, in any way, for the action of players, this does not mean that it is a poor model. Many are the real world applications of cooperative game theory illustrated in the literature through games in characteristic function form. This model has been extensively used to study applicative scenarios in economics and social sciences (Aumann and Hart, 2002), but also, is interesting in distributed AI, multi-agent systems, electronic commerce and Internet related issues (Ieong and Shoham, 2005; Conitzer and Sandholm, 2004; Papadimitriou, 2001).

Also, coalitional games themselves have been classified in two families. The distinction is based on the mechanisms underlying the payoff distribution. To illustrate, we talked about a characteristic function that assigns to every coalition a set of the payoff vectors for which the coalition is effective. Sometimes, the characteristic function returns a set of payoff vectors or consequences, and these are the only payoff distributions admitted for that coalition. In this case, we talk about Coalitional Games *with non-transferable utility* (or NTU Games). In other circumstances the characteristic function associates to every coalition a worth—a single real number—that is the total amount of worth that the coalition is able to achieve and that is freely distributable amongst coalition's members. In this case we talk about Coalitional Games *with transferable utility* (or TU Games).

As seen before, a fundamental problem for coalitional games is that several outcomes might be associated with a given game and, hence, the relevant question comes into play about the outcomes to be looked at as those most properly capturing the rational behavior of the players. This matter has been extensively studied in economics and social sciences (Aumann and Hart, 2002). And, in fact, various solution concepts have been proposed for cooperative games to individuate worth distributions that embody some rational concept of stability, i.e., that are somehow "immune" by deviations caused by groups of players who may decide to leave the grand-coalition to form subcoalitions in order to claim for higher worths, and fairness. Well-known and widely-accepted solution concepts on which we will put our attention in this thesis are the *core*, the *kernel*, the *bargaining set*, and the *nucleolus* (Osborne and Rubinstein, 1994; Aumann and Hart, 2002; Shubik, 1981). Each solution concept defines a set of outcomes which are referred to with the name of the underlying concept—e.g., the "core of a game" is the set of those outcomes satisfying the conditions associated with the concept of core.

1.3.1 The computational complexity perspective

Recently, Deng and Papadimitriou (1994) reconsidered the definition of such solution concepts from a computer science perspective, by arguing that decisions taken by realistic agents cannot involve unbounded resources to support reasoning (Deng and Papadimitriou, 1994; Papadimitriou and Yannakakis, 1994; Simon, 1972, 1996), and by suggesting to formally capture such a bounded rationality principle by assessing the amount of resources needed to compute solution concepts in terms of their computational complexity (Deng and Papadimitriou, 1994; Kalai and Stanford, 1988; Simon, 1996).

Knowing the complexity of the solution concepts is not only related to the possible modeling of agent with bounded rationality using these concepts. It is also worthwhile to asses solution complexity from a bare practical viewpoint, inasmuch as coalitional games and their solution concepts have been used to model scenarios more oriented to applications in computer science and operational research. And hence, if we know that a solution concept of a particular coalitional game represent the set of solutions for that scenario, it will be of primary interest to known whether evaluating that solution concept is indeed computationally feasible or not.

For example, through coalitional games, one may model the traffic through autonomous systems over the Internet (Papadimitriou, 2001; Markakis and Saberi, 2005), the value of agents in recommendation systems (Kleinberg et al., 2001), coalition formation in multi-agent systems (Sandholm et al., 1999), the problem of facility location (such as hospitals, libraries, etc.) (Goemans and Skutella, 2004), supply chain management (Cachon and Netessine, 2004; Thun, 2003, 2005), alliances between liner shipping (Song and Panayides, 2002), distributed vehicle routing (Sandholm and Lesser, 1997), and many others.

Often, obtaining intractability results in the previous scenarios is a bad news because this means that finding a good or stable solution will be very difficult. But an intractability result is not always to be considered necessarily harmful. In the context of voting systems for example, the celebrated Gibber-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) demonstrated that every voting systems with at leats three candidates satisfying non-imposition (for every candidate, there exist votes that would make that candidate win) and non-dictatorship (the election rule does not simply always choose the most-preferred candidate of a single fixed voter) is manipulable, meaning that a subset of the agents can report their preferences insincerely to make the outcome more favorable to them. It was shown that these manipulations on voting systems are not simply achieved, and, it in this case, the more the manipulation is difficult, even intractable, the better it is (see e.g., Xia et al., 2009).

In order to deal with the computational complexity of solution concept related problems, the issue of how to represent a coalitional game comes into play. The main issue here is to decide whether to explicitly represent the entire worth/consequence function for all possible coalitions or not. Note that, from a computational perspective, to explicitly represent these associations of coalitions with their worths or consequences is unfeasible, since listing all those associations would require exponential space in the number of players. This problem is practically relevant in several application cases as, for instance, with typical Internet-based ones. Moreover, from a computational complexity perspective, having a so huge representation of the game may alter the results of complexity analysis (since it is always carried out with respect to the problem's input size).

These are the reasons why several compact representation schemes for coalitional games have been proposed in the literature. Among them we cite weighted voting games (Elkind et al., 2007), graph games (Deng and Papadimitriou, 1994), and marginal contribution networks (Ieong and Shoham, 2005).

Within the context of such compact representation schemes of games, the very aim of this thesis is to study the computational complexity of solution concepts in TU, NTU, and constrained coalitional games, where constrained games is a particular NTU framework that we propose.

1.4 Contributions

Given a game and a solution concept, relevant problems are deciding whether a payoff vector belongs to the solution concept and deciding whether that solution concept admits a vector at all since there are solution concepts that, for some games, can be empty.

1.4.1 TU Games

Ieong and Shoham (2005) studied the computational complexity of deciding whether the core of a coalitional game represented through a marginal contribution network is non-empty. They proved that it is co-**NP**-hard, but they left as on open question the membership for this problem.

Deng and Papadimitriou (1994) proved in their work many complexity results about solution concepts in graph games, but for the bargaining set and the kernel they conjectured the complexity ($\mathbf{\Pi_2^P}$-complete and **NP**-hard respectively). Moreover their polynomial time algorithm for computing the nucleolus works only on a specific class of graph games and they asked for complexity results of solution concepts for a more general class of compact representation and not only for graph games.

We have closed those open problems for every "compact" representation, hence fulfilling the request of Deng and Papadimitriou (1994), in particular we have proposed a general compact representation scheme for coalitional games (both TU and NTU) (Malizia et al., 2007), where it is just required for the characteristic function to be computable in **FNP**, that is, computable in polynomial-time by a non-deterministic Turing transducer (Papadimitriou, 1994). We have then obtained several complexity results, summarized below. We have proved that deciding whether a payoff vector belongs to:

- the core of any TU game represented in compact form is a problem in co-**NP** (Malizia et al., 2007), hence closing the open problem of Ieong and Shoham (2005);
- the bargaining set of any TU game represented in compact form is a problem in $\mathbf{\Pi_2^P}$ and is $\mathbf{\Pi_2^P}$-complete for graph games (Greco et al., 2009b), hence proving Deng and Papadimitriou's (1994) conjecture;
- the kernel of any TU game represented in compact form is a problem in $\mathbf{\Delta_2^P}$ and is $\mathbf{\Delta_2^P}$-complete for graph games (Greco et al., 2009b), hence proving Deng and Papadimitriou's (1994) conjecture even with a tighter result;
- the nucleolus of any TU game represented in compact form is a problem in $\mathbf{\Delta_2^P}$ and is $\mathbf{\Delta_2^P}$-complete for graph games in the general case (Greco et al., 2009b).

We have not studied the non-emptiness problems for the bargaining set, the kernel, and the nucleolus, since these solution concepts are always non-empty in TU games (see, e.g., Osborne and Rubinstein, 1994; Maschler, 1992).

Note that, while the hardness results are mainly combinatorial contributions (for they rely on the definition of rather elaborate reductions), the membership results appear algorithmic in their nature. Indeed, they are achieved by showing that the various solution concepts on compact games can ultimately be defined in terms of suitable linear programs over exponentially many inequalities (succinctly specified, in their turn). In particular, those results are established by providing complexity bounds on several problems related to succinctly specified linear programs, which are of independent interest, and whose complexity for programs with polynomially many constraints is well-known instead (see e.g., Schrijver, 1998; Papadimitriou and Steiglitz, 1998; Grötschel et al., 1993).

The results obtained show that nothing has to be paid, in this context, for the succinctness of the specifications, since all the membership results that hold for graph games also hold for any class of compact games. Surprisingly, this is true not only for games whose worth functions can be computed by a deterministic polynomial-time oracle, but even for the cases where **FNP** oracles are used instead.

As last part of the research on TU games, we have investigated suitable specializations of graph games, by looking for tractable classes based on exploiting some of their graph invariants. In particular, we focused on the acyclicity property, motivated by the fact that many **NP**-hard problems in different application areas ranging, e.g., from AI (Pearson and Jeavons,

1997) and Database Theory (Bernstein and Goodman, 1981) to Game theory (Daskalakis and Papadimitriou, 2006), are known to be efficiently solvable when restricted to instances whose underlying structures can be modeled via *acyclic* graphs or *nearly-acyclic* ones, such as those graphs having *bounded treewidth* (Robertson and Seymour, 1984).

In particular, Ieong and Shoham (2005) observed that deciding the membership of an outcome into the core and deciding its non-emptiness are feasible in polynomial time on bounded treewidth marginal contribution nets. We have continued along this line of research showing that:

- deciding whether a payoff vector belongs to the kernel of a graph game whose graph has bounded treewidth is feasible in deterministic polynomial time (Greco et al., 2009b).

Interestingly, the above result has been established by showing how the kernel can be expressed in terms of optimization problems over *Monadic Second Order Logic (MSO)* formulae. This was not observed in earlier literature. Thus, on graphs having bounded treewidth, tractability emerges as a consequence of Courcelle's Theorem (Courcelle, 1990) and of its generalization to optimization problems due to Arnborg, Lagergren, and Seese (1991).

1.4.2 NTU Games

We have devoted part of our work also to NTU games. For the particular definition of NTU games given by Osborne and Rubinstein (1994), we have proposed a representation scheme, called NTU marginal contribution networks, extending marginal contribution networks to the NTU setting (Malizia et al., 2007), and we have studied the complexity of solutions concept on this particular scheme.

In the NTU setting there is no widely accepted definition for the kernel nor for the nucleolus, so we have focused our analysis on the core and the bargaining set. More specifically, checking whether a payoff vector belongs to:

- the core of any NTU game represented in compact form is a problem in co-**NP** and is co-**NP**-complete for NTU marginal contribution networks;
- the bargaining set of any NTU game represented in compact form is a problem in $\boldsymbol{\Pi_2^P}$ and is $\boldsymbol{\Pi_2^P}$-complete for NTU marginal contribution networks.

We have also analyzed the complexity of non-emptiness problems for the same solution concepts obtaining that deciding the non-emptiness of:

- the core of any NTU game represented in compact form is a problem in $\boldsymbol{\Sigma_2^P}$ and is $\boldsymbol{\Sigma_2^P}$-complete for NTU marginal contribution networks (Malizia et al., 2007);
- the bargaining set of any NTU game represented in compact form is a problem in $\boldsymbol{\Sigma_3^P}$ and is $\boldsymbol{\Sigma_3^P}$-complete for NTU marginal contribution networks.

1.4.3 Constrained Games

We have noticed that in several applicative scenarios, assuming complete transferability of the worth amongst coalition's member is not adequate to model the analyzed context that, often, looked closely at, would be better modeled through an NTU framework. This is common since sometimes we face with market regulation authorities, or more simply with discrete domains that need models that are, in some sense, constrained to the peculiar features of the studied case.

To put these observations in a formal framework, we have proposed *constrained games* as a particular case of NTU games. Constrained games are NTU games defined upon TU games, whose characteristic function is represented by means of a value (the worth of the same coalition in the underlying TU game) and a set of constraints expressed as *mixed-integer linear (in)equalities*. In this way, in a constrained game, a coalition S is effective for a payoff vector x if it is effective for x in the underlying TU game and x satisfies the constraints.

Imposing linear constraints on the outcomes of games is an approach that has been explored by several authors in the context of strategic games (e.g., Charnes, 1953; Semple, 1997; Ryan, 1998), but in the context of cooperative games, instead, this approach has received considerably less attention and, indeed, no general framework was proposed in the literature and no analysis of its properties was conducted so far.

Being based on imposing linear (in)equalities, the proposed framework provides a unifying view of earlier approaches to define constrained coalitional games. Moreover, by using integer variables, also non-convex imputation sets can be represented. This feature may be relevant in several application scenarios, as it allows us to model in a natural way all those situations where arbitrary fractions of the objects to be distributed are meaningless, and thus not acceptable as possible outcomes of the game.

In fact, the framework we have developed shares the spirit of the recent arguments of Shoham (2008), who advocated the use of a broader vocabulary than the fairly terse one characterizing the early foundations of the game theory, and with those proposals that reconsidered basic concepts of cooperative games in the light of a modeling perspective that is closer to the requirements of the computer science applications.

Very seminal and influential proposals of this type give rise, in particular, to coalitional skill games (Bachrach and Rosenschein, 2008), qualitative coalitional games (Wooldridge and Dunne, 2004), coalitional resource games (Wooldridge and Dunne, 2006), Bayesian coalitional games (Ieong and Shoham, 2008), multi-attribute coalitional games (Ieong and Shoham, 2006), temporal qualitative coalitional games (Ågotnes et al., 2006), and cooperative Boolean games (Dunne et al., 2008).

Differently from such recent approaches, however, our constrained games are rather close to standard TU formalizations, since non-transferable conditions are just imposed on top of them.

In our work, we have conducted a systematic study of constrained games within the proposed framework, that has been analyzed with respect to the structural and the computational properties of some relevant solution concepts in order to asses whether the issuing of constraints affects relationships among solution concepts or their computational complexity (compared to standard TU games) (Greco et al., 2009a).

1.4.4 Outline of the thesis

We are now ready to embark on the full details of the aforementioned results. In Chapter 2 we will give some preliminaries on cooperative game theory, both the TU and NTU framework, and complexity theory. In the last part of it we will define what is for us a “compact” representations, since this definition will accompany us along the whole thesis.

After that, we will devote the three subsequent chapters to the analysis of the three frameworks: in Chapter 3 we will study TU games, in Chapter 4 we will analyze NTU games, while in Chapter 5 we will formally define our framework for constrained games that will be subsequently studied in that same Chapter. Finally we will draw our conclusions in Chapter 6. In the appendix we will report the details of our study on compact linear programs needed for the membership results for TU games, and other details for other issues.

2

Preliminaries

Cooperative game theory provides—under the concept of *coalitional games*—an elegant framework for modeling multi-agent systems where agents might collaborate with other agents, by forming *coalitions* in order to guarantee certain outcomes. Coalitional games are very often classified in different species based on the mechanisms underlying the payoff distribution and come in two guises (Osborne and Rubinstein, 1994): Coalitional Games *with transferable utility* (or TU Games), where there is no constraints whatsoever over the way coalitional worths can be distributed among coalition members, and Coalitional Games *with non-transferable utility* (or NTU Games), where a coalition guarantees a specific set of consequences (payoff distributions) on which its players have specific preferences.

In the cooperative framework that we are going to analyze, to every coalition is assigned, through a function called *characteristic function*, the set of outcomes for which that coalition is *effective*[1]. Note that the characteristic function does not account for the actions taken by players, nor the players' influence on the payoff of coalitions they do not belong to, but it lists (or encode in a specific way) the outcomes for that coalition. Games modeled in this way are called games in *characteristic function form* (von Neumann and Morgenstern, 1953; Luce and Raiffa, 1957).

A fundamental problem for coalitional games is that several outcomes might be associated with a given game and, hence, the relevant question comes into play about the outcomes to be looked at as those most properly capturing the rational behavior of the players. This matter has been extensively studied in economics and social sciences (Aumann and Hart, 2002). And, in fact, various *solution concepts* have been proposed in the literature (see, e.g., Shubik, 1981) to individuate outcomes that embody some rational concept of stability, i.e., that are somehow "immune" by deviations caused by groups of players who may decide to leave the grand-coalition to form sub-coalitions in order to claim for higher worths, and fairness. Well-known and widely-accepted so-

[1] For the different notions of effectiveness refer to Section 1.3.

lution concepts are the *Shapely value*, the *stable sets*, the *core*, the *kernel*, the *bargaining set*, and the *nucleolus*. Each solution concept defines a set of outcomes which are referred to with the name of the underlying concept—e.g., the "core of a game" is the set of those outcomes satisfying the conditions associated with the concept of core.

Now we give some common notions needed for both the TU and the NTU framework. A coalition S is a non-empty subset of N, where N is the set of all players. For any coalition $S \subseteq N$, $|S|$ denotes the cardinality of S and $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space whose axes are labeled with the members of S. A payoff vector $x \in \mathbb{R}^S$ is the vector $(x_i)_{i \in S}$, also called an S-*vector*, whose $|S|$ components are labeled through players in S. For any pair of players i and j of $\mathcal{G}$, we denote by $\mathcal{I}_{i,j}$ the set of all coalitions containing player i but not player j.

2.1 Transferable utility games

We now recall the definitions and properties for TU games and solutions concepts that will be subject of our research. These definitions and properties may be found in any text on cooperative game theory (see, e.g., Osborne and Rubinstein, 1994; Aumann and Hart, 2002).

In TU games, the characteristic function associates with each coalition a real number that is the total amount of worth that coalition is able to achieve and that is freely distributable amongst coalition's members.

Definition 2.1 (TU Game). *A* Coalitional Game with transferable utility (TU Game) $\mathcal{G}$ *is a pair* $\langle N, v \rangle$ *where:*

- N *is the finite set of players;*
- v *is a function that associates with every coalition* S *a real number* $v(S)$ *(the* worth *of* S*)* $(v\colon 2^N \mapsto \mathbb{R})$*.*

In TU games, a vector $x \in \mathbb{R}^S$ is called an *S-feasible payoff vector* if $\sum_{i \in S} x_i = v(S)$. The value $\sum_{i \in S} x_i$ will be denoted by $x(S)$ in the following. A *feasible payoff profile* is an N-feasible payoff vector. An N-feasible vector x that is also *individually rational* (i.e., such that $x_i \geq v(\{i\})$, for each player $i \in N$) is called an *imputation* of $\mathcal{G}$. The set of all the imputations of $\mathcal{G}$ is denoted by $X(\mathcal{G})$. The *excess* of the coalition S at the imputation $x \in X(\mathcal{G})$, denoted by $e(S, x)$, is defined as $v(S) - x(S)$, and intuitively it measures the dissatisfaction of S at x. The *surplus* $s_{i,j}(x)$ of i against j at x is $s_{i,j}(x) = \max_{S \in \mathcal{I}_{i,j}} e(S, x)$.

Definition 2.2 (Core). *The* core $\mathscr{C}(\mathcal{G})$ *of a TU game* $\mathcal{G} = \langle N, v \rangle$ *is the set of all imputations*[2] x *for which there is no coalition* S *and an* S*-feasible payoff vector* y *such that* $y_i > x_i$ *for all* $i \in S$*.*

[2] In the literature, feasible payoff profiles are sometimes considered in place of imputations. In fact, it is easily checked that the two forms are equivalent as far as the definition of the core is concerned.

A definition that is obviously equivalent is that the core is the set of feasible payoff profile $(x_i)_{i \in N}$ for which $x(S) \leq v(S)$ for all coalitions S. Thus the core is the set of payoff profiles satisfying a system of weak linear inequalities and hence is closed and convex.

Example 2.3. Let $\mathcal{G} = \langle N, v \rangle$ be a TU game with $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, and $v(\{1, 2, 3\}) = 42$. Consider the imputation $x = (4, 14, 24)$. Since $v(\{2, 3\}) = 40 > 38 = x(\{2, 3\})$ such imputation x is not in $\mathscr{C}(\mathcal{G})$. Consider coalitions $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{2, 3\}$ and the worths associated with them by the characteristic function. An imputation x to be in $\mathscr{C}(\mathcal{G})$ have to satisfy three conditions:

$$\begin{aligned} x_1 + x_2 &\geq 20 \\ x_1 + x_3 &\geq 30 \\ x_2 + x_3 &\geq 40. \end{aligned}$$

Summing these inequalities we obtain that $2x_1 + 2x_2 + 2x_3 \geq 90$, implying that $x_1 + x_2 + x_3 \geq 45$. Thus the core of $\mathcal{G}$ is empty because the grand-coalition would need to receive 45 instead of $v(N) = 42$ in order to satisfy the claims of S_1, S_2 and S_3. In fact, if we consider the game $\mathcal{G}' = \langle N, v' \rangle$, whose worth function v' is the same as that of $\mathcal{G}$ except for the grand-coalition for which $v'(N) = 45$, the imputation $x' = (5, 15, 25)$ is easily verified to be in $\mathscr{C}(\mathcal{G}')$.

Let x be an imputation. We say that (y, S) is an *objection of player i against player j to x*, if $S \in \mathcal{I}_{i,j}$, and y is an S-feasible payoff vector such that $y_k > x_k$ for all $k \in S$. If (y, S) is an objection of player i against player j to x, then we say i can object against j to x through S.

A *counterobjection to the objection (y, S) of i against j* is a pair (z, T) where $T \in \mathcal{I}_{j,i}$, and z is a T-feasible payoff vector such that $z_k \geq x_k$ for all $k \in T \setminus S$ and $z_k \geq y_k$ for all $k \in T \cap S$. If (z, T) is a counterobjection to the objection (y, S) of i against j, we say that j can counterobject to (y, S) through T.

If there does not exist any counterobjection to objection (y, S), we say that (y, S) is a *justified objection*.

Definition 2.4 (Bargaining Set). *The* bargaining set $\mathscr{B}(\mathcal{G})$ *of a TU game $\mathcal{G} = \langle N, v \rangle$ is the set of all imputations to which there is no justified objection.*

Example 2.5. Let $\mathcal{G} = \langle N, v \rangle$ be a TU game with $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, and $v(\{1, 2, 3\}) = 42$. Consider now the bargaining set of $\mathcal{G}$. An imputation for this game is $x = (8, 10, 24)$. An objection of player 3 against player 1 to x is $((12, 28), \{2, 3\})$. Player 1 can counterobject to this objection using $((8, 12), \{1, 2\})$. Another objection of player 3 against player 1 to x is $((14, 26), \{2, 3\})$. In this case, player 1 cannot counterobject. The reason why this holds is that coalition $\{1, 2\}$ receives a payoff 20 and this is not sufficient for player 1 to counterobject since she needs at least 8 for herself and at least 14 for player 2, in order to respond to the proposal of player 3. Therefore imputation x does not belong to $\mathscr{B}(\mathcal{G})$. The intuitive reason is that player 1 receives too much, according to this profile. Consider now the imputation $x' = (4, 14, 24)$. We focus on the objections of player 1 against player 2. We note that in order to object player 1 has to form the coalition $S = \{1, 2\}$. The excess $e(S, x)$ of S at x is 2, hence players 1 and 2 have the possibility to distribute among themselves a payoff of 2 to make the objection. But player 3 can always counterobject to player 1 because she can form the coalition $T = \{2, 3\}$ whose excess at x is 2 and hence she can always match the proposal made to player 2 by player 1 in order to object. A similar argumentation holds for every objection of every player against any other. Thus $x' \in \mathscr{B}(\mathcal{G})$.

Definition 2.6 (Kernel). *The* kernel $\mathscr{K}(\mathcal{G})$ *of a TU game* $\mathcal{G} = \langle N, v\rangle$ *is the set:*

$$\mathscr{K}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v(\{j\}), \forall i, j \in N, i \neq j\}.$$

Example 2.7. Let $\mathcal{G} = \langle N, v\rangle$ be a TU game with $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, and $v(\{1, 2, 3\}) = 42$. It is easily verified that the imputation $x = (4, 14, 24)$ is in the kernel of $\mathcal{G}$. At first we note that every player in N receives in x a payoff strictly greater than what she is able to obtain by acting on her own. For this reason, in order for x to belong to $\mathscr{K}(\mathcal{G})$ it must be the case that $s_{i,j}(x) \leq s_{j,i}(x)$ for all distinct players i and j. By the definition of the worth function, the maximum excess that a coalition S including i and excluding j can achieve is obtained by the coalition $S \in \mathcal{I}_{i,j}$ such that $|S| = 2$. By this, $s_{i,j}(x) = 2$ for all pairs of different players i, j. Thus, $x \in \mathscr{K}(\mathcal{G})$.

For any imputation x, define $\theta(x)$ as the vector where the various excesses of all coalitions (but the empty one) are arranged in non-increasing order:

$$\theta(x) = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^{|N|}-1}, x)).$$

Let $\theta(x)[i]$ denote the i-th element of $\theta(x)$. For a pair of imputations x and y, we say that $\theta(x)$ is *lexicographically smaller* than $\theta(y)$, denoted by $\theta(x) \prec \theta(y)$, if there exists a positive integer q such that $\theta(x)[i] = \theta(y)[i]$ for all $i < q$ and $\theta(x)[q] < \theta(y)[q]$.

Definition 2.8 (Nucleolus). *The* nucleolus $\mathscr{N}(\mathcal{G})$ *of a TU game* $\mathcal{G} = \langle N, v\rangle$ *is the set*

$$\mathscr{N}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid \nexists y \in X(\mathcal{G}) \text{ such that } \theta(y) \prec \theta(x)\}.$$

Example 2.9. Let $\mathcal{G} = \langle N, v\rangle$ be a TU game with $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, and

$v(\{1,2,3\}) = 42$. Consider the imputation $x = (4, 14, 24)$. We can note that, by the definition of the characteristic function, all coalitions with two players have an excess of 2 at x. Moreover all other coalitions have a lower excess at x. If we transfer some payoff, say $\delta > 0$, from a player p_1 to a different player p_2 in order to obtain an imputation $x' \neq x$, then all the coalitions S including p_2 would have an excess at x' that is lower than the one at x, in particular it would hold that $e(S, x') = e(S, x) - \delta < e(S, x)$. But all the coalitions T including p_1 would have an excess at x' that is greater than the one at x, in particular it would hold that $e(T, x') = e(T, x) + \delta > e(T, x)$. Hence we have that $\theta(x) \prec \theta(x')$ holds for every payoff transfers $\delta > 0$ between different players, implying that $x \in \mathscr{N}(\mathcal{G})$.

Proposition 2.10. *Let $\mathcal{G}$ be a TU game. Then:*

1. $|\mathscr{N}(\mathcal{G})| = 1$;
2. $\mathscr{N}(\mathcal{G}) \subseteq \mathscr{K}(\mathcal{G})$ *(hence,* $\mathscr{K}(\mathcal{G}) \neq \varnothing$*);*
3. $\mathscr{K}(\mathcal{G}) \subseteq \mathscr{B}(\mathcal{G})$ *(hence* $\mathscr{B}(\mathcal{G}) \neq \varnothing$*);*
4. $\mathscr{C}(\mathcal{G}) \subseteq \mathscr{B}(\mathcal{G})$*; and,*
5. $\mathscr{C}(\mathcal{G}) \neq \varnothing$ *implies* $\mathscr{N}(\mathcal{G}) \subseteq \mathscr{C}(\mathcal{G})$.

2.2 Non-transferable utility games

NTU games, are those cooperative games in which the characteristic function associates with every coalition a set of worth distributions (called in this framework also *consequences*) for which that coalition is effective. There are two different definitions of NTU games in the literature. In the first one, being also the classical one (Aumann and Peleg, 1960), an NTU game is a pair $\langle N, V\rangle$ where N is the set of players, and V is the consequence function associating each coalition S with a subset of $\mathbb{R}^S$ which satisfies the following conditions:

1. $V(S)$ is convex and closed, for each $S \subseteq N$, with $V(\varnothing) = \mathbb{R}^N$;[3]
2. if $x \in V(S)$, $y \in \mathbb{R}^S$ and $(\forall i \in S)(y_i \leq x_i)$, then $y \in V(S)$, for each $S \subseteq N$.

According to this view, an NTU game is actually a game equipped with the concepts of *convexity*, *closeness*, and *comprehensiveness*. This last property is that of condition 2 in which is stated that if S is effective for a given payoff vector x, then it is also effective for any payoff vector which assigns a smaller or equal payoff to all its members)[4].

[3] In some text, $V(\varnothing) = \varnothing$ is required instead of $V(\varnothing) = \mathbb{R}^N$; see, e.g., Peleg and Sudhölter, 2007.

[4] In the original paper by Aumann and Peleg (1960) a form of additivity was also required, that is, $V(S_1 \cup S_2) \supset V(S_1) \times V(S_2)$, for each $S_1, S_2 \subseteq N$ with $S_1 \cap S_2 = \varnothing$. This condition means that if two disjoint coalitions, S_1 and S_2 are effective for the outcomes $x \in \mathbb{R}^{S_1}$ and $y \in \mathbb{R}^{S_2}$ respectively, then the coalition

The other perspective from which NTU games are modeled, is that of Osborne and Rubinstein (1994), in which the consequence function associates with each coalition a set of consequence and does not need to satisfy any specific property.

Definition 2.11. *A* Coalitional Game with non-transferable utility (NTU) *is a four-tuple* $\langle N, X, V, (\succsim_i)_{i\in N}\rangle$*, where:*

- N *is a finite set of players;*
- X *is the set of all possible consequences;*
- V *is a function that assigns, to any coalition* $S \subseteq N$ *of players, a set of consequences* $V(S) \subseteq X$ *(*$V : 2^N \mapsto 2^X$*);*
- $(\succsim_i)_{i\in N}$ *is the set of all preference relations* $\succsim_i$ *on* X*, for each player* $i \in N$*.*

Every preference relation $\succsim_i$ is a complete reflexive transitive binary relation on X (Osborne and Rubinstein, 1994), and if $x \succsim_i y$ but not $y \succsim_i x$ then we write $x \succ_i y$ and intuitively it means that player i strictly prefer consequence x to y.

It is easy to see that the second definition is more general than the first one, and in fact every NTU game with a consequence function satisfying the requirements of Aumann and Paleg is also an NTU game for the Osborne and Rubinstein's definition, but the converse is not always true.

It is also straightforward to show that coalitional games with transferable utility can be seen as special cases of coalitional games with non-transferable utility (Osborne and Rubinstein, 1994). Indeed, given a TU game $\mathcal{G} = \langle N, v\rangle$, its equivalent formulation as an NTU game is that in which the set of players remains the same, and the consequence function is such that

$$V(S) = \left\{ x \in \mathbb{R}^S \,\middle|\, \sum_{i\in S} x_i \le v(S) \right\}.$$

The following definitions are given for the NTU framework of Aumann and Peleg, but them are easily adaptable to that one of Osborne and Rubinstein. Actually, in order to obtain the definitions for the Osborne and Rubinstein's framework it suffices to substitute every payoff comparison of the form $x_i \ge y_i$ with that one $x \succsim_i y$ using the preference relation of player i, peculiar of the other framework.

The following definition, that in some way represent the concept of "imputation" for NTU games, is due to Peleg (1963).

Definition 2.12. *Let* $\mathcal{G}$ *be an NTU game. A consequence* $x \in V(N)$ *is an* individually rational payoff configuration (i.r.p.c.) *if, for all players* $i \in N$*,*

$S_1 \cup S_2$ must be effective for the outcome $z \in \mathbb{R}^{S_1 \cup S_2}$ such that $z_i = x_i$ for all players $i \in S_1$ and $z_i = y_i$ for all players $i \in S_2$. However, this condition is clarified in that paper not to be actually required (in fact, Peleg 1963 gets rid of this further condition).

$x_i \geq z_i$ for all $z \in V(\{i\})$, and there does not exist a consequence $y \in V(N)$ such that $y_i \geq x_i$ for all players $i \in N$.

Definition 2.13 (Core). *The* core $\mathscr{C}(\mathcal{G})$ *of an NTU game $\mathcal{G} = \langle N, V\rangle$ is the set of all i.r.p.c.'s x such that there is not a coalition $S \subseteq N$ and a consequence $y \in V(S)$ such that $y_k \geq x_k$ for all players $k \in S$. If such coalition S and consequence y exist, we say that S* has an objection *to x.*

By the previous definition follows that the core of a game is the set of all i.r.p.c.'s such that no coalition has an objection to them.

Now have a look at the following example. It is given in the Osborne and Rubinstein's framework since in this way we can have an arbitrary consequence function that have not to satisfy any particular requirement. However consequences will be payoff vectors as in the Aumann and Peleg's framework.

Example 2.14. Let $\mathcal{G} = \langle N, X, V, (\succsim_i)_{i\in N}\rangle$ be an NTU game with $N = \{1, 2, 3\}$, the consequences are conveniently represented by payoff vectors (as in the Aumann and Peleg's framework) and in particular $V(\{1\}) = V(\{2\}) = V(\{3\}) = \{(0, 0, 0)\}$, $V(\{1, 2\}) = \{(10, 10, 0); (5, 15, 0)\}$, $V(\{1, 3\}) = \{(10, 0, 20); (15, 0, 15)\}$, $V(\{2, 3\}) = \{(0, 10, 30); (0, 12, 28)\}$, $V(\{1, 2, 3\}) = \{(8, 10, 24); (4, 14, 24)\}$, and the preference relation is for each player the usual one that the more one gets the better is. Consider the i.r.p.c. $x = (8, 10, 24)$. It is not in $\mathscr{C}(\mathcal{G})$ since coalition $S = \{2, 3\}$ has an objection to x (note that $(0, 12, 28) \in V(S)$ and $y \succ_2 x$ and $y \succ_3 x$). Consider the i.r.p.c. $x' = (4, 14, 24)$. It is not in $\mathscr{C}(\mathcal{G})$ since coalition $S = \{1, 2\}$ has an objection to x (note that $(5, 15, 0) \in V(S)$ and $y \succ_1 x$ and $y \succ_2 x$). By this $\mathscr{C}(\mathcal{G})$ is empty. Consider the game $\mathcal{G}' = \langle N, X, V', (\succsim_i)_{i\in N}\rangle$ in which the consequence function V' is the same as in $\mathcal{G}$ except for $V'(N)$ that contains also $(5, 15, 25)$: this i.r.p.c. is in $\mathscr{C}(\mathcal{G}')$.

Definition 2.15. *Let $\mathcal{G} = \langle N, V\rangle$ be an NTU game, and x be an i.r.p.c. (y, S) is an* objection of player i against player j to x, *if $S \in \mathcal{I}_{i,j}$, and $y \in V(S)$ is a consequence such that $y_k > x_k$ for all $k \in S$. We say that this objection is* through S.

Definition 2.16. *Let $\mathcal{G} = \langle N, V\rangle$ be an NTU game, x be an i.r.p.c., and (y, S) be an objection of player i against player j to x. (z, T) is a* counterobjection to the objection (y, S) of i against j, *if $T \in \mathcal{I}_{j,i}$, and $z \in V(T)$ is a consequence such that $z_k \geq x_k$ for all $k \in T \setminus S$ and $z_k \geq y_k$ for all $k \in T \cap S$. We say that this counterobjection is* through T.

For an NTU game $\mathcal{G} = \langle N, V\rangle$, an i.r.p.c. x and an objection (y, S) of player i against player j to x, we call (y, S) a *justified objection* if there does not exist any counterobjection to (y, S) by j.

Definition 2.17 (Bargaining Set). *Let $\mathcal{G} = \langle N, V\rangle$ be an NTU game. The* bargaining set $\mathscr{B}(\mathcal{G})$ *of $\mathcal{G}$ is the set of all i.r.p.c.'s x such that there is no justified objection to x of any player i against any other player j.*

Also in the following example we adopt the Osborne and Rubinstein's framework with the consequences being payoff vectors for simplicity.

Example 2.18. Let $\mathcal{G} = \langle N, X, V, (\succsim_i)_{i \in N} \rangle$ be an NTU game with $N = \{1, 2, 3\}$, the consequences are conveniently represented by payoff vectors (as in the Aumann and Peleg's framework) and in particular $V(\{1\}) = V(\{2\}) = V(\{3\}) = \{(0,0,0)\}$, $V(\{1,2\}) = \{(10,10,0);(5,15,0)\}$, $V(\{1, 3\}) = \{(10,0,20);(15,0,15)\}$, $V(\{2,3\}) = \{(0,10,30);(0,12,28)\}$, $V(\{1,2,3\}) = \{(8,10,24);(4,14,24)\}$, and the preference relation is for each player the usual one that the more one gets the better is. Consider the i.r.p.c. $x = (8,10,24)$. Player 3 can object against player 1 to x through $((0,12,28), \{2,3\})$. Unlike Example 2.5, player 1 cannot counterobject, (she cannot answer with $((8,12,0),\{1,2\})$, because consequence $(8,12,0) \notin V(\{1,2\})$ and the payoffs are not redistributable), moreover $(10,10,0)$ and $(5,15,0)$ are useless to counterobject to player 3; therefore, x is outside the bargaining set of $\mathcal{G}$. Consider the i.r.p.c. $x' = (4,14,24)$, and the objection $((5,15,0),\{1,2\})$ of player 1 against player 3 to x'. Note that neither $(0,10,30)$ nor $(0,12,28)$ can be used by player 3 in order to counterobject to player 1, so x' is outside the bargaining set of $\mathcal{G}$ too. As a matter of fact, for this game, the bargaining set is empty.

2.3 Computational Complexity

For the sake of completeness, we recall here some basic definitions about complexity theory, by referring the reader to any specific text (see e.g., Johnson, 1990; Papadimitriou, 1994) for more on this.

Decision problems are maps from strings (encoding the input instance over a suitable alphabet) to the set {"*yes*", "*no*"}. A (possibly nondeterministic) Turing machine M answers a decision problem if on a given input x, (i) at least one branch of M halts in an accepting state if and only if x is a "yes" instance, and (ii) all the branches of M halt in some rejecting state if and only if x is a "no" instance.

The class $\mathbf{P}$ is the set of decision problems that can be answered by a deterministic Turing machine in polynomial time. The classes $\boldsymbol{\Sigma}_{\boldsymbol{k}}^{\boldsymbol{P}}$ and $\boldsymbol{\Pi}_{\boldsymbol{k}}^{\boldsymbol{P}}$, forming the *polynomial hierarchy*, are defined as follows: $\boldsymbol{\Sigma}_{\mathbf{0}}^{\boldsymbol{P}} = \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{P}} = \mathbf{P}$ and for all $k \geq 1$, $\boldsymbol{\Sigma}_{\boldsymbol{k}}^{\boldsymbol{P}} = \mathbf{NP}^{\Sigma_{k-1}^P}$, $\boldsymbol{\Delta}_{\boldsymbol{k}}^{\boldsymbol{P}} = \mathbf{P}^{\Sigma_{k-1}^P}$, and $\boldsymbol{\Pi}_{\boldsymbol{k}}^{\boldsymbol{P}} = \text{co-}\boldsymbol{\Sigma}_{\boldsymbol{k}}^{\boldsymbol{P}}$ where co-$\boldsymbol{\Sigma}_{\boldsymbol{k}}^{\boldsymbol{P}}$ denotes the class of problems whose complementary version is solvable in $\boldsymbol{\Sigma}_{\boldsymbol{k}}^{\boldsymbol{P}}$. Here, $\boldsymbol{\Sigma}_{\boldsymbol{k}}^{\boldsymbol{P}}$ (resp. $\boldsymbol{\Delta}_{\boldsymbol{k}}^{\boldsymbol{P}}$) models computability by a nondeterministic (resp. deterministic) polynomial-time Turing machine that may use an oracle, that is, loosely speaking, a subprogram that can be run with no computational cost, for solving a problem in $\boldsymbol{\Sigma}_{\boldsymbol{k-1}}^{\boldsymbol{P}}$. In particular, the class $\boldsymbol{\Sigma}_{\mathbf{1}}^{\boldsymbol{P}}$ of decision problems that can be solved by a nondeterministic Turing machine in polynomial time is also denoted by $\mathbf{NP}$, while the class $\boldsymbol{\Pi}_{\mathbf{1}}^{\boldsymbol{P}}$ of decision problems whose complementary problem is in $\mathbf{NP}$, is denoted by co-$\mathbf{NP}$. Moreover, the class $\mathbf{D^P}$ is the class of problems defined as a conjunction of two independent problems, one from **NP** and one from co-**NP**, respectively.

To conclude, let us briefly recall the notion of reduction for decision problems. A decision problem A_1 is *polynomially reducible* to a decision problem A_2 if there is a polynomial time computable function h such that, for every x, $h(x)$ is defined and A_1 outputs "yes" on input x if and only if A_2 outputs "yes" on input $h(x)$. A decision problem A is *complete* for the class $\mathcal{C}$ of the polynomial hierarchy (beyond **P**) if A belongs to $\mathcal{C}$ and every problem in $\mathcal{C}$ is polynomially reducible to A.

2.4 Compact representations

In order to deal with the computational complexity of solution concept related problems the issue of how to represent a coalitional game comes into play. The main issue here is to decide whether to explicitly represent the entire worth/consequence function for all possible coalitions or not. Note that, from a computational perspective, to explicitly represent these associations of coalitions with their worths or consequences is unfeasible, since listing all those associations would require exponential space in the number of players. This problem is practically relevant in several application cases as, for instance, with typical Internet-based ones. Moreover, from a computational complexity perspective, having a so huge representation of the game may alter the results of complexity analysis (since it is always carried out with respect to the problem's input size).

Several compact representation schemes for coalitional games have been proposed in the literature. Among them we cite weighted voting games (Elkind et al., 2007), graph games (Deng and Papadimitriou, 1994), and marginal contribution networks (Ieong and Shoham, 2005).

2.4.1 Graph games

Graph games is a representation scheme proposed by Deng and Papadimitriou (1994). They consider games where players are encoded as vertices in an edge-weighted graph $G = \langle (N, E), w\rangle$, and the worth of a coalition S is computed as the sum of the weights of the edges connecting players in S. That is, if we denote by $edges(S)$ the set $\{\{p, q\} \in E \mid \{p, q\} \subseteq S\}$, the worth of coalition S is $v(S) = \sum_{e \in edges(S)} w(e)$.

Example 2.19. Consider the graph game in Figure 2.1 formed by four players, $N = \{a, b, c, d\}$. The coalition $\{a, b\}$ gets a worth $v(\{a, b\}) = 2$, while the coalition $\{a, b, d\}$ gets a worth $v(\{a, b, d\}) = 3 + 2 - 1 = 4$.

In their work, Deng and Papadimitriou (1994) extended their representation scheme to hypergraphs. In that case the hyperedge weighting function is the basis on which the worth function is built. If we denote by $\mathcal{H}$ the set of the

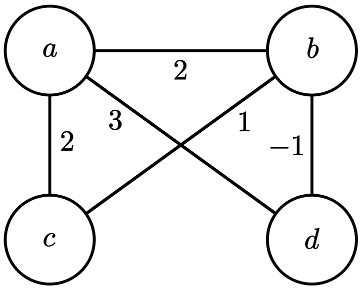


Fig. 2.1. The graph game in Example 2.19.

hyperedges of hypergraph G, and by $hedges(S)$ the set $\{h \in \mathcal{H} \mid h \subseteq S\}$, the worth of coalition S is $v(S) = \sum_{h \in hedges(S)} w(h)$.

It is worth to note that graph games is a representation scheme that is not fully expressive, that is there are coalitional games whose worth function cannot be represented through a graph game. The lost of fully expressiveness is the cost to be paid for the representation succinctness of graph games.

2.4.2 Marginal contribution networks

Marginal contribution networks (also MC-nets) for TU games were proposed by Ieong and Shoham (2005).

A marginal contribution network is constituted by a set of rules. Rules in a marginal contribution network are in the form

$$pattern \rightarrow value$$

where a *pattern* is a conjunction that may include both positive and negative literals, with each literal denoting a player, and *value* is the additive contribution associated with the pattern. A rule is said to *apply* to a coalition S if all the players whose literals occurs positively in the pattern are also in S and all the players whose literals occurs negatively in the pattern do not belong to S. When more than one rule applies to a coalition, the value for that coalition is given by the contribution of all those rules, i.e., by the sum of their values. Vice versa, if no rule applies to a given coalition, then the value for that coalition is set to zero by default.

Example 2.20. Consider the following marginal contribution network formed by the following three rules:

$$\begin{aligned} a \wedge b &\rightarrow 5 \\ b &\rightarrow 2 \\ a \wedge \neg b &\rightarrow 3. \end{aligned}$$

We obtain that $v(\{a\}) = 3$ (the third rule applies), $v(\{b\}) = 2$ (the second rule applies), and $v(\{a, b\}) = 5 + 2 = 7$ (both the first and the second rules apply, but not the third one).

It is easy to check that, for any graph game $\mathcal{G}$ there is an equivalent MC-nets representation having the same size, while the converse is not true in general (Ieong and Shoham, 2005). In particular, given a graph game it is representable through a marginal contribution network whose rules are all formed by at most two positive literals.

This representation scheme is very powerful and, contrary to graph games, MC-nets is proven to be fully expressive (Ieong and Shoham, 2005), that is it allows to represent any TU coalitional game. Using MC-nets, games can be, but not necessarily are, more succinct than the explicit representation of the characteristic function, where all the 2^n values of the worth function should be explicitly listed. There are games for which their representation through MC-nets is exponentially more succinct than previous proposals, such as the multi-issue representation proposed by Conitzer and Sandholm (2004). But there are also games where the size of any possible MC-nets encoding has almost the same size as the explicit characteristic function form, and this is the cost to be paid for the fully expressiveness of marginal contribution networks.

2.4.3 A general framework for compact representation schemes

Next, we propose a more general compact representation scheme for coalitional games (both TU and Osborne and Rubinstein's NTU frameworks), where it is just required for the worth (resp., consequences) function to be computable in **FNP**, that is, computable in polynomial-time by a non-deterministic Turing transducer (Papadimitriou, 1994). Carrying out the complexity analysis of solution concepts for this general scheme accounts for the request of Deng and Papadimitriou (1994) for an analysis on a representation scheme more general than graph games.

In the following we denote by $||x||$ the size of the object x. This size can be interpreted as the number of bits, or the number of cell on the tape of a Turing machine, needed to represent it.

Formally, let $\mathcal{C}$ be a class of games with transferable (resp., non-transferable) utility as defined by a certain given encoding scheme. For any TU (resp., NTU) game $\mathcal{G}$, denote by $v_{\mathcal{G}}$ its *worth* function (resp., by $V_{\mathcal{G}}$ its *consequence* function). Then, define the *worth* (resp., *consequence*) relation for the class $\mathcal{C}$ as the set of tuples $W_{\mathcal{C}} = \{\langle \mathcal{G}, S, w \rangle \mid \mathcal{G} \in \mathcal{C}, v_{\mathcal{G}}(S) = w\}$ (resp., $W_{\mathcal{C}} = \{\langle \mathcal{G}, S, w \rangle \mid \mathcal{G} \in \mathcal{C}, w \in V_{\mathcal{G}}(S)\}$).

We say that $W_{\mathcal{C}}$ is polynomial-time computable if there is a positive integer k and a deterministic polynomial-time transducer M that, given any game encoding $\mathcal{G} \in \mathcal{C}$ and a coalition S of players of $\mathcal{G}$, outputs a value w (resp., all consequences w) such that $\langle \mathcal{G}, S, w \rangle \in W_{\mathcal{C}}$ in at most $||\langle \mathcal{G}, S \rangle||^k$ steps.

We say that $W_{\mathcal{C}}$ is non-deterministically polynomial-time computable if there is a positive integer k such that $W_{\mathcal{C}}$ is k-balanced and k-decidable, as defined below. A worth (resp., consequence) relation $W_{\mathcal{C}}$ is k-balanced if $||w|| \leq ||\langle \mathcal{G}, S \rangle||^k$, while it is said k-decidable if there is a non-deterministic Turing machine that decides $W_{\mathcal{C}}$ in at most $||\langle \mathcal{G}, S, w \rangle||^k$ time. It then follows that there is a non-deterministic Turing transducer M that may compute in $O(||\langle \mathcal{G}, S \rangle||^k)$ time the worth $v(S)$ (resp., some consequence in $V(S)$) of any coalition S of players of $\mathcal{G}$. Indeed, M guesses such a value w and a witness y of the correctness of this value (note that $W_{\mathcal{C}} \in \mathbf{NP}$), and then verifies in *deterministic* polynomial-time that $\langle \mathcal{G}, S, w \rangle \in W_{\mathcal{C}}$, possibly exploiting the witness (also, certificate) y.

Observe that such a machine M is independent of the game, which is just given as one of its inputs. In fact, we say that such an M computes the worth function of any game in the class $\mathcal{C}$.

Definition 2.21. *Let $\mathcal{C}(\mathcal{R})$ be the class of all TU (resp., NTU) games encoded according to some representation $\mathcal{R}$.*

We say that $\mathcal{R}$ is a polynomial-time compact representation and that is an **FP** *representation if the* worth relation *(resp.,* consequence relation*) for $\mathcal{C}(\mathcal{R})$ is polynomial-time computable.*

We say that $\mathcal{R}$ is a non-deterministic polynomial-time compact representation and that is an **FNP** *representation if the* worth relation *(resp.,* consequence relation*) for $\mathcal{C}(\mathcal{R})$ is non-deterministically polynomial-time computable.*

For instance, both the above mentioned schemes of marginal contribution networks and graph games are polynomial-time compact representations since, given a game $\mathcal{G}$ encoded either as a weighted graph, or as a marginal contribution network, and given any coalition S, the worth of S in $\mathcal{G}$ can be computed in deterministic polynomial-time in the size of $\mathcal{G}$ and S. While the NTU MC-nets that will be proposed in Chapter 4 is a non-deterministic polynomial-time compact representation.

3

TU Games

In this chapter we are going to study the computational complexity of solution concepts in TU games. Our hardness results will be given on a quite simple representation scheme for TU games, that is graph games. Since every graph game can be represented by a marginal contribution network in almost the same size, hardness results holds straightforward also for marginal contribution networks.

Deng and Papadimitriou (1994) conjectured two complexity results, one for the bargaining set and one for the kernel. In particular they conjectured that deciding whether an imputation belongs to the bargaining set of a graph game is $\mathbf{\Pi_2^P}$-complete, while deciding whether an imputation belongs to the kernel of a graph game is **NP**-hard. In our research we will prove that these conjectures hold, and for the kernel we will give an even tighter results. Despite the prominence of studying the computational properties of solution concepts has assumed in the last few years, the questions raised by Deng and Papadimitriou were still long-standing open problems.

For the hardness of the nucleolus it is worth to note that, Deng and Papadimitriou proposed in their paper (1994) a polynomial-time computable closed-form characterization for the Shapely value, and showed that this value coincides with the nucleolus, but this holds only when each of its components is non-negative. We will prove that in the general case deciding whether an imputation is the nucleolus of a graph game is an harder problem (until $\mathbf{P}\neq\mathbf{NP}$).

Our membership results hold for any **FNP** representation. In particular our proofs will be given for **FP** representation for simplicity. At the end we will illustrate how the proofs can be modified in order to hold also for **FNP** representation.

3.1 Hardness results

3.1.1 The Bargaining Set

It has been suggested by Maschler (1992) that *computing* the bargaining set might be intrinsically more complex than computing the core. The result presented in this section, together with those reported in Section 3.3.1 (stating that checking whether an imputation is in the core is feasible in **NP**), provides some fresh evidence that this is indeed the case. We show that the imputation checking problem for the bargaining set is $\mathbf{\Pi_2^P}$-hard, even over graph games. The reduction is from the validity of quantified Boolean formulae.

Let $\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ be an $\mathbf{NQBF}_{2,\forall}$ formula, i.e., a quantified Boolean formula over the variables $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\}$, where $\phi(\boldsymbol{\alpha},\boldsymbol{\beta}) = c_1 \wedge \cdots \wedge c_m$ is a **3CNF** formula, and where each universally quantified variable $\alpha_k \in \boldsymbol{\alpha}$ occurs only in the two clauses $c_{i(k)} = (\alpha_k \vee \neg\beta_k)$ and $c_{\bar{i}(k)} = (\neg\alpha_k \vee \beta_k)$—intuitively, each variable α_k enforces the truth-value of a corresponding variable β_k, which thus plays its role in the formula $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$. Based on $\boldsymbol{\Phi}$, we define the weighted graph $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w\rangle$ such that (see Figure 3.1 for an illustration):

- The set $N_{\mathbf{BS}}$ of the nodes (i.e., players) includes: a *clause player* c_j, for each clause c_j; a *literal player* $\ell_{i,j}$, for each literal ℓ_i occurring in c_j; and, two special players "*chall*" and "*sat*".
- The set $E_{\mathbf{BS}}$ of edges includes three kinds of edges.
  (Positive edges): an edge $\{c_j, \ell_{i,j}\}$ with $w(\{c_j, \ell_{i,j}\}) = 1$, for each literal ℓ_i occurring in the clause c_j; and an edge $\{\mathit{chall}, \ell_{i,j}\}$ with $w(\{\mathit{chall}, \ell_{i,j}\}) = 1$, for each literal ℓ_i of the form α_i or $\neg\alpha_i$ (i.e., built over a universally quantified variable) occurring in c_j.
  ("Penalty" edges): an edge $\{\gamma_{i,j}, \neg\gamma_{i,j'}\}$ with $w(\{\gamma_{i,j}, \neg\gamma_{i,j'}\}) = -m - 1$, for each variable γ_i (either $\gamma_i = \alpha_i$ or $\gamma_i = \beta_i$) occurring in c_j and $c_{j'}$; an edge $\{\ell_{i,j}, \ell_{i',j}\}$ with $w(\ell_{i,j}, \ell_{i',j}) = -m - 1$, for each pair of literals ℓ_i and $\ell_{i'}$ occurring in c_j; an edge $\{\mathit{chall}, \ell_{i,j}\}$ with $w(\{\mathit{chall}, \ell_{i,j}\}) = -m - 1$, for each literal ℓ_i of the form β_i or $\neg\beta_i$ (i.e., built over an existentially quantified variable) occurring in c_j; and, an edge $\{\mathit{chall}, c_j\}$ with $w(\{\mathit{chall}, c_j\}) = -m - 1$, for each clause c_j.
  ("Normalizer" edge): the edge $\{\mathit{chall}, \mathit{sat}\}$ with $w(\{\mathit{chall}, \mathit{sat}\}) = n - 1 + m - \sum_{e \in E_{\mathbf{BS}} | e \neq \{\mathit{chall}, \mathit{sat}\}} w(e)$.

Note that the size of the representation of all the weights is polynomial in the number of variables and clauses of $\boldsymbol{\Phi}$. Moreover, the following properties hold.

Lemma 3.1. *Let* $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w\rangle$ *be the graph game associated with the* $\mathbf{NQBF}_{2,\forall}$ *formula* $\boldsymbol{\Phi}$*. Then:*

(A) $D \leq m$;
(B) $w(\{\mathit{chall}, \mathit{sat}\}) > 2 \times m$;

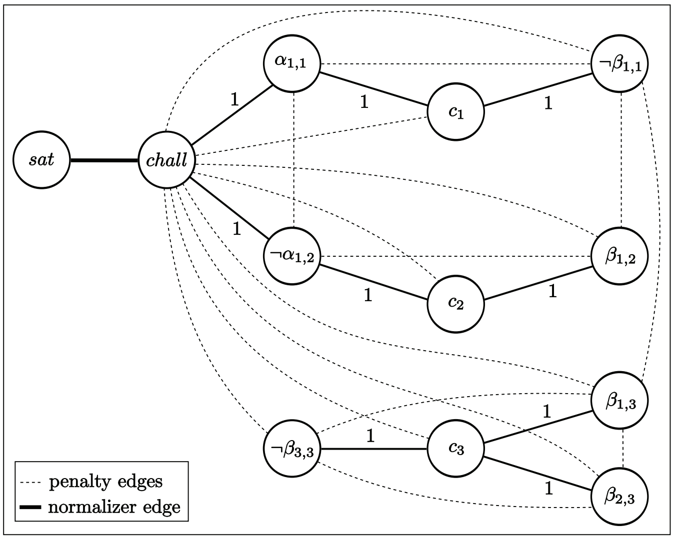


Fig. 3.1. The game $\boldsymbol{BS}(\widehat{\boldsymbol{\Phi}})$, where $\widehat{\boldsymbol{\Phi}} = (\forall\alpha_1)(\exists\beta_1,\beta_2,\beta_3)(\alpha_1 \vee \neg\beta_1) \wedge (\neg\alpha_1 \vee \beta_1) \wedge (\beta_1 \vee \beta_2 \vee \neg\beta_3)$.

(C)$D + w(e) < 0$, *for each penalty edge* $e \in E_{\mathbf{BS}}$*; and,*
(D)$m \geq 2 \times n$,

where $D = \max_{\{chall,sat\} \not\subseteq S \subseteq N} v(S)$ *denotes the maximum worth over all the coalitions not covering the edge* $\{chall, sat\}$*.*

Proof. The fact (A) that $D \leq m$ is immediate by construction. Moreover, the weight of each penalty edge is $-m-1$ and, hence, (C) holds, too. Eventually, (D) (i.e., $m \geq 2 \times n$) is also immediate since $\boldsymbol{\Phi}$ is a $\mathbf{NQBF}_{2,\forall}$ formula.

Let us, hence, focus on (B) by observing that $\sum_{e \in E_{\mathbf{BS}} | e \neq \{chall,sat\}} w(e) \leq -m-1$, for $E_{\mathbf{BS}}$ contains more penalty edges than positive edges. Thus, $w(\{chall, sat\}) = n-1+m-\sum_{e \in E | e \neq \{chall,sat\}} w(e) \geq n-1+m+(m+1) > 2 \times m$. $\square$

Theorem 3.2. *Let* $\mathcal{G}$ *be a graph game, and* x *be an imputation of* $\mathcal{G}$*. Then, deciding whether* x *belongs to* $\mathscr{B}(\mathcal{G})$ *is* $\mathbf{\Pi_2^P}$-hard.

Proof. Deciding the validity of $\mathbf{NQBF}_{2,\forall}$ formulae is $\mathbf{\Pi_2^P}$-complete (Schaefer, 2001). Thus, given an $\mathbf{NQBF}_{2,\forall}$ formula $\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ (where $\boldsymbol{\alpha} = \{\alpha_1,\ldots,\alpha_n\}$, $\boldsymbol{\beta} = \{\beta_1,\ldots,\beta_r\}$, and $\phi(\boldsymbol{\alpha},\boldsymbol{\beta}) = c_1 \wedge \cdots \wedge c_m$), consider the graph game $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w \rangle$ and the imputation x that assigns m to *sat*, $n-1$ to *chall*, and 0 to all other players. Note that x is an imputation, since $v(N) = m+n-1$ because of the weight of the edge $\{chall, sat\}$ (recall that $w(\{chall, sat\}) = m+n-1-\sum_{e \in E_{\mathbf{BS}} | e \neq \{chall,sat\}} w(e)$).

Beforehand, we note that the following properties hold on $\boldsymbol{BS}(\boldsymbol{\Phi})$ and x.

Property 3.2.(1). *No player has a justified objection against a clause or a literal player.*
Indeed, any clause player c_j receives 0 in x and is such that $v(\{c_j\}) = 0$. Therefore, she can counterobject to any objection through the singleton coalition $\{c_j\}$. Similarly, any literal player $\ell_{i,j}$ receives 0 in x and is such that $v(\{\ell_{i,j}\}) = 0$, and can counterobject through $\{\ell_{i,j}\}$.

Property 3.2.(2). *No player has a justified objection against chall.*
Assume that a player $p \in N_{\mathbf{BS}}$ wants to object against *chall* through a coalition S. If $v(S) < 0$ then p cannot object against anyone. Indeed, she has to propose an S-feasible vector y such that $y_k > x_k$ for all player $k \in S$, and hence $v(S) = y(S) > x(S)$, but $x(S) \geq 0$ by definition, so player p cannot fulfill this requirements if $v(S) < 0$. Then, because of Lemma 3.1.(C), S does not include penalty edges. In particular, for each universally quantified variable α_k, it holds that $|S \cap \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)}\}| \leq 1$, where $c_{i(k)}$ and $c_{\bar{i}(k)}$ are the two clauses where this variable occurs (recall that $\boldsymbol{\Phi}$ is an $\mathbf{NQBF}_{2,\forall}$ formula).
Consider, then, the coalition $T \subseteq \{chall\} \cup \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)} \mid 1 \leq k \leq n\}$ such that $|T| = n+1$, $T \cap S = \varnothing$, and $|T \cap \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)}\}| = 1$, for each $1 \leq k \leq n$. Note that $v(T) = n$ and $x(T) = x_{chall} = n-1$. Then, consider the vector z such that $z_{chall} = x_{chall} = n-1$ and $z_q = \frac{1}{n} > x_q = 0$ for each $q \in T$ with $q \neq chall$, and observe that $z(T) = v(T)$. Eventually, since $T \cap S = \varnothing$, (z, T) is a counterobjection to any objection of p against *chall* through S.

Property 3.2.(3). *No player different from chall has a justified objection against sat.*
Suppose that a player $p \neq chall$ has an objection (y, S) against *sat* to x. Since $sat \notin S$, it is the case that $\{chall, sat\} \not\subseteq S$ and hence, from Lemma 3.1.(A), that $v(S) \leq m$. Then, we claim that $(z, \{sat, chall\})$ is a counterobjection to (y, S) of p against *sat*, where z is a feasible distribution that assigns m to *sat* and $w(\{chall, sat\}) - m$ to *chall*. In fact, by Lemma 3.1.(B), *chall* receives a payoff strictly greater than m (i.e., $z_{chall} > m$). Then, note that $z_{sat} = x_{sat}$ and let us distinguish two cases. In the case where $chall \notin S$, we have that $z_{chall} > m \geq n-1 = x_{chall}$ (recall that $m \geq 2 \times n$, by Lemma 3.1.(D)). Instead, in the case where $chall \in S$, we have that $z_{chall} > m$ while $y_{chall} \leq v(S) \leq m$, and hence $z_{chall} > y_{chall}$. It follows that in both cases $(z, \{sat, chall\})$ is a counterobjection to (y, S).

In the light of the properties above, we can limit our attention on the objections of *chall* against *sat*. Consider an objection (y, S) of *chall* against *sat* to x. In particular, y_{chall} must be greater than $x_{chall} = n-1$ and $y_q > 0 = x_q$ for each $q \in S$ with $q \neq chall$. Thus, $y(S) = v(S) > n-1$.

Then, because of Lemma 3.1.(C), no penalty edge is covered by S. Therefore, given that $chall \in S$, we have that S must contain exactly one player per universal variable so that $v(S) = n$, i.e., $|S| = n+1$ and

$|S \cap \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)}\}| = 1$, for each $1 \leq k \leq n$. Eventually, y has to be a vector such that $y_{chall} > n-1$, $y(S) = n$, and $y_q > 0$, for each $q \in S$ with $q \neq chall$. Note that the coalition S encodes an assignment σ_S for the variables in $\boldsymbol{\alpha}$ such that α_k evaluates to false (resp., true) in σ_S if $\alpha_{k,i(k)}$ (resp., $\neg\alpha_{k,\bar{i}(k)}$) occurs in S—note that in this correspondence the truth values are inverted with respect to the membership of the corresponding literal player in S.

Moreover, note that for each truth assignment σ for the variables in $\boldsymbol{\alpha}$, we may immediately build a coalition S and a vector y such that (y, S) is an objection of *chall* against *sat*, and $\sigma_S = \sigma$. Therefore, objections of *chall* against *sat* are in correspondence with truth assignments for universally quantified variables.

We are now ready to show that $\boldsymbol{\Phi}$ is valid if and only if x is in $\mathscr{B}(\boldsymbol{BS}(\boldsymbol{\Phi}))$:

($\Rightarrow$) Suppose that $\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is valid, and let (y, S) be any objection of *chall* against *sat* to x. We show that this objection is not justified, because *sat* has a counterobjection (z, T). Recall first that S encodes an assignment σ_S over the variables in $\boldsymbol{\alpha}$. Then, let σ be a satisfying assignment over the variables in $\boldsymbol{\Phi}$ such that α_k evaluates to true in σ if and only if it evaluates to true in σ_S; indeed, σ exists since $\boldsymbol{\Phi}$ is valid. Based on σ, let us construct the coalition T such that $T \subseteq \{sat\} \cup \{\ell_{i,j} \mid \ell_i \text{ evaluates to true in } \sigma \wedge c_j \text{ is a clause where } \ell_i \text{ occurs}\} \cup \{c_1, \ldots, c_m\}$, $T \cap S = \varnothing$, $|T| = 2m+1$, and $v(T) = m$. In particular, T can be such that $v(T) = m$, precisely because σ is a satisfying assignment and by construction of σ_S. Moreover, consider the vector z such that $z_{sat} = v(T) = x_{sat} = m$ and $z_q = x_q = 0$ for each $q \in T$ with $q \neq sat$ (and, hence, $q \notin S$). By construction (z, T) is a counterobjection to (y, S), which is therefore not justified in its turn.

($\Leftarrow$) Let σ be an assignment over the variables in $\boldsymbol{\alpha}$ witnessing that $\boldsymbol{\Phi}$ is not valid. Let (y, S) be a arbitrary objection of *chall* against *sat* to x such that $\sigma_S = \sigma$. We claim that (y, S) is justified. Indeed, assume for the sake of contradiction, that (y, S) is not justified and let (z, T) be a counterobjection. Since $sat \in T \setminus S$ and since we must have that $z_{sat} \geq x_{sat} = m$, because of Lemma 3.1.(A) we actually have that $z_{sat} = x_{sat}$. In fact, since m is the maximum available payoff over all the coalitions not including both *chall* and *sat*, $z(T) = v(T) = m$ and the fact that (z, T) is a counterobjection to (y, S) entail that $S \cap T = \varnothing$ and that T contains all clause players, exactly one literal player per clause, so that $|T| = 2m+1$. Observe now that T encodes a satisfying truth value assignment σ_T for $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$, because $v(T) > 0$ and hence T does not cover any penalty edge. More precisely, we let each variable γ_i (either $\gamma_i = \alpha_i$ or $\gamma_i = \beta_i$) to evaluate true (resp., false) in σ_T if $\gamma_{i,j}$ (resp., $\neg\gamma_{i,j}$) occurs in T, for some clause c_j. Let $\sigma_T^{\boldsymbol{\alpha}}$ denote the restriction of σ_T over the variables in $\boldsymbol{\alpha}$. Then, since $S \cap T = \varnothing$ and given the definition of σ_S, we have that

$\sigma_S = \sigma_T^{\boldsymbol{\alpha}}$. Clearly, this contradicts the fact that $\sigma = \sigma_S$ witnesses that $\boldsymbol{\Phi}$ is not valid. □

3.1.2 The Kernel

We now show that checking whether an imputation belongs to the kernel is $\boldsymbol{\Delta_2^P}$-hard on graph games. The proof is based on a reduction from the problem of the lexicographically maximum satisfying assignment for Boolean formulae.

Let $\phi = c_1 \wedge \cdots \wedge c_m$ be a **3CNF** Boolean formula, that is, a Boolean formula in conjunctive normal form over the set $\{\alpha_1, \ldots, \alpha_n\}$ of variables that are lexicographically ordered (according to their indices) where each clause contains three literals (positive or negated variables) at most. Assume, w.l.o.g., that there is a clause in ϕ containing at least two literals. Based on ϕ, we build in polynomial time the weighted graph $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w\rangle$ such that (see Figure 3.2 for an illustration):

- The set $N_{\mathbf{K}}$ of nodes (i.e, players) includes: a *variable player* α_i, for each variable α_i in ϕ; a *clause player* c_j, for each clause c_j in ϕ; a *literal player* $\ell_{i,j}$ (either $\ell_{i,j} = \alpha_{i,j}$ or $\ell_{i,j} = \neg\alpha_{i,j}$), for each literal ℓ_i ($\ell_i = \alpha_i$ or $\ell_i = \neg\alpha_i$, respectively) occurring in c_j; and, two special players "*chall*" and "*sat*".
- The set $E_{\mathbf{K}}$ consists of the following three types of edges.
  (Positive edges): an edge $\{c_j, \ell_{i,j}\}$ with $w(\{c_j, \ell_{i,j}\}) = 2^{n+3}$, for each literal ℓ_i occurring in c_j; an edge $\{chall, \alpha_i\}$ with $w(\{chall, \alpha_i\}) = 2^i$, for each $1 \le i \le n$; an edge $\{sat, \alpha_i\}$ with $w(\{sat, \alpha_i\}) = 2^i$, for each $2 \le i \le n$; the edge $\{sat, \alpha_1\}$ with $w(\{sat, \alpha_1\}) = 2^1 + 2^0$.
  ("Penalty" edges): an edge $\{\ell_{i,j}, \ell_{i',j}\}$ with $w(\{\ell_{i,j}, \ell_{i',j}\}) = -2^{m+n+7}$, for each pair of literals ℓ_i and $\ell_{i'}$ occurring in c_j; an edge $\{\alpha_{i,j}, \neg\alpha_{i,j'}\}$ with $w(\{\alpha_{i,j}, \neg\alpha_{i,j'}\}) = -2^{m+n+7}$, for each variable α_i occurring positively in c_j and negated in $c_{j'}$; an edge $\{\alpha_i, \neg\alpha_{i,j}\}$ with $w(\{\alpha_i, \neg\alpha_{i,j}\}) = -2^{m+n+7}$, for each variable α_i occurring negated in c_j.
  ("Normalizer" edge): the edge $\{chall, sat\}$ with $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e)$.

Note that the size of the representation of all the weights is polynomial in the number of variables and clauses of ϕ. In addition, two crucial properties of the construction are stated below.

Lemma 3.3. *Let $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w\rangle$ be the graph game associated with the* **3CNF** *formula ϕ. Then:*

(A)$w(\{chall, sat\}) \ge D + 1$*; and,*
(B)$D + w(e) < 0$*, for each penalty edge $e \in E_{\mathbf{K}}$,*

where $D = \max_{\{chall, sat\} \not\subseteq S \subseteq N} v(S)$ denotes the maximum worth over all the coalitions not covering the edge $\{chall, sat\}$.

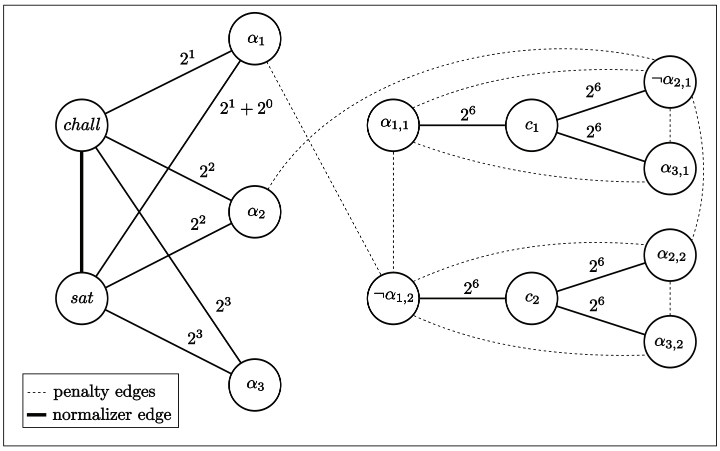


Fig. 3.2. The game $\boldsymbol{K}(\widehat{\phi})$, where $\widehat{\phi} = (\alpha_1 \vee \neg\alpha_2 \vee \alpha_3) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \alpha_3)$.

Proof. Let $P = \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}, w(e)>0} w(e)$ be the sum of all the positive edges in $\boldsymbol{K}(\phi)$. Let us firstly observe that:

$$P \leq 3 \times m \times 2^{n+3} + 2 \times \sum_{i=1}^{n} 2^i + 2^0 \leq 2^{m+n+5} + 2^{n+2} + 2^0 \leq 2^{m+n+6}.$$

Thus, $2^{m+n+7} \geq 2 \times P$ holds. Moreover, observe that $\boldsymbol{K}(\phi)$ contains at least one penalty edge, since w.l.o.g. there is a clause in ϕ containing at least two literals. Hence, $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e) \geq 1 + 2^{m+n+7} - P$. It follows that $w(\{chall, sat\}) \geq 1 + 2 \times P - P = 1 + P$. Eventually, $P \geq D$ holds by definition of D and, therefore, $w(\{chall, sat\}) \geq 1 + D$, which proves (A).

As for (B), given that $D > 0$ and $P \geq D$, we may note that $2^{m+n+7} \geq 2 \times P$ implies $2^{m+n+7} > D$. □

Theorem 3.4. *Let $\mathcal{G}$ be a graph game, and x be an imputation of $\mathcal{G}$. Then, deciding whether x belongs to $\mathscr{K}(\mathcal{G})$ is $\boldsymbol{\Delta_2^P}$-hard.*

Proof. Let $\phi = c_1 \wedge \cdots \wedge c_m$ be a satisfiable **3CNF** formula over a set $\{\alpha_1, \ldots, \alpha_n\}$ of variables that are lexicographically ordered (according to their indices). The problem of deciding whether α_1 (that is the lexicographically least significant variable) is true in the *lexicographically maximum satisfying assignment* for ϕ is a well-known $\boldsymbol{\Delta_2^P}$-complete problem (Krentel, 1986).

Consider the graph game $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w \rangle$, and the imputation x that assigns 0 to all players of $\boldsymbol{K}(\phi)$, but to *sat*, which receives 1. In fact, x is an imputation since $v(N) = 1$ because of the weight of the edge $\{chall, sat\}$ (recall that $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e)$).

Beforehand, observe that by Definition 2.6 and since *sat* is the only player that receives in x a payoff strictly greater than her worth as a singleton coalition, $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if $\max_{S \in \mathcal{I}_{i,sat}} e(S,x) \leq \max_{S \in \mathcal{I}_{sat,i}} e(S,x)$, for each player $i \neq sat$. The following property further restricts the coalitions of interest and, indeed, suffices to conclude that $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if

$$\max_{S \in \mathcal{I}_{chall,sat}} e(S,x) \leq \max_{S \in \mathcal{I}_{sat,chall}} e(S,x). \tag{3.1}$$

Property 3.4.(1): *For each player $i \neq chall$ it holds that*

$$\max_{S \in \mathcal{I}_{i,sat}} e(S,x) \leq \max_{S \in \mathcal{I}_{sat,i}} e(S,x).$$

Let S be an arbitrary coalition in $\mathcal{I}_{i,sat}$ with $i \neq chall$ and consider the coalition $T = \{chall, sat\} \in \mathcal{I}_{sat,i}$. Note that $e(T,x) = v(T) - x(T) = v(\{chall, sat\}) - 1$ while $e(S,x) = v(S)$. By Lemma 3.3.(A), we know that $v(\{chall, sat\}) \geq v(S)+1$. Thus is, $e(T,x) \geq e(S,x)$ holds for all coalitions $S \in \mathcal{I}_{i,sat}$ with $i \neq chall$.

Now, we are going to characterize the two terms $\max_{S \in \mathcal{I}_{chall,sat}} e(S,x)$ and $\max_{S \in \mathcal{I}_{sat,chall}} e(S,x)$ occurring in Equation 3.1 by establishing a connection with satisfying assignments for ϕ. In particular, for any truth assignment σ, we denote by $\sigma \models \phi$ the fact that σ satisfies ϕ, and by $\sigma(\alpha_i) = \texttt{true}$ (resp., $\sigma(\alpha_i) = \texttt{false}$) the fact that α_i evaluates to true (resp., false) in σ.

Property 3.4.(2): *The following condition holds*

$$\max_{S \in \mathcal{I}_{chall,sat}} e(S,x) = m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \texttt{true}} 2^i.$$

Let us firstly note that $\max_{S \in \mathcal{I}_{chall,sat}} e(S,x) = \max_{S \in \mathcal{I}_{chall,sat}} v(S)$, by construction of the imputation x. Let S_* be the coalition having maximum worth over all the coalitions in $\mathcal{I}_{chall,sat}$. Because of Lemma 3.3.(B) and since $v(\{chall\}) = 0$, S_* cannot cover any penalty edge, for otherwise S_* would not be a coalition with maximum worth amongst those belonging to $\mathcal{I}_{chall,sat}$. Thus, *(i)* for each variable player $\alpha_i \in S_*$, no literal player of the form $\neg\alpha_{i,j}$ is in S_*; *(ii)* for each clause player $c_j \in S_*$, at most one literal player of the form $\ell_{i,j}$ is in S_*; and, *(iii)* for each variable α_i, S_* contains no pair of literals of the form $\alpha_{i,j}$ and $\neg\alpha_{i,j'}$.
It follows that the worth of S_* is such that: $v(S_*) = |C| \times 2^{n+3} + \sum_{\alpha_i \in S_*} 2^i$, where C is the set of the clause players $c_j \in S_*$ for which exactly one literal player $\ell_{i,j}$ is in S_*; in particular, recall that 2^i is the weight associated with the edge $\{chall, \alpha_i\}$, while 2^{n+3} is the weight associated with each edge of the form $\{c_j, \ell_{i,j}\}$. Let, now, $\widehat{\sigma}$ be an assignment such that $\widehat{\sigma}(\alpha_i) = \texttt{true}$ (resp., $\widehat{\sigma}(\alpha_i) = \texttt{false}$) if $\alpha_{i,j}$ (resp., $\neg\alpha_{i,j}$) occurs in S_* for some clause c_j. Note that $\widehat{\sigma}$ may be a partial assignment, over a set of variables $\widehat{\alpha} \subseteq$

$\{\alpha_1, \ldots, \alpha_n\}$; however, because of *(iii)* above, $\widehat{\sigma}$ is coherent and satisfies all the clauses whose players are in C. Eventually, since ϕ is satisfiable and since $2^{n+3} > \sum_{i=1}^{n} 2^i$, S_* because of *(ii)* will certainly contain all the m clause players (i.e., $\widehat{\sigma}$ is a satisfying assignment for ϕ). That is, $v(S_*) = m \times 2^{n+3} + \sum_{\alpha_i \in S_*} 2^i$.
Observe now that if $\alpha_{i,j}$ is in S_*, then α_i is in S_* as well, since this leads to maximize the worth of S_*. Moreover, if $\neg\alpha_{i,j}$ is in S_*, then α_i is not in S_* because of *(i)*. Thus, the assignment σ_{S_*} such that $\sigma_{S_*}(\alpha_i) = \texttt{true}$ (resp., $\sigma_{S_*}(\alpha_i) = \texttt{false}$) if α_i occurs (resp., not occurs) in S_* coincides with $\widehat{\sigma}$ when restricted over the domain of the variables in $\widehat{\alpha}$. Therefore, σ_{S_*} is in turn a satisfying assignment, and we have that:

$$v(S_*) = m \times 2^{n+3} + \sum_{\alpha_i \mid \sigma_{S_*}(\alpha_i) = \texttt{true}} 2^i \leq m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i \mid \sigma(\alpha_i) = \texttt{true}} 2^i.$$

We conclude the proof by showing that the above inequality cannot be strict. Indeed, assume, for the sake of contradiction, that a satisfying assignment $\overline{\sigma}$ exists for ϕ such that $v(S_*) < m \times 2^{n+3} + \sum_{\alpha_i \mid \overline{\sigma}(\alpha_i) = \texttt{true}} 2^i$. Based on $\overline{\sigma}$, we can build a coalition $\overline{S}$ such that $\{chall, c_1, \ldots, c_m\} \subseteq \overline{S}$; $\alpha_i \in \overline{S}$, for each α_i such that $\overline{\sigma}(\alpha_i) = \texttt{true}$; exactly one literal $\ell_{i,j}$ is in $\overline{S}$, for each clause c_j that is satisfied by $\ell_{i,j}$ according to the truth values defined in $\overline{\sigma}$; and no further player is in $\overline{S}$.
Given that $\overline{\sigma}$ is a satisfying assignment, no penalty edge is covered by $\overline{S}$. In particular, $v(\overline{S}) = m \times 2^{n+3} + \sum_{\alpha_i \in \overline{S}} 2^i$ and, hence, $v(\overline{S}) = m \times 2^{n+3} + \sum_{\alpha_i \mid \overline{\sigma}(\alpha_i) = \texttt{true}} 2^i$. But, this is impossible since we would have a coalition $\overline{S} \in \mathcal{I}_{chall,sat}$ such that $v(\overline{S}) > v(S_*) = \max_{S \in \mathcal{I}_{chall,sat}} v(S)$.

Property 3.4.(3): *The following condition holds*

$$\max_{S \in \mathcal{I}_{sat,chall}} e(S,x) =$$
$$m \times 2^{n+3} + \max_{\sigma \models \phi} \left(|\{\alpha_1 \mid \sigma(\alpha_1) = \texttt{true}\}| + \sum_{\alpha_i \mid \sigma(\alpha_i) = \texttt{true}} 2^i \right) - 1.$$

The property can be proven precisely along the same line of reasoning as in the proof of Property 3.4.(2). The differences are that: $x(S) = 1$ holds for each S with $sat \in S$; and that the weight associated with the edge $\{sat, \alpha_i\}$ is 2^i, for each $2 \leq i \leq n$, while $2^1 + 2^0$ for the case where $i = 1$. In particular, $|\{\alpha_1 \mid \sigma(\alpha_1) = \texttt{true}\}|$ precisely encodes the fact that an unitary weight has to be added to any assignment where α_1 evaluates to true.

We can now rewrite Equation 3.1 in the light of the above two properties, and conclude that $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if:

$$1 + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i \leq \max_{\sigma \models \phi} \left(\sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i + |\{\alpha_1 \mid \sigma(\alpha_1) = \mathtt{true}\}| \right),$$

that is, $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if α_1 is true in the lexicographically maximum satisfying assignment for ϕ. $\square$

3.1.3 The Nucleolus

In this section, we show that checking whether an imputation is in the nucleolus is $\boldsymbol{\Delta_2^P}$-hard, even on graph games. The proof is based on extending the reduction used to show the $\boldsymbol{\Delta_2^P}$-hardness of the kernel.

Let $\phi = c_1 \wedge \cdots \wedge c_m$ be a **3CNF** Boolean formula over the set $\{\alpha_1, \ldots, \alpha_n\}$ of variables, and let $\boldsymbol{K}(\phi)$ be the game associated with it according to the reduction in Section 3.1.2. Let $N_k = N_{\mathbf{K}} \setminus \{sat\}$, let $\overline{N}_k$ be the set of players $\{\overline{p} \mid p \in N_k \wedge p \neq \alpha_1\}$ containing a copy of each player in N_k but α_1, and let $N_r = \{a, \overline{a}, b, \overline{b}\}$. Moreover, let $E_k = \{\{p,q\} \mid \{p,q\} \in E_{\mathbf{K}} \wedge \{p,q\} \subseteq N_k\}$.

Based on ϕ, we define the weighted graph $\boldsymbol{N}(\phi) = \langle (N_{\mathbf{N}}, E_{\mathbf{N}}), w \rangle$ such that (see Figure 3.3 for an illustration):

- The set $N_{\mathbf{N}}$ of nodes coincides with $N_k \cup \overline{N}_k \cup N_r$.
- The set $E_{\mathbf{N}}$ of edges coincides with $E_k \cup \{\{\overline{p}, \overline{q}\} \mid \{p,q\} \in E_k \wedge \{\overline{p}, \overline{q}\} \subseteq \overline{N}_k\} \cup \{\{\alpha_1, \overline{q}\} \mid \{\alpha_1, q\} \in E_k \wedge \overline{q} \in \overline{N}_k\} \cup \{\{p, \overline{q}\} \mid p \in N_k \setminus \{\alpha_1\} \wedge \overline{q} \in \overline{N}_k\} \cup \{\{a,b\}, \{\overline{a}, \overline{b}\}, \{b, \overline{b}\}\}$. Moreover, weights are such that edges can be partitioned into three groups:
  (Positive edges): $w(\{c_j, \ell_{i,j}\}) = w(\{\overline{c}_j, \overline{\ell}_{i,j}\}) = 2^{n+3}$, for each literal ℓ_i occurring in c_j; $w(\{chall, \alpha_i\}) = w(\{\overline{chall}, \overline{\alpha}_i\}) = 2^i$, for each $2 \leq i \leq n$; and, $w(\{chall, \alpha_1\}) = w(\{\overline{chall}, \alpha_1\}) = 2^1$.
  ("Penalty" edges): $w(\{\ell_{i,j}, \ell_{i',j}\}) = w(\{\overline{\ell}_{i,j}, \overline{\ell}_{i',j}\}) = -2^{m+n+7}$, for each pair of literals ℓ_i and $\ell_{i'}$ occurring in c_j; $w(\{\alpha_{i,j}, \neg\alpha_{i,j'}\}) = w(\{\overline{\alpha}_{i,j}, \overline{\neg\alpha}_{i,j'}\}) = -2^{m+n+7}$, for each variable α_i occurring positively in c_j and negated in $c_{j'}$; $w(\{\alpha_i, \neg\alpha_{i,j}\}) = w(\{\overline{\alpha}_i, \overline{\neg\alpha}_{i,j}\}) = -2^{m+n+7}$, for each variable α_i with $i \neq 1$ occurring negated in c_j; $w(\{\alpha_1, \neg\alpha_{1,j}\}) = w(\{\alpha_1, \overline{\neg\alpha}_{1,j}\}) = -2^{m+n+7}$, for each clause c_j where α_1 negatively occurs; and $w(\{p, \overline{q}\}) = -2^{m+n+7}$, for each pair of players $p \neq \alpha_1$ and $\overline{q}$, with $p \in N_k$ and $\overline{q} \in \overline{N}_k$.
  ("Normalizer" edges): Let $\Delta = 1 - \sum_{e \in E_{\mathbf{N}} | e \subseteq N_k \cup \overline{N}_k} w(e)$. Then, $w(\{a, b\}) = w(\{\overline{a}, \overline{b}\}) = \Delta + 2$ and $w(\{b, \overline{b}\}) = -\Delta - 4$.

Note that the size of the representation of all the weights is polynomial in the number of variables and clauses of ϕ. In addition, two important properties of the construction are stated below.

Lemma 3.5. *Let $\boldsymbol{N}(\phi) = \langle (N_{\mathbf{N}}, E_{\mathbf{N}}), w \rangle$ be the graph game associated with the* **3CNF** *formula ϕ. Then:*

(A)$\Delta > 1$*; and,*

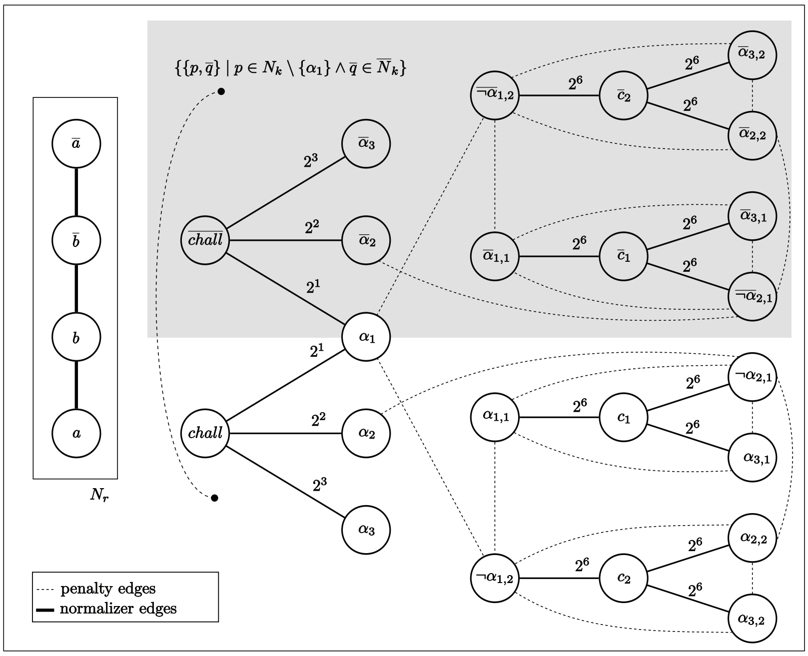


Fig. 3.3. The game $\boldsymbol{N}(\widehat{\phi})$, where $\widehat{\phi} = (\alpha_1 \vee \neg\alpha_2 \vee \alpha_3) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \alpha_3)$. Nodes in $\overline{N}_K$ are reported in the gray area.

(B)$D + w(e) < 0$, *for each penalty edge,*

where $D = \max_{S \subseteq N_k \cup \overline{N}_k} v(S)$ *denotes the maximum worth over all the coalitions of players in* $N_k \cup \overline{N}_k$.

Proof. Let $P = \sum_{e \in E_{\mathbf{N}}, e \subseteq N_k \cup \overline{N}_k, w(e)>0} w(e)$. Then, let us observe that $\Delta = 1 - \sum_{e \in E_{\mathbf{N}}, e \subseteq N_k \cup \overline{N}_k} w(e) \geq 1 + 2^{m+n+7} - P$, and that the following bound holds:

$$P + 1 \leq 2 \times \left(3 \times m \times 2^{n+3} + \sum_{i=1}^{n} 2^i\right) + 1 \leq$$
$$2 \times \left(2^{m+n+5} + 2^{n+1}\right) + 1 \leq 2^{m+n+6} + 2^{n+2} + 1 \leq 2^{m+n+7}.$$

Therefore, $P < 2^{m+n+7}$ and hence $\Delta \geq 1 + 2^{m+n+7} - P > 1$, which proves (A). Eventually, as for (B), note that $D \leq P$. Thus, $2^{m+n+7} > D$. $\square$

Theorem 3.6. *Let $\mathcal{G}$ be a graph game, and x be an imputation of $\mathcal{G}$. Then, deciding whether x belongs to $\mathscr{N}(\mathcal{G})$ is* $\mathbf{\Delta_2^P}$-hard.

Proof. Let $\phi = c_1 \wedge \cdots \wedge c_m$ be a satisfiable **3CNF** formula over a set $\{\alpha_1, \ldots, \alpha_n\}$ of variables that are lexicographically ordered (according to their indices), and consider again the $\mathbf{\Delta_2^P}$-complete problem of deciding whether α_1 is true in the *lexicographically maximum satisfying assignment* for ϕ. Here, we also assume w.l.o.g. that the assignment where all the variables evaluate false is not satisfying.

Consider the graph game $\boldsymbol{N}(\phi) = \langle (N_{\mathbf{N}}, E_{\mathbf{N}}), w\rangle$, and the imputation x that assigns 0 to all players of $\boldsymbol{N}(\phi)$, but to α_1, which receives 1. In fact, x is an imputation since $v(N_{\mathbf{N}}) = 1$ because of the weight of the edges $\{a, b\}$, $\{\overline{a}, \overline{b}\}$, and $\{b, \overline{b}\}$ (observe that that $v(\{a, b, \overline{a}, \overline{b}\}) = \Delta = 1 - \sum_{e \in E_{\mathbf{N}}, e \subseteq N_k \cup \overline{N}_k} w(e)$ and that there is no edge between a node in $\{a, b, \overline{a}, \overline{b}\}$ and a node in $N_k \cup \overline{N}_k$).

Let us first investigate the structure of the worth function.

Property 3.6.(1): *Let S be a coalition such that $S \subseteq N_k \cup \overline{N}_k$ and $v(S) > 0$. Then, $S \cap (N_k \setminus \{\alpha_1\}) \neq \varnothing$ if and only if $S \cap \overline{N}_k = \varnothing$.*
Indeed, in the light of Lemma 3.5.(B), S cannot contain penalty edges. Thus, in particular, there is no pair $p \neq \alpha_1$ and $\overline{q}$ of players in S such that $p \in N_k$ and $\overline{q} \in \overline{N}_k$.

A coalition built from players in N_k is said *primal*, whereas a coalition built from players in $\overline{N}_k \cup \{\alpha_1\}$ is said *dual*. In fact, from the definition of the game $\boldsymbol{N}(\phi)$, it is immediate to observe that there is a one-to-one correspondence between primal and dual coalitions. In particular, for each primal coalition S, let $\overline{S} = \{\overline{p} \mid p \in S \wedge p \neq \alpha_1\} \cup \{\alpha_1 \mid \alpha_1 \in S\}$ denote its corresponding dual coalition, and observe that $v(S) = v(\overline{S})$ holds. Moreover, for each dual coalition $\overline{S}$, let $S = \{p \mid \overline{p} \in \overline{S}\} \cup \{\alpha_1 \mid \alpha_1 \in \overline{S}\}$ denote its corresponding primal coalition, and observe that $v(\overline{S}) = v(S)$ holds.

Below, we shall state further properties of the construction, by focusing, w.l.o.g., on primal coalitions only. As usual, for any assignment σ, we denote by $\sigma \models \phi$ the fact that σ satisfies ϕ, and by $\sigma(\alpha_i) = \texttt{true}$ (resp., $\sigma(\alpha_i) = \texttt{false}$) the fact that α_i evaluates to true (resp., false) w.r.t. σ. Moreover, for any coalition S, let σ_S denote the assignment such that $\sigma_S(\alpha_i) = \texttt{true}$ (resp., $\sigma_S(\alpha_i) = \texttt{false}$) if α_i occurs (resp., does not occur) in S.

Property 3.6.(2): $\max_{S \subseteq N_k} v(S) = m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i \mid \sigma(\alpha_i) = \texttt{true}} 2^i$. *Moreover, let $S_* \subseteq N_k$ be a coalition such that $v(S_*) = \max_{S \subseteq N_k} v(S)$, and let σ_* be the lexicographically maximum satisfying assignment. Then, $chall \in S_*$ and $\sigma_{S_*} = \sigma_*$.*
The property can be shown by following the same line of reasoning as in the proof of Property 3.4.(2), after having observed that there is no player *sat* in the game $\boldsymbol{N}(\phi)$. For completeness, this is illustrated below.
Let S_* be the coalition having maximum worth over all the coalitions with players in N_k. Because of Lemma 3.5.(B), S_* cannot cover any penalty edge. Thus, the worth of S_* can be written as $v(S_*) = |C| \times 2^{n+3} + \sum_{\{chall, \alpha_i\} \subseteq S_*} 2^i$, where C is the set of the clause players $c_j \in S_*$ for which exactly one literal player $\ell_{i,j}$ is in S_*; in particular, 2^i is the weight associated with the edge $\{chall, \alpha_i\}$, while 2^{n+3} is the weight associated

with each edge of the form $\{c_j, \ell_{i,j}\}$. Now, given that the assignment where all the variables evaluate false is not satisfying and since ϕ is satisfiable, we always have that $chall \in S_*$.
Eventually, since ϕ is satisfiable and since $2^{n+3} > \sum_{i=1}^{n} 2^i$, S_* will certainly contain all the m clause players. That is, $v(S_*) = m \times 2^{n+3} + \sum_{\alpha_i \in S_*} 2^i$. Moreover, the assignment σ_{S_*} associated with S_* is satisfying precisely because, in order to avoid covering penalty edges and to maximize the worth of S_*, it conforms with the truth assignment induced by the selection of the literal players that satisfy all the clauses of ϕ. Therefore, it holds that:

$$v(S_*) = m \times 2^{n+3} + \sum_{\alpha_i | \sigma_{S_*}(\alpha_i)=\mathtt{true}} 2^i \leq m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i)=\mathtt{true}} 2^i.$$

To conclude the proof, we now just claim that σ_{S_*} is the lexicographically maximum satisfying assignment. Indeed, assume, for the sake of contradiction, that a satisfying assignment $\overline{\sigma}$ exists for ϕ such that $v(S_*) < m \times 2^{n+3} + \sum_{\alpha_i | \overline{\sigma}(\alpha_i)=\mathtt{true}} 2^i$. Then, based on $\overline{\sigma}$, we can build a coalition $\overline{S}$ such that $\{chall, c_1, \ldots, c_m\} \subseteq \overline{S}$; $\alpha_i \in \overline{S}$, for each α_i such that $\overline{\sigma}(\alpha_i) = \mathtt{true}$; exactly one literal $\ell_{i,j}$ is in $\overline{S}$, for each clause c_j that is satisfied by $\ell_{i,j}$ according to the truth values defined in $\overline{\sigma}$; and no further player is in $\overline{S}$. Given that $\overline{\sigma}$ is a satisfying assignment, no penalty edge is covered by $\overline{S}$. In particular, $v(\overline{S}) = m \times 2^{n+3} + \sum_{\alpha_i \in \overline{S}} 2^i$ and, hence, $v(\overline{S}) = m \times 2^{n+3} + \sum_{\alpha_i | \overline{\sigma}(\alpha_i)=\mathtt{true}} 2^i$, which is impossible since we would have a coalition $\overline{S} \subseteq N_k$ such that $v(\overline{S}) > v(S_*) = \max_{S \subseteq N_k} v(S)$.

Property 3.6.(3): *Let $S_* \subseteq N_k$ be a coalition with $v(S_*) = \max_{S \subseteq N_k} v(S)$. Then, for each coalition $S \subseteq N_k \cup \overline{N}_k$ with $S \neq S_*$ and $S \neq \overline{S}_*$ it holds that $v(S_*) = v(\overline{S}_*) > v(S)$, and moreover for each imputation y it holds that $e(S_*, y) = e(\overline{S}_*, y) > e(S, y)$.*
Because of Property 3.6.(2), $v(S_*) = m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i)=\mathtt{true}} 2^i$. Note first that a coalition S such that $S \cap (N_k \setminus \{\alpha_1\}) \neq \varnothing$ and $S \cap \overline{N}_k \neq \varnothing$ is such that $v(S) < 0$, by Lemma 3.5.(B). Thus, $e(S, y) = v(S) - y(S) < 0$. Instead, $e(S_*, y) = e(\overline{S}_*, y) > 0$ holds.
Consider, now, a primal coalition $S \subseteq N_k$ with $S \neq S_*$—precisely the same arguments apply to the case of dual coalitions. Let σ_* denote the lexicographically maximum satisfying assignment for ϕ, and let us distinguish two cases. In the case where a satisfying assignment for ϕ exists which is different from σ_*, it can be shown that the maximum possible worth of S coincides with $m \times 2^{n+3} + \max_{\sigma \models \phi, \sigma \neq \sigma_*} \sum_{\alpha_i | \sigma(\alpha_i)=\mathtt{true}} 2^i$ (with the same line of reasoning as in the proof of Property 3.6.(2)). Then, $v(S) \leq m \times 2^{n+3} + \max_{\sigma \models \phi, \sigma \neq \sigma_*} \sum_{\alpha_i | \sigma(\alpha_i)=\mathtt{true}} 2^i \leq m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i)=\mathtt{true}} 2^i - 2 = v(S_*) - 2 = v(\overline{S}_*) - 2$. It follows that:

$$e(\overline{S}_*, y) = e(S_*, y) = v(S_*) - y(S_*) \geq$$
$$v(S_*) - 1 \geq v(S) + 1 > v(S) \geq e(S, y).$$

Instead, in the case where σ_* is the only satisfying assignment for ϕ, it must be the case that $v(S) \leq (m-1) \times 2^{n+3} + \sum_{i=1}^{n} 2^i$. Again, we derive that $v(S) \leq v(S_*) - 2 = v(\overline{S}_*) - 2$, which suffices to prove that $e(\overline{S}_*, y) = e(S_*, y) > e(S, y)$ holds, as for we have illustrated above.

The subsequent step is then to characterize the worth of primal (and dual) coalitions extended with players in $N_r = \{a, b, \overline{a}, \overline{b}\}$.

Property 3.6.(4): *Let $S \subseteq N_k \cup \overline{N}_k \cup N_r$. Then, $v(S) = v(S \cap N_r) + v(S \cap (N_k \cup \overline{N}_k))$.*
Indeed, this follows from the fact that there is no edge between any node in $\{a, b, \overline{a}, \overline{b}\}$ and any node in $N_k \cup \overline{N}_k$.

Property 3.6.(5): *Let $S_* \subseteq N_k$ be a coalition with $v(S_*) = \max_{S \subseteq N_k} v(S)$. Then, the eight coalitions $S_1 = S_* \cup \{a, b\}$, $S_2 = S_* \cup \{\overline{a}, \overline{b}\}$, $S_3 = \overline{S}_* \cup \{a, b\}$, $S_4 = \overline{S}_* \cup \{\overline{a}\,\overline{b}\}$, $S_5 = S_1 \cup \{\overline{a}\}$, $S_6 = S_2 \cup \{a\}$, $S_7 = S_3 \cup \{\overline{a}\}$, and $S_8 = S_4 \cup \{a\}$ are such that $v(S_1) = \cdots = v(S_8) = \max_{S \subseteq N_k \cup \overline{N}_k \cup N_r} v(S) = v(S_*) + \Delta + 2$.*
Observe that $w(\{a, b\}) = w(\{\overline{a}, \overline{b}\}) = w(\{a, b, \overline{a}\}) = w(\{\overline{a}, \overline{b}, a\}) = max_{S \subseteq N_r} v(S) = \Delta + 2$ and that $\Delta > 1$ holds by Lemma 3.5.(A). Then, the result follows from Property 3.6.(1), (3), and (4).

Property 3.6.(6): *For each imputation y and each coalition $S \notin \{S_1, \ldots, S_8\}$, it holds that $e(S_i, y) > e(S, y)$, for each $1 \leq i \leq 8$.*
Consider a coalition S and note that, due to Property 3.6.(4), $e(S, y) = e(S \cap N_r, y) + e(S \cap (N_k \cup \overline{N}_k), y)$, for each imputation y. Observe that for each coalition $T \subseteq N_r$ with $v(T) \neq v(\{a, b\})$, $v(T) \leq v(\{a, b\}) - 2 = v(\{a, b, \overline{a}\}) - 2 = v(\{\overline{a}, \overline{b}\}) - 2 = v(\{\overline{a}, \overline{b}, a\}) - 2$ holds. Combined with Property 3.6.(3), this latter result entails that if $S \notin \{S_1, \ldots, S_8\}$, then, for each $S' \in \{\{a, b\}, \{a, b, \overline{a}\}, \{\overline{a}, \overline{b}\}, \{\overline{a}, \overline{b}, a\}\}$ and $S'' \in \{S_*, \overline{S}_*\}$, either $e(S \cap N_r, y) < e(S', y)$ or $e(S \cap (N_k \cup \overline{N}_k), y) < e(S'', y)$ holds. Thus, $e(S, y) = e(S \cap N_r, y) + e(S \cap (N_k \cup \overline{N}_k), y) < e(S', y) + e(S'', y) = e(S_i, y)$, for each $1 \leq i \leq 8$.

Armed with these properties, we now show that α_1 is true in the lexicographically maximum satisfying assignment for ϕ if and only if x is in the nucleolus.

($\Rightarrow$) Assume that α_1 is true in the lexicographically maximum satisfying assignment for ϕ, and let S_* be such that $v(S_*) = \max_{S \subseteq N_k} v(S)$. Recall that given the correspondence between primal and dual coalitions, $v(S_*) = v(\overline{S}_*)$ holds. Consider then the coalitions $S_1, \ldots, S_8$ of Property 3.6.(5) and note that, by Property 3.6.(2), $S_1 \cap \cdots \cap S_8 = \{\alpha_1\}$ holds. Also, note that $e(S_i, x) = v(S_i) - 1$ holds for each $1 \leq i \leq 8$. Therefore, if we consider an imputation $y \neq x$ (so that $y_{\alpha_1} < 1$), a coalition S_h (with

$1 \leq h \leq 8$) will certainly exists such that $e(S_h, y) > e(S_h, x)$. In fact, because of Property 3.6.(6), $S_1, \ldots, S_8$ are those coalitions on which the maximum excess is achieved, independently on the specific imputation being considered. Hence, x is the nucleolus.

($\Leftarrow$) Assume that α_1 is false in the lexicographically maximum satisfying assignment for ϕ, and let S_* be such that $v(S_*) = v(\overline{S}_*) = \max_{S \subseteq N_k} v(S)$. Because of Property 3.6.(2), it is the case that $\alpha_1 \notin S_*$ and $\alpha_1 \notin \overline{S}_*$. Thus, $e(S_*, x) = e(\overline{S}_*, x) = v(S_*) = v(\overline{S}_*)$. And, $e(S_i, x) = e(S_*, x) + \Delta + 2 = e(\overline{S}_*, x) + \Delta + 2$ for $1 \leq i \leq 8$. Consider now the imputation y such that $y_{chall} = y_{\overline{chall}} = \frac{1}{2}$. Due to Property 3.6.(2), $chall \in S_*$ (and $\overline{chall} \in \overline{S}_*$, by duality). Thus, $e(S_*, y) = e(\overline{S}_*, y) = v(S_*) - \frac{1}{2} = v(\overline{S}_*) - \frac{1}{2} = e(S_*, x) - \frac{1}{2} = e(\overline{S}_*, x) - \frac{1}{2}$. That is, $e(S_*, y) < e(S_*, x)$ and $e(\overline{S}_*, y) < e(\overline{S}_*, x)$. It follows that $e(S_i, y) = e(S_*, y) + \Delta + 2 < e(S_*, x) + \Delta + 2 = e(S_i, x)$ for each $1 \leq i \leq 8$. Eventually, because of Property 3.6.(6), the latter entails that x is not the nucleolus. □

3.2 Reasoning on succinctly specified linear programs

In order to derive membership results, presented in the next section and corresponding to the hardness results proved above, we need to develop some technical machinery. In particular, we need upper bounds on the complexity of several reasoning tasks related to linear programs that are succinctly specified over exponentially many inequalities. In the present section we will give only preliminary definitions and a summary of the results. Full details will be reported in Appendix A.

Let $Ax \leq b$ be a system of linear inequalities, where $A \in \mathbb{R}^{m \times n}$ is a matrix with entries in $\mathbb{R}$, m rows and n column, while $b \in \mathbb{R}^m$ is a (column) vector of m components (one for each row of the matrix A). The i-th row of a matrix $A \in \mathbb{R}^{m \times n}$ is denoted by $A_{i,\bullet}$. We denote by $\Omega(Ax \leq b)$, called the *feasible region* of $Ax \leq b$, the set of vectors in $\mathbb{R}^n$ satisfying the system $Ax \leq b$. By its definition, $\Omega(Ax \leq b)$ is a *polyhedron* of $\mathbb{R}^n$, that is a convex region delimited by the hyperplanes associated with the linear inequalities of the linear system. We are interested in systems whose feasible region is a *polytope* that is a *bounded* polyhedron. An inequality of the system $Ax \leq b$ is said to be an *implied equality* if it is satisfied as equality by all vectors of the feasible region. By $ie(Ax \leq b)$ we denote the set of all implied equalities of $Ax \leq b$. With $dim(Ax \leq b)$ we denote the dimension of the feasible region of the linear system.

There are several reasoning tasks (decision or computation problems) over $Ax \leq b$ that one frequently encounters:

MEMBERSHIP: Given a vector $\bar{x} \in \mathbb{R}^n$, is $\bar{x} \in \Omega(Ax \leq b)$?

NONEMPTINESS: Is $\Omega(Ax \leq b) \neq \varnothing$?

And, assuming that $\Omega(Ax \leq b)$ is non empty:

DIMENSION: Given a natural number $k \geq 0$, is $dim(Ax \leq b) \leq n - k$?

AFFINEHULLCOMPUTATION: Compute the set $ie(Ax \leq b)$ of all the implied equalities.

OPTIMALVALUECOMPUTATION: Given a vector $c \in \mathbb{R}^n$, compute the real number $\min\{c^T\bar{x} \mid \bar{x} \in \Omega(Ax \leq b)\}$.

FEASIBLEVECTORCOMPUTATION: Compute a vector $x \in \Omega(Ax \leq b)$.

OPTIMALVECTORCOMPUTATION: Given a vector $c \in \mathbb{R}^n$, compute a vector $x_* \in \Omega(Ax \leq b)$ such that $c^T x_* = \min\{c^T\bar{x} \mid \bar{x} \in \Omega(Ax \leq b)\}$.

For any real number $x \in \mathbb{R}$, let us denote by $||x||$ its *encoding length* (see e.g., Papadimitriou and Steiglitz, 1998; Grötschel et al., 1993, for the description of some encoding techniques). It is well-known that all the above problems are feasible in deterministic polynomial time whenever the system $Ax \leq b$ is explicitly given, so that the size of the input is the encoding length $||Ax \leq b||$, which is $O(m \times (n+1) \times \max_i\{\max_j ||A_{i,j}||, ||b_i||\})$ (cf., e.g., Schrijver, 1998; Papadimitriou and Steiglitz, 1998; Grötschel et al., 1993).

In this section, we are instead interested in analyzing the complexity of these problems with respect to a scenario where the system of linear inequalities is specified in succinct form, as to conveniently encode problems over exponentially many inequalities.

Definition 3.7. *A function $\mathcal{L} : \{1, \ldots, m\} \mapsto \mathbb{R}^n \times \mathbb{R}$ is a* succinct specification *for the system of linear inequalities $Ax \leq b$ if:*

- *$\mathcal{L}(i) = (A_{i,\bullet}, b_i)$, for each $1 \leq i \leq m$;*
- *there is a fixed natural number $k_1 > 0$ such that the encoding length of $\mathcal{L}$ is*
$$||\mathcal{L}|| = \left(\log_2(m) \times n \times \max_i\{\max_j ||A_{i,j}||, ||b_i||\}\right)^{k_1} ; \textit{ and,}$$
- *there is a fixed natural number $k_2 > 0$ such that $\mathcal{L}(i)$ can be computed in time $O(||\mathcal{L}||^{k_2})$, for each $1 \leq i \leq m$.*

The system $Ax \leq b$ will be denoted by $\mathcal{L}[x]$ in the following.

For each problem $\mathcal{P}$ illustrated above, let SUCCINT-$\mathcal{P}$ be the problem $\mathcal{P}$ whose input is given in terms of a succinct specification (plus the encoding length of possible additional input parameters).

A summary of the results is illustrated in Figure 3.4. Note that all these problems fall in the first level of the polynomial hierarchy; in particular, computation problems are feasible with transducers that use polynomially many **NP** oracle calls.

Problem	**Result**
Membership	in co-**NP**
NonEmptiness	in co-**NP**
Dimension	in **NP**
AffineHullComputation	in $\mathbf{F\Delta_2^P}$
OptimalValueComputation	in $\mathbf{F\Delta_2^P}$
FeasibleVectorComputation	in $\mathbf{F\Delta_2^P}$
OptimalVectorComputation	in $\mathbf{F\Delta_2^P}$

Fig. 3.4. Summary of Complexity Results for Linear Programming Problems.

3.3 Membership results

In this section, we show that the various hardness results derived in Section 3.1 are tight, since the corresponding membership results can be established. In fact, such memberships are proven to hold over any polynomial-time compact representation. As we noticed in Section 2.4.3, graph games and marginal contribution networks are **FP** representation and hence all the membership results we shall provide in this setting immediately holds for such special kinds of compact games.

3.3.1 The Core

Theorem 3.8. *Let $\mathcal{C}$ be a class of TU compact games. For any $\mathcal{G} \in \mathcal{C}$, deciding whether an imputation x belongs to $\mathscr{C}(\mathcal{G})$ is feasible in* co-**NP**. *Also, deciding whether $\mathscr{C}(\mathcal{G}) \neq \varnothing$ is feasible in* co-**NP**.

Proof. Recall that the core of a coalitional game with transferable utility and n players is defined by a set of inequalities over n variables and, hence, it is a polytope in $\mathbb{R}^n$. Note moreover that it can be specified in succinct form. Given the imputation x, we firstly observe that we can check whether x does not belong to $\mathscr{C}(\mathcal{G})$ by simply solving the Succinct-Membership problem in co-**NP** over such polytope (cf. Theorem A.2). Eventually, deciding whether $\mathscr{C}(\mathcal{G}) \neq \varnothing$ is feasible in co-**NP**, since it just amount at solving the Succinct-NonEmptiness problem (cf. Theorem A.4). □

The previous proof for the non-emptiness of the core is based on succinct linear programs. In particular, before that, we devised a constructive geometrical proof for the same problem. The interested reader is referred to Appendix B for details.

Corollary 3.9. *Let $\mathcal{G}$ be a TU game represented through a marginal contribution network. Deciding whether $\mathscr{C}(\mathcal{G})$ is non-empty is* co-**NP**-complete.

Proof. The hardness of the problem follows from the fact that deciding whether the core of a graph game is non-empty is co-**NP**-complete (Deng and Papadimitriou, 1994) and marginal contribution networks can represent such games (Ieong and Shoham, 2005).

The membership directly follows from Theorem 3.8 and the fact that marginal contribution networks is a polynomial-time compact representation. □

Corollary 3.9 closes a problem left open by Ieong and Shoham (2005).

3.3.2 The Bargaining Set

As for the result pertaining checking whether an imputation belongs to the bargaining set, it has already been conjectured that this problem is in $\mathbf{\Pi_2^P}$ for graph games (Deng and Papadimitriou, 1994). Indeed, it has been observed that one may decide that an imputation x is not in the bargaining set, by firstly guessing in **NP** the objection (y, S), and then by calling a co-**NP** oracle, which checks that there is no counterobjection (z, T) to (y, S). However, this argument assumes to restrict ourselves to deal with real values with fixed-precision.

Our main achievement in this section is to show that membership in $\mathbf{\Pi_2^P}$ can be established, independently of the precision adopted to represent the real values of interest in the game. To this end, we shall provide a useful characterization of a player i having a justified objection against some player j. The result is in the spirit of one of Maschler (1966) and connects the existence of a justified objection of i against j to some algebraic conditions to hold on coalitions that i and j (may) participate to.

Lemma 3.10. *Let $\mathcal{G}$ be a TU game, and x be an imputation of $\mathcal{G}$. Then, player i has a justified objection against player j to x through coalition $S \in \mathcal{I}_{i,j}$ if and only if there exists a vector $y \in \mathbb{R}^S$ such that:*

(1) *$y(S) = v(S)$;*
(2) *$y_k > x_k$, for each $k \in S$; and,*
(3) *$v(T) < y(T \cap S) + x(T \setminus S)$, for each $T \in \mathcal{I}_{j,i}$.*

Proof.

($\Rightarrow$) Assume that player i has a justified objection against player j to x through coalition $S \in \mathcal{I}_{i,j}$. Let (y, S) be such justified objection and note that, by definition, $y(S) = v(S)$ and $y_k > x_k$ for each $k \in S$ hold. Assume now, for the sake of contradiction, that (3) above does not hold. Let $\bar{T} \in \mathcal{I}_{j,i}$ be such that $v(\bar{T}) \geq y(\bar{T} \cap S) + x(\bar{T} \setminus S)$. Based on $\bar{T}$, let us build a vector $z \in \mathbb{R}^{\bar{T}}$ such that: *(i)* $z_k = x_k + \delta$ for each $k \in \bar{T} \setminus S$; and, *(ii)* $z_k = y_k + \delta$ for each $k \in \bar{T} \cap S$, where

$$\delta = \frac{v(\bar{T}) - y(\bar{T} \cap S) - x(\bar{T} \setminus S)}{|\bar{T}|} \geq 0.$$

Note that $z(\bar{T}) = v(\bar{T})$ holds by construction. Hence, $(z, \bar{T})$ is a counterobjection to (y, S): a contradiction.

($\Leftarrow$) Assume that there is a vector $y \in \mathbb{R}^S$ such that all the three conditions above hold. Consider the pair (y, S) (with $S \in \mathcal{I}_{i,j}$) and note that due to (1) and (2), (y, S) is in fact an objection of player i against player j. We now claim that (y, S) is justified. Indeed, assume for the sake of contradiction that a counterobjection $(z, \bar{T})$ (with $\bar{T} \in \mathcal{I}_{j,i}$) exists such that: $z(\bar{T}) = v(\bar{T})$, $z_k \geq x_k$ for each $k \in \bar{T} \setminus S$, and $z_k \geq y_k$ for each $k \in \bar{T} \cap S$. In this case, it holds that $v(\bar{T}) = z(\bar{T}) \geq y(T \cap S) + x(T \setminus S)$, which contradicts (3). □

Theorem 3.11. *Let $\mathcal{C}$ be a class of TU compact games. For any $\mathcal{G} \in \mathcal{C}$, deciding whether an imputation x belongs to $\mathscr{B}(\mathcal{G})$ is feasible in $\mathbf{\Pi_2^P}$.*

Proof. Consider the complementary problem of deciding whether $x \notin \mathscr{B}(\mathcal{G})$. In the light of Lemma 3.10, $x \notin \mathscr{B}(\mathcal{G})$ if and only if there exists two player i and j, and a coalition $S \in \mathcal{I}_{i,j}$ such that the set

$$
\begin{aligned}
W(i,j,S) = \{y \in \mathbb{R}^S \mid y(S) = v(S) \wedge \\
y_k > x_k,\ \forall k \in S \wedge v(T) < y(T \cap S) + x(T \setminus S),\ \forall T \in \mathcal{I}_{j,i}\}
\end{aligned}
$$

is not empty. Thus, the problem can be solved by firstly guessing in **NP** the players i and j, and the set S. Indeed, it is immediate to check that $W(i,j,S)$ is a polytope that can be specified in succinct form, and that the size of this representation is polynomial in the size $||\mathcal{G}||$. Thus, non-emptiness can be checked by means of a co-**NP** oracle (cf. Theorem A.4). □

Corollary 3.12. *Let $\mathcal{G}$ be a graph game, and x be an imputation. Deciding whether $x \in \mathscr{B}(\mathcal{G})$ is* $\mathbf{\Pi_2^P}$-complete.

Corollary 3.12 shows that the conjecture posed by Deng and Papadimitriou (1994) actually holds.

3.3.3 The Kernel

Theorem 3.13. *Let $\mathcal{C}$ be a class of TU compact games. For any $\mathcal{G} \in \mathcal{C}$, deciding whether an imputation x belongs to $\mathscr{K}(\mathcal{G})$ is feasible in $\mathbf{\Delta_2^P}$.*

Proof. Given the imputation x, we firstly observe that for each pair of players i and j, the value $s_{i,j}(x)$ can be computed by means of a binary search over the range of the possible values for the worth functions, by using an **NP** oracle. Indeed, for any value h in this range, we can decide in **NP** whether there is a coalition S such that $e(S, x) > h$, by guessing the coalition S, and by checking in polynomial time that $e(S, x) > h$. By the definition of excess, to evaluate $e(S, x) = v(S) - x(S)$ we have to compute the worth $v(S)$ and this is feasible in polynomial time since $\mathcal{G} \in \mathcal{C}$. In particular, since $v(S)$ requires polynomially many bits for its representation in the size $||\mathcal{G}||$, then the range to explore in order to find a coalition such that $e(S, x) > h$ is exponential in

$||\mathcal{G}||$. Yet, a binary search on this range again requires at most polynomially many steps.

Eventually, we can compute in polynomial time the value $v(\{i\})$ for each player $i \in \{1, \ldots, n\}$. Thus, after polynomially-many oracle calls, we may check in polynomial time that for each pair of players i and j such that $x_j \neq v(\{j\})$, it is the case that $s_{i,j}(x) \leq s_{j,i}(x)$. □

Corollary 3.14. *Let $\mathcal{G}$ be a graph game, and x be an imputation. Deciding whether $x \in \mathscr{K}(\mathcal{G})$ is $\mathbf{\Delta_2^P}$-*complete.

Deng and Papadimitriou (1994) conjectured that checking whether an imputation belongs to the kernel of a graph game is **NP**-hard. Corollary 3.14 shows that the conjecture actually holds, in particular by giving a tighter result.

3.3.4 The Nucleolus

Maschler et al. (1979) showed that the nucleolus besides being the imputation lexicographically minimizing the vector of excesses in a game, it can be equivalently characterized as the lexicographic center of the game.

Intuitively, the procedure they describe to obtain the lexicographic center of a game is as follows. First, we find all the imputations that minimize the maximum excess; in general these will form a nonempty compact convex set. Then we put aside those coalitions whose excess never goes below this minimum in this set, and "reminimize" the maximum excess over the remaining coalitions in order to select the set of all imputations minimizing the second maximum excess. This gives us, in general, a nonempty compact convex *subset* of the previous set, as well as some new coalitions whose excess cannot be further reduced. These coalitions in turn are put aside, and the process is repeated until there are no coalitions left.

Maschler et al. (1979) showed that the previous procedure ends with a region comprising a unique imputation that they proved to be the nucleolus. Obviously, the previous procedure needs a finite number of "minimization" steps, and hence it halts in finite time. More interestingly, they also noted that if at each step we put aside all coalitions whose excess is *constant* over the selected region, and not only those coalitions whose excess is the maximum over the selected region, then the aforementioned procedure needs at most n minimization steps, where n is the number of players of the game. Intuitively this holds because by putting aside all those coalitions, the dimension of the minimized region decreases at least of 1 at each minimization step, and hence in at most n steps we obtain a region containing only one point which is the nucleolus. This is a great enhancement over the "standard" procedure since, in general, when putting aside only coalitions with the maximum excess over the minimized region, it can happen that only one coalition is put aside at each step, thus generally requiring an exponential number of steps to terminate.

Let us analyze now this procedure more formally. Let $\mathcal{G} = \langle N, v \rangle$ be a TU game. The Maschler et al.'s procedure can be rewritten in a linear programming form, and next we show, at first, a rewriting of that presented by Paulusma (2001). The whole algorithm computes the nucleolus of $\mathcal{G}$ by solving a sequence of linear programming problems. Let $\mathtt{LP}_1$ be the first of such problems, and $\mathcal{F}_0 = \{\varnothing, N\}$. We have that:

$$\begin{aligned} \mathtt{LP}_1 = \{\min \epsilon \mid \ & x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_0 \\ & x \in X(\mathcal{G})\}. \end{aligned}$$

Let us have a look at the constraints of $\mathtt{LP}_1$. It is straightforward to note that for this program the feasible region shrinks with the decreasing of the value ϵ. Moreover if we rewrite "$x(S) \geq v(S) - \epsilon$" into "$\epsilon \geq x(S) - v(S)$" we can easily observe that the feasible region becomes empty whenever for ϵ is chosen a value, say $\bar{\epsilon}$, for which exists a coalition S such that $\bar{\epsilon} < e(S, x)$ for all $x \in X(\mathcal{G})$.

Let ϵ_1 denote the optimum value of ϵ in $\mathtt{LP}_1$, and let V_1 be the set of optimal solutions of $\mathtt{LP}_1$. By the previous observation and since the objective function of $\mathtt{LP}_1$ is only "$\min \epsilon$", then V_1 equals the set of all feasible solutions when $\epsilon = \epsilon_1$ (note that for every $\bar{\epsilon} < \epsilon_1$ the feasible region is empty and hence $\bar{\epsilon}$ is not the optimum since there is no feasible solution for which that "optimum" is reached). By this, we have that $V_1 \subseteq X(\mathcal{G})$ is the set of all imputations minimizing the maximum excess. This is precisely what is needed by the first step of the Maschler et al.'s procedure.

The second step of the procedure requires that all coalitions having a constant excess over V_1 (and hence also those coalitions having the maximum excess) are put aside from the minimization process, call this set $\mathcal{F}_1$. It is easy to see that $\mathcal{F}_0 \subseteq \mathcal{F}_1$ and that $X(\mathcal{G}) \supseteq V_1$ holds. The linear program at the second step is:

$$\begin{aligned} \mathtt{LP}_2 = \{\min \epsilon \mid \ & x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_1 \\ & x \in V_1\}. \end{aligned}$$

The optimal value ϵ_2 of $\mathtt{LP}_2$ is such that $\epsilon_1 > \epsilon_2$ because all the coalitions not having an excess smaller that ϵ_1 over $X(\mathcal{G})$ belong to $\mathcal{F}_1$ (are not optimized) and, hence, we can decrease ϵ further below ϵ_1. Note that ϵ_2 is not the absolute second minimum maximum excess. Rather, it is the minimum maximum excess reachable over V_1 by those coalitions not belonging to $\mathcal{F}_1$ (those coalitions whose excess is not a constant over V_1). This means that, in principle, there may exist a coalition $\bar{S} \in \mathcal{F}_1$ such that $\epsilon_1 > e(\bar{S}, x) = c > \epsilon_2$ for all $x \in V_1$ where c is a constant. Even if such a coalition $\bar{S}$ actually exists, it does not matter neither to know $\bar{S}$, nor to compute the value $e(\bar{S}, x)$, because this value is a constant over V_1 and hence it does not affect the selection of the nucleolus among the imputations in V_1.

The third step is a new minimization over the region of optimal solutions V_2 excluding the coalitions in $\mathcal{F}_2$ being those ones whose excess is a constant over V_2. It is easy to see that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ and $X(\mathcal{G}) \supseteq V_1 \supseteq V_2$. The procedure is carried on, step by step, until we arrive to a linear program

$$\begin{aligned} \mathtt{LP}_{k_*} = \{\min \epsilon \;\mid\; & x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_{k_*-1} \\ & x \in V_{k_*-1}\}, \end{aligned}$$

such that V_{k_*} is a singleton. As said before, Maschler et al. (1979) proved that $V_{k_*} = \mathscr{N}(\mathcal{G})$ and that $k_* \leq |N|$.

Now, let us focus our attention on the sets V_k's, starting with V_1. We know that V_1 equals the feasible region of $\mathtt{LP}_1$ when $\epsilon = \epsilon_1$, that is

$$V_1 = \{x \in \mathbb{R}^n | x \in X(\mathcal{G}) \wedge x(S) \geq v(S) - \epsilon_1, \forall S \subseteq N, S \notin \mathcal{F}_0\}.$$

Since the feasible region of $\mathtt{LP}_1$ is empty when $\epsilon < \epsilon_1$, there is a set $\Lambda_1 \subseteq (2^N \setminus \mathcal{F}_0)$ such that, for all coalitions $S \in \Lambda_1$, it holds that $x(S) = v(S) - \epsilon_1$ for all imputation in V_1 (that is $e(S,x) = \epsilon_1$ for all imputations belonging to V_1). We can rewrite V_1 as:

$$\begin{aligned} V_1 = \{x \in \mathbb{R}^n | x \in X(\mathcal{G}) \wedge x(S) = v(S) - \epsilon_1, \forall S \in \Lambda_1 \wedge \\ x(S) \geq v(S) - \epsilon_1, \forall S \subseteq N, S \notin (\Lambda_1 \cup \mathcal{F}_0)\}. \end{aligned}$$

By definition of Λ_1, $\Lambda_1 \subseteq \mathcal{F}_1$, and those coalitions S such that $e(S,x)$ is constant over V_1 but $e(S,x) \neq \epsilon_1$ belong to $\mathcal{F}_1 \setminus \Lambda_1$. Let $x_1 \in V_1$ be an imputation being an optimal solution of $\mathtt{LP}_1$. We can further rewrite V_1 as:

$$\begin{aligned} V_1 = \{x \in \mathbb{R}^n | x \in X(\mathcal{G}) \wedge x(S) = v(S) - \epsilon_1, \forall S \in \Lambda_1 \wedge \\ x(S) = v(S) - e(S, x_1), \forall S \in (\mathcal{F}_1 \setminus \Lambda_1) \wedge \\ x(S) \geq v(S) - \epsilon_1, \forall S \subseteq N, S \notin \mathcal{F}_1\}. \end{aligned}$$

We substitute this characterization of the set V_1 in $\mathtt{LP}_2$, and we obtain

$$\begin{aligned} \mathtt{LP}_2 = \{\min \epsilon \;\mid\; & x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_1 \\ & x(S) = v(S) - \epsilon_1, && \forall S \in \Lambda_1 \\ & x(S) = v(S) - e(S, x_1), && \forall S \in (\mathcal{F}_1 \setminus \Lambda_1) \\ & x(S) \geq v(S) - \epsilon_1, && \forall S \subseteq N, S \notin \mathcal{F}_1 \\ & x \in X(\mathcal{G})\}. \end{aligned}$$

Since all variables of players belonging to coalitions $S \notin \mathcal{F}_1$ are subject to optimization in $\mathtt{LP}_2$, and ϵ_2 will be smaller than ϵ_1, we can completely get rid of constraints "$x(S) \geq v(S) - \epsilon_1, \forall S \subseteq N, S \notin \mathcal{F}_1$". Moreover, by applying a result of Granot et al. (1998, Lemma 4.3), we can remove also constraints "$x(S) = v(S) - e(S, x_1), \forall S \in (\mathcal{F}_1 \setminus \Lambda_1)$" without altering the result of the computation. Eventually, the expression of the linear program can be simplified to

$$\begin{aligned}
\mathtt{LP}_2 = \{\min \epsilon \mid\ & x(S) = v(S) - \epsilon_1, && \forall S \in \Lambda_1 \\
& x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_1 \\
& x \in X(\mathcal{G})\}.
\end{aligned}$$

The argumentations carried out for V_1 and $\mathtt{LP}_2$ hold in general and can be extended inductively to every step of the procedure, obtaining the following linear program that have to be solved at the generic step k:

$$\begin{aligned}
\mathtt{LP}_k = \{\min \epsilon \mid\ & x(S) = v(S) - \epsilon_r, && \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1 \\
& x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_{k-1} \\
& x \in X(\mathcal{G})\},
\end{aligned}$$

where ϵ_r is the optimum of $\mathtt{LP}_r$, $\Lambda_r = \{S \subseteq N \mid x(S) = v(S) - \epsilon_r, \forall x \in V_r\}$ with $V_r = \{x \mid x \in X(\mathcal{G}) \wedge (x, \epsilon_r)$ is an optimal solution to $\mathtt{LP}_r\}$, and $\mathcal{F}_r = \{S \subseteq N \mid x(S) = y(S), \forall x, y \in V_r\}$ with $\mathcal{F}_0 = \{\varnothing, N\}$.

By solving all the k_* linear programs we can compute the nucleolus of $\mathcal{G}$. There are some issues to deal with in order to solve this sequence of linear programs. At the generic step k we have that sets $\mathcal{F}_{k-1}$ and Λ_r (for all $1 \leq r \leq k-1$) contain exponentially many elements in the number of players. And, as already noted by Paulusma (2001), these difficulties make the problem intractable. In fact, no polynomial time algorithms are known for computing the nucleolus in general. Neither such an algorithm may exist (until $\mathbf{P} \neq \mathbf{NP}$) since the complexity of deciding whether an imputation of a graph game is the nucleolus (that is $\mathbf{\Delta_2^P}$-hard, see Theorem 3.6) is an insurmountable lower bound. Hence, the best that can be done is devising a general algorithm to compute the nucleolus of a game working in $\mathbf{F\Delta_2^P}$, and this is our aim.

Note that there are polynomial algorithms for particular classes of games. For instance, there exist efficient algorithms for computing the nucleolus of standard tree games (Megiddo, 1978; Granot et al., 1996), the nucleolus of convex games (Kuipers, 1996), and the nucleolus of assignment games (Solymosi and Raghavan, 1994).

In order not to exceed $\mathbf{F\Delta_2^P}$, we cannot explicitly represent $\mathcal{F}_{k-1}$ nor Λ_r for all $1 \leq r \leq k-1$, and we have to find a smart way to represent them in polynomial space and decide if a coalition belongs to them in polynomial time.

With regard to Λ_r, it can be shown that (see Appendix C for the details) the set of constraints "$x(S) = v(S) - \epsilon_r, \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1$" in the linear program $\mathtt{LP}_k$, can be substituted by an appropriate *polynomial* subset of them, more specifically a *basis* $\mathcal{B}_{k-1}$ of the implied equalities of V_{k-1}, without altering the computation of the nucleolus.

Now, let us turn our attention to $\mathcal{F}_{k-1}$. Instead of explicitly listing all coalitions belonging to $\mathcal{F}_{k-1}$, we can devise a polynomial-time test in order to decide whether $S \notin \mathcal{F}_{k-1}$ or not.

Denote by $\mathbb{1}_S$ the *indicator vector* of S, that is, a vector in $\mathbb{N}^n$ such that its component $\mathbb{1}_{S_i}$ is 1 if and only if $i \in S$, and 0 otherwise. Let $\mathcal{S}$ be a set of coalitions. We say that a coalition S is a linear combination of $\mathcal{S}$ if the vector $\mathbb{1}_S$ can be written as a linear combination of the indicator vectors of the coalitions in $\mathcal{S}$.

It can be shown that (see Appendix C for the details) a coalition $S \in \mathcal{F}_{k-1}$ if and only if S is a linear combination of the coalitions whose related constraints are selected to be included in $\mathcal{B}_{k-1}$. We now are ready to state our result.

Theorem 3.15. *Let $\mathcal{C}$ be a class of compact games. Then, computing the nucleolus of $\mathcal{G}$ is feasible in* $\mathbf{F\Delta_2^P}$.

Proof. Consider the following sequence of linear programming problems $\mathtt{LP}_k$, for $k > 0$:

$$\begin{aligned}
\mathtt{LP}_k = \{\min \epsilon \mid \; & x(S) = v(S) - \epsilon_r, && \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1 && (3.2)\\
& x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_{k-1} && (3.3)\\
& x \in X(\mathcal{G})\}. && && (3.4)
\end{aligned}$$

Following Maschler et al. (1979), it can be shown that there is an index $k_* \leq |N|$ such that $\mathtt{LP}_{k_*}$ has exactly one optimal solution (x_*, ϵ_{k_*}) where x_* is the nucleolus of $\mathcal{G}$.

Consider the generic step $1 < k \leq k_*$. We know that V_{k-1} equals the feasible region of $\mathtt{LP}_{k-1}$ and hence it can be represented in succinct form (its representation comes from the constraints of $\mathtt{LP}_{k-1}$, that in their turn can be represented in succinct form, with $\epsilon = \epsilon_{k-1}$). By this, we can compute a (polynomial) basis $\mathcal{B}_{k-1}$ for the implied equalities of V_{k-1} in $\mathbf{F\Delta_2^P}$ (see Theorem A.9). Moreover we can avoid to represent explicitly $\mathcal{F}_{k-1}$, needed to represent constraints (3.3). In fact, we can check whether $S \in \mathcal{F}_{k-1}$ by checking whether the indicator vector $\mathbb{1}_S$ can be written as a linear combination of the coalitions whose constraints have been selected to be included in $\mathcal{B}_{k-1}$. Note that, since B_{k-1} contains only polynomially many constraints, checking this linear dependency can be carried out in polynomial time.

By this, the program $\mathtt{LP}_k$ can be represented in succinct form by substituting all constraints (3.2) with the computed basis $\mathcal{B}_{k-1}$, and by using the linear dependency test instead of the explicit representation of $\mathcal{F}_{k-1}$. Hence, we can compute the optimal value of $\mathtt{LP}_k$ in $\mathbf{F\Delta_2^P}$ (see Theorem A.11). After having computed the optimal value ϵ_k of $\mathtt{LP}_k$, we can compute the dimension of V_k in **NP** (see Theorem A.6) (recall that V_k can be represented succinctly). If the dimension is greater than 0, we start the next optimization step. Note that the current optimization step has required computations confined in $\mathbf{F\Delta_2^P}$, and a similar argumentation shows that this holds for the first step ($k = 1$) as well, since the program $\mathtt{LP}_1$, not being characterized by any particular constraint, can be obviously represented in succinct form.

Instead, if the dimension of V_k is 0 then the procedure is almost over (we are actually at step k_*) since the unique optimal (and feasible) vector belonging to V_k is the nucleolus. We lastly need only to compute this feasible vector, which can be done in $\mathbf{F}\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$ (see Theorem A.12).

To conclude, recall that $k_* \leq |N|$ and that each step is feasible in $\mathbf{F}\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$, hence the whole procedure is in $\mathbf{F}\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$ □

Corollary 3.16. *Let $\mathcal{C}$ be a class of compact games. Then, deciding whether an imputation x belongs to $\mathscr{N}(\mathcal{G})$ is feasible in $\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$.*

Corollary 3.17. *Let $\mathcal{G}$ be a graph game, and x be an imputation. Deciding whether $x \in \mathscr{N}(\mathcal{G})$ is $\boldsymbol{\Delta}_{\mathbf{2}}^{\boldsymbol{P}}$-*complete.

3.3.5 More powerful worth functions

In all the above illustrated membership results, we have assumed that worth functions are polynomially time computable and, within this setting, we have shown that various hardness results are indeed tight. Thus, the reader might naturally believe that, by considering more powerful worth functions (e.g., functions computable in **FNP**), the complexity for the problems will consistently increase.

Surprisingly, this is not the case. Indeed, we can show that nothing has to be paid if more powerful worth functions are considered that encode, for instance, **NP**-complete problems that reflect the results of complex algorithmic procedures as those arising in allocation, scheduling and routing scenarios. This is the case when we consider non-deterministic polynomial time compact representation. Thus, whenever we guess in **NP** some coalition S for a game $\mathcal{G}$ with an **FNP** representation, we can at the same time (and with no further efforts) guess the worth w and a certificate c, which may subsequently be used to verify in polynomial time that $v_{\mathcal{G}}(S) = w$ (where $v_{\mathcal{G}}$ denote the worth function of $\mathcal{G}$). In practice, computing the value $v_{\mathcal{G}}(S)$ comes for free, whenever the coalition S is guessed beforehand. Interestingly, the reader may check that in all the membership results provided in this section, the value of any worth function $v_{\mathcal{G}}(S)$ is used precisely just after that S is guessed. Hence, all our memberships immediately hold over such more powerful worth functions.

Theorem 3.18. *The membership results in Section 3.3 hold on any* **FNP** *compact representation $\mathcal{R}$.*

3.4 Tractable classes of games

Many **NP**-hard problems in different application areas are known to be efficiently solvable when restricted to instances that can be modeled via (nearly)acyclic graphs. Indeed, on these kinds of instances, solutions can usually be computed via dynamic programming, by incrementally processing the acyclic (hyper)graph, according to some of its topological orderings.

In this section, we shall investigate on whether some of the solutions concepts for coalitional games are tractable when restricted to (nearly)acyclic instances. In particular, we shall consider the most powerful generalization of graph acyclicity, as for it is provided via the notion of *tree decomposition* (Robertson and Seymour, 1984).

3.4.1 Treewidth and monadic second order logic

Treewidth. A *tree decomposition* of a graph $G = (N, E)$ is a pair $\langle T, \chi \rangle$, where $T = (V, F)$ is a tree, and χ is a labeling function assigning to each vertex $p \in N$ a set of vertices $\chi(p) \subseteq V$, such that the following conditions are satisfied:

(1) for each node b of G, there exists $p \in V$ such that $b \in \chi(p)$;
(2) for each edge $(b, d) \in E$, there exists $p \in V$ such that $\{b, d\} \subseteq \chi(p)$; and,
(3) for each node b of G, the set $\{p \in V \mid b \in \chi(p)\}$ induces a connected subtree.

The *width* of $\langle T, \chi \rangle$ is the number $\max_{p \in V}(|\chi(p)| - 1)$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all its tree decompositions. It is well-known that a graph G is acyclic if and only if $tw(G) = 1$. Deciding whether a given graph has treewidth bounded by a fixed natural number k is known to be feasible in linear time (Bodlaender, 1996).

MSO. Monadic Second Order (MSO) Logic formulae on graphs are made up of the logical connectors $\vee$, $\wedge$, and $\neg$, the membership relation $\in$, the quantifiers $\exists$ and $\forall$, and vertices variables and vertex sets variables—in addition, it is often convenient to use symbols like $\subseteq$, $\subset$, $\cap$, $\cup$, and $\rightarrow$ with their usual meaning, as abbreviations.

Courcelle (1990) and other authors considered an extension of MSO, called MSO_2, where variables for edge sets are also allowed. The fact that an MSO_2 sentence ϕ holds over a graph G is denoted by $G \models \phi$.

The relationship between treewidth and MSO_2 is illustrated next.

Proposition 3.19 (Courcelle, 1990). *For a fixed constant k, let $\mathcal{C}_k$ be a class of graphs having treewidth bounded by k. Then, for a fixed* MSO_2 *sentence ϕ, deciding whether $G \models \phi$, for each $G \in \mathcal{C}_k$, is feasible in linear time.*

An important generalization of MSO_2 formulae to *optimization* problems was presented by Arnborg et al. (1991). Next, we state a simplified version of these kinds of problems. Optimization problems are defined over MSO_2 formulae containing free variables and over graphs that are weighted on both nodes and edges.

Let $G = \langle (N, E), f_N, f_E \rangle$ be a weighted graph where f_N and f_E are the lists of weights associated with nodes and edges, respectively. Then, $f_N(v)$ (resp., $f_E(e)$) denotes the weight associated with $v \in N$ (resp., $e \in E$).

Let $\phi(X, Y)$ be an MSO_2 formula over the graph (N, E), where X and Y are the free variables occurring in ϕ, with X (resp., Y) being a vertex (resp., edge) set variable. For a pair of interpretations $\langle z_N, z_E\rangle$ mapping X to subsets of N and Y to subsets of E, we denote by $\phi[\langle z_N, z_E\rangle]$ the MSO_2 formula (without free variables) where the sets X and Y are replaced by the sets $z_N(X)$ and $z_E(Y)$, respectively.

A *solution* to ϕ over G is a pair of interpretations $\langle z_N, z_E\rangle$ such that $(N, E) \models \phi[\langle z_N, z_E\rangle]$ holds. The *cost* of $\langle z_N, z_E\rangle$ is the value $\sum_{x\in z_N(X)} f_N(x) + \sum_{y\in z_E(Y)} f_E(y)$. A solution of minimum cost is said *optimal*.

For a positive constant k, let hereafter $\mathcal{C}_k$ be a class of graphs having treewidth bounded by k.

Theorem 3.20 (simplified from Arnborg et al., 1991). *Let ϕ be a fixed* MSO_2 *sentence and let $G = \langle(N, E), f_N, f_E\rangle$ be a weighted graph such that $(N, E) \in \mathcal{C}_k$. Then, computing an optimal solution to ϕ over G is feasible in deterministic polynomial time (w.r.t. $||G||$).*

3.4.2 $\mathbf{MSO_2}$ and bounded treewidth graph games

We are now in the position of stating a tractability result about the kernel over bounded treewidth graph games.

Theorem 3.21. *Let $\mathcal{G} = \langle(N, E), w\rangle$ be a graph game such that $(N, E) \in \mathcal{C}_k$, and let x be an imputation of $\mathcal{G}$. Then, deciding whether $x \in \mathscr{K}(\mathcal{G})$ is feasible in* $\mathbf{P}$.

Proof. Firstly, consider the problem of computing the coalition over which the maximum excess at x is achieved. For each $X \subseteq N$ and $Y \subseteq E$, consider the following MSO_2 formula, stating that Y is the set of all those edges $e \in E$ such that $e \subseteq X$:

$$\begin{aligned} \mathit{proj}(X, Y) \equiv \forall v, v' \Big(\{v, v'\} \in Y \rightarrow \{v, v'\} \subseteq X\Big) \wedge \\ \forall v, v' \Big(\{v, v'\} \subseteq X \wedge \{v, v'\} \in E \rightarrow \{v, v'\} \in Y\Big). \end{aligned}$$

Let w_E and w_N be such that $w_E(\{v, v'\}) = -w(\{v, v'\})$ and $w_N(v) = x_v$, and observe that $\max_{S\subseteq N} e(S, x) = -\min_{S\subseteq N}(x(S) - v(S))$ coincides with the cost of an optimal solution to $\mathit{proj}(X, Y)$ over $\langle(N, E), w_N, w_E\rangle$.

Recall, now, that $x \in \mathscr{K}(\mathcal{G})$ if and only if, for each pair of players $i \neq j$, $s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v(\{j\})$, where $s_{i,j}(x) = \max_{S\in\mathcal{I}_{i,j}} e(S, x)$. In fact, one may modify the weights of i and j in $\mathcal{G}$ so that, in any optimal solution of the above formula, $i \in X$ and $j \notin X$, so that $\max_{S\subseteq N} e(S, x)$ coincides with $s_{i,j}$. Hence, by Theorem 3.20 and the above MSO_2 formula, $s_{i,j}$ (and $s_{j,i}$, too) is computable in polynomial time. By checking this condition for each pair i, j, membership of x in $\mathscr{K}(\mathcal{G})$ can be decided in polynomial time. $\square$

4

NTU games

Many works in the literature deals with complexity analysis of TU games, but there are very few ones studying NTU games. In this chapter we will characterize the computational complexity of the core and the bargaining set for NTU games in the framework proposed by Deng and Papadimitriou (1994). We concentrate our study only on these two solution concepts since these are the two ones for which there is a widely accepted definition for NTU games.

At first we introduce a compact representation scheme for such NTU games, called NTU marginal contribution networks, and then we will carry out a complexity analysis whose hardness results will be given on NTU MC-nets, while membership results will be given for any NTU non-deterministic polynomial-time compact representation (see Section 2.4.3). The two relevant problems for these solution concepts for the NTU framework are (i) deciding whether an i.r.p.c. belongs to them and (ii) decide whether such an i.r.p.c. exists at all.

4.1 NTU marginal contribution networks

NTU marginal contribution networks is a representation scheme for NTU games in the framework of Osborne and Rubinstein (1994). NTU MC-nets is an extension of standard marginal contribution networks (Ieong and Shoham, 2005) to NTU games in order to represent these games in compact form.

Also in this case, games are described by associating player patterns with corresponding consequences, in particular the consequences of the game are represented by payoff vectors whose payoffs cannot be redistributed amongst players. In this way, the preference relations for all players are based on payoffs comparison such that for all players the more one gets the better is. The payoff a player gets in the consequences of a coalition is characterized by the sums of the contributions of the rules whose patterns are satisfied by the coalition the player belongs to.

In particular, an NTU marginal contribution network for a game with non-transferable utility is a finite set of rules in the form

$$pattern \rightarrow (consequences\_addendum)^{+},$$

where *pattern* is a conjunction of positive and negative player literals, and $(consequences\_addendum)^{+}$ is a non-empty set of consequence addenda for a set of players occurring as positive literals in *pattern*.

A consequence addendum is a vector in the form $[p_1{+}{=}a_1, \ldots, p_m{+}{=}a_m]$ that specifies for each player $p_1, \ldots, p_m$ the increment (positive or negative) $a_1, \ldots, a_m$ with which the rule contributes in the construction of a consequence (that is a payoff vector) for the coalition triggering the rule. In any rule many consequence addenda can be associated with the pattern, and when this is the case only one consequence addendum is chosen for the construction of a single consequence (different addenda can be chosen when building different consequences). The set of all consequences $V(S)$ for a coalition S is equal to the set of all possible payoff vectors that can be built by choosing one consequence addendum from each rule triggered by S. The set X of all consequences is simply the set constituted by all consequences that can be build for all coalitions.

For each player p there is a default implicitly specified rule which is triggered by the player p alone and assigns her the (initial) value 0. Then, the only consequence of coalitions that do not meet any (non-default) rule is the payoff vector assigning 0 to all their players.

Example 4.1. Let us consider a game involving players a, b and c. Consider the following NTU marginal contribution network with rules:

$$\begin{aligned} a \wedge b &\rightarrow [a{+}{=}1], [a{+}{=}2, b{+}{=}1] \\ b \wedge \neg c &\rightarrow [b{+}{=}4]. \end{aligned}$$

Then, the set of consequences $V(\{a\})$ is the singleton $\{(0,0,0)\}$, since only the implicit default rule applies. On the other hand, for instance, $V(\{a,b\}) = \{(1,4,0),(2,5,0)\}$, $V\{b,c\} = \{(0,0,0)\}$, and $V(\{a,b,c\}) = \{(1,0,0),(2,1,0)\}$. The set X of all consequences is $X = \bigcup_{S \subseteq N} V(S)$.

NTU marginal contribution networks is a non-deterministic polynomial-time compact representation. Indeed, given a coalition S a non-deterministic Turing transducer may compute a consequence belonging to $V(S)$ by guessing the consequence x and for each rule triggered by S the consequence addendum needed to build x. Obviously the number of consequence addenda to guess is polynomial and summing up their values is feasible in deterministic polynomial time. Hence, NTU MC-nets is an **FNP** representation scheme for NTU games in the Osborne and Rubinstein's form.

4.2 Hardness results

4.2.1 The Core

Definition 4.2. *Let $\phi(\boldsymbol{\alpha})$ be a Boolean formula in DNF. $\boldsymbol{C_c}(\phi(\boldsymbol{\alpha}))$ is the coalitional game with non-transferable utility obtained from $\phi(\boldsymbol{\alpha})$ in the following way:*

- *for each variable α_i there are two players α_i^T and α_i^F, and we call them* variable players*;*
- *players "unsat", and "sat";*
- *the set of all possible consequences, the consequence function $V(\cdot)$, and the preference relations are defined by the following NTU marginal contribution network (generic rules, that are those rules including indexed players as α_i^T, are meant to be replicated for each specific player):*

$$unsat \wedge \neg sat \rightarrow [unsat{+}{=}1] \tag{4.1}$$

$$\left.\begin{aligned} &unsat \wedge \alpha_i^T \wedge \neg sat \rightarrow [\alpha_i^T{+}{=}1] \\ &unsat \wedge \alpha_i^F \wedge \neg sat \rightarrow [\alpha_i^F{+}{=}1] \end{aligned}\right\} \tag{4.2}$$

$$unsat \wedge \neg\alpha_i^T \wedge \neg\alpha_i^F \wedge \neg sat \rightarrow [unsat{+}{=}-B] \tag{4.3}$$

$$\left.\begin{aligned} &\textit{(for each disjunct } d_i \textit{ of } \phi(\boldsymbol{\alpha})\textit{)}, \\ &unsat \wedge p(\ell_{i,1}) \wedge \cdots \wedge p(\ell_{i,m_i}) \wedge \neg sat \rightarrow [unsat{+}{=}-B] \end{aligned}\right\} \tag{4.4}$$

$$sat \rightarrow [sat{+}{=}1] \tag{4.5}$$

where m_i is the number of literals in disjunct d_i, function $p(\cdot)$ maps a literal to its corresponding player (e.g. $p(\alpha_i) = \alpha_i^T$, $p(\neg\alpha_i) = \alpha_i^F$), B is a big value, larger than every possible positive value that can be obtained by summing up values other than B through rules in $\boldsymbol{C_c}(\phi(\boldsymbol{\alpha}))$, and the preference relations are based on individual payoffs (the more the better).

Example 4.3. Let $\phi(\boldsymbol{\alpha}) = (\alpha_1 \wedge \alpha_2) \vee (\neg\alpha_1 \wedge \alpha_2 \wedge \neg\alpha_3)$ be a Boolean formula in DNF. Then, the coalitional game $\boldsymbol{C_c}(\phi(\boldsymbol{\alpha}))$ with non-transferable utility has eight players, that are $N = \{unsat, \alpha_1^T, \alpha_1^F, \alpha_2^T, \alpha_2^F, \alpha_3^T, \alpha_3^F, sat\}$. The next NTU marginal contribution network generates the set of all possible consequences, the consequence function $V(\cdot)$ and the preference relations of $\boldsymbol{C_c}(\phi(\boldsymbol{\alpha}))$:

$$unsat \wedge \neg sat \rightarrow [unsat{+}{=}1]$$
$$unsat \wedge \alpha_1^T \wedge \neg sat \rightarrow [\alpha_1^T{+}{=}1]$$
$$unsat \wedge \alpha_1^F \wedge \neg sat \rightarrow [\alpha_1^F{+}{=}1]$$
$$unsat \wedge \alpha_2^T \wedge \neg sat \rightarrow [\alpha_2^T{+}{=}1]$$
$$unsat \wedge \alpha_2^F \wedge \neg sat \rightarrow [\alpha_2^F{+}{=}1]$$
$$unsat \wedge \alpha_3^T \wedge \neg sat \rightarrow [\alpha_3^T{+}{=}1]$$
$$unsat \wedge \alpha_3^F \wedge \neg sat \rightarrow [\alpha_3^F{+}{=}1]$$

$$
\begin{aligned}
&unsat \wedge \neg\alpha_1^T \wedge \neg\alpha_1^F \wedge \neg sat \rightarrow [unsat{+}{=}-2]\\
&unsat \wedge \neg\alpha_2^T \wedge \neg\alpha_2^F \wedge \neg sat \rightarrow [unsat{+}{=}-2]\\
&unsat \wedge \neg\alpha_3^T \wedge \neg\alpha_3^F \wedge \neg sat \rightarrow [unsat{+}{=}-2]\\
&unsat \wedge \alpha_1^T \wedge \alpha_2^T \wedge \neg sat \rightarrow [unsat{+}{=}-2]\\
&unsat \wedge \alpha_1^F \wedge \alpha_2^T \wedge \alpha_3^F \wedge \neg sat \rightarrow [unsat{+}{=}-2]\\
&sat \rightarrow [sat{+}{=}1].
\end{aligned}
$$

Note that $B = 2$ since, even applying all rules with positive values, no player receives more than 1.

Theorem 4.4. *Let $\mathcal{G}$ be an NTU game represented trough an NTU marginal contribution network, and x be an i.r.p.c. Deciding whether x belongs to $\mathscr{C}(\mathcal{G})$ is* co-**NP**-hard*.*

Proof. In order to prove this result we will show a polynomial-time reduction from **TAUT**(ology). Let $\phi(\boldsymbol{\alpha})$ be an instance of **TAUT** with $\phi(\boldsymbol{\alpha})$ in DNF, and consider the NTU game $\boldsymbol{C_c}(\phi(\boldsymbol{\alpha}))$ obtained from $\phi(\boldsymbol{\alpha})$. Note that $\boldsymbol{C_c}(\phi(\boldsymbol{\alpha}))$ can be computed in polynomial-time. Let x be the i.r.p.c. assigning 1 to *sat* and 0 to all other players. We will show that $\phi(\boldsymbol{\alpha})$ is a tautology if and only if x is in $\mathscr{C}(\boldsymbol{C_c}(\phi(\boldsymbol{\alpha})))$.

Property 4.4.(1): *Every coalition firing a penalty rule (that is a rule assigning $-B$ to unsat) has not objections to x.*
Let S be a coalition firing a penalty rule. Then, all consequences of $V(S)$ assign a negative payoff to *unsat*, and hence she will not prefer them to x in which she receives 0.

Property 4.4.(2): *Every coalition containing sat has not objections to x.*
If *sat* belongs to a coalition S only rule (4.5) applies. Then, *sat* would not striclty prefer consequences of $V(S)$ because she would get a payoff equal to what she receives in x (that is 1).

Property 4.4.(3): *Coalitions excluding unsat has not objections to x.*
Note that, all rules giving a positive increment to players (and hence letting them preferring the new consequences) require *unsat* in the coalition, except for rule (4.5) (requiring the presence of *sat*) but we know that coalitions containing *sat* have not objections to x (Property 4.4.(2)).

By the previous properties it is enough to analyze only objections of coalitions S containing *unsat*, excluding *sat*, and including at least one variable player per variable (in order to avoid penalty rules (4.3)). Moreover note that, if coalition S has an objection to x and includes both variable players, α_i^T and α_i^F, for a variable α_i, then also coalition S', excluding from S only one player between α_i^T and α_i^F, has an objection to x. In fact, let $y \in V(S)$ be a consequence such that $y \succ_k x$ for all players $k \in S$ and consider the consequence y' equals to y except for the payoff of player p excluded from S in S' that receives 0 in y' instead of 1 as in y. By construction $y' \in V(S')$ and $y' \succ_k x$

for all players $k \in S'$. So we consider only objections of coalitions containing exactly one variable player per variable of $\phi(\boldsymbol{\alpha})$. By this, such a coalition S can be seen as the truth-value assignment σ_S for all Boolean variables of $\phi(\boldsymbol{\alpha})$. In particular, if $\alpha_i^T \in S$ let $\sigma_S[\alpha_i] = true$, and if $\alpha_i^F \in S$ let $\sigma_S[\alpha_i] = false$.

($\Rightarrow$) Assume that $\phi(\boldsymbol{\alpha})$ is a tautology, and, hence, for all truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ variables, the formula $\phi(\boldsymbol{\alpha})$ is satisfied. We are going to prove that x belongs to $\mathscr{C}(\boldsymbol{C_c}(\boldsymbol{\Phi}))$. We know that a coalition S in order to have a chance to have an objection to x must contain *unsat*, exactly one variable player per variable, and must exclude *sat*.
Since $\phi(\boldsymbol{\alpha})$ is a tautology, there is no way to select variable players to include in S in order to avoid that the corresponding truth-value assignment σ_S satisfies all disjuncts of $\phi(\boldsymbol{\alpha})$, and hence coalition S fires penalty rules (4.4) implying that it has not an objection to x. By this, the i.r.p.c. x belongs to $\mathscr{C}(\boldsymbol{C_c}(\boldsymbol{\Phi}))$.

($\Leftarrow$) Assume that $\phi(\boldsymbol{\alpha})$ is not a tautology, we will show that the i.r.p.c. x does not belong to $\mathscr{C}(\boldsymbol{C_c}(\boldsymbol{\Phi}))$. Consider a truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ variables such that $\phi(\boldsymbol{\alpha})$ is not satisfied by $\sigma_{\boldsymbol{\alpha}}$ (note that such a truth-value assignment exists since $\phi(\boldsymbol{\alpha})$ is not a tautology).
Consider the coalition S containing *unsat*, and all those variable player encoding $\sigma_{\boldsymbol{\alpha}}$, that is $\sigma_S = \sigma_{\boldsymbol{\alpha}}$. Such a coalition S does not fire penalty rules (4.4).
Note that there is only one consequence $y \in V(S)$. By rules (4.2) variable players in S receive 1 in y instead of 0, and by rule (4.1) *unsat* in S receives 1 in y instead of 0. Therefore, such a coalition S has an objection to x, and hence x is not in $\mathscr{C}(\boldsymbol{C_c}(\boldsymbol{\Phi}))$. $\square$

Definition 4.5. *Let $\boldsymbol{\Phi} = (\exists\boldsymbol{\alpha})(\forall\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ be an instance of* $\mathbf{QBF}_{2,\exists}$ *where $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is in DNF. $\boldsymbol{C_{ne}}(\boldsymbol{\Phi})$ is the coalitional game with non-transferable utility obtained from $\boldsymbol{\Phi}$ in the following way:*

- *for each existentially quantified variable α_i there are two players α_i^T and α_i^F, and we call them* existential players*;*
- *for each universally quantified variable β_i there are two players β_i^T and β_i^F, and we call them* universal players*;*
- *players "unsat", and "sat";*
- *the set of all possible consequences, the consequence function $V(\cdot)$, and the preference relations are defined by the following NTU marginal contribution network (generic rules, that are those rules including indexed players as α_i^T or β_i^T, are meant to be replicated for each specific player):*

$$unsat \wedge \neg sat \rightarrow [unsat{+}{=}1] \tag{4.6}$$

$$\left.\begin{array}{l} unsat \wedge \alpha_i^T \wedge \neg\alpha_i^F \wedge \neg sat \rightarrow [\alpha_i^T{+}{=}1] \\ unsat \wedge \neg\alpha_i^T \wedge \alpha_i^F \wedge \neg sat \rightarrow [\alpha_i^F{+}{=}1] \end{array}\right\} \tag{4.7}$$

$$unsat \wedge \neg\alpha_i^T \wedge \neg\alpha_i^F \wedge \neg sat \rightarrow [unsat{+}{=}-B] \tag{4.8}$$

$$\left.\begin{array}{l} unsat \wedge \beta_i^T \wedge \neg sat \rightarrow [\beta_i^T \mathrel{+}= 1] \\ unsat \wedge \beta_i^F \wedge \neg sat \rightarrow [\beta_i^F \mathrel{+}= 1] \end{array}\right\} \tag{4.9}$$

$$unsat \wedge \neg\beta_i^T \wedge \neg\beta_i^F \wedge \neg sat \rightarrow [unsat \mathrel{+}= -B] \tag{4.10}$$

$$\left.\begin{array}{l} \textit{(for each disjunct } d_i \textit{ of } \phi(\boldsymbol{\alpha}, \boldsymbol{\beta})\textit{)}, \\ unsat \wedge p(\ell_{i,1}) \wedge \cdots \wedge p(\ell_{i,m_i}) \wedge \neg sat \rightarrow [unsat \mathrel{+}= -B] \end{array}\right\} \tag{4.11}$$

$$sat \rightarrow [sat \mathrel{+}= 1] \tag{4.12}$$

$$\alpha_i^T \wedge \alpha_i^F \rightarrow [\alpha_i^T \mathrel{+}= 1], [\alpha_i^F \mathrel{+}= 1] \tag{4.13}$$

where m_i is the number of literals in disjunct d_i, function $p(\cdot)$ maps a literal to its corresponding player (e.g. $p(\alpha_i) = \alpha_i^T$, $p(\neg\beta_i) = \beta_i^F$), B is a big value, larger than every possible positive value that can be obtained by summing up values other than B through rules in $\boldsymbol{C_{ne}(\Phi)}$, and the preference relations are based on individual payoffs (the more the better).

Example 4.6. Let $\boldsymbol{\Phi} = (\exists\alpha_1, \alpha_2)(\forall\beta_1, \beta_2)(\alpha_1 \wedge \beta_1) \vee (\neg\alpha_1 \wedge \alpha_2 \wedge \neg\beta_2)$ be a quantified Boolean formula instance of $\mathbf{QBF}_{2,\exists}$ in DNF. Then, the coalitional game $\boldsymbol{C_{ne}(\Phi)}$ with non-transferable utility has ten players, that are $N = \{unsat, \alpha_1^T, \alpha_1^F, \alpha_2^T, \alpha_2^F, \beta_1^T, \beta_1^F, \beta_2^T, \beta_2^F, sat\}$. The next NTU marginal contribution network generates the set of all possible consequences, the consequence function $V(\cdot)$ and the preference relations of $\boldsymbol{C_{ne}(\Phi)}$:

$$\begin{array}{l}
unsat \wedge \neg sat \rightarrow [unsat \mathrel{+}= 1] \\
unsat \wedge \alpha_1^T \wedge \neg\alpha_1^F \wedge \neg sat \rightarrow [\alpha_1^T \mathrel{+}= 1] \\
unsat \wedge \neg\alpha_1^T \wedge \alpha_1^F \wedge \neg sat \rightarrow [\alpha_1^F \mathrel{+}= 1] \\
unsat \wedge \alpha_2^T \wedge \neg\alpha_2^F \wedge \neg sat \rightarrow [\alpha_2^T \mathrel{+}= 1] \\
unsat \wedge \neg\alpha_2^T \wedge \alpha_2^F \wedge \neg sat \rightarrow [\alpha_2^F \mathrel{+}= 1] \\
unsat \wedge \neg\alpha_1^T \wedge \neg\alpha_1^F \wedge \neg sat \rightarrow [unsat \mathrel{+}= -3] \\
unsat \wedge \neg\alpha_2^T \wedge \neg\alpha_2^F \wedge \neg sat \rightarrow [unsat \mathrel{+}= -3] \\
unsat \wedge \beta_1^T \wedge \neg sat \rightarrow [\beta_1^T \mathrel{+}= 1] \\
unsat \wedge \beta_1^F \wedge \neg sat \rightarrow [\beta_1^F \mathrel{+}= 1] \\
unsat \wedge \beta_2^T \wedge \neg sat \rightarrow [\beta_2^T \mathrel{+}= 1] \\
unsat \wedge \beta_2^F \wedge \neg sat \rightarrow [\beta_2^F \mathrel{+}= 1] \\
unsat \wedge \neg\beta_1^T \wedge \neg\beta_1^F \wedge \neg sat \rightarrow [unsat \mathrel{+}= -3] \\
unsat \wedge \neg\beta_2^T \wedge \neg\beta_2^F \wedge \neg sat \rightarrow [unsat \mathrel{+}= -3] \\
unsat \wedge \alpha_1^T \wedge \beta_1^T \wedge \neg sat \rightarrow [unsat \mathrel{+}= -3] \\
unsat \wedge \alpha_1^F \wedge \alpha_2^T \wedge \beta_2^F \wedge \neg sat \rightarrow [unsat \mathrel{+}= -3]
\end{array}$$

$$sat \rightarrow [sat{+}{=}1]$$
$$\alpha_1^T \wedge \alpha_1^F \rightarrow [\alpha_1^T{+}{=}1], [\alpha_1^F{+}{=}1]$$
$$\alpha_2^T \wedge \alpha_2^F \rightarrow [\alpha_2^T{+}{=}1], [\alpha_2^F{+}{=}1].$$

Note that $B = 3$ since, even applying all rules with positive values, no player receives more than 2.

Theorem 4.7. *Let $\mathcal{G}$ be an NTU game represented trough an NTU marginal contribution network. Deciding whether $\mathscr{C}(\mathcal{G})$ is non-empty is* $\boldsymbol{\Sigma_2^P}$-hard.

Proof. In order to prove this result we will show a polynomial-time reduction from $\mathbf{QBF}_{2,\exists}$. Let $\boldsymbol{\Phi} = \exists \boldsymbol{\alpha} \forall \boldsymbol{\beta} \phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an instance of $\mathbf{QBF}_{2,\exists}$ with $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in DNF, and consider the NTU game $\boldsymbol{C_{ne}}(\boldsymbol{\Phi})$ obtained from $\boldsymbol{\Phi}$. Note that $\boldsymbol{C_{ne}}(\boldsymbol{\Phi})$ can be computed in polynomial-time. We will show that $\boldsymbol{\Phi}$ is valid if and only if $\boldsymbol{C_{ne}}(\boldsymbol{\Phi})$ has a non-empty core.

At first note that every i.r.p.c. of $\boldsymbol{C_{ne}}(\boldsymbol{\Phi})$ assigns 1 to *sat*, and for every pair of existential players, α_i^T and α_i^F, it assigns 1 to only one of them (note that, for the grand-coalition, only rules (4.12), (4.13) apply). Other players receive 0.

Property 4.7.(1): *Every coalition S firing a penalty rule (that is a rule assigning $-B$ to unsat) has not objections to any i.r.p.c.*
Let S be a coalition firing a penalty rule. Then, all consequences of $V(S)$ assign a negative payoff to *unsat*, and hence she will not prefer them to any i.r.p.c. in which she receives 0.

Property 4.7.(2): *Every coalition containing sat has not objections to any i.r.p.c.*
If *sat* belongs to a coalition S only rules of type (4.12) and (4.13) apply. Then, *sat* does not strictly prefer consequences of $V(S)$ because she get a payoff equal to what she receives in all i.r.p.c.'s (that is 1).

Property 4.7.(3): *Every coalition containing a pair of existential players for an existentially quantified variable has not objections to any i.r.p.c.*
Let S be a coalition containing both existential players, α_i^T and α_i^F, for an existentially quantified variable. Players α_i^T and α_i^F do not receive in any consequence of $V(S)$ a payoff strictly preferable for both of them to any i.r.p.c. In fact, the pair of existential players meets a rule (4.13), and that rule build a consequence y assigning a payoff 1 to only one of them. Consider a particular i.r.p.c. x, two are the cases: (i) the payoffs they receive in y are the same as in x, or (ii) the payoffs are inverted. In case (i) S has not an objection to x because both existential players do not strictly prefer the new consequence; in case (ii) an existential player of the pair strictly prefer the new consequence but the other one does not. Note that this holds for every i.r.p.c.

Property 4.7.(4): *Every coalition excluding unsat has not objections to any i.r.p.c.*

All rules giving a positive increment to players (and hence letting them preferring the new consequences) require *unsat* in the coalition, except for rules (4.12) and (4.13) but we know that nor coalitions containing *sat* neither coalitions containing a pair of existential players have objections to any i.r.p.c. (Property 4.7.(2) and Property 4.7.(3)).

By the previous properties it is enough to analyze only coalitions containing *unsat*, excluding *sat*, and including at least one player per existentially and universally quantified variable (in order to avoid penalty rules (4.8) and (4.10)). Recall that if such a coalition contains both existential players, α_i^T and α_i^F, for a variable α_i then it has not an objection to any i.r.p.c. (Property 4.7.(3)). Moreover note that if coalition S has an objection to an i.r.p.c. x and contains both universal players β_i^T and β_i^F for a variable β_i, then also coalition S' excluding only one player between β_i^T and β_i^F has an objection to x. In fact, let $y \in V(S)$ be a consequence such that $y \succ_k x$ for all players $k \in S$ and consider the consequence y' equals to y except for the payoff of player p excluded from S in S' that receives 0 in y' instead of 1 as in y. By construction $y' \in V(S')$ and $y' \succ_k x$ for all players $k \in S'$. So we consider only objections of coalitions containing exactly one existential and universal player per existentially and universally quantified variable of $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$. By this, such a coalition S can be seen as the truth-value assignment σ_S for all Boolean variables of $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$. In particular, if $\alpha_i^T \in S$ let $\sigma_S[\alpha_i] = true$, if $\alpha_i^F \in S$ let $\sigma_S[\alpha_i] = false$, if $\beta_i^T \in S$ let $\sigma_S[\beta_i] = true$, and if $\beta_i^F \in S$ let $\sigma_S[\beta_i] = false$. Given a truth-value assignment σ_S derived from coalition S we denote by $\sigma_S^{\boldsymbol{\alpha}}$ and $\sigma_S^{\boldsymbol{\beta}}$ the projections of the entire truth-value assignment on the existentially or universally quantified variable respectively only.

($\Rightarrow$) Assume that $\boldsymbol{\Phi}$ is valid, and, hence, there exists a truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for variables $\boldsymbol{\alpha}$ such that $(\exists \boldsymbol{\beta})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid. Consider the i.r.p.c. x assigning 1 to *sat*, to all opposite existential players of $\sigma_{\boldsymbol{\alpha}}$, that is if $\sigma_{\boldsymbol{\alpha}}[\alpha_i] = true$ then $x_{\alpha_i^F} = 1$ and if $\sigma_{\boldsymbol{\alpha}}[\alpha_i] = false$ then $x_{\alpha_i^T} = 1$, and assigns 0 to all other players. We are going to prove that x belongs to $\mathscr{C}(\boldsymbol{C_{ne}}(\boldsymbol{\Phi}))$ and hence $\boldsymbol{C_{ne}}(\boldsymbol{\Phi})$ has a non-empty core.
Consider a coalition S containing *unsat*, exactly one existential and universal player per existentially and universally quantified variable, excluding *sat* and consider the unique consequence $y \in V(S)$. If in S there is an existential player receiving 1 in x then S has not an objection to x because that existential player would not strictly prefer the new consequence y (she receives 1 in S in y by rule (4.7) but this is not sufficient). So we can consider only that coalition S (and the related consequence) containing all those existential players getting 0 in x, that is S is such that $\sigma_S^{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}}$.
Since $(\exists \boldsymbol{\beta})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid, there is no way to select universal players to include in S avoiding that the corresponding truth-value assignment σ_S satisfies all disjunct of $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Hence coalition S fires penalty

rules (4.11) and has not an objection to x. By this, the i.r.p.c. x belongs to $\mathscr{C}(\boldsymbol{C_{ne}}(\boldsymbol{\Phi}))$, that results to be non-empty.

($\Leftarrow$) Assume that $\boldsymbol{\Phi}$ is not valid, meaning that $\neg\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is valid, we will show that $\mathscr{C}(\boldsymbol{C_{ne}}(\boldsymbol{\Phi}))$ is empty. Consider a truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ variables, and the related i.r.p.c. x, that is the consequence assigning 1 to *sat* and to all opposite existential players of $\sigma_{\boldsymbol{\alpha}}$. Since $\neg\boldsymbol{\Phi}$ is valid, given the truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for the existentially quantified variables there is an assignment $\sigma_{\boldsymbol{\beta}}$ for the universally quantified variables such that $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is not satisfied by $\sigma_{\boldsymbol{\beta}}$. Consider the coalition S containing *unsat*, all those existential player receiving 0 in x (so that $\sigma_S^{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}}$), and all those universal players such that $\sigma_S^{\boldsymbol{\beta}} = \sigma_{\boldsymbol{\beta}}$. We have that coalition S does not fire any penalty rule (4.11).
Let y be the unique consequence of $V(S)$ for such a coalition S. By rules (4.7) and (4.9) existential and universal players in S receive 1 in y instead of 0, and by rule (4.6) *unsat* in S receives 1 in y instead of 0. Therefore, such a coalition S has an objection to x, and hence x is not in $\mathscr{C}(\boldsymbol{C_{ne}}(\boldsymbol{\Phi}))$. Note that this is true for all i.r.p.c.'s, since we are assuming that $(\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is valid, then $\mathscr{C}(\boldsymbol{C_{ne}}(\boldsymbol{\Phi}))$ is empty. □

4.2.2 The Bargaining Set

Definition 4.8. *Let $\Phi = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ be an instance of* $\mathbf{QBF}_{2,\forall}$*, where $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is in CNF. $\boldsymbol{BS_c}(\boldsymbol{\Phi})$ is the coalitional game with non-transferable utility obtained from $\boldsymbol{\Phi}$ as follows:*

- *for each universally quantified variable α_i there are two players α_i^T and α_i^F, and we call them* universal players*;*
- *for each existentially quantified variable β_i there are two players β_i^T and β_i^F, and we call them* existential players*;*
- *players "chall" and "sat";*
- *the set of all possible consequences, the consequence function $V(\cdot)$, and the preference relations are defined by the way of the following NTU marginal contribution network (generic rules, that are those rules including indexed players as α_i^T or β_i^T, are meant to be replicated for each specific player):*

$$chall \wedge \neg sat \rightarrow [chall{+}{=}1] \tag{4.14}$$

$$\left.\begin{array}{l} chall \wedge \alpha_i^T \wedge \neg sat \rightarrow [\alpha_i^T{+}{=}1] \\ chall \wedge \alpha_i^F \wedge \neg sat \rightarrow [\alpha_i^F{+}{=}1] \end{array}\right\} \tag{4.15}$$

$$chall \wedge \alpha_i^T \wedge \alpha_i^F \wedge \neg sat \rightarrow [chall{+}{=}-B, \alpha_i^T{+}{=}-B, \alpha_i^F{+}{=}-B] \tag{4.16}$$

$$\left.\begin{array}{l} chall \wedge \beta_i^T \wedge \neg sat \rightarrow [chall{+}{=}-B, \beta_i^T{+}{=}-B] \\ chall \wedge \beta_i^F \wedge \neg sat \rightarrow [chall{+}{=}-B, \beta_i^F{+}{=}-B] \end{array}\right\} \tag{4.17}$$

$$sat \rightarrow [sat{+}{=}1] \tag{4.18}$$

$$sat \wedge \beta_i^T \wedge \beta_i^F \wedge \neg chall \rightarrow [sat{+}{=}-B, \beta_i^T{+}{=}-B, \beta_i^F{+}{=}-B] \quad (4.19)$$

$$\left.\begin{array}{l}\textit{(for each clause } c_i \textit{ of } \phi(\boldsymbol{\alpha}, \boldsymbol{\beta})\textit{)}\\ sat \wedge \neg p(\ell_{i,1}) \wedge \cdots \wedge \neg p(\ell_{i,m_i}) \wedge \neg chall \rightarrow [sat{+}{=}-B]\end{array}\right\} \quad (4.20)$$

where m_i is the number of literals in clause c_i, function $p(\cdot)$ maps a literal to its corresponding player (e.g. $p(\alpha_i) = \alpha_i^T$, $p(\neg\beta_i) = \beta_i^F$), B is a big value, larger than every possible positive value that can be obtained by summing up values other than B through rules in $\boldsymbol{BS_c}(\boldsymbol{\Phi})$, and the preference relations are naturally based on payoffs (the more the better).

Example 4.9. Let $\boldsymbol{\Phi} = (\forall\alpha_1, \alpha_2)(\exists\beta_1, \beta_2)(\alpha_1 \vee \neg\beta_1) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \neg\beta_2)$ be a quantified Boolean formula instance of $\mathbf{QBF}_{2,\forall}$ in CNF. Then, the coalitional game $\boldsymbol{BS_c}(\boldsymbol{\Phi})$ with non-transferable utility has ten players, that are $N = \{chall, \alpha_1^T, \alpha_1^F, \alpha_2^T, \alpha_2^F, \beta_1^T, \beta_1^F, \beta_2^T, \beta_2^F, sat\}$. The next NTU marginal contribution network generates the set of all possible consequences, the consequence function $V(\cdot)$ and the preference relations of $\boldsymbol{BS_c}(\boldsymbol{\Phi})$:

$$\begin{array}{l}
chall \wedge \neg sat \rightarrow [chall{+}{=}1]\\
chall \wedge \alpha_1^T \wedge \neg sat \rightarrow [\alpha_1^T{+}{=}1]\\
chall \wedge \alpha_1^F \wedge \neg sat \rightarrow [\alpha_1^F{+}{=}1]\\
chall \wedge \alpha_2^T \wedge \neg sat \rightarrow [\alpha_2^T{+}{=}1]\\
chall \wedge \alpha_2^F \wedge \neg sat \rightarrow [\alpha_2^F{+}{=}1]\\
chall \wedge \alpha_1^T \wedge \alpha_1^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \alpha_1^T{+}{=}-2, \alpha_1^F{+}{=}-2]\\
chall \wedge \alpha_2^T \wedge \alpha_2^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \alpha_2^T{+}{=}-2, \alpha_2^F{+}{=}-2]\\
chall \wedge \beta_1^T \wedge \neg sat \rightarrow [chall{+}{=}-2, \beta_1^T{+}{=}-2]\\
chall \wedge \beta_1^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \beta_1^F{+}{=}-2]\\
chall \wedge \beta_2^T \wedge \neg sat \rightarrow [chall{+}{=}-2, \beta_2^T{+}{=}-2]\\
chall \wedge \beta_2^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \beta_2^F{+}{=}-2]\\
sat \rightarrow [sat{+}{=}1]\\
sat \wedge \beta_1^T \wedge \beta_1^F \wedge \neg chall \rightarrow [sat{+}{=}-2, \beta_1^T{+}{=}-2, \beta_1^F{+}{=}-2]\\
sat \wedge \beta_2^T \wedge \beta_2^F \wedge \neg chall \rightarrow [sat{+}{=}-2, \beta_2^T{+}{=}-2, \beta_2^F{+}{=}-2]\\
sat \wedge \neg\alpha_1^T \wedge \neg\beta_1^F \wedge \neg chall \rightarrow [sat{+}{=}-2]\\
sat \wedge \neg\alpha_1^F \wedge \neg\alpha_2^T \wedge \neg\beta_2^F \wedge \neg chall \rightarrow [sat{+}{=}-2].
\end{array}$$

Note that $B = 2$ since, even applying all rules with positive values, no player receives more than 1.

Theorem 4.10. *Let $\mathcal{G}$ be an NTU game represented through an NTU marginal contribution network, and x be an i.r.p.c. Deciding whether x is in $\mathscr{B}(\mathcal{G})$ is* $\boldsymbol{\Pi_2^P}$-hard.

Proof. In order to prove this result we will show a polynomial-time reduction from $\mathbf{QBF}_{2,\forall}$. Let $\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ be an instance of $\mathbf{QBF}_{2,\forall}$ where $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is in CNF, and consider the game $\boldsymbol{BS_c}(\boldsymbol{\Phi})$ obtained from $\boldsymbol{\Phi}$. Note that $\boldsymbol{BS_c}(\boldsymbol{\Phi})$ can be computed in polynomial-time. Let x be the i.r.p.c. that assigns 1 to *sat* and 0 to all other players. We will show that $\boldsymbol{\Phi}$ is valid if and only if x belongs to $\mathscr{B}(\boldsymbol{BS_c}(\boldsymbol{\Phi}))$. We begin by illustrating some properties of this game.

Property 4.10.(1): *No player can object or counterobject through a coalition firing a penalty rule giving her $-B$.*
Actually such a rule does not permit to build a consequence letting that player to have enough payoff to object against anyone neither to counterobject to anyone because she receives a non negative payoff in x and with such a penalty rule fired she would receive less than 0.

Property 4.10.(2): *No player has a justified objection against chall, or against a universal player or against an existential player to x.*
Universal and existential players can counterobject to everyone by forming a singleton coalition since the unique consequence for that coalition give them 0 as payoff (that is what they receive in x). Player *chall* can counterobject to anyone because the singleton coalition formed by herself has a unique consequence giving her 1, that is even better than what she receives in x.

Property 4.10.(3): *An existential player cannot object against anyone.*
Since an existential player gets 0 in x and she receives 0 in every consequence of every coalition she can form, she has not the possibility to object against anyone, because in order to object she needs to strictly prefer her new proposal to x.

By the previous properties, to prove this theorem it is enough to analyze only objections of *chall* against *sat*, and those of universal players against *sat*. When *chall* objects against *sat*, she forms a coalition S excluding existential players and including at most one universal player per universally quantified variable (in order to avoid penalties from rules (4.17) and (4.16)), and the only consequence $y \in V(S)$ obtained is such that $y_k = 1$ for all players $k \in S$ (see rules (4.14) and (4.15)). Note that for *sat* to counterobject, she must form a coalition T having among its consequence one that gives her 1 (rule (4.18)) and moreover avoids for her to take a penalty from one of the rules (4.20); therefore, she must form a coalition including as many players as possible. Rules (4.19) forces *sat* to take at most one existential player per existentially quantified variable, but she has no other constraints on coalition formation. However, *sat* cannot have in T any players that also belong to S, because in $V(T)$ there is no consequence giving them 1 as payoff (see rules (4.18), (4.19), and (4.20)). If *chall* includes in S none of the universal players for a universally quantified variable, then *sat* can insert both universal players for that variable. Therefore, the strongest objections *chall* can raise against *sat* are objections through coalitions containing exactly one universal player per universally quantified variable and then the best *sat* can do is to counterobject through a coalition containing exactly one universal player per universally quantified variable.

For these reasons the coalition S formed by *chall* to object against *sat*, and the coalition T formed by *sat* to counterobject to *chall* can be seen as truth-value assignments. For the coalition S the associated truth-value assignment σ_S is only on universally quantified variables of $\boldsymbol{\Phi}$, in particular (note the inversion) if $\alpha_i^T \in S$ let $\sigma_S[\alpha_i] = false$, and if $\alpha_i^F \in S$ let $\sigma_S[\alpha_i] = true$. For the coalition T the associated truth-value assignment σ_T is on all variables of $\boldsymbol{\Phi}$, in particular if $\alpha_i^T \in T$ let $\sigma_T[\alpha_i] = true$, if $\alpha_i^F \in T$ let $\sigma_T[\alpha_i] = false$, if $\beta_i^T \in T$ let $\sigma_T[\beta_i] = true$, and if $\beta_i^F \in T$ let $\sigma_T[\beta_i] = false$. Given a truth-value assignment σ_T derived from coalition T, we denote by $\sigma_T^{\boldsymbol{\alpha}}$ and $\sigma_T^{\boldsymbol{\beta}}$ the projections of the entire truth-value assignment on existentially or universally quantified variables only respectively.

A similar reasoning can be drawn for objections of a universal player against *sat*.

($\Rightarrow$) Assume that $\boldsymbol{\Phi}$ is valid and, hence, for all truth-value assignments $\sigma_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ variables, the quantified Boolean formula $(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid, that is there exists a truth-value assignment $\sigma_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ variables that satisfies $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$.
Consider an objection (y, S) of *chall* against *sat* to x, where S contains exactly one universal player per variable. Note that there is only one consequence $y \in V(S)$. We know that a truth-value assignment σ_S is associated to S. Player *sat* can counterobject to (y, S) because she can form a coalition T containing all those universal players not belonging to S, hence $\sigma_T^{\boldsymbol{\alpha}} = \sigma_S$, and the set of existential players such that $\sigma_T^{\boldsymbol{\beta}} = \sigma_{\boldsymbol{\beta}}$ avoiding penalty rules (4.20) (note that $\sigma_T^{\boldsymbol{\beta}}$ satisfies $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_S]$). Let z be the only consequence of $V(T)$ for the aforementioned coalition T. We have that (z, T) is a counterobjection to (y, S) by *sat*. Since we are assuming $\boldsymbol{\Phi}$ valid, for every coalition S (and the related unique consequence) formed by *chall* to object against *sat* there is a counterobjection by *sat* (because for every coalition S there is a coalition T such that $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_S]$ is satisfied by $\sigma_T^{\boldsymbol{\beta}}$), and thus *chall* has no justified objection against *sat* to x. A similar reasoning shows that no universal player has justified objection against *sat*. Thus, x is in $\mathscr{B}(\boldsymbol{BS_c}(\boldsymbol{\Phi}))$.

($\Leftarrow$) Assume that $\boldsymbol{\Phi}$ is not valid, meaning that $\neg\boldsymbol{\Phi} = (\exists\boldsymbol{\alpha})(\forall\boldsymbol{\beta})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is valid, and let $\sigma_{\boldsymbol{\alpha}}$ a truth-value assignment for $\boldsymbol{\alpha}$ variables witnessing the validity of $\neg\boldsymbol{\Phi}$. Consider the objection (y, S) of player *chall* against player *sat* to x where S is such that $\sigma_S = \sigma_{\boldsymbol{\alpha}}$ (and y is the only consequence belonging to $V(S)$).
In order to have a chance to counterobject, player *sat* must form a coalition T containing universal players not in S, that is $\sigma_T^{\boldsymbol{\alpha}} = \sigma_S$. Since $(\forall\boldsymbol{\beta})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid, there is no way for *sat* to select existential players to include in T to avoid penalty rules (4.20), because for

every coalition T the related truth-value assignment $\sigma_T^{\boldsymbol{\beta}}$ does not satisfy $\phi(\boldsymbol{\alpha},\boldsymbol{\beta})[\boldsymbol{\alpha}/\sigma_S]$. Hence *sat* cannot counterobject to (y,S) through any coalition T, implying that (y,S) is a justified objection of player *chall* against *sat* to x, which therefore does not belong to $\mathscr{B}(\boldsymbol{BS_c}(\boldsymbol{\Phi}))$. □

Definition 4.11. *Let $\boldsymbol{\Phi} = (\exists\boldsymbol{\alpha})(\forall\boldsymbol{\beta})(\exists\boldsymbol{\gamma})\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ be an instance of $\mathbf{QBF}_{3,\exists}$ where $\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ is in CNF. $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$ is the coalitional game with non-transferable utility obtained from $\boldsymbol{\Phi}$ in the following way:*

- *for each existentially quantified variable α_i there are two players α_i^T and α_i^F, and we call them* α-existential players*;*
- *for each universally quantified variable β_i there are two players β_i^T and β_i^F, and we call them* universal players*;*
- *for each existentially quantified variable γ_i there are two players γ_i^T and γ_i^F, and we call them* γ-existential players*;*
- *players "chall", and "sat";*
- *the set of all possible consequences, the consequence function $V(\cdot)$, and the preference relations are defined by the following NTU marginal contribution network (generic rules, that are those rules including indexed players as α_i^T, β_i^T or γ_i^T are meant to be replicated for each specific player):*

$$chall \wedge \neg sat \to [chall{+}{=}1] \tag{4.21}$$

$$\left.\begin{aligned} &chall \wedge \beta_i^T \wedge \neg sat \to [\beta_i^T{+}{=}1] \\ &chall \wedge \beta_i^F \wedge \neg sat \to [\beta_i^F{+}{=}1] \end{aligned}\right\} \tag{4.22}$$

$$chall \wedge \beta_i^T \wedge \beta_i^F \wedge \neg sat \to [chall{+}{=}-B, \beta_i^T{+}{=}-B, \beta_i^F{+}{=}-B] \tag{4.23}$$

$$\left.\begin{aligned} &chall \wedge \gamma_i^T \wedge \neg sat \to [chall{+}{=}-B, \gamma_i^T{+}{=}-B] \\ &chall \wedge \gamma_i^F \wedge \neg sat \to [chall{+}{=}-B, \gamma_i^F{+}{=}-B] \end{aligned}\right\} \tag{4.24}$$

$$sat \to [sat{+}{=}1] \tag{4.25}$$

$$sat \wedge \gamma_i^T \wedge \gamma_i^F \wedge \neg chall \to [sat{+}{=}-B, \gamma_i^T{+}{=}-B, \gamma_i^F{+}{=}-B] \tag{4.26}$$

$$\left.\begin{aligned} &\textit{(for each clause } c_i \textit{ of } \phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\textit{)} \\ &sat \wedge \neg p(\ell_{i,1}) \wedge \cdots \wedge \neg p(\ell_{i,m_i}) \wedge \neg chall \to [sat{+}{=}-B] \end{aligned}\right\} \tag{4.27}$$

$$\alpha_i^T \wedge \alpha_i^F \to [\alpha_i^T{+}{=}1], [\alpha_i^F{+}{=}1] \tag{4.28}$$

where m_i is the number of literals in clause c_i, function $p(\cdot)$ maps a literal to its corresponding player (e.g. $p(\alpha_i) = \alpha_i^T$, $p(\neg\beta_i) = \beta_i^F$), B is a big value, larger than every possible positive value that can be obtained by summing up values other than B through rules in $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$, and the preference relations are based on individual payoffs (the more the better).

Example 4.12. Let $\boldsymbol{\Phi} = (\exists\alpha_1,\alpha_2)(\forall\beta_1,\beta_2)(\exists\gamma_1,\gamma_2)(\alpha_1\vee\neg\beta_2\vee\gamma_1)\wedge(\neg\alpha_2\vee\beta_1\vee\neg\gamma_2)$ be a quantified Boolean formula instance of $\mathbf{QBF}_{3,\exists}$ in CNF. Then, the coalitional game $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$ with non-transferable utility has fourteen players, that are $N = \{chall, \alpha_1^T, \alpha_1^F, \alpha_2^T, \alpha_2^F, \beta_1^T, \beta_1^F, \beta_2^T, \beta_2^F, \gamma_1^T, \gamma_1^F, \gamma_2^T, \gamma_2^F, sat\}$. The next NTU marginal contribution network generates the set of all possible consequences, the consequence function $V(\cdot)$, and the preference relations of $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$:

$$\begin{aligned}
&chall \wedge \neg sat \rightarrow [chall{+}{=}1]\\
&chall \wedge \beta_1^T \wedge \neg sat \rightarrow [\beta_1^T{+}{=}1]\\
&chall \wedge \beta_1^F \wedge \neg sat \rightarrow [\beta_1^F{+}{=}1]\\
&chall \wedge \beta_2^T \wedge \neg sat \rightarrow [\beta_2^T{+}{=}1]\\
&chall \wedge \beta_2^F \wedge \neg sat \rightarrow [\beta_2^F{+}{=}1]\\
&chall \wedge \beta_1^T \wedge \beta_1^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \beta_1^T{+}{=}-2, \beta_1^F{+}{=}-2]\\
&chall \wedge \beta_2^T \wedge \beta_2^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \beta_2^T{+}{=}-2, \beta_2^F{+}{=}-2]\\
&chall \wedge \gamma_1^T \wedge \neg sat \rightarrow [chall{+}{=}-2, \gamma_1^T{+}{=}-2]\\
&chall \wedge \gamma_1^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \gamma_1^F{+}{=}-2]\\
&chall \wedge \gamma_2^T \wedge \neg sat \rightarrow [chall{+}{=}-2, \gamma_2^T{+}{=}-2]\\
&chall \wedge \gamma_2^F \wedge \neg sat \rightarrow [chall{+}{=}-2, \gamma_2^F{+}{=}-2]\\
&sat \rightarrow [sat{+}{=}1]\\
&sat \wedge \gamma_1^T \wedge \gamma_1^F \wedge \neg chall \rightarrow [sat{+}{=}-2, \gamma_1^T{+}{=}-2, \gamma_1^F{+}{=}-2]\\
&sat \wedge \gamma_2^T \wedge \gamma_2^F \wedge \neg chall \rightarrow [sat{+}{=}-2, \gamma_2^T{+}{=}-2, \gamma_2^F{+}{=}-2]\\
&sat \wedge \neg\alpha_1^T \wedge \neg\beta_2^F \wedge \neg\gamma_1^T \wedge \neg chall \rightarrow [sat{+}{=}-2]\\
&sat \wedge \neg\alpha_2^F \wedge \neg\beta_1^T \wedge \neg\gamma_2^F \wedge \neg chall \rightarrow [sat{+}{=}-2]\\
&\alpha_1^T \wedge \alpha_1^F \rightarrow [\alpha_1^T{+}{=}1], [\alpha_1^F{+}{=}1]\\
&\alpha_2^T \wedge \alpha_2^F \rightarrow [\alpha_2^T{+}{=}1], [\alpha_2^F{+}{=}1]
\end{aligned}$$

Note that $B = 3$ since, even applying all rules with positive values, no player receives more than 2.

Theorem 4.13. *Let $\mathcal{G}$ be an NTU game represented as an NTU marginal contribution network. Deciding wether $\mathscr{B}(\mathcal{G})$ is non-empty is* $\boldsymbol{\Sigma_3^P}$-hard.

Proof. In order to prove this result we will show a polynomial-time reduction from $\mathbf{QBF}_{3,\exists}$. Let $\boldsymbol{\Phi} = (\exists\boldsymbol{\alpha})(\forall\boldsymbol{\beta})(\exists\boldsymbol{\gamma})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be an instance of $\mathbf{QBF}_{3,\exists}$ where $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is in CNF and consider the NTU game $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$ obtained from $\boldsymbol{\Phi}$. Note that $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$ can be computed in polynomial-time. We will show that $\boldsymbol{\Phi}$ is valid if and only if $\mathscr{B}(\boldsymbol{BS_{ne}}(\boldsymbol{\Phi}))$ is non-empty. As usual, we begin by discussing some preliminary facts. Every i.r.p.c. of $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$ assigns 1 to *sat* and for every pair of α-existential players, α_i^T and α_i^F, it assigns 1 to only one of them (note that, for the grand-coalition, only rules (4.28) and (4.25) apply), and it assigns 0 to all other players. By this, a truth-value assignment σ_x for $\boldsymbol{\alpha}$ variables can be associated with every i.r.p.c. x. In particular, let x be an i.r.p.c. of $\boldsymbol{BS_{ne}(\Phi)}$, (note the inversion) if $x_{\alpha_i^T} = 1$ let $\sigma_x[\alpha_i] = false$, and if $x_{\alpha_i^F} = 1$ let $\sigma_x[\alpha_i] = true$.

Property 4.13.(1): *No player can object or counterobject through a coalition firing a penalty rule giving her $-B$.*
In fact such a rule does not permit to build a consequence letting that player to have enough payoff to object against anyone neither to counterobject to anyone because she receives a non negative payoff in x and with such a penalty rule fired she would receive less than 0.

Property 4.13.(2): *No player has a justified objection against chall, or a against universal player, or against an α-existential player or against a γ-existential player to x.*
Universal and γ-existential players can counterobject to everyone by forming a singleton coalition since the unique consequence for that coalition give them 0 as payoff (that is what they receive in x). Player *chall* can counterobject to everyone because the singleton coalition formed by herself has a unique consequence giving her 1, that is even better than what she receives in x. Consider now the case of α-existential players. Let p be an α-existential player taking 0 in x. She can counterobject by forming the singleton coalition for which the unique consequence gives her 0. Let p be an α-existential player taking 1 in x, r be a generic player and (y, S) be an objection of r against p to x. Let q be the opposite α-existential player of p, note that q cannot be in S together with r. Actually the consequence y cannot give to q a strictly preferable payoff because q can receive 1 only in a consequence of a coalition including also p (see rule (4.28)), but player r cannot include p in S because she is objecting against p. By this p can counterobject by forming the coalition $\{p, q\}$ and the unique consequence of $\{p, q\}$ guarantees her to receive 1 and 0 to q (as in x).

Property 4.13.(3): *An α-existential player cannot object against anyone.*
Let p be an α-existential player taking 0 in x; in order to object, she must form a coalition S that provides a consequence with strictly better payoffs to her and to the other players in S. To receive more than in x, p must include in S her opposite α-existential player q such that the necessary rule (4.28) applies. But that rule gives 1 to just one player amongst p and q and hence if p takes 1 then q, taking 1 in x, will take only 0 in the objection; hence p cannot object against anyone. Consider an α-existential player taking 1 in x. Since she receives 1 in x she cannot object against anyone because she cannot form a coalition with a consequence giving her a strictly preferable payoff.

Property 4.13.(4): *A γ-existential player cannot object against anyone.*
Since a γ-existential player gets 0 in x and she receives 0 in every consequence of every coalition she can form, she has not the possibility to

object against anyone, because in order to object she needs to strictly prefer her new proposal to x.

By the previous properties, to prove this theorem it is sufficient to consider the objections of *chall* against *sat*, and those of universal players against *sat*. When *chall* objects against *sat* to an i.r.p.c. x, she forms a coalition S excluding γ-existential players, including at most one universal player per universally quantified variable (in order to avoid penalties from rules (4.24) and (4.23)), and the only consequence $y \in V(S)$ obtained is such that $y_k = 1$ for all players $k \in S$ (see rules (4.21) and (4.22)). Note that for *sat* to counterobject, she must form a coalition T having between its consequences one that gives her 1 (rule (4.25)) and, moreover, avoids for her to take a penalty from a rule (4.27); therefore, she must form a coalition including as many player as possible. Rules (4.26) force *sat* to take at most one γ-existential player per $\boldsymbol{\gamma}$ variable, and since the unique consequence belonging to $V(T)$ is such that everyone, but *sat*, gets 0 (see rules (4.25), (4.26), and (4.27)) she cannot have in T any α-existential player taking 1 in x; but she has no other constraints on coalition formation. However, *sat* cannot include in T any players belonging to S because in $V(T)$ there is no consequence giving them 1 as payoff. If *chall* does not include in S any universal player for a universally quantified variable, then *sat* can include both universal players for that variable. Therefore, the strongest objections *chall* can raise against *sat* are objections through coalitions containing exactly one universal player associated to a universally quantified variable and then the best *sat* can do is to counterobject through a coalition containing exactly one universal player per universally quantified variable and all α-existential players taking 0 in x.

For these reasons the coalition S formed by *chall* to object against *sat* to x, and the coalition T formed by *sat* to counterobject to *chall* can be seen as truth-value assignments. For the coalition S the associated truth-value assignment σ_S is only on universally quantified variables of $\boldsymbol{\Phi}$, in particular (note the inversion) if $\beta_i^T \in S$ let $\sigma_S[\beta_i] = false$, and if $\beta_i^F \in S$ let $\sigma_S[\beta_i] = true$. For the coalition T the associated truth-value assignment σ_T is on all variables of $\boldsymbol{\Phi}$, in particular if $\alpha_i^T \in T$ let $\sigma_T[\alpha_i] = true$, if $\alpha_i^F \in T$ let $\sigma_T[\alpha_i] = false$, if $\beta_i^T \in T$ let $\sigma_T[\beta_i] = true$, if $\beta_i^F \in T$ let $\sigma_T[\beta_i] = false$, if $\gamma_i^T \in T$ let $\sigma_T[\gamma_i] = true$, and if $\gamma_i^F \in T$ let $\sigma_T[\gamma_i] = false$. Given a truth-value assignment σ_T derived from coalition T, we denote by $\sigma_T^{\boldsymbol{\alpha}}$, $\sigma_T^{\boldsymbol{\beta}}$ and σ_T^{γ} the projections of the entire truth-value assignment on $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ or $\boldsymbol{\gamma}$ variables only respectively.

A similar reasoning can be drawn for objections of a universal player against *sat*.

($\Rightarrow$) Assume that $\boldsymbol{\Phi}$ is valid and, hence, there exists a truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ variables such that $(\forall\boldsymbol{\beta})(\exists\boldsymbol{\gamma})\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid. Consider the i.r.p.c. x such that $\sigma_x = \sigma_{\boldsymbol{\alpha}}$. We are going to prove that x belongs to $\mathscr{B}(\boldsymbol{BS_{ne}}(\boldsymbol{\Phi}))$, and hence $\boldsymbol{BS_{ne}}(\boldsymbol{\Phi})$ has a non-empty bargaining set.

Consider one of the strongest objection (y, S) of *chall* against *sat* to x (note that there is only one consequence $y \in V(S)$). We know that with coalition S is associated a truth-value assignment σ_S. We can prove that *sat* can counterobject to *chall*. Indeed, since $(\forall\boldsymbol{\beta})(\exists\boldsymbol{\gamma})\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid then there exists a truth-value assignment $\sigma_{\boldsymbol{\gamma}}$ for $\boldsymbol{\gamma}$ variables that satisfies $\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_x, \boldsymbol{\beta}/\sigma_S]$. Then, *sat* can form a coalition T with those α-existential players taking 0 in x, that is $\sigma_T^{\boldsymbol{\alpha}} = \sigma_x$, those universal players not in S, that is $\sigma_T^{\boldsymbol{\beta}} = \sigma_S$, and those γ-existential players such that $\sigma_T^{\boldsymbol{\gamma}} = \sigma_{\boldsymbol{\beta}}$, and this coalition avoids all penalties from rules (4.27). Let z be the unique consequence such that $z \in V(T)$ for the aforementioned coalition T. We have that (z, T) is a counterobjection to (y, S) by *sat*. This holds for every coalition S (and the related unique consequence) *chall* can form to object since $(\forall\boldsymbol{\beta})(\exists\boldsymbol{\gamma})\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}]$ is valid. A similar reasoning applies to a possible objection of a universal player against *sat*. It follows that x belongs to $\mathscr{B}(\boldsymbol{BS_{ne}}(\boldsymbol{\Phi}))$, which is thus non-empty.

($\Leftarrow$) Assume that $\boldsymbol{\Phi}$ is not valid, meaning that $\neg\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})(\forall\boldsymbol{\gamma})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ is valid, we will show that $\mathscr{B}(\boldsymbol{BS_{ne}}(\boldsymbol{\Phi}))$ is empty. Consider a truth-value assignment $\sigma_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha}$ variables, and the related i.r.p.c. x such that $\sigma_x = \sigma_{\boldsymbol{\alpha}}$. Since $\neg\boldsymbol{\Phi}$ is valid, then there exists a truth-value assignment $\sigma_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ variables such that $(\forall\boldsymbol{\gamma})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}, \boldsymbol{\beta}/\sigma_{\boldsymbol{\beta}}]$ is valid.

We note that *chall* has a justified objection against *sat* to x. Indeed, *chall* can object with (y, S) where S is a coalition containing those universal players such that $\sigma_S = \sigma_{\boldsymbol{\beta}}$ and y is the unique consequence of $V(S)$. Player *sat* cannot counterobject to that objection, because in order to have a chance to do so she should form a coalition with those α-existential players taking 0 in x, that is $\sigma_T^{\boldsymbol{\alpha}} = \sigma_x$, and those universal players not in S that is $\sigma_T^{\boldsymbol{\beta}} = \sigma_S$. But we know that $(\forall\boldsymbol{\gamma})\neg\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}, \boldsymbol{\beta}/\sigma_{\boldsymbol{\beta}}]$ is valid and, therefore, for every set of γ-existential players *sat* can choose the related truth-value assignment $\sigma_T^{\boldsymbol{\gamma}}$ does not satisfy $\phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})[\boldsymbol{\alpha}/\sigma_{\boldsymbol{\alpha}}, \boldsymbol{\beta}/\sigma_{\boldsymbol{\beta}}]$. Hence, at least one penalty rule (4.27) applies, and it follows that *sat* cannot counterobject. Since $\neg\boldsymbol{\Phi}$ is valid, this holds for all truth assignment $\sigma_{\boldsymbol{\alpha}}$ and hence for all i.r.p.c.'s. Thus no i.r.p.c. is in $\mathscr{B}(\boldsymbol{BS_{ne}}(\boldsymbol{\Phi}))$, that turns thus out to be empty. $\square$

4.3 Membership results

4.3.1 The Core

Theorem 4.14. *Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation and let $\mathcal{G}$ be an NTU game in $\mathcal{C}(\mathcal{R})$. Moreover, let x be an i.r.p.c. Deciding whether x belongs to $\mathscr{C}(\mathcal{G})$ is in* co-**NP**.

Proof. We know that an i.r.p.c. x is not in $\mathscr{C}(\mathcal{G})$ if and only if there is a coalition S having an objection to it. We show that this can be checked in

NP. Indeed, a non-deterministic Turing machine M may guess a coalition S and a consequence $y \in V(S)$ and check in polynomial-time that all players of S strictly prefer y to x. Note that, the machine M besides S and y must guess a certificate C_y in order to check in polynomial-time that $\langle \mathcal{G}, S, y \rangle \in W_{\mathcal{C}}(\mathcal{R})$, and this is feasible for M since we are assuming that $\mathcal{G}$ is in $\mathcal{C}(\mathcal{R})$, where $\mathcal{R}$ is an **FNP** representation. □

Corollary 4.15. *Let $\mathcal{G}$ be an NTU game defined by an NTU marginal contribution network, and x be an i.r.p.c. for $\mathcal{G}$. Deciding whether x belongs to the core of $\mathcal{G}$ is* co-**NP**-complete*.*

Theorem 4.16. *Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation and let $\mathcal{G}$ be an NTU game in $\mathcal{C}(\mathcal{R})$. Deciding whether $\mathscr{C}(\mathcal{G})$ is non-empty is in $\mathbf{\Sigma_2^P}$.*

Proof. Since $\mathcal{G}$ is in $\mathcal{C}(\mathcal{R})$ and $\mathcal{R}$ is an **FNP** representation, a non-deterministic Turing machine M may guess an i.r.p.c. x and a certificate C_x in order to check in polynomial-time that $\langle \mathcal{G}, N, x \rangle \in W_{\mathcal{C}}(\mathcal{R})$. Moreover for every player $p \in N$ it can ask to an **NP** oracle whether there exists a consequence $z \in V\{p\}$ such that $z \succ_p k$ and whether there exists a consequence $y \in V(N)$ such that $y \succ_k x$ for all player $k \in N$, in order to check in polynomial-time whether x is indeed an i.r.p.c. Then, after Theorem 4.14, M can check that x is in $\mathscr{C}(\mathcal{G})$ by asking it to an **NP** oracle. □

Corollary 4.17. *Let $\mathcal{G}$ be an NTU game defined by an NTU marginal contribution network. Deciding whether $\mathscr{C}(\mathcal{G})$ is non-empty is $\mathbf{\Sigma_2^P}$-*complete*.*

4.3.2 The Bargaining Set

Lemma 4.18. *Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation, and let $\mathcal{G}$ be an NTU game in $\mathcal{C}(\mathcal{R})$. Moreover, let x be an i.r.p.c., i and j be two players, $S \in \mathcal{I}_{i,j}$ be a coalition, and $y \in V(S)$ be a consequence. Deciding whether (y, S) is a justified objection of player i against player j to x is in* co-**NP**.

Proof. An **NP** machine M may show that (y, S) is not a justified objection by player i against player j to x through coalition S as follows: M guesses a coalition $T \in \mathcal{I}_{j,i}$, a consequence z and a certificate C_z to check in polynoamil-time that $\langle \mathcal{G}, T, z \rangle \in W_{\mathcal{C}(\mathcal{R})}$. Then, it checks in deterministic polynomial-time that $z \succsim_k y$ for all players $k \in T \cap S$, and that $z \succsim_k x$ for all player $k \in T \setminus S$. □

Theorem 4.19. *Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation, and let $\mathcal{G}$ be an NTU game in $\mathcal{C}(\mathcal{R})$ and x be an i.r.p.c. Deciding whether x belongs to $\mathscr{B}(\mathcal{G})$ is in $\mathbf{\Pi_2^P}$.*

Proof. We show that deciding whether x does not belong to $\mathscr{B}(\mathcal{G})$ is in $\mathbf{\Sigma_2^P}$. An **NP** machine M with an oracle in **NP** may guess two player i and j, a coalition $S \in \mathcal{I}_{i,j}$, a consequence y, and a certificate C_y to check in polynomial-time that $\langle \mathcal{G}, S, y \rangle \in W_{\mathcal{C}(\mathcal{R})}$. Then, after Lemma 4.18, M can ask to its oracle to check that (y, S) is a justified objection of player i against player j to x. □

Corollary 4.20. *Let $\mathcal{G}$ be an NTU game defined by an NTU marginal contribution network, and x be an i.r.p.c. for $\mathcal{G}$. Deciding whether x belongs to $\mathscr{B}(\mathcal{G})$ is* $\mathbf{\Pi_2^P}$-complete.

Theorem 4.21. *Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation, and let $\mathcal{G}$ be an NTU game in $\mathcal{C}(\mathcal{R})$. Deciding whether $\mathscr{B}(\mathcal{G})$ is non-empty is in $\mathbf{\Sigma_3^P}$.*

Proof. An **NP** machine M with a $\mathbf{\Sigma_2^P}$ oracle in order to answer "yes" guesses an i.r.p.c. x, and a certificate C_x to check in polynomial-time that $\langle \mathcal{G}, N, x \rangle \in W_{\mathcal{C}(\mathcal{R})}$. Moreover for every player $p \in N$ it can ask to its oracle whether there exists a consequence $z \in V\{p\}$ such that $z \succ_p k$ and whether there exists a consequence $y \in V(N)$ such that $y \succ_k x$ for all player $k \in N$, in order to check in polynomial-time whether x is indeed an i.r.p.c. (note that for M is sufficient an oracle in **NP** to obtains answer to these questions). Then, after Theorem 4.19, M can ask to its oracle to check that x belongs to the bargaining set of $\mathcal{G}$. □

Corollary 4.22. *Let $\mathcal{G}$ be an NTU game defined by an NTU marginal contribution network. Deciding whether $\mathscr{B}(\mathcal{G})$ is non-empty is* $\mathbf{\Sigma_3^P}$-complete.

5

Constrained Games

In several applicative scenarios, assuming complete transferability of the worth amongst coalition's member is not adequate to model the analyzed context that, often, looked closely at would be better modeled through an NTU framework. This is common since sometimes we face with market regulation authorities, or more simply with discrete domains that need models that are, in some sense, constrained to the peculiar features of the studied case. The NTU framework that attracted more attention in the literature is that of Aumann and Peleg (1960). But we saw in Section 2.2 that such an TNU framework does not allow to specify arbitrary consequence functions, since the characteristic functions are required to have the properties of convexity, closeness, and comprehensiveness. However this view appears not appropriate to model applicative scenarios where the aforementioned requirements do not naturally hold. So the NTU framework of Osborne and Rubinstein (1994) seems to be a more appropriate general framework.

Specifying such NTU games of the Osborne and Rubinstein's framework can be very difficult when we have to explicitly associate the coalitions with the set of its consequences if it is convex. Nonetheless such situations are very common. To overcome this issue, we propose a new way to represent NTU games that is a combination of the two NTU frameworks. The outcomes in this new type of games are payoff vectors, as in the framework of Aumann and Peleg, but the characteristic functions have not to fulfil any particular property. This is achieved by issuing constraint on standard TU games.

5.1 Constrained games

This approach to define non-transferable specifications, based on enhancing TU games with application-oriented *constraints* to be issued on the game, has been sometimes adopted in the literature.

In fact, the first occurrence of the name "constrained games" is due to Aumann and Dreze (1974), who considered games with *coalition structures*, where players are partitioned in groups $\mathcal{S}_1, \ldots, \mathcal{S}_k$, and where any outcome $(x_i)_{i \in N}$ must allocate the total payoff $v(\mathcal{S}_j)$ exactly amongst the members of each group $\mathcal{S}_j$, that is, satisfying the equalities $\sum_{i \in \mathcal{S}_j} x_i = v(\mathcal{S}_j)$, for $1 \leq j \leq k$. However, Aumann and Dreze noticed in their turn that considering constraints over TU games was not a novel idea, since the *core* and the *nucleolus* (which are two prominent solution concepts for TU games) were defined by Gillies (1959) and Schmeidler (1969), respectively, on games with outcomes restricted to convex subsets of $\mathbb{R}^N$.

Recently, constrained games have been reconsidered under the pragmatic perspective of modeling some relevant application scenarios, such as price formation (Byford, 2007) and autonomic wireless networks (Jiang and Baras, 2007).

However, as a matter of fact, they received considerably less attention over the years, if compared with the axiomatic approaches defining NTU games. In particular, no general framework was proposed in the literature and no systematic study of the (analytical, as well as computational) properties of this kind of approaches was conducted so far.

We precisely embark on such a systematic formalization of constrained games, and we investigate a framework for coalitional games that retains the nice properties of the transferable setting (e.g., by smoothly inheriting the definition of various solution concepts for TU games), though allowing to specify non-transferable conditions on arbitrary sets of players, via a set of constraints expressed as *mixed-integer linear (in)equalities*[1]. In particular:

(1) Being based on imposing linear (in)equalities, the proposed framework will provide a unifying view of earlier approaches; and,
(2) By allowing the use of integer variables, it will improve their expressiveness in that admissible outcomes might be possibly restricted over *non-convex* regions.

An intuitive exemplification of the framework we are going to study in this Chapter is illustrated below.

Example 5.1. Three brothers, Tim, John and Jim, aged 10, 8 and 5, resp., have collected into a piggy money-box all the small Euro coins (values 1, 2, 5, and 10 cents) that Mom every week has given to each of them since the age of four. Now, the time has come to break the money-box and divide its content.

In order to avoid quarrels among the kids, Mom decides that the distribution has to go with their ages, so that Tim will deserve at least 10/8 the money John will get and John, in its turn, will receive at least 8/5 of Jim's money share. (Jim is not very happy with that, but agrees to comply with Mom's rule). The money-box gets broken and the little treasure of seven Euros and ninety Euro cents, as resulting from the available coin set including

[1] A good source for basic notions and results on mixed-integer linear programming is the book by Nemhauser and Wolsey (1988).

one-hundred 1-cent coins, seventy 2-cent coins, fifty 5-cents coins, and thirty 10-cent coins, can then be divided amongst the kids.

Note that this scenario is based on the non-transferable condition that the treasure cannot freely be distributed amongst the brothers. The specific distribution rule, however, does not fit the classical NTU formalization of Aumann and Peleg (1960).

Instead, it is easily seen that the scenario can be modeled by means of a set of linear (in)equalities, with a few variables taking values from the set $\mathbb{Z}$ of integer numbers.

In this example, admissible outcomes can indeed be identified as the solutions to the following system of mixed-integer linear (in)equalities (the three brothers Tim, John and Jim are denoted by using the indexes 1, 2 and 3, respectively):

$$\begin{cases} x_i = 1 \times \alpha_1^i + 2 \times \alpha_2^i + 5 \times \alpha_5^i + 10 \times \alpha_{10}^i, \forall 1 \leq i \leq 3 \\ \alpha_1^1 + \alpha_1^2 + \alpha_1^3 = 100 \\ \alpha_2^1 + \alpha_2^2 + \alpha_2^3 = 70 \\ \alpha_5^1 + \alpha_5^2 + \alpha_5^3 = 50 \\ \alpha_{10}^1 + \alpha_{10}^2 + \alpha_{10}^3 = 30 \\ x_1 \geq 10/8 \times x_2 \\ x_2 \geq 8/5 \times x_3 \\ \alpha_1^i, \alpha_2^i, \alpha_5^i, \alpha_{10}^i \geq 0, \forall 1 \leq i \leq 3 \\ x_i \in \mathbb{R},\ \alpha_1^i, \alpha_2^i, \alpha_5^i, \alpha_{10}^i \in \mathbb{Z}, \forall 1 \leq i \leq 3 \end{cases}$$

Note that in the above system, the auxiliary variables α_j^i denote the number of coins of value j taken by player i, the first five equalities encode restrictions on the domains of the variables as defined by the available coin set, and the subsequent two inequalities encode Mom's rule (which can be seen, for instance, as playing the role of a central market regulation authority).

Despite the intuitiveness of the modeling approach adopted in Example 5.1, there is no reference framework in the literature accounting for it, both because of the specificity of Mom's rule and because money distribution is constrained by the available coin set, so that those allowed outcomes do not form a convex set.

Proposing and investigating a framework that may serve to model such kinds of scenarios is our main contribution. We define a formal framework for constrained games based on mixed-integer linear (in)equalities, discussing its modeling capabilities. We define the core and the bargaining set for this type of games and we study the relationship between them and with their TU counterpart. Then, we assess the impact of adding constraints on the computational complexity underlying the two analyzed solution concepts.

5.2 Constrained games via mixed-integer linear (in)equalities

Assume that a TU game $\mathcal{G}$ is given and consider the problem of modeling and dealing with constraints to be imposed on feasible worth distributions amongst players in $\mathcal{G}$. These constraints might be well implied by the very nature of the domain at hand (e.g., when the worth is not arbitrarily divisible), or because they reflect some hard preferences expressed by the players or by some regulation authority—recall again Example 5.1.

Our approach to encode application-oriented constraints "within" a classical coalitional TU game setting relies on defining a set of *mixed-integer linear (in)equalities*, which have to be satisfied by the payoff vectors of the given game. In the following, this approach is introduced by discussing some examples, and it is subsequently formalized.

5.2.1 Modeling capabilities

We start by recalling here that a mixed-integer linear (in)equality is a linear (in)equality where some of the variables on which it is defined is constrained to take values from the set $\mathbb{Z}$ of integers. For a set $\mathtt{LC}$ of mixed-integer linear (in)equalities, we denote by $\mathit{real}(\mathtt{LC})$ and $\mathit{int}(\mathtt{LC})$ the sets of all the variables in $\mathtt{LC}$ defined over $\mathbb{R}$ and $\mathbb{Z}$, respectively.

Note that by considering mixed-integer linear (in)equalities we both abstract and generalize earlier approaches to define constrained games. Indeed, on the one hand, given that constraints are defined in terms of an arbitrary set of linear (in)equalities, we may shape the set of allowed imputations over arbitrary convex and compact subsets, thereby sharing the modeling perspective of some foundational works on constrained games (Gillies, 1959; Schmeidler, 1969) and abstracting those approaches that consider specific set of (in)equalities (such as Aumann and Dreze, 1974; Byford, 2007; Jiang and Baras, 2007). In particular, this capability might be exploited to:

(1) State hard preferences on the worth distributions.
As an example, consider a game $\mathcal{G} = \langle N, v \rangle$ over the set of players $N = \{1, 2, 3, 4\}$, and where $v(N) = 10$. Assume that players 3 and 4 together pretend to get more than a half of the worth. Then, this requirement can be modelled as

$$\begin{cases} x_3 + x_4 \geq 5 \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \end{cases}$$

On the other hand, by allowing integer variables, completely novel modeling capabilities emerge in our setting w.r.t. earlier approaches. Indeed, integer variables can be used to isolate non-convex regions, which might be needed to model specific application requirements that are NTU in their very nature, as exemplified below.

(2) Restrict worth functions over specific domains.
When domains are required to be integer intervals, this is rather obvious. For instance, assume that x_3 should take values from the domain $\{4, 5, 6, 7, 8, 9, 10\}$. Then, we may simply consider the following constraints:

$$\begin{cases} 4 \geq x_3 \geq 10 \\ x_1, x_2, x_4 \in \mathbb{R} \\ x_3 \in \mathbb{Z} \end{cases}$$

Instead, in the general case, additional variables have to be used. For instance, assume that player 2 wants either to take the whole worth for herself (even when forming coalitions with other players) or, whenever this is not possible, to get nothing. This can be modelled by a few constraints over an additional variable w:

$$\begin{cases} x_2 = v(N) \times w \\ 0 \leq w \leq 1 \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \\ w \in \mathbb{Z} \end{cases}$$

Note that Example 5.1 basically presents a more realistic case, where several additional variables are used to restrict money distributions to the available coin set.

(3) Consider alternative scenarios.
By allowing integer variables, we may easily model alternative preferences of the players. For instance, we may have that players 1 and 2 have to get together no more than 3, *or* that 2 and 3 have to get together no more than 5. This can be modelled as:

$$\begin{cases} x_1 + x_2 \leq 3 + U \times y \\ x_2 + x_3 \leq 5 + U \times (1 - y) \\ 0 \leq y \leq 1 \\ x_1, x_2, x_3 \in \mathbb{R} \\ y \in \mathbb{Z} \end{cases}$$

where U is an upper bound on the worth of any coalition. In fact, with simple manipulations, one may easily specify other kinds of alternatives, e.g., the fact that at least (or at most) k given constraints have to be satisfied.

5.2.2 Putting it all together

Let us now proceed with our formalization. Let $\mathcal{G} = \langle N, v \rangle$ be a TU game, and let $\texttt{LC}$ be a set of mixed-integer linear (in)equalities. Define $\Omega(\texttt{LC})$ as the set of all the solutions to $\texttt{LC}$. And, for a coalition $S \subseteq N$, let $\Omega(\texttt{LC})[S]$ denote the projection of $\Omega(\texttt{LC})$ on the subspace associated with payoff domains for players in S; that is, a vector of reals $\hat{x}$ with index set S belongs to $\Omega(\texttt{LC})[S]$

if and only if there is a vector $\hat{y} \in \mathbb{R}^{real(\texttt{LC}) \cup int(\texttt{LC})}$ in $\Omega(\texttt{LC})$ such that $\hat{x}_i = \hat{y}_i$ holds, for each $i \in S$.

Intuitively, a constrained game on LC is defined by restricting the set of the possible outcomes of an underlying TU game $\mathcal{G}$ to those imputations of $\mathcal{G}$ belonging to the solution space of the mixed-integer linear-inequalities system projected onto the subspace associated with player variables—recall that further auxiliary variables may occur in LC. The main concepts for constrained games are formalized below.

Definition 5.2. *A* constrained game *is a pair* $(\mathcal{G}, \texttt{LC})$*, also denoted by* $\mathcal{G}|_{\texttt{LC}}$*, where* $\mathcal{G} = \langle N, v\rangle$ *is a TU game and* LC *is a set of mixed-integer linear (in)equalities. An* imputation *of* $\mathcal{G}|_{\texttt{LC}}$ *is a vector* $x \in \Omega(\texttt{LC})[N]$ *such that* $x(N) = v(N)$ *and* $x_i \geq v(\{i\})$ *holds, for all* $i \in N$ *(thus,* $x \in X(\mathcal{G}) \cap \Omega(\texttt{LC})[N]$*). The set of all* imputations *of* $\mathcal{G}|_{\texttt{LC}}$ *is denoted by* $X(\mathcal{G}|_{\texttt{LC}})$*.*

We recall here that we are considering the use of integer variables in the definition of the various constraints. It follows that restricting $X(\mathcal{G})$ over the solution set of such constraints may cause losing its convexity. Thus, while for a TU game $\mathcal{G}$ it is immediate that either $|X(\mathcal{G})| \leq 1$ or $X(\mathcal{G})$ contains an infinite number of imputations, by adding constraints we may deal with imputation sets of arbitrary sizes. The following easy properties point out these characteristics of constrained games.

Proposition 5.3. *Let* $\mathcal{G} = \langle N, v\rangle$ *be a TU game and let* $\mathcal{X} \subseteq X(\mathcal{G})$ *be an arbitrary (finite) set of imputations. Then, there is a (finite) set of constraints* LC *such that* $X(\mathcal{G}|_{\texttt{LC}}) = \mathcal{X}$*.*

Proof. Consider the game $\mathcal{G} = \langle N, v\rangle$ and the set $\mathcal{X} = \{\hat{x}^1, \ldots, \hat{x}^k\}$ of imputations of $\mathcal{G}$. The claim immediately follows by considering the following set of constraints:

$$\texttt{LC} = \begin{cases} x_i = \hat{x}_i^1 \times y_1 + \cdots + \hat{x}_i^k \times y_k, \ \forall 1 \leq i \leq n \\ 0 \leq y_j \leq 1, \ \forall 1 \leq j \leq k \\ y_1 + \cdots + y_k = 1 \\ x_1, \ldots, x_n \in \mathbb{R} \\ y_1, \ldots, y_k \in \mathbb{Z} \end{cases}$$

□

In addition, integer variables might well be used to succinctly specify exponentially many imputations via polynomially many (in)equalities.

Proposition 5.4. *There exists a class* $\mathcal{C} = \{\mathcal{G}|^n_{\texttt{LC}}\}_{n>0}$ *of constrained games such that each game* $\mathcal{G}|^n_{\texttt{LC}}$ *is over* $n+1$ *players,* LC *consists of* $2 \times n$ *inequalities, and* $|X(\mathcal{G}|_{\texttt{LC}})| = 2^n$*.*

Proof. Consider the class $\mathcal{C} = \{\mathcal{G}|^n_{\texttt{LC}}\}_{n>0}$ where the constrained game $\mathcal{G}|^n_{\texttt{LC}} = \langle N, v\rangle$ is such that $N = \{1, \ldots, n, n+1\}$, $v(N) = n$, $v(\{i\}) = 0$, for each i, and $\texttt{LC} = \{0 \leq x_i \leq 1, x_i \in \mathbb{Z}, \forall 1 \leq i \leq n\}$. It can be easily checked that $|X(\mathcal{G}|^n_{\texttt{LC}})| = 2^n$. □

We believe that this framework is rather appealing from a knowledge representation perspective. Indeed, one may exploit constrained games to naturally model scenarios where non-transferable conditions emerge, even though not complying with the requirements of NTU games according to Aumann and Peleg (1960). Moreover, the constrained framework allows us to devise very compact specifications of the desired restrictions on how utilities may be transferred among coalition members. Consequently, various circumstances can be envisaged where the usage of constrained games is a natural choice, e.g., whenever the worth to be distributed among the agents comes as a set of indivisible goods, as exemplified below.

*Example 5.5 (***Distributing indivisible goods).** A certain region of the country characterizes itself to host several producers of two kinds of goods, named α and β. For each producer $i \in \{1, \dots, n\}$, let α_i and β_i denote the quantity of α and β pieces produced by i, respectively. By assembling together one piece α and one piece β, a novel kind of indivisible good can be obtained. In fact, commercializing the assembled product is a much more advantageous business than selling α and β separately. Therefore, an agreement is found amongst producers in the area in order to assemble the pieces of α and β that are overall available, provided that the resulting units of the assembled product are (fairly) distributed amongst the involved producers, which would like to independently commercialize them.

This scenario can easily be modelled within our framework as follows. Firstly, we associate with every coalition $S \subseteq \{1, \dots, n\}$, the number of pieces of the assembled product that S can produce. Thus, we just define:

$$v(S) = \min(\sum_{i \in S} \alpha_i, \sum_{i \in S} \beta_i).$$

Then, since the assembled product is indivisible, any possible worth distribution is a vector of non-negative integers, which can immediately be modelled via the following set of constraints $\mathtt{LC} = \{x_i \geq 0, x_i \in \mathbb{Z},\ \forall 1 \leq i \leq n\}$. In particular, $\Omega(\mathtt{LC})$ is not a convex region, so that earlier modeling perspectives on NTU games, such as those reported in the handbook edited by Aumann and Hart (2002), do not apply here.

There are cases, however, where the worth might practically be assumed to be divisible, but specific constraints regulate its actual distribution. Notably, even in these cases, integer variables may play a crucial role as illustrated next.

*Example 5.6 (***Services composition).** Assume that a service T can be acquired for 100 dollars—for the sake of simplicity, we assume that money is divisible, for otherwise worth distributions might simply be restricted over a discrete domain as in Example 5.5 or in Example 5.1. To supply the service T implies executing m tasks, named $t_1, \dots, t_m$.

Assume also that there is a set $\{1, \ldots, n\}$ of agents each one capable of carrying out some of those m tasks, and let s^i_j denote the ability of agent i to perform the task t_j ($s^i_j = 1$ means that agent i is able to perform t_j, whereas $s^i_j = 0$ means that she is not capable). Thus, a coalition $S \subseteq \{1, \ldots, n\}$ is capable of supporting the service T in the case where $\sum_{i \in S} s^i_j \geq 1$, for each $j \in \{1, \ldots, m\}$.

Assume, moreover, that in order to complete T, not only all of its tasks must be completed, but agents contributing to T must be in the conditions to exchange some partial results returned by performing required tasks. Establishing a communication infrastructure guaranteing the needed result transfers to take place has a specific cost for each coalition S, which we denote by $c(S) < 100$. Hence, the amount of money that might eventually be distributed amongst players in S is described by the following worth function:

$$v(S) = \begin{cases} 100 - c(S) & \text{if } \sum_{i \in S} s^i_j \geq 1,\ \forall j \in \{1, \ldots, m\} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the above scenario defines a classical TU game $\mathcal{G} = \langle \{1, \ldots, n\}, v \rangle$. However, things may be significantly different if we assume that each agent i has to sustain an internal cost, say c^i_j, whenever actually performing the task t_j, and that she hence may decide not to perform the task at all. Indeed, in this case, letting $\gamma^i_j \in \{0, 1\}$ be a variable denoting whether i is actually performing t_j, it is natural to state that the total internal cost for agent i (which is given by the expression $\sum_{j=1}^{m} \gamma^i_j \times c^i_j$) should not exceed what the agent gets from the worth distribution. Hence, utilities cannot be freely distributed and, for a proper modeling of this more realistic scenario, the game has to be enriched with the following set of constraints:

$$\texttt{LC} = \begin{cases} \displaystyle\sum_{i \in N} \gamma^i_j \geq 1,\ \forall j \in \{1, \ldots, m\} \\ \displaystyle x_i \geq \sum_{j=1}^{m} \gamma^i_j \times c^i_j, \forall i \in \{1, \ldots, n\} \\ x_1, \ldots, x_n \in \mathbb{R} \\ \gamma^i_j \in \mathbb{Z}, \forall i \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, m\} \end{cases}$$

With respect to this formalization, it is worthwhile noting that if $\Omega(\texttt{LC})$ is empty then the service cannot be provided at all, and indeed $X(\mathcal{G}|_{\texttt{LC}})$ would be empty in its turn. Otherwise, i.e., if $\Omega(\texttt{LC}) \neq \varnothing$, the imputations of $\mathcal{G}|_{\texttt{LC}}$ correspond to those worth distributions associated with some legal staffing for the tasks, rather than to all arbitrary possible worth distributions (as it would be in the plain TU case).

Observe that one might supposedly try to encode the constraints in LC directly in the definition of the worth function, instead of using a separate component thereof, as done in the framework we are proposing here. For instance, one may add to the condition for $v(S) = 100 - c(S)$ the requirement that there exists an element $\hat{x} \in \Omega(\mathtt{LC})$ such that $S \supseteq \{i \mid \exists j \in \{1, \ldots, m\}$ s.t. $\gamma_j^i \neq 0$ in $\hat{x}\}$. This way, we can ensure that the payoff $100 - c(S)$ is assigned to each coalition S which is formed by all the players that can perform some task conforming with cost constraints, and that can jointly complete T. This refined modeling perspective, however, does not prescribe how the payoff $100 - c(S)$ has to be actually distributed amongst the players in S. In fact, while focusing on accurately modeling the worth function, it cannot guarantee that the outcome of the game (according to any chosen solution concept) fulfils the desired constraints on the distribution of payoffs for the single players. In other words, adding constraints to the worth function may be useful in certain cases for the careful modeling purposes, but cannot in general replace the use of external constraints to constrain allowed worth distributions.

As an important remark, we note here that the "structure" of the above example may well be used as a guideline in the formalization of other application scenarios. Indeed, the basic idea has been the one of using mixed-integer linear inequalities to define solutions for combinatorial problems, together with their associated feasible worth distributions (reflecting, e.g., the costs of such solutions). Thus, while we have contextualized the approach to the case of a staffing problem, very similar encodings can be used to define constrained games suited to deal with scheduling and planning problems, just to cite a few.

We close the section by noticing that there are cases in which the worth consists of indivisible goods, *and* constraints of different nature are defined on feasible worth distributions. A simple scenario of this kind is, for instance, the one illustrated in Example 5.1.

5.3 Solution concepts for constrained games

To define the core and the bargaining set of $\mathcal{G}|_{\mathtt{LC}}$, we just need to consider (in their respective definitions) the imputations and S-feasible payoff vectors of $\mathcal{G}|_{\mathtt{LC}}$ in place of the imputations and S-feasible payoff vectors of the TU game $\mathcal{G}$. We need only to define now what is an S-feasible payoff vectors for constrained games. A vector $x \in \mathbb{R}^S$ is an S-feasible payoff vector *w.r.t.* LC if $x(S) \leq v(S)$ and $x \in \Omega(\mathtt{LC})[S]$—recall that $\Omega(\mathtt{LC})[S]$ denotes the projection of $\Omega(\mathtt{LC})$ on the subspace associated with payoff domains for players in S.

For a constrained game $\mathcal{G}|_{\mathtt{LC}}$, we denote by $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$, and $\mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$, its core, and bargaining set, respectively.

Example 5.7. Consider the TU game $\mathcal{G} = \langle N, v\rangle$ where $N = \{1, 2, 3\}$, $v(\{1, 2, 3\}) = 3$, $v(\{1, 2\}) = 1$, and $v(S) = 0$ for each $S \subset N$ with $S \neq \{1, 2\}$. Moreover, consider the following set of constraints:

$$\texttt{LC} = \begin{cases} \alpha + \beta = 1 \\ x_3 \geq 2 \times \alpha \\ x_3 \times \beta \leq 1 \\ \alpha, \beta \geq 0 \\ \alpha, \beta \in \mathbb{Z} \\ x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

It is immediate to check that the set of all possible imputations for the constrained game is $X(\mathcal{G}|_{\texttt{LC}}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3 \wedge x_1, x_2, x_3 \geq 0 \wedge (x_3 \leq 1 \vee x_3 \geq 2)\}$, which corresponds to the grey area depicted in Figure 5.1(a). Note that this area is not convex. Now, we focus on the solution concepts $\mathscr{C}(\mathcal{G}|_{\texttt{LC}})$, and $\mathscr{B}(\mathcal{G}|_{\texttt{LC}})$.

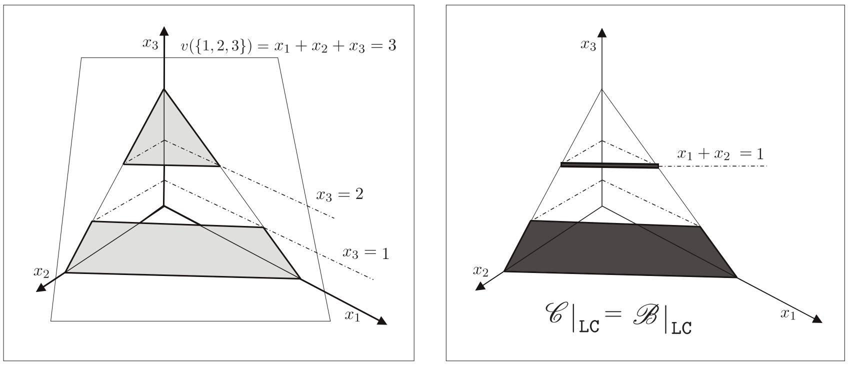
Core. Consider the coalition $S = \{1, 2\}$. Since the set of all S-feasible payoff vectors w.r.t. LC is $\{x \in \mathbb{R}^S | x_1 + x_2 \leq 1\}$, no imputation in $X(\mathcal{G}|_{\texttt{LC}})$ assigning a total worth strictly smaller than 1 to S belongs to $\mathscr{C}(\mathcal{G}|_{\texttt{LC}})$. By this every imputation in the core is such that $x_3 \leq 2$, and hence $x_3 \leq 1 \vee x_3 = 2$, due to the constraints. Thus, $\mathscr{C}(\mathcal{G}|_{\texttt{LC}}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3 \wedge x_1, x_2, x_3 \geq 0 \wedge (x_3 \leq 1 \vee x_3 = 2)\}$, as depicted in dark grey area in Figure 5.1(b).

Bargaining Set. Consider an arbitrary imputation $x \in X(\mathcal{G}|_{\texttt{LC}})$. An objection of player i against player j to x is a pair (y, S), where $i \in S$ and $j \notin S$, such that y is an S-feasible payoff vector w.r.t. LC and $y_k > x_k$ for all $k \in S$. Given that $v(S) = 0$ for each $S \subset N$ but for $S = \{1, 2\}$, any possible objection to x is restricted to those of player 1 (or 2) against 3 (through the coalition $\{1, 2\}$). Let, in particular, (y, S) with $S = \{1, 2\}$ be one such an objection to x of player 1 against 3, and assume that $x_1 + x_2 < 1$ (which implies $x_3 > 2$, because x is an imputation). Then, player 3 cannot counterobject to (y, S) as a singleton since $x_3 > 2$ and $v(\{3\}) = 0$, and she cannot counterobject through coalition $\{2, 3\}$ since $x_2 + x_3 > 2$ and $v(\{2, 3\}) = 0$. Hence, player 1 has a justified objection against player 3 to any imputation x with $x_1 + x_2 < 1$ (or equivalently with $x_3 > 2$). It follows that an imputation x such that $x_3 > 2$ cannot belong to the bargaining set. Consider, instead, an imputation x such that $x_3 \leq 2$ (that is, $x_3 \leq 1 \vee x_3 = 2$, due to the constraints). Then, $x_1 + x_2 \geq 1$ holds and, hence, no objection is possible at all to x. Thus, $\mathscr{B}(\mathcal{G}|_{\texttt{LC}}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3 \wedge x_1, x_2, x_3 \geq 0 \wedge (x_3 \leq 1 \vee x_3 = 2)\}$, which happens to coincide with the core.

In the remainder of this section, we will study the properties of the solution concepts exemplified above.

5.3.1 Properties of solution concepts

We start our investigation of the properties of constrained games by assessing whether an outcome that is stable (under the above solution concepts) in a TU game remains stable when constraints are issued.



(a) Constrained imputations.

(b) Constrained Core and Bargaining Set.

Fig. 5.1. Exemplification of the Solution Concepts.

Proposition 5.8. *For any constrained game $\mathcal{G}|_{\texttt{LC}}$, $\mathscr{C}(\mathcal{G}) \cap X(\mathcal{G}|_{\texttt{LC}}) \subseteq \mathscr{C}(\mathcal{G}|_{\texttt{LC}})$ holds.*

Proof. Let x be an imputation belonging to $\mathscr{C}(\mathcal{G}) \cap X(\mathcal{G}|_{\texttt{LC}})$. Consider now the constrained game $\mathcal{G}|_{\texttt{LC}}$, and assume by contradiction that $x \notin \mathscr{C}(\mathcal{G}|_{\texttt{LC}})$. Then there exist a coalition S and an S-feasible payoff vector w.r.t. LC y such that $y_i > x_i$ for all $i \in S$. The vector y is such that $y(S) \leq v(S)$ and satisfies all the constraints. If $y(S) = v(S)$ then y is S-feasible also in $\mathcal{G}$ (and no constraint have to be satisfied) and hence $x \notin \mathscr{C}(\mathcal{G})$: a contradiction. If $y(S) < v(S)$ we can build an S-feasible vector, y', for $\mathcal{G}$ in this way. Let $v(S) - y(S) = \epsilon > 0$, and $y'_i = y_i + \epsilon/|S|$. Thus $y'_i > x_i$ for all players $i \in S$ and hence $x \notin \mathscr{C}(\mathcal{G})$: a contradiction. □

Proposition 5.9. *There exists a constrained game $\bar{\mathcal{G}}|_{\texttt{LC}}$ (with int(LC) $= \varnothing$) such that $\mathscr{C}(\bar{\mathcal{G}}|_{\texttt{LC}}) \not\subseteq \mathscr{C}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\texttt{LC}})$, with $\mathscr{C}(\bar{\mathcal{G}}|_{\texttt{LC}}) = X(\bar{\mathcal{G}}|_{\texttt{LC}}) \neq \varnothing$.*

Proof. Consider the game $\bar{\mathcal{G}}$ with players $\{1, 2, 3\}$ and where $v(N) = v(\{1, 2\}) = v(\{2, 3\}) = 2$, $v(\{1, 3\}) = 1$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. Also consider the constraints:

$$\texttt{LC} = \begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \\ x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

Clearly, the imputation $\hat{x}$ where $\hat{x}_1 = 0$ and $\hat{x}_2 = \hat{x}_3 = 1$ is the only element of $X(\bar{\mathcal{G}}|_{\texttt{LC}})$. Moreover, there is no coalition S and a vector y such that y is S-feasible w.r.t. LC and where $y_k > \hat{x}_k$, for each $k \in S$. Thus, $\hat{x}$ trivially belongs to $\mathscr{C}(\bar{\mathcal{G}}|_{\texttt{LC}})$.

Now consider $\bar{\mathcal{G}}$ and the coalition $S = \{1, 2\}$. The vector $(y_1, y_2) = (\frac{1}{2}, 1 + \frac{1}{2})$ is an S-feasible payoff vector in $\bar{\mathcal{G}}$ and is such that $y_1 > \hat{x}_1$ and $y_2 > \hat{x}_2$. Thus $\hat{x} \notin \mathscr{C}(\bar{\mathcal{G}})$ (and hence $\hat{x} \notin \mathscr{C}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\mathtt{LC}})$). □

Proposition 5.10. *There exists a constrained game $\bar{\mathcal{G}}|_{\mathtt{LC}}$ (with int$(\mathtt{LC}) = \varnothing$) such that*

- $\mathscr{B}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\mathtt{LC}}) \neq \varnothing$ *and* $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}}) = \varnothing$ *and, thus*
- $\mathscr{B}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\mathtt{LC}}) \not\subseteq \mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}})$.

Proof. Consider a game $\bar{\mathcal{G}}$ over players $\{1, 2, 3, 4\}$ and where worths are such that: $v(N) = 3$, $v(\{1, 2\}) = 2$, $v(\{2, 3, 4\}) = 4$, $v(\{1, 3, 4\}) = 3$, $v(\{2\}) = 1$, and $v(S) = 0$ for any other coalition $S \subset N$. Let $\hat{x}$ be the imputation with $\hat{x}_1 = 0$ and $\hat{x}_2 = \hat{x}_3 = \hat{x}_4 = 1$.

Claim A: $\hat{x} \in \mathscr{B}(\bar{\mathcal{G}})$.

We start by noticing that each possible objection (y, S) to $\hat{x}$ must be such that either $S = \{1, 2\}$ (objection of either 1 or 2 against 3 or 4), or $S = \{2, 3, 4\}$ (objection of someone against 1), or $S = \{1, 3, 4\}$ (objection of someone against 2); indeed, these are the only three coalitions for which $v(S) > \hat{x}(S)$ holds, and hence for which an S-feasible vector y may exist with $y(S) \leq v(S)$ and $y_k > \hat{x}_k$, for each $k \in S$. However, $(z, \{1\})$ is a trivial counterobjection to $(y, \{2, 3, 4\})$ for any y and with $z_1 = 0$, since $v(\{1\}) = 0$. Similarly, $(z, \{2\})$ is a trivial counterobjection to $(y, \{1, 3, 4\})$ for any y and with $z_2 = 1$, since $v(\{2\}) = 1$.

Hence, let us now consider an objection of the form $(y, \{1, 2\})$ of player 1 (against 3 or 4) where $y_1 + y_2 = v(\{1, 2\}) = 2$, $y_2 > 1$ and $y_1 > 0$. In particular, note that $y_2 < 2$ and $y_1 < 1$ hold. Then, consider a pair $(z, \{2, 3, 4\})$ with $z_3 = z_4 = 1$ and $z_2 = 2$, and note that: $z(\{2, 3, 4\}) = v(\{2, 3, 4\}) = 4$, $z_3 \geq \hat{x}_3$, $z_4 \geq \hat{x}_4$, and $z_2 = 2 \geq y_2$. Therefore, $(z, \{2, 3, 4\})$ is a counterobjection to $(y, \{1, 2\})$ (of 1 against 3 or 4).

Consider, instead, the case where $(y, \{1, 2\})$ is an objection of player 2 (against 3 or 4). Then, consider the pair $(z, \{1, 3, 4\})$ with $z_1 = z_3 = z_4 = 1$, and note that $z(\{1, 3, 4\}) = v(\{1, 3, 4\}) = 3$, $z_3 \geq \hat{x}_3$, $z_4 \geq \hat{x}_4$, and $z_1 = 1 \geq y_1$. Thus, $(z, \{1, 3, 4\})$ is a counterobjection to $(y, \{1, 2\})$ (of 2 against 3 or 4). It follows that $\hat{x} \in \mathscr{B}(\bar{\mathcal{G}})$.

Let us now consider the following set of constraints:

$$\mathtt{LC} = \begin{cases} x_2 + x_3 + x_4 = 3 \\ x_1 + x_3 = 1 \\ x_1 + x_4 = 1 \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \end{cases}$$

It is easy to check that $\hat{x} \in X(\bar{\mathcal{G}}|_{\mathtt{LC}})$. Observe in particular that $\hat{x}$ is the only imputation in $X(\bar{\mathcal{G}}|_{\mathtt{LC}})$, since the grand-coalition imposes $x_1 + x_2 + x_3 + x_4 = 3$. Thus, to prove that $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}}) = \varnothing$, we have just to show that $\hat{x}$ is not contained in $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}})$.

Consider the objection $(y, \{1,2\})$ of player 1 against 3, where $y_1 = \frac{1}{3}$ and $y_2 = \frac{5}{3}$. Note that $y_1 + y_2 = v(\{1,2\}) = 2$, $y_2 > 1$ and $y_1 > 0$, and thus it is $\{1,2\}$-feasible w.r.t. $\mathtt{LC}$. Moreover, recall that $v(\{3\}) = v(\{3,4\}) = v(\{2,3\}) = 0$. It follows that counterobjections of 3 against 1 to $(y, \{1,2\})$ may be only constructed over the coalition $\{2,3,4\}$. Let $(z, \{2,3,4\})$ be such a counterobjection. For player 2, which belongs to the intersection of these two coalitions, $z_2 \geq y_2 > 1$ holds. Because of the constraint $z_2 + z_3 + z_4 = 3$, this entails $z_3 + z_4 < 2$. However, this is impossible since we should have also $z_3 \geq x_3 = 1$ and $z_4 \geq x_4 = 1$.

Thus, there are no possible counterobjections to the objection $(y, \{1,2\})$ to $\hat{x}$. It follows that the only candidate $\hat{x}$ does not belong to $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}})$ and, hence, $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}}) = \varnothing$, even though $\mathscr{B}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\mathtt{LC}}) \neq \varnothing$. □

Proposition 5.11. *There exists a constrained game $\bar{\mathcal{G}}|_{\mathtt{LC}}$ (with int$(\mathtt{LC}) = \varnothing$) such that $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}}) \not\subseteq \mathscr{B}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\mathtt{LC}})$, with $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}}) = X(\bar{\mathcal{G}}|_{\mathtt{LC}}) \neq \varnothing$.*

Proof. Consider the game $\bar{\mathcal{G}}$ over players $\{1,2,3\}$ and where $v(N) = v(\{1,2\}) = v(\{2,3\}) = 2$, $v(\{1,3\}) = 1$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. Also consider the constraints:

$$\mathtt{LC} = \begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \\ x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

Clearly, the imputation $\hat{x}$ where $\hat{x}_1 = 0$ and $\hat{x}_2 = \hat{x}_3 = 1$ is the only element of $X(\bar{\mathcal{G}}|_{\mathtt{LC}})$. Moreover, there is no pair (y, S) such that y is S-feasible w.r.t. $\mathtt{LC}$ and where $y_k > \hat{x}_k$, for each $k \in S$. Thus, $\hat{x}$ trivially belongs to $\mathscr{B}(\bar{\mathcal{G}}|_{\mathtt{LC}})$.

However, $\hat{x}$ does not belong to $\mathscr{B}(\bar{\mathcal{G}})$. To see why this is the case, consider an objection $(y, \{1,2\})$ of player 1 against player 3, with $y_1 = \frac{1}{2}$, $y_2 = 1 + \frac{1}{2}$. Then, one may immediately check that there is no counterobjection to $(y, \{1,2\})$ Thus, $\hat{x} \notin \mathscr{B}(\bar{\mathcal{G}})$ (and hence $\hat{x} \notin \mathscr{B}(\bar{\mathcal{G}}) \cap X(\bar{\mathcal{G}}|_{\mathtt{LC}})$). □

We are now going to prove that property (4) of Proposition 2.10 also holds on constrained games.

Proposition 5.12. *For any constrained game $\mathcal{G}|_{\mathtt{LC}}$, it holds that $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}}) \subseteq \mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$.*

Proof. Consider an imputation $x \in \mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ and assume by contradiction that $x \notin \mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$. By this there exist an objection (y, S) to x. It must be the case that y is an S-feasible payoff vector w.r.t. $\mathtt{LC}$ and $y_k > x_k$, for each $k \in S$. This implies that $x \notin \mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$: a contradiction. Thus, $x \in \mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$. □

Let us now briefly discuss the matter of whether solution concepts for constrained games might be empty or not. To this end, we note that Proposition 5.10 has already evidenced that there is a constrained game with an empty bargaining set. By Proposition 5.12 constrained games with empty bargaining set have an empty core too.

5.4 Complexity Analysis

In this section, we shall look at the various solution concepts for constrained coalitional games from a computational viewpoint. In particular, our aim is to shed light on the impact of issuing constraints w.r.t. the intrinsic complexity of these notions, and to assess whether any price has to be paid for the increased expressiveness of constrained games.

In the analysis that follows, we assume that constrained games are provided as a pair $\langle \mathcal{G}, \mathtt{LC} \rangle$, where $\mathcal{G}$ is the TU game over which is built the constrained game, and $\mathtt{LC}$ is an explicit and complete representation of all the constraints issued on $\mathcal{G}$.

In the following, all hardness results will be shown to hold in the simplest case of (deterministic) polynomial time compact representation for $\mathcal{G}$. Moreover, for the sake of presentation and readability, membership results are firstly proven for deterministic oracles, and then extended to the more general case of **FNP** representation schemes for $\mathcal{G}$.

Importantly, such membership results do not a-priori assume any bound on the representation size of real numbers. To this end, some non-trivial technical matters will be faced next, to show that algorithms can safely work with as few as polynomially many bits, for any solution concept considered in this Chapter.

5.4.1 Hardness results

In this section, we shall establish our hardness results. In particular, in order to highlight the intrinsic difficulty for the various notions, constructions are reported over kinds of worth functions that are "simple" not only from a computational viewpoint, in that they are given via oracles computable in polynomial time, but also from an algebraic viewpoint, as they induce *cohesive* games.

We recall here that a game is cohesive if its worth function v is such that for each partition $\mathcal{S}$ of the players in N, $v(N) \geq \sum_{S \in \mathcal{S}} v(S)$ holds—a condition often imposed in order to guarantee that the grand-coalition forms up. In the following we speak of cohesive constrained games when a constrained game is built upon a TU cohesive game.

In order to establish hardness results, we exploit a number of reductions that refer to Boolean formulae. Let φ be a Boolean formula, and let $\mathit{vars}(\varphi) = \{W_1, \ldots, W_n\}$ be the set of Boolean variables occurring in φ. Recall that a literal is either a Boolean variable W_i or its negation $\overline{W_i}$. The former is called a positive literal, while the latter is called a negative literal. We denote by $\overline{\mathit{vars}}(\varphi) = \{\overline{W_i} \mid W_i \in \mathit{vars}(\varphi)\}$ the set of negative literals for the variables occurring in φ. Literals are associated with game players in most proofs. For a set of players S, define $\sigma(S)$ to be the truth assignment where $W_i \in \mathit{vars}(\varphi)$ is true if W_i occurs in S, and false, otherwise. The fact that $\sigma(S)$ satisfies φ is denoted by $\sigma(S) \models \varphi$. Moreover, we say that a coalition S is *consistent* w.r.t.

a set of variables $Y \subseteq vars(\varphi)$ if, for each $W_i \in Y$, $|\{W_i, \overline{W_i}\} \cap S| = 1$ holds. In the case where $Y = vars(\varphi)$, we simply say that S is consistent.

The Bargaining Set

Theorem 5.13. *Let $\mathcal{G}$ be a TU game and x be an imputation. Deciding whether x belongs to $\mathscr{B}(\mathcal{G})$ is* $\mathbf{\Pi_2^P}$-hard, *even for cohesive games (and hence also for cohesive constrained games).*

Proof. We show a polynomial-time reduction from the problem of deciding whether a quantified Boolean formula $H = \forall Y_1, \ldots, Y_n \exists Z_1, \ldots, Z_q \Phi$ is valid, which is a well-known $\mathbf{\Pi_2^P}$-complete problem (Johnson, 1990). Let $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$ and $\mathbf{Z} = \{Z_1, \ldots, Z_q\}$ be the sets of universally and existentially quantified variables, respectively.

Based on H, we build a game $\mathcal{G}(H) = \langle N, v\rangle$, where $N = vars(\Phi) \cup \overline{vars}(\Phi) \cup \{a, a'\}$ and where, for each set of players S, v is such that:

$$v(S) = \begin{cases} 2 & \text{if } S = N, \\ 1 & \text{if } |S| = n \text{ and } S \text{ is consistent w.r.t. } \{Y_1, \ldots, Y_n\}, \\ 1 & \text{if } S \text{ is consistent, } |\{a, a'\} \cap S| = 1, \text{ and } \sigma(S) \models \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{x}$ be the imputation with $\hat{x}_a = \hat{x}_{a'} = 1$ and $\hat{x}_p = 0$, for each other player p. Let us study the structure of a possible objection to $\hat{x}$. Recall that (y, S) is an objection of player i against player j to $\hat{x}$ if and only if: $i \in S$, $j \notin S$, $y(S) \leq v(S)$ and $y_k > \hat{x}_k$, for each $k \in S$. Thus, we observe that $v(S) = 1$ must hold, in order to improve the payoffs, and because a worth value equal to 2 can only be obtained by the grand-coalition. In addition, since $\hat{x}_a = \hat{x}_{a'} = 1$ and since $y(S) \leq v(S) = 1$, it must also be the case that $\{a, a'\} \cap S = \varnothing$, because these players get 1 in the current imputation $\hat{x}$, yet. Due to the definition of the worth function, these entail that S is consistent over the variables $\mathbf{Y}$, i.e., it is a set of n players corresponding to literals over the universally quantified variables. We thus associate with each possible objection (y, S) to $\hat{x}$ (where $y(S) \leq 1$ and $y_k > 0$, $\forall k \in S$) a truth-value assignment to the variables in $\mathbf{Y}$ such that, $\forall 1 \leq i \leq n$, Y_i is assigned *false* if $Y_i \in S$ (and *true* if $\bar{Y}_i \in S$). According to our notations, this means that such an objection is associated with the truth-value assignment $\sigma(\mathbf{Y} \setminus S)$. But we can define the converse, as well. For any truth-value assignment $\sigma_{\mathbf{Y}}$ for the universally quantified variables, its associated objection is the pair (y, S) such that $S = \{Y_k \in \mathbf{Y} \mid \sigma_{\mathbf{Y}}(Y_k) = \mathit{false}\}$, and $y_{Y_k} = \frac{1}{|S|}$, for each $Y_k \in S$. Note that y is an S-feasible payoff vector and hence (y, S) is an objection of any player in S against player a to $\hat{x}$, and that $\sigma(\mathbf{Y} \setminus S) = \sigma_{\mathbf{Y}}$.

Indeed, if (y, S) is an objection against a player $j \notin \{a, a'\}$, then $(z, \{j\})$ with $z_j = 0$ is a trivial counterobjection, since $v(\{j\}) = \hat{x}_j = 0$ and $j \notin S$. It follows that the set of objections that are possibly justified has to be restricted to those against player a or player a'. Next, we consider the case of an objection against a, but precisely the same arguments hold for objections against a'. Let (y, S) be an objection of i against player a to $\hat{x}$. Any counterobjection (z, T) must now be with $a \in T$ (and $i \notin T$). Thus, in order to have $z_a \geq \hat{x}_a = 1$, it must be the case that $v(T) \geq 1$ and, actually, that $v(T) = 1$, due to the definition of the worth function. In particular, the latter entails that, for each player $p \neq a$ with $p \in T$, it holds that $z_p = 0$. Thus, $T \cap S$ must be empty, because all members of S get something strictly greater than 0 according to y, T is consistent, and $\sigma(T) \models \Phi$. In particular, since $T \cap S = \varnothing$, according to this satisfying assignment, Y_i is true in $\sigma(T)$ if and only if $Y_i \notin S$. It follows that $\sigma(T)$ coincides with $\sigma(\mathbf{Y} \setminus S)$ over the universally quantified variables, and thus it is in fact an extension of this truth-value assignment to the set of all variables occurring in the formula Φ. Conversely, note that every satisfying assignment for Φ that extends $\sigma(\mathbf{Y} \setminus S)$ corresponds with such a counterobjection to (y, S).

After the above observations, we show that: $\hat{x} \in \mathscr{B}(\mathcal{G}(H))$ *if and only if H is valid.*

($\Rightarrow$) Assume that that $\hat{x} \in \mathscr{B}(\mathcal{G}(H))$. Let $\sigma_{\mathbf{Y}}$ be any truth-value assignment for the universally quantified variables, and let (y, S) be the objection to $\hat{x}$ such that $\sigma(\mathbf{Y} \setminus S) = \sigma_{\mathbf{Y}}$. Since $\hat{x} \in \mathscr{B}(\mathcal{G}(H))$, there exists a valid counterobjection (z, T) to (y, S), and we have seen above that its corresponding truth-value assignment $\sigma(T)$ is an extension of $\sigma(\mathbf{Y} \setminus S)$ to the set of all variables *vars*(Φ), and that $\sigma(T) \models \Phi$. It follows that H is valid.

($\Leftarrow$) Assume that $\hat{x} \notin \mathscr{B}(\mathcal{G}(H))$. Then, there is a justified objection (y, S) against a (or a') to $\hat{x}$. It follows from the discussion above that there is no truth-value assignment that is able to extend the assignment $\sigma(\mathbf{Y} \setminus S)$ to all the universally quantified variables, and to satisfy Φ. Indeed, such an extension would be associated with a counterobjection to (y, S). It follows that H is not valid.

To conclude the proof, note that the game is cohesive. Indeed, for each coalition S where $v(S) = 1$, it is the case that $|S \cap \{Y_1, \ldots, Y_n, \bar{Y}_1, \ldots, \bar{Y}_n\}| = n$. Thus, given any three coalitions S_1, S_2 and S_3 with $v(S_1) = v(S_2) = v(S_3) = 1$, it must be the case that two of them overlap over some players. Therefore, any partition $\mathcal{S}$ of N contains at most two coalitions getting a worth greater than 0, and the result follows since $v(N) = 2$. $\square$

Theorem 5.14. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game. Deciding whether $\mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$ is non-empty is* $\mathbf{\Sigma_3^P}$-hard*, even for cohesive constrained games.*

Proof. Deciding the validity of the formula $P = \exists X_1, \ldots, X_m \forall Y_1, \ldots, Y_n \exists Z_1, \ldots, Z_q \Phi$ is a well-known $\mathbf{\Sigma_3^P}$-complete problem (Johnson, 1990).

Based on P, we build in polynomial time a game $\mathcal{G}(P) = \langle N, v\rangle$, where $N = \mathit{vars}(\Phi) \cup \overline{\mathit{vars}}(\Phi) \cup \{a, w\}$ and where, for each set of players S, v is such that:

$$v(S) = \begin{cases} m+1 & \text{if } S = N \\ 1 & \text{if } w \in S \wedge |S| = n+1 \wedge \\ & \quad S \text{ is consistent w.r.t. } \{Y_1, \ldots, Y_n\}, \\ 1 & \text{if } S = \{X_i,\ \bar{X}_i\}, \text{ for some } i \\ 1 & \text{if } a \in S \wedge S \text{ is consistent } \wedge \\ & \quad\quad \sigma(S) \models \varPhi, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We also build in polynomial time a set $\mathtt{LC}$ that, for each $1 \leq i \leq m$, contains the following constraints:

$$\mathtt{LC} = \begin{cases} x_{X_i} + x_{\bar{X}_i} = 1 \\ x_{X_i} \geq 0 \\ x_{\bar{X}_i} \geq 0 \\ x_a = 1 \\ x_{X_i}, x_{\bar{X}_i} \in \mathbb{Z} \end{cases}$$

and note that $\mathtt{LC}$ forces variables x_{X_i} and $x_{\bar{X}_i}$ to take distinct values from the set $\{0, 1\}$. As a result, we associate any imputation x for the constrained game $\mathcal{G}(P)|_{\mathtt{LC}}$ with an assignment $\sigma(x)$ to the variables in $\{X_1, \ldots, X_m\}$ such that X_i is true in $\sigma(x)$ if and only if $x_{X_i} = 0$—please note that we are associating 0 with true, here. In addition, $\hat{x}_a = 1$ holds in any imputation $\hat{x}$.

Thus, consider an arbitrary imputation $\hat{x}$ where $\hat{x}_a = 1$. Any objection (y, S) to $\hat{x}$ must be such that $v(S) = 1$ (which is indeed the maximum available worth over each coalition $S \subset N$) and there is no player $i \in S$ with $\hat{x}_i = 1$. It follows that objections are necessarily of the form (y, S) where $w \in S$, $|S| = n + 1$, and S is consistent w.r.t. $\{Y_1, \ldots, Y_n\}$. In other words, any objection (y, S) to $\hat{x}$ is univocally associated with a truth-value assignment to the universally quantified variables $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$. Let $\sigma(\mathbf{Y} \setminus S)$ be this assignment, that is, we set $Y_j = \mathit{true}$ if and only if $Y_j \notin S$, for any $1 \leq j \leq n$. We define also the converse: given any truth value assignment $\sigma_{\mathbf{Y}}$ to the universally quantified variables, its associated objection is the pair (y, S) such that $S = \{Y_j \mid \sigma_{\mathbf{Y}}(Y_j) = \mathit{false}\} \cup \{w\}$, and $y_k = \frac{1}{|S|}$, for every $k \in S$.

Now, if (y, S) is an objection against a player $j \notin \{a, X_1, \ldots, X_m, \bar{X}_1, \ldots, \bar{X}_m\}$, then $(z, \{j\})$ with $z_j = 0$ is a trivial counterobjection. Indeed, such a player j does not belong to S and may be either an element of $\{Y_1, \ldots, Y_n\}$ or an element of $\{Z_1, \ldots, Z_q\}$. In either cases, $v(\{j\}) = \hat{x}_j = 0$. Instead, if (y, S) is an objection against a player X_i (or, $\bar{X}_i$), then $(z, \{X_i, \bar{X}_i\})$ with $z_{X_i} = \hat{x}_{X_i}$ and $z_{\bar{X}_i} = \hat{x}_{\bar{X}_i}$ is again a counterobjection, because $v(\{X_i, \bar{X}_i\}) = z(\{X_i, \bar{X}_i\}) = 1$ and $\{X_i, \bar{X}_i\} \cap S = \varnothing$. It follows that the set of objections that are possibly justified has to be restricted to the objections against player a. Let (y, S) be an objection of a player $i \in S$ against player a to $\hat{x}$. Any counterobjection (z, T) must have $a \in T$. Thus, in order to have $z_a \geq \hat{x}_a = 1$, it must be the case that $v(T) \geq 1$ and, actually, that $v(T) = 1$, due to the definition of the worth function. In particular, the latter entails that for each player $p \neq a$ with $p \in T$, it holds that $z_p = 0$. Thus, $T \cap S$ must be empty and, in particular, the only possibility is that T is consistent, and $\sigma(T) \models \Phi$. Eventually, since $T \cap S = \varnothing$ and $\hat{x}_p = 0$ for each $p \in T$ with $p \neq a$, we have that $\sigma(T)$ is a satisfying assignment for Φ where: X_i is true in $\sigma(T)$ if and only if X_i is true in $\sigma(\hat{x})$; Y_i is true in $\sigma(T)$ if and only if $Y_i \in T$ and hence $Y_i \notin S$, and thus Y_i is true in $\sigma(\mathbf{Y} \setminus S)$. That is, $\sigma(T)$ is a complete assignment for Φ that extends both partial assignments $\sigma(\hat{x})$ and $\sigma(\mathbf{Y} \setminus S)$.

By exploiting the above observations, we may show that: $\mathscr{B}(\mathcal{G}(P)|_{\mathtt{LC}}) \neq \varnothing$ *if and only if* P *is valid.*

($\Rightarrow$) Assume that there exists $\hat{x} \in \mathscr{B}(\mathcal{G}(P)|_{\mathtt{LC}})$. We have seen that such an imputation $\hat{x}$ is associated with a truth-value assignment $\sigma(\hat{x})$ to the variables in $\{X_1, \ldots, X_m\}$. Moreover, we have seen that every assignment $\sigma_{\mathbf{Y}}$ to the universally quantified variables corresponds to a possible objection (y, S) to $\hat{x}$, and since $\hat{x}$ belongs to the bargaining set there must exist a valid counterobjection (z, T) to (y, S) associated with a satisfying truth-value assignment for Φ that extends both partial assignments $\sigma(\hat{x})$ and $\sigma_{\mathbf{Y}}$. This means that $\hat{x}$ is a witness of the validity of P.

($\Leftarrow$) If P is valid then there is an assignment $\sigma_{\mathbf{X}}$ to the variables in $\{X_1, \ldots, X_m\}$ that witnesses its validity. Consider the imputation $\hat{x}$ such that, $\forall 1 \leq i \leq m$, $\hat{x}_{X_i} = 0$ and $\hat{x}_{\bar{X}_i} = 1$ if $\sigma_{\mathbf{X}}(X_i) = \mathit{True}$, and $\hat{x}_{X_i} = 1$ and $\hat{x}_{\bar{X}_i} = 0$ otherwise. Moreover, $\hat{x}_a = 1$ and all other players get 0. Since, for every extension of σ_X to the universally quantified variables (corresponding to a possible objection (y, S) to $\hat{x}$), there exists a further extension to all variables that satisfies Φ (corresponding to a counterobjection to (y, S)), it follows that $\hat{x} \in \mathscr{B}(\mathcal{G}(P)|_{\mathtt{LC}})$.

Finally, note that the game is cohesive. Consider, indeed, any partition $\mathcal{S}$ of the players in N, and a coalition $S \in \mathcal{S}$ where $v(S) = 1$. In the case where $a \in S$ and S is consistent w.r.t. $\mathit{vars}(\Phi)$, then there cannot exist any other coalition $S' \in \mathcal{S}$ with $v(S') = 1$ and with $S' = \{X_i, \bar{X}_i\}$ for some i. In addition, there can exist at most one further coalition $S'' \in \mathcal{S}$ with $v(S'') = 1$ (for $|S''| = n + 1$, $w \in S''$, and S'' is consistent w.r.t. $\{Y_1, \ldots, Y_n\}$). Thus, $\sum_{S \in \mathcal{S}} v(S) \leq 2$. Similarly, if there is no coalition $S \in \mathcal{S}$ such that $a \in S$ and S is consistent w.r.t. $\mathit{vars}(\Phi)$, then $\sum_{S \in \mathcal{S}} v(S) \leq m + 1$. Indeed, $\mathcal{S}$ might contain the coalitions $\{X_1, \bar{X}_1\}, \ldots, \{X_m, \bar{X}_m\}$, plus at most one coalition that is consistent w.r.t. $\{Y_1, \ldots, Y_n\}$ and which gets worth 1. In particular, $\mathcal{S}$ cannot contain two coalitions S' and S'' consistent w.r.t. $\{Y_1, \ldots, Y_n\}$ and with $v(S') = v(S'') = 1$, as w should be contained in both. □

The core

Theorem 5.15. *Let* $\mathcal{G}$ *be a TU game and* x *be an imputation. Deciding whether* x *belongs to* $\mathscr{C}(\mathcal{G})$ *is* co-**NP**-hard*, even for cohesive TU games (and hence also for cohesive constrained games).*

Proof. Recall that deciding whether a Boolean formula Φ over the variables $X_1, \ldots, X_n$ is not satisfiable, i.e., deciding whether there exists no truth assignment to the variables making Φ true, is a co-**NP**-complete problem (Johnson, 1990).

Given such a formula Φ, we build in polynomial time the TU game $\mathcal{G}(\Phi) = \langle N, v\rangle$, where $N = \mathit{vars}(\Phi) \cup \{w, e\}$ and where, for each set of players S, v is such that:

$$v(S) = \begin{cases} 1 & \text{if } S = N, \\ 1 & \text{if } e \notin S \wedge w \in S \wedge \sigma(S) \models \Phi, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Consider, now, the vector $\hat{x}$ where $\hat{x}_e = 1$ and $\hat{x}_p = 0$ for each other player p, and note that $\hat{x}$ is an imputation. We claim that: $\hat{x} \in \mathscr{C}(\mathcal{G}(\Phi))$ *if and only if Φ is not satisfiable.*

($\Rightarrow$) $\hat{x} \in \mathscr{C}(\mathcal{G}(\Phi))$ implies that there is no coalition S and an S-feasible payoff vector y with $y_i > x_i$, for each $i \in S$. Consider any coalition/assignment $\hat{S}$ such that $e \notin \hat{S}$ and $w \in \hat{S}$, and observe that $\hat{x}(\hat{S}) = 0$. Since $\hat{x} \in \mathscr{C}(\mathcal{G}(\Phi))$, we have that $v(\hat{S}) = 0$ for all $\hat{S}$, that is $\sigma(\hat{S})$ does not satisfy Φ, by definition of the worth function. Given that there is a one-to-one correspondence between coalitions $\hat{S}$ (with $e \notin \hat{S}$ and $w \in \hat{S}$) and truth assignments for Φ, we conclude that Φ is not satisfiable.

($\Leftarrow$) Assume that $\hat{x} \notin \mathscr{C}(\mathcal{G}(\Phi))$. Then, there must exist a coalition $\hat{S} \subset N$ with an $\hat{S}$-feasible payoff vector y such that $y_i > x_i$ for each $i \in \hat{S}$, and hence with $\hat{x}(\hat{S}) < y(\hat{S}) \leq v(\hat{S})$, which is only possible if $\hat{x}(\hat{S}) = 0$ and $v(\hat{S}) = 1$. By construction of the worth function, it follows that $e \notin \hat{S}$, $w \in \hat{S}$ and $\sigma(\hat{S}) \models \Phi$. That is, Φ is satisfiable.

Finally, observe that the role of player w is to guarantee that the game is cohesive. Indeed, for any partition $\mathcal{S}$ of N, there is at most one set $S \in \mathcal{S}$ that contains w, and hence that may get 1 as its coalition worth. □

When considering constrained games and arbitrary input vectors (i.e., not necessarily imputations), checking whether an imputation belongs to the core turns out to be slightly more difficult than in the previous case.

Theorem 5.16. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game and x be a vector. Deciding whether x belongs to $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is* $\mathbf{D^P}$-hard*, even for cohesive constrained games.*

Proof. Given a pair of Boolean formulae (Φ, Φ'), deciding whether Φ is not satisfiable *and* Φ' is satisfiable is a prototypical $\mathbf{D^P}$-complete problem (Johnson, 1990). Assume, w.l.o.g., that $\Phi' = c_1 \wedge \ldots \wedge c_m$, with $c_i = t_{i,1} \vee t_{i,2} \vee t_{i,3}$, for each $i \in \{1, \ldots, m\}$. That is, Φ' is in conjunctive normal form and every clause contains exactly three literals. Moreover, let $\{Y_1, \ldots, Y_\ell\}$ be its set of Boolean variables.

Consider the TU game $\mathcal{G}(\Phi) = \langle N, v\rangle$ built in the proof of Theorem 5.15, and let us equip $\mathcal{G}(\Phi)$ with the following set of constraints:

$$\texttt{LC} = \begin{cases} 1 \geq T_{Y_j} \geq 0, \forall j \in \{1, \ldots, \ell\} \\ \rho(t_{i,1}) + \rho(t_{i,2}) + \rho(t_{i,3}) \geq 1, \forall i \in \{1, \ldots, m\} \\ x_p \in \mathbb{R}, \forall p \in N \\ T_{Y_j} \in \mathbb{Z}, \forall j \in \{1, \ldots, \ell\} \end{cases}$$

where $\rho(t_{i,h})$ denotes the expression $1-T_{t_{i,h}}$ if $t_{i,h}$ is a negative literal, and the expression $T_{t_{i,h}}$ if $t_{i,h}$ is a positive literal. Note that, for each $j \in \{1, \ldots, \ell\}$, T_{Y_j} is constrained over the domain $\{0, 1\}$ as to encode the truth value of the Boolean variable Y_j. Clearly, LC can be computed in polynomial time from Φ.

It is immediate to check that $\Omega(\texttt{LC}) \neq \varnothing$ if, and only if, Φ' is satisfiable. Consider, now, the vector $\hat{x}$ such that $\hat{x}_e = 1$ and $\hat{x}_p = 0$ for each other player p. Then, $\hat{x}$ is in the core of $\mathcal{G}(\Phi)|_{\texttt{LC}}$ if and only if $\hat{x}$ is an imputation that moreover satisfies the conditions in Definition 2.2. In fact, to verify that $\hat{x}$ is an imputation for this constrained game amounts to check whether LC admits a solution, i.e., whether Φ' is satisfiable. Indeed, since players in $\mathcal{G}(\Phi)$ are not constrained in LC, $\hat{x}$ is always an imputation, unless $\Omega(\texttt{LC}) = \varnothing$. In addition, to verify that conditions in Definition 2.2 are satisfied amounts to check that Φ is not satisfiable instead, as discussed in the proof of Theorem 5.15, which closes the proof. □

Theorem 5.17. *Let $\mathcal{G}|_{\texttt{LC}}$ be a constrained game. Deciding whether $\mathscr{C}(\mathcal{G}|_{\texttt{LC}})$ is non-empty is* $\mathbf{\Sigma_2^P}$-hard*, even for cohesive constrained games.*

Proof. Deciding whether a quantified Boolean formula $F = \exists X_1, \ldots, X_n \forall Y_1, \ldots, Y_q \Phi$ is valid is a well-known $\mathbf{\Sigma_2^P}$-complete problem (Johnson, 1990). Without loss of generality, we assume $n \geq 2$.

Based on F, we build in polynomial time the game $\mathcal{G}(F) = \langle N, v\rangle$, where $N = \mathit{vars}(\Phi) \cup \overline{\mathit{vars}}(\Phi)$ and where, for each set of players S, v is such that:

$$v(S) = \begin{cases} n & \text{if } S = N, \\ 1 & \text{if } S \text{ is consistent and } \sigma(S) \not\models \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we build in polynomial time a set LC that, for each $1 \leq i \leq n$, contains the following constraints:

$$\texttt{LC} = \begin{cases} x_{X_i} + x_{\bar{X}_i} = 1 \\ x_{X_i} \geq 0 \\ x_{\bar{X}_i} \geq 0 \\ x_{X_i}, x_{\bar{X}_i} \in \mathbb{Z} \end{cases}$$

Beforehand, note that LC forces variables x_{X_i} and $x_{\bar{X}_i}$ to take distinct values from the set $\{0, 1\}$. As a result, since $v(N) = n$, any imputation x for the constrained game $\mathcal{G}(F)|_{\texttt{LC}}$ does not distribute any worth to the players associated with the variables in $\{Y_1, \ldots, Y_q\}$. Hence, such an imputation x can be associated with an assignment $\sigma(x)$ to the variables in $\{X_1, \ldots, X_n\}$ such that X_i is true in $\sigma(x)$ if and only if $x_{X_i} = 0$—please, note that we are associating 0 with true, here.

Then, we claim that: $\hat{x} \in \mathscr{C}(\mathcal{G}(F)|_{\mathtt{LC}})$ *if and only if* $\sigma(\hat{x})$ *witnesses the validity of* F*, i.e., for each truth assignment* σ_Y *to the variables in* $\{Y_1, \ldots, Y_q\}$*,* $\sigma(\hat{x}) \cup \sigma_Y \models \Phi$*.*

($\Rightarrow$) $\hat{x} \in \mathscr{C}(\mathcal{G}(F)|_{\mathtt{LC}})$ implies that $\forall S \subseteq N$, ther is no S-feasible payoff vector w.r.t. `LC` such that $y_i > x_i$ for each $i \in S$. In particular, consider a coalition $\bar{S}$ such that $\bar{S}$ is consistent. Note that $\hat{x}(\bar{S}) \geq 1$ if and only if $\sigma(\bar{S} \cap \{X_1, \ldots, X_n\}) \neq \sigma(\hat{x})$. Thus, it is relevant to consider the case where $\sigma(\bar{S} \cap \{X_1, \ldots, X_n\}) = \sigma(\hat{x})$, for which $\bar{S}$ basically encodes an assignment to all the variables in Φ, which conforms with $\sigma(\hat{x})$ over the variables in $\{X_1, \ldots, X_n\}$. In this case, since $\hat{x}(\bar{S}) = 0$, we must have that $\sigma(\bar{S}) \models \Phi$ for otherwise $v(\bar{S}) = 1$ would hold, and hence there would be an $\bar{S}$-feasible payoff vector w.r.t. `LC`, y, such that $y_i > x_i$, for each $i \in S$, implying that $x \notin \mathscr{C}(\mathcal{G}(F)|_{\mathtt{LC}})$. Thus, $\sigma(\hat{x})$ witnesses the validity of F, given that $\sigma(\bar{S}) \models \Phi$ holds for each coalition $\bar{S}$ encoding an assignment conforming with $\sigma(\hat{x})$, that is, for every assignment to $\{Y_1, \ldots, Y_q\}$.

($\Leftarrow$) Assume that $\hat{x} \notin \mathscr{C}(\mathcal{G}(F)|_{\mathtt{LC}})$. Then, there is a coalition $\bar{S}$ with an $\bar{S}$-feasible payoff vector w.r.t. `LC`, y, such that $y_i > x_i$ for each $i \in S$ and hence with $0 = \hat{x}(\bar{S}) < y(S) \leq v(\bar{S}) = 1$. In particular, $v(\bar{S}) = 1$ entails that $\bar{S}$ is consistent and $\hat{x}(\bar{S}) = 0$ entails that $\sigma(\bar{S} \cap \{X_1, \ldots, X_n\}) = \sigma(\hat{x})$. Hence, by construction of the worth function, $\bar{S}$ must be such that $\sigma(\bar{S}) \not\models \Phi$. That is, the truth assignment $\sigma_Y = \sigma(\bar{S} \cap \{Y_1, \ldots, Y_q\})$ to the variables in $\{Y_1, \ldots, Y_q\}$ is such that $\sigma(\hat{x}) \cup \sigma_Y \not\models \Phi$.

In the light of the claim above, the result follows since $\mathscr{C}(\mathcal{G}(F)|_{\mathtt{LC}}) \neq \varnothing$ if and only if there is a truth assignment to the variables in $\{X_1, \ldots, X_n\}$ witnessing the validity of F.

To conclude the proof, we notice that $\mathcal{G}(F)|_{\mathtt{LC}}$ is cohesive. Indeed, for each coalition S where $v(S) = 1$, it is the case that $|S \cap (vars(\Phi) \cup \overline{vars}(\Phi))| = n+q$. Thus, given any three coalitions S_1, S_2 and S_3 with $v(S_1) = v(S_2) = v(S_3) = 1$, it must be the case that two of them overlap over some players. Thus, any partition $\mathcal{S}$ of N contains at most two coalitions getting worth 1, and the result follows since we assumed $n \geq 2$. $\square$

5.4.2 Membership results

We now complete the picture of the complexity arising in the context of constrained games, by proving membership results. We recall here that we shall refer to constrained games $\mathcal{G}|_{\mathtt{LC}}$ where $\mathcal{G}$ is represented through an **FP** scheme. We defer the extension to the case where $\mathcal{G}$ is represented through an **FNP** scheme to a section at the end.

Lemma 5.18. *Let* $\mathcal{G}|_{\mathtt{LC}}$ *be a constrained game. Deciding whether a vector is an imputation for* $\mathcal{G}|_{\mathtt{LC}}$ *is in* **NP**. *(In particular, the problem is in* **P** *for TU games.)*

Proof. Let $\mathcal{G} = \langle N, v\rangle$ be a TU game and let $\mathtt{LC}$ be a set of constraints. Let $\hat{x}$ be a vector assigning a payoff value to each player in N. We can check in polynomial time that $\hat{x}(N) = v(N)$ and that $\hat{x}_i \geq v(\{i\})$, for each $i \in N$. This suffices to decide whether $\hat{x}$ is an imputation for the TU game $\mathcal{G}$.

In fact, if the above check fails, then we conclude that $\hat{x}$ is not an imputation for $\mathcal{G}|_{\mathtt{LC}}$. Otherwise, consider the set of linear inequalities $\mathtt{LC}'$ derived from $\mathtt{LC}$ by replacing all variables in N by their values according to $\hat{x}$. Note that $\mathtt{LC}'$ is a mixed integer linear program defined over the variables (if any) in $\mathit{real}(\mathtt{LC}) \cup \mathit{int}(\mathtt{LC}) \setminus N$, and that $\hat{x}$ is an imputation for $\mathcal{G}|_{\mathtt{LC}}$ if and only if $\mathtt{LC}'$ is satisfiable. By well-known results on mixed integer linear programming (see, e.g., Nemhauser and Wolsey, 1988), $\mathtt{LC}'$ admits a solution if and only if it admits a solution that can be represented with polynomially many bits (w.r.t. the size of $\mathtt{LC}'$). Thus, the problem can be solved by firstly guessing in **NP** a vector $\hat{x}'$ assigning a value to each variable in $\mathit{real}(\mathtt{LC}) \cup \mathit{int}(\mathtt{LC}) \setminus N$, and by subsequently checking whether $\hat{x}'$ satisfies all the constraints in $\mathtt{LC}'$ (which is feasible in polynomial time). □

In the following we will use this notation. Let $S \subseteq N$ be a set of players and let y_S be the set of variables $\{y_k \mid k \in S\}$. We denote by $\mathtt{LC}[y_S]$ the copy of the system of linear inequalities $\mathtt{LC}$ where every player variable x_i is renamed as y_i, and every other variable v occurring in $\mathtt{LC}$ is renamed as v_{y_S}.

The Bargaining Set

Lemma 5.19. *Let $\mathcal{G} = \langle N, v\rangle$ be a TU game, $\mathtt{LC}$ be a set of constraints, and $\hat{x}$ be an imputation for $\mathcal{G}|_{\mathtt{LC}}$ that does not belong to $\mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$. Then, there exists a justified objection to $\hat{x}$ that is representable with polynomially many bits.*

Proof. In the following $n = \| \mathcal{G}|_{\mathtt{LC}} \| + \| \hat{x} \|$ denotes the size of the input.

Since $\hat{x} \notin \mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$, there are two players i and j, a coalition S with $i \in S$ and $j \notin S$, and an S-feasible vector w.r.t. $\mathtt{LC}$, $\hat{y}$, such that $(\hat{y}, S)$ is a justified objection of i against j to $\hat{x}$.

Consider the system $\mathtt{LC}'$ containing all the constraints of $\mathtt{LC}[y_S]$ plus the following ones: $y(S) \leq v(S)$ and $y_i > x_i$ for each $i \in S$. The solutions of $\mathtt{LC}'$ are all the objections of i against j to $\hat{x}$. Since $\hat{y}$ is a justified objection we know that $\mathtt{LC}'$ admits solutions, and among them there are also justified objections. The issue here is that, in principle, we have not guarantee that exists a justified objection representable with polynomially many bits.

Denote by $\Omega(i,j,S)$ the set of vectors in $\mathbb{R}^S$ obtained as the projection on the index set S of all vectors solutions of $\mathtt{LC}'$.

Now, we want to exclude from $\Omega(i,j,S)$ all objections of i against j to $\hat{x}$ that are not justified, in order to show next that that set containing only justified objections includes a justified objection representable with polynomially many bits.

For any $T \subseteq N$ with $j \in T$ and $i \notin T$, let $\mathtt{LC}^{-}(i,j,S,T)$ be the system including the inequalities of the system $\mathtt{LC}[y_S]$ and those of the system $\mathtt{LC}[z_T]$, plus the inequalities $y(S) \leq v(S)$, $\forall k \in S$, $y_k > \hat{x}_k$, $z(T) \leq v(T)$, $\forall k \in T \cap S$, $z_k \geq y_k$, and $\forall k \in T \setminus S$, $z_k \geq \hat{x}_k$. Let $\Omega^-_{i,j,S}(T)$ be the set of the projections over S of the solutions of $\mathtt{LC}^-(i,j,S,T)$. Thus, $\Omega^-_{i,j,S}(T)$ contains all vectors y over the index set S such that there exists a counterobjection (z,T) to the objection (y,S) of i against j to $\hat{x}$. It follows that the set of all vectors y such that (y,S) is a justified objection of i against j to $\hat{x}$ is the set:

$$\Omega'(i,j,S) = \Omega(i,j,S) \setminus \bigcup_{T \mid i \notin T \wedge j \in T} \Omega^-_{i,j,S}(T).$$

Obviously, we have that $\Omega'(i,j,S)$ is not a convex polytope in general. Of course, $\Omega'(i,j,S) \neq \varnothing$, because we know that $\hat{y} \in \Omega'(i,j,S)$.

Note that $\Omega'(i,j,S)$ contains only and all the justified objection of i against j to $\hat{x}$. By this, if we find a vector in $\Omega'(i,j,S)$ representable with polynomially many bits it is a justified objection representable with polynomially many bits belonging also to $\Omega(i,j,S)$.

If $\Omega'(i,j,S)$ contains isolated points, then they are representable with polynomially many bits because these isolated points are obtained directly from the constraints of the game, and we have that LC is completely and explicitly represented. Hence those points are the justified objections we are looking for.

If $\Omega'(i,j,S)$ does not contains isolated points, then let P be an arbitrary maximal convex set such that $P \subseteq \Omega'(i,j,S)$. Note that P is not empty and it is a convex polytope, whose vertices are given by the intersections of some halfspaces (corresponding to inequalities) determining the boundaries of $\Omega(i,j,S)$ or $\Omega^-_{i,j,S}(T)$, for some coalitions T such that $i \notin T$ and $j \in T$. Hence, such vertices can be represented with polynomially many bits.

Moreover, note that P is not guaranteed to be closed, i.e., to contain its own boundaries. Indeed, if P is a closed polytope, then each vertex of P is a justified objection that can be represented with polynomially many bits. However, portions of the boundaries of P (and hence some of its vertices) may occur in a set $\Omega^-_{i,j,S}(T)$, for some coalitions T such that $i \notin T$ and $j \in T$; then, these portions do not belong to P, which happens whenever the inequalities defining $\Omega^-_{i,j,S}(T)$ are not strict (that is $\Omega^-_{i,j,S}(T)$ contains its own boundaries). Therefore, to show that there exists a justified objection that can be represented with polynomially many bits, it remains to consider the case where P does not include any of its vertices. In this case, we can distinguish two scenarios, depicted in Figure 5.2: (i) some part of the boundary of P belongs to it, and (ii) otherwise.

(i) Consider a (possibly open) maximal segment belonging to P and actually lying on its boundary. Let y^a and y^b be the endpoints of the segment, and note that they are representable with polynomially many bits, precisely for the same reason as the vertices of P. Let $\hat{y}'$ be the middle point of the segment (i.e., $\hat{y}'_k = \frac{y^a_k + y^b_k}{2}$, $\forall k \in S$). Then, we have that $\hat{y}'$ belongs to P and that it can be represented with polynomially many bits as well.

(ii) In this case, all points of P are interior points. Then, the polytope P has at least three vertices y^a, y^b, and y^c that do not belong to the same (open) facet of P. Consider the open segment with endpoints y^a and y^b. Then, either all its points belong to P, or it is part of the boundary of P and thus none of its points belong to P. In the former case, we set $\hat{y}'$ to be the middle point y_m of this segment, as in item (i) above. In the latter case, the open segment with endpoints y_m and y^c is included in P, otherwise it should be part of the boundary, and hence there must a be a facet including all these points (including y^a and y^b), which contradicts our choice of y^a, y^b, and y^c. Then, we set $\hat{y}'$ to be the middle point of this segment. Again, note that $\hat{y}'$ may be represented with polynomially many bits, because this is the case for the vertices y^a, y^b, and y^c.

In summary, we have shown that there exists a point $\hat{y}'$ in P and thus in $\Omega'(i, j, S)$ that can be represented with $p'(n)$ bits, where $p'(n)$ is a polynomial (greater than $p(n)$, in general). It follows that $(\hat{y}', S)$ is a justified objection of i against j to $\hat{x}$ that can be represented with polynomially many bits. □

Armed with the above lemma, we can prove the following result.

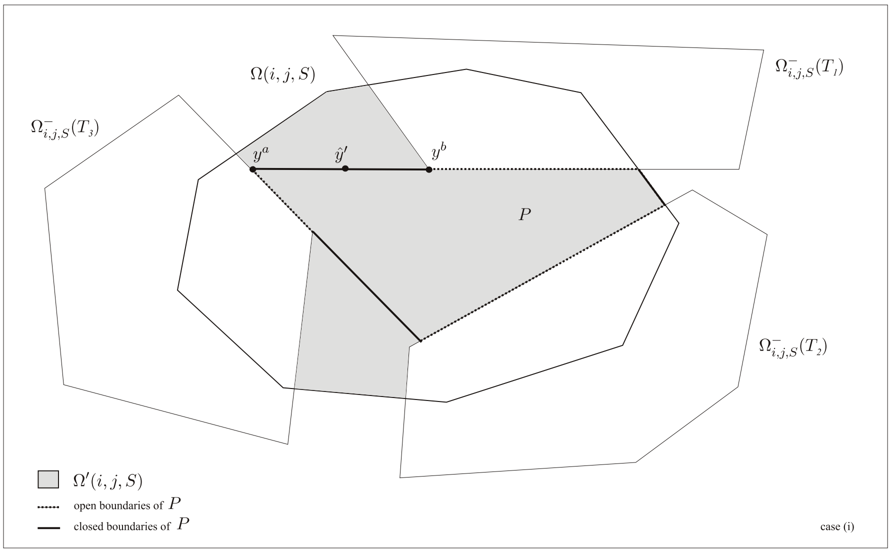
Theorem 5.20. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game and x be a vector. Deciding whether x belongs to $\mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$ is in $\mathbf{\Pi_2^P}$.*

Proof. We show that the complementary problem of deciding whether a vector $\hat{x}$ is not in the bargaining set of a given constrained game $\mathcal{G}|_{\mathtt{LC}}$ is in $\mathbf{\Sigma_2^P}$. We may start checking whether $\hat{x}$ is an imputation or not with an **NP** oracle call (cf. Lemma 5.18). If this is the case, after Lemma 5.19 we may guess in **NP** an objection to $\hat{x}$, that is, a coalition S, two players $i \in S$ and $j \notin S$, a vector $\hat{y}$ such that $(\hat{y}, S)$ is an objection of i against j to $\hat{x}$. In fact, within the same non-deterministic step, we also guess an assignment to the auxiliary variables in *int*(LC). Then, we check that (1) $\hat{y}$ is S-feasible; (2) $y_k > \hat{x}_k$, $\forall k \in S$; and, (3) there is no counterobjection (z, T) of i against j to y.

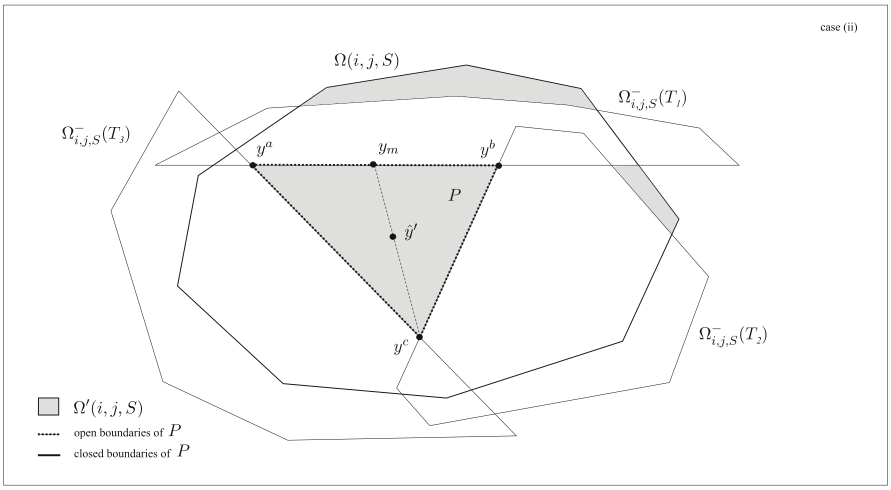
Note that steps (1) and (2) are feasible in polynomial time, once we have $\hat{x}$, $\hat{y}$, S, and the assignment to the integer variables, because we have to check the satisfiability of a standard linear program.

Check (3) requires instead a co-**NP** oracle call. In particular, the oracle works by checking the complementary condition in **NP**, i.e., it guesses a coalition T with $j \in T$ and $i \notin T$, and an assignment to the variables in *int*(LC). Then, it checks in polynomial time whether there is some vector z such that: (1') $z(T) \leq v(T)$, (2') z satisfies all the constraints in LC, (3') $z_k \geq y_k$ for each $k \in T \cap S$, and (4') $z_k \geq x_k$ for each $k \in T \setminus S$. Again, since integer variables were fixed, this amounts to solving a standard linear program. □

Corollary 5.21. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game and x be a vector. Deciding whether x belongs to $\mathscr{B}(\mathcal{G}|_{\mathtt{LC}})$ is $\mathbf{\Pi_2^P}$-complete.*



(a) Some part of the boundary of P belongs to it (case (i)).



(b) No part of the boundary of P belongs to it (case (ii)).

Fig. 5.2. Illustration of Lemma 5.19: a justified objection $\hat{y}'$ representable with polynomially many bits, in a scenario where T_1, T_2 and T_3 are the possible coalitions to counterobject. In this illustration, for each $h \in \{1,2,3\}$, $\Omega^-_{i,j,S}(T_h)$ is a closed polytope (none of its defining inequalities is strict). Thus, those parts of the boundaries of P falling over these polytopes do not belong to P.

Lemma 5.22. *Let $\mathcal{G} = \langle N, v\rangle$ be a TU game and* LC *a set of constraints. If $\mathscr{B}(\mathcal{G}|_{\texttt{LC}}) \neq \varnothing$, then there is an imputation $\hat{x} \in \mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ that is representable with polynomially many bits.*

Proof. Let i and j be two players in N. For every pair of sets of players S and T with $i \in S \setminus T$ and $j \in T \setminus S$, let $\texttt{LC}^-(i, j, S, T)$ be the system of mixed integer inequalities including the inequalities of the systems $\texttt{LC}[x_N]$, $\texttt{LC}[y_S]$, and $\texttt{LC}[z_T]$, plus the inequalities $y(S) \le v(S)$, $y_k > x_k$ for each $k \in S$, $z(T) \le v(T)$, $z_k \ge y_k$ for each $k \in T \cap S$, and $z_k \ge x_k$ for each $k \in T \setminus S$. That is, we proceed in the same way as in the proof of Lemma 5.19, but here the components of vector x are variables of this linear program, while in the previous lemma they were fixed values (the components of the vector $\hat{x}$ in that Lemma). Let $\Omega^-_{i,j,S}(T)$ be the set of the projections over x_N and y_S of the solutions of $\texttt{LC}^-(i, j, S, T)$. Thus, $\Omega^-_{i,j,S}(T)$ contains all pairs $\langle x, y\rangle$ such that there exists a counterobjection (z, T) to the objection (y, S) of i against j to x. Moreover, let $\Omega(i, j, S)$ be the set of all pairs $\langle x, y\rangle$ such that x is an imputation and y is an S-feasible vector w.r.t. LC with $i \in S$ and $j \notin S$. Then:

$$\Omega'(i, j, S) = \Omega(i, j, S) \setminus \bigcup_{T | i \notin T \wedge j \in T} \Omega^-_{i,j,S}(T)$$

is the set of all pairs $\langle x, y\rangle$ such that x is an imputation and (y, S) is a justified objection of i against j to x. Therefore, if $\Omega''(i, j, S)$ denotes the projection of such pairs onto the first component, that is, onto the index set of x_N, we get:

$$\mathscr{B}(\mathcal{G}|_{\texttt{LC}}) = X(\mathcal{G}|_{\texttt{LC}}) \setminus \bigcup_{S \subseteq N \wedge i \in S \wedge j \notin S} \Omega''(i, j, S).$$

Indeed, we obtain all and only those imputations without any associated justified objection from some player i against them. Of course, $\mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ is not a convex polytope in general. Since we are assuming $\mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ to be non-empty, then by following the same arguments as in the proof of Lemma 5.19 we obtain that there exists a point $\hat{x}' \in \mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ that is representable with polynomially many bits. □

Theorem 5.23. *Let $\mathcal{G}|_{\texttt{LC}}$ be a constrained game. Deciding whether $\mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ is non-empty is in $\boldsymbol{\Sigma_3^P}$.*

Proof. We know from Lemma 5.22 that, if the bargaining set is not empty, then there exists a vector $\hat{x} \in \mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ which can be represented with polynomially many bits. Therefore, we can decide the non-emptiness of the bargaining set by firstly guessing in **NP** such a vector $\hat{x}$. Then, we may call an **NP** oracle to check that $\hat{x}$ is an imputation (cf. Lemma 5.18), and finally we can verify that $\hat{x}$ is indeed in the bargaining set with a further call to a $\boldsymbol{\Pi_2^P}$ oracle (cf. Theorem 5.20). □

Corollary 5.24. *Let $\mathcal{G}|_{\texttt{LC}}$ be a constrained game. Deciding whether $\mathscr{B}(\mathcal{G}|_{\texttt{LC}})$ is non-empty is $\boldsymbol{\Sigma_3^P}$-*complete.

The Core

Theorem 5.25. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game and x be a vector. Deciding whether x belongs to $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is in $\mathbf{D^P}$.*

Proof. At first we have to check that $\hat{x}$ is actually an imputation. By Theorem 5.18 it is feasible in **NP** for constrained games.

Then we have to check that $\hat{x}$ is actually in the core. If it is not the case then there exist a coalition S and an S-feasible payoff vector w.r.t. LC such that $y_i > x_i$ for each $i \in S$. Consider the system $\mathtt{LC}'$ containing all the constraints of $\mathtt{LC}[y_S]$ plus the following ones: $y(S) \leq v(S)$ and $y_i > x_i$ for each $i \in S$. Note that all solutions of $\mathtt{LC}'$ are S-feasible payoff vector w.r.t. LC proving that $\hat{x} \notin \mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$. By standard results on mixed integer linear programming (see, e.g., Nemhauser and Wolsey, 1988) $\mathtt{LC}'$ admits a solution if and only if it admits a solution that can be represented with polynomially many bits (w.r.t. the size of $\mathtt{LC}'$). By this we can check whether $\hat{x}$ does not belongs to $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ by guessing in **NP** such a coalition S, and a solution $\hat{y}$ of $\mathtt{LC}'$, and then by checking in polynomial time that $\hat{y}$ is indeed a solution of $\mathtt{LC}'$. Hence checking whether $\hat{x}$ does belong to $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is feasible in co-**NP**. Thus the whole problem is feasible in $\mathbf{D^P}$. □

Corollary 5.26. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game and x be a vector. Deciding whether x belongs to $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is $\mathbf{D^P}$-complete.*

Lemma 5.27. *Let $\mathcal{G} = \langle N, v\rangle$ be a TU game and* LC *a set of constraints. If $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}}) \neq \varnothing$, then there is an imputation $\hat{x} \in \mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ that is representable with polynomially many bits.*

Proof. Let $S \subseteq N$ be a coalition, and $\mathtt{LC}^-(S)$ be the system of mixed integer inequalities including the inequalities of the systems $\mathtt{LC}[x_N]$ and $\mathtt{LC}[y_S]$, plus the inequalities $y(S) \leq v(S)$, and $y_i > x_i$ for each $i \in S$ (note that in $\mathtt{LC}^-(S)$ the components of vector x are variables coming from system $\mathtt{LC}[x_N]$). Let $\Omega^-(S)$ be the set of the projections over x_N of the solutions of $\mathtt{LC}^-(S)$. We have that $\Omega^-(S)$ is the set of all vector x not belonging to $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ because there exist an S-feasible payoff vector w.r.t. LC, y, such that $y_i > x_i$ for each $i \in S$. By this, we get

$$\mathscr{C}(\mathcal{G}|_{\mathtt{LC}}) = X(\mathcal{G}|_{\mathtt{LC}}) \setminus \bigcup_{S \subseteq N} \Omega^-(S),$$

Of course, $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is not a convex polytope in general. Since we are assuming $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ to be non-empty, then by following the same arguments as in the proof of Lemma 5.19 we obtain that there exists a point $\hat{x}' \in \mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ that is representable with polynomially many bits. □

Theorem 5.28. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game. Deciding whether $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is non-empty is in $\mathbf{\Sigma_2^P}$.*

Proof. We know from Lemma 5.27 that, if the core is not empty, then there exists a vector $\hat{x} \in \mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ which can be represented with polynomially many bits. Therefore, we can decide the non-emptiness of the core by guessing in **NP** at first such a vector $\hat{x}$. Then, we may call an **NP** oracle to check that $\hat{x}$ is an imputation (cf. Lemma 5.18), and finally we can verify that $\hat{x}$ is indeed in the core with a further call to a co-**NP** oracle (cf. the proof of Theorem 5.20). □

Corollary 5.29. *Let $\mathcal{G}|_{\mathtt{LC}}$ be a constrained game. Deciding whether $\mathscr{C}(\mathcal{G}|_{\mathtt{LC}})$ is non-empty is $\mathbf{\Sigma_2^P}$*-complete.

5.4.3 More powerful worth functions

In all the above illustrated membership results, we have assumed that worth functions are polynomially time computable and, within this setting, we have shown that various hardness results are indeed tight. Also in this case, as for the TU one, we can show that nothing has to be paid if the constrained game has the underlying TU game represented in a non-deterministic polynomial time compact representation. This holds because whenever we guess in **NP** some coalition S for a game $\mathcal{G}|_{\mathtt{LC}}$ whose underlying TU game $\mathcal{G}$ has an **FNP** representation, we can at the same time (and with no further efforts) guess the worth w and a certificate c, which may subsequently be used to verify in polynomial time that $v_{\mathcal{G}}(S) = w$ (where $v_{\mathcal{G}}$ is the worth function of $\mathcal{G}$). In practice, computing the value $v_{\mathcal{G}}(S)$ comes for free, whenever the coalition S is guessed beforehand. Also in this case, the reader may check that in all the membership results provided in this section, the value of any worth function $v_{\mathcal{G}}(S)$ is used precisely just after that S is guessed. Hence, all our memberships immediately hold over such more powerful worth functions.

Theorem 5.30. *The membership results in Section 5.4.2 hold on any* **FNP** *compact representation $\mathcal{R}$.*

6

Conclusions and future research

In this Ph.D. thesis we have carried out a thorough analysis of computational issues related to solution concepts in compact coalitional games. We have illustrated that coalitional games come in two flavors: games with transferable utility and games with non-transferable utility, and for both of them we have studied the computational complexity underlying computing with their solution concepts. In particular for graph games, a particular representation scheme for TU games, we have closed long-standing open problems stated by Deng and Papadimitriou (1994).

Moreover, our membership results have been given for a broad class of compact coalitional games whose worth and consequence functions is just required to be computable in **FNP**, and are not tailored to any specific representation. In this respect, our results are rather general indeed. In particular, these general results, close a problem about the core of TU games left open by Ieong and Shoham (2005).

To obtain our membership results for TU games, we had to deal with linear programs in succinct form, that are linear programs with exponentially many constraints in the number of variables, but that are represented with a size only polynomial in the number of variables. We have provided complexity bounds on several problems related to succinctly specified linear programs, which are of independent interest, and whose complexity for programs with polynomially many constraints is well-known instead.

A deep analysis of the NTU framework seems to be innovative, since works of this type are rather rare in the literature. We have proposed a representation scheme, the NTU marginal contribution networks, to model NTU games in the framework of Osborne and Rubinstein (1994), and we have analyzed the complexity of solution concepts for this type of game identifying an increasing in the complexity of the problems with respect to the TU case.

After observing that in many real life scenarios the utility among players is rarely completely transferable, we have realized the need for an handy and expressive representation scheme for NTU games. Thinking that TU games would be a good modeling tool if they could be tailored to particular scenarios, such as regulation authorities, discrete domains etc., we have proposed a new kind of games, that we called *constrained games*. Such games are standard TU games over which application specific constraints are imposed that have to be satisfied by the outcomes of the game. We have chosen for the constraints to be mixed integer linear (in)equalities, showing that this type of constraints are very expressive and allow to capture several interesting application scenarios. We have conducted a structural and computational analysis in order to asses whether the issuing of constraints affects relationships among solution concepts or their computational complexity.

All the aforementioned complexity results are reported in Figures 6.1 and 6.2.

As the reader may note, all these results indicate the intractability of analyzed problems. These results may provide clues on the source of the complexity to devise algorithms, and it is moreover sensible to ask whether there exist *natural* representation schemes on which the problems analyzed here turn out to be tractable.

In this respect, we have provided a result of tractability for the kernel in graph games, highlighting that the problem is easy, unlike the general case, when the graph of the game is characterized by bounded treewidth. We have obtained this results by introducing a logic-based approach to isolate classes of tractable games for the kernel, which was not explored in earlier literature. An avenue of further research is to apply this approach to trace the tractability frontier for other solution concepts.

Also the constrained games here proposed, in their early stage, can be a basis for a fruitful future research. In fact, an interesting avenue of further research may be the one of considering more expressive kinds of constraints formulated, for instance, via logic-based languages and where preference criteria can be adopted in the place of hard constraints. Being an NTU framework that is somewhat between pure TU and NTU games, it may be interesting to single out definitions of excess in this setting, perhaps in the spirit of that of Kalai (1975), in order to define and study other solution concepts as the kernel and the nucleolus in constrained games.

Besides this, constrained games can be seen as a general framework to model, through game theory, specific real life scenarios that do not naturally fall in the sometimes rigid requirements of standard game theoretic models. It would be interesting to push further these ideas forward towards developing new types of games that can be used to model interaction and cooperation in distributed computation environments that have been becoming more and more common and pervasive.

Problem	TU	NTU	Constrained
CORE-MEMBERSHIP	co-**NP**-complete	co-**NP**-complete	$\mathbf{D^P}$-complete
KERNEL-MEMBERSHIP	$\mathbf{\Delta_2^P}$-complete	n/a$^\diamond$	-
BARGAININGSET-MEMBERSHIP	$\mathbf{\Pi_2^P}$-complete	$\mathbf{\Pi_2^P}$-complete	$\mathbf{\Pi_2^P}$-complete
NUCLEOLUS-MEMBERSHIP	$\mathbf{\Delta_2^P}$-complete	n/a$^\diamond$	-
CORE-NONEMPTINESS	co-**NP**-complete	$\mathbf{\Sigma_2^P}$-complete	$\mathbf{\Sigma_2^P}$-complete
BARGAININGSET-NONEMPTINESS	trivial	$\mathbf{\Sigma_3^P}$-complete	$\mathbf{\Sigma_3^P}$-complete

Fig. 6.1. Complexity Results for TU, NTU and Constrained Games. Hardness results for TU games hold on graph games, for NTU games hold on NTU marginal contribution networks, and for constrained games hold even on cohesive games with worth functions computable in **FP**. Membership results hold on representation in **FNP** without any assumption on the representation of real numbers. $^\diamond$For NTU games there are no widely accepted definitions for the kernel and the nucleolus.

Problem	Result
MEMBERSHIP	in co-**NP**
NONEMPTINESS	in co-**NP**
DIMENSION	in **NP**
AFFINEHULLCOMPUTATION	in $\mathbf{F\Delta_2^P}$
OPTIMALVALUECOMPUTATION	in $\mathbf{F\Delta_2^P}$
FEASIBLEVECTORCOMPUTATION	in $\mathbf{F\Delta_2^P}$
OPTIMALVECTORCOMPUTATION	in $\mathbf{F\Delta_2^P}$

Fig. 6.2. Summary of Complexity Results for Succinct Linear Programming Problems.

A

Reasoning on succinctly specified linear programs (Reprise)

A.1 Elements of polyhedral geometry

A.1.1 Preliminaries

Any vector $x \in \mathbb{R}^n$ will be viewed in this section as a column vector (i.e., an $n \times 1$ matrix). For a column (resp., row) vector z, the corresponding row (resp., column) vector obtained as the *transposition* of z is denoted by z^T. Thus, for a row vector x^T (with $x \in \mathbb{R}^n$) and a column vector $y \in \mathbb{R}^n$, $x^T y$ is their *inner product* $\sum_{i=1}^n x_i y_i$.

For any pair of natural numbers $m, n > 0$, we denote by $\mathbb{R}^{m \times n}$ the set of all $m \times n$ matrices with entries in $\mathbb{R}$. The i-th row of a matrix $A \in \mathbb{R}^{m \times n}$ is denoted by $A_{i,\bullet}$ (note that $A_{i,\bullet}^T \in \mathbb{R}^n$), the j-th column of A is denoted by $A_{\bullet,j}$ (note that $A_{\bullet,j} \in \mathbb{R}^m$), and the i-th entry of $A_{\bullet,j}$ is denoted by $A_{i,j}$ (note that $A_{i,j} \in \mathbb{R}$).

A vector $x \in \mathbb{R}^n$ is a *linear combination* of the vectors $v^1, \ldots, v^k \in \mathbb{R}^n$, if there are k real numbers $\lambda^1, \ldots, \lambda^k$ such that $x = \sum_{h=1}^k \lambda^h v^h$; the combination is *proper* if neither $\lambda^h = 0$ for each $h \in \{1, \ldots, k\}$, nor $\lambda^{\bar{h}} = 1$ for an $\bar{h} \in \{1, \ldots, k\}$ and $\lambda_{h'} = 0$ for each $h' \neq \bar{h}$.

A subset $S \subseteq \mathbb{R}^n$ is *linear independent* if none of its members is a proper linear combination of the vectors in S; the *rank* of S, denoted by *rank*(S), is the cardinality of the largest linear independent subset of S.

A vector $x \in \mathbb{R}^n$ is an *affine combination* of the vectors $v^1, \ldots, v^k \in \mathbb{R}^n$, if there are k real numbers $\lambda^1, \ldots, \lambda^k$ such that $x = \sum_{h=1}^k \lambda^h v^h$ and $\sum_{h=1}^k \lambda^h = 1$. The *affine hull* of a non-empty set $S \subseteq \mathbb{R}^n$, denoted by *aff*(S), is the set of all the vectors in $\mathbb{R}^n$ that are affine combinations of finitely many vectors of S.

A.1.2 Polyhedral geometry

If $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then $Ax \leq b$ is called a *system of linear inequalities*; any vector $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} \leq b$ is a *feasible solution* to the system, and the set of all the feasible solutions is a *polyhedron*, which will be denoted by $\Omega(Ax \leq b)$ in the following. A polyhedron is *bounded* if there is a real number $k \in \mathbb{R}$ such that, for each $x \in \Omega(Ax \leq b)$ and each $1 \leq j \leq n$, it holds that $-k \leq x_j \leq k$. A bounded polyhedron is also called a *polytope*.

For a vector $\bar{x} \in \Omega(Ax \leq b)$, we denote by $act(\bar{x}) = \{`A_{i,\cdot}x \leq b_i' \mid A_{i,\cdot}\bar{x} = b_i\}$ the set of the inequalities that are satisfied as equalities at $\bar{x}$; we say hereafter that they are *active* at $\bar{x}$. A vector $\bar{x} \in \Omega(Ax \leq b)$ is a *basic feasible solution* if $rank(\{A_{i,\cdot}^T \mid `A_{i,\cdot}x \leq b_i' \in act(\bar{x})\}) = n$. It is well-known that basic feasible solutions can geometrically be interpreted as the *vertices* of the polyhedron $\Omega(Ax \leq b)$, and that every polytope is associated with at least one basic feasible solution.

For a system $Ax \leq b$, we say that the inequality $A_{i,\cdot}x \leq b_i$ is an *implied equality* if, for each $\bar{x} \in \Omega(Ax \leq b)$, $`A_{i,\cdot}x \leq b_i' \in act(\bar{x})$. We denote by $ie(Ax \leq b)$ the set of all the implied equalities.

The affine hull $aff(Ax \leq b)$ of the polyhedron $\Omega(Ax \leq b)$ is known to coincide with the set

$$\{x \in \mathbb{R}^n \mid A_{i,\cdot}x = b_i, \quad \text{for each implied equality } `A_{i,\cdot}x \leq b_i' \in ie(Ax \leq b)\},$$

which can geometrically be interpreted as the smallest affine subspace of $\mathbb{R}^n$ containing $\Omega(Ax \leq b)$.

The *dimension* of the polyhedron $\Omega(Ax \leq b)$ is the natural number

$$dim(Ax \leq b) = n - rank(\{A_{i,\cdot}^T \mid `A_{i,\cdot}x \leq b_i' \in ie(Ax \leq b)\}) \geq 0.$$

A.1.3 Linear programming

Given a system $Ax \leq b$ and a vector $c \in \mathbb{R}^n$, a feasible solution $x_* \in \Omega(Ax \leq b)$ such that $c^T x_* = \min\{c^T x \mid x \in \Omega(Ax \leq b)\}$ is said *optimal* (w.r.t. $Ax \leq b$ and c). Computing an optimal solution is usually called a *linear programming problem*.

The fundamental theorem of linear programming states that if $Ax \leq b$ has a basic feasible solution, then either the linear program is unbounded below, or there is an optimal solution that is a basic feasible solution. In particular, if $\Omega(Ax \leq b)$ is a polytope, then the linear program has an optimal solution that is a basic feasible solution.

A.2 Computational problems over succinct polytopes

Let $Ax \leq b$ be a system of linear inequalities such that $\Omega(Ax \leq b)$ is a polytope. There are several reasoning tasks (decision or computation problems) over $Ax \leq b$ that one frequently encounters:

MEMBERSHIP: Given a vector $\bar{x} \in \mathbb{R}^n$, is $\bar{x} \in \Omega(Ax \leq b)$?

NonEmptiness: Is $\Omega(Ax \leq b) \neq \varnothing$?

And, assuming that $\Omega(Ax \leq b)$ is non empty:

Dimension: Given a natural number $k \geq 0$, is $dim(Ax \leq b) \leq n - k$?

AffineHullComputation: Compute the set $ie(Ax \leq b)$ of all the implied equalities.

OptimalValueComputation: Given a vector $c \in \mathbb{R}^n$, compute the real number $\min\{c^T \bar{x} \mid \bar{x} \in \Omega(Ax \leq b)\}$.

FeasibleVectorComputation: Compute a vector $x \in \Omega(Ax \leq b)$.

OptimalVectorComputation: Given a vector $c \in \mathbb{R}^n$, compute a vector $x_* \in \Omega(Ax \leq b)$ such that $c^T x_* = \min\{c^T \bar{x} \mid \bar{x} \in \Omega(Ax \leq b)\}$.

For any real number $x \in \mathbb{R}$, let us denote by $||x||$ its *encoding length* (see e.g., Papadimitriou and Steiglitz, 1998; Grötschel et al., 1993, for the description of some encoding techniques). It is well-known that all the above problems are feasible in polynomial time whenever the system $Ax \leq b$ is explicitly given, so that the size of the input is the encoding length $||Ax \leq b||$, which is $O(m \times (n+1) \times \max_i\{\max_j ||A_{i,j}||, ||b_i||\})$ (cf., e.g., Schrijver, 1998; Papadimitriou and Steiglitz, 1998; Grötschel et al., 1993).

In this section, we are instead interested in analyzing the complexity of these problems with respect to a scenario where the system of linear inequalities is specified in succinct form, as to conveniently encode problems over exponentially many inequalities.

Definition A.1. *A function* $\mathcal{L} : \{1, \ldots, m\} \mapsto \mathbb{R}^n \times \mathbb{R}$ *is a* succinct specification *for the system of linear inequalities* $Ax \leq b$ *if:*

- $\mathcal{L}(i) = (A_{i,\bullet}, b_i)$, *for each* $1 \leq i \leq m$;
- *there is a fixed natural number* $k_1 > 0$ *such that the encoding length of* $\mathcal{L}$ *is*
$$||\mathcal{L}|| = \left(\log_2(m) \times n \times \max_i\{\max_j ||A_{i,j}||, ||b_i||\}\right)^{k_1}; \text{ and,}$$
- *there is a fixed natural number* $k_2 > 0$ *such that* $\mathcal{L}(i)$ *can be computed in time* $O(||\mathcal{L}||^{k_2})$, *for each* $1 \leq i \leq m$.

The system $Ax \leq b$ *will be denoted by* $\mathcal{L}[x]$ *in the following.* □

For each problem $\mathcal{P}$ illustrated above, let Succint-$\mathcal{P}$ be the problem $\mathcal{P}$ whose input is given in terms of a succinct specification (plus the encoding length of possible additional input parameters). We next illustrate the complexity figures emerging with such succinct specifications. In fact, our focus in this section is just on membership results, given that, for the purposes of

Problem	Result
Membership	in co-**NP**
NonEmptiness	in co-**NP**
Dimension	in **NP**
AffineHullComputation	in $\mathbf{F\Delta_2^P}$
OptimalValueComputation	in $\mathbf{F\Delta_2^P}$
FeasibleVectorComputation	in $\mathbf{F\Delta_2^P}$
OptimalVectorComputation	in $\mathbf{F\Delta_2^P}$

Fig. A.1. Summary of Complexity Results for Linear Programming Problems.

our investigation, we are interested in providing upper bounds on the computational resources needed for solving these problems.

A summary of the results is illustrated in Figure A.1. Note that all these problems fall in the first level of the polynomial hierarchy; in particular, computation problems are feasible with transducers that use polynomially many **NP** oracle calls.

A.3 Proofs of membership results

We next provide detailed proofs for the various results.

Theorem A.2. Succint-Membership *is in* co-**NP**.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities, and let $\bar{x} \in \mathbb{R}^n$. Note that the size of the input is the sum of the encoding lengths of $\mathcal{L}$ and $\bar{x}$, that is, $||\mathcal{L}|| + ||\bar{x}||$.

Consider the complementary problem of checking whether $\bar{x} \notin \Omega(\mathcal{L}[x])$. This problem can be solved by guessing an index $i \in \{1, \ldots, m\}$, computing $\mathcal{L}(i) = (A_{i,\bullet}, b_i)$, and by subsequently checking that $A_{i,\bullet}\bar{x} > b_i$. In particular, note that representing i requires $\lceil \log_2(m) \rceil \leq ||\mathcal{L}||$ bits, and thus this guess is actually in **NP**. Moreover, $\mathcal{L}(i) = (A_{i,\bullet}, b_i)$ can be computed in time polynomial in the size $||\mathcal{L}||$, and checking whether $A_{i,\bullet}\bar{x} > b_i$ holds is feasible in polynomial time in $||\mathcal{L}|| + ||\bar{x}||$. □

In order to deal with the complexity of the non-emptiness problem, we need to recall a basic combinatorial result on convex sets, which is due to Helly.

Proposition A.3 (Helly's Theorem; Helly 1923; Rabin 1955). *Given a collection $\mathcal{C} = \{c_1, \ldots, c_h\}$ of convex subsets of $\mathbb{R}^n$, $\bigcap_{c_i \in \mathcal{C}} c_i = \varnothing$ implies the existence of a collection $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'| \leq n+1$ and $\bigcap_{c_i \in \mathcal{C}'} c_i = \varnothing$.*

Theorem A.4. Succint-NonEmptiness *is in* co-**NP**.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities, and observe that the set $\{x \in \mathbb{R}^n \mid A_{i,\bullet}x \leq b_i\}$, where $\mathcal{L}(i) = (A_{i,\bullet}, b_i)$, is convex for every i.

Consider then the complementary problem of deciding whether $\Omega(\mathcal{L}[x]) = \varnothing$. By Helly's Theorem, we may observe that $\Omega(\mathcal{L}[x]) = \varnothing$ if and only if there is a set of (at most) $n+1$ indices $I = \{i_1, \ldots, i_{n+1}\} \subseteq \{1, \ldots, m\}$ such that the set $\{x \in \mathbb{R}^n \mid A_{i,\bullet}x \leq b_i, \forall i \in I\}$ is empty. Therefore, this complementary problem can be solved by guessing in **NP** at first the set I (indeed, I can be represented with $(n+1) \times \lceil \log_2(m) \rceil$ bits), and then by checking that $P = \{x \in \mathbb{R}^n \mid A_{i,\bullet}x \leq b_i, \forall i \in I\}$ is empty. In particular, P is a polyhedron defined by (at most) $n+1$ inequalities and, hence, its non-emptiness can be checked in polynomial time in the size of its (explicit) complete representation, by standard results of linear programming (see, e.g., Papadimitriou and Steiglitz, 1998; Grötschel et al., 1993). In turn, the representation of this polyhedron is polynomial in $||\mathcal{L}||$. □

Let us now study the problem of checking the dimension of a given polytope. To this end, we need a simple characterization of implied equalities.

Lemma A.5. *Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities such that $\Omega(\mathcal{L}[x]) \neq \varnothing$. Then, '$A_{\bar{i},\bullet}x \leq b_{\bar{i}}$' $\in ie(\mathcal{L}[x])$ if and only if there is a set $W_{ie}(\bar{i}) = \{i_1, \ldots, i_{n+1}\} \subseteq \{1, \ldots, m\}$ such that $\{x \in \mathbb{R}^n \mid A_{i,\bullet}x \leq b_i, \forall i \in W_{ie}(\bar{i})$ with $i \neq \bar{i} \wedge A_{\bar{i},\bullet}x < b_{\bar{i}}\} = \varnothing$ and $\bar{i} \in W_{ie}(\bar{i})$.*

Proof. Recall that $A_{\bar{i},\bullet}x \leq b_{\bar{i}}$ is an implied equality if for each $\bar{x} \in \Omega(\mathcal{L}[x])$, $A_{\bar{i},\bullet}\bar{x} = b_{\bar{i}}$ holds. Consider the family $\mathcal{C}$ of convex sets: $c_i = \{x \in \mathbb{R}^n \mid A_{i,\bullet}x \leq b_i\}$, $\forall i \neq \bar{i}$; and $c_{\bar{i}} = \{x \in \mathbb{R}^n \mid A_{\bar{i},\bullet}x < b_{\bar{i}}\}$.

We first claim that $A_{\bar{i},\bullet}x \leq b_{\bar{i}}$ is an implied equality if and only if $\bigcap_{c_i \in \mathcal{C}} c_i = \varnothing$. Indeed, if $A_{\bar{i},\bullet}x \leq b_{\bar{i}}$ is an implied equality, then we immediately have that $\bigcap_{c_i \in \mathcal{C}} c_i = \varnothing$. Moreover, for the other side, if $\bigcap_{c_i \in \mathcal{C}} c_i = \varnothing$ (while $\Omega(Ax \leq b) \neq \varnothing$), then $\forall \bar{x} \in \Omega(\mathcal{L}[x])$ it is the case that $A_{\bar{i},\bullet}\bar{x} = b_{\bar{i}}$ holds; that is, $A_{\bar{i},\bullet}x \leq b_{\bar{i}}$ is an implied equality.

Then, we can exploit Helly's Theorem, and observe that $A_{\bar{i},\bullet}x \leq b_{\bar{i}}$ is an implied equality if and only if there are at most $n+1$ indices $W_{ie}(\bar{i}) = \{i_1, \ldots, i_{n+1}\} \subseteq \{1, \ldots, m\}$ such that $\bar{i} \in W_{ie}(\bar{i})$ and $\{x \in \mathbb{R}^n \mid A_{i,\bullet}x \leq b_i, \forall i \in W_{ie}(\bar{i})$ with $i \neq \bar{i} \wedge A_{\bar{i},\bullet}x < b_{\bar{i}}\}$ is empty. In particular, note that $\bar{i}$ occurs in $W_{ie}(\bar{i})$, since $\Omega(\mathcal{L}[x]) \neq \varnothing$. □

Below, a set $W_{ie}(\bar{i})$ with the properties stated in Lemma A.5 is called a *supporting set*. In fact, checking whether a set is supporting is feasible in polynomial time (w.r.t. $||\mathcal{L}||$), since this amounts to deciding the satisfiability of a set of linear inequalities (where one therein is strict). In particular, it is well-known (Papadimitriou and Steiglitz, 1998) that any strict inequality of the form $A_{\bar{i},\bullet}x < b_{\bar{i}}$ can equivalently be replaced by the inequality $A_{\bar{i},\bullet}x \leq b_{\bar{i}} + \varepsilon$, for a suitable constant ε such that $||\varepsilon||$ is polynomial in the size of the system (induced by the rows associated with the indices in $W_{ie}(\bar{i})$).

Theorem A.6. SUCCINT-DIMENSION *is in* **NP**.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities (with $\Omega(\mathcal{L}[x]) \neq \emptyset$), and let k be a natural number. Thus, the size of the input is $||\mathcal{L}|| + ||k||$.

By definition, $dim(\mathcal{L}[x]) \leq n-k$ if and only if there is a set $I \subseteq \{1,\ldots,m\}$ with $|I| = k$ and such that: *(1)* '$A_{i,\bullet}x \leq b_i$' $\in ie(\mathcal{L}[x])$, for each $i \in I$; and, *(2)* $rank(\{A^T_{i,\bullet} \mid i \in I\}) = k$.

Therefore, the problem can be solved by guessing in **NP** the set I together with k supporting sets $I_1,\ldots,I_k$ and then checking that: *(1)* $I_i = W_{ie}(i)$, for each $i \in I$ (cf. Lemma A.5); and *(2)* that $rank(\{A^T_{i,\bullet} \mid i \in I\}) = k$. In particular, note that *(2)* is feasible in polynomial time, that representing I requires $k \times \lceil \log_2(m) \rceil$ bits, and that representing $I_1,\ldots,I_k$ requires $k \times (n+1) \times \lceil \log_2(m) \rceil$ bits (that is, a number of bits polynomial in $||\mathcal{L}|| + ||x||$). □

Let us now focus on the computation problems AFFINEHULLCOMPUTATION, OPTIMALVALUECOMPUTATION, FEASIBLEVECTORCOMPUTATION and OPTIMALVECTORCOMPUTATION.

As far as AFFINEHULLCOMPUTATION is concerned, we observe that over succinctly specified linear programs the set of implied equalities might be of exponential size. To avoid representing this set, we next discuss a useful characterization for affine hulls. For each $h \in \{1,\ldots,m\}$, let $[A_{h,\bullet}, b_h]^T$ denote the vector in $\mathbb{R}^{n+1}$ whose first n components are those of $A^T_{h,\bullet}$ and where the last component is b_h. Then, a set $\mathcal{B} \subseteq ie(Ax \leq b)$ such that

$$|\mathcal{B}| = rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in \mathcal{B}\}) = \\ rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in ie(Ax \leq b)\})$$

is called a *basis of the implied equalities of* $Ax \leq b$.

Lemma A.7. *Let $\mathcal{B}$ be a basis of the implied equalities of $Ax \leq b$. Then:*

(1) '$A_{\bar{i},\bullet}x \leq b_{\bar{i}}$' $\in ie(Ax \leq b)$ *if and only if* $|\mathcal{B}| = rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in \mathcal{B}\} \cup \{\, [A_{\bar{i},\bullet}, b_{\bar{i}}]^T\})$;
(2) $|\mathcal{B}| = rank(\{A^T_{i,\bullet} \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in ie(Ax \leq b)\})$*; and,*
(3) $|\mathcal{B}| = n - dim(Ax \leq b)$.

Proof.

(1) Assume that '$A_{\bar{i},\bullet}x \leq b_{\bar{i}}$' $\in ie(Ax \leq b)$. Since $\mathcal{B} \subseteq ie(Ax \leq b)$ and $|\mathcal{B}| = rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in \mathcal{B}\}) = rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in ie(Ax \leq b)\})$, we immediately have that $|\mathcal{B}| = rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in \mathcal{B}\} \cup \{\, [A_{\bar{i},\bullet}, b_{\bar{i}}]^T\})$.
For the other side, in the case where $|\mathcal{B}| = rank(\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in \mathcal{B}\} \cup \{\, [A_{\bar{i},\bullet}, b_{\bar{i}}]^T\})$, we conclude that $[A_{\bar{i},\bullet}, b_{\bar{i}}]^T$ can be written as a (proper) linear combination of the vectors in $\{[A_{i,\bullet}, b_i]^T \mid \text{'}A_{i,\bullet}x \leq b_i\text{'} \in \mathcal{B}\}$. Moreover, for each $\bar{x} \in \Omega(Ax \leq b)$ and for each '$A_{i,\bullet}x \leq b_i$' $\in \mathcal{B}$, we know that $A_i\bar{x} = b_i$ holds since $\mathcal{B} \subseteq ie(Ax \leq b)$. Therefore, it is easy to see that $A_{\bar{i}}\bar{x} = b_{\bar{i}}$ holds as well, i.e., '$A_{\bar{i},\bullet}x \leq b_{\bar{i}}$' $\in ie(Ax \leq b)$.

(2) By definition of basis, $|\mathcal{B}| = \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in \mathcal{B}) = \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in ie(Ax \leq b)\})$. Thus, it holds that:

$$|\mathcal{B}| = \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in ie(Ax \leq b)\}) \geq \mathit{rank}(\{A_{i,\bullet}^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in ie(Ax \leq b)\}).$$

Assume for the sake of contradiction that:

$$|\mathcal{B}| = \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in ie(Ax \leq b)\}) > \mathit{rank}(\{A_{i,\bullet}^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in ie(Ax \leq b)\}).$$

Then, for each ‘$A_{\bar{i},\bullet}x \leq b_{\bar{i}}$’ $\in \mathcal{B}$, it is the case that $A_{\bar{i},\bullet}^T$ can be written as a proper linear combination of the vectors $\{A_{i,\bullet}^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in \mathcal{B}\}$, i.e., $A_{\bar{i},\bullet}^T = \lambda^1 A_{i_1,\bullet}^T + \cdots + \lambda^h A_{i_h,\bullet}^T$ where $i_1, \ldots, i_h$ are the indices of the inequalities in $\mathcal{B}$. Given that $\mathcal{B}$ is a set of implied equalities, we also have that for each $\bar{x} \in \Omega(Ax \leq b)$, $A_{\bar{i},\bullet}\bar{x} = \lambda^1 b_{i_1} + \cdots + \lambda^h b_{i_h} = b_{\bar{i}}$. Thus, $|\mathcal{B}| > \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in \mathcal{B}\})$ which is impossible, since $|\mathcal{B}| = \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in \mathcal{B}) = \mathit{rank}(\{[A_{i,\bullet}, b_i]^T \mid \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in ie(Ax \leq b)\})$, by definition of basis.

(3) The result immediately follows from (2) and the definition of dimension of a polytope. □

Corollary A.8. *Let $\mathcal{B}$ be a basis of the implied equalities of $Ax \leq b$. Then, $\mathit{aff}(Ax \leq b) = \{x \in \mathbb{R}^n \mid A_{i,\bullet}x = b_i,\ \forall \text{‘}A_{i,\bullet}x \leq b_i\text{’} \in \mathcal{B}\}$.*

In the light of the above corollary, the Succinct-AffineHullComputation problem can then be restated as the problem of computing a basis of the implied equalities.

Theorem A.9. Succinct-AffineHullComputation *is in* $\mathbf{F\Delta_2^P}$.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities (with $\Omega(\mathcal{L}[x]) \neq \varnothing$)). At first we have to compute the dimension of $\Omega(\mathcal{L}[x])$. To this end, it is relevant to observe that we can decide in **NP** whether $\mathit{dim}(\mathcal{L}[x]) \leq n - k$, for any given natural number k (cf. Succint-Dimension in Theorem A.6). Thus, in order to compute the dimension, we may implement a binary search by using Succint-Dimension as an oracle in **NP**. Hence, computing $\mathit{dim}(\mathcal{L}[x])$ is feasible in $\mathbf{F\Delta_2^P}$ (actually, with $\lceil \log_2(n) \rceil$ oracle calls, at most).

Eventually, in order to solve Succinct-AffineHullComputation, we have to compute a basis $\mathcal{B}$ of the implied equalities of $\mathcal{L}[x]$. To this end, consider the problem of deciding whether there is a basis containing a certain set S of inequalities plus an inequality not in S whose index is less or equal than an integer h. This problem can be solved by guessing in **NP** the set $\mathcal{B} \supset S$, and then (in the light of Lemma A.7.(3)) by checking that $|\mathcal{B}| =$

$n - dim(\mathcal{L}[x]) = rank(\{[A_{i,\boldsymbol{\cdot}}, b_i]^T \mid \text{`}A_{i,\boldsymbol{\cdot}}x \leq b_i\text{'} \in \mathcal{B})$ holds, which is feasible in polynomial time; finally, we have also to check that $\mathcal{B}$ contains an inequality '$A_{\bar{i},\boldsymbol{\cdot}}x \leq b_{\bar{i}}$' $\notin S$ such that $\bar{i} \leq h$. Note that, besides $\mathcal{B}$, all the supporting sets of the inequalities guessed in $\mathcal{B}$ must be also guessed. This is needed since we have to check in polynomial time that $\mathcal{B}$ is actually constituted only by implied equalities, or, in other words, we have to check that $\mathcal{B} \subseteq ie(Ax \leq b)$ holds. Recall that checking whether a set is supporting is feasible in polynomial time.

Given this **NP** oracle we may compute a basis by starting with $S = \varnothing$ and by performing a binary search over all the possible indices h. If the oracle answers true, then S will be updated as to include the current index and a novel iteration of binary search is performed in order to find a new index to include in S. When S reach the appropriate cardinality to be a basis the procedure is stopped. Thus, the whole procedure is feasible with polynomially many **NP** oracle calls. □

For the remaining results on computation problems, we need to provide a bound on the size of basic feasible solutions.

Lemma A.10. *Let $\mathcal{L}$ be a succinct specification for system of linear inequalities. Then, there is a constant $k > 0$ such that for each basic feasible solution $\bar{x} \in \Omega(\mathcal{L}[x])$, $||\bar{x}|| \leq ||\mathcal{L}||^k$.*

Proof. Since $\bar{x}$ is a basic feasible solution, there is a set $I \subseteq \{1, \ldots, m\}$ with $|I| = n$ and such that: $\{\bar{x}\} = \{x \in \mathbb{R}^n \mid A_{i,\boldsymbol{\cdot}}x = b_i,\ \forall i \in I\}$. Therefore, the encoding length of $\bar{x}$ is bounded by a polynomial of the size of the inequalities defining the polyhedron $\{x \in \mathbb{R}^n \mid A_{i,\boldsymbol{\cdot}}x \leq b_i,\ \forall i \in I\}$ (cf. Papadimitriou and Steiglitz, 1998), which in turn is bounded by $||\mathcal{L}||^k$ for some constant $k > 0$, by definition of succinct specification. □

Theorem A.11. Succint-OptimalValueComputation *is in* $\mathbf{F\Delta_2^P}$.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities such that $\Omega(\mathcal{L}[x]) \neq \varnothing$ is a polytope, and let $c \in \mathbb{R}^n$. The size of the input is then $||\mathcal{L}|| + ||c||$.

Consider at first the decision problem of deciding whether $\min\{c^T\bar{x} \mid \bar{x} \in \Omega(\mathcal{L}[x])\} \leq k$. The problem can be solved by considering the specification $\mathcal{L}'$ over $m + 1$ constraints such that $\mathcal{L}'(i) = \mathcal{L}(i)$ $(1 \leq i \leq m)$, plus $\mathcal{L}'(m + 1) = (c^T, k)$, and by deciding whether $\Omega(\mathcal{L}'[x])$ is non empty. This tasks is in fact feasible in co-**NP** (cf. Theorem A.4), and $\mathcal{L}'$ can clearly be built in polynomial time.

Observe now that $\Omega(\mathcal{L}[x])$ is a polytope and, hence, the optimal value is certainly achieved over some basic feasible solution. In particular, in the light of Lemma A.10, the encoding of the value $\min\{c^T\bar{x} \mid \bar{x} \in \Omega(Ax \leq b)\}$ is a polynomial in $||\mathcal{L}|| + ||c||$. Thus, $\min\{c^T\bar{x} \mid \bar{x} \in \Omega(Ax \leq b)\}$ can be computed by means of a binary search over the range of all the possible values, where at each iteration, deciding whether $\min\{c^T\bar{x} \mid \bar{x} \in \Omega(\mathcal{L}[x])\} \leq k$ is faced by means of the co-**NP** oracle discussed above. The result then follows since the binary search will converge after a polynomial number of steps. □

Theorem A.12. SUCCINT-FEASIBLEVECTORCOMPUTATION *is in* $\mathbf{F\Delta_2^P}$.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities such that $\Omega(\mathcal{L}[x])$ is a polytope. Consider the vector $c \in \mathbb{R}^n$ whose components are defined as follows: $c_1 = \min\{x_1 \mid x \in \Omega(\mathcal{L}[x])\}$ and $c_j = \min\{x_j \mid x_j \in \Omega(\mathcal{L}[x]) \wedge x_1 = c_1 \wedge \cdots \wedge x_{j-1} = c_{j-1}\}$ for each $2 \leq j \leq n$. Note that the various components can incrementally (i.e., from c_1 to c_n) be computed in $\mathbf{F\Delta_2^P}$ according to the procedure discussed in the proof of Theorem A.11. This completes the proof, since c is a feasible solution. □

Theorem A.13. SUCCINT-OPTIMALVECTORCOMPUTATION *is in* $\mathbf{F\Delta_2^P}$.

Proof. Let $\mathcal{L}$ be a succinct specification for a system of linear inequalities such that $\Omega(\mathcal{L}[x])$ is a polytope. In order to solve the problem, we firstly may compute in $\mathbf{F\Delta_2^P}$ the value $v^* = \min\{c^T\bar{x} \mid \bar{x} \in \Omega(\mathcal{L}[x])\}$ according to the procedure discussed in the proof of Theorem A.11.

Eventually, we may just consider the specification $\mathcal{L}'$ over $m+2$ constraints such that $\mathcal{L}'(i) = \mathcal{L}(i)$ $(1 \leq i \leq m)$, plus $\mathcal{L}'(m+1) = (c^T, v^*)$ and $\mathcal{L}'(m+2) = (-c^T, -v^*)$. Clearly enough, any feasible vector in $\Omega(\mathcal{L}'[x])$ is, by construction, an optimal solution for the linear program over $\mathcal{L}[x]$ and whose aim is to minimize $c^T x$. Thus, the result follows since we can compute in $\mathbf{F\Delta_2^P}$ an arbitrary vector of $\Omega(\mathcal{L}'[x])$ (cf. Theorem A.12). □

B

Infeasibility certificates and the complexity of the core

In Section 3.3.1 we proved that deciding the non-emptiness of the core of a TU game represented in compact form is feasible in co-**NP**. That proof descends from a result we have on succinct linear programs (in particular the problem of deciding the non-emptiness of the feasible region of a linear program encoded in a succinct way), that relies in its turn on Helly's theorem (Helly, 1923; Rabin, 1955).

Historically, such a proof was not the first one we devised for this problem. That time, in the early stage of our research, we have still not analyzed succinct linear programs, and the result for the non-emptiness of the core was obtained through nice geometrical argumentations on polyhedral sets. We next account for that result in its primal form, reporting it here as it appeared, without any modification, in the proceedings of the conference in which that (important) result was presented (Malizia et al., 2007).

In the following we will use the characterization of the core that defines it as the set of all imputations satisfying the following inequalities:

$$\sum_{i \in S} x_i \geq v(S), \quad \forall S \subseteq N \wedge S \neq \varnothing \tag{B.1}$$

$$\sum_{i \in N} x_i \leq v(N). \tag{B.2}$$

where the last inequality, combined with its opposite in (B.1), enforces the feasibility of computed profiles.

B.1 Separating polyhedra

Because of (B.1) and (B.2), the core of a coalitional game with transferable payoffs and n players is a polyhedral set of $\mathbb{R}^n$. In this section, we prove some nice properties of polyhedral sets that will be useful to deal with such games.

B.1.1 Preliminaries on polyhedral sets

We next give some useful definitions and facts about polyhedral sets. We refer the interested reader to any text on this subject for further readings (see, e.g., Grünbaum, 1967; Brøndsted, 1983)).

Let $n > 0$ be any natural number. A *Polyhedral Set* (or *Polyhedron*) P of $\mathbb{R}^n$ is the intersection of a finite set $\mathcal{S}$ of closed halfspaces of $\mathbb{R}^n$. Note that in this paper we always assume, unless otherwise stated, that $n > 0$. We denote this polyhedron by $\mathrm{Pol}(\mathcal{S})$ and we denote $\mathcal{S}$ by $\mathrm{Half}(P)$.

Recall that a *hyperplane* H of $\mathbb{R}^n$ is a set of points $\{x \in \mathbb{R}^n | a^T x = b\}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The closed *halfspace* H^+ is the set of points $\{x \in \mathbb{R}^n | a^T x \geq b\}$. We say that these points *satisfy* H^+. We denote the points that do not satisfy this halfspace by H^-, i.e., $H^- = \mathbb{R}^n \setminus H^+ = \{x \in \mathbb{R}^n | a^T x < b\}$. Note that H^- is an open halfspace. We say that H *determines* H^+ and H^-. Define the *opposite* of H as the set of points $\bar{H} = \{x \in \mathbb{R}^n | a'^T x = b'\}$, where $a' = -1 \cdot a$ and $b' = -1 \cdot b$. Note that $\bar{H}^+ = H^- \cup H$, since it is the set of points $\{x \in \mathbb{R}^n | a^T x \leq b\}$.

Let P be a polyhedron and H a hyperplane. Then, H *cuts* P if both H^+ and H^- contain points of P, and we say that H *passes through* P, if there is a non-empty touching set $C = H \cap P$. Furthermore, we say that H *supports* P, or that it is a *supporting hyperplane* for P, if H does not cut P, but passes through P, i.e., it just touches P, as the only common points of H and P are those in their intersection C.

Moreover, we say that H^+ is a *supporting halfspace* for P if H is a supporting hyperplane for P and $P \subseteq H^+$. Note that $P \subseteq \mathrm{Pol}(\mathcal{S})$ for any set of halfspaces $\mathcal{S} \subseteq \mathrm{Half}(P)$, since the latter polyhedron is obtained from the intersection of a smaller set of halfspaces than P. We say that such a polyhedron is a *supporting polyhedron* for P.

Recall that, for any set $A \subseteq \mathbb{R}^n$, its dimension $\dim(A)$ is the dimension of its affine hull. For instance, if A consists of two points, or it is a segment, its affine hull is a line and thus $\dim(A) = 1$. By definition, $\dim(\varnothing) = -1$, while single points have dimension 0.

A set $F \subseteq P$ is a *face* of P if either $F = \varnothing$, or $F = P$, or if there exists a supporting hyperplane H_F of P such that F is their touching set, i.e., $F = H_F \cap P$. In the latter case, we say that F is a *proper face* of P. A *facet* of P is a proper face of P having the largest possible dimension, that is, whose dimension is $\dim(P) - 1$.

The following facts are well known (Grünbaum, 1967):

1. For any facet F of P, there is a halfspace $H^+ \in \mathrm{Half}(P)$ such that $F = H^+ \cap P$. We say that H^+ *generates* F.
2. For any proper face F of P, there is a facet F' of P such that $F \subseteq F'$.
3. If F and F' are two proper faces of P and $F \subset F'$, then $\dim(F) < \dim(F')$.

Lemma B.1. *Let P be a polyhedron of $\mathbb{R}^n$ with $\dim(P) = n$, and H_F^+ a supporting halfspace of P whose touching set is F. Then, there exists a set of*

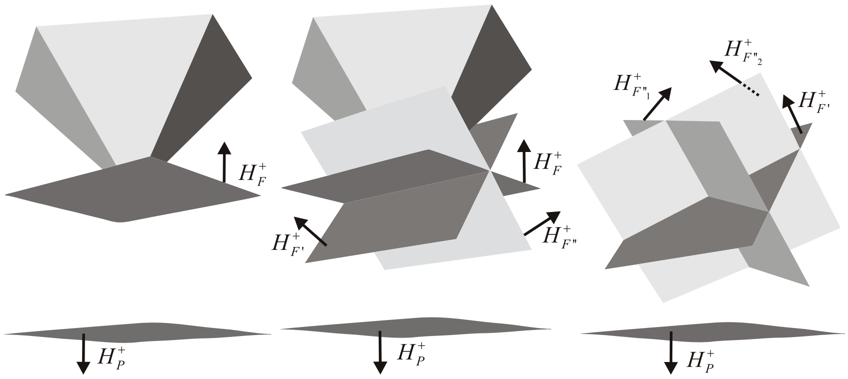


Fig. B.1. Construction of an infeasibility certificate for the core.

halfspaces $\mathcal{H}_F \subseteq \mathrm{Half}(P)$ *such that* $|\mathcal{H}_F| \leq n - \dim(F)$*,* H_F^+ *is a supporting halfspace of* $Pol(\mathcal{H}_F)$*, and their touching set* C *is such that* $F \subseteq C$*.*

Proof. (Rough Sketch.) The proof is by induction. *Base case*: If $\dim(F) = n-1$ we have that the touching face $F = H_F^+ \cap P$ is a facet of P. Thus, from Fact 1, F is generated by some halfspace $H^+ \subseteq \mathrm{Half}(P)$ such that $H^+ \cap P = F$, as for H_F. Since $\dim(F) = \dim(H) = \dim(H_F) = n-1$, it easily follows that in fact $H = H_F$ holds. Thus, H_F^+ is trivially a supporting halfspace of H^+, and this case is proved: just take $\mathcal{H}_F = \{H^+\}$ and note that $|\mathcal{H}_F| = 1$.

Inductive step: By the induction hypothesis, the property holds for any supporting halfspace $H_{F'}^+$ of P such that its touching face F' has a dimension $d \leq \dim(F') \leq n-1$, for some $d > 0$. We show that it also holds for any supporting halfspace H_F^+ of P, whose touching face F has a dimension $\dim(F) = d-1$.

For space limitations, we just give the proof idea, with the help of Figure B.1. Since F is not a facet, from Fact 2 there exists a facet F' of P such that $F \subset F'$. In the three-dimensional example shown in Figure B.1, F is the vertex at the bottom of the diamond, and F' is some facet on its “dark side.” Let $C = H_F \cap H_{F'}$, and consider the rotation of H_F about C on the opposite direction w.r.t. $H_{F'}$ that first touches P, say $H_{F''}$. As shown in Figure B.1, the face F''—an edge of the diamond—properly includes F and its dimension is at least $d > \dim(F)$. It can be shown that, given two halfspaces obtained in such a way, it holds that H_F^+ is a supporting halfspace of the polyhedron $\mathrm{Pol}(H_{F'}^+, H_{F''}^+)$, called its roof. Formally, the proof proceeds by exploiting the induction hypothesis. Intuitively, consider $H_{F''}$: we want a set $\mathcal{H}_F$ supported by H_F^+ and consisting of just halfspaces taken from $\mathrm{Half}(P)$, and $H_{F''}^+$ does not belong to this set, because it does not generate a facet of P. However, we can see that it is a supporting halfspace for $H_{F''_1}^+ \cap H_{F''_2}^+$—its roof, which correspond to faces having higher dimension than F''. In the running example, they are both facets of the diamond, and hence the property immediately holds (base case). In general, the procedure may continue, encountering each time at least one facet and one more face with a higher dimension than the current one. Eventually, in our example we get $\mathcal{H}_F = \{H^+_{F'}, H^+_{F''_1}, H^+_{F''_2}\}$. Moreover, recall that $\dim(F') = n-1$ and $\dim(F'') > \dim(F) = d-1$. Then, by the induction hypothesis, $|\mathcal{H}_F| \leq |\mathcal{H}_{F'}| + |\mathcal{H}_{F''}| = 1 + |\mathcal{H}_{F''}| \leq 1 + n - dim(F'') \leq 1 + n - d = n - \dim(F)$. □

B.2 Small emptiness certificates for the core

In this section, we prove our main results on TU games.

With a little abuse of notations, since coalitions correspond to the inequalities (B.1) and hence with the associated halfspaces of $\mathbb{R}^n$, we use hereafter interchangeably these terms.

Definition B.2. *Let $\mathcal{G} = \langle N, v\rangle$ be a game with transferable payoffs. A coalition set $\mathcal{S} \subseteq 2^N$ is a* certificate of emptiness *(or* infeasibility certificate*) for the core of $\mathcal{G}$ if the intersection of* Pol($\mathcal{S}$) *with the grand-coalition halfspace* (B.2) *is empty.*

The definition above is motivated by the following observation. Let P be the polyhedron of $\mathbb{R}^n$ obtained as the intersection of all halfspaces (B.1). Since $\mathcal{S}$ is a subset of all possible coalitions, $P \subseteq \mathrm{Pol}(\mathcal{S})$. Therefore, if the intersection of Pol($\mathcal{S}$) with the grand-coalition halfspace (B.2) is empty, the intersection of this halfspace with P is empty, as well.

Theorem B.3. *Let $\mathcal{G} = \langle N, v\rangle$ be a game with transferable payoffs. If the core of $\mathcal{G}$ is empty, there is a certificate of emptiness $\mathcal{S}$ for it such that $|\mathcal{S}| \leq |N|$.*

Proof. (Sketch.) Let $n = |N|$ and P be the polyhedron of $\mathbb{R}^n$ obtained as the intersection of all halfspaces (B.1). Since we are not considering the feasibility constraint (B.2), there is no upper-bound on the values of any variable x_i, and thus it is easy to see that $P \neq \varnothing$ and $\dim(P) = n$.

Let H^+_P be the halfspace defined by the grand-coalition inequality (B.2). If the core of $\mathcal{G}$ is empty, the whole set of inequalities has no solution, that is, $P \cap H^+_P = \varnothing$.

Let $\bar{H}^+_F$ be the halfspace parallel to H^+_P that first touches P, that is, the smallest relaxation of H^+_P that intersect P. Consider the opposite H^+_F of $\bar{H}^+_F$, as shown in Figure B.1, on the left. By construction, $H^+_P \cap H^+_F = \varnothing$, $H_F = \bar{H}_F$ is a supporting hyperplane of P, and H^+_F is a supporting halfspace of P. Let F be the touching set of H_F with P, and let $d = \dim(F)$. In Figure B.1, it is the vertex at the bottom of the diamond P. From Lemma B.1, there is a set of halfspaces $\mathcal{S} \subseteq \mathrm{Half}(P)$, with $|\mathcal{S}| \leq n - d$, such that H^+_F is a supporting halfspace for Pol($\mathcal{S}$). It follows that $H^+_P \cap \mathrm{Pol}(\mathcal{S}) = \varnothing$, whence $\mathcal{S}$ is an infeasibility certificate for the core of $\mathcal{G}$. Finally, note that the largest cardinality of $\mathcal{S}$ is n, and corresponds to the case $\dim(F) = 0$, that is, to the case where the face F is just a vertex. Therefore the maximum cardinality of the certificate is n. In our three-dimensional example, the certificate is $\{H^+_{F'}, H^+_{F''_1}, H^+_{F''_2}\}$, as shown in Figure B.1, on the right. □

Note that the above proof is constructive and has a nice geometrical interpretation. Exploiting the above property, we can now state the precise complexity of core non-emptiness for any (non-deterministic) polynomial-time compact representation.

Theorem B.4. *Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation. Given any coalitional game with transferable payoffs $\mathcal{G} \in \mathcal{C}(\mathcal{R})$, deciding whether the core of $\mathcal{G}$ is not empty is* co-**NP**-*complete. Hardness holds even if $\mathcal{R}$ is a deterministic polynomial-time compact representation.*

Proof. Hardness trivially follows from the hardness results in (Deng and Papadimitriou, 1994).

Membership in co-**NP**. Let $\mathcal{G} = \langle N, v\rangle$ be a game with transferable payoffs. If its core *is* empty, from Theorem B.3, there is an infeasibility certificate $\mathcal{S}$, with $|\mathcal{S}| \leq n$, where n is the number of players of $\mathcal{G}$. For the sake of presentation, let us briefly sketch the case of a polynomial-time deterministic representation $\mathcal{R}$. In this case, a non-deterministic Turing machine in polynomial time may check that the core is empty performing the following operations: (i) guess of the set $\mathcal{S}$, i.e., of the coalitions of players corresponding to the halfspaces in $\mathcal{S}$; (ii) computation in deterministic polynomial time of the worth $v(S)$, for each $S \in \mathcal{S}$, and for the grand-coalition N; and (iii) checking that $\mathrm{Pol}(\mathcal{S}) \cap H^+_P = \varnothing$, where H^+_P is the halfspace defined by the grand-coalition inequality (B.2). Note that the last step is feasible in polynomial time, as we have to solve a system consisting of just $n+1$ linear inequalities.

The case of a non-deterministic polynomial-time compact representation $\mathcal{R}$ is a simple variation where, at step (ii), for each $S \in \mathcal{S}$, the machine should also guess the value $w = v(S)$ and a witness y that $\langle \mathcal{G}, S, w\rangle \in W_{\mathcal{C}(\mathcal{R})}$. □

Corollary B.5. *Given a coalitional game with transferable payoffs encoded as a marginal contribution net, deciding whether its core is not empty is* co-**NP**-*complete.*

C

Details for the nucleolus computation algorithm depicted in Section 3.3.4

We are going to give some details on two important points of the algorithm, proposed in Section 3.3.4, to compute the nucleolus of a game represented in any **FP** scheme (and, after the discussion of Section 3.3.5, also in any **FNP** scheme).

Let $\mathcal{G} = \langle N, v \rangle$ be a TU game. Recall the sequence, for $1 \leq k \leq k_*$, of linear programs $\mathtt{LP}_k$ that have to be solved in order to compute the nucleolus of $\mathcal{G}$:

$$\begin{aligned}
\mathtt{LP}_k = \{\min \epsilon \mid\ & x(S) = v(S) - \epsilon_r, && \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1 && \text{(C.1)}\\
& x(S) \geq v(S) - \epsilon, && \forall S \subseteq N, S \notin \mathcal{F}_{k-1} && \text{(C.2)}\\
& x \in X(\mathcal{G})\}. && && \text{(C.3)}
\end{aligned}$$

We claimed in Section 3.3.4 that there is the possibility of avoiding the explicit and complete representation of sets Λ_r, for all $1 \leq r \leq k_1$, and $\mathcal{F}_{k-1}$. We said that all the exponentially many constraints (C.1) can be substituted by a *polynomial* subset of them without altering the result of the computation, and that we can decide in polynomial time whether a coalition S belongs to $\mathcal{F}_{k-1}$ giving us the opportunity of not explicitly represent that set. These two tricks allow us to encode the program $\mathtt{LP}_k$ in succinct form.

In the following, our argumentations will rely heavily on concepts of polyhedral geometry illustrated throughout Appendix A, where the reader is referred especially for what concerns the concept of *basis* of the implied equalities of a system of linear inequalities.

C.1 Dealing with Λ_r

Consider at first the issue of Λ_r. Let us focus our attention on the set V_{k-1} of optimal solutions of $\mathtt{LP}_{k-1}$. We know that V_{k-1} equals the feasible region of $\mathtt{LP}_{k-1}$ with $\epsilon = \epsilon_{k-1}$, where ϵ_{k-1} is the optimal value of $\mathtt{LP}_{k-1}$. Hence we have that

$$
\begin{aligned}
V_{k-1} = \{x \in \mathbb{R}^n | x \in X(\mathcal{G}) \wedge \\
& x(S) \geq v(S) - \epsilon_r, && \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1 \wedge \\
& x(S) \geq v(S) - \epsilon_r, && \forall S \in (\mathcal{F}_r \setminus \Lambda_r), \forall 1 \leq r \leq k-1 \wedge \\
& x(S) \geq v(S) - \epsilon_{k-1}, && \forall S \subseteq N, S \notin \mathcal{F}_{k-1}\}.
\end{aligned}
$$

In each step r, the coalitions belonging to Λ_r are those ones whose excess is constant and is ϵ_r. Hence, the set $ie(V_{k-1})$ of the implied equalities of V_{k-1} is formed up exactly by those inequalities regarding the coalitions belonging to some Λ_r for some r. We highlight this fact by rewriting V_{k-1} in

$$
\begin{aligned}
V_{k-1} = \{x \in \mathbb{R}^n | x \in X(\mathcal{G}) \wedge \\
& x(S) = v(S) - \epsilon_r, && \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1 \wedge \\
& x(S) \geq v(S) - \epsilon_r, && \forall S \in (\mathcal{F}_r \setminus \Lambda_r), \forall 1 \leq r \leq k-1 \wedge \\
& x(S) \geq v(S) - \epsilon_{k-1}, && \forall S \subseteq N, S \notin \mathcal{F}_{k-1}\}.
\end{aligned}
\tag{C.4}
$$

We note that all the implied equalities of V_{k-1} are also constraints in the linear program $\mathtt{LP}_k$. In principle $ie(V_{k-1})$ may contain exponentially many implied equalities. We can avoid to represent explicitly $ie(V_{k-1})$ by substituting it with one of its (polynomial) basis, say $\mathcal{B}_{k-1}$, and obtain a new linear program called $\mathtt{LP}'_k$. In this way we can replace the whole set of constraints (C.1) with an object having only polynomial size.

We ask, now, whether this substitution alters the computation of the nucleolus. Since $\mathcal{B}_{k-1}$ is a basis of $ie(V_{k-1})$, it is a maximal subset of linear independent implied equalities of $ie(V_{k-1})$. Hence all implied equalities in $ie(V_{k-1}) \setminus \mathcal{B}_{k-1}$ can be written as a linear combination of those belonging to $\mathcal{B}_{k-1}$. By this, all the constraints (C.1) in $\mathtt{LP}_k$ are linear combination of that belonging to $\mathcal{B}_{k-1}$. This means that a vector satisfies the constraints (C.1) of $\mathtt{LP}_k$ if and only if it satisfies the polynomially many ones of $\mathcal{B}_{k-1}$. Thus the substitution of the constraints (C.1) of the linear program $\mathtt{LP}_k$ with the constraints of $\mathcal{B}_{k-1}$ does not alter its feasible region implying that the program $\mathtt{LP}'_k$ is totally equivalent to $\mathtt{LP}_k$.

C.2 Dealing with $\mathcal{F}_{k-1}$

Let us turn now our attention to the issue of $\mathcal{F}_{k-1}$. We denote by $\mathbb{1}_S$ the *indicator vector* of S, that is a vector in $\mathbb{N}^n$ such that its component $\mathbb{1}_{S_i}$ is 1 if and only if $i \in S$, and 0 otherwise. Let $\mathcal{S}$ be a set of coalitions. We say that a coalition S is a linear combination of $\mathcal{S}$ if the vector $\mathbb{1}_S$ can be written as a linear combination of the indicator vectors of the coalitions in $\mathcal{S}$.

Note that our linear programs have all their inequalities related to some coalition. Let I be a set of inequalities coming from our linear programs and hence that are related to coalitions. We denote by $\mathcal{C}(I)$ the set of coalitions whose related inequalities (or constraints) belong to I. Let $\mathbb{I}_{k-1}$ be the set

$\{\mathbb{1}_S | S \in \mathcal{C}(\mathcal{B}_{k-1})\}$ where $\mathcal{B}_{k-1}$ is a basis of $ie(V_{k-1})$. We can state the following result.

Lemma C.1. *Let $\mathcal{B}_{k-1}$ be a basis of $ie(V_{k-1})$. $S \in \mathcal{F}_{k-1}$ if and only if it is a linear combination of $\mathbb{I}_{k-1}$.*

Proof.

($\Rightarrow$) Let $x_{k-1} \in V_{k-1}$ be an optimal point of the program $\mathtt{LP}_k$. The coalitions $S \in (\mathcal{F}_r \setminus \Lambda_r)$, for all $1 \leq r \leq k-1$, are those ones with a constant excess, different from ϵ_r and in particular strictly less than ϵ_r, for all $x \in V_r \supseteq V_{k-1}$. Their excess is such that the equality $x(S) = v(S) - e(S, x_{k-1})$ holds for all $x \in V_r$. We rewrite the set V_{k-1} (cf. equation (C.4)) in the following form in order to highlight also these new equalities:

$$\begin{aligned} V'_{k-1} = \{x \in \mathbb{R}^n | x \in X(\mathcal{G}) \wedge \\ & x(S) = v(S) - \epsilon_r, && \forall S \in \Lambda_r, \forall 1 \leq r \leq k-1 \wedge \\ & x(S) = v(S) - e(S, x_{k-1}), && \forall S \in (\mathcal{F}_r \setminus \Lambda_r), \\ & && \forall 1 \leq r \leq k-1 \wedge \\ & x(S) \geq v(S) - \epsilon_{k-1}, && \forall S \subseteq N, S \notin \mathcal{F}_{k-1}\}. \end{aligned} \tag{C.5}$$

Obviously, even if the formulations of V_{k-1} and V'_{k-1} are different, these two sets contain exactly the same points. The equality $V_{k-1} = V'_{k-1}$ implies that $dim(V_{k-1}) = dim(V'_{k-1})$. Note that the coefficient of the variables in the inequalities $ie(V_{k-1})$ and $ie(V'_{k-1})$ are only 0's and 1's, as indicator vectors. And in fact from this observation and the definition of dimension, $rank(\{\mathbb{1}_S | S \in \mathcal{C}(ie(V_{k-1}))\}) = rank(\{\mathbb{1}_S | S \in \mathcal{C}(ie(V'_{k-1}))\})$ holds. By Lemma A.7.(2) $rank(\mathbb{I}_{k-1}) = rank(\{\mathbb{1}_S | S \in \mathcal{C}(ie(V_{k-1}))\})$. By this, and the fact that $\mathcal{C}(ie(V'_{k-1})) = \mathcal{F}_{k-1}$, it follows that

$$rank(\mathbb{I}_{k-1}) = rank(\{\mathbb{1}_S | S \in \mathcal{F}_{k-1}\}).$$

Thus every coalition $S \in \mathcal{F}_{k-1}$ is a linear combination of $\mathbb{I}_{k-1}$.

($\Leftarrow$) Let $T^1, \ldots, T^q$ be the coalitions such that $\mathbb{1}_{T^i} \in \mathbb{I}_{k-1}$, and assume that S is a linear combination of $\mathbb{I}_{k-1}$. Then, it is the case that exist real numbers $\lambda_1, \ldots, \lambda_q$ such that

$$\mathbb{1}_S = \sum_{i=1}^{q} \lambda_i \cdot \mathbb{1}_{T^i}.$$

This implies that $\sum_{i \in S} x_i = \sum_{i=1}^{q} (\lambda_i \cdot \sum_{j \in T^i} x_j)$ (or that is the same $x(S) = \sum_{i=1}^{q} \lambda_i \cdot x(T^i)$). Since $x(T^i)$ is a constant for all $x \in V_{k-1}$, and for all $1 \leq i \leq q$, then $x(S)$ is also a constant, for all $x \in V_{k-1}$. Hence, $e(S, x) = v(S) - x(S)$ is a constant for all $x \in V_{k-1}$, and thus by definition of $\mathcal{F}_{k-1}$ the coalition S belongs to it. $\square$

By the previous Lemma, we can check whether a coalition S belongs to $\mathcal{F}_{k-1}$ by simply checking whether S is a linear combination of $\mathbb{I}_k$. And since $\mathbb{I}_{k-1}$ contains only polynomially many indicator vectors, the linear dependency test is feasible in polynomial time.

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