UNIVERSITÀ DELLA CALABRIA

Università degli Studi della Calabria

Dipartimento di Matematica e Informatica

Dottorato di Ricerca in Matematica e Informatica
XXXIII ciclo

Titolo Tesi

Surfaces with Prym-canonical hyperplane sections

Settore Scientifico Disciplinare: MAT/03 Geometria

Supervisori:

Ch.mo Prof. Ciro Ciliberto

Prof.ssa Concettina Galati

Cosupervisore:

Ch.mo Prof. Andreas Leopold Knutsen

Dottorando:

Dott.ssa
Martina Anelli

GALATI
CONCETTINA
17.05.2021
13:05:21
UTC

Contents

Abstract

Uno dei principali problemi della geometria algebrica è la classificazione delle varietà algebriche a meno di isomorfismi o di equivalenza birazionale. Mentre il problema della classificazione di curve algebriche è essenzialmente risolto, il problema della classificazione di superfici presenta ancora qualche area sconosciuta.

L'argomento di ricerca discusso in questa tesi rientra in quello della classificazione di superfici proiettive complesse con sezioni iperpiane Prym-canoniche. I soli esempi conosciuti di questo tipo di superficie sono la superficie di Enriques ed una superficie in $\mathbb{P}^5$ di grado 10 ottenuta come immersione dello scoppiamento di $\mathbb{P}^2$ nei 10 nodi di una curva piana razionale irriducibile di grado 6.

Noi diciamo che una superficie X ha sezioni iperpiane Prym-canoniche se può essere realizzata birazionalmente in qualche spazio proiettivo $\mathbb{P}^{g-1}$, per $g \geq 5$, tale che una generica sezione iperpiana C di X è una curva liscia di genere g immersa Prym-canonicamente.

Mostreremo che una superficie con sezioni iperpiane Prym-canoniche può essere birazionalmente equivalente o ad una superficie di Enriques o a $\mathbb{P}^2$, ed in tal caso può contenere soltanto punti doppi razionali come singolarità, oppure ad una superficie rigata su una curva base di genere $q \geq 0$. In quest'ultimo caso, la somma dei generi geometrici delle singolarità di X è uguale al genere geometrico della curva base q. La proprietà generale di queste superfici è che, se $\pi : X' \to X$ è la risoluzione minimale delle singolarità di X, allora esiste solo un divisore antibicanonico effettivo su X' il cui supporto è contenuto nel luogo eccezionale di π.

Dal momento che le superfici di Enriques sono già state studiate da diversi autori, costruiremo nuove superfici con sezioni iperpiane Prym-canoniche birazionalmente equivalenti a superfici rigate o a $\mathbb{P}^2$. Il metodo per costruire esempi di questo tipo di superfici consiste nel trovare sistemi lineari L'' su superfici minimali X'' (superfici rigate o $\mathbb{P}^2$) tali che, dopo aver scoppiato tutti i punti base di L'' per ottenere X', la trasformata stretta L' di L'' è disgiunta dal solo divisore antibicanonico di X' mentre il divisore anticanonico di X' ristretto ad una generica curva di L' è un divisore di torsione non-nullo.

Introduction

One of the main problems of algebraic geometry is to classify algebraic varieties up to isomorphisms or to birational equivalence. While the classification of algebraic curves is essentially complete, the problem of classification of surfaces still presents some unknown areas. In 1914, Enriques classified all the smooth surfaces into four large classes; two surfaces in the same class have the same invariant, called "Kodaira dimension".

The research problem discussed in this thesis is the classification of complex projective surfaces with Prym-canonical hyperplane sections. The idea of this work is inspired by the PhD Thesis of Dick Epema entitled "Surfaces with canonical hyperplane sections" (see [13], [14]).

We say that a surface X has Prym-canonical hyperplane sections if it can be birationally realized in some projective space $\mathbb{P}^{g-1}$, for $g \geq 5$, such that a general hyperplane section C of X is a smooth Prym-canonical embedded curve of genus g.

We will show that a surface with Prym-canonical hyperplane sections can be birationally equivalent to an Enriques surface or to $\mathbb{P}^2$ or to a ruled surface over a base curve of genus $q \geq 0$. In the last case, the sum of the geometric genus of the singularities of X is equal to the genus of the base curve q while, in the other cases, the surface X can only contain rational double points as singularities. The general property of these surfaces is that, if $\pi : X' \to X$ is the minimal resolution of the singularities of X, then there exists only one effective antibicanonical divisor on X' with support contained in the exceptional locus of π.

Since Enriques surfaces have already been studied in details by several authors, we will only construct new surfaces with Prym-canonical hyperplane sections birationally equivalent to ruled surfaces or $\mathbb{P}^2$. The method to construct examples of this type of surfaces consists in constructing linear systems L'' on minimal surfaces X'' (ruled surfaces or $\mathbb{P}^2$) such that, after blowing up all the base points of L'' to obtain X', the strict transform L' of L'' is disjoint from the only antibicanonical divisor of X' while the anticanonical divisor of X' restricted to a general curve of L' is a non-zero two torsion divisor.

More precisely, in Chapter 1 we will study general properties of surfaces with Prym-canonical hyperplane sections. After a brief introduction, we will focus on the study of the possible singularities on this type of surfaces.

In Chapters 2 and 3, we will construct new examples of these surfaces, respectively birationally equivalent to ruled surfaces or $\mathbb{P}^2$. In particular, in Chapter 2, we will recall some properties regarding ruled surfaces and, after studying the possible non-rational singularities on them, we will construct other new examples. We will compute how many moduli one of these new examples depends on, concluding that it can be a degeneration of a generic Enriques surface. Moreover, we will give a list in terms of invariants of some possible surface with Prym-canonical hyperplane sections.

In Chapter 3, we will review the only known example of surface with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$ and we will construct a new example. Also in this case, we will give a list in terms of numerical values of some possible surface with Prym-canonical hyperplane sections.

In the last Chapter, we will only consider surfaces with Prym-canonical hyperplane sections with specific assumptions. Concerning these, we will give an upper bound to the number of moduli of these surfaces birationally equivalent to rational ruled surfaces. Whereas, for the case of surfaces birationally equivalent to non-rational ruled surfaces, we will only give an upper bound for the number of moduli of their hyperplane sections. This is still a work in progress.

Terminology

In the next sections, we will work over $\mathbb{C}$, the field of the complex numbers.

Later on, we will denote a divisor and its associated line bundle with the same symbol.

We will use the classical language of algebraic geometry. Below, we list some symbols and notation:

K_X = canonical divisor of a smooth variety X;

$\sim$ = linear equivalence of divisors;

$\equiv$ = numerical equivalence of divisors;

g^r_d = a linear system of dimension r and degree d on a curve;

$C \cdot D$, D^2 = intersection, respectively self-intersection number of divisors C, D on a surface;

$|D|$ = the complete linear system of which D is a member;

$H^i(F)$ = the $i-th$ cohomology group of the coherent sheaf F;

$h^i(F)$ = dim $H^i(F)$;

$\mathcal{I}_Z$ = ideal of a set Z of points;

Cliff(C) = Clifford index of a smooth curve C;

$q(X)$ = irregularity $h^1(\mathcal{O}_X)$ of a variety X;

$p_a(X)$ = arithmetic genus of a variety X;

$p_g(X)$ = geometric genus of a variety X;

$\wedge^2\mathcal{E}$ = second exterior power of the sheaf $\mathcal{E}$;

$\mathbb{P}_X(\mathcal{E})$ = projective space bundle associated with a locally free coherent sheaf $\mathcal{E}$ on a variety X;

supp(D), supp(F) = support of a divisor D respectively a sheaf F;

$(-i) -$ curve = curve isomorphic to $\mathbb{P}^1$ such that its self-intersection number is equal to $-i$;

$\mathbb{N}$ = set of natural numbers, including 0;

$\mathbb{N}_{>0}$ = set of natural numbers without 0;

$N_{Y|X}$ = normal sheaf of a nonsingular subvariety Y on a nonsingular variety X;

Pic(X) = group of isomorphism classes of invertible sheaves on X, called Picard group of X;

Pic$^0(X)$ = subgroup of $Pic(X)$ of invertible sheaf of degree 0;

Sing(X) = set of singular points of a variety X;

$R^i f_* \mathcal{F}$ = i-th direct image of a sheaf $\mathcal{F}$ relative to the morphism f;

$S^a \mathcal{E}$ = $a-$th symmetric product of the sheaf $\mathcal{E}$, with $a \in \mathbb{N}_{>0}$.

1 Surfaces with Prym-canonical hyperplane sections

In this first chapter, we want to study the general properties of surfaces with Prym-canonical hyperplane sections, that are the subject of our work. In particular, we will give some information about the singularities that occur on them.

1.1 Properties of surfaces with Prym-canonical hyperplane sections

Let us start with the definition of Prym-canonical curve. By *curve* we will mean an irreducible projective algebraic variety of dimension 1 over $\mathbb{C}$.

Definition 1.1. *A divisor $\alpha \in \mathrm{Pic}^0(C)$ such that $\alpha \nsim 0$ but $2\alpha \sim 0$ is called* non-zero two torsion divisor*.*

Definition 1.2. *Let $g \geq 3$. A* Prym curve *is a pair (C, α), where C is a smooth genus $g = p_g(C)$ curve and α is a non-zero $2-$torsion point of* $\mathrm{Pic}^0(C)$*.*

We can consider the so called *Prym-canonical map*, that is the rational map

$$\phi_{|K_C \otimes \alpha|} : C \to \mathbb{P}^{g-2}$$

defined by $|K_C \otimes \alpha|$. The pair $(C, K_C \otimes \alpha)$ is called *Prym-canonical curve.*

Remark 1.3. *If $g = p_g(C) = 3$, then the map $\phi_{|K_C \otimes \alpha|} : C \to \mathbb{P}^1$ cannot be an embedding.*
Let C be a smooth curve of genus $g = 4$. If we suppose that the morphism $\phi_{|K_C \otimes \alpha|} : C \to \overline{C} \subset \mathbb{P}^2$ is an embedding, then $deg(\overline{C}) = deg(K_C \otimes \alpha) = 2g - 2 = 6$. Since $\overline{C}$ is smooth, we have that $p_g(\overline{C}) = p_a(\overline{C}) = \frac{5 \cdot 4}{2} = 10$ by the Plücker formula. This is a contradiction, so the Prym-canonical map can be an embedding only if $g \geq 5$.

In the following, we will assume that the Prym-canonical map is an embedding, so $g \geq 5$.

We recall Lemma 2.1 proved in [6], that we will use later.

Lemma 1.4. *The Prym-canonical system $|K_C \otimes \alpha|$ is base point free if and only if, for any point $p \in C$, the linear system $|p + \alpha| = \varnothing$. Moreover $|p + \alpha| \neq \varnothing$ if and only if C is hyperelliptic and $\alpha \sim \mathcal{O}_C(p - q)$, with p and q ramification points of the g^1_2.*
If $|K_C \otimes \alpha|$ is base point free, then the divisor $K_C \otimes \alpha$ is very ample (i.e. the morphism $\phi_{|K_C \otimes \alpha|} : C \to \mathbb{P}^{g-2}$ is an embedding) if and only if $|p + q + \alpha| = \varnothing$, for any two points $p, q \in C$ (also $p = q$). Moreover $|p + q + \alpha| \neq \varnothing$ if and only if C has a g^1_4 and $\alpha \sim \mathcal{O}_C(p + q - x - y)$, where $2(p + q)$ and $2(x + y)$ are members of the g^1_4.

Let X be a *projective algebraic surface*, that is an irreducible projective algebraic variety of dimension 2 over $\mathbb{C}$.

Definition 1.5. *A surface X is called* surface with Prym-canonical hyperplane sections *if there is an embedding $i_L : X \hookrightarrow \mathbb{P}^{g-1}$ defined by the linear system L such that, if C is a general hyperplane section of X, then the restriction $i_L|_C : C \hookrightarrow \mathbb{P}^{g-2}$ is a Prym-canonical embedding.*

We will simply write X instead of $i_L(X)$. Because a general hyperplane section $C \subseteq X$ is Prym-canonically embedded in $\mathbb{P}^{g-2}$, then $p_g(C) = g$.

By definition, the linear system $|K_C \otimes \alpha|$ associated with $i_L|_C$ is complete. We can prove that L is also complete. Indeed, if $V \subseteq H^0(\mathcal{O}_X(1))$ corresponds to the morphism i_L and $V_C = H^0(\mathcal{O}_C(1))$ (complete) corresponds to the morphism $i_L|_C$, then we have a surjection $V \twoheadrightarrow V_C$, whose kernel is $\mathbb{C}$. So $\dim V = 1 + \dim V_C = 1 + h^0(\mathcal{O}_C(1))$. From the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_C(1) \to 0,$$

we have that $1 + h^0(\mathcal{O}_C(1)) \geq h^0(\mathcal{O}_X(1))$, then

$$\dim V \geq h^0(\mathcal{O}_X(1)).$$

On the other hand, because $V \subseteq H^0(\mathcal{O}_X(1))$, we have

$$\dim V \leq h^0(\mathcal{O}_X(1)).$$

Thus $V = H^0(\mathcal{O}_X(1))$ and L is also complete.

Remarks 1.6.

- *The generic hyperplane section of X is irreducible and smooth, whence X has at most isolated singularities.*
- *Let $\pi : X' \to X$ be the minimal resolution of singularities of X and let $C' = \pi^* C$ be the inverse image of a general hyperplane section. Then π is an isomorphism in a neighbourhood of C, so a general hyperplane section C' of X' is also Prym-canonical embedded, i.e. $\mathcal{O}_{X'}(C') \otimes \mathcal{O}_{C'} \cong \mathcal{O}_{C'}(K_{C'} + B)$, with B a non trivial two-torsion element of $\mathrm{Pic}^0(C')$. By the adjunction formula and since X' is smooth, we can say that $B = -K_{X'}|_{C'}$, in particular $K_{X'} \cdot C' = 0$.*

We recall the definition of Clifford index.

Recall 1.7. *Let C be a smooth curve of genus $g \geq 2$. The* Clifford index of a line bundle *A on C is defined by*

$$\mathrm{Cliff}(A) = \deg(A) - 2(h^0(A) - 1).$$

The Clifford index of *C is defined by*

$$\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(A) \mid h^0(A) \geq 2,\ h^1(A) \geq 2\}.$$

If C is a smooth curve of genus g, then $\text{Cliff}(C) \leq |\frac{g-1}{2}|$. If C is general, equality holds.

If we suppose that $i_{L|_C} : C \hookrightarrow \mathbb{P}^{g-2}$ is a Prym-canonical embedding and the *Clifford index* $\text{Cliff}(C) \geq 3$ (if C is a general curve, then it happens if and only if $g \geq 7$), then C is projectively normal with respect to the given embedding. This is a consequence of the following Theorem, applied to the case $B = K_C \otimes \alpha$ (see [18], Theorem 1):

Theorem 1.8. *Let C be a smooth irreducible complex projective curve of genus $g \geq 2$ and let B be a very ample line bundle on C. If $\deg(B) \geq 2g+1-2h^1(\mathcal{O}_C(B))-\text{Cliff}(C)$, then B embeds C as a projectively normal curve.*

In general, there are also projectively normal curves $C \subseteq \mathbb{P}^{g-2}$ with $\text{Cliff}(C) < 3$. From now on, we will assume that C embedded in $\mathbb{P}^{g-2}$ is projectively normal.

We will see a first important result on surfaces with Prym-canonical hyperplane sections after the following recall.

Recall 1.9. *Let X' be a smooth surface. We recall that the* Kodaira dimension *of X' is*

$$\kappa(X') = max\{\dim \ \phi_{|nK_{X'}|}(X'), n \in \mathbb{N}\},$$

where $\phi_{|nK_{X'}|}$ is the map associated with the pluricanonical linear system.

Theorem 1.10. *Let X be a surface with Prym-canonical hyperplane section C of genus $g \geq 5$ and let $\pi : X' \to X$ be the minimal resolution of its singularities. If C is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$, then:*

1. *$h^1(\mathcal{O}_X(n)) = 0$ and $h^2(\mathcal{O}_X(n)) = 0$ for any $n \geq 0$, in particular $h^1(\mathcal{O}_X) = 0$ and $h^2(\mathcal{O}_X) = 0$, whence $p_a(X) = 0$;*
2. *X is projectively normal with respect to its embedding in $\mathbb{P}^{g-1}$;*
3. *the Kodaira dimension $\kappa(X')$ equals to $-\infty$ or 0; in the second case, X' is a minimal Enriques surface;*
4. *$\deg(X) = 2g - 2$.*

Proof. 1. By assumption, C is a projectively normal curve, that means that

$$H^0(\mathcal{O}_{\mathbb{P}^{g-2}}(n)) \twoheadrightarrow H^0(\mathcal{O}_C(n)),$$

for all $n \geq 0$. Thus also $H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) \twoheadrightarrow H^0(\mathcal{O}_C(n))$, for any $n \geq 0$. Since $H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) \twoheadrightarrow H^0(\mathcal{O}_C(n))$ is equal to $H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) \to H^0(\mathcal{O}_X(n))$ composed with $H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_C(n))$, we have that the map $H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_C(n))$ is surjective for any $n \geq 0$.

Let us consider the exact sequence

$$0 \to \mathcal{O}_X(n-1) \to \mathcal{O}_X(n) \to \mathcal{O}_C(n) \to 0. \tag{1}$$

Then, the second part of the long exact sequence associated with (1) is

$$0 \to H^1(\mathcal{O}_X(n-1)) \to H^1(\mathcal{O}_X(n)) \to H^1(\mathcal{O}_C(n)) \to H^2(\mathcal{O}_X(n-1)) \to$$
$$\to H^2(\mathcal{O}_X(n)) \to 0. \qquad (2)$$

By Serre's Theorem, there is an integer n_0 such that $h^i(\mathcal{O}_X(n)) = 0$ for each $i > 0$ and each $n \geq n_0$. Then $h^1(\mathcal{O}_X(n)) = 0$ for a sufficiently large n. From the exact sequence (2), also $h^1(\mathcal{O}_X(n-1)) = 0$. Applying descending induction on n, we obtain that $h^1(\mathcal{O}_X(n)) = 0$ for any $n \geq 0$.

It is clear that $H^1(\mathcal{O}_C(n)) = H^1(\mathcal{O}_C(n(K_C + \alpha)))$. For $n = 1$, we have that $h^1(\mathcal{O}_C(K_C + \alpha)) = h^0(\mathcal{O}_C(-\alpha)) = 0$. Since $\deg(K_C + \alpha) = 2g - 2$, then $\deg(n(K_C + \alpha)) > 2g - 2$ for $n \geq 2$ and it is first cohomology has dimension 0 (see [20], Example $IV.$1.3.4). Again by Serre's Theorem, we have $h^2(\mathcal{O}_X(n)) = 0$ for $n >> 0$. Applying descending induction on n, from the long exact sequence (2) we can conclude that $h^2(\mathcal{O}_X(n)) = 0$ for any $n \geq 0$.

2. To prove that X is projectively normal, it is enough to show that X is normal and that the map $H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) \to H^0(\mathcal{O}_X(n))$ is surjective for any $n \geq 0$.

   Let $\eta : \widetilde{X} \to X$ be the normalization of X. We consider the following exact sequence on X:
$$0 \to \mathcal{O}_X \to \eta_* \mathcal{O}_{\widetilde{X}} \to F \to 0, \qquad (3)$$
   where $\text{supp}(F) \subset \text{Sing}(X)$. Since X has isolated singularities ($\text{Sing}(X) = \{x_1, ..., x_r\}$), then $F \cong H^0(F) = \oplus_{i=1}^r (\widetilde{\mathcal{O}_i}/\mathcal{O}_i)$, where $\mathcal{O}_i$ is the local ring of x_i on X and $(\eta_* \mathcal{O}_{\widetilde{X}})_{x_i} = \widetilde{\mathcal{O}_i}$ is the normalization of $\mathcal{O}_i$ in the function field of X, with $\text{Sing}(X) = \{x_1, ..., x_r\}$.

   We know that $H^0(\mathcal{O}_X) = k$ because X is irreducible. By the properties of push-forward and because $\widetilde{X}$ is still irreducible, it is obvious that $H^0(\eta_* \mathcal{O}_{\widetilde{X}}) \cong H^0(\mathcal{O}_{\widetilde{X}}) \cong k$. Moreover $h^1(\mathcal{O}_X) = 0$ by the previous part of this Proposition. For the long exact sequence associated with (3) we have that $h^0(F) = 0$ and by definition of F it is true that $\widetilde{\mathcal{O}_i} \cong \mathcal{O}_i$, for any $i = 1, ..., r$. We conclude that X is normal.

   The surjectivity of $H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) \to H^0(\mathcal{O}_X(n))$ is trivial for $n = 0$. Let us consider the following diagram, where H is a general hyperplane in $\mathbb{P}^{g-1}$ and $C = X \cap H$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n-1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) & \longrightarrow & H^0(\mathcal{O}_H(n)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle r_3} & & \\ 0 & \longrightarrow & H^0(\mathcal{O}_X(n-1)) & \longrightarrow & H^0(\mathcal{O}_X(n)) & \longrightarrow & H^0(\mathcal{O}_C(n)) & \longrightarrow & 0 \end{array}$$

   The bottom row is the first part of the long exact sequence associated with (1). We observe that r_3 is surjective because C is projectively normal and r_1 is surjective by the inductive hypothesis. Then r_2 is also surjective and the claim is proved.

3. Consider the following exact sequence:

$$0 \to \mathcal{O}_{X'}(-C' + mK_{X'}) \to \mathcal{O}_{X'}(mK_{X'}) \to \mathcal{O}_{C'}(mK_{X'})) \to 0.$$

Since $-K_{X'}|_{C'} \sim \alpha$, for α a non-zero two torsion element of C', then we have that $(-C' + mK_{X'}) \cdot C' = -C'^2 = 2 - 2g < 0$. Whence $h^0(\mathcal{O}_{X'}(-C' + mK_{X'})) = 0$ otherwise, if this divisor was effective, it would be a fixed component of $|C'|$ that is a linear system without base locus by definition. At the same time

$$h^0(\mathcal{O}_{C'}(m(K_{X'}))) = \begin{cases} 0 & if \quad m \quad odd \\ 1 & if \quad m \quad even. \end{cases}$$

Consequently the plurigenus $P_m(X') := h^0(\mathcal{O}_{X'}(mK_{X'})) \leq h^0(\mathcal{O}_{C'}(mK_{X'})) \leq 1$. Then the Kodaira dimension $\kappa(X') = -\infty$ or 0.

Let $\kappa(X')$ be 0. Then X' is minimal because if it were not so, there would be a $(-1)-$curve E' on X' and we would find a contradiction. Indeed, let $m > 0$ be such that $|mK_{X'}| \neq \varnothing$ (it can only happen if m is even by the previous remark). Since $\kappa(X') = 0$, then $|mK_{X'}|$ consists of one effective divisor D'. It is obvious that E' is a component of D'. Now $\mathcal{O}_{C'}(D') \cong \mathcal{O}_{C'}(mK_{X'}) \cong \mathcal{O}_{C'}$ because m is even and $-K_{X'}|_{C'}$ is a non-zero two torsion element. Therefore D' and consequently E' are contracted to a point on X by π, contradicting the minimality of the resolution π.

It is true that $12K_{X'} \sim 0$ because X' is minimal and $\kappa(X') = 0$ (see [20], Theorem V.6.3). By the classification of minimal surfaces, there is a smallest $m \geq 1$ such that $mK_{X'} \sim 0$: the possibilities are $m = \{1, 2, 3, 4, 6\}$. If $m = 1$, then $K_{X'} \sim 0$, whence $\mathcal{O}_{C'}(C') \cong \mathcal{O}_{C'}(K_{C'} - K_{X'}) \cong \mathcal{O}_{C'}(K_{C'})$. This is not valid because C' is a Prym-canonical curve. So we exclude the cases in which X' is a $K3$ surface or an abelian surface. Hence X' can be either an Enriques surface or a hyperelliptic surface. If X' was a hyperelliptic surface, it would not contain curves with negative self-intersection, then we would have $X = X'$ smooth by Mumford's Theorem (see [25], Chapter 1). By definition, a hyperelliptic surface is irregular, contradicting the first point of this proposition. In conclusion X' is a minimal Enriques surface.

4. We have $\deg(X) = \deg(C) = C^2 = \deg(K_C + \alpha) = 2g - 2$.

□

1.2 Singularities on a surface with Prym-canonical hyperplane sections

We want to determine the possible singularities on a surface X with Prym-canonical hyperplane sections. By the previous section, we know that such a surface is birationally equivalent to either an Enriques surface, or a ruled or rational surface.

Let us see a preliminary lemma before studying the antibicanonical divisor on X'.

Lemma 1.11. *Let X be a surface with Prym-canonical hyperplane section C and let $\pi : X' \to X$ be the minimal resolution of singularities of X. Then $\pi_*(2K_{X'}) \sim 0$.*

Proof. Let $C_m \in |mC|$ be smooth satisfing $C_m \cap \mathrm{Sing}(X) = \varnothing$. We put $C'_m = \pi^*(C_m)$ and we want to show that $2K_{X'}|_{C'_m} \sim 0$ for any $m \geq 1$.
By Chow's moving Lemma (see [20], pag. 427), we have that $-K_{X'} \cdot C'_m$ is rationally equivalent to $-mK_{X'} \cdot C'$, so $-K_{X'} \cdot C'_m \sim -mK_{X'} \cdot C'$ in $A(X')$, where $A(X')$ is the Chow ring of X', i.e. the ring of the cycles of codimension 1 and 2 on X' modulo rational equivalence.
Among the properties of the Chow ring, we can consider the following (see [20], pag. 426): if $i : C'_m \hookrightarrow X'$ is a morphism of varieties, then $i^* : A(X') \to A(C'_m)$ is a ring homomorphism. Since i^* is a homomorphism, then $i^*(0) = [\mathcal{O}_{C'_m}]$.
By definition, $-K_{X'}|_{C'}$ is a non-trivial two torsion element, so, if m is even, we have that $-mK_{X'}|_{C'} \sim 0$. Thus $-2K_{X'} \cdot C'_m \sim -2mK_{X'} \cdot C' \sim 0$ in $A(X')$ and, by the previous remark, we have that $[\mathcal{O}_{C'_m}] = i^*(0) = i^*(-2K_{X'} \cdot C'_m) = [\mathcal{O}_{C'_m}(-2K_{X'})]$. So $-2K_{X'}|_{C'_m} \sim 0$ for any $m \geq 1$ (rational equivalence and linear equivalence coincide on a curve).

We also observe that $\pi_*(2K_{X'})|_{C_m} \sim 0$. Indeed, since π is an isomorphism in a neighbourhood of C'_m, then $\pi_*\mathcal{O}_{C'_m} \cong \mathcal{O}_{C_m}$ and $\mathcal{O}_{C'_m} \cong \pi^*(\mathcal{O}_{C_m})$. Using the projection formula (see [20], Exe $II.5.1$) and the previous results, we obtain that $\mathcal{O}_{C_m} \cong \pi_*(\mathcal{O}_{C'_m}) \cong \pi_*(2K_{X'} \otimes \mathcal{O}_{C'_m}) = \pi_*(2K_{X'} \otimes \pi^*\mathcal{O}_{C_m}) \cong \pi_*(2K_{X'}) \otimes \mathcal{O}_{C_m}$.

By a known result of Zariski ([33], Theorem 4), if $\pi_*(2K_{X'})$ is a Weil divisor on X (not necessarily effective), then the existence of an effective divisor D_m on C_m such that $\pi_*(2K_{X'})|_{C_m} \sim D_m$, for every m sufficiently large, implies the existence of an effective Weil divisor D on X such that $D \sim \pi_*(2K_{X'})$ (linear equivalence of Weil divisor is possible because $\mathrm{codim}(Sing(X)) \geq 2$). In our case $D_m = 0$ for any m, so there is a divisor D such that $D|_{C_m} \sim 0$. Since C_m is very ample on X, then $D \sim 0$. So $\pi_*(2K_{X'}) \sim 0$. This proves the lemma.

□

Theorem 1.12. *Let X be a surface with Prym-canonical hyperplane section C and let $\pi : X' \to X$ be the minimal resolution of singularities of X.*
Then $\dim|-2K_{X'}| = 0$.
In particular, if W' is the effective antibicanonical divisor on X', then either $W' \sim 0$ or $\mathrm{supp}(W') = \pi^{-1}(x_1, ..., x_r)$ *for certain singularities $x_i \in X$, for $i = 1, .., r$.*

Proof. Since $\pi_*(2K_{X'}) \sim 0$ by the previous lemma, then either $2K_{X'} \sim 0$ or there is a bicanonical divisor $2K_{X'}$ on X' with support in $\pi^{-1}(\mathrm{Sing}(X))$. In the latter case, let $2K_{X'} = \sum m_i F_i - \sum n_j G_j$ be the decomposition in reduced and irreducible components, with $m_i, n_j \in \mathbb{N}_{>0}$ and $F_i \neq G_j$, for all i, j. Let $F = \sum m_i F_i$ and $G = \sum n_j G_j$.
Suppose $F \neq 0$. By Mumford's Theorem (see [25], Chapter 1), we have that $F_i^2 < 0$ for any i and, because the intersection form on $\pi^{-1}(\mathrm{Sing}(X))$ is negative definite, also $F^2 < 0$. So there is an i_0 such that $F \cdot F_{i_0} < 0$. Up to renaming the index, we suppose

that $i_0 = 1$. It is obvious that $F_1 \cdot G \geq 0$ because $F_1 \neq G_j$, for any j. Since F_1 is an irreducible component, then

$$0 \leq p_a(F_1) = 1 + \frac{1}{2}F_1 \cdot (F_1 + K_{X'}) = 1 + \frac{1}{2}F_1^2 + \frac{1}{4}F_1 \cdot F - \frac{1}{4}F_1 \cdot G.$$

If $F_1^2 \leq -2$, then $p_a(F_1) < 0$, so we necessarily have $F_1^2 = -1$. Thus F_1 is a $(-1)-$curve, contradicting the minimality of π. Hence $F = 0$.
We conclude that either $2K_{X'} \sim 0$ or there are effective antibicanonical divisors with support in $\pi^{-1}(\mathrm{Sing}(X))$. Then $\dim|-2K_{X'}| = 0$.

Let $|-2K_{X'}| = \{W'\}$. To conclude the proof, we have only to show that, if $x \in \mathrm{Sing}(X)$ is such that $\pi^{-1}(x)$ meets $\mathrm{supp}(W')$, then $\pi^{-1}(x)$ does not contain curves which are not part of $\mathrm{supp}(W')$.
Suppose that there is an irreducible curve $E \subset \pi^{-1}(x)$ which is not part of $\mathrm{supp}(W')$. Since X is normal by Theorem 1.10, Point 2., then $\pi^{-1}(x)$ is connected, so we can assume that E intersects W' and $E \cdot W' > 0$. Then

$$0 \leq p_a(E) = 1 + \frac{1}{2}E^2 + \frac{1}{2}E \cdot K_{X'} = 1 + \frac{1}{2}E^2 - \frac{1}{4}E \cdot W'.$$

Again by Mumford's Theorem, we have $E^2 < 0$. So the only possible case for which the previous inequality is valid is: $E \cdot W' = 2$, $E^2 = -1$ and $p_a(E) = 0$. This contradicts the minimality of π. □

Remark 1.13. *By the previous Theorem, we observe that, if X is smooth, then $W' \sim 0$. Since $p_a(X) = p_g(X) = 0$ by Theorem 1.10, Point* 1.*, then X is an Enriques surface by [20], Theorem* V.6.3.

We recall the following definition.

Definition 1.14. *Let X be a normal surface and $x \in X$ a singularity. Let $\pi : X' \to X$ be the minimal resolution of x. Then we call* (geometric) genus of x *the number $p_g(x) = \dim_k(R^1\pi_*\mathcal{O}_{X'})_x$.*
If $p_g(x) = 0$, then x is called rational singularity.

Finally we can determine the geometric genus of the singularities that occur on X surface with Prym-canonical hyperplane sections.

We recall that the Kodaira dimension $\kappa(X') = -\infty$ or 0 by Theorem 1.10. Moreover, if it is equal to 0, then X' is a minimal Enriques surface (see Theorem 1.10, Case 3.). Instead, since the Kodaira dimension is a birational invariant for smooth varieties, if $\kappa(X') = -\infty$, then the minimal model of X' is a ruled surface or $\mathbb{P}^2$ (see [20], Theorem V.6.1).

Proposition 1.15. *Let X be a surface with Prym-canonical hyperplane sections such that a general hyperplane section C is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$ and let $\mathrm{Sing}(X) = \{x_1, ..., x_r\}$ be the locus of the singular points of X. Let $\pi : X' \to X$ be the minimal resolution of singularities of X. Thus:*

- *if X is birationally equivalent to an Enriques surface or $\mathbb{P}^2$, then all singularities of X are rational;*

- *if X is birationally equivalent to a ruled surface X'' over a base curve of genus $q \geq 0$, then $\sum_{i=1}^{r} p_g(x_i) = q$.*

Proof. These results can be obtained using the following exact sequence, which one gets from the Leray spectral sequence for the sheaf $\mathcal{O}_{X'}$ and the morphism π (see [19], pag. 462):

$$0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{X'}) \to H^0(R^1\pi_*\mathcal{O}_{X'}) \to H^2(\mathcal{O}_X) \to ...$$

We know that $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ by Theorem 1.10, so

$$h^0(R^1\pi_*\mathcal{O}_{X'}) = h^1(\mathcal{O}_{X'}).$$

By Theorem 1.10, Point 3., the surface X is birationally equivalent to either an Enriques surface or $\mathbb{P}^2$ or a ruled surface.

- If X is birationally equivalent to an Enriques surface, then X' is a minimal Enriques surface by Theorem 1.10, Point 3., so $h^1(\mathcal{O}_{X'}) = 0$ by definition. Thus $\sum_{i=1}^{r} p_g(x_i) = h^0(R^1\pi_*\mathcal{O}_{X'}) = 0$, whence X can only contain rational singularities.

  If X is birationally equivalent to $\mathbb{P}^2$, it is rational, as X'. By definition, X' is smooth. Then it is clear that $h^1(\mathcal{O}_{X'}) = 0$ and $\sum_{i=1}^{r} p_g(x_i) = 0$. This proves the first part of proposition.

- If X is birationally equivalent to a ruled surface X'' over a base curve of genus $q \geq 0$, also X' is birationally equivalent to X'' (relatively minimal model of X'). It is a known result that $h^1(\mathcal{O}_{X''}) = q(X'') = q$. Since the irregularity q is a birational invariant for smooth varieties, then $q(X') = q(X'') = q$. So $\sum_{i=1}^{r} p_g(x_i) = h^0(R^1\pi_*\mathcal{O}_{X'}) = h^1(\mathcal{O}_{X'}) = q$.

□

Let us give some definition that we will use in the following lemma.

Let $\pi : X' \to X$ be the minimal resolution of singularities of X normal surface. Let $x \in X$ be a singular point. Then $E = \pi^{-1}(x)$ is a connected curve. Let $E = \cup_{i=1}^{n} E_i$ be its decomposition in irreducible components.

Definition 1.16. *There is a unique non-zero effective divisor $Z_0 = \sum a_i E_i$ such that $Z_0 \cdot E_i \leq 0$, for any $i = 1, ..., n$, and if Z is another divisor satisfying the above conditions, then $Z \leq Z_0$. We call Z_0 the* fundamental cycle associated with *x.*

The following proposition gives us equivalent definitions of rational singularity (see [24], Proposition-Definition 2.1.).

Proposition 1.17. *The singularity $x \in X$ is a* rational singularity *if one of the following equivalent conditions holds:*

- $p_g(x) = 0$*;*
- *for every effective divisor D supported on E, $h^1(\mathcal{O}_D) = 0$;*
- *for every effective divisor D supported on E, $p_a(D) \leq 0$;*
- $p_a(Z_0) = 0$*, for Z_0 the fundamental cycle associated with x.*

Definition 1.18. *A rational singularity with fundamental cycle Z_0 is called a* rational n-point *if $-Z_0^2 = n$.*

Let us see another result about the singularities that occur on X.

Lemma 1.19. *Let X be a surface with Prym-canonical hyperplane section C and let $\pi : X' \to X$ be the minimal resolution of singularities of X. If W' is the unique effective antibicanonical divisor on X', then:*
a singularity $x \in X$ such that $\pi^{-1}(x)$ does not meet $\operatorname{supp}(W')$ *is a rational double point.*

Proof. Let $x \in X$ be a singularity such that $\pi^{-1}(x)$ does not meet $\operatorname{supp}(W')$. Let T be an irreducible component of the connected component $\pi^{-1}(x)$, then $T \cdot W' = 0$. So

$$0 \leq p_a(T) = 1 + \frac{1}{2}T^2 + \frac{1}{2}T \cdot K_{X'} = 1 + \frac{1}{2}T^2 - \frac{1}{4}T \cdot W' = 1 + \frac{1}{2}T^2.$$

By Mumford's Theorem $T^2 < 0$, so the only possible case for which the inequality above is valid is: $T^2 = -2$ and $p_a(T) = 0$. Then all the irreducible components of $\pi^{-1}(x)$ are smooth rational curves with self-intersection -2. In analogy with the previous notation, we can call these curves E_i, for $i = 1, ..., n$.

We can prove that x must be a rational singularity using Proposition 1.17. Let $Z_0 = \sum_{i=1}^n a_i \cdot E_i$, for $a_i \geq 0$ not all zero. First of all, $p_a(Z_0) = 1 + \frac{1}{2}Z_0^2 + \frac{1}{2}Z_0 \cdot K_{X'}$. Now $Z_0 \cdot K_{X'} = -\frac{1}{2}Z_0 \cdot W' = -\frac{1}{2}\sum_{i=1}^n a_i E_i \cdot W' = 0$, while $Z_0^2 < 0$ since Z_0 is contracted by π. So $p_a(Z_0) < 1$.
By [30], Lemma 1.1, we have that $p_a(E_i) \leq p_a(Z_0)$ for every E_i contained in Z_0. Since $p_a(E_i) = 0$ as computed before, then the only possible case is $p_a(Z_0) = 0$.

In conclusion, $x \in X$ is a rational double point. Indeed, by the adjunction formula, the self-intersection $Z_0^2 = 2p_a(Z_0) - 2 - Z_0 \cdot K_{X'} = 2p_a(Z_0) - 2 = -2$. □

Remark 1.20. *We have already seen that, if X is birationally equivalent to an Enriques surface, then X can only contain rational singularities.*
Moreover, by the previous lemma, we conclude that if X is birationally equivalent to an Enriques surface, then it can only contain rational double points as singularities. Indeed, if X' is a minimal Enriques surface, then $2K_{X'} \sim 0$. Thus the unique antibicanonical divisor W' on X' is linearly equivalent to 0. Therefore, $\pi^{-1}(x)$ does not meet $\operatorname{supp}(W')$*, for every singularity $x \in X$. The thesis follows by the previous lemma.*

1.3 Blowing up of minimal models of surfaces with Prym-canonical hyperplane sections

Let $\phi : X' \to X''$ be a relatively minimal model for X', where X' is the minimal resolution of the singularities of X, as seen before. Let W' be the only effective antibicanonical divisor of X' (see Lemma 1.12) and we put

$$W'' = \phi_* W' \in |-2K_{X''}|.$$

Remark 1.21. *In the following, we will talk about the connected components of a divisor D on a surface Y. By connected component we mean a component that cannot be represented as the union of two or more disjoint non-empty closed subsets.*

Proposition 1.22. *Let $X' = X'_s \xrightarrow{\phi_{s-1}} X'_{s-1} \to ... \xrightarrow{\phi_1} X'_1 = X''$ be the factorization of ϕ in such a way that ϕ_1 blows up the points on X'' in which ϕ^{-1} is not defined, ϕ_2 blows up the points in which $\phi^{-1} \circ \phi_1$ is not defined etc. and let $W'_t = (\phi_t \circ ... \circ \phi_{s-1})_*(W') \in |-2K_{X'_t}|$, for $t = 1, ..., s-1$.*

1. *All points blown up by ϕ_t on X'_t lie on* $\mathrm{supp}(W'_t)$, *for $t = 1, ..., s-1$.*
2. *If ϕ_t blows up the points $P_{t,1}, ..., P_{t,i(t)} \in X'_t$, which have multiplicity μ_{t,i_t} on W'_t, if $\phi_t^{-1}(P_{t,i_t}) = E_{t+1,i_t}$ and if $\widetilde{W'_t}$ is the strict transform of W'_t on X'_{t+1}, then $\mu_{t,i_t} \geq 2$ and*
$$W'_{t+1} = \widetilde{W'_t} + \sum_{i_t=1}^{i(t)} (\mu_{t,i_t} - 2) E_{t+1,i_t},$$
for $t = 1, ..., s-1$ and $i_t = 1, ..., i(t)$.
3. *The number of connected components of the antibicanonical divisor W' on X' compared to the number of connected components of W'' on X'' can at most increase by the number of the double points blown up by ϕ that lie on two or more irreducible components of W''.*

Proof. 1. Let $E \subset X'_{t+1}$ be a $(-1)-$curve such that $\phi_t(E)$ is a point P on X'_t, for $t = 1, ..., s-1$. Let $W'_{t+1} = m \cdot E + D'_{t+1}$, for $m \geq 0$ and E not a component of D'_{t+1} (we observe that, if $m = 0$, then W'_{t+1} does not contain E). By the adjunction formula, the intersection $E \cdot W'_{t+1} = -2E \cdot K_{X'_{t+1}} = 2$, so $E \cdot D'_{t+1} = E \cdot W'_{t+1} - mE^2 = 2 + m > 0$. Then E and D'_{t+1} have intersection but ϕ_t blows down E in a point P, so $P \in \mathrm{supp}((\phi_t)_* D'_{t+1}) = \mathrm{supp}(W'_t)$.

2. By the previous point of this Proposition, the intersection $E \cdot D'_{t+1} \geq 2$, for any $t = 1, ..., s-1$. Since E is contracted to a point $P \in (\phi_t)_* W'_{t+1} = W'_t$, then the multiplicity μ of P on W'_t is at least 2. This is valid for any point P_{t,i_t} on W'_t blown up by ϕ_t, for any $t = 1, ..., s-1$ and for any $i_t = 1, ..., i(t)$. Then $\mu_{t,i_t} \geq 2$ for any t and for any i_t.

Now we may assume $t = 1$ and $i(t) = 1$, so ϕ_1 blows up only one point $P \in W_1'$ with multiplicity μ to a curve E on X_2'. The general formula for the canonical divisor is $K_{X_2'} \sim \phi_1^* K_{X_1'} + E$ (see [20], Proposition V.3.3). Then $-2K_{X_2'} \sim \phi_1^*(-2K_{X_1'}) - 2E$ implies that $W_2' \sim \phi_1^*(W_1') - 2E$. Observing that $\phi_1^*(W_1') \sim \widetilde{W_1'} + \mu E$, where $\widetilde{W_1'}$ is the strict transform of W_1', then the desired formula is proved.

3. It is enough to work with $W_1' := W''$ and W_2'. Furthermore, we can assume that ϕ_1 blows up only one point P with multiplicity μ on W_1' to $E \in X_2'$ and that W_1' is connected.

   If P lies on only one irreducible component of $\mathrm{supp}(W_1')$, then the strict transform $\widetilde{W_1'}$ is still connected and E intersects $\widetilde{W_1'}$. By the previous point of this Theorem, W_2' is also connected for any multiplicity μ. The number of connected components of W_1' and W_2' is the same.

   If P lies on two or more irreducible components of $\mathrm{supp}(W_1')$, then $\widetilde{W_1'}$ can be disconnected. If $\mu > 2$, then E is a component of W_2' by the previous point of this Proposition and it is clear that E intersects all connected components of $\widetilde{W_1'}$, so W_2' is still connected. Again in this case, the number of connected components of W_1' and W_2' is the same.

   Instead, if $\mu = 2$, by the previous point of this Proposition we have that E is not a component of W_2', so W_2' can be disconnected. More exactly, the number of connected components of W_2' can increase by one. The result follows.

□

If L is the $(g-1)-$dimensional complete linear system of hyperplane sections of X and $L' = \pi^* L$, we put $L'' = \phi_* L'$, with $i_{L''}(X'') = i_L(X)$, where i_L and $i_{L''}$ are respectively the maps associated with L and L''.

Lemma 1.23. *With the same notation as before, a general $C' \in L'$ is disjoint from W'.*

Proof. If i_L and $i_{L'}$ are the maps associated respectively with the linear systems L and L', then $i_L \circ \pi = i_{L'}$. By Theorem 1.12, we have that $\mathrm{supp}(W') = \pi^{-1}(x_1, ..., x_r)$, for certain singularities $x_i \in X$. So π blows down W' and so does $i_{L'}$. If C' is a general divisor of L', then $C' \cdot W' = 0$ and C' is disjoint from W'. □

As a consequence X' is obtained blowing up X'' along all the base points of L''. Indeed, if not, L' would have another base point P to blow up and this would contradict the previous lemma (the proof is similar to Proposition 1.22, Case 1.). By Proposition 1.22, all the base points of L'' lie on $\mathrm{supp}(W'')$. In particular, L'' can have infinitely near base points. We will call *base point of order t* a base point of L'' that lies on X_t', with the notation of Proposition 1.22.

In the following, we will study the possible base points of L'', where they are situated on X''.

Proposition 1.24. *As in Proposition 1.22, let* $X' = X'_s \xrightarrow{\phi_{s-1}} X'_{s-1} \to ... \xrightarrow{\phi_1} X'_1 = X''$ *be the factorization of* ϕ *in such a way that* ϕ_1 *blows up the points on* X'' *in which* ϕ^{-1} *is not defined,* ϕ_2 *blows up the points in which* $\phi^{-1} \circ \phi_1$ *is not defined etc. and let* $W'_t = (\phi_t \circ ... \circ \phi_{s-1})_*(W') \in |-2K_{X'_t}|$, *for* $t = 1, ..., s-1$. *Then:*

1. *for every* $t = 1, ..., s-1$, *the strict transform* $C'_{t+1} \in X'_{t+1}$ *of a general* $C'' \in L''$ *does not intersect* $\text{supp}(W'_{t+1})$ *outside its base locus.*
2. *if* $P \in X'_t$ *is a base point of* L'' *lying on only one irreducible component* D'_t *of* W'_t *on which it has multiplicity* 2, *then the possible base points infinitely near to* P *lie on the strict transform of* D'_t *by* ϕ_t*;*
3. *if* $P \in X'_t$ *is a base point of* L'' *of order* t, *for* $t = 1, ..., s-1$, *and if the multiplicity of* P *on* W'_t *is* $\mu \geq 3$, *then there must be base points infinitely near to* P *of order* j, *at least for every* $t \leq j \leq [\frac{\mu+1}{2}] + t$.

Proof. 1. If the strict transform $C'_{t+1} \in X'_{t+1}$ of a general $C'' \in L''$ had a variable intersection with $\text{supp}(W'_{t+1})$, for $t = 1, ..., s-1$, i.e. if C'_{t+1} had intersection with $\text{supp}(W'_{t+1})$ outside its base points, these intersection points would not be blown up by ϕ, so the intersection would survive between C' and $\text{supp}(W')$, contradicting the previous point of this Proposition.

2. We suppose that ϕ_t blows up a point $P \in X'_t$ of multiplicity $\mu = 2$ on W'_t such that $\phi_t^{-1}(P) = E \in X'_{t+1}$. In particular, we suppose that P lies on only one irreducible component D'_t of W'_t.

   By Proposition 1.22, we have that $W'_{t+1} = \widetilde{W'_t} + \sum_{i_t=1}^{i(t)} (\mu_{t,i_t} - 2)E_{t+1,i_t}$, so E is not a component of W'_{t+1} because the multiplicity of P is $\mu = 2$. Always by Proposition 1.22, we have that the base points of L'' on X'_{t+1} must lie on W'_{t+1}, so the only possible base point on E (the only fixed direction in P) is the point $E \cap W'_{t+1} = E \cap D'_{t+1}$, where D'_{t+1} is the strict transform of D'_t.

3. We can assume that $P \in W'_1$, so we fix $t = 1$. By Proposition 1.22, the divisor W'_2 contains $E = \phi_1^{-1}(P)$ with multiplicity $\mu - 2 \geq 1$. The strict transform $C'_2 \in X'_2$ of a general $C'' \in L''$ intersects E according to the directions of C'' in P, so the intersection points of C'_2 with E are the base points of L'' of order 2 with multiplicity at least $\mu - 2$ for W'_2. Iterating this process, if $Q \in C'_2 \cap E$, then W'_3 contains $F = \phi_2^{-1}(Q)$ with multiplicity at least $(\mu - 2) - 2$. Again $C'_3 \in X'_3$ intersects F in the base points of L'' of order 3 with multiplicity at least $(\mu-2)-2$ for W'_3. We conclude that there must be base points infinitely near to P of order j, at least for every $j \leq [\frac{\mu+1}{2}]$. If $t > 1$ and $P \in X'_t$, we only translate this process from t.

□

2 Surfaces with Prym-canonical hyperplane sections birationally equivalent to ruled surfaces

In this chapter, we will study in more detail surfaces with Prym-canonical hyperplane sections birationally equivalent to rational or non-rational ruled surfaces. After studying the possible non-rational singularities that occur on them, we will construct examples of this type of surfaces.

Let X be a surface with Prym-canonical hyperplane sections. As before, we will suppose that a general hyperplane section C of X is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$. We will denote with $\pi : X' \to X$ the minimal resolution of the singularities of X and with $\phi : X' \to X''$ be a relatively minimal model for X'. We will assume that X'' is a rational or non-rational ruled surface.

2.1 Notation and basic facts regarding ruled surfaces

We will first fix some notation and state some facts about minimal smooth ruled surfaces in which we will follow [20], Chapter V.2.

Let Y be such a surface, then there is a natural map $p : Y \to \Gamma$, where Γ is the smooth base curve of genus $p_g(\Gamma) := q \geq 0$. If $q = 0$, then Y is a rational ruled surface. Every fibre $f_y := p^{-1}(y)$ is isomorphic to $\mathbb{P}^1$, for every point $y \in \Gamma$. We will denote with f the generic fibre of Y.

We can find the proof of the following Proposition in [20], Proposition V.2.8:

Proposition 2.1. *If $p : Y \to \Gamma$ is a ruled surface, it is possible to write $Y \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a normalized locally free sheaf of rank 2 on Γ (we recall that* normalized *means that $h^0(\Gamma, \mathcal{E}) \neq 0$, while $h^0(\Gamma, \mathcal{E} \otimes L) = 0$ for every invertible sheaf L of negative degree on Γ). In this case the integer $e = -\deg(\mathcal{E})$ is an invariant of Y. Furthermore there is a section $\sigma_0 : \Gamma \to Y$ with image C_0 such that $\mathcal{O}_Y(C_0) = \mathcal{O}_Y(1)$.*

Let D be the divisor on Γ corresponding to the invertible sheaf $\wedge^2\mathcal{E}$, i.e.

$$\wedge^2(\mathcal{E}) = \mathcal{O}_\Gamma(D),$$

for $D \in \mathrm{Div}(\Gamma)$. The integer

$$e = -\deg D.$$

We recall the following exact sequence:

$$0 \to \mathcal{O}_\Gamma \to \mathcal{E} \to \mathcal{O}_\Gamma(D) \to 0. \tag{4}$$

Definition 2.2. *Let $\mathcal{E}$ be a normalized locally free sheaf of rank 2 on Γ. We say that $\mathcal{E}$ is* decomposable *if $\mathcal{E} \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$.*
In particular, we will say that a ruled surface Y is decomposable *if $Y = \mathbb{P}_\Gamma(\mathcal{E})$, for $\mathcal{E}$ a decomposable sheaf.*

By [20], Theorem $V.2.12$, the invariant $e \geq 0$ if $\mathcal{E}$ is decomposable. Moreover, if $q = 0$, then $e \geq 0$ (see [20], Corollary $V.2.13$).

As showed in Proposition [20], $V.2.9$, the intersection $C_0^2 = -e$.
If $\mathcal{E}$ is decomposable, then Y contains disjoint sections and we will denote by C_1 a fixed section of p disjoint from C_0. The intersection $C_1^2 = e$.

Every divisor on Y is of the form

$$aC_0 + \Delta \cdot f, \text{ for } a \in \mathbb{Z} \text{ and } \Delta \in \mathrm{Pic}(\Gamma). \tag{5}$$

In particular, the canonical divisor $K_Y \sim -2C_0 + (K_\Gamma + D) \cdot f$.

Lemma 2.3. *If $Y = \mathbb{P}_\Gamma(\mathcal{E})$, with $\mathcal{E}$ decomposable, then the section $C_1 \sim C_0 - D \cdot f$ (see [15], Remark 27).*

The next lemmas give simple techniques to compute the dimensions of linear systems on ruled surfaces.

Lemma 2.4. *If $aC_0 + \Delta \cdot f$ is a divisor on $Y \cong \mathbb{P}_\Gamma(\mathcal{E})$ with $a \geq 0$, then*

$$h^0(Y, \mathcal{O}_Y(aC_0 + \Delta \cdot f)) = h^0(\Gamma, S^a\mathcal{E} \otimes \mathcal{O}_\Gamma(\Delta)),$$

where $S^a\mathcal{E}$ is the $a-th$ symmetric product of $\mathcal{E}$.

Proof. By Lemma $V.2.4$ of [20], we know that

$$H^0(Y, \mathcal{O}_Y(aC_0 + \Delta \cdot f)) \cong H^0(\Gamma, p_*\mathcal{O}_Y(aC_0 + \Delta \cdot f)),$$

where $p : Y \to \Gamma$ is the natural map associated with the ruled surface Y.
Using the projection formula (see [20], Exercise $III.8.3$), we can compute that

$$p_*\mathcal{O}_Y(aC_0 + \Delta \cdot f) \cong p_*(\mathcal{O}_Y(aC_0) \otimes \mathcal{O}_Y(\Delta \cdot f)) \cong p_*(\mathcal{O}_Y(aC_0) \otimes p^*\mathcal{O}_\Gamma(\Delta)) \cong$$

$$\cong p_*(\mathcal{O}_Y(aC_0)) \otimes \mathcal{O}_\Gamma(\Delta).$$

By [20], Exercise $III.8.4$, we have that $p_*(\mathcal{O}_Y(aC_0)) \cong S^a\mathcal{E}$. The claim is proved. □

We can find the proof of the following Lemma in [15], Lemma 35:

Lemma 2.5. *Let $|aC_0 + \Delta \cdot f|$ be an $a-$secant linear system on a decomposable ruled surface $Y \cong \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$. Then, for $i \geq 0$, we have that*

$$h^i(\mathcal{O}_Y(aC_0 + \Delta \cdot f)) = \sum_{k=0}^{a} h^i(\mathcal{O}_\Gamma(\Delta + kD)).$$

We also recall the following Theorem (see [20], $V.2.15$):

Theorem 2.6. *If Y is a ruled surface over an elliptic curve Γ corresponding to an indecomposable $\mathcal{E}$, then $e = 0$ or -1 and there is exactly one such ruled surface over Γ for each of these two values of e, that we respectively call Y_0 and Y_{-1}.*
In particular, if $e = 0$, then $D \sim 0$.

Notation 2.7. *From now on, we will identify C_0 with Γ thanks to the isomorphism $i : \Gamma \to C_0$. So we will use the following notation: if $F \in \mathrm{Pic}(X'')$, then $\mathcal{O}_\Gamma(F) := i^*(\mathcal{O}_{C_0}(F))$.*

Remark 2.8. *If $N_{C_0/X''}$ is the* normal sheaf *of C_0 in X'', we observe that $\mathcal{O}_{C_0}(C_0) \cong N_{C_0/X''} \cong \mathcal{O}_\Gamma(D)$. So $D \sim C_0|_{C_0}$ using the identification of Notation 2.7.*

Finally we state a Proposition that we will use later ([13], Proposition $II.1.4$):

Proposition 2.9. *Let $Y = \mathbb{P}_\Gamma(\mathcal{E})$ be the minimal ruled surface over a smooth curve Γ of genus $q \geq 1$ associated with a locally free sheaf $\mathcal{E}$ on Γ of rank 2 which is a nontrivial extension of $\mathcal{O}_\Gamma(-K_\Gamma)$ by $\mathcal{O}_\Gamma$. Let G be an irreducible, reduced curve on Y, $G \neq C_0$. Then $G \cap C_0 \neq \varnothing$.*

2.2 Study of possible non-rational singularities

If X'' is a rational ruled surface, then X contains only rational singularities (see Proposition 1.15).
By Theorem 1.12, the surface X' contains a unique effective antibicanonical divisor W'. Because $\phi_* W' \in |-2K_{X''}|$, also the antibicanonical system of X'' is nonempty. We will determine the possible antibicanonical divisors on a minimal non-rational ruled surface X'', which we will use to study the possible non-rational singularities on X. Before this, we recall a Lemma (see [29], Lemma 17.19.4).

Lemma 2.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $F_2 \to F_1 \to F \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X-$modules. For each $n \geq 1$, there is an exact sequence $F_2 \otimes S^{n-1}F_1 \to S^n F_1 \to S^n F \to 0$.*

Remark 2.11. *Applying the previous Lemma to the exact sequence (4), we obtain the following sequence*

$$S^{n-1}\mathcal{E} \otimes \mathcal{O}_\Gamma \to S^n\mathcal{E} \to \mathcal{O}_\Gamma(nD) \to 0.$$

It is obvious that this sequence is also exact on the left, thus

$$0 \to S^{n-1}\mathcal{E} \to S^n\mathcal{E} \to \mathcal{O}_\Gamma(nD) \to 0. \tag{6}$$

Proposition 2.12. *Let $Y = \mathbb{P}_\Gamma(\mathcal{E})$ be a minimal ruled surface over a smooth curve Γ of genus $q \geq 1$ and assume $\mathcal{E}$ to be normalized. If $|-2K_Y| \neq \varnothing$, we have the following possibilities:*

I. if $\mathcal{E}$ is indecomposable and $q \geq 2$, then $\dim|-2K_Y| = 0$*; moreover*

I.1 if $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) > 0$, then $|-2K_Y| = \{4C_0 - 4K_\Gamma \cdot f\}$, $e = -2$ and $q = 2$.
In this case $2D \sim 2K_\Gamma$;

I.2 if $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) = 0$, then $|-2K_Y| = \{4C_0\}$ and $e = 2q-2$.
In particular $2D \sim -2K_\Gamma$;

II. if $\mathcal{E}$ is decomposable and $q \geq 2$, then $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) = 0$; moreover

II.1 if $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) = 0$, then $\dim|-2K_Y| = \dim|-2K_\Gamma - 2D|$ *, so every antibicanonical divisor is of the form $4C_0 + D_0 \cdot f$, for $D_0 \in |-2K_\Gamma - 2D|$ and $4C_0$ a fixed component.*
In this case we have $e \geq 2q-2$;

II.2 if $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) > 0$, then $\dim|-2K_Y| > \dim|-2K_\Gamma - 2D|$.
In this case $e \geq 4q-4$;

III. if $\mathcal{E}$ is indecomposable and $q = 1$, we have the two following possibilities:

III.1 if $Y = Y_0$, then $D \sim 0$, $\dim|-2K_Y| = 0$ *and $|-2K_Y| = \{4C_0\}$;*

III.2 if $Y = Y_{-1}$, then $\dim|-2K_Y| = 1$*; every divisor in $|-2K_Y|$ is isomorphic to the base curve Γ, except the non-reduced elements, precisely three, that are irreducible;*

IV. if $\mathcal{E}$ is decomposable and $q = 1$, then $\mathcal{E} \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$, for $D \in Div(\Gamma)$ with $e = -\deg(D) \geq 0$. *Thus:*

IV.1 if $e = 0$ and $D \sim 0$, then $Y = \Gamma \times \mathbb{P}^1$ and $\dim|-2K_Y| = 4$*, so any antibicanonical divisor is formed by four fibres of the projection $Y \to \mathbb{P}^1$;*

IV.2 if $e = 0$, $D \nsim 0$ and $2D \nsim 0$, then $\dim|-2K_Y| = 0$ *and $|-2K_Y| = \{2C_0 + 2C_1\}$;*

IV.3 if $e = 0$, $D \nsim 0$ and $2D \sim 0$, then $\dim|-2K_Y| = 2$ *and $-2K_Y \sim 4C_0$;*

IV.4 if $e > 0$, then $\dim|-2K_Y| = 3e$ *and any antibicanonical divisor has $2C_0$ as fixed component.*

Proof.

I.,II. Let $q \geq 2$. We have already seen that $-2K_Y \sim 4C_0 - 2(K_\Gamma + D) \cdot f$. Using Lemma 2.4, the dimension

$$\dim|-2K_Y| = h^0(\mathcal{O}_Y(4C_0 + 2(-K_\Gamma - D) \cdot f)) - 1 =$$
$$= h^0(\Gamma, S^4\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D)) - 1.$$

We assume $|-2K_Y| \neq \varnothing$, so $h^0(\Gamma, S^4\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D)) \geq 1$. Tensoring with $\mathcal{O}_\Gamma(-2K_\Gamma - 2D)$ the exact sequence (6) with $n = 4$, we obtain

$$0 \to S^3\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D) \to S^4\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D) \to \mathcal{O}_\Gamma(2D - 2K_\Gamma) \to 0. \quad (7)$$

- We assume that $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) > 0$. Then $\deg(2D-2K_\Gamma) = -2e-4q+4 \geq 0$ and $e \leq -2q+2$. If we work with $q \geq 2$, then $e < 0$, so $\mathcal{E}$ is necessarily indecomposable. Moreover $e \geq -q$ by [20], Exe. V.2.5. Thus the only possible value of q such that $-q \leq e \leq -2q+2$ is $q=2$. In this case $e=-2$.

  Tensoring with $\mathcal{O}_\Gamma(-2K_\Gamma-2D)$ the exact sequence (6) with $n=3$, we obtain

  $$0 \to S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2D-2K_\Gamma) \to S^3\mathcal{E}\otimes\mathcal{O}_\Gamma(-2D-2K_\Gamma) \to \mathcal{O}_\Gamma(D-2K_\Gamma) \to 0.$$

  Since $\deg(D-2K_\Gamma) = -e-4q+4 < 0$, we have that $h^0(\mathcal{O}_\Gamma(D-2K_\Gamma)) = 0$, therefore we deduce that

  $$h^0(S^3\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = h^0(S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)).$$

  In the same way we obtain that

  $$h^0(S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)).$$

  Since $\deg(-2K_\Gamma-2D) = -4q+4+2e = -8$ and $\mathcal{E}$ is normalized, it is obvious that $h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = 0$. Then

  $$h^0(S^3\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = 0.$$

  By assumption $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) > 0$. Since $\deg(2D-2K_\Gamma) = 0$, this implies that $2D \sim 2K_\Gamma$ and $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) = 1$.

  Now $h^0(S^4\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) \geq 1$ by hypothesis and moreover $h^0(S^4\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) \leq h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) = 1$ from the exact sequence (7). So the equality holds and

  $$\dim|-2K_Y| = h^0(S^4\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) - 1 = 0.$$

  Since $2D \sim 2K_\Gamma$, then $|-2K_Y| = \{4C_0 - 4K_\Gamma\cdot f\}$ (**CASE I.1**).

- We assume that $h^0(\mathcal{O}_\Gamma(2D-2K_\Gamma)) = 0$, then

  $$h^0(S^4\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = h^0(S^3\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D))$$

  from the exact sequence (7).

  Tensoring with $\mathcal{O}_\Gamma(-2K_\Gamma-2D)$ the exact sequence (6) with $n=3$, we obtain:

  $$0 \to S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D) \to S^3\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D) \to \mathcal{O}_\Gamma(D-2K_\Gamma) \to 0.$$

  If $h^0(\mathcal{O}_\Gamma(D-2K_\Gamma)) > 0$, then $\deg(D-2K_\Gamma) = -e+4-4q \geq 0$, so $e \leq 4-4q < 0$ since $q \geq 2$. Thus $\mathcal{E}$ is indecomposable and by [20], Exe.

V.2.5, we have that $-q \leq e$. A value $q \geq 2$ such that $-q \leq e \leq 4-4q$ does not exist. Then $h^0(\mathcal{O}_\Gamma(D-2K_\Gamma))=0$ and

$$h^0(S^3\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = h^0(S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)).$$

Let us consider the exact sequence (6) with $n=2$ already tensored with $\mathcal{O}_\Gamma(-2K_\Gamma-2D)$:

$$0\to\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)\to S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)\to\mathcal{O}_\Gamma(-2K_\Gamma)\to 0.$$

Since $\deg(-2K_\Gamma)<0$, then $h^0(\mathcal{O}_\Gamma(-2K_\Gamma))=0$ and

$$h^0(S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)).$$

As seen before, since $\dim|-2K_Y|\neq 0$ by hypothesis, then

$$h^0(S^4\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)) = h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D))>0.$$

Since $\mathcal{E}$ is normalized, then $\deg(-2K_\Gamma-2D) = -4q+4+2e \geq 0$, so $e\geq 2q-2$.

Finally, let us consider the exact sequence (4) tensored with $\mathcal{O}_\Gamma(-2K_\Gamma-2D)$:

$$0\to\mathcal{O}_\Gamma(-2K_\Gamma-2D)\to\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D)\to\mathcal{O}_\Gamma(-2K_\Gamma-D)\to 0. \quad (8)$$

* If $\mathcal{E}$ is indecomposable, by [20], Theorem *V*.2.12 we have that $e\leq 2q-2$, so e must be equal to $2q-2$. Since $q\geq 2$ by assumption, then $\deg(-2K_\Gamma-D) = -4q+4+e = -2q+2<0$, so $h^0(\mathcal{O}_\Gamma(-2K_\Gamma-D))=0$. We conclude that

  $$\dim|-2K_Y| = h^0(S^4\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D))-1 = h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma-2D))+$$

  $$-1 = h^0(\mathcal{O}_\Gamma(-2K_\Gamma-2D))-1 = \dim|-2K_\Gamma-2D|.$$

  At this point it is clear that $4C_0$ is a fixed component of $|-2K_Y|$. Since $\deg(-2K_\Gamma-2D)=0$ and $h^0(\mathcal{O}_\Gamma(-2K_\Gamma-2D))>0$, then $2D\sim -2K_\Gamma$ and $-2K_Y\sim 4C_0$. Therefore

  $$\dim|-2K_Y| = \dim|-2K_\Gamma-2D| = 0$$

  and $|-2K_Y| = \{4C_0\}$ (**CASE I.2**).

* Let $\mathcal{E}$ be decomposable. As seen before we have that $e\geq 2q-2$.

· Let us suppose that $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) = 0$, so

$$h^0(\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D)) = h^0(\mathcal{O}_\Gamma(-2K_\Gamma - 2D)).$$

Also in this case we have that

$$\dim |-2K_Y| = \dim |-2K_\Gamma - 2D|$$

and $4C_0$ is a fixed component of $|-2K_Y|$ (**CASE II.1**).

· Let us suppose that $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) > 0$. Then $\deg(-2K_\Gamma - D) = 4 - 4q + e \geq 0$, so $e \geq 4q - 4$. We observe that, since $\deg(3K_\Gamma + 2D) = 3(2q-2) - 2e \leq 6q - 6 + 2(4 - 4q) = = 2 - 2q < 0$, then $h^1(\mathcal{O}_\Gamma(-2K_\Gamma - 2D)) = h^0(\mathcal{O}_\Gamma(3K_\Gamma + 2D)) = 0$. Hence, using the exact sequence (8) and the previous remarks, we have that $\dim |-2K_Y| = h^0(S^4\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D)) - 1 = = h^0(\mathcal{E} \otimes \mathcal{O}_\Gamma(-2K_\Gamma - 2D)) - 1 = h^0(\mathcal{O}_\Gamma(-2K_\Gamma - 2D)) + h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) - 1$.
So

$$\dim |-2K_Y| = \dim |-2K_\Gamma - 2D| + h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) >$$

$$> \dim |-2K_\Gamma - 2D|.$$

We have obtained **CASE II.2**.

III. As seen in Theorem 2.6, there are two different types of elliptic ruled surface with $\mathcal{E}$ indecomposable: Y_0 and Y_{-1}. In both cases, since $p_g(\Gamma) = 1$, it is clear that $K_\Gamma \sim 0$, so $-2K_Y \sim 4C_0 - 2D \cdot f$, for $Y = Y_i$, with $i = \{0, -1\}$.

- **CASE III.1**
  If $Y = Y_0$, then $D \sim 0$ (see Theorem 2.6) and $-2K_Y \sim 4C_0$. Let us suppose that there is $W \in |-2K_Y|$ effective such that $W \neq 4C_0$. If C_0 was not a component of W, then $C_0 \cdot W = 4C_0^2 = 0$ and $W \cap C_0 = \varnothing$ which is impossible by Proposition 2.9. Then $C_0 \subset W$ and $W - C_0 \sim 3C_0$. Again, if C_0 was not a component of $W - C_0$, then $C_0 \cdot (W - C_0) = 3C_0^2 = 0$ and $(W - C_0) \cap C_0 = \varnothing$ which is impossible by Proposition 2.9. So $C_0 \subset (W - C_0)$ and $W - 2C_0 \sim 2C_0$. Another time, if C_0 was not a component of $W - 2C_0$, then $C_0 \cdot (W - 2C_0) = 2C_0^2 = 0$ and $W \cap C_0 = \varnothing$ which is impossible by Proposition 2.9. Then $C_0 \subset (W - 2C_0)$ and $W - 3C_0 \sim C_0$. Since $h^0(\mathcal{O}_Y(C_0)) = h^0(\Gamma, \mathcal{E}) = 1$, then $|C_0| = \{C_0\}$ and $W - 3C_0 = C_0$, so $W = 4C_0$. Then
  $$|-2K_Y| = \{4C_0\}.$$

- **CASE III.2**
  If $Y = Y_{-1}$, it was shown in Chapter 2 Section 5 of [5] that
  $$\dim |-2K_Y| = 1.$$

We denote by $\oplus$ the group operation on Γ and by e_0 the neutral element. Recall that the group law is defined by

$$x \oplus y \sim x + y - e_0,$$

for $x, y \in \Gamma$. It is known that $Y_{-1} := Sym^2(\Gamma)$, that is the second symmetric product of Γ. In particular, for every $x \in \Gamma$, we have a morphism

$$\begin{aligned} \varepsilon_x : \quad \Gamma &\longrightarrow Y_{-1} \\ e &\longrightarrow e + (e \oplus x). \end{aligned}$$

Let c_x be the image curve in Y_{-1}. Now ε_x is an isomorphism onto its image except precisely when x is one of the three non-zero $2-$torsion points of Γ. In the latter three cases, the map ε_η realizes Γ as an unramified double cover of its image curve c_η, for $\eta \in \Gamma$ a non-zero 2 torsion element. Indeed $c_\eta = \Gamma/\langle\eta\rangle$. Moreover, the diagonal $\Delta := c_{e_0}$ is linearly equivalent to $-2K_{Y_{-1}}$ (for more details see [5], Chapter 2, and [7]).
Therefore every divisor in $|-2K_Y|$ is isomorphic to Γ, hence smooth and irreducible, except precisely three irreducible and non-reduced elements, each of multiplicity 2, isomorphic to Γ modulo a non-zero two torsion element, as seen before.

IV. If $\mathcal{E}$ is decomposable, then $Y = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$, for $D \in \mathrm{Div}(\Gamma)$. By [20], Theorem V.2.12, we have that $e \geq 0$. Let $q = p_g(\Gamma) = 1$, so $K_\Gamma \sim 0$. By the previous remarks, we have that

$$-2K_Y \sim 4C_0 - 2D \cdot f \sim 2C_0 + 2C_1.$$

By Lemma 2.5, we have that

$$h^0(\mathcal{O}_Y(-2K_Y)) = h^0(\mathcal{O}_Y(4C_0 - 2D \cdot f)) =$$

$$= h^0(\mathcal{O}_\Gamma(2D)) + h^0(\mathcal{O}_\Gamma(D)) + h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-2D)).$$

- **CASE IV.1**
  If $e = 0$ and $D \sim 0$, then

  $$\dim|-2K_Y| = h^0(\mathcal{O}_Y(-2K_Y)) - 1 = 4.$$

  As observed in [20], Example V.2.11.1, we have that $Y = \Gamma \times \mathbb{P}^1$. So $-2K_Y = -2pr_1^*K_\Gamma - 2pr_2^*K_{\mathbb{P}^1} = -2pr_2^*K_{\mathbb{P}^1}$, where $pr_1 : Y \to \Gamma$ and $pr_2 : Y \to \mathbb{P}^1$ are the two projection maps. Then any antibicanonical divisor is formed by four fibres of the projection $pr_2 : Y \to \mathbb{P}^1$.

- **CASE IV.2**
  If $e = 0$, $D \nsim 0$ and $2D \nsim 0$, then

  $$\dim|-2K_Y| = h^0(\mathcal{O}_Y(-2K_Y)) - 1 = 0.$$

Using Lemma 2.5, we have that

$$h^0(\mathcal{O}_Y(-2K_Y - 2C_0)) = h^0(\mathcal{O}_Y(2C_0 - 2D \cdot f)) =$$
$$= h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-2D)) = 1.$$

Since $h^0(\mathcal{O}_Y(-2K_Y)) = h^0(\mathcal{O}_Y(-2K_Y - 2C_0)) = 1$, then $2C_0$ is a fixed component of the linear system $|-2K_Y|$. Hence the only possibility is

$$|-2K_Y| = \{2C_0 + 2C_1\}.$$

- **CASE IV.3**
  If $e = 0$, $D \nsim 0$ and $2D \sim 0$, then

  $$\dim|-2K_Y| = h^0(\mathcal{O}_Y(-2K_Y)) - 1 = 2$$

  and $-2K_Y \sim 4C_0$.
- **CASE IV.4**
  If $e > 0$, then $\deg(D) = -e < 0$, so, using Lemma 2.5 and Riemann-Roch's Theorem, we have that

  $$\dim|-2K_Y| = h^0(\mathcal{O}_Y(-2K_Y)) - 1 =$$
  $$= h^0(\mathcal{O}_\Gamma(2D)) + h^0(\mathcal{O}_\Gamma(D)) + h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-2D)) - 1 =$$
  $$= h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-2D)) - 1 = 1 + e + 2e - 1 = 3e.$$

  Moreover, always by Lemma 2.5, we can compute that

  $$h^0(\mathcal{O}_Y(-2K_Y - 2C_0)) = h^0(\mathcal{O}_Y(2C_0 - 2D \cdot f)) =$$
  $$= h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-2D)) = 3e + 1 = h^0(\mathcal{O}_Y(-2K_Y)),$$

  while

  $$h^0(\mathcal{O}_Y(-2K_Y - 3C_0)) = h^0(\mathcal{O}_Y(C_0 - 2D \cdot f)) =$$
  $$= h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-2D)) = 3e \neq h^0(\mathcal{O}_Y(-2K_Y)).$$

  We conclude that $2C_0$ is a fixed component of $|-2K_Y|$, leaving as variable part the system $|2C_0 - 2D \cdot f| = |2C_1|$.

□

We can show that not all the cases of minimal ruled surfaces Y of the type of Proposition 2.12 can be minimal models of surfaces with Prym-canonical hyperplane sections.

Lemma 2.13. *Let $X'' = \mathbb{P}_\Gamma(\mathcal{E})$ be a minimal ruled surface over a smooth elliptic curve Γ. We assume that $\mathcal{E}$ is normalized and decomposable, i.e. $\mathcal{E} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$, for $D \in \mathrm{Div}(\Gamma)$ and $e = -\deg(D) \geq 0$.*
If $e = 0$ and $D \sim 0$ (CASE IV.1 of Proposition 2.12) or if $e = 0$, $D \nsim 0$ and $2D \nsim 0$ (CASE IV.2 of Proposition 2.12), then X'' cannot be the minimal model of a surface with Prym-canonical hyperplane sections.

Proof. We analyze the two cases separately.

- Let $D \sim 0$.

  Let us suppose that $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)) = \Gamma \times \mathbb{P}^1$ is the minimal model of a surface X with Prym-canonical hyperplane sections. With the same notation as before, we have that $\pi : X' \to X$ is the minimal resolution of the singularities of X.

  We exclude the case $X'' = X'$ because $\dim |-2K_{X''}| = 4$ by Proposition 2.12 and this would contradict Theorem 1.12. Then there are always base points of L'' to blow up and $X'' \neq X'$.

  Since $K_\Gamma \sim 0$, we can write $-2K_{X''} \sim 4C_0$, where $C_0 \cong \Gamma$ (see Notation 2.7). So there are at most four disjoint connected components in W'' (the fibres can coincide).

  We put $W'' = C_{0,1} + C_{0,2} + C_{0,3} + C_{0,4}$, where $C_{0,1}, C_{0,2}, C_{0,3}, C_{0,4} \in |C_0|$ not necessarily disjoint. We have observed that the base locus of L'' lies on $\mathrm{supp}(W'')$ (see Proposition 1.22), and, as seen before, there are always base points of L'' to blow up. Moreover C'' and W'' intersect only on the base locus of L'' by Proposition 1.22 and Proposition 1.24. Since $W'' \sim 4C_0$, then C'' intersects C_0 and consequently $C_{0,i}$, for $i = 1, 2, 3, 4$. The intersection points between C'' and $C_{0,i}$, for $i = 1, 2, 3, 4$, are base points of L'', i.e. the points to blow up.

  We recall that the multiplicity of any blown up point on W'' by ϕ is at least two (see Proposition 1.22). If we suppose that W'' is formed by four disjoint fibres either W'' contains three disjoint components of which only one is non-reduced counted with multiplicity 2 or W'' has two disjoint components isomorphic to Γ, one reduced and the other counted with multiplicity 3, then the multiplicity of the blown up points by ϕ on the reduced components of W'' is one. We exclude these three cases.

  It remains to analyze the case in which W'' has two disjoint components non-reduced, counted with multiplicity 2, whose supports are called respectively $C_{0,1}$ and $C_{0,2}$, or W'' has only one irreducible component of multiplicity 4, whose support $\overline{C_0}$ is isomorphic to Γ.

  We observe that $-K_{X''} = -pr_1^* K_\Gamma - pr_2^* K_{\mathbb{P}^1} = pr_2^*(-K_{\mathbb{P}^1})$, i.e. every anticanonical divisor is formed by two fibres of the projection $pr_2 : X'' \to \mathbb{P}^1$. To obtain X', we blow up the base points of L'' that lie on $\mathrm{supp}(W'') = C_{0,1} + C_{0,2}$ or $\mathrm{supp}(W'') = \overline{C_0}$ respectively. In the first case, since $C_{0,1} + C_{0,2} \in |-K_{X''}|$ and it contains the points to blow up, then $\widetilde{C_{0,1}} + \widetilde{C_{0,2}} \in |-K_{X'}|$ and $h^0(\mathcal{O}_{X'}(-K_{X'})) > 0$. Similarly, if $W'' = 4\overline{C_0}$, then $2\overline{C_0} \in |-K_{X''}|$ and $h^0(\mathcal{O}_{X'}(-K_{X'})) > 0$. Moreover, in both cases, $h^0(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$ (the anticanonical divisor does not contain C'). From the exact sequence

$$0 \to \mathcal{O}_{X'}(-K_{X'} - C') \to \mathcal{O}_{X'}(-K_{X'}) \to \mathcal{O}_{C'}(-K_{X'}) \to 0,$$

we deduce that $h^0(\mathcal{O}_{C'}(-K_{X'})) > 0$. Since X has Prym-canonical hyperplane sections by hypothesis and X' is the minimal resolution of its singularities, then $C' \cdot (-K_{X'}) = 0$, for C' a general hyperplane section of X'. So $h^0(\mathcal{O}_{C'}(-K_{X'})) = 1$ and $\mathcal{O}_{C'}(-K_{X'}) \cong \mathcal{O}_{C'}$. Then C' is a canonical embedded curve and X has canonical hyperplane sections. We also exclude these cases.

In conclusion $X'' = \Gamma \times \mathbb{P}^1$ cannot be the minimal model of a surface X with Prym-canonical hyperplane sections.

- Let D be a non-zero torsion divisor of order different from two (Proposition 2.12, CASE IV.2). Then there are two disjoint connected components in W'', $2C_0$ and $2C_1$.

  By Proposition 1.22, since $2C_0$ and $2C_1$ are irreducible and non-reduced with multiplicity 2, we have that $W' \sim 2\widetilde{C_0} + 2\widetilde{C_1}$, where $\widetilde{C_0}$ and $\widetilde{C_1}$ are respectively the strict transforms of C_0 and C_1. In this case W' has two disjoint connected components.

  Moreover we have that $-K_{X''} = \{C_0 + C_1\}$ (see [13], Proposition II.2.1). To obtain X', we blow up the base points of L'' that lie on $\operatorname{supp}(W'') = C_0 + C_1$. Since $C_0 + C_1 \in |-K_{X''}|$ and it contains the points to blow up, then $h^0(\mathcal{O}_{X'}(-K_{X'})) > 0$. As seen before, we obtain that $h^0(\mathcal{O}_{C'}(-K_{X'})) > 0$. Since X has Prym-canonical hyperplane sections by hypothesis and X' is the minimal resolution of its singularities, then $C' \cdot (-K_{X'}) = 0$, for C' a general hyperplane section of X'. So $h^0(\mathcal{O}_{C'}(-K_{X'})) = 1$ and $\mathcal{O}_{C'}(-K_{X'}) \cong \mathcal{O}_{C'}$. Then X has canonical hyperplane sections. Similarly, if $X'' = X'$, i.e. L'' has not base points to blow up and $\phi : X' \to X''$ is an isomorphism, then X has canonical hyperplane sections.

  In conclusion, the surface $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$, for D a non zero torsion divisor, not of order two, cannot be the minimal model of a surface X with Prym-canonical hyperplane sections.

□

Finally, we can determine the number of non-rational singularities on a surface X with Prym-canonical hyperplane sections birationally equivalent to a non-rational ruled surface.

Proposition 2.14. *Let X be a surface with Prym-canonical hyperplane sections birationally equivalent to a ruled surface $X'' = \mathbb{P}_\Gamma(\mathcal{E})$ over a base curve Γ of genus $q \geq 2$. If $W'' \in |-2K_{X''}|$ effective, then it contains only one connected component, except for the case $\mathcal{E}$ decomposable and $e = 4q-4$, where there is the possibility that W'' contains two disjoint connected components.*

Proof. By Proposition 2.12 and Lemma 2.13, we have four different cases to study.

- If $\mathcal{E}$ is indecomposable and $|-2K_{X''}| = \{4C_0\}$ (Proposition 2.12, CASE I.2), then $W'' = 4C_0$ is connected (it is irreducible and non-reduced).

- If $\mathcal{E}$ is indecomposable and $|-2K_{X''}| = \{4C_0 - 4K_\Gamma \cdot f\}$ (Proposition 2.12, CASE I.1), we can prove that W'' contains only one connected component. Let us suppose that $\Delta := 4C_0 - 4K_\Gamma \cdot f$ can be factorized as $\Delta_1 + ... + \Delta_k + F_1 + ... + F_l$, where Δ_i are irreducible curves, for $i = 1, ..., k$, and F_j fibers, for $j = 1, ..., l$. Since $\Delta \equiv 4C_0 - 8f$, then $\Delta_1 + ... + \Delta_k \equiv 4C_0 - (8+l)f$. So every Δ_i is of the form $\alpha_i C_0 - \beta_i \cdot f$, with $\sum_{i=1}^k \deg(\beta_i) = 8 + l$ and $\alpha_i > 0$ for any i because we assume that Δ_i is effective (we also exclude $\alpha = 0$ because $\Delta_i \neq F_j$, for any i, j).

  - Since $\Delta \equiv 4C_0 - 8f$, then $k \leq 4$. Let us start with the case in which $\alpha_i = 1$, for $i = 1, ..., 4$. Since C_0 is irreducible and $C_0^2 = -e$, for $e = -2$ (see Proposition 2.12, CASE I.1), then $C_0 \cdot \Delta_i = C_0(C_0 - \beta_i \cdot f) = 2 - \deg(\beta_i) \geq 0$, so $\deg(\beta_i) \leq 2$, for any $i = 1, ..., 4$. Then $8 = \sum_{i=1}^4 2 \geq \sum_{i=1}^4 \deg(\beta_i) = 8 + l$, so $l = 0$ and $\deg(\beta_i) = 2$, for any i.
    Therefore, for $i \neq j$, the intersection $\Delta_i \cdot \Delta_j = (C_0 - \beta_i \cdot f) \cdot (C_0 - \beta_j \cdot f) = C_0^2 - \deg(\beta_i) - \deg(\beta_j) = 2 - 2 - 2 < 0$. Thus $\Delta_i = \Delta_j$ (Δ_i and Δ_j are irreducible, so they cannot have common components) and $\Delta = 4\Delta_i$. So the antibicanonical divisor is irreducible and non-reduced, in particular connected.
  - If $k = 3$, then $\alpha_1 = 2$ and $\alpha_i = 1$, for $i = 2, 3$. Since C_0 is irreducible and $C_0^2 = -e$, with $e = -2$ as seen in Proposition 2.12, CASE I.1, then $C_0 \cdot \Delta_j \geq 0$, for $j = 1, 2, 3$. In particular, we have $C_0 \cdot \Delta_1 = C_0(2C_0 - \beta_1 \cdot f) = 4 - \deg(\beta_1) \geq 0$, so $\deg(\beta_1) \leq 4$, and $C_0 \cdot \Delta_i = C_0(C_0 - \beta_i \cdot f) = 2 - \deg(\beta_i) \geq 0$, so $\deg(\beta_i) \leq 2$, for $i = 2, 3$. Then $8 = 4 + 2 + 2 \geq \sum_{i=1}^3 \deg(\beta_i) = 8 + l$, so $l = 0$ and $\deg(\beta_1) = 4$ while $\deg(\beta_i) = 2$, for $i = 2, 3$. So the intersection $\Delta_1 \cdot \Delta_2 = (2C_0 - \beta_1 \cdot f) \cdot (C_0 - \beta_2 \cdot f) = 2C_0^2 - \deg(\beta_1) - 2\deg(\beta_2) = 4 - 4 - 4 < 0$. This is impossible because Δ_1 and Δ_2 have not common components because they are irreducible and distinct by hypothesis. So we exclude this case.
  - If $k = 2$, then either $\alpha_1 = \alpha_2 = 2$ or $\alpha_1 = 3$ and $\alpha_2 = 1$.
    * If $\alpha_1 = \alpha_2 = 2$, then, as observed in the previous cases, the intersection $C_0 \cdot \Delta_i = C_0(2C_0 - \beta_i \cdot f) = 4 - \deg(\beta_i) \geq 0$, for $i = 1, 2$, so $\deg(\beta_i) \leq 4$, for any i. As before, we can show that $\deg(\beta_i) = 4$, for $i = 1, 2$, and $l = 0$.
      Then $\Delta_1 \cdot \Delta_2 = 4C_0^2 - 2\deg(\beta_1) - 2\deg(\beta_2) = -4e - 8 - 8 < 0$ because $e = -2$ (see Proposition 2.12, CASE I.1). Since Δ_1 and Δ_2 are irreducible, they do not have common components, so $\Delta_1 = \Delta_2$ and $\Delta = 2\Delta_i$, that is connected.
    * If $\alpha_1 = 3$ and $\alpha_2 = 1$, then, with the same techniques as before, we have that $C_0 \cdot \Delta_1 = C_0(3C_0 - \beta_1 \cdot f) = 6 - \deg(\beta_1) \geq 0$, so $\deg(\beta_1) \leq 6$, and $C_0 \cdot \Delta_2 = C_0(C_0 - \beta_2 \cdot f) = 2 - \deg(\beta_2) \geq 0$, so $\deg(\beta_2) \leq 2$. Then $8 = 6 + 2 \geq \deg(\beta_1) + \deg(\beta_2) = 8 + l$, so $l = 0$, $\deg(\beta_1) = 6$ and $\deg(\beta_2) = 2$. Since $e = -2$ and $C_0^2 = -e$, the intersection $\Delta_1 \cdot \Delta_2 = (3C_0 - \beta_1 \cdot f) \cdot (C_0 - \beta_2 \cdot f) = 3C_0^2 - \deg(\beta_1) - 3\deg(\beta_2) < 0$. This is

impossible because Δ_1 and Δ_2 have not common components because they are irreducible and distinct by hypothesis. So we also exclude this case.

– Finally, if $k = 1$, then $\alpha_1 = 4$. Since C_0 is irreducible and $C_0^2 = -e$, with $e = -2$ as seen in Proposition 2.12, CASE I.1, then $C_0 \cdot \Delta_1 = C_0 \cdot (4C_0 - \beta_1 \cdot f) = 8 - \deg(\beta_1) \geq 0$, so $\deg(\beta_1) \leq 8$. Since $8 \geq \deg(\beta_1) = 8 + l$, then $l = 0$ and $\deg(\beta_1) = 8$. Then $\Delta = \Delta_1$ and it is an irreducible component by construction.

Then the claim is proved: in any analyzed case W'' contains only one connected component.

- If $\mathcal{E}$ is decomposable and $\dim |-2K_{X''}| = \dim |-2K_\Gamma - 2D| \geq 0$ (see Proposition 2.12, CASE II.1), then $-2K_{X''} \sim 4C_0 + (-2K_\Gamma - 2D) \cdot f$, with $4C_0$ a fixed component. We observe that, since $e \geq 2q - 2$, then $\deg(-2K_\Gamma - 2D) = -2(2q-2) + 2e \geq 0$. In particular, if $e = 2q - 2$ and $\deg(-2K_\Gamma - 2D) = 0$, then $-2K_\Gamma - 2D \sim 0$ since $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - 2D)) > 0$. Therefore $(-2K_\Gamma - 2D) \cdot f$ is the variable part of $-2K_{X''}$. Since $C_0 \cdot f = 1$, we have that the fixed part $4C_0$ and the variable part $(-2K_\Gamma - 2D) \cdot f$ have intersection, so there is only one connected component in W''.

- If $\mathcal{E}$ is decomposable and $\dim |-2K_{X''}| > \dim |-2K_\Gamma - 2D|$, then $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) > 0$ and $e \geq 4q - 4$ (see Proposition 2.12, CASE II.2). We recall that $-2K_{X''} \sim 4C_0 + (-2K_\Gamma - 2D) \cdot f$, where $\deg(-2K_\Gamma - 2D) = -2(2q-2) + 2e \geq 4 - 4q + 2(4q-4) = 4q - 4 > 0$ since $q \geq 2$.

  Let us see which is the fixed component of $|-2K_{X''}|$. Since $C_0^2 = -e \leq 4 - 4q$, then $C_0(4C_0 + (-2K_\Gamma - 2D) \cdot f) = -4e + 4 - 4q + 2e < 0$, again $C_0(3C_0 + (-2K_\Gamma - 2D) \cdot f) = -3e + 4 - 4q + 2e < 0$ and finally $C_0(2C_0 + (-2K_\Gamma - 2D) \cdot f) = -2e + 4 - 4q + 2e < 0$. Thus $3C_0$ is a fixed component of $|-2K_{X''}| = |4C_0 + (-2K_\Gamma - 2D) \cdot f|$. Let us check if C_0 is also a fixed component of $|C_0 + (-2K_\Gamma - 2D) \cdot f|$.

  By Lemma 2.5 we have that

$$h^0(\mathcal{O}_{X''}(C_0 + (-2K_\Gamma - 2D) \cdot f)) = h^0(\mathcal{O}_\Gamma(-2K_\Gamma - 2D)) + h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D))$$

  and

$$h^0(\mathcal{O}_{X''}((-2K_\Gamma - 2D) \cdot f)) = h^0(\mathcal{O}_\Gamma(-2K_\Gamma - 2D)).$$

  Since $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) > 0$, then

$$h^0(\mathcal{O}_{X''}(C_0 + (-2K_\Gamma - 2D) \cdot f)) > h^0(\mathcal{O}_{X''}((-2K_\Gamma - 2D) \cdot f))$$

  and C_0 is not a fixed component of $|C_0 + (-2K_\Gamma - 2D) \cdot f|$, in particular $4C_0$ is not a fixed component of $|-2K_{X''}| = |4C_0 + (-2K_\Gamma - 2D) \cdot f|$ while $3C_0$ is.

  So the following cases may occur:

- if $4C_0$ is a component of $W'' \sim 4C_0 + (-2K_\Gamma - 2D) \cdot f$, then $4C_0$ intersects $(-2K_\Gamma - 2D) \cdot f$ since $C_0 \cdot f = 1$, so W'' has only one connected component;
- if $e > 4q - 4$ and $3C_0$ is a component of $W'' \sim 4C_0 + (-2K_\Gamma - 2D) \cdot f$, then $3C_0 \cdot (C_0 + (-2K_\Gamma - 2D) \cdot f) = -3e + 3\deg(-2K_\Gamma - 2D) = -3e + 3(4 - 4q + 2e) = -12q + 12 + 3e > 12 - 12q + 3(4q - 4) = 0$, so W'' has only one connected component;
- finally, we suppose that $e = 4q - 4$ and $3C_0$ is a component of $W'' \sim 4C_0 + (-2K_\Gamma - 2D) \cdot f$. By Remark 2.8, we have that $D \sim C_0|_{C_0}$. Since $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) > 0$ (we are in CASE II.2 of Proposition 2.12) and $\deg(-D - 2K_\Gamma) = e - 4q + 4 = 0$, then $D \sim -2K_\Gamma$. The restriction

  $$(C_0 + (-2K_\Gamma - 2D) \cdot f)|_{C_0} = C_0|_{C_0} + (-2K_\Gamma - 2D) \sim D - 2K_\Gamma - 2D \sim 0,$$

  so $3C_0$ and $C_0 + (-2K_\Gamma - 2D) \cdot f$ are the two disjoint connected components of W''.

□

Corollary 2.15. *Let X be a surface with Prym-canonical hyperplane sections birationally equivalent to a ruled surface X'' over a base curve Γ of genus $q \geq 2$. Let us suppose that a general hyperplane section C of X is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$. If the number of connected components of W' and W'' is the same, then either X contains exactly one non-rational singularity x, whose $p_g(x) = q$ or, if $\mathcal{E}$ is decomposable and $e = 4q - 4$, there is also the possibility that there are two singularities x_1 and x_2 such that $\pi^{-1}(x_i) \cap \mathrm{supp}(W') \neq \varnothing$ and $p_g(x_1) + p_g(x_2) = q$.*

Proof. By Lemma 1.19 we know that the singularities of the type $x \in X$ such that $\pi^{-1}(x)$ does not intersect $\mathrm{supp}(W')$ are rational double points. Moreover, by Theorem 1.12, we have that $\mathrm{supp}(W') = \pi^{-1}(x_1, ..., x_r)$, for certain singularities in X, so there are r connected components in $\mathrm{supp}(W')$, for $r = 0, 1, 2, ...$ By Proposition 1.15, we know that, if the singular locus $\mathrm{Sing}(X) = \{x_1, ..., x_s\}$, for $r \leq s$, then $\sum_{i=1}^{s} p_g(x_i) = q$.

Let us suppose that $r := \{$number of connected components of $W'\}$ is the same as the number of connected components of W'' (see Proposition 1.22, Case 3.).
By Proposition 2.14, we have $W'' \in |-2K_{X''}|$ is connected, except in the case in which $\mathcal{E}$ is decomposable and $e = 4q - 4$. Since we suppose that W' has the same number of connected component of W'', if W'' is connected, then X has only one singularity x such that $\pi^{-1}(x) \cap \mathrm{supp}(W') \neq \varnothing$. By Proposition 1.15, this singularity is non-rational with $p_g(x) = q$. Instead, if W'' has two disjoint connected components, since W' is contracted by π, we have two singularities x_1 and x_2 on X such that $\pi^{-1}(x_1) \cup \pi^{-1}(x_2) = \mathrm{supp}(W')$ and $p_g(x_1) + p_g(x_2) = q$ (see Propositon 1.15, Theorem 1.12 and Lemma 1.19). □

Remark 2.16. *In the hypothesis of the previous Corollary, if $q = 1$, whatever the number of connected components of W'' and W' we have that there is only one non-rational singularity x on X, whose geometric genus $p_g(x) = 1$ (see Proposition 1.15).*

Remark 2.17. *In the hypothesis of the above Corollary, if $q \geq 2$ and the number of connected components of W' and W'' is not the same, we do not know how many non-rational singularities there are on X.*

Remark 2.18. *If a general hyperplane section C of X is not projectively normal, then Proposition 1.15 is not valid. Anyway we can conclude that Corollary 2.15 is still valid, except for the fact that we do not know the geometric genus of the singularities.*

2.3 Construction of surfaces with Prym-canonical hyperplane sections birationally equivalent to non-rational ruled surfaces

In this section, we will construct examples of surfaces with Prym-canonical hyperplane sections birationally equivalent to ruled surfaces over a base curve of genus $q \geq 1$. Before this, we will study some general properties regarding minimal models of this type of surfaces.

First of all we can list some notation that we will use in the following.

- Let X be a surface with Prym canonical hyperplane sections embedded in $\mathbb{P}^{g-1}$, for $g \geq 5$.
- Let $\pi : X' \to X$ be the minimal resolution of the singularities of X and let $\phi : X' \to X''$ be a relatively minimal model of X'. We assume that X'' is a ruled surface, so $X'' \cong \mathbb{P}_\Gamma(\mathcal{E})$, with $\mathcal{E}$ normalized, $\Lambda^2\mathcal{E} = \mathcal{O}_\Gamma(D)$, for $D \in \operatorname{Div}(\Gamma)$, and $e = -\deg(D)$. Let $p : X'' \to \Gamma$ be the natural projection of X'' onto its base curve Γ. We also assume that $p_g(\Gamma) = q \geq 1$, so X'' is a non-rational ruled surface.
- If L is the $(g-1)-$dimensional complete linear system of hyperplane sections of X and $C \in L$ is its general curve, let L' be the complete linear system of hyperplane sections of X' such that $i_{L'} : X' \to \mathbb{P}^{g-1}$ and $C' \in L'$ is its general curve. Then $L'' = \phi_* L'$ is a linear system on X'', with $C'' \in L''$ a general curve. Also $i_{L''} : X'' \dashrightarrow \mathbb{P}^{g-1}$, with
$$i_{L''}(X'') = i_L(X).$$
- We have already seen that the divisors on X'' are of the form $aC_0 + \Delta \cdot f$, for $a \in \mathbb{Z}$ and $\Delta \in \operatorname{Pic}(\Gamma)$. Then $L'' \subseteq |aC_0 + \Delta \cdot f|$, for some a, Δ. In particular, we suppose that L'' has base points P_{t,i_t} with multiplicity r_{t,i_t}, for $t = 1, ..., s-1$ and $i_t = 1, ..., i(t)$, with possible infinitely near base points. We can observe that X' is the blowing up of X'' along the base points of L'', that lie on $\operatorname{supp}(W'') \subset X''$ by Proposition 1.22. In the following, we will assume that $|aC_0 + \Delta \cdot f|$ is base-point free (we will see in Proposition 2.23 when this occurs). If also L'' does not have base points, then $L'' = |aC_0 + \Delta \cdot f| = L'$ and $X' = X''$.

How do we construct surfaces X with Prym-canonical hyperplane sections birationally equivalent to non-rational ruled surfaces? Let us give an outline of the program.

1. Let X'' be a minimal ruled surface over a base curve of genus $q \geq 1$ such that $|-2K_{X''}| \neq \varnothing$ and let W'' be a fixed effective antibicanonical divisor.
2. Let us fix a linear subsystem $L'' \subseteq |aC_0 + \Delta \cdot f|$, for $a \geq 1$ and $\Delta \in \mathrm{Pic}(\Gamma)$, such that the base points of L'' lie on the supp(W'') and L'' and W'' do not intersect outside the base points of L''. We suppose that $|aC_0 + \Delta \cdot f|$ is base-point free.
3. We call $\phi : X' \to X''$ the blowing up of X'' along the base points of L''. Let L' be the strict transform of L'' and let W' be an antibicanonical divisor of X'. If a general $C' \in L'$ is disjoint from W', i.e. $-2K_{X'}|_{C'} \sim 0$, and if $-K_{X'}|_{C'}$ is a non-zero torsion divisor of $\mathrm{Pic}(C')$, then X' is the required surface.

2.3.1 Preliminaries on minimal models

We can replace X'' by a more suitable minimal model in which the transform of the linear system L'' has a simpler form.

Proposition 2.19. *Let $X'' = \mathbb{P}_\Gamma(\mathcal{E})$ be a ruled surface over a smooth curve Γ of genus $q \geq 1$ and let us suppose that X'' is the minimal model of a surface X with Prym-canonical hyperplane sections. Let e be the invariant of X''. Let $L'' \subseteq |aC_0+\Delta\cdot f|$ be the $(g-1)-$dimensional linear system of hyperplane sections of X''.*

a. We have that $e \geq 2q-2$, except for the cases $I.1$ and $III.2$ of Proposition 2.12, that correspond to ruled surfaces with $\mathcal{E}$ indecomposable.

Moreover $e = 2q-2$ if and only if $2D \sim -2K_\Gamma$.

b. We can assume that $\mathcal{E}$ is decomposable, i.e. $\mathcal{E} \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$, for $D \in \mathrm{Div}(\Gamma)$, and a general curve $C'' \in L''$ is disjoint from C_0 after possibly replacing X'' by another relatively minimal model of X'. In particular, we can assume that the base points P_{t,i_t} of L'' do not belong to C_0 and to its strict transforms, for $t = 1, ..., s-1$ and $i_t = 1, ..., i(t)$.

Then we have the following relations:

b.1 $r_{t,i_t} \leq a-1$, *for* $t = 1, ..., s-1$ *and* $i_t = 1, ..., i(t)$;

b.2 $\Delta \sim -aD$, *whence* $L'' \subseteq |aC_1|$, *for* $a \geq 1$;

b.3 $\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} = a(e-2q+2)$;

b.4 $g = 1 + \frac{1}{2}a^2e - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2$;

b.5 $1 + a^2(q-1) + \frac{1}{2}a(e-2q+2) \leq g \leq 1 + \frac{1}{2}a^2e - \frac{1}{2}a(e-2q+2)$;

b.6 if L'' has base points, the number of conditions imposed by them is exactly the expected number.

Proof. a. As seen in Proposition 2.12, we always have $e \geq 2q-2$, except for the CASE $I.1$, where $e=-2<2q-2=2$, and CASE $III.2$, where $e=-1$ and $2q-2=0$. Both cases correspond to ruled surfaces with $\mathcal{E}$ indecomposable.

If $2D \sim -2K_\Gamma$, then $\deg(2D)=-2e=\deg(-2K_\Gamma)=4-4q$, so $e=2q-2$.

Let $q \geq 2$. If $\mathcal{E}$ is indecomposable and $e=2q-2$, then we are in CASE $I.2$ of Proposition 2.12. As computed in that Proposition, we have that $2D \sim -2K_\Gamma$.

Again, let $q \geq 2$. If $\mathcal{E}$ is decomposable and $e=2q-2$, then we are in CASE $II.1$ of Proposition 2.12. Thus we have that $h^0(\mathcal{O}_\Gamma(-2K_\Gamma-2D))>0$. Since $\deg(-2K_\Gamma-2D)=0$, then $2D \sim -2K_\Gamma$.

If $q=1$ and $X''=Y_0$ (CASE $III.1$ of Proposition 2.12), then $0=e=2q-2$ and $2D \sim -2K_\Gamma \sim 0$. On the other hand, if $X''=Y_{-1}$ (CASE $III.2$ of Proposition 2.12), then $-1=e \neq 2q-2=0$. Since $\deg(2D)=2$ and $\deg(-2K_\Gamma)=0$, then $2D \nsim -2K_\Gamma$. The thesis is satisfied also in the case $q=1$ and $\mathcal{E}$ indecomposable.

Finally, we assume $q=1$ and $\mathcal{E}$ decomposable. If $e=2q-2=0$, then we exclude CASE $IV.1$ and $IV.2$ of Proposition 2.12 by Lemma 2.13. So there is only one case to consider: $D \nsim 0$ but $2D \sim 0$ (CASE $IV.3$ of Proposition 2.12). Whence it is clear that $2D \sim -2K_\Gamma \sim 0$.

In conclusion $e=2q-2 \Leftrightarrow 2D \sim -2K_\Gamma$.

b. In general, any decomposable ruled surface is obtained from an indecomposable one applying a finite number of elementary transformations at points on C_0 (see [15], Corollary 48). Moreover, if the minimal resolution of singularities $X'=\mathbb{P}_\Gamma(\mathcal{E})$, with $\mathcal{E}$ indecomposable, then the minimal model X'' is birationally equivalent to $\mathbb{P}_\Gamma(\mathcal{E}')$, for $\mathcal{E}'$ decomposable. So we can suppose that the minimal model is decomposable up to birational equivalence and a finite number of elementary transformations at points on C_0.

Let us suppose that L'' has base points on C_0, in particular we call P one of them. If $C'' \in L''$ is a general curve, then $C'' \cdot C_0 > 0$. We can apply an elementary transformation of X'' centered in P to obtain another relatively minimal model $\widetilde{X''}$. If $\widetilde{C''}$ and $\widetilde{C_0}$ are the strict transforms respectively of C'' and C_0, since we blow up the point P of intersection, then $\widetilde{C''} \cdot \widetilde{C_0} < C'' \cdot C_0$. Repeating this process a finite number of times, we can assume that L'' has not base points on C_0. Moreover, since we assume that X'' is decomposable and we blow up points of C_0, then even the new minimal model is decomposable (see [15], Theorem 50).

We can also prove that a general curve $C'' \in L''$ does not intersect C_0 outside the base points. Indeed, after the blowing up of X'' along the base points of L'', we obtain X' a surface whose hyperplane sections are Prym-canonical embedded. With the notation of Proposition 1.22, since

$$W'' := W_1' \sim 4C_0 - (2K_\Gamma + 2D) \cdot f \equiv 4C_0 + (4-4q+2e)f,$$

then

$$W_2' \equiv 4\widetilde{C_0} + (4 - 4q + 2e) \cdot \widetilde{f} + \sum_{i_1=1}^{i(1)} (\mu_{1,i_1} - 2) E_{2,i_1},$$

where $\widetilde{C_0}$ and $\widetilde{f}$ are the strict transforms respectively of C_0 and a generic fibre f of X'', μ_{1,i_1} is the multiplicity of the base point P_{1,i_1} on W'' and E_{2,i_1} is the exceptional divisor associated with P_{1,i_1}, for $i_1 = 1, ..., i(1)$. We observe that

$$C_2' \cdot W_2' = 4C_2' \cdot \widetilde{C_0} + (4 - 4q + 2e) C_2' \cdot \widetilde{f} + \sum_{i_1=1}^{i(1)} (\mu_{1,i_1} - 2) C_2' \cdot E_{2,i_1},$$

where C_2' is the strict transform of C''. Since $e \geq 2q - 2$ by Case a. of this Proposition, then $(4-4q+2e) \geq 0$ and consequently $(4-4q+2e)C_2' \cdot \widetilde{f} \geq 0$. Again, as observed in Proposition 1.22, we have that $\mu_{1,i_1} \geq 2$ for any $i_1 = 1, ..., i(1)$, so $\sum_{i_1=1}^{i(1)} (\mu_{1,i_1} - 2) C_2' \cdot E_{2,i_1} \geq 0$. If C'' and C_0 intersected outside the base points, then C_2' and $\widetilde{C_0}$ would have intersection. Hence $C_2' \cdot W_2' > 0$. Iterating this process, we would have that $W_t' \cdot C_t' > 0$, for $t = 1, .., s$. This would contradict the Lemma 1.23.

We conclude that C'' and C_0 have not intersection points.

With the previous assumptions, let us see the relations between the numbers a, e, q, g and r_{t,i_t}, for $t = 1, ..., s - 1$ and $i_t = 1, ..., i(t)$.

- b.1 First of all we observe that, if $P_{t_2,i_{t_2}}$ is a base point of L'' infinitely near to the base point $P_{t_1,i_{t_1}} \in X''$, for $t_1, t_2 \in \{1, ..., s-1\}$, $i_{t_1} = \{1, ..., i(t_1)\}$ and $i_{t_2} = \{1, ..., i(t_2)\}$, then $r_{t_1,i_{t_1}} \geq r_{t_2,i_{t_2}}$. So it is enough to show that the multiplicity $r := r_{1,1} < a$, for a base point $P := P_{1,1} \in X''$ that is not infinitely near to another point.

  Let f_0 be the fibre on X'' passing through P. Since $P \in C''$ with multiplicity r, then $r \leq C'' \cdot f_0$. On the other hand, since $C'' \in L'' \subseteq |aC_0 + \Delta \cdot f|$, we have that $C'' \cdot f_0 = (aC_0 + \Delta \cdot f) \cdot f_0 = a$. Then $r \leq a$. Let us suppose that $r = a$, hence C'' and f_0 intersect only in P, so, after blowing up this point, the strict transforms $\widetilde{C''}$ and $\widetilde{f_0}$ respectively of C'' and f_0 are disjoint and, in particular, $\widetilde{f_0}^2 = -1$. To obtain X' we blow up only base points of L'' and since $\widetilde{C''}$ and $\widetilde{f_0}$ are disjoint, we do not blow up other points on $\widetilde{f_0}$, so, if f_0' is the strict transform of f_0 on X', also $(f_0')^2 = -1$. Then f_0' is a $(-1)-$curve on X'. We also observe that C' and f_0' are still disjoint, so f_0' is contracted by $i_{L'} = i_L \circ \pi$ on X. This contradicts the minimality of the resolution π, so $r < a$.

- b.2 As observed before, we can assume that a general curve $C'' \in L''$ is disjoint from C_0. On one hand

$$\mathcal{O}_{X''}(C'') \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_{C_0} \cong \mathcal{O}_\Gamma$$

with the identification of Notation 2.7. On the other hand, using Remark 2.8, we have that

$$\mathcal{O}_{X''}(C'') \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_{X''}(aC_0 + \Delta \cdot f) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_{\Gamma}(aD + \Delta).$$

Hence

$$\Delta \sim -aD.$$

Then $C'' \sim aC_0 - aD \cdot f \sim aC_1$ and

$$L'' \subseteq |aC_1|.$$

b.3 We claim that

$$2\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} = C'' \cdot W'',$$

for r_{t,i_t} the multiplicities of all the base points of L''. Since $C' \cdot W' := C'_s \cdot W'_s = 0$ by Lemma 1.23, it is sufficient to show that

$$C'_t \cdot W'_t = C'_{t+1} \cdot W'_{t+1} + 2\sum_{i_t=1}^{i(t)} r_{t,i_t},$$

for $t = 1, ..., s-1$, so $C'_{s-1} \cdot W'_{s-1} = 2\sum_{i_{s-1}=1}^{i(s-1)} r_{s-1,i_{s-1}}$, $C'_{s-2} \cdot W'_{s-2} = 2\sum_{i_{s-1}=1}^{i(s-1)} r_{s-1,i_{s-1}} + 2\sum_{i_{s-2}=1}^{i(s-2)} r_{s-2,i_{s-2}}$ and so on and the claim is proved.

Let us assume that $t = 1$. By Proposition 1.22, we know that $W'_2 \sim \phi_1^* W'_1 - 2\sum_{i_1=1}^{i(1)} E_{2,i_1}$. If $C'_2 \in X'_2$ is the strict transform of $C'' := C'_1$, then $C'_2 \sim \phi_1^* C'_1 - \sum_{i_1=1}^{i(1)} r_{1,i_1} E_{2,i_1}$. So

$$W'_2 \cdot C'_2 = (\phi_1^* W'_1 - 2\sum_{i_1=1}^{i(1)} E_{2,i_1}) \cdot (\phi_1^* C'_1 - \sum_{i_1=1}^{i(1)} r_{1,i_1} E_{2,i_1}) = W'_1 \cdot C'_1 - 2\sum_{i_1=1}^{i(1)} r_{1,i_1}$$

as asserted. Hence the claim is proved.

Moreover, using the previous point of this Proposition, we know that $C'' \cdot W'' = aC_1 \cdot (4C_0 - (2K_\Gamma + 2D) \cdot f) = a \cdot \deg(-2K_\Gamma - 2D) = a(4 - 4q + 2e)$. We can conclude that $2\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} = C'' \cdot W'' = a(4 - 4q + 2e)$. So the thesis is satisfied.

b.4 By the adjunction formula, the self-intersection $(C')^2 = 2g - 2 - C' \cdot K_{X'} = 2g - 2$ since $-K_{X'}|_{C'}$ is a two torsion element. If C' is the strict transform of C'', i.e. $C' = \phi^* C'' - \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} E_{t+1,i_t}$, and $C'' \sim aC_1$ by the point $b.2$ of this Proposition, then $(C')^2 = (\phi^* C'')^2 - \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = (C'')^2 - \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = (aC_1)^2 - \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = a^2 e - \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2$. Unifying the two results we have that $2g - 2 = a^2 e - \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2$ and $g = 1 + \frac{1}{2}a^2 e - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2$.

b.5 It is clear that

$$\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 \geq \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}.$$

By the points $b.3$ and $b.4$ of this Proposition, we also have that

$$g = 1 + \frac{1}{2}a^2 e - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 \leq 1 + \frac{1}{2}a^2 e - \frac{1}{2}a(e-2q+2).$$

This gives the upper bound to g.
Similarly, $\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2$ reaches its maximum value when every r_{t,i_t} is equal to $(a-1)$ (see point $b.1$ of this Proposition). Then

$$\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 \leq \sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}(a-1) = a(a-1)(e-2q+2)$$

using the point $b.3$ of this Proposition. So

$$g = 1 + \frac{1}{2}a^2 e - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 \geq 1 + \frac{1}{2}a^2 e - \frac{1}{2}a(a-1)(e-2q+2) =$$

$$= 1 + \frac{1}{2}a^2 e + \frac{1}{2}a(e-2q+2) - \frac{1}{2}a^2(e-2q+2) = 1 + a^2(q-1) + \frac{1}{2}a(e-2q+2).$$

This gives the lower bound to g.

b.6 Since we assume that $X'' = \mathbb{P}_\Gamma(\mathcal{E})$, with $\mathcal{E}$ decomposable, then the invariant $e \geq 2q-2$ by Case a. of this Proposition. Moreover the coefficient $a \geq 1$ by the previous assumptions and since $\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} = a(e-2q+2)$ by the point $b.3$ of this Proposition, then L'' has not base points if and only if $e = 2q-2$.
However, let us assume that L'' has base points, so $e > 2q-2$. If the base points were independent, we would have

$$\dim L'' = \dim |aC_1| - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}(r_{t,i_t}+1).$$

We know that

$$\dim |aC_1| = h^0(\mathcal{O}_{X''}(aC_1)) - 1 = h^0(\mathcal{O}_{X''}(aC_0 - aD\cdot f)) - 1.$$

By Lemma 2.5, we have that

$$h^0(\mathcal{O}_{X''}(aC_0 - aD\cdot f)) = h^0(\mathcal{O}_\Gamma(-aD)) + h^0(\mathcal{O}_\Gamma(-(a-1)D)) + \ldots + h^0(\mathcal{O}_\Gamma).$$

Using the Riemann-Roch Theorem, the cohomology

$$h^0(\mathcal{O}_\Gamma(-mD)) = h^0(\mathcal{O}_\Gamma(K_\Gamma + mD)) + me + 1 - q,$$

for $1 \leq m \leq a$. Since $\deg(K_\Gamma + mD) = 2q - 2 - me$ and $e > 2q - 2$ as observed before, we have that $\deg(K_\Gamma + mD) < 2q - 2 - m(2q - 2) \leq 0$, so $h^0(\mathcal{O}_\Gamma(K_\Gamma + mD)) = 0$. Then

$$\dim |aC_1| = h^0(\mathcal{O}_\Gamma(-aD)) + h^0(\mathcal{O}_\Gamma(-(a-1)D)) + ... + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma) - 1 =$$

$$= (ae + 1 - q) + ((a-1)e + 1 - q) + ... + (e + 1 - q) = (1 + .. + a)e + a(1 - q).$$

Furthermore, by the point $b.3$ of this Proposition, we know that $-\frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} = -\frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 - \frac{1}{2}a(e-2q+2)$. In conclusion, the expected dimension of L'' is

$$\dim |aC_1| - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}(r_{t,i_t} + 1) =$$

$$= (1 + ... + a)e + a(1 - q) - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 - \frac{1}{2}a(e - 2q + 2) =$$

$$= \frac{a(a+1)}{2}e + a(1 - q) - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 - \frac{1}{2}ae + a(q - 1) =$$

$$= \frac{1}{2}a^2 e - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2.$$

Since $\dim L'' = (g - 1)$ by hypothesis and $g - 1 = \frac{1}{2}a^2 e - \frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2$ by the point $b.4$ of this Proposition, we conclude the Theorem.

□

Remark 2.20. *From now on, we will assume the conclusions of Proposition 2.19 always satisfied.*

Since we suppose that $\mathcal{E}$ is decomposable, by Case a. of the previous Proposition we have that the invariant $e \geq 2q - 2$, for $q \geq 1$. It is not difficult to prove that, when $a \geq 2$, the linear system $|aC_1|$ is always base-point free, except for only one case. In addition, there are also cases in which the linear system $|C_1|$ is base-point free. Before showing this, we recall some preliminary results.

Proposition 2.21 ([15], Proposition 36)**.** *Let $|aC_0 + \Delta \cdot f|$ be an $a-$secant linear system on a decomposable ruled surface $Y \cong \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$. Then this linear system is base-point free on Y if and only if $|\Delta|$ and $|\Delta + aD|$ are base-point free on Γ.*

Proposition 2.22 ([15], Corollary 13)**.** *Let $Y = \mathbb{P}_\Gamma(\mathcal{E})$ be a ruled surface with $\Lambda^2(\mathcal{E}) = \mathcal{O}_\Gamma(D)$ and $|H| = |C_0 + \Delta \cdot f|$ a complete linear system on Y. Then $|H|$ is base-point free if and only if, for all the points $P \in \Gamma$, we have*

$$h^0(\mathcal{O}_Y(H - P \cdot f)) = h^0(\mathcal{O}_Y(H)) - 2.$$

Proposition 2.23. *If the conclusions of Proposition 2.19 are satisfied, in particular $\mathcal{E}$ is decomposable and $e \geq 2q-2$, we have that*

1. *if $a \geq 2$, then the linear system $|aC_1|$ is base-point free , except for the case in which $q = 1$, $e = 0$ and $-aD \nsim 0$ but $-2aD \sim 0$;*

2. *let us consider the linear system $|aC_1|$, with $a = 1$:*

   - *if $e > 2q-2$, the linear system $|C_1|$ is base-point free, except for the case in which $e = 2q-1$ and there is a point $P \in \Gamma$ such that $K_\Gamma + D + P \sim 0$;*
   - * *if $e = 2q-2$ and $q = 1$, the linear system $|C_1|$ is base-point free if and only if $K_\Gamma + D \sim 0$;*
     * *if $e = 2q-2$, $q > 1$ and $K_\Gamma + D \sim 0$, then $|C_1|$ is base-point free;*
     * *if $e = 2q-2$, $q > 1$ and $K_\Gamma + D \sim \alpha$, for $\alpha \in \mathrm{Pic}^0(\Gamma)$ such that $\alpha \nsim 0$ but $2\alpha \sim 0$, then $|C_1|$ is base-point free if and only if $h^0(\mathcal{O}_\Gamma(P+\alpha)) = 0$, for any point $P \in \Gamma$.*

Proof. 1. Let us consider the linear system $|aC_1|$ with $a \geq 2$. By Proposition 2.21, a linear system of the type $|aC_0 + \Delta \cdot f|$ on X'' is base-point free if and only if the linear systems $|\Delta|$ and $|\Delta + aD|$ on Γ are base-point free.

In our case $|aC_1| = |aC_0 - aD \cdot f|$, so $\Delta \sim -aD$. It is clear that $|\Delta + aD|$ is base-point free.

By [20], Corollary $IV.3.2$, the linear system $|\Delta|$ is base-point free if $\deg(\Delta) \geq 2q$. In our case, $\deg(\Delta) = \deg(-aD) = ae$. Since $a \geq 2$ by hypothesis, then $\deg(\Delta) \geq 2e$. By Proposition 2.19, Case $a.$, since we suppose that $\mathcal{E}$ is decomposable, then $e \geq 2q-2$ and, in particular, $\deg(\Delta) \geq 2(2q-2) = 4q-4 = 2q + (2q-4)$.

We observe that $2q - 4 \geq 0$ if and only if $q > 1$. Accordingly, if $q > 1$, we have that $\deg(\Delta) \geq 2q$ and $|aC_1|$ is base-point free.

We analyze the case $q = 1$ separately. By [20], Proposition $IV.3.1$, the linear system $|-aD|$ is base-point free if and only if, for every point $P \in \Gamma$, we have that

$$\dim|-aD| - 1 = \dim|-aD-P|.$$

We observe that

$$h^0(\mathcal{O}_\Gamma(-aD-P)) = h^0(\mathcal{O}_\Gamma(K_\Gamma + aD + P)) + ae - 1 + 1 - q \tag{9}$$

while

$$h^0(\mathcal{O}_\Gamma(-aD)) = h^0(\mathcal{O}_\Gamma(K_\Gamma + aD)) + ae + 1 - q. \tag{10}$$

We distinguish two cases:

- $q = 1$ and $e > 0$; we have that $\deg(K_\Gamma + aD + P) = -ae + 1 < 0$ since $a \geq 2$, so $h^0(\mathcal{O}_\Gamma(K_\Gamma + aD + P)) = 0$ and $h^0(\mathcal{O}_\Gamma(-aD-P)) = ae - q$ from (9). Similarly, we have that $\deg(K_\Gamma + aD) = -ae < 0$, then $h^0(\mathcal{O}_\Gamma(K_\Gamma + aD)) = 0$ and $h^0(\mathcal{O}_\Gamma(-aD)) = ae + 1 - q$ from (10).

  Hence $h^0(\mathcal{O}_\Gamma(-aD-P)) = h^0(\mathcal{O}_\Gamma(-aD)) - 1$, that is equivalent to the equality $\dim|-aD| - 1 = \dim|-aD-P|$, for every point $P \in \Gamma$;

- $q = 1$ and $e = 0$; since $\deg(D) = 0$, then $\deg(-aD - P) = -1$, so $h^0(\mathcal{O}_\Gamma(-aD - P)) = 0$. By Proposition 2.19, since $q = 1$ and $e = 0$, then $2D \sim -2K_\Gamma \sim 0$, so either $D \sim 0$ or D is a non-zero two torsion element on Γ. So we have the two following possibilities:
  - ⋄ $aD \sim 0$; in this case $h^0(\mathcal{O}_\Gamma(-aD)) = 1$, so $\dim|-aD| - 1 = \dim|-aD - P|$, for every point P on Γ;
  - ⋄ $aD \nsim 0$ and $2aD \sim 0$; we have that $h^0(\mathcal{O}_\Gamma(-aD)) = 0$, so $\dim|-aD| - 1 \neq \dim|-aD - P|$.

In conclusion, if $q = 1$ and $a \geq 2$, then the linear system $|aC_1|$ is always base-point free, except for the case $e = 0$ and $-aD$ a non-zero two-torsion element in $\mathrm{Pic}^0(\Gamma)$.

2. Let us consider the linear system $|C_1|$. By Proposition 2.22, a linear system of the type $|C_0 + \Delta \cdot f|$ is base-point free if and only if, for all points $P \in \Gamma$, we have that $h^0(\mathcal{O}_{X''}(C_0 + (\Delta - P) \cdot f)) = h^0(\mathcal{O}_{X''}(C_0 + \Delta \cdot f)) - 2$. In our case we want to show when $|C_1| = |C_0 - D \cdot f|$ is base-point free. Since we assume that $\mathcal{E}$ is decomposable, then $e \geq 2q - 2$ by Case a. of Proposition 2.19.

   We distinguish between two cases.

   - Let $e > 2q - 2$.

     Using Lemma 2.5, we have that
$$h^0(\mathcal{O}_{X''}(C_1)) = h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma).$$
     Thanks to the Riemann-Roch Theorem we know that
$$h^0(\mathcal{O}_\Gamma(-D)) = h^0(\mathcal{O}_\Gamma(K_\Gamma + D)) + \deg(-D) + 1 - q.$$
     Since $\deg(-D) = e > 2q - 2$, then $\deg(K_\Gamma + D) = 2q - 2 - e < 0$, so $h^0(\mathcal{O}_\Gamma(-D)) = e + 1 - q$. Hence
$$h^0(\mathcal{O}_{X''}(C_1)) = e + 2 - q.$$

     As before, we can compute that
$$h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) = h^0(\mathcal{O}_\Gamma(-D - P)) + h^0(\mathcal{O}_\Gamma(-P)).$$
     Since $\deg(-P) < 0$, then $h^0(\mathcal{O}_\Gamma(-P)) = 0$, so $h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) = h^0(\mathcal{O}_\Gamma(-D-P))$. We observe that $\deg(-D-P) = e-1 > 2q-2-1 = 2q-3$.

     – If $\deg(-D - P) > 2q - 2$, i.e. $e > 2q - 1$, then $h^1(\mathcal{O}_\Gamma(-D - P)) = 0$. By the Riemann-Roch Theorem, we have that $h^0(\mathcal{O}_\Gamma(-D - P)) = e - 1 + 1 - q = e - q$. In this case
$$h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) = e - q = h^0(\mathcal{O}_{X''}(C_1)) - 2.$$
       As observed before, the linear system $|C_1|$ is base-point free.

– If $\deg(-D-P) = 2q-2$, i.e $e = 2q-1$, then $h^1(\mathcal{O}_\Gamma(-D-P)) = h^0(\mathcal{O}_\Gamma(K_\Gamma + D + P))$ by Serre Duality.
We know that $\deg(K_\Gamma + D + P) = 2q - 2 - e + 1 = 0$. If there exists a point $P \in \Gamma$ such that $K_\Gamma + D + P \sim 0$, then $h^0(\mathcal{O}_\Gamma(K_\Gamma + D + P)) = h^1(\mathcal{O}_\Gamma(-D-P)) = 1$. In this case, by the Riemann-Roch Theorem, we have that

$$h^0(\mathcal{O}_\Gamma(-D-P)) = 1 + e - 1 + 1 - q = e + 1 - q,$$

so

$$h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) \neq h^0(\mathcal{O}_{X''}(C_1)) - 2$$

for a certain $P \in \Gamma$, thus $|C_1|$ is not base-point free. On the contrary, if $P \in \Gamma$ such that $K_\Gamma + D + P \sim 0$ does not exist, i.e. $K_\Gamma + D + P$ is a torsion element, then $h^0(\mathcal{O}_\Gamma(K_\Gamma + D + P)) = h^1(\mathcal{O}_\Gamma(-D-P)) = 0$ and, by the Riemann-Roch Theorem, we have that

$$h^0(\mathcal{O}_\Gamma(-D-P)) = e - 1 + 1 - q = e - q.$$

In this case

$$h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) = h^0(\mathcal{O}_{X''}(C_1)) - 2$$

and the linear system $|C_1|$ is base-point free.

• Let $e = 2q - 2$.
We know that $h^0(\mathcal{O}_{X''}(C_1)) = h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma)$ by Lemma 2.5. Using the Riemann-Roch Theorem, we have that $h^0(\mathcal{O}_\Gamma(-D)) = h^0(\mathcal{O}_\Gamma(K_\Gamma + D)) + \deg(-D) + 1 - q = h^0(\mathcal{O}_\Gamma(K_\Gamma + D)) + e + 1 - q$. We observe that $\deg(K_\Gamma + D) = 2q - 2 - e = 0$. Then

$$h^0(\mathcal{O}_{X''}(C_1)) = h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma) =$$

$$= \begin{cases} q & if \;\; K_\Gamma + D \nsim 0 \\ q+1 & if \;\; K_\Gamma + D \sim 0. \end{cases}$$

Similarly we have that $h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) = h^0(\mathcal{O}_\Gamma(-D-P)) + h^0(\mathcal{O}_\Gamma(-P))$. Since $\deg(-P) < 0$, then $h^0(\mathcal{O}_\Gamma(-P)) = 0$ and

$$h^0(\mathcal{O}_{X''}(C_1 - P \cdot f)) = h^0(\mathcal{O}_\Gamma(-D-P)).$$

 * If $q = 1$, since $\deg(-D-P) = 2q - 2 - 1 < 0$, then $h^0(\mathcal{O}_\Gamma(-D-P)) = 0$. Therefore $h^0(\mathcal{O}_{X''}(C_1)) - 2 = h^0(\mathcal{O}_{X''}(C_1 - P \cdot f))$ if and only if $K_\Gamma + D \sim 0$. We conclude that the linear system $|C_1|$ is base-point free if and only if $K_\Gamma + D \sim 0$.
 * Let $q > 1$ and $K_\Gamma + D \sim 0$. Since $|K_\Gamma|$ is base-point free (see [20], Lemma IV.5.1), we have that $h^0(\mathcal{O}_\Gamma(-D-P)) = h^0(\mathcal{O}_\Gamma(K_\Gamma - P)) = q - 1$ by [20], Proposition IV.3.1. Thus $h^0(\mathcal{O}_{X''}(C_1)) - 2 = h^0(\mathcal{O}_{X''}(C_1 - P \cdot f))$ and the linear system $|C_1|$ is base-point free.

* Let $q > 1$ and $D \sim -K_\Gamma + \alpha$, where $\alpha \in \mathrm{Pic}^0(\Gamma)$ is such that $\alpha \nsim 0$ and $2\alpha \sim 0$ by Case a. of Proposition 2.19. We observe that $h^1(\mathcal{O}_\Gamma(-D - P)) = h^1(\mathcal{O}_\Gamma(K_\Gamma - \alpha - P)) = h^0(\mathcal{O}_\Gamma(P + \alpha))$ by Serre Duality.

  If a point $P \in \Gamma$ such that $h^0(\mathcal{O}_\Gamma(P + \alpha)) > 0$ exists, then $h^0(\mathcal{O}_\Gamma(-D - P)) = h^0(\mathcal{O}_\Gamma(P + \alpha)) + 2q - 2 - 1 + 1 - q \geq q - 1$. So $h^0(\mathcal{O}_{X''}(C_1)) - 2 \neq h^0(\mathcal{O}_{X''}(C_1 - P \cdot f))$ and $|C_1|$ is not base point free.

  Instead, if $h^0(\mathcal{O}_\Gamma(P + \alpha)) = 0$ for any point $P \in \Gamma$, then $h^0(\mathcal{O}_\Gamma(-D - P)) = q - 2$ and $h^0(\mathcal{O}_{X''}(C_1)) - 2 = h^0(\mathcal{O}_{X''}(C_1 - P \cdot f))$. In this case $|C_1|$ is base-point free.

□

Remark 2.24. *In the hypothesis of the previous Proposition, if $p_g(\Gamma) := q > 1$ and $h^0(\mathcal{O}_\Gamma(P + \alpha)) > 0$, for $P \in \Gamma$ a point and α a non-zero torsion element of order 2 in $\mathrm{Pic}^0(\Gamma)$, then $h^0(\mathcal{O}_\Gamma(P + \alpha)) = 1$; indeed, if there was a divisor $D \in \Gamma$ such that $deg(D) = 1$ and $h^0(\mathcal{O}_\Gamma(D)) \geq 2$, then $p_g(\Gamma) = 0$ but we know that $p_g(\Gamma) = q > 1$. Then there is only one point $Q \in \Gamma$ such that $P + \alpha \sim Q$. In addition $2P \sim 2Q$. Since Γ is not rational, then the linear system $|2P|$ on Γ has degree 2 and dimension 1, so Γ is hyperelliptic.*

If we assume that Γ is not hyperelliptic, then $h^0(\mathcal{O}_\Gamma(P + \alpha)) = 0$, for any point $P \in \Gamma$ and for any α a non-zero two torsion element on Γ.

2.3.2 Examples

In this paragraph, we will construct new examples of surfaces with Prym-canonical hyperplane sections.
First of all, we prove that, if $L'' \subseteq |aC_1|$ with $a = 1$, then X is a cone over a Prym-canonically embedded curve with only one singularity.

Proposition 2.25. *We suppose that X'' is a minimal model of a surface X with Prym-canonical hyperplane sections and $\pi : X' \to X$ is the minimal resolution of the singularities of X. Moreover we assume that X'' satisfies the conclusions of Proposition 2.19.*
If $a = 1$, i.e. if the fibres of X'' are transformed into straight lines on X, then $X' = X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$, with $D \sim -K_\Gamma + \alpha$, where α is a non-trivial two torsion divisor. In addition $g = q \geq 5$ and $L'' = |C_1|$. Thus X is a cone over the Prym-canonically embedded curve with only one singularity.

Proof. By Proposition 2.19, Case $b.1$, we know that, if $a = 1$, then $r_{t,i_t} \leq 0$ for any $t = 1, ..., s - 1$ and $i_t = 1, ..., i(t)$, so there are no base points for L'' and $X' = X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$, for $D \in \mathrm{Pic}(\Gamma)$.
Consequently $L'' = |C_1|$ and, by Case $b.3$ of Proposition 2.19, we have that $0 = \sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = a(e - 2q + 2)$, so $e = 2q - 2$.

We also observe that $g = 1+\frac{1}{2}a^2e-\frac{1}{2}\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 1+\frac{1}{2}(2q-2) = q$ by Proposition 2.19, Case $b.4$ and the previous remarks. Since we suppose that X is a surface with Prym-canonical hyperplane sections, then a general hyperplane section C'' of X'' is a Prym-canonically embedded curve, i.e. $i_{L''}|_{C''} : C'' \hookrightarrow \mathbb{P}^{g-2}$ is a Prym-canonical embedding, for C'' a general curve in L''. We have already observed that $g \geq 5$ (see Remark 1.3).

Again by the assumptions on X'', we have that $\mathcal{O}_{C''}(-2K_{X''}) \cong \mathcal{O}_{C''}$ while $\mathcal{O}_{C''}(-K_{X''}) \ncong \mathcal{O}_{C''}$, where $C'' \in L''$ is a general curve in the linear system L''. We know that $\dim|-2K_{X''}| = 0$, as proved in Theorem 1.12. Indeed $W'' := -2K_{X''} \sim 4C_0 - (2K_\Gamma + 2D)\cdot f$. By Proposition 2.19, Case $a.$, since $e = 2q-2$, then $2D \sim -2K_\Gamma$, so $-2K_{X''} \sim 4C_0$. Moreover, because $-2K_\Gamma - D \sim 2D - D = D$ and $\deg(D) = -e = 2-2q < 0$, then $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) = 0$, hence we are in CASE $II.1$ of Proposition 2.12 and $|W''| = \{4C_0\}$. Since W'' is connected and contracted by $i_{L''}$, then the image of W'' is a single point.

It is obvious that $C''\cdot W'' = 0$, so also $C''\cdot(-K_{X''}) = 0$.
If $D \sim -K_\Gamma$, then $-K_{X''} \sim 2C_0$ and $h^0(\mathcal{O}_{X''}(-K_{X''})) = h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(D)) + h^0(\mathcal{O}_\Gamma(2D))$ by Lemma 2.5. Since $\deg(D) = 2-2q < 0$, then $h^0(\mathcal{O}_{X''}(-K_{X''})) = h^0(\mathcal{O}_\Gamma) = 1$ and $-K_{X''} \sim 2C_0$ is effective. Then $\mathcal{O}_{C''}(-K_{X''}) \cong \mathcal{O}_{C''}$ but this contradict the hypothesis.
Then $D \sim -K_\Gamma + \alpha$, where α is a non-trivial two torsion divisor since $2D \sim -2K_\Gamma$.
We can verify that, in this case, it is true that $\mathcal{O}_{C''}(-K_{X''}) \ncong \mathcal{O}_{C''}$. Indeed, we know that $h^0(\mathcal{O}_{X''}(-K_{X''})) = h^0(\mathcal{O}_{X''}(2C_0-(K_\Gamma+D)\cdot f)) = h^0(\mathcal{O}_\Gamma(-\alpha))+h^0(\mathcal{O}_\Gamma(-K_\Gamma))+h^0(\mathcal{O}_\Gamma(-2K_\Gamma+\alpha))$ by Lemma 2.5. Since α is a non-zero torsion element and $\deg(-K_\Gamma) = 2-2q < 0$, then $h^0(\mathcal{O}_{X''}(-K_{X''})) = 0$, so $-K_{X''}$ is not effective. Moreover, since C'' is disjoint from C_0 by Proposition 2.19, Case $b.$, we have that $C_0|_{C''} \cong \mathcal{O}_{C''}$. Then $\mathcal{O}_{C''}(-K_{X''}) \cong \mathcal{O}_{C''}(2C_0 - (K_\Gamma + D)\cdot f) \cong \mathcal{O}_{C''}(2C_0) \otimes \mathcal{O}_{C''}(-(K_\Gamma+D)\cdot f) \cong \mathcal{O}_{C''}(-(K_\Gamma+D)\cdot f)$. Since $C'' \sim C_1$ is a section of X'', then $C'' \cong \Gamma$ and, since $C''\cdot f = 1$, then $\mathcal{O}_{C''}(-(K_\Gamma+D)\cdot f) \cong \mathcal{O}_\Gamma(-(K_\Gamma+D)) \cong \mathcal{O}_\Gamma(-\alpha) \ncong \mathcal{O}_{C''}$. The claim is proved.

The images of fibres of X'' by $i_{L''}$ are lines since $C''\cdot f = C_1\cdot f = 1$. Therefore $X = i_{L''}(X'')$ is a cone since $i_{L''}$ contracts $W'' \sim 4C_0$ in a single point. Thus X has only one singularity because the generic hyperplane section is smooth by hypothesis. The base curve of X is the image of a generic section $C_1 \in X''$ disjoint from C_0. Since $C = i_{L''}(C_1)$ is a section of X and X has Prym-canonical hyperplane sections by hypothesis, then C is a Prym-canonically embedded curve. □

Before proving a Corollary of this Proposition, we recall a Theorem whose proof is contained in [15], Theorem 33.

Theorem 2.26. *Let Y be a decomposable ruled surface and let $|H| = |C_0 + \beta\cdot f|$ be a complete linear system on Y. Then:*

- *if β is very ample and $\beta + D$ is base-point free, then $|H|$ defines an isomorphism in $Y - C_0$;*

- *if $\beta + D$ is very ample and β is base-point free, then $|H|$ defines an isomorphism in $Y - C_1$;*
- *$|H|$ is very ample if and only if β and $\beta + D$ are very ample.*

Corollary 2.27. *Let $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$ be a minimal ruled surface over a base curve Γ of genus $q \geq 5$, with $D \sim -K_\Gamma + \alpha$, where α is a not trivial two torsion divisor, and let $L'' = |C_1|$. If Γ is non-hyperelliptic and it does not admit g^1_4, then a general hyperplane section of X'' is Prym-canonically embedded and the map $i_{L''} : X'' \dashrightarrow \mathbb{P}^{q-1}$ defined by the linear system L'' is such that $X = i_{L''}(X'')$ is a cone on a Prym-canonically embedded curve. In addition, X has only one singularity and if a general hyperplane section C of X is projectively normal, then the geometric genus of the only singularity is q.*

Proof. We observe that $e = -\deg(D) = \deg(K_\Gamma - \alpha) = 2q - 2$, with $q \geq 5$. Since Γ is non-hyperelliptic, by Remark 2.24 we have that $h^0(\mathcal{O}_\Gamma(P + \alpha)) = 0$ for any point $P \in \Gamma$. By Proposition 2.23, the linear system $L'' = |C_1|$ is base-point free.
Using Lemma 2.5, we have that $h^0(\mathcal{O}_{X''}(L'')) = h^0(\mathcal{O}_{X''}(C_0 - D \cdot f)) = h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(-D)) = 1 + h^0(\mathcal{O}_\Gamma(K_\Gamma + D)) + 2q - 2 + 1 - q = h^0(\mathcal{O}_\Gamma(\alpha)) + q = q \geq 5$, so the image of X'' by the map $i_{L''}$ defined by L'' is contained in $\mathbb{P}^{q-1}$.
By Bertini's Theorem (see [20], Corollary $III.10.9$), the generic curve $C'' \in L''$ is smooth.

Since $D \sim -K_\Gamma + \alpha$ by hypothesis, then $2D \sim -2K_\Gamma$ and $W'' \sim 4C_0$. Again, since $-2K_\Gamma - D \sim D$ and $\deg(D) < 0$, then $h^0(\mathcal{O}_\Gamma(-2K_\Gamma - D)) = 0$ and we are in CASE $II.1$ of Proposition 2.12, so $\dim|-2K_{X''}| = 0$ and $W'' = 4C_0$. Since $C'' \cdot W'' = (C_0 - D \cdot f) \cdot 4C_0 = 4C_0^2 - 4\deg(D) = 0$ and W'' is effective, then $\mathcal{O}_{C''}(W'') \cong \mathcal{O}_{C''}$. On the other hand, as showed in the proof of Proposition 2.25, we have that $\mathcal{O}_{C''}(-K_{X''}) \ncong \mathcal{O}_{C''}$.

We know that $i_{L''}$ contracts $W'' = 4C_0$. Our claim is to show that L'' defines an isomorphism in $X'' - C_0$, so $i_{L''}|_{C''}$ is an embedding for a general $C'' \in L''$, the general hyperplane section C'' of X'' is Prym-canonically embedded and X has Prym-canonical hyperplane sections.

By Theorem 2.26, the linear system L'' defines an isomorphism in $X'' - C_0$ if $-D$ is very ample and $|-D + D|$ is base-point free. The second hypothesis is always satisfied, while, by [20], Proposition $IV.3.1$, the divisor $-D$ is very ample if and only if, for every two points $P, Q \in \Gamma$, we have that

$$\dim|-D - P - Q| = \dim|-D| - 2.$$

By the Riemann-Roch Theorem, the dimension

$$\dim|-D| = h^0(\mathcal{O}_\Gamma(K_\Gamma - \alpha)) - 1 = h^0(\mathcal{O}_\Gamma(\alpha)) + 2q - 2 + 1 - q - 1 = q - 2.$$

Then, it is sufficient to show that $\dim|-D-P-Q| = q-4$, so $-D$ is very ample and the claim is proved. By the Riemann-Roch Theorem, we have that

$\dim|-D-P-Q| = h^0(\mathcal{O}_\Gamma(K_\Gamma-\alpha-P-Q))-1 = h^0(\mathcal{O}_\Gamma(P+Q+\alpha))+2q-4+1-q-1 = h^0(\mathcal{O}_\Gamma(P+Q+\alpha))+q-4$.
If $h^0(\mathcal{O}_\Gamma(P+Q+\alpha)) = 0$ for any $P, Q \in \Gamma$, then the claim is proved. On the contrary we suppose that $h^0(\mathcal{O}_\Gamma(P+Q+\alpha)) > 0$ for some $P, Q \in \Gamma$. Hence there are two points T and S (with T not necessarily different from S) such that $P+Q+\alpha \sim T+S$, so $2P+2Q \sim 2T+2S$.
Since we suppose that Γ does not admit g^1_4 and $2P+2Q$ and $2T+2S$ define a linear system of this type, then we exclude this case. So $h^0(\mathcal{O}_\Gamma(P+Q+\alpha)) = 0$ for any $P, Q \in \Gamma$ and the claim is proved. In conclusion, the general hyperplane section of X'' is Prym-canonically embedded.

In the previous Proposition, we have already observed that $X = i_{L''}(X'')$ is a cone. Since X'' has hyperplane sections that are Prym-canonical embedded and the base curve of X is the image of a generic section C_1 disjoint from C_0, then the base curve of X is a Prym-canonically embedded curve. In particular, X is a cone with Prym-canonical hyperplane sections. Thus X has only one singularity because the generic hyperplane section is smooth by hypothesis. The only singularity is the image of W'' by $i_{L''}$.
Moreover, if a general hyperplane section C of X is projectively normal, then, by Proposition 1.15, the only singularity of X has geometric genus equal to q. □

Remark 2.28. *We assume that $X' = X''$ is a minimal ruled surface, that is the minimal resolution of the singularities of a surface with Prym-canonical hyperplane sections.*
If $a \geq 2$, then the images of fibres of X'' by the map $i_{L''}$ associated with L'' are not lines since $C'' \cdot f = (aC_0 + \Delta \cdot f) \cdot f = a > 1$, for C'' a general curve in L''. Furthermore there is not another family of rational curves on X'' mapped into lines by $i_{L''}$ because the genus of the base curve Γ is $q > 0$. Indeed, by the Riemann-Hurwitz formula (see [20], Corollary IV.2.4), there is not a curve of genus 0 (not equal to a fibre) on a ruled surface over a base curve of genus $q > 0$. Hence we never obtain $X = i_{L''}(X'')$ as a cone.

Now we focus our attention on surfaces with Prym-canonical hyperplane sections birationally equivalent to ruled surfaces X'' over a non-hyperelliptic base curve of genus $q \geq 3$ (not cone over a Prym-canonically embedded curve) and with only one non-rational singularity. In the first phase, we will assume that $X'' = X'$.

First of all, we recall a Theorem that we will use later (see [15], Theorem 38).

Theorem 2.29. *Let $|mC_0 + \beta \cdot f|$ be a $m-$secant linear system on a decomposable ruled surface Y. It defines a rational map $\varepsilon : Y \to \mathbb{P}^N$. Then:*

- *if β, $\beta + D$, $\beta + (m-1)D$ and $\beta + mD$ are base-point free and*
  - *$\beta+(m-k)D$ is very ample for some $k \in \{0, ..., m\}$, then ε is an isomorphism in $Y - (C_0 \cup C_1)$;*
  - *β is very ample, then ε is an isomorphism in $Y - C_0$;*

– $\beta + mD$ *is very ample, then* ε *is an isomorphism in* $Y - C_1$*;*

- $|mC_0 + \beta \cdot f|$ *is very ample if and only if* β, $\beta + D$, $\beta + (m-1)D$ *and* $\beta + mD$ *are base-point free and* β *and* $\beta + mD$ *are very ample.*

Theorem 2.30. *Let* $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$ *be a minimal ruled surface over a non-hyperelliptic smooth base curve* Γ *of genus* $q \geq 3$ *and let* $L'' = |aC_1|$ *be a base-point free linear system with* $a \geq 2$*. If* $D \sim -K_\Gamma + \alpha$*, for* α *a non-zero two-torsion divisor, then the image* $X = i_{L''}(X'') \subseteq \mathbb{P}^{a^2(q-1)}$ *of the morphism associated with* L'' *has Prym-canonical hyperplane sections and only one singularity. In particular, if a general hyperplane section* C *of* X *is projectively normal, then the geometric genus of the only singularity* x *is* $p_g(x) = q$*.*

Proof. Let Γ be a smooth non-hyperelliptic curve of genus $q \geq 3$ and let K_Γ be its canonical divisor. Set $\mathcal{E} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$, for $D \in \mathrm{Div}(\Gamma)$ with $e = -\deg(D) > 0$, and let $X'' = \mathbb{P}_\Gamma(\mathcal{E})$. Let $p : X'' \to \Gamma$ be the canonical surjection on the base curve of X''. Fix a linear system $L'' \subseteq |aC_1|$ on X'' and suppose that $a \geq 2$. We recall that $C_1 \sim C_0 - D \cdot f$ is a fixed section of p disjoint from C_0. By Proposition 2.23, the linear system $|aC_1|$ is base-point free because $q > 1$. Let us suppose that also L'' does not have base points, so $L'' = |aC_1|$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 0$. Since we want to construct X'' whose hyperplane sections are Prym-canonical embedded, we fix $e = 2q - 2$, so it is satisfied the relation $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = a(e - 2q + 2)$, as in Proposition 2.19, Case $b.3$.
Since the linear system L'' is base-point free and $(C'')^2 > 0$, then, by Bertini's Theorem (see [20], Corollary $III.10.9$), the generic curve $C'' \in L''$ is smooth and irreducible.

CLAIM 1 : If $D \sim -K_\Gamma + \alpha$, for α a non trivial two-torsion element on Γ, hence $2D \sim -2K_\Gamma$, then $-K_{X''}$ is not effective, while $-2K_{X''}$ is.

Proof. We know that $h^0(\mathcal{O}_{X''}(-2K_{X''})) = h^0(\mathcal{O}_{X''}(4C_0)) = h^0(\mathcal{O}_\Gamma) + h^0(\mathcal{O}_\Gamma(D)) + h^0(\mathcal{O}_\Gamma(2D)) + h^0(\mathcal{O}_\Gamma(3D)) + h^0(\mathcal{O}_\Gamma(4D))$ by [15], Lemma 35. Because $\deg(D) = 2 - 2q < 0$, then $h^0(\mathcal{O}_{X''}(-2K_{X''})) = 1$ and $-2K_{X''}$ is effective.

On the other hand, we have that $h^0(\mathcal{O}_{X''}(-K_{X''})) = h^0(\mathcal{O}_{X''}(2C_0 - (K_\Gamma + D) \cdot f)) = h^0(\mathcal{O}_\Gamma(-K_\Gamma - D)) + h^0(\mathcal{O}_\Gamma(-K_\Gamma)) + h^0(\mathcal{O}_\Gamma(-K_\Gamma + D))$ by [15], Lemma 35. Since $\deg(D - K_\Gamma) = 4 - 4q < 0$ and $\deg(-K_\Gamma) < 0$, then $h^0(-K_{X''}) = h^0(\mathcal{O}_\Gamma(-K_\Gamma - D)) = h^0(\mathcal{O}_\Gamma(-\alpha)) = 0$, so $-K_{X''}$ is not effective. □

CLAIM 2 : We prove that $\mathcal{O}_{C''}(-K_{X''}) \ncong \mathcal{O}_{C''}$ and $\mathcal{O}_{C''}(-2K_{X''}) \cong \mathcal{O}_{C''}$, where $C'' \in L''$ is a general curve.

Proof. Since $-2K_{X''}$ is effective as seen in Claim 1 and $C'' \cdot (-2K_{X''}) = (aC_0 - aD \cdot f) \cdot (4C_0) = 4aC_0^2 - 4a\deg(D) = 0$, then $\mathcal{O}_{C''}(-2K_{X''}) \cong \mathcal{O}_{C''}$.

Clearly also $C'' \cdot (-K_{X''}) = 0$. Since $h^0(\mathcal{O}_{X''}(-K_{X''})) = 0$ as seen in Claim 1, if we prove that $h^1(\mathcal{O}_{X''}(-K_{X''} - C'')) = 0$, then, from the exact sequence

$$0 \to \mathcal{O}_{X''}(-K_{X''} - C'') \to \mathcal{O}_{X''}(-K_{X''}) \to \mathcal{O}_{C''}(-K_{X''}) \to 0,$$

we have that $h^0(\mathcal{O}_{C''}(-K_{X''})) = 0$, which implies that $\mathcal{O}_{C''}(-K_{X''}) \ncong \mathcal{O}_{C''}$.

By Serre Duality, we know that $h^1(\mathcal{O}_{X''}(-K_{X''} - C'')) = h^1(\mathcal{O}_{X''}(2K_{X''} + C''))$. If we prove that $K_{X''} + C''$ is ample, then, by the Kodaira vanishing Theorem (see [20], Remark $III.7.15$), we have that $h^1(\mathcal{O}_{X''}(2K_{X''} + C'')) = 0$ and the claim is proved.
By [20], Proposition $V.2.20$, a divisor $D \equiv bC_0 + cf$ on X'', for $b, c \in \mathbb{Z}$, is ample if and only if $b > 0$ and $c > be$, for e the invariant of X''. In our case $K_{X''} + C'' \equiv aC_0 + a(2q-2)f - 2C_0 = (a-2)C_0 + a(2q-2)f$. If $a > 2$, then $a - 2 > 0$ and $c = a(2q-2) > (a-2)(2q-2) = be$, so $K_{X''} + C''$ is ample and the claim is satisfied. Instead, if $a = 2$, then $K_{X''} + C''$ is not ample because $a - 2 = 0$. About that, let us suppose that $\mathcal{O}_{C''}(-K_{X''}) \cong \mathcal{O}_{C''}$. Then the image of X'' by $i_{L''}$ is a surface with canonical hyperplane sections. By [13], Corollary 5.4, X'' contains only one effective anticanonical divisor. This contradicts Claim 1. The claim is also satisfied for $a = 2$. □

It is clear that $\dim(L'') \geq 0$ since C_1 is effective and, in particular, $aC_1 \sim C''$ is effective, for $a \geq 2$. More precisely we have that $h^0(\mathcal{O}_{X''}(L'')) = h^0(\mathcal{O}_{X''}(aC_0 - aD \cdot f)) = h^0(\mathcal{O}_\Gamma(-aD)) + h^0(\mathcal{O}_\Gamma((-a+1)D)) + ... + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma)$ by Lemma 2.5.
We know that $h^0(\mathcal{O}_\Gamma) = 1$. Using the Riemann-Roch Theorem, we also have that $h^0(\mathcal{O}_\Gamma(-D)) = h^0(\mathcal{O}_\Gamma(\alpha)) + 2q - 2 + 1 - q = (2q-2) + 1 - q$ and $h^0(\mathcal{O}_\Gamma(-mD)) = m(2q-2) + 1 - q$, for any $m > 1$. Hence

$$h^0(\mathcal{O}_{X''}(L'')) = (1 + ... + a)(2q-2) + a(1-q) + 1 =$$

$$= a(a+1)(q-1) + a(1-q) + 1 = a^2(q-1) + 1.$$

So $X = i_{L''}(X'')$ is contained in $\mathbb{P}^{a^2(q-1)}$, for $a^2(q-1) \geq 4 \cdot 2 = 8$.

CLAIM 3 : It remains to show that L'' defines a birational morphism $i_{L''}$, in particular an isomorphism outside C_0. So $i_{L''}|_{C''}$ is a Prym-canonical embedding, for $C'' \in L''$ a general divisor.

Proof. We can prove that L'' defines a birational map, in particular an isomorphism outside C_0, using Theorem 2.29. Indeed, this happens if $-aD$ is very ample and if $|-aD + D|$, $|-aD + (a-1)D| = |-D|$ and $|-aD + aD|$ are base-point free.
The last case is trivial. By [20], Corollary $IV.3.2$, since $\deg(-aD) = a(2q-2) \geq 4q-4 = 2q + (2q-4) \geq 2q+1$, then $-aD$ is very ample. Again by [20], Corollary $IV.3.2$, if $a \geq 3$, since $\deg(-aD+D) = a(2q-2) + (2-2q) = (a-1)(2q-2) \geq 4q-4 = 2q + (2q-4) \geq 2q$, then $|-aD + D|$ is base-point free.
It remains to show that $|-D|$ is base-point free. By [20], Proposition $IV.3.1$, it happens if and only if $\dim|-D-P| = \dim|-D| - 1$, for any point $P \in \Gamma$.

We have that $\dim|-D| = h^0(\mathcal{O}_\Gamma(K_\Gamma - \alpha)) - 1 = h^0(\mathcal{O}_\Gamma(\alpha)) + 2q - 2 + 1 - q - 1 = q - 2$. On the other hand, we observe that $\dim|-D-P| = h^0(\mathcal{O}_\Gamma(K_\Gamma - \alpha - P)) - 1 = h^0(\mathcal{O}_\Gamma(P+\alpha)) + 2q - 2 - 1 + 1 - q - 1 = h^0(\mathcal{O}_\Gamma(P+\alpha)) + q - 3$. By Remark 2.24, since we assume that Γ is not hyperelliptic, then $h^0(\mathcal{O}_\Gamma(P+\alpha)) = 0$ for any point $P \in \Gamma$. So $|-D|$ is base-point free and L'' defines an isomorphism outside C_0, called $i_{L''}$.

A general $C'' \sim aC_1$ in L'' is disjoint from C_0 by definition, then

$$i_{L''}|_{C''} : C'' \to \mathbb{P}^{a^2(q-1)-1}$$

is an embedding. By the adjunction formula, we know that

$$L''|_{C''} \cong K_{C''} - K_{X''}|_{C''}$$

but in Claim 2 we have already proved that $-K_{X''}|_{C''}$ is a non trivial two-torsion divisor, so $i_{L''}|_{C''}$ is a Prym-canonical embedding. □

We observe that, since $i_{L''|_{C''}} : C'' \to \mathbb{P}^{g-2}$ by definition of Prym-canonical map, then $g = g(C'') = a^2(q-1)+1$.
The image $x \in X$ of $W'' \in |-2K_{X''}|$ by $i_{L''}$ is a singular point (we observe that $\dim|-2K_{X''}| = 0$ and $W'' = 4C_0$ is connected). There are not other possible singularities because L'' is an isomorphism outside C_0. We have found examples of surfaces in $\mathbb{P}^{a^2(q-1)}$ with Prym-canonical hyperplane sections birationally equivalent to non-rational ruled surfaces, for $a \geq 2$ and $q \geq 3$.
If a general hyperplane section C of X is projectively normal, then, by Proposition 1.15, the singularity x has geometric genus equal to q.

□

Remarks 2.31.

- *We do not consider the case $q = 2$ in the previous Theorem since we suppose that Γ is non-hyperelliptic.*
- *We suppose that X'' is a ruled surface over a base curve Γ of genus $q = 1$. Moreover we suppose that X is a surface with Prym-canonical hyperplane sections and $\pi : X'' \to X$ is the minimal resolution of the singularities of X. We assume that the linear system $L'' = |aC_1|$, for $a \geq 2$, is base-point free. Thus $\sum_{t=1}^{s-1}\sum_{i_t=1}^{i(t)} r_{t,i_t} = 0$ and, if X'' and L'' satisfy the conclusions of Proposition 2.19, then $e = 2q-2$ by Case b.3. In particular, a general $C'' \in L''$ has genus $g(C'') = 1 + \frac{1}{2}(C'')^2 + \frac{1}{2}C'' \cdot K_{X''} = 1 + \frac{1}{2}a^2 e = 1$. So we exclude this case since we have already observed that $g(C'') > 4$ (see Remark 1.3).*

Before constructing another example, we can list in terms of the invariants (q, g, a, e) some of the possible surfaces X with Prym-canonical hyperplane sections whose minimal models X'' satisfy the conclusions of Proposition 2.19.

Proposition 2.32. *Let X be a surface with Prym-canonical hyperplane sections birationally equivalent to a ruled surface X'' over a base curve of genus $q \geq 2$, not a cone. Let us suppose that X'' satisfies the conclusions of Proposition 2.19. Using the same notation as before, there are the following numerical possibilities for $g \leq 10$ (it is possible to continue for all $g > 10$):*

1. *if $g = \{5, 6, 7, 8\}$, then $(a, q, e) = (2, 2, g - 3)$ and L'' has, respectively, $\{0, 2, 4, 6\}$ simple base points on X'';*

2. *if $g = 9$, then either $(a, q, e) = (2, 2, 6)$ or $(a, q, e) = (2, 3, 4)$. In the first case L'' has 8 simple base points while in the second case there are not base points;*

3. *if $g = 10$, either $(a, q, e) = (2, 2, 7)$ or $(a, q, e) = (2, 3, 5)$ or $(a, q, e) = (3, 2, 2)$. Moreover L'' has, respectively, $\{10, 2, 0\}$ simple base points on X''.*

Proof. Let $a \geq 2$ otherwise X is a cone (see Proposition 2.25). By hypothesis we have that $q \geq 2$ and $\mathcal{E}$ is decomposable, so $e \geq 2q - 2 \geq 2$ as showed in Proposition 2.19, Case *a*.. By Proposition 2.19, Case *b*.5, we have $1 + a^2(q-1) + \frac{1}{2}a(e - 2q + 2) \leq g$, hence $g \geq 1 + a^2(q-1) \geq 1 + 4(1) = 5$.

We list all the possible relations in terms of the numbers q, a, e up to $\mathbb{P}^{g-1} = \mathbb{P}^9$ (always using Proposition 2.19).

1. **CASE g = 5** We look for values of a, e and q such that

$$1 + a^2(q-1) + \frac{1}{2}a(e - 2q + 2) \leq 5 = g \leq 1 + \frac{1}{2}a^2 e - \frac{1}{2}a(e - 2q + 2)$$

(Proposition 2.19, Case *b*.5). The only couples (a, q) that satisfy the previous relation are $(a, q) = \{(2, 2), (2, 3)\}$.

If $a = 2$ and $q = 2$, then

$$5 + (e - 2) \leq 5 \leq 1 + 2e - (e - 2),$$

so $e = 5 - 3 = 2$. Moreover, by Proposition 2.19, Case *b*.3, we have that $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(2-2) = 0$, i.e. there are not base points on L'' and $X' = X''$. Therefore $L'' = |2C_1|$.

Instead, if $(a, q) = (2, 3)$, then

$$1 + 8 + (e - 4) \leq 5 \leq 1 + 2e - (e - 4) \quad \Leftrightarrow \quad e = 0.$$

Since we know that $e \geq 2q - 2 = 4$, then we exclude this possibility.

CASE g = 6 By Proposition 2.19, Case *b*.5, we have that

$$1 + a^2(q-1) + \frac{1}{2}a(e - 2q + 2) \leq 6 = g \leq 1 + \frac{1}{2}a^2 e - \frac{1}{2}a(e - 2q + 2).$$

The only couples (a, q) that satisfy the previous relation are $(a, q) = \{(2, 2), (2, 3)\}$.

If $a = 2$ and $q = 2$, then

$$1 + 4 + (e - 2) \leq 6 \leq 1 + 2e - (e - 2),$$

so $6 = g = 3 + e$ and $e = g - 3 = 3$. Furthermore $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(3-2) = 2$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 2 + a^2 e - 2g = 2 + 12 - 12 = 2$ respectively by Case

$b.3$ and Case $b.4$ of Proposition 2.19. Then there are two simple base points for $L'' \subset |2C_1|$.

Instead, if $(a, q) = (2, 3)$, then

$$1 + 8 + (e - 4) \le 6 \le 1 + 2e - (e - 4) \quad \Leftrightarrow \quad e = 1 \not\ge 2q - 2.$$

So we exclude this possibility.

CASE g = 7 If $(a, q) = (2, 3)$, then

$$1 + a^2(q - 1) + (e - 2q + 2) = 9 - 4 + e \le 7 \le 1 + 2e - (e - 4) \Leftrightarrow e = 2 \not\ge 2q - 2,$$

so we exclude this case. In this way it is possible to show that the unique acceptable couple is $(a, q) = (2, 2)$. In this case

$$1 + 4 + (e - 2) \le 7 \le 1 + 2e - (e - 2),$$

so $7 = g = 3 + e$ and $e = g - 3 = 4$. Moreover $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(4 - 2) = 4$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 2 + 16 - 14 = 4$ by Proposition 2.19, Case $b.3$ and $b.4$, so there are four simple base points for $L'' \subset |2C_1|$.

CASE g = 8 As seen before, the only values that satisfy the conclusions of Proposition 2.19 are $(a, q) = (2, 2)$ and $e = g - 3 = 5$. Moreover $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(5 - 2) = 6$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 2 + 20 - 16 = 6$ by Proposition 2.19, Case $b.3$ and $b.4$, so there are six simple base points for $L'' \subset |2C_1|$.

2. **CASE g = 9** We have to analyze different cases.

   If $(a, q) = (2, 2)$, then

$$1 + 4 + (e - 2) \le 9 \le 1 + 2e - (e - 2) \quad \Leftrightarrow \quad e = 9 - 3 = 6.$$

   Furthermore $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(6 - 2) = 8$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 2 + 24 - 18 = 8$, i.e. we have eight simple base points for a generic $C'' \in L'' \subset |2C_1|$.

   If $(a, q) = (2, 3)$, then

$$1 + 8 + (e - 4) \le 9 \le 1 + 2e - (e - 4) \quad \Leftrightarrow \quad e = 9 - 5 = 4.$$

   In this case $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(4 - 4) = 0$, so $L'' = |2C_1|$ does not have base points and $X' = X''$.

   We consider the couple $(a, q) = (3, 2)$. Then

$$1 + 9 + \frac{3}{2}(e - 2) \le 9 \le 1 + \frac{9}{2}e - \frac{3}{2}(e - 2) \quad \Rightarrow \quad e \le \frac{4}{3} < 2 = 2q - 2.$$

   So we esclude this case and similarly all the other cases with $a > 2$ and $q > 2$.

3. **CASE g = 10** We have three possible cases to analyze:

- if $(a, q) = (2, 2)$, then

$$1 + 4 + (e - 2) \leq 10 \leq 1 + 2e - (e - 2) \quad \Leftrightarrow \quad e = 10 - 3 = 7.$$

Furthermore $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(7 - 2) = 10$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 2 + 28 - 20 = 10$, i.e. $L'' \subset |2C_1|$ has ten simple base points;

- if $(a, q) = (2, 3)$, then

$$1 + 8 + (e - 4) \leq 10 \leq 1 + 2e - (e - 4) \quad \Leftrightarrow \quad e = 10 - 5 = 5.$$

Since $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 2(5-4) = 2$ and $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t}^2 = 2+20-20 = 2$, then $L'' \subset |2C_1|$ has two simple base points;

- if $(a, q) = (3, 2)$, then

$$1 + 9 + \frac{3}{2}(e - 2) \leq 10 \leq 1 + \frac{9}{2}e - \frac{3}{2}(e - 2) \quad \Leftrightarrow \quad 7 + \frac{3}{2}e \leq 10 \leq 4 + 3e.$$

Then the only acceptable value of e such that $e \geq 2q - 2$ is $e = 2$. Since $\sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 3(2 - 2) = 0$, then $X' = X''$ and $L'' = |3C_1|$.

As in the previous cases, we exclude all the other couples (a, q) with $a \geq 3$ and $q \geq 3$ because they do not satisfy the conclusions of Proposition 2.19.

We can continue with the same procedure for all $g > 10$. □

Remark 2.33. *We do not assure the existence of surfaces corresponding to the different values of g, q, a, e found in the previous Proposition, they are at most possible.*

Remark 2.34. *If we assume that the number of connected components of W' and W'' is the same, then, by Corollary 2.15 and Remark 2.18, we know that, if the surfaces X of Proposition 2.32 exist, then they have exactly one non-rational singularity x and the only case in which there could be two non-rational singularities is for $e = 4q - 4$. Among the cases considered in the previous Proposition, the surface X could have two non-rational singularities if and only if $g = 7$, $a = 2$, $q = 2$ and $e = 4$.*

The following Proposition is analogous to Proposition 2.32, only assuming $q = 1$.

Proposition 2.35. *Let X be a surface with Prym-canonical hyperplane sections birationally equivalent to a ruled surface X'' over a base curve of genus $q = 1$, not a cone. Let us suppose that X'' satisfies the conclusions of Proposition 2.19. Using the same notation as before, if $g = 5$, then there are the following numerical possibilities for the invariants of X'':*

- $(a, e) = (2, 4)$ *and L'' has 8 simple base points on X'';*
- $(a, e) = (3, 2)$ *and L'' has 2 simple base points and 2 double base points on X'';*
- $(a, e) = (4, 1)$ *and L'' has 2 double base points on X'';*

- $(a,e)=(4,2)$ *and* L'' *has* 2 *double base points and* 1 *base point of multiplicity four on* X''*;*

- $(a,e)=(5,1)$ *and* L'' *has* 1 *simple base point and* 1 *base point of multiplicity four on* X''*.*

Proof. Let $a \geq 2$ otherwise X is a cone (see Proposition 2.25). Since $q=1$ and we suppose that $\mathcal{E}$ is decomposable, then, by Proposition 2.19, Case $b.5$ and Case $a.$, we have that

$$g \geq 1+\frac{1}{2}ae \geq 1+\frac{1}{2}a(2q-2)=1.$$

More precisely, by Remark 1.3, we know that $g>4$. In the following, we will only consider $g=5$.
The only pairs (a,e) that satisfy the condition $b.5$ of Proposition 2.19 are

$$\{(2,4),(3,2),(4,1),(4,2),(5,1),(6,1),(7,1),(8,1)\}.$$

It is not difficult to verify that there are no multiplicities r_{t,i_t}, for $t=1,...,s-1$ and $i_t=1,...,i(t)$, that satisfy Case $b.3$ and Case $b.4$ of Proposition 2.19 assuming that $(a,e)=\{(6,1),(7,1),(8,1)\}$. Hence we exclude these three cases.
In the other situations, to find the multiplicities r_{t,i_t} of all the base points of L'' is easy using the conditions of Proposition 2.19. □

Remark 2.36. *We can find other numerical invariants of possible surfaces* X *with Prym-canonical hyperplane sections for any* $g \geq 6$ *using the same techniques of the previous Proposition.*

Remark 2.37. *We can apply Remark 2.33 also to the previous Proposition.*

Remark 2.38. *By Corollary 2.15 and Remark 2.18, we know that, if the surfaces* X *of the previous Proposition exist, then they have only one non-rational singularity* x*.*

We can construct an example of surface with Prym-canonical hyperplane sections birationally equivalent to an elliptic ruled surface X'' such that $X'' \neq X'$. If we fix $g=10$ and $a=3$, then, by Proposition 2.19, Case $b.5$, we expect that $3 \leq e \leq 6$ using the same procedure of the proof of Proposition 2.35.
We will use $e=3$ to construct a new example.

Example 2.39. *Let* Γ *be an elliptic curve and let* $X''=\mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$ *be a minimal ruled surface with base curve* Γ*, for* $D \in \mathrm{Div}(\Gamma)$*. If* Q_i*, for* $i=1,2,3$*, is a general point on* Γ *and* $\alpha,\beta \in \Gamma$ *are two points such that* $\alpha-\beta$ *is a non zero two torsion element of* Γ*, then we assume that* $D=-Q_1-Q_2-Q_3-\alpha+\beta$*. So* $e=-\deg(D)=3$*.*

We consider the linear system $|3C_1|=|3C_0+3(Q_1+Q_2+Q_3+\alpha-\beta)\cdot f|$*. As seen in Proposition 2.23, this linear system is base-point free, so, by Bertini's Theorem, its general element* $\mathcal{L}$ *is smooth and, since* $\mathcal{L}^2=9C_1^2=9e>0$*, it is also irreducible.*

We call $f_i := Q_i \cdot f$, for $i = 1, 2, 3$. For any fibre f, we have that $\mathcal{L} \cdot f = 3$, so we can fix the 9 points

$$Z := \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}\}$$

of intersection between $\mathcal{L}$ and f_i, for $i = 1, 2, 3$. Since $3C_1$ is disjoint from C_0, we can assume that $Z \cap C_0 = \varnothing$. So we can consider the linear system $L'' \subset |3C_1|$ with Z as base locus. In particular, we suppose that every curve $C'' \in L''$ simply passes through the 9 points, so $\mathcal{L}$ is an element of L''. By Lemma 2.5, we have that $h^0(\mathcal{O}_{X''}(3C_0 + 3(Q_1 + Q_2 + Q_3 + \alpha - \beta) \cdot f)) = h^0(\mathcal{O}_\Gamma(3(Q_1 + Q_2 + Q_3 + \alpha - \beta))) + h^0(\mathcal{O}_\Gamma(2(Q_1+Q_2+Q_3))) + h^0(\mathcal{O}_\Gamma(Q_1+Q_2+Q_3+\alpha-\beta)) + h^0(\mathcal{O}_\Gamma) = 1+3+6+9 = 19$, so $\dim L'' \geq 18 - 9 = 9$. Since $\mathcal{L} \in L''$ and smoothness is an open condition, then the general element C'' of L'' is smooth.

We know that $-K_{X''} \sim 2C_0 - D \cdot f$ and $h^0(\mathcal{O}_{X''}(-K_{X''})) = 4$ by Lemma 2.5.
Given $B := 2C_0 + (Q_1 + Q_2 + Q_3) \cdot f$ an effective divisor on X'', then $Z \subset B$, i.e $h^0(\mathcal{O}_{X''}((2C_0 + (Q_1 + Q_2 + Q_3) \cdot f) \otimes \mathcal{I}_Z)) > 0$. On the other hand, we can show that $h^0(\mathcal{O}_{X''}(-K_{X''}) \otimes \mathcal{I}_Z)) = 0$. Indeed, if we suppose that there is an effective divisor $T \in |-K_{X''} \otimes \mathcal{I}_Z|$, then $T \cdot C_0 = 2C_0^2 - \deg(D) = 2(-3) + 3 < 0$, so C_0 is a fixed component of T. Moreover $T \cdot f = 2$ but T contains 3 points of the fibres f_i, so it also contains f_1, f_2, f_3. Then

$$|-K_{X''} \otimes \mathcal{I}_Z| = |(2C_0 - D \cdot f) \otimes \mathcal{I}_Z| = C_0 + f_1 + f_2 + f_3 + |C_0 + (\alpha - \beta) \cdot f|,$$

so

$$\dim |-K_{X''} \otimes \mathcal{I}_Z| = \dim |C_0 + (\alpha - \beta) \cdot f|.$$

By Lemma 2.5, we have that $h^0(\mathcal{O}_{X''}(C_0 + (\alpha - \beta) \cdot f)) = h^0(\mathcal{O}_\Gamma(\alpha - \beta)) + h^0(\mathcal{O}_\Gamma(\alpha - \beta - Q_1 - Q_2 - Q_3 - \alpha + \beta)) = 0$. Hence T effective does not exist.

From Proposition 2.12, Case IV.4, it follows that $h^0(\mathcal{O}_{X''}(-2K_{X''})) = 10$ and, since $-2K_{X''} \sim 4C_0 + 2(Q_1 + Q_2 + Q_3) \cdot f$ and $4C_0 + 2(Q_1 + Q_2 + Q_3) \cdot f$ contains Z with multiplicity 2, then also $h^0(\mathcal{O}_{X''}(-2K_{X''}) \otimes \mathcal{I}_Z) > 0$, in particular $h^0(\mathcal{O}_{X''}(-2K_{X''}) \otimes \mathcal{I}_Z^2) > 0$.

Let $\phi : X' \to X''$ be the blowing up of X'' along the 9 points defining Z. Let $E_{i,j} \in X'$ be the exceptional divisor of $x_{i,j}$, for $i, j \in \{1, 2, 3\}$, and let $\widetilde{f_i}$ be the strict transform of f_i, for $i = 1, 2, 3$. With abuse of notation, we call $C_0 := \phi^*(C_0)$ (we remark that $\phi^*(C_0)$ is the strict transform of C_0 since $Z \cap C_0 = \varnothing$), $\phi^*(\alpha \cdot f) := f_\alpha$ and $\phi^*(\beta \cdot f) := f_\beta$. Let L' be such that $L'' = \phi_* L'$. Then the strict transform $C' \in L'$ of a general $C'' \in L''$ is of the form

$$\begin{aligned} C' = \phi^*(C'') - E_{1,1} - \dots - E_{3,3} &\sim 3\phi^*(C_0) + 3\sum_{i=1}^{3} \phi^*(f_i) + 3\phi^*(\alpha \cdot f) - 3\phi^*(\beta \cdot f) + \\ -E_{1,1} - \dots - E_{3,3} = 3C_0 &+ 3\sum_{i=1}^{3} \widetilde{f}_i + 3(E_{1,1} + \dots + E_{3,3}) + \\ +3f_\alpha - 3f_\beta - E_{1,1} - \dots - E_{3,3} &= 3C_0 + 3\sum_{i=1}^{3} \widetilde{f}_i + 3f_\alpha - 3f_\beta + 2(E_{1,1} + \dots + E_{3,3}). \end{aligned}$$

Moreover, using [20], Proposition V.3.3, we have that

$$\begin{aligned} -K_{X'} = \phi^*(-K_{X''}) - E_{1,1} - ... - E_{3,3} &\sim 2\phi^*(C_0) + \sum_{i=1}^{3} \phi^*(f_i) + \phi^*(\alpha \cdot f) + \\ -\phi^*(\beta \cdot f) - E_{1,1} - ... - E_{3,3} &= 2C_0 + \sum_{i=1}^{3} \widetilde{f}_i + E_{1,1} + ... + E_{3,3} + \\ +f_\alpha - f_\beta - E_{1,1} - ... - E_{3,3} &= 2C_0 + \sum_{i=1}^{3} \widetilde{f}_i + f_\alpha - f_\beta. \end{aligned}$$

At the same way, we find that every antibicanonical divisor of X' is of the form $-2K_{X'} \sim 4C_0 + 2\widetilde{f}_1 + 2\widetilde{f}_2 + 2\widetilde{f}_3$. We put

$$W' = 4C_0 + 2\widetilde{f}_1 + 2\widetilde{f}_2 + 2\widetilde{f}_3 \in |-2K_{X'}|.$$

It is clear that $h^0(\mathcal{O}_{X'}(-K_{X'})) = h^0(\mathcal{O}_{X''}(-K_{X''} \otimes \mathcal{I}_Z)) = 0$ while $h^0(\mathcal{O}_{X'}(-2K_{X'})) = h^0(\mathcal{O}_{X''}(-2K_{X''} \otimes \mathcal{I}_Z^2)) > 0$.

Step by step, we can prove that the general hyperplane section C' of X' is a Prym-canonical embedded curve.

CLAIM 1 : *We have that $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$ and $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$, where $C' \in L'$ is a general curve. In particular, $-K_{X'}|_{C'}$ is a non-zero two torsion divisor.*

Proof. The intersection

$$C' \cdot (-K_{X'}) = 6C_0^2 + 6\sum_{i=1}^{3} C_0 \cdot \widetilde{f}_i + 3\sum_{i=1}^{3}(C_0 \cdot \widetilde{f}_i + \widetilde{f}_i^{\,2}) + 2[\widetilde{f}_1 \cdot (E_{1,1} + E_{1,2} + E_{1,3}) +$$

$$+\widetilde{f}_2 \cdot (E_{2,1}+E_{2,2}+E_{2,3}) + \widetilde{f}_3 \cdot (E_{3,1}+E_{3,2}+E_{3,3})] = 6(-3)+6(3)+3(3-9)+2(3+3+3) = 0$$

and clearly also $C' \cdot (-2K_{X'}) = 0$.
Since $-2K_{X'}$ is effective, then the antibicanonical divisor of X' is contracted by $i_{L'}$, in particular $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$.
On the contrary, we have that $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$, so $-K_{X'}$ is not effective. Using the exact sequence

$$0 \to \mathcal{O}_{X'}(-K_{X'} - C') \to \mathcal{O}_{X'}(-K_{X'}) \to \mathcal{O}_{C'}(-K_{X'}) \to 0, \tag{11}$$

if we prove that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$, then $h^0(\mathcal{O}_{C'}(-K_{X'})) = 0$. This would imply that $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$.

By Serre Duality, we have that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = h^1(\mathcal{O}_{X'}(2K_{X'} + C'))$. If we prove that $K_{X'} + C'$ is big and nef, then, by the Kawamata-Viehweg vanishing Theorem (see [22] and [32]), the first cohomology $h^1(\mathcal{O}_{X'}(2K_{X'} + C')) = 0$.

In our case, since $2(\alpha - \beta) \sim 0$, then $2f_\alpha - 2f_\beta \sim 0$ and the divisor

$$K_{X'} + C' \sim C_0 + 2\widetilde{f}_1 + 2\widetilde{f}_2 + 2\widetilde{f}_3 + 2E_{1,1} + ... + 2E_{3,3}.$$

We recall that a divisor θ on a surface Y is nef if and only if, for any effective curve δ on Y, the intersection $\theta \cdot \delta \geq 0$. Moreover, it is also big if $\theta^2 > 0$.
Since $K_{X'} + C'$ is written as sum of irreducible and effective curves, then, to prove that $K_{X'} + C'$ is nef, it is enough to prove that $(K_{X'} + C') \cdot \delta \geq 0$, for any irreducible component δ. Therefore, for $i, j \in \{1, 2, 3\}$, we have:

$$(K_{X'} + C') \cdot C_0 = C_0^2 + 2(C_0 \cdot (\widetilde{f}_1 + \widetilde{f}_2 + \widetilde{f}_3)) = -3 + 6 = 3 > 0;$$

$$(K_{X'} + C') \cdot \widetilde{f}_i = C_0 \cdot \widetilde{f}_i + 2\widetilde{f}_i^{\,2} + \widetilde{f}_i \cdot (2E_{i,1} + 2E_{i,2} + 2E_{i,3}) = 1 + 2(-3) + 2(3) = 1 > 0;$$

$$(K_{X'} + C') \cdot E_{i,j} = 2\widetilde{f}_i \cdot E_{i,j} + 2E_{i,j}^2 = 0.$$

Then $K_{X'} + C'$ is nef. Because we have strictly positive intersections between $K_{X'} + C'$ and its components, then $K_{X'} + C'$ is also big. So the claim is satisfied. □

It is not difficult to compute that $C'^2 = 18$, *so, by the adjunction formula, the genus* $g(C') = 1 + \frac{1}{2}(C'^2 + K_{X'} \cdot C') = 10$. *We also observe that* C' *is smooth because it is the strict transform of a general element* C'' *of* L'', *that is smooth. Since* $-K_{X'}|_{C'}$ *is a non-zero two torsion divisor as seen in Claim* 1, *we have that* $L'|_{C'} = |K_{C'} - K_{X'}|_{C'}|$ *defines a Prym-canonical map*

$$i_{L'|_{C'}} : C' \dashrightarrow \mathbb{P}^8.$$

CLAIM 2 : *The rational map* $i_{L'|_{C'}} : C' \dashrightarrow \mathbb{P}^8$ *is an embedding, for any general curve* $C' \in L'$.

Proof. We know that $(C', -K_{X'}|_{C'})$ is a Prym curve of genus 10 and we can show that the Prym-canonical system $L'|_{C'}$ is base point free. Indeed, by [20], Proposition $IV.3.1$, the linear system $L'|_{C'}$ is base-point free if and only if, for every point $P \in C'$,

$$\dim\ L'|_{C'} - 1 = \dim\ |C'|_{C'} - P|.$$

By the Riemann-Roch Theorem, we have that $\dim\ L'|_{C'} - 1 = h^0(\mathcal{O}_{C'}(C')) - 2 = h^1(\mathcal{O}_{C'}(C')) + 18 + 1 - 10 - 2$. By Serre Duality and the adjunction formula, we observe that $h^1(\mathcal{O}_{C'}(C')) = h^0(\mathcal{O}_{C'}(K_{C'} - C')) = h^0(\mathcal{O}_{C'}(K_{X'}))$. This is equal to 0 because $-K_{X'}|_{C'}$ is a non-zero two torsion divisor. Then

$$\dim\ L'|_{C'} - 1 = 7.$$

On the other hand, the dimension $\dim\ |C'|_{C'} - P| = h^0(\mathcal{O}_{C'}(C' - P)) - 1 = h^1(\mathcal{O}_{C'}(C' - P)) + 17 + 1 - 10 - 1$ by the Riemann-Roch Theorem. By Serre Duality and the adjunction formula, we have that $h^1(\mathcal{O}_{C'}(C' - P)) = h^0(\mathcal{O}_{C'}(K_{C'} - C' + P) = h^0(\mathcal{O}_{C'}(P + K_{X'}))$.
Since $-K_{X'}|_{C'}$ is a non-zero two torsion divisor, if we suppose that $h^0(\mathcal{O}_{C'}(P + K_{X'})) > 0$ for some point $P \in C'$, then C' is hyperelliptic by Remark 2.24. Since $C' \cdot \widetilde{f} = 3$, where $\widetilde{f}$ is the pullback of a general fibre f of X'', then C' is also a covering $3:1$ of the elliptic curve Γ. So C' is a double cover of $\mathbb{P}^1$ and a triple cover of Γ. This is not possible by

Castelnuovo-Severi inequality otherwise we would have $10 = g(C') \leq 2 \cdot 0 + 3 \cdot 1 + 1 \cdot 2 = 5$ (we will recall it in Recall 2.40).
Thus $h^0(\mathcal{O}_{C'}(P + K_{X'})) = 0$ for any point $P \in C'$ and

$$\dim |C'|_{C'} - P| = 7.$$

We have proved that $L'|_{C'}$ is base-point free.

Thanks to [6], Corollary 2.2, we know that, if C' is not bielliptic, then $L'|_{C'}$ is an embedding. Since C' is a triple cover of Γ as observed before, we have that C' cannot be bielliptic again by Castelnuovo-Severi inequality otherwise we would have $10 \leq 2 \cdot 1 + 3 \cdot 1 + 1 \cdot 2 = 7$. Then the claim is proved. □

At this point, since $L'|_{C'}$ is base-point free, it is clear that L' is also base-point free. Since the restriction $L'|_{C'}$ defines an embedding for each generic curve $C' \in L'$, then $i_{L'}$ is a birational map, generically $1:1$.

Before we have showed that $\dim(L'') = \dim(L') \geq 9$. From the exact sequence

$$0 \to \mathcal{O}_{X'}(C' - C') \to \mathcal{O}_{X'}(C') \to \mathcal{O}_{C'}(C') \to 0,$$

we conclude that $h^0(\mathcal{O}_{X'}(C')) \leq 10$ since $\mathcal{O}_{X'}(C' - C') \cong \mathcal{O}_{X'}$ and $h^0(\mathcal{O}_{C'}(C')) = 9$. So we have that $h^0(\mathcal{O}_{X'}(C')) = 10$.

Then X' has hyperplane sections that are Prym-canonical embedded and, in particular, we have found a new surface $X = i_{L'}(X') \subset \mathbb{P}^9$ with Prym-canonical hyperplane sections. Since W' is connected, then its image $x \in X$ of W' is a singular point. There are other possible rational double singularities on X whose exceptional divisors on X' do not intersect W'.
If a general hyperplane section C of X is projectively normal, then, by Proposition 1.15, the geometric genus $p_g(x)$ is equal to 1.

Recall 2.40. *We recall the* Castelnuovo-Severi inequality for Riemann surfaces, *discussed in [1].*

Let W_g be a compact Riemann surface of genus g. We suppose that W_g covers two Riemann surfaces, W_h of genus h in m sheets and W_k of genus k in n sheets. Again we suppose that the two coverings admit no common non-trivial factorization.

Then the Castelnuovo-Severi inequality states:

$$g \leq mh + nk + (m - 1)(n - 1).$$

Remark 2.41. *We can compute how many moduli the couple (X'', L'') of previous example depends on.*

The choice of the elliptic curve Γ depends on one parameter. In addition we fix a divisor $D = -Q_1 - Q_2 - Q_3 - \alpha + \beta$ of degree -3, where Q_i is a general point on Γ, for $i = 1, 2, 3$, and $\alpha - \beta$ is a non-zero two torsion element of Γ.
We know that there are only three non-zero two torsion points on Γ. Furthermore we observe that $|Q_1 + Q_2 + Q_3|$ is a linear system of dimension 2, so the choice of $\mathcal{O}_\Gamma(D)$ depends on $3 - 2 = 1$ parameter.
Moreover, every automorphism of Γ lifts to an automorphism of X'' that means that, if $\gamma : \Gamma \to \Gamma$ is an automorphism, then $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)) \cong \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(\gamma^*(D)))$. The group $\mathrm{Aut}(\Gamma)$ has dimension 1.

To construct the surface with Prym-canonical hyperplane sections of the previous example, we also fix a linear system $L'' \subset |3C_1| = |3C_0 - 3D \cdot f|$ with 9 simple base points. The linear system $|3C_1|$ depends on the parameters fixed before. The 9 simple base points are the points of intersection between a general element $\mathcal{L} \in |3C_1|$ and the three fibres $f_i := Q_i \cdot f$, for $i = 1, 2, 3$. The choice of the effective divisor in a linear system of the type $|Q_1 + Q_2 + Q_3|$ that defines the three fibres f_1, f_2, f_3 depends on 2 parameters. In addition, as seen in the previous example, the nine points $\{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}\}$ impose independent conditions on the linear system L'', so they depend on 9 parameters.

The choice of the pair (X'', L'') depends on $1 + 1 - 1 + 2 + 9 = 12$ parameters.

We know that $\dim(\mathrm{Aut}(\mathbb{P}^3)) = 15$. We can consider X'' as the blowing up of the vertex of the cone $C_{X''}$ on a plane cubic of $\mathbb{P}^3$. If C_Γ is the base curve of $C_{X''}$, there are ∞^8 plane cubics isomorphic to C_Γ. Since we can choose the vertex among all the possible points of $\mathbb{P}^3$ obtaining always isomorphic cones, then there are $\infty^{(8+3)} = \infty^{11}$ isomorphic cones of the type of $C_{X''}$ in $\mathbb{P}^3$.
Thus there are ∞^4 automorphism of $\mathbb{P}^3$ that fix X'' so, in conclusion, the couple (X'', L'') depends on $12 - 4 = 8$ parameters.

Since the surface constructed in the previous example depends on 8 moduli while a general Enriques surface depends on 10 moduli, then the generic Enriques surface can degenerate to one of these surfaces since they depend on less parameters.

2.4 Construction of surfaces with Prym-canonical hyperplane sections birationally equivalent to rational ruled surfaces

In this section, we will construct examples of surfaces with Prym-canonical hyperplane sections birationally equivalent to rational ruled surfaces. We will use the same notation of the previous paragraph, so the base curve Γ of the ruled surface X'' has genus $q = 0$.

We recall that a surface X birationally equivalent to a rational ruled surface contains only rational singularities (see Proposition 1.15).

We can replace X'' by a more suitable minimal model, as done in Proposition 2.19. The proof is similar, except for the point a. since we have already observed that $e \geq 0$ always, then the case $e = 2q-2 = -2$ cannot happen as well as $2D \sim -2K_\Gamma$. Moreover, if $q = 0$, then $\mathcal{E}$ is always decomposable. Anyway we will assume the conclusions of Proposition 2.19 always satisfied.

Remark 2.42. *If the conclusions of Proposition 2.19 are satisfied, then we exclude the case $X' = X''$. Indeed, by Case b.3 of Proposition 2.19, we would have $0 = \sum_{t=1}^{s-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = a(e+2)$. Since $a \geq 1$ and $e \geq 0$, we have to exclude this possibility.*

We will use the same techniques as before to construct surfaces X with Prym-canonical hyperplane sections. Since we consider a linear system $L'' \subseteq |aC_1|$, then it is important to show that $|aC_1|$ is always base-point free on X'' rational ruled surface.

Proposition 2.43. *Let $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$, with $p_g(\Gamma) := q = 0$ and $D \in \mathrm{Div}(\Gamma)$. If $C_1 \sim C_0 - D \cdot f$ is a fixed section of X'' disjoint from C_0, then the linear system $|aC_1|$ on X'' is base point free for any $a \geq 1$.*

Proof. By Proposition 2.22, the linear system $|C_1|$ is base-point free if and only if, for any point $P \in \Gamma$, we have

$$h^0(\mathcal{O}_{X''}(C_0 - D \cdot f - P \cdot f)) = h^0(\mathcal{O}_{X''}(C_0 - D \cdot f)) - 2.$$

By Lemma 2.5, we know that $h^0(\mathcal{O}_{X''}(C_0 - D \cdot f - P \cdot f)) = h^0(\mathcal{O}_\Gamma(-D - P)) + h^0(\mathcal{O}_\Gamma(-P)) = e - 1 + 1 = e$.
On the other hand $h^0(\mathcal{O}_{X''}(C_0 - D \cdot f)) = h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma(-D + D)) = e + 1 + 1$.
So the claim is proved and $|C_1|$ is always base point free.

Consequently it is obvious that $|aC_1|$ is also base-point free, for any $a \geq 1$. □

Remark 2.44. *To construct examples of surfaces with Prym-canonical hyperplane sections, we fix a linear system $L'' \subseteq |aC_1|$, for $a \geq 1$. If $a = 1$, then $\sum_{s=1}^{t-1} \sum_{i_t=1}^{i(t)} r_{t,i_t} = 0$ by Case b.1 of Proposition 2.19. But this would imply that $e = -2$ by Proposition 2.19, Case b.3. If X'' is a rational ruled surface, this is not possible.*
Hence we assume $a \geq 2$.

Before constructing a new example, let us recall a definition that we will use later.

Definition 2.45. *Let C be an algebraic curve. A* correspondence *T is a curve in the product $C \times C$.*
Let us suppose that T does not contain curves of the type $C_x := x \times C$, for $x \in C$. We call $T(x)$ the divisor corresponding to *x* in the correspondence *T, for $x \in C$, that is the projection of the intersection between T and C_x on the second factor of C_x.*
A correspondence T on a curve C is said valence correspondence with valence *$c \in \mathbb{Z}$ if, for any point $x \in C$, we have that the class of linear equivalence of the divisor $cx + T(x)$ does not depend on x. A correspondence without valence is said* singular correspondence.

Remark 2.46. *A valence correspondence belongs to the subgroup of Neron-Severi of $C \times C$ generated by $C \times q$, $p \times C$ and by the diagonal class, for p and q points of C. A singular correspondence is identified with a class of curves that does not belong to this subgroup.*

Example 2.47. *Let Γ be a rational smooth curve and let $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$ be a minimal ruled surface with base curve Γ, for $D \in Div(\Gamma)$. We assume that $e = 4$, so* $\deg(D) = -4$. *Hence X'' is a Hirzebruch surface F_4.*

We know that $-K_{X''} \sim 2C_0 - (K_\Gamma + D) \cdot f$, where $\deg(-K_\Gamma - D) = 2 + 4 = 6$. *Since all divisors of the same degree on Γ are linearly equivalent, then we put*

$$-K_{X''} = 2C_0 + 2\sum_{i=1}^{3} F_i,$$

where F_1, F_2 and F_3 are distinct and fixed fibres. We also set

$$W'' = 4C_0 + 3F_0 + 3\sum_{i=1}^{3} F_i \in |-2K_{X''}|,$$

where F_0 is a generic fibre distinct from F_i, for $i = 1, 2, 3$.

We consider the linear system $|4C_1| = |4C_0 - 4D \cdot f|$ on X''. Every element in $|4C_1|$ intersects every fibre of X'' in 4 points since $4C_1 \cdot f = 4$. In addition, we know that $h^0(\mathcal{O}_{X''}(4C_1)) = h^0(\mathcal{O}_\Gamma(-4D)) + h^0(\mathcal{O}_\Gamma(-3D)) + h^0(\mathcal{O}_\Gamma(-2D)) + h^0(\mathcal{O}_\Gamma(-D)) + h^0(\mathcal{O}_\Gamma) = 17 + 13 + 9 + 5 + 1 = 45$ by Lemma 2.5.

CLAIM 1 : *There is a smooth curve $\mathcal{L} \in |4C_1|$ such that $\mathcal{L}$ is tangent to F_i, for $i = 0, 1, 2, 3$, respectively in two points.*

Proof. We can assume that $X'' = F_4$ is the blowing up of the vertex of a cone in $\mathbb{P}^5$ on a rational normal curve of $\mathbb{P}^4$. It is clear that C_0 is the exceptional divisor associated with the vertex.
Let $E = F_0 + F_1 + F_2 + F_3$ be the curve intersection between the cone and a hyperplane H_0 of $\mathbb{P}^5$ passing through the vertex of the cone. With abuse of notation, the total transform of E on X'' is $E = C_0 + \sum_{i=0}^{3} F_i$.
Let H_1 and H_2 be two generic hyperplane sections of X'' (they do not intersect C_0). Then

$$H_1 \cap E = \{x_{0,1}, x_{1,1}, x_{2,1}, x_{3,1}\},$$

where $x_{i,1} \in F_i$, for $i = 0, 1, 2, 3$, and

$$H_2 \cap E = \{x_{0,2}, x_{1,2}, x_{2,2}, x_{3,2}\},$$

where $x_{i,2} \in F_i$, for $i = 0, 1, 2, 3$.
Let L_1 be the pencil of hyperplane sections of X'' generated by H_0, hence by E, and by H_1 and let L_2 be the pencil of hyperplane sections of X'' generated by H_0, hence

by E, and by H_2. It is clear that $\{x_{0,1}, x_{1,1}, x_{2,1}, x_{3,1}\}$ is the base locus of L_1 and $\{x_{0,2}, x_{1,2}, x_{2,2}, x_{3,2}\}$ is the base locus of L_2.

Since $|3C_1|$ is base-point free by Proposition 2.43, then it contains a general curve G that is smooth by Bertini's Theorem. Furthermore, since G is general, it does not contain $\{x_{0,j}, x_{1,j}, x_{2,j}, x_{3,j}\}$, for $j = 1, 2$.
Let $d_j, e_j \in L_j$, for $j = 1, 2$, be generic divisors in the two pencils L_j. They are also smooth curves since E, $H_1 \cap X''$ and $H_2 \cap X''$ simply pass through the base locus. Then $d_j + e_j$ contains $\{x_{0,j}, x_{1,j}, x_{2,j}, x_{3,j}\}$ with multiplicity 2, for $j = 1, 2$.
Let us consider a pencil $\mathcal{P}$ generated by $d_1 + e_1 + d_2 + e_2$ and $E + G$. By Bertini's Theorem, its curves may have singular points only on the base locus of the pencil. At this point we observe that $G \cdot (d_1 + e_1 + d_2 + e_2) = 48$ since d_j and e_j are hyperplane sections and $G \sim 3C_1$, where C_1 is a rational normal curve of degree 4. These 48 points are base points for the pencil, different from $x_{i,j}$, for $i = 0, ..., 3$ and $j = 1, 2$. Now $E + G$ has only 16 singular points since E has only 4 singular points on C_0, while G is smooth and disjoint from C_0 for its generality and $E \cdot G \sim (C_0 + 4F) \cdot 3C_1 = 12$. These points are different from the 48 base points for the generality of the hyperplane sections d_j and e_j, for $j = 1, 2$. Then $E + G$ is smooth in the 48 base points. The same is true for $d_1 + e_1 + d_2 + e_2$. Hence also a general divisor $\mathcal{L}$ in the pencil $\mathcal{P}$ is smooth in the 48 base points.
On the other hand $(d_1+e_1+d_2+e_2)|_E = \{x_{0,1}, x_{0,2}, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}\}$. These are other 8 base points for $\mathcal{P}$. Since $d_1+e_1+d_2+e_2$ passes through $x_{i,j}$, for $i = 0, 1, 2, 3$ and $j = 1, 2$, with multiplicity 2 and since $E+G$ simply passes through the eight points (E contains the fibres F_0, F_1, F_2 and F_3 while G does not contain these 8 points), then a general curve $\mathcal{L}$ is smooth in these 8 points and, in particular, it is tangent to F_0, F_1, F_2, F_3 in $\{x_{0,1}, x_{0,2}, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}\}$.
Finally, we observe that d_j and e_j are general hyperplane sections of X'', so they are linearly equivalent to C_1 and furthermore $d_1 + e_1 + d_2 + e_2 \sim 4C_1$. Similarly, we have $G \sim 3C_1$ and $E \sim C_0 + 4f$, so $E + G \sim 4C_1$. Then we have found a smooth curve $\mathcal{L} \in |4C_1|$ tangent to F_i in $x_{i,1}$ and $x_{i,2}$, for $i = 0, 1, 2, 3$. □

In the Figure 1 below, we will analyze what happens blowing up all the intersection points between $\mathcal{L}$ and W'', also infinitely near. We observe that, since $\mathcal{L} \sim 4C_1$ and $4C_1$ is disjoint from C_0, then $\mathcal{L}$ does not intersect C_0. With abuse of notation, we will call the strict transforms of C_0, F_i and $\mathcal{L}$ with the same names.
In the figure, we only focus on F_1, it is the same for F_2, F_3 and F_0.

- *STEP 1 We blow up the intersection points $x_{i,1}$ and $x_{i,2}$ on X'', for $i = 1, 2, 3$. On*
$$X''_1 := Bl_{x_{1,1},x_{1,2},x_{2,1},x_{2,2},x_{3,1},x_{3,2}}(X''),$$
we have that F_i is a $(-2)-$curve while the exceptional divisors $J_{i,1}$ and $J_{i,2}$ are $(-1)-$curves, for any i. Moreover, the antibicanonical divisor
$$W''_1 = 4C_0 + 3F_0 + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2}) \in |-2K_{X''_1}|.$$

- *STEP 2 In X_1'', the curve $\mathcal{L}$ simply passes through the infinitely near base points $y_{i,1}$ and $y_{i,2}$, for $i = 1, 2, 3$. We also blow up these six points. On*

$$X_2'' := Bl_{y_{1,1},y_{1,2},y_{2,1},y_{2,2},y_{3,1},y_{3,2}}(X_1''),$$

we have that F_i is a $(-4)-$curve, $J_{i,1}$ and $J_{i,2}$ are $(-2)-$curves and $E_{i,1}$ and $E_{i,2}$ are $(-1)-$curves, for any i. Furthermore, the antibicanonical divisor

$$W_2'' = 4C_0 + 3F_0 + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) \in |-2K_{X_2''}|.$$

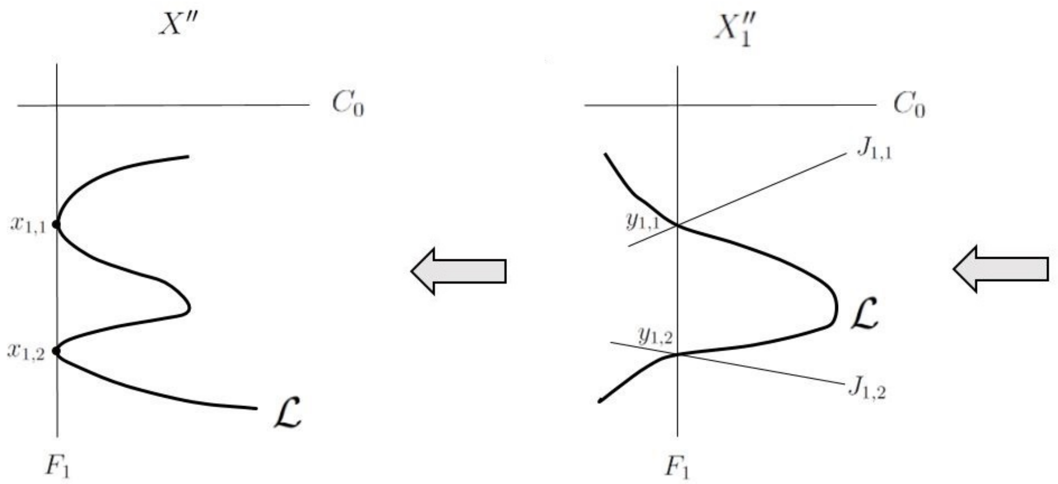
- *STEP 3 Again $\mathcal{L}$ intersects the exceptional divisors $E_{i,1}$ and $E_{i,2}$ respectively in $z_{i,1}$ and $z_{i,2}$ on X_2'', for $i = 1, 2, 3$. Let*

$$Y = Bl_{z_{1,1},z_{1,2},z_{2,1},z_{2,2},z_{3,1},z_{3,2}}(X_2'').$$

On Y, we have that F_i is a $(-4)-$curve, $J_{i,1}$ and $J_{i,2}$ are $(-2)-$curves, $E_{i,1}$ and $E_{i,2}$ are $(-2)-$curves and $B_{i,1}$ and $B_{i,2}$ are $(-1)-$curves, for $i = 1, 2, 3$. Now the antibicanonical divisor

$$-2K_Y \sim 4C_0 + 3F_0 + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2})$$

and we fix $W_Y'' = W_2''$, so there are not other infinitely near intersection points to blow up belonging to $\mathcal{L}$ and F_i, for $i = 1, 2, 3$.



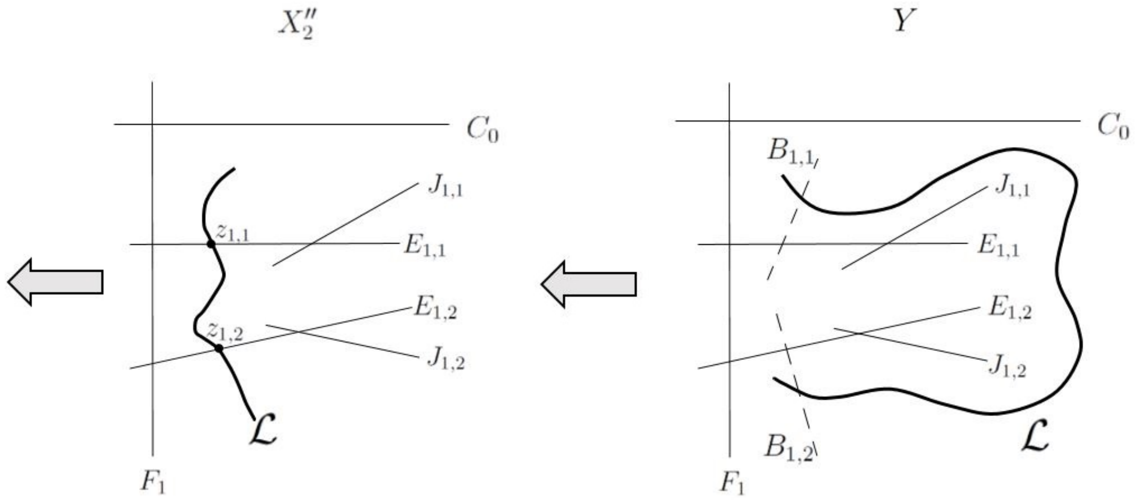


Figure 1: Example of blowing up of the intersection points bewteen $\mathcal{L}$ and W''

With the same techniques as before, we also blow up $\{x_{0,1}, x_{0,2}, y_{0,1}, y_{0,2}, z_{0,1}, z_{0,2}\} \in \mathcal{L} \cap F_0$ *(as seen in the previous figure, they are infinitely near points). We define*

$$X' := Bl_{z_{0,1},z_{0,2}}(Bl_{y_{0,1},y_{0,2}}(Bl_{x_{0,1},x_{0,2}}(Y))).$$

In X'*, there are no intersection points between* $\mathcal{L}$ *and* $-2K_{X'}$*.*

After observing how the blowing up works, we consider $L'' \subset |4C_1|$ *as the linear system of the curves of* $|4C_1|$ *simply passing through*

$$Z := \{x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}, z_{i,1}, z_{i,2}\}, \ \ for \ \ i = 0, 1, 2, 3.$$

Then $\dim L'' \geq \dim |4C_1| - 6 \cdot 4 = 44 - 6 \cdot 4 = 20$*. It is clear that* $\mathcal{L}$ *is an element of* L'' *and since smoothness is an open condition, then the general element* C'' *of* L'' *is smooth.*

With the same notation as before, we can assume that

$$-K_Y = 2C_0 + \sum_{i=1}^{3}(2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2})$$

and, as observed above, we have that

$$W''_Y = 4C_0 + 3F_0 + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) \in |-2K_Y|.$$

CLAIM 2 : *The divisor* $-K_Y = 2C_0 + \sum_{i=1}^{3}(2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2})$ *is the only one effective curve in its linear system.*

Proof. We will prove that $2C_0 + \sum_{i=1}^3 (2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2})$ is a fixed component of $|-K_Y|$. This is true because $-K_Y \cdot C_0 = -8+6 < 0$, so C_0 is a fixed component of $-K_Y$; again $(-K_Y - C_0)\cdot F_i = 1-8+2+2 < 0$, for $i = 1,2,3$, so F_1, F_2 and F_3 are fixed components of $-K_Y - C_0$; iterating this process, we have that $(-K_Y - C_0 - \sum_{i=1}^3 F_i)\cdot E_{i,j} = 1+1-4+1 < 0$, for $i = 1,2,3$ and $j = 1,2$, and again $(-K_Y - C_0 - \sum_{i=1}^3 (F_i + E_{i,1} + E_{i,2}))\cdot J_{i,j} = -2+1 < 0$, for $i = 1,2,3$ and $j = 1,2$. Continuing with this computation, $(-K_Y - C_0 - \sum_{i=1}^3 (F_i + E_{i,1} + E_{i,2} + J_{i,1} + J_{i,2}))\cdot C_0 = (C_0 + \sum_{i=1}^3 (F_i + E_{i,1} + E_{i,2} + B_{i,1} + B_{i,2}))\cdot C_0 = -4+3 < 0$, so C_0 is a fixed component of $-K_Y$ with multiplicity 2; moreover $(-K_Y - 2C_0 - \sum_{i=1}^3 (F_i + E_{i,1} + E_{i,2} + J_{i,1} + J_{i,2}))\cdot F_i = \sum_{i=1}^3 (F_i + E_{i,1} + E_{i,2} + B_{i,1} + B_{i,2})\cdot F_i = -4+2 < 0$, for $i = 1,2,3$, $(-K_Y - 2C_0 - \sum_{i=1}^3 (2F_i + E_{i,1} + E_{i,2} + J_{i,1} + J_{i,2}))\cdot E_{i,j} = (\sum_{i=1}^3 (E_{i,1} + E_{i,2} + B_{i,1} + B_{i,2}))\cdot E_{i,j} = -2+1 < 0$, for $i = 1,2,3$ and $j = 1,2$, and finally $(-K_Y - 2C_0 - \sum_{i=1}^3 (2F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2}))\cdot B_{i,j} = \sum_{i=1}^3 (B_{i,1} + B_{i,2})\cdot B_{i,j} = -1 < 0$, for $i = 1,2,3$ and $j = 1,2$. Then $2C_0 + \sum_{i=1}^3 (2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2})$ is a fixed component of $|-K_Y|$ and, more precisely, $-K_Y$ is the only one effective divisor in its linear system. □

CLAIM 3 : *The part* $4C_0 + \sum_{i=1}^3 (3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2})$ *is the fixed part of* $|-2K_Y|$ *while its variable part is* $3F_0$. *In particular, the dimension* $\dim|-2K_Y| = 3$.

Proof. The thesis is a consequence of the following computations: $-2K_Y \cdot C_0 = -16 + 3 + 9 < 0$, so C_0 is a fixed component of $-2K_Y$; $(-2K_Y - C_0)\cdot F_i = 3-12+2+2 < 0$, for $i = 1,2,3$, so F_1, F_2 and F_3 are fixed components of $-2K_Y - C_0$; $(-2K_Y - C_0 - \sum_{i=1}^3 F_i)\cdot F_i = 3-8+2+2 < 0$, for $i = 1,2,3$, so F_1, F_2 and F_3 are fixed components of $-2K_Y - C_0$ with multiplicity 2; again $(-2K_Y - C_0 - \sum_{i=1}^3 2F_i)\cdot E_{i,j} = 1+1-4 < 0$, for $i = 1,2,3$ and $j = 1,2$, and $(-2K_Y - C_0 - \sum_{i=1}^3 (2F_i + E_{i,1} + E_{i,2}))\cdot J_{i,j} = -2+1 < 0$, for $i = 1,2,3$ and $j = 1,2$. Continuing with this computations, $(-2K_Y - C_0 - \sum_{i=1}^3 (2F_i + E_{i,1} + E_{i,2} + J_{i,1} + J_{i,2}))\cdot E_{i,j} = (3C_0 + 3F_0 + \sum_{i=1}^3 (F_i + E_{i,1} + E_{i,2}))\cdot E_{i,j} = -2+1 < 0$, for $i = 1,2,3$ and $j = 1,2$; $(-2K_Y - C_0 - \sum_{i=1}^3 (2F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2}))\cdot F_i = (3C_0 + 3F_0 + \sum_{i=1}^3 F_i)\cdot F_i = -4+3 < 0$, for $i = 1,2,3$; $(-2K_Y - C_0 - \sum_{i=1}^3 (3F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2}))\cdot C_0 = (3C_0 + 3F_0)\cdot C_0 = -12+3 < 0$; $(-2K_Y - 2C_0 - \sum_{i=1}^3 (3F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2}))\cdot C_0 = (2C_0 + 3F_0)\cdot C_0 = -8+3 < 0$ and finally $(-2K_Y - 3C_0 - \sum_{i=1}^3 (3F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2}))\cdot C_0 = (C_0 + 3F_0)\cdot C_0 = -4+3 < 0$. In conclusion, we have that $4C_0 + \sum_{i=1}^3 (3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2})$ is a fixed component of $|-2K_Y|$, while the variable part is $3F_0$, indeed $3F_0 \cdot F_0 = 0$ and $|3F_0|$ is base point free, so F_0 is not another fixed component. Therefore $h^0(\mathcal{O}_Y(-2K_Y)) = h^0(\mathcal{O}_Y(3F_0)) = h^0(\mathcal{O}_\Gamma(3)) = 4$. □

After obtaining Y, *similarly we blow up* $x_{0,1}, x_{0,2} \in F_0$ *and the infinitely near base points* $y_{0,1}, y_{0,2}, z_{0,1}$ *and* $z_{0,2}$. *Then it is clear that, if* $Y_1 = Bl_{x_{0,1},x_{0,2}}(Y)$ *and* $\phi_1 : Y_1 \to Y$, *we have that*

$$-K_{Y_1} \sim 2C_0 + \sum_{i=1}^3 (2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2}) - J_{0,1} - J_{0,2}$$

and we put

$$W_1' = \phi_1^*(W_Y'') - 2J_{0,1} - 2J_{0,2} =$$

$$= 4C_0 + 3F_0 + J_{0,1} + J_{0,2} + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) \in |-2K_{Y_1}|.$$

Again, if $Y_2 = Bl_{y_{0,1},y_{0,2}}(Y_1)$ *and* $\phi_2 : Y_2 \to Y_1$*, we have that*

$$-K_{Y_2} \sim 2C_0 + \sum_{i=1}^{3}(2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2}) - J_{0,1} - J_{0,2} - 2E_{0,1} - 2E_{0,2}$$

and we put

$$W_2' = \phi_2^*(W_1') - 2E_{0,1} - 2E_{0,2} =$$

$$= 4C_0 + \sum_{i=0}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) \in |-2K_{Y_2}|.$$

Finally, if $X' = Bl_{z_{0,1},z_{0,2}}(Y_2)$ *and* $\phi' : X' \to Y_2$*, then*

$$-K_{X'} \sim 2C_0 + \sum_{i=1}^{3}(2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2}) +$$

$$-J_{0,1} - J_{0,2} - 2E_{0,1} - 2E_{0,2} - 3B_{0,1} - 3B_{0,2}$$

is not effective. Instead $W' = \phi'^*(W_2') - 2B_{0,1} - 2B_{0,2} \in |-2K_{X'}|$ *is of the type*

$$W' = 4C_0 + \sum_{i=0}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2})$$

and it is effective.

CLAIM 4 : *The divisor* $W' \in |-2K_{X'}|$ *is the only effective curve in its linear system.*

Proof. Since $-2K_{X'} \cdot C_0 = -16 + 3 + 9 < 0$, then C_0 is a fixed component of $-2K_{X'}$; $(-2K_{X'} - C_0) \cdot F_i = 3 - 12 + 2 + 2 < 0$, for $i = 0, 1, 2, 3$, so F_0, F_1, F_2 and F_3 are fixed components of $-2K_{X'} - C_0$; again $(-2K_{X'} - C_0 - \sum_{i=0}^{3} F_i) \cdot F_i = 3 - 8 + 2 + 2 < 0$, so F_0, F_1, F_2 and F_3 are fixed components of $-2K_{X'} - C_0$ with multiplicity 2; $(-2K_{X'} - C_0 - \sum_{i=0}^{3} 2F_i) \cdot E_{i,j} = 1 + 1 - 4 < 0$, for $i = 0, 1, 2, 3$ and $j = 1, 2$; again $(-2K_{X'} - C_0 - \sum_{i=0}^{3}(2F_i + E_{i,1} + E_{i,2})) \cdot J_{i,j} = -2 + 1 < 0$, for $i = 0, 1, 2, 3$ and $j = 1, 2$. Continuing with this computations, we still have that $(-2K_{X'} - C_0 - \sum_{i=0}^{3}(2F_i + E_{i,1} + E_{i,2} + J_{i,1} + J_{i,2})) \cdot E_{i,j} = (3C_0 + \sum_{i=0}^{3}(F_i + E_{i,1} + E_{i,2})) \cdot E_{i,j} = -2 + 1 < 0$; $(-2K_{X'} - C_0 - \sum_{i=0}^{3}(2F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2})) \cdot F_i = (3C_0 + \sum_{i=0}^{3} F_i) \cdot F_i = -4 + 3 < 0$, for $i = 0, 1, 2, 3$; moreover $(-2K_{X'} - C_0 - \sum_{i=0}^{3}(3F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2})) \cdot C_0 = 3C_0 \cdot C_0 = -12 < 0$; $(-2K_{X'} - 2C_0 - \sum_{i=0}^{3}(3F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2})) \cdot C_0 = 2C_0 \cdot C_0 = -8 < 0$ and finally $(-2K_{X'} - 3C_0 - \sum_{i=0}^{3}(3F_i + 2E_{i,1} + 2E_{i,2} + J_{i,1} + J_{i,2})) \cdot C_0 = C_0 \cdot C_0 = -4 < 0$. Then all $4C_0 + \sum_{i=0}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2})$ is a fixed component of $|-2K_{X'}|$ and consequently it is the only one effective divisor in its linear system. □

Since all the divisors on Γ of the same degree are linearly equivalent, then we have that $-4D \cdot f \sim 4\sum_{i=0}^{3} F_i$, so

$$C'' \sim 4C_0 + 4\sum_{i=0}^{3} F_i,$$

where C'' is a general element in L''. Then, its strict transform C' on X' is linearly equivalent to

$$C' \sim 4C_0 + \sum_{i=0}^{3}(4F_i + 3J_{i,1} + 3J_{i,2} + 6E_{i,1} + 6E_{i,2} + 5B_{i,1} + 5B_{i,2}).$$

Step by step, we can prove that a general hyperplane section C' of X' is a Prym-canonically embedded curve.

CLAIM 5 : *We have that $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$ while $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$. In particular, $-K_{X'}|_{C'}$ is a non-zero two torsion divisor.*

Proof. It is easy to prove that

$$C' \cdot W' = (4C_0 + \sum_{i=0}^{3}(4F_i + 3J_{i,1} + 3J_{i,2} + 6E_{i,1} + 6E_{i,2} + 5B_{i,1} + 5B_{i,2}))\cdot$$

$$\cdot(4C_0 + \sum_{i=0}^{3} 3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) =$$

$$= -64 + 16 + 48 + 4(12 - 48 + 18 + 18) + 8(6 - 6) + 8(6 - 24 + 10 + 8) = 0.$$

Since W' is effective, then it is contracted by the map defined by L', in particular $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$.
It is clear that also $C' \cdot (-K_{X'}) = 0$ but this time we have that $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$. Using the long exact sequence associated with

$$0 \to \mathcal{O}_{X'}(-K_{X'} - C') \to \mathcal{O}_{X'}(-K_{X'}) \to \mathcal{O}_{C'}(-K_{X'}) \to 0, \quad (12)$$

if we prove that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$, then $h^0(\mathcal{O}_{C'}(-K_{X'})) = 0$ and $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$.
By Serre Duality, we know that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = h^1(\mathcal{O}_{X'}(2K_{X'} + C'))$. If we prove that $K_{X'}+C'$ is big and nef, then, by the Kawamata-Viehweg vanishing Theorem (see [22] and [32]), the first cohomology $h^1(\mathcal{O}_{X'}(2K_{X'} + C')) = 0$.
Now

$$K_{X'} + C' = 2C_0 + \sum_{i=1}^{3}(2F_i + 2J_{i,1} + 2J_{i,2} + 4E_{i,1} + 4E_{i,2} + 4B_{i,1} + 4B_{i,2})+$$

$$+4F_0 + 4J_{0,1} + 4J_{0,2} + 8E_{0,1} + 8E_{0,2} + 8B_{0,1} + 8B_{0,2}.$$

Since $K_{X'}+C'$ is written as sum of irreducible and effective curves, then, to prove that $K_{X'}+C'$ is nef, it is enough to prove that $(K_{X'}+C')\cdot\delta\geq 0$, for any its irreducible component δ. Then, for $i\in\{1,2,3\}$ and $j\in\{1,2\}$, we have:

$$(K_{X'}+C')\cdot C_0=-8+6+4=2>0; \qquad (K_{X'}+C')\cdot F_i=2-8+4+4>0;$$

$$(K_{X'}+C')\cdot J_{i,j}=-4+4=0; \qquad (K_{X'}+C')\cdot E_{i,j}=2+2-8+4=0;$$

$$(K_{X'}+C')\cdot B_{i,j}=-4+4=0; \qquad (K_{X'}+C')\cdot F_0=2-16+8+8>0;$$

$$(K_{X'}+C')\cdot J_{0,j}=-8+8=0; \qquad (K_{X'}+C')\cdot E_{0,j}=4+4-16+8=0;$$

$$(K_{X'}+C')\cdot B_{0,j}=-8+8=0;$$

Therefore $K_{X'}+C'$ is nef. Because we have strictly positive intersections between $K_{X'}+C'$ and its components, then $K_{X'}+C'$ is also big. So the claim is satisfied. □

It is not difficult to compute that $C'^2=40$*, so, by the adjunction formula, the genus* $g(C')=1+\frac{1}{2}(C'^2)=21$*. We also observe that* C' *is smooth because it is the strict transform of a general element* C'' *of* L''*, that is smooth. Since* $-K_{X'}|_{C'}$ *is a non-zero two torsion divisor as seen in Claim* 5*, we have that* $L'|_{C'}=|K_{C'}-K_{X'}|_{C'}|$ *defines a Prym-canonical map*

$$i_{L'|_{C'}}:C'\dashrightarrow\mathbb{P}^{19}.$$

Moreover we have showed that $\dim(L'')=\dim(L')\geq 20$*. From the exact sequence*

$$0\to\mathcal{O}_{X'}(C'-C')\to\mathcal{O}_{X'}(C')\to\mathcal{O}_{C'}(C')\to 0,$$

we conclude that $h^0(\mathcal{O}_{X'}(C'))\leq 21$ *since* $\mathcal{O}_{X'}(C'-C')\cong\mathcal{O}_{X'}$ *and* $h^0(\mathcal{O}_{C'}(C'))=20$*. So we have that* $h^0(\mathcal{O}_{X'}(C'))=21$ *and* $i_{L'}(X')\subseteq\mathbb{P}^{20}$.

CLAIM 6 : *The rational map* $i_{L'|_{C'}}:C'\dashrightarrow\mathbb{P}^{19}$ *is an embedding, for any general curve* $C'\in L'$.

Proof. We have that $(C',-K_{X'}|_{C'})$ is a Prym curve of genus 21 and we can show that the Prym-canonical system $L'|_{C'}$ is base point free. Indeed, by [20], Proposition *IV*.3.1, the linear system $L'|_{C'}$ is base-point free if and only if, for every point $P\in C'$,

$$\dim\ L'|_{C'}-1=\dim\ |C'|_{C'}-P|.$$

Since $L'|_{C'}=|K_{C'}-K_{X'}|_{C'}|$, then $\dim\ L'|_{C'}=g-2=19$.
On the other hand, we have that $\dim\ |C'|_{C'}-P|=h^0(\mathcal{O}_{C'}(C'-P))-1=h^1(\mathcal{O}_{C'}(C'-P))+39+1-21-1$ by the Riemann-Roch Theorem. By Serre Duality and the adjunction formula, we know that $h^1(\mathcal{O}_{C'}(C'-P))=h^0(\mathcal{O}_{C'}(K_{C'}-C'+P)=h^0(\mathcal{O}_{C'}(P+K_{X'}))$.
By Remark 2.24, since $-K_{X'}|_{C'}$ is a non-zero two torsion divisor, if we suppose that $h^0(\mathcal{O}_{C'}(P+K_{X'}))>0$ for some point $P\in C'$, then C' is hyperelliptic.
If we show that C' is not hyperelliptic, then $h^0(\mathcal{O}_{C'}(P+K_{X'}))=0$ for any point $P\in C'$ and $L'|_{C'}$ is base-point free.

Let us suppose that C' is hyperelliptic. Since we know that $C' \in X'$ is isomorphic to $C'' \sim 4C_1$ on X'', then C'' is also hyperelliptic.
The self-intersection $C''^2 = (4C_1)^2 = 64$. Furthermore C'' is nef since $C'' \sim 4C_0 + 4\sum_{i=0}^{3} F_i$ and $(4C_0 + 4\sum_{i=0}^{3} F_i) \cdot C_0 = 0$ and $(4C_0 + 4\sum_{i=0}^{3} F_i) \cdot F_i = 4$, for $i = 0, 1, 2, 3$. Moreover, by [20], Proposition $IV.5.2$, we have that $|K_{C''}|$ is not very ample, precisely it does not separate any pair of points p and q such that $p + q$ is a member of the g_2^1 on C''. By the adjunction formula, we also have that $|K_{X''} + C''|$ does not separate such p and q.
By [27], Theorem 1., there is an effective divisor E on X'' passing through p and q, for any couple of point p and q not separated by $|K_{X''} + C''|$, such that either

$$C'' \cdot E = 0 \quad \textit{and} \quad E^2 = -1 \ \textit{or} \ -2 \tag{13}$$
$$\textit{or} \ C'' \cdot E = 1 \quad \textit{and} \quad E^2 = -1 \ \textit{or} \ 0 \tag{14}$$
$$\textit{or} \ C'' \cdot E = 2 \quad \textit{and} \quad E^2 = 0. \tag{15}$$

Since $C'' \sim 4C_1$, then we exclude the two cases $C'' \cdot E = 1$ and $C'' \cdot E = 2$, for any E. On the other hand, by Bertini's Theorem, C'' is smooth and, since $(C'')^2 > 0$ as seen before, it is also irreducible. If we suppose that $C'' \cdot E = 0$, since C'' is nef, then the intersections bewteen C'' and the irreducible components of E are also 0. Since $C''^2 > 0$, then E does not contain C''. Since p and q belong to E and C'' (they define the g_2^1 of C''), then $C'' \cdot E$ cannot be 0. Thus E cannot exist and we exclude the case C'' hyperelliptic. The claim is proved, hence $L'|_{C'}$ is base-point free.

Furthermore, we can prove that $L'|_{C'}$ defines a birational map. Indeed, if this did not happen, we would have C' bielliptic and the image of X' via the map associated with L' would be a surface in $\mathbb{P}^{20}$ with elliptic sections (see [6], Corollary 2.2). Since $20 > 9$, then the surface image in $\mathbb{P}^{20}$ could not be a Del Pezzo surface but it would be an elliptic cone. Anyway X' is a rational surface, so it cannot cover an elliptic cone. Then $L'|_{C'}$ defines a birational map.

More precisely, we can also show that $L'|_{C'}$ defines an embedding, for any general $C' \in L'$. By [6], Lemma 2.1, we know that $L'|_{C'}$ does not separate p and q (possibly infinitely near) if and only if C' has a g_4^1 and $-K_{X'}|_{C'} \sim \mathcal{O}_{C'}(p + q - x - y)$, where $2(p + q)$ and $2(x + y)$ are members of the g_4^1.

We know that $C' \cong C''$ and $C'' \sim 4C_1$ has a g_4^1 defined by the fibres of the ruled surface X''. This is the only one. Indeed, if C' had two g_4^1, then there would be a map $\psi : C' \to \mathbb{P}^1 \times \mathbb{P}^1$. If ψ was a birational map, the image curve would be of the type $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then its geometric genus would be at most $(4-1)(4-1) = 9$. Since C' has genus 21, this case is excluded. It is easy to show that ψ would be a map $2:1$ on a curve D. The image curve D would be a curve of type $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, so its geometric genus would be $g(D) \leq 1$. Since C' is non-hyperelliptic as seen before, then $g(D) = 1$ and C' is bielliptic. Then C' admits a singular correspondence. By Corollary 2.2 of [8], the map determined by the linear system $|C'|$ is not birational, in particular it is $2:1$ on a surface with elliptic sections. We have already excluded this possibility, so C'' has only one g_4^1.

It is clear that $C'' \cap F_0 \sim C'' \cap F_1 \sim C'' \cap F_2 \sim C'' \cap F_3$ and we know that C'' is tangent in two points to these four fibres. So we have four pairs of points $(p, q) \in F_0$, $(x, y) \in F_1$, $(z, w) \in F_2$ and $(a, b) \in F_3$ such that $2(p+q) \sim 2(x+y)$ and so on for all the possible cases. Since C' is the strict transform of C'', it has the same characteristics of C'' and, after the blowing up, the four pairs of points that satisfy this property are the intersection points between C' and $B_{i,j}$, for $i = 0, 1, 2, 3$ and $j = 1, 2$. Now, with abuse of notation and using the expression of $-K_{X'}$ seen before, we have that

$$-K_{X'}|_{C'} = \sum_{i=1}^{3}(B_{i,1} + B_{i,2})|_{C'} - (3B_{0,1} + 3B_{0,2})|_{C'} =$$

$$= x + y + w + z + a + b - 3p - 3q \sim x + y - w - z + a + b - p - q.$$

At this point, we observe that

$$x + y - w - z + a + b - p - q \nsim x + y - p - q$$

otherwise, if $a + b - w - z \sim 0$, then C' would have a g_2^1. Hence $L'|_{C'}$ separate each pair of points and it defines an embedding. □

At this point, since $L'|_{C'}$ is base-point free, it is clear that L' is also base-point free. Since the restriction $L'|_{C'}$ defines an embedding for each generic curve $C' \in L'$, then $i_{L'}$ is a birational map, generically $1:1$.
Then X' has hyperplane sections that are Prym-canonical embedded. In particular, $i_{L'}(X')$ is a surface with Prym-canonical hyperplane sections.

We have found a new surface $X = i_{L'}(X') \subset \mathbb{P}^{20}$ with Prym-canonical hyperplane sections. Since W' is connected, then the image $x \in X$ of W' is a rational singular point (see Proposition 1.15). There are other possible rational double singularities on X whose exceptional divisors on X' do not intersect $-2K_{X'}$.

3 More surfaces with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$

In this chapter we will study in more detail surfaces with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$. Observing that many of the previous results are still valid, we will construct new examples of this type of surfaces.

3.1 The only known example of a surface with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$

First of all we recall the notation, similar to the previous section.

- Let X be the surface with Prym-canonical hyperplane sections embedded in $\mathbb{P}^{g-1}$, for $g \geq 5$.

- Let $\pi : X' \to X$ be the minimal resolution of the singularities of X and let $X'' = \mathbb{P}^2$ be the minimal model of X'. We denote by ϕ the map $\phi : X' \to \mathbb{P}^2$.
- Let L be the $(g-1)-$dimensional complete linear system of hyperplane sections of X and let C be its general curve of genus g. If $L' = \pi^* L$ is the complete linear system of hyperplane sections of X' and C' is its general curve, then $L'' = \phi_* L'$ with general curve C'' defines a map $i_{L''} : X'' \dashrightarrow \mathbb{P}^{g-1}$ such that $i_{L''}(X'') = i_L(X)$.
- The linear system L'' on $\mathbb{P}^2$ has base points P_{t,i_t} of multiplicity r_{t,i_t}, for $t = 1, ..., s-1$ and $i = 1, ..., i(t)$, that lie on $\text{supp}(W'') \subset X''$, with possible infinitely near base points. Since ϕ blows up the base points of L'', we have that L'' and W'' do not intersect outside the base points of L''.

We know that $\text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ and we can take the class l of a line as generator (see [20], Example V.1.4.2). Moreover we know that $K_{\mathbb{P}^2} \sim -3l$ (see [20], Example V.1.4.4), so $-2K_{\mathbb{P}^2} \sim 6l$ is effective and $\dim |6l| = \binom{6+2}{2} - 1 = 27$.

Again we observe that, if C' is the strict transform of C'' on X', for C'' a general curve in L'', and W' is the antibicanonical divisor of X', then $W'|_{C'} \sim 0$ while $-K_{X'}|_{C'}$ is a non-zero two torsion divisor of $\text{Pic}(C')$.

Remark 3.1. *The case $X' = X'' = \mathbb{P}^2$ is not possible because there are not linear systems on $\mathbb{P}^2$ that contract effective curves while, if $\pi : X' \to X$ is the minimal resolution of the singularities of X and X has Prym-canonical hyperplane sections, then L' contracts at least W'.*

The only known example of surface with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$ is the following.

Example 3.2. *Let $X' := \widetilde{\mathbb{P}^2} = Bl_{x_1,...,x_{10}}\mathbb{P}^2$ be the blowing up of $X'' = \mathbb{P}^2$ at 10 nodes $\{x_1, ..., x_{10}\}$ of an irreducible rational plane curve J of degree 6 with ordinary nodes as singularities. Rational plane sextic of this kind were intensively studied by Coble (see [9], [12]).*
We recall that the Severi variety *is the Zariski closure of the following set:*

$$V_{n,g} = \overline{\{C \subset \mathbb{P}^2 \mid C \, is \, a \, nodal \, plane \, curve \, of \, degree \, n \, and \, geometric \, genus \, g\}},$$

where with nodal plane curve we mean an irreducible and reduced curve having exactly

$$\delta = p_a(C) - g = \frac{(n-1)(n-2)}{2} - g$$

nodes as singularities (see [10]). If $\delta \le p_a(C)$ (so $g \ge 0$), then there is a curve C of this type. In our case, the irreducible sextic J with $\delta = 10$ nodes in $\{x_1, ..., x_{10}\}$ exists since $p_a(J) = 10$.
If we suppose that another irreducible sextic $J' \neq J$ passing with multiplicity 2 through the nodes $\{x_1, ..., x_{10}\}$ of J exists, then either $J \subseteq J'$ or $J' \subseteq J$ since $J \cdot J' =$

$6 \cdot 6 - 2 \cdot 2 \cdot 10 < 0$. *But J and J' are irreducible by assumption, then $J = J'$ is the unique irreducible sextic with nodes in $\{x_1, ..., x_{10}\}$.*

By [20], Proposition V.3.3, we have that

$$-K_{X'} \sim 3l - \sum_{i=1}^{10} E_i \text{ and } -2K_{X'} \sim 6l - 2\sum_{i=1}^{10} E_i,$$

where l is the pullback of the class l of a line of $\mathbb{P}^2$ on X' with abuse of notation and E_i is the exceptional divisor of x_i, for $i = 1, ..., 10$.

CLAIM 1 : *The linear system $|-2K_{X'}| \neq \varnothing$ while $|-K_{X'}| = \varnothing$.*

Proof. With abuse of notation, let $J = 6l - 2\sum_{i=1}^{10} E_i$ be the strict transform of J on X'. Since J is effective and $J \in |-2K_{X'}|$, then it is clear that $|-2K_{X'}| \neq \varnothing$.
On the other hand we can show that $|-K_{X'}| = \varnothing$. Indeed, by [16], Example 9.3.3(c), the nodes of J impose independent conditions on curves of degree $k \geq 6 - 3 = 3$, in particular they impose independent conditions on cubics. Since we know that $-K_{X'} \sim 3l - \sum_{i=1}^{10} E_i$, then $|-K_{X'}| = \varnothing$ since $h^0(\mathcal{O}_{X'}(3l - \sum_{i=1}^{10} E_i)) = \binom{3+2}{2} - 10 = 0$. □

Let L'' be the linear system of $\mathbb{P}^2$ of the curves C'' of degree 10 passing through the nodes $\{x_1, ..., x_{10}\}$ of J with multiplicity 3. By Plücker's Formula, we have that

$$p_a(C'') = \frac{9 \cdot 8}{2} = 36 \text{ and } p_g(C'') = \frac{9 \cdot 8}{2} - 10\frac{3 \cdot 2}{2} = 6.$$

After the blowing up, we can write that

$$C' \sim 10l - 3\sum_{i=1}^{10} E_i,$$

where C' is a generic curve of L' and E_i is the exceptional divisor associated with x_i, for $i = 1, ..., 10$. We observe that $h^0(\mathcal{O}_{X'}(C')) \geq \binom{10+2}{2} - 10 \cdot \frac{3 \cdot 4}{2} = 66 - 60 = 6$.
With abuse of notation, we will indicate with $J = 6l - 2\sum_{i=1}^{10} E_i$ the strict transform on X' of $J \in \mathbb{P}^2$.

CLAIM 2 : *The dimension* $\dim L' = h^0(\mathcal{O}_{X'}(C')) - 1 = 5$, *so the surface image $X = i_{L'}(X')$ is contained in $\mathbb{P}^{g-1} = \mathbb{P}^5$.*

Proof. We use the following exact sequence:

$$0 \to \mathcal{O}_{X'}(4l - \sum_{i=1}^{10} E_i) \to \mathcal{O}_{X'}(10l - 3\sum_{i=1}^{10} E_i) \to \mathcal{O}_J(10l - 3\sum_{i=1}^{10} E_i) \to 0. \tag{16}$$

Since the nodes of J impose independent conditions on curves of degree $k \geq 3$ (see [16], Example 9.3.3(c)), then $h^0(\mathcal{O}_{X'}(4l - \sum_{i=1}^{10} E_i)) = \binom{4+2}{2} - 10 = 5$.
Since $J \cdot (10l - 3\sum_{i=1}^{10} E_i) = 60 - 60 = 0$, then $h^0(\mathcal{O}_J(10l - 3\sum_{i=1}^{10} E_i)) = 1$. From the exact sequence (16), we have that $h^0(\mathcal{O}_{X'}(10l - 3\sum_{i=1}^{10} E_i)) \leq 6$. Since we already know that $h^0(\mathcal{O}_{X'}(10l - 3\sum_{i=1}^{10} E_i)) \geq 6$, then the claim is proved. □

CLAIM 3 : *There are irreducible curves of degree* 10 *with exactly* 10 *triple points in the nodes of* J.

Proof. We observe that curves of the type $J+D$, with $D \in |4l - \sum_{i=1}^{10} E_i|$, are contained in $L' = |10l - 3\sum_{i=1}^{10} E_i|$.
As proved in Claim 2, we have that $\dim L' = 5$ while $\dim |4l - \sum_{i=1}^{10} E_i| = 4$, so the reducible curves $J + D$ do not define all the linear system of the curves of degree 10. As consequence of Bertini's Theorem (see [2], pag. 1), the generic curve C' of L' is irreducible (indeed the curves of the linear system L' with fixed part J define a sublinear system and moreover the sublinear system is not composed by a pencil, even more so the linear system L').
Since special curves of L' of the type $J + D$, with $D \in |4l - \sum_{i=1}^{10} E_i|$, have exactly triple points in the 10 nodes of J, then the generic curves of the linear system L' have the same property. Thus irreducible curves of degree 10 with exactly triple points in the 10 nodes of J exist. □

Step by step, we can prove that a general hyperplane section C' *of* X' *is a Prym-canonical embedded curve.*

Since $C' \cdot (-2K_{X'}) = 0$ *and* J *is irreducible and effective, then* L' *contracts* J *in a single point, so*

$$\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}.$$

Therefore X *has a rational singularity, since* J *is rational and irreducible, and of multiplicity* 4 *because the fundamental cycle* $Z_0 = J$ *and* $Z_0^2 = J^2 = -4$.

CLAIM 4 : *We have that* $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$ *and in particular* $-K_{X'}|_{C'}$ *is a non-zero two torsion divisor.*

Proof. It is sufficient to show that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$, indeed, using the long exact sequence associated with

$$0 \to \mathcal{O}_{X'}(-K_{X'} - C') \to \mathcal{O}_{X'}(-K_{X'}) \to \mathcal{O}_{C'}(-K_{X'}) \to 0, \tag{17}$$

it is easy to see that $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$ since $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$.
First of all we have that $-K_{X'} - C' \sim -7l + 2\sum_{i=1}^{10} E_i$, so it is not effective and $h^0(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$. Moreover, by Serre Duality, we have that $h^2(\mathcal{O}_{X'}(-K_{X'} - C')) = h^0(\mathcal{O}_{X'}(2K_{X'} + C')) = h^0(\mathcal{O}_{X'}(4l - \sum_{i=1}^{10} E_i)) = \binom{4+2}{2} - 10 = 5$ (see [16], Example 9.3.3(c): the nodes of J impose independent conditions on curves of degree $k \geq 3$). Thus, using the Riemann-Roch Theorem, we have that $-h^1(\mathcal{O}_{X'}(-K_{X'} - C')) + 5 = \frac{1}{2}(-7l + 2\sum_{i=1}^{10} E_i)(-4l + \sum_{i=1}^{10} E_i) + 1 - 0 = 5$, so $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$.
In conclusion, since $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$, then $-K_{X'}|_{C'}$ is a non-zero two torsion divisor. □

CLAIM 5 : *The linear system* L' *is base-point free.*

Proof. Let $\overline{X'} = Bl_{x_{11}}(X')$, where x_{11} is a point of a general $C' \in L'$. If $\overline{L'} = |10l - 3\sum_{i=1}^{10} E_i - E_{11}|$, where E_{11} is the exceptional divisor associated with x_{11}, then L' is base point free if and only if

$$\dim(\overline{L'}) = \dim(L') - 1,$$

for any point $x_{11} \in C'$, for a general $C' \in L'$.

We have already proved that $\dim(L') = 5$ while $\dim(\overline{L'}) \geq \binom{12}{2} - 10\frac{3 \cdot 4}{2} - 1 - 1 = 4$. We observe that, since $x_{11} \in C'$ and C' and J are disjoint by assumption, then $\mathcal{O}_{\overline{J}}(\overline{C'}) \cong \mathcal{O}_{\mathbb{P}^1}$, where $\overline{C'}$ is a general curve in $\overline{L'}$ and $\overline{J}$ is the strict transform of J on $\overline{X'}$. Similarly to the exact sequence (16), we have the following:

$$0 \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{10} E_i - E_{11}) \to \mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11}) \to \mathcal{O}_{\mathbb{P}^1} \to 0; \qquad (18)$$

$$0 \to \mathcal{O}_{\overline{X'}}(3l - \sum_{i=1}^{9} E_i) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{10} E_i - E_{11}) \to \mathcal{O}_{l-E_{10}-E_{11}}(4l - \sum_{i=1}^{10} E_i - E_{11}) \to 0 \qquad (19)$$

We know that $h^0(\mathcal{O}_{\overline{X'}}(3l - \sum_{i=1}^{9} E_i)) = 1$ while $h^1(\mathcal{O}_{\overline{X'}}(3l - \sum_{i=1}^{9} E_i)) = 0$. Since $(l - E_{10} - E_{11}) \cdot (4l - \sum_{i=1}^{10} E_i - E_{11}) = 2$, then $h^0(\mathcal{O}_{l-E_{10}-E_{11}}(4l - \sum_{i=1}^{10} E_i - E_{11})) = 3$. From the exact sequence (19), we conclude that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{10} E_i - E_{11})) = 4$. Furthermore, from the exact sequence (18), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11})) \leq 4 + 1$. Since we know that $h^0(\mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11})) \geq 5$ as observed before, then the equality holds.
We conclude that L' is base-point free since the above proof does not depend on the choice of the point $x_{11} \in C'$. □

We have showed that L' is base-point free, so the generic curve $C' \in L'$ is smooth by Bertini's Theorem.

CLAIM 6 : *The linear system L' defines an embedding outside the contracted curve J.*

Proof. Let $x_{11}, x_{12} \in X' - J$.
If $x_{11} \neq x_{12}$, let $\overline{L'} = |10l - 3\sum_{i=1}^{10} E_i - E_{11} - E_{12}|$ be a linear system in $\overline{X'} = Bl_{x_{11},x_{12}}X'$, where E_{11} and E_{12} are the exceptional divisors associated with two points x_{11} and x_{12}. If x_{11} and x_{12} are infinitely near, we consider $\overline{L'} = |10l - 3\sum_{i=1}^{10} E_i - E_{11} - 2E_{12}|$ in $\overline{X'} = Bl_{x_{12}}(Bl_{x_{11}}(X'))$.
We have that L' defines an embedding outside the contracted curve J if and only if

$$\dim \overline{L'} = \dim(L') - 2,$$

for any x_{11} and x_{12} distinct or infinitely near not belonging to J.

Anyway we have already proved that $\dim(L') = 5$ while $\dim(\overline{L'}) \geq \binom{12}{2} - 10\frac{3 \cdot 4}{2} - 2 - 1 = 3$.

At this point we observe three things:

- if $\overline{C'}$ is a general curve in $\overline{L'}$ and $\overline{J}$ is the strict transform of J on $\overline{X'}$, then $\overline{C'} \cdot \overline{J} = 0$ since C' and J are disjoint and we choose x_{11} and x_{12} not belonging to J, for x_{11} and x_{12} distinct or infinitely near;

- the 10 nodes $\{x_1, ..., x_{10}\}$ of J are not all belonging to a line, so there exists one of them not aligned with x_{11} and x_{12}. We suppose that, up to renaming the terms, $\{x_{10}, x_{11}, x_{12}\}$ are not aligned. Similarly, if x_{11} and x_{12} are infinitely near, we suppose that the only one line passing through x_{11} and x_{12} does not contain x_{10}.

- let us consider a generic Halphen pencil of sextic with 9 base points of multiplicity 2. The base points always lie on a unique cubic curve. If we consider a generic Halphen pencil, then the cubic is smooth.

  As showed in [9], pag. 12, in a pencil of sextic with nodes in 9 points there are exactly twelve sextics with 10 nodes and clearly they admit a cubic through 9 of them.

  Hence we assume that the sextic J with 10 nodes $\{x_1, ..., x_{10}\}$ belongs to a generic Halphen pencil, so that a smooth cubic passing through $\{x_1, ..., x_9\}$ exists.

◇ We prove the claim for $x_{11} \neq x_{12}$. We can consider the following exact sequence:

$$0 \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{10} E_i - E_{11} - E_{12}) \to \mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11} - E_{12}) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \tag{20}$$

1. Let us suppose that $x_{11}, x_{12} \notin 3l - \sum_{i=1}^{9} E_i$. Then we consider the exact sequence:

$$0 \to \mathcal{O}_{\overline{X'}}(l - E_{10} - E_{11} - E_{12}) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i) \to$$
$$\to \mathcal{O}_{3l-\sum_{i=1}^{9} E_i}(4l - \sum_{i=1}^{12} E_i) \to 0. \tag{21}$$

   It is clear that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i)) \geq \binom{6}{2} - 12 = 3$.
   On the other hand, since we choose E_{10}, E_{11} and E_{12} not aligned, then we have that $h^0(\mathcal{O}_{\overline{X'}}(l - E_{10} - E_{11} - E_{12})) = 0$. From the exact sequence (21), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i)) \leq h^0(\mathcal{O}_{3l-\sum_{i=1}^{9} E_i}(4l - \sum_{i=1}^{12} E_i)) = 3 + 1 - 1 = 3$. Then the equality holds.

2. If only x_{11} (or equivalently x_{12}) belongs to $3l - \sum_{i=1}^{9} E_i$, then we consider

$$0 \to \mathcal{O}_{\overline{X'}}(l - E_{10} - E_{12}) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i) \to \mathcal{O}_{3l-\sum_{i=1}^{9} E_i - E_{11}}(4l - \sum_{i=1}^{12} E_i) \to 0. \tag{22}$$

We know that $h^0(\mathcal{O}_{\overline{X'}}(l - E_{10} - E_{12})) = 1$ and $h^1(\mathcal{O}_{\overline{X'}}(l - E_{10} - E_{12})) = 0$. Using Riemann-Roch's Theorem, we obtain that $h^0(\mathcal{O}_{3l-\sum_{i=1}^{9} E_i - E_{11}}(4l - \sum_{i=1}^{12} E_i)) = 2$. Again, from the exact sequence (22), we have that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i)) = 3$.

3. If both x_{11} and x_{12} belong to $3l - \sum_{i=1}^{9} E_i$, then we consider

$$0 \to \mathcal{O}_{\overline{X'}}(l - E_{10}) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i) \to \mathcal{O}_{3l-\sum_{i=1}^{9} E_i - E_{11} - E_{12}}(4l - \sum_{i=1}^{12} E_i) \to 0 \tag{23}$$

We know that $h^0(\mathcal{O}_{\overline{X'}}(l - E_{10})) = 2$ while $h^1(\mathcal{O}_{\overline{X'}}(l - E_{10})) = 0$. Using Riemann-Roch's Theorem, we have that $h^0(\mathcal{O}_{3l-\sum_{i=1}^{9} E_i - E_{11} - E_{12}}(4l - \sum_{i=1}^{12} E_i)) = 1$. Then $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i)) = 3$ from the exact sequence (23).

In all three cases we have $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{12} E_i)) = 3$.

Moreover, since $h^0(\mathcal{O}_{\mathbb{P}^1}) = 1$, we conclude from the exact sequence (20) that $h^0(\mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11} - E_{12})) \leq 4$ but we know that $h^0(\mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11} - E_{12})) \geq 4$, so equality holds and the claim is proved for x_{11} and x_{12} distinct.

◇ At this point, let x_{11} and x_{12} be infinitely near. So we can consider the following exact sequence:

$$0 \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{10} E_i - E_{11} - 2E_{12}) \to \mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11} - 2E_{12}) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \tag{24}$$

1. Let us suppose $x_{11} \notin 3l - \sum_{i=1}^{9} E_i$ and consequently also x_{12}. Then we consider the exact sequence:

$$0 \to \mathcal{O}_{\overline{X'}}(l - E_{10} - E_{11} - 2E_{12}) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12}) \to$$
$$\to \mathcal{O}_{3l-\sum_{i=1}^{9} E_i}(4l - \sum_{i=1}^{11} E_i - 2E_{12}) \to 0. \tag{25}$$

It is clear that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) \geq \binom{6}{2} - 12 = 3$.
On the other hand, by the previous assumption, we have that $h^0(\mathcal{O}_{\overline{X'}}(l - E_{10} - E_{11} - 2E_{12})) = 0$. From the exact sequence (25), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) \leq h^0(\mathcal{O}_{3l-\sum_{i=1}^{9} E_i}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) = 3 + 1 - 1 = 3$. Then the equality holds.

2. If $x_{11} \in 3l - \sum_{i=1}^{9} E_i$ but $x_{12} \notin 3l - \sum_{i=1}^{9} E_i$, then we consider

$$0 \to \mathcal{O}_{\overline{X'}}(l - E_{10} - 2E_{12}) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12}) \to$$
$$\mathcal{O}_{3l - \sum_{i=1}^{9} E_i - E_{11}}(4l - \sum_{i=1}^{11} E_i - 2E_{12}) \to 0. \quad (26)$$

If a line passes through x_{12}, then it also contains x_{11}. Then, as in the previous point, we have that $h^0(\mathcal{O}_{\overline{X'}}(l - E_{10} - 2E_{12})) = 0$. Using the Riemann-Roch Theorem, we also have that $h^0(\mathcal{O}_{3l-\sum_{i=1}^{9} E_i - E_{11}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) = (12 - 9 - 2 + 2) + 1 - 1 = 3$. From the exact sequence (26), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) \leq 3$ but we know that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) \geq 3$, so equality holds.

3. If both x_{11} and x_{12} belong to $3l - \sum_{i=1}^{9} E_i$, then we consider

$$0 \to \mathcal{O}_{\overline{X'}}(l - E_{10}) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12}) \to$$
$$\to \mathcal{O}_{3l - \sum_{i=1}^{9} E_i - E_{11} - 2E_{12}}(4l - \sum_{i=1}^{11} E_i - 2E_{12}) \to 0. \quad (27)$$

We know that $h^0(\mathcal{O}_{\overline{X'}}(l - E_{10})) = 2$ while $h^1(\mathcal{O}_{\overline{X'}}(l - E_{10})) = 0$. Using the Riemann-Roch Theorem, we have that $h^0(\mathcal{O}_{3l-\sum_{i=1}^{9} E_i - E_{11} - 2E_{12}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) = (12-9-2+2-4+2)+1-1 = 1$. From the exact sequence (27), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) = 3$.

In all three cases we have $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=1}^{11} E_i - 2E_{12})) = 3$.

We conclude from the exact sequence (24) that $h^0(\mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11} - 2E_{12})) \leq 4$ since $h^0(\mathcal{O}_{\mathbb{P}^1}) = 1$ but we know that $h^0(\mathcal{O}_{\overline{X'}}(10l - 3\sum_{i=1}^{10} E_i - E_{11} - 2E_{12})) \geq 4$, so equality holds and the claim is proved.

□

In conclusion, $X \subset \mathbb{P}^5$ is a surface with Prym-canonical hyperplane sections of degree $\deg(X) = \frac{C'^2}{\deg i_{L'}} = 10$ *birationally equivalent to $\mathbb{P}^2$ with only one singularity, that is a quadruple rational singularity.*

3.2 Construction of other rational surfaces with Prym-canonical hyperplane sections

In this section we will construct surfaces with Prym-canonical hyperplane sections whose minimal model is $X'' = \mathbb{P}^2$ using techniques similar to Example 3.2.

Remark 3.3. *With the same notation as before, assuming that* $X' = Bl_{x_1,...,x_r}(\mathbb{P}^2)$, *where* $\{x_1, ..., x_r\} \in \mathbb{P}^2$ *are the base points of* L'', *we have that*

$$-K_{X'} \sim 3l - \sum_{i=1}^{r} E_i \quad and \quad -2K_{X'} \sim 6l - 2\sum_{i=1}^{r} E_i,$$

*by [20], Proposition V.*3.3*,*
Moreover, if we suppose that X' *is a surface whose general hyperplane section is Prym-canonical embedded, then* $C' \cdot (-K_{X'}) = 0$*, with* $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$ *and* $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$*, for* C' *a general curve in* L'*. So we need to have* $-K_{X'}$ *not effective otherwise* $\mathcal{O}_{C'}(-K_{X'}) \cong \mathcal{O}_{C'}$*.*

A first step to construct surfaces with Prym-canonical hyperplane sections is to assume that $-2K_{X'}$ *is effective while* $-K_{X'}$ *is not, i.e.* $h^0(\mathcal{O}_{X'}(-2K_{X'})) > 0$ *while* $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$*. If we suppose that* $-2K_{X'}$ *is irreducible, then this happens if and only if* $r = 10$ *(see Claim 1 of Example 3.2). Indeed, if* $1 \leq r < 10$*, then* $h^0(\mathcal{O}_{X'}(-K_{X'})) \geq \binom{3+2}{2} - r = 10 - r > 0$ *and therefore* $-K_{X'}$ *would be effective. On the contrary, if* $r > 10$*, then* $-2K_{X'}$ *would be reducible since an effective irreducible nodal curve exists if and only if the number of nodes r is at most equal to the arithmetic genus of the curve (if* $r > 10$*, then* $p_a(-2K_{X'}) = \frac{(6-1)(6-2)}{2} = 10$ *and* $p_g(-2K_{X''}) < 0$*). Instead, as seen in Example 3.2, if* $r = 10$ *then the desired property is valid.*

We can see a general result regarding surfaces with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$.

Lemma 3.4. *Let* X *be a surface with Prym-canonical hyperplane sections whose minimal model is* $X'' = \mathbb{P}^2$ *and let* X' *be the minimal resolution of the singularities of* X*. If we also assume that* $-2K_{X'}$ *is irreducible and effective,* $-K_{X'}$ *is not effective and* C' *is a general divisor of* L'*, then* $C' \sim bl - \sum_{i=1}^{10} r_i E_i$ *is such that* $3b = \sum_{i=1}^{10} r_i$*, where* $r_i > 0$ *for any* i*.*

Proof. If $C'' \in X''$ is a generic divisor in L'', then $C'' \sim bl$ has base points $\{x_1, ..., x_k\}$ on X'', for $b \geq 1$ and $k \geq 1$ (see Remark 3.1). Let r_i, for $i = 1, ..., k$, be the multiplicity of x_i on C'', so it is clear that $r_i > 0$ for any i. The assumptions imply $k = 10$ (see Remark 3.3).
The map $\phi : X' \to X''$ is the blowing up of X'' along $\{x_1, ..., x_{10}\}$. So $-2K_{X'} = 6l - 2\sum_{i=1}^{10} E_i$ (as seen in Example 3.2, it is the only effective and irreducible divisor in its class of linear equivalence). On the other hand we have that the strict transform $C' \in X'$ of C'' is $C' \sim bl - \sum_{i=1}^{10} r_i E_i$.

Since we have that $C' \cdot (-2K_{X'}) = 0$ by hypothesis, then

$$0 = C' \cdot (-2K_{X'}) = (bl - \sum_{i=1}^{10} r_i E_i) \cdot (6l - 2\sum_{i=1}^{10} E_i) = 2(3b - \sum_{i=1}^{10} r_i),$$

so the claim is satisfied. □

Let us see a new example of surface with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$.

Example 3.5. *Let $X'' = \mathbb{P}^2$ be such that $-2K_{X''}$ is an irreducible sextic with 10 nodes $\{x_1, ..., x_{10}\}$. Let L'' be a linear system of curves of degree 18 with base points $\{x_1, ..., x_{10}\} \in X''$ of multiplicity respectively $r_i = 4$, for $i = 1, 2, 3$, and $r_i = 6$, for $i = 4, ..., 10$. Let $X' = Bl_{\{x_1,...,x_{10}\}}(\mathbb{P}^2)$ be the blowing up of X'' along the base points of L''. As observed before, the anticanonical divisor $-K_{X'}$ is not effective.*
Let

$$C' \sim 18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i$$

be a general curve in L', where E_i is the exceptional divisor associated with x_i, for $i = 1, ..., 10$. It is obvious that

$$\deg(C'|_{C'}) = 18^2 - 3 \cdot 16 - 7 \cdot 36 = 24.$$

We have that $h^0(\mathcal{O}_{X'}(C')) \geq \binom{20}{2} - 4\frac{4\cdot 5}{2} - 7\frac{6\cdot 7}{2} = 13$, so $i_{L'}(X') = X \subset \mathbb{P}^r$, for $r \geq 12$.

Since $-2K_{X'} \sim J = 6l - 2\sum_{i=1}^{10} E_i$ is effective and $C' \cdot (-2K_{X'}) = 0$ by construction, then $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$. So L' contracts J in a single point since J is irreducible. Moreover, since J is also rational, then $i_{L'}(J)$ is a rational singularity of multiplicity 4 because the fundamental cycle $Z_0 = J$ is such that $Z_0^2 = J^2 = -4$.

CLAIM 1 : *The dimension $\dim(L') = 12$, so $i_{L'}(X') = X \subseteq \mathbb{P}^{12}$.*

Proof. If $J = 6l - 2\sum_{i=1}^{10} E_i \in X'$, we can consider the exact sequence

$$0 \to \mathcal{O}_{X'}(-J) \to \mathcal{O}_{X'} \to \mathcal{O}_J \to 0.$$

Tensoring with $\mathcal{O}_{X'}(C')$, we obtain

$$0 \to \mathcal{O}_{X'}(C' - J) \to \mathcal{O}_{X'}(C') \to \mathcal{O}_J(C') \to 0. \tag{28}$$

Since $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$, $J \in |-2K_{X'}|$ and J is rational, then $\mathcal{O}_J(C') \cong \mathcal{O}_J \cong \mathcal{O}_{\mathbb{P}^1}$. We can rewrite (28) as

$$0 \to \mathcal{O}_{X'}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i) \to \mathcal{O}_{X'}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \tag{29}$$

Similarly we obtain that

$$0 \to \mathcal{O}_{X'}(C' - 2J) \to \mathcal{O}_{X'}(C' - J) \to \mathcal{O}_J(C' - J) \to 0,$$

so

$$0 \to \mathcal{O}_{X'}(6l - 2\sum_{i=4}^{10} E_i) \to \mathcal{O}_{X'}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i) \to$$

$$\to \mathcal{O}_J(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i) \to 0. \tag{30}$$

It is possible to choose a quintuple of points among the 10 nodes $\{x_1, ..., x_{10}\}$ of J such that three of these points are not aligned, then an irreducible conic passing through this quintuple of points exists. Up to renaming the nodes of J, we suppose that a conic passing through $\{x_4, ..., x_8\}$ exists.
So let us consider the following exact sequences:

$$0 \to \mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - 2E_9 - 2E_{10}) \to \mathcal{O}_{X'}(6l - 2\sum_{i=4}^{10} E_i) \to \mathcal{O}_{2l - \sum_{i=4}^{8} E_i}(6l - 2\sum_{i=4}^{8} E_i) \to 0; \tag{31}$$

$$0 \to \mathcal{O}_{X'}(3l - \sum_{i=4}^{10} E_i) \to \mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i) \to$$

$$\to \mathcal{O}_{l - E_9 - E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i) \to 0. \tag{32}$$

We have that $h^0(\mathcal{O}_{X'}(3l - \sum_{i=4}^{10} E_i)) = \binom{5}{2} - 7 = 3$ by [16], Example 9.3.3(c). Since it is an effective divisor on X' and it has the expected dimension, then $h^1(\mathcal{O}_{X'}(3l - \sum_{i=4}^{10} E_i)) = 0$.
Since $l - E_9 - E_{10}$ is rational and $(l - E_9 - E_{10}) \cdot (4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i) = 0$, then $h^0(\mathcal{O}_{l-E_9-E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i)) = 1$ and $h^1(\mathcal{O}_{l-E_9-E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i)) = 0$.
From the exact sequence (32), we conclude that $h^0(\mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i)) = 3 + 1 = 4$ and $h^1(\mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i)) = 0$.

We observe that, since J and $2l - \sum_{i=4}^{8} E_i$ are rational, then, by Serre Duality, we have that

$$h^1(\mathcal{O}_J(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i)) = h^0(\mathcal{O}_J(-2-4)) = 0$$

and

$$h^1(\mathcal{O}_{2l - \sum_{i=4}^{8} E_i}(6l - 2\sum_{i=4}^{10} E_i)) = h^0(\mathcal{O}_{2l - \sum_{i=4}^{8}}(-2-2)) = 0.$$

Using the Riemann-Roch Theorem, we obtain that

$$h^0(\mathcal{O}_J(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i)) = 4 + 1 = 5$$

and

$$h^0(\mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(6l - 2\sum_{i=4}^{10} E_i)) = 2 + 1 = 3.$$

From the exact sequence (31), we conclude that $h^0(\mathcal{O}_{X'}(6l - 2\sum_{i=4}^{10} E_i)) = 7$ and $h^1(\mathcal{O}_{X'}(6l - 2\sum_{i=4}^{10} E_i)) = 0$. Again, from the exact sequence (30), we have that $h^0(\mathcal{O}_{X'}(12l-2\sum_{i=1}^{3} E_i-4\sum_{i=4}^{10} E_i)) = 12$ and $h^1(\mathcal{O}_{X'}(12l-2\sum_{i=1}^{3} E_i-4\sum_{i=4}^{10} E_i)) = 0$. Finally, from the exact sequence (29), we obtain that

$$h^0(\mathcal{O}_{X'}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i)) = 12 + 1 = 13.$$

Then the claim is proved. □

Step by step we want to show that a general hyperplane section of X' is a Prym-canonically embedded curve.

CLAIM 2 : *We prove that $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$.*

Proof. If we show that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$, then, using the long exact sequence associated with

$$0 \to \mathcal{O}_{X'}(-K_{X'} - C') \to \mathcal{O}_{X'}(-K_{X'}) \to \mathcal{O}_{C'}(-K_{X'}) \to 0$$

and observing that $-K_{X'}$ is not effective, we have that $h^0(\mathcal{O}_{C'}(-K_{X'})) = 0$, thus $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$.
Since

$$-K_{X'} - C' \sim -15l + 3\sum_{i=1}^{3} E_i + 5\sum_{i=4}^{10} E_i,$$

then it is not effective and $h^0(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$. By Serre Duality, we have that $h^2(\mathcal{O}_{X'}(-K_{X'} - C')) = h^0(\mathcal{O}_{X'}(2K_{X'} + C')) = h^0(\mathcal{O}_{X'}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i)) = 12$ as proved in Claim 1.
Using the Riemann-Roch Theorem, we conclude that $-h^1(\mathcal{O}_{X'}(-K_{X'} - C')) + {}+h^2(\mathcal{O}_{X'}(-K_{X'}-C')) = -h^1(\mathcal{O}_{X'}(-K_{X'}-C'))+12 = \frac{1}{2}(-15l+3\sum_{i=1}^{3} E_i+5\sum_{i=4}^{10} E_i)\cdot {}\cdot(-12l + 2\sum_{i=1}^{3} E_i + 4\sum_{i=4}^{10} E_i) + 1 - 0 = 12$, so $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$ and the claim is proved. □

CLAIM 3 : *There are irreducible curves of degree* 18 *with exactly* 3 *quadruple points and* 7 *points of multiplicity six in the ten nodes of J.*

Proof. We observe that curves of the type $J+D$, with $D \in |12l-2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i|$ and J fixed part, are contained in $L' = |18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i|$.
As proved in Claim 1, we have that $\dim|18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i| = 12$ while $\dim|12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i| = 11$, so the reducible curves $J + D$ do not define all the linear system of the curves of degree 18. As consequence of Bertini's Theorem (see [2], pag. 1), the generic curve of L' is irreducible (indeed the curves of the linear system L' with fixed part J define a sublinear system and moreover the sublinear system is not composed by a pencil, even more so the linear system L').
Also curves of the type $2J + F$, with $F \in |6l - 2\sum_{i=4}^{10} E_i|$ and $2J$ fixed part, are contained in L'. Since these special curves of L' have exactly quadruple points in three of the 10 nodes of J and points of multiplicity 6 in seven of the 10 nodes of J, then the generic curves of the linear system L' have the same property. Thus irreducible curves of degree 18 with exactly quadruple points in three of the 10 nodes of J and points of multiplicity 6 in the remaining nodes of J exist. □

Therefore the arithmetic genus, that is equal to the geometric genus of C, is

$$g(C') = \frac{17 \cdot 16}{2} - 3\frac{4 \cdot 3}{2} - 7\frac{6 \cdot 5}{2} = 13$$

by the Plücker Formula.

It remains to show that L' defines an embedding outside the contracted curve J, in particular L defines an embedding.

CLAIM 4 : *The linear system L' is base-point free.*

Proof. Let $\overline{X'} = Bl_{x_{11}}(X')$, where x_{11} is a point of a general $C' \in L'$. If $\overline{L'} = |18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11}|$, for E_{11} the exceptional divisor associated with x_{11}, then L' is base point free if and only if

$$\dim(\overline{L'}) = \dim(L') - 1,$$

for any point $x_{11} \in C'$, for a general $C' \in L'$.

We have already proved that $\dim(L') = 12$ while $\dim(\overline{L'}) \geq \binom{20}{2} - 3\frac{4 \cdot 5}{2} - 7\frac{6 \cdot 7}{2} - 1 - 1 = 11$. We observe that, since $x_{11} \in C'$ and C' and J are disjoint by assumption, then $\mathcal{O}_{\overline{J}}(\overline{C'}) \cong \mathcal{O}_{\mathbb{P}^1}$, where $\overline{C'}$ is a general curve in $\overline{L'}$ and $\overline{J}$ is the strict transform of J on $\overline{X'}$.
Similarly to the exact sequences (29), (30), we have the following:

$$0 \to \mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11}) \to \mathcal{O}_{\overline{X'}}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11}) \to \mathcal{O}_{\mathbb{P}^1} \to 0 \tag{33}$$

$$0 \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to \mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11}) \to$$

$$\to \mathcal{O}_{\overline{J}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11}) \to 0. \tag{34}$$

As in Claim 1, we suppose that an irreducible conic passing through $\{x_4, ..., x_8\}$ exists.

1. If $x_{11} \in 2l - \sum_{i=4}^{8} E_i$, we can consider the exact sequence

$$0 \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i) \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to$$

$$\to \mathcal{O}_{2l-\sum_{i=4}^{8} E_i - E_{11}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to 0. \tag{35}$$

From the exact sequence (32), we know that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i)) = 4$ and $h^1(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i)) = 0$. Since $h^0(\mathcal{O}_{2l-\sum_{i=4}^{8} E_i - E_{11}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 2$ by Riemann-Roch's Theorem, then $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 6$ and $h^1(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 0$ from the exact sequence (35).

2. If $x_{11} \notin 2l - \sum_{i=4}^{8} E_i$, then we consider the following

$$0 \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2E_9 - 2E_{10} - E_{11}) \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to$$

$$\to \mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to 0. \tag{36}$$

   (a) If $x_{11} \in l - E_9 - E_{10}$, we have that

$$0 \to \mathcal{O}_{\overline{X'}}(3l - \sum_{i=4}^{10} E_i) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to$$

$$\to \mathcal{O}_{l-E_9-E_{10}-E_{11}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to 0. \tag{37}$$

   By [16], Example 9.3.3(c), we have that $h^0(\mathcal{O}_{\overline{X'}}(3l - \sum_{i=4}^{10} E_i)) = 3$. On the other hand, since the product $(l - E_9 - E_{10} - E_{11}) \cdot (4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) = -1$, then $h^0(\mathcal{O}_{l-E_9-E_{10}-E_{11}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 0$. From the exact sequence (37), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 3$.

(b) If $x_{11} \notin l - E_9 - E_{10}$, we can consider the following exact sequences

$$0 \to \mathcal{O}_{\overline{X'}}(3l - \sum_{i=4}^{11} E_i) \to \mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to$$
$$\to \mathcal{O}_{l-E_9-E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to 0; \qquad (38)$$

$$0 \to \mathcal{O}_{\overline{X'}}(l - \sum_{i=9}^{11} E_i) \to \mathcal{O}_{\overline{X'}}(3l - \sum_{i=4}^{11} E_i) \to \mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(3l - \sum_{i=4}^{11} E_i) \to 0. \qquad (39)$$

By assumption we have that $h^0(\mathcal{O}_{\overline{X'}}(l - \sum_{i=9}^{11} E_i)) = 0$. Because $h^0(\mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(3l - \sum_{i=4}^{11} E_i)) = 2$, then, from the exact sequence (39) we conclude that $h^0(\mathcal{O}_{\overline{X'}}(3l - \sum_{i=4}^{11} E_i)) \le 2$. Since $h^0(\mathcal{O}_{\overline{X'}}(3l - \sum_{i=4}^{11} E_i)) \ge \binom{5}{2} - 8 = 2$, then equality holds.

Moreover $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) \ge \binom{6}{2} - 6 - 3 - 3 = 3$. Since $h^0(\mathcal{O}_{l-E_9-E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 1$, then, from the exact sequence (38), we have that $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) \le 3$, so equality holds.

In both previous cases (a) and (b), we have found $h^0(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 3$. Since it is the expected dimension, then $h^1(\mathcal{O}_{\overline{X'}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 0$.

Using the Riemann-Roch Theorem, we have that $h^0(\mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 3$. So we obtain that $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 6$ and $h^1(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 0$ from the exact sequence (36).

In both cases we have that $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 6$.

With the same techniques as before, from the exact sequence (34) we obtain that $h^0(\mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11})) = 11$ and $h^1(\mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11})) = 0$ and finally, from the exact sequence (33), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11})) = 12$. Then we can conclude that L' is base-point free. □

By Bertini's Theorem, since L' is base-point free, then the generic $C' \in L'$ is smooth.

CLAIM 5 : *The linear system L' defines an embedding outside the contracted curve J.*

Proof. It is sufficient to show that $\dim \overline{L'} = \dim(L') - 2$, where either $\overline{L'} = |18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11} - E_{12}|$, for E_{11} and E_{12} the exceptional divisors associated with any two distinct points x_{11} and x_{12} not belonging to J, or $\overline{L'} = |18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}|$, for E_{11} and E_{12} the exceptional divisors associated with any two points x_{11} and x_{12} infinitely near not belonging to J.

We have already proved that $\dim(L') = 12$. Moreover we have that $\dim(\overline{L'}) \geq \binom{20}{2} - 3\frac{4\cdot5}{2} - 7\frac{6\cdot7}{2} - 2 - 1 = 10$. If $\overline{C'}$ is a general curve in $\overline{L'}$ and $\overline{J}$ is the strict transform of J on $\overline{X'}$, then $\overline{C'} \cdot \overline{J} = 0$ since we choose x_{11} and x_{12} not belonging to J.

- ⋄ Let x_{11} and x_{12} be distinct. We can consider the following exact sequences:

$$0 \to \mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to$$

$$\to \mathcal{O}_{\overline{X'}}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \mathcal{O}_{\mathbb{P}^1} \to 0; \tag{40}$$

$$0 \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to$$

$$\to \mathcal{O}_{\overline{J}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to 0 \tag{41}$$

$$0 \to \mathcal{O}_{\overline{X'}}(2\sum_{i=1}^{3} E_i - E_{11} - E_{12}) \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to$$

$$\to \mathcal{O}_{\overline{J}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to 0. \tag{42}$$

It is clear that $h^0(\mathcal{O}_{\overline{X'}}(2\sum_{i=1}^{3} E_i - E_{11} - E_{12})) = 0$. Again, by Serre Duality, we have that $h^2(\mathcal{O}_{\overline{X'}}(2\sum_{i=1}^{3} E_i - E_{11} - E_{12})) = h^0(\mathcal{O}_{\overline{X'}}(-3l + \sum_{i=1}^{12} E_i - 2\sum_{i=1}^{3} E_i + E_{11} + E_{12})) = 0$. Using the Riemann-Roch Theorem, we obtain that $-h^1(2\sum_{i=1}^{3} E_i - E_{11} - E_{12})) = \frac{1}{2}(2\sum_{i=1}^{3} E_i - E_{11} - E_{12}) \cdot (3l + \sum_{i=1}^{3} E_i - \sum_{i=4}^{10} E_i - 2E_{11} - 2E_{12}) + 1 = -4$.

Since $\overline{J} \cdot (6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}) = 8$, then $h^0(\mathcal{O}_{\overline{J}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 9$. Moreover $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12})) \geq \binom{8}{2} - 7\cdot3 - 2 = 5$. To show that equality holds, it is sufficient to prove that $h^1(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0$.

We observe that curves of the type $(3l - \sum_{i=4}^{12} E_i) + F$, with $F \in |3l - \sum_{i=4}^{10} E_i|$ and $3l - \sum_{i=4}^{12} E_i$ fixed part, are contained in $|6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}|$.

We have that $\dim |6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}| \geq 5$ while $\dim |3l - \sum_{i=4}^{10} E_i| = \binom{5}{2} - 7 = \frac{5\cdot4}{2} - 7 = 3$, so the reducible curves of the type $(3l - \sum_{i=4}^{12} E_i) + F$ do not define all the linear system of the curves of degree 6. As consequence of Bertini's

Theorem (see [2], pag. 1), the generic curve D of $|6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}|$ is irreducible (indeed the curves of the linear system $|6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}|$ with fixed part $3l - \sum_{i=4}^{12} E_i$ define a sublinear system and moreover the sublinear system is not composed by a pencil, even more so the linear system $|6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12}|$).

Let us consider the exact sequence

$$0 \to \mathcal{O}_{\overline{X'}} \to \mathcal{O}_{\overline{X'}}(D) \to \mathcal{O}_D(D) \to 0. \tag{43}$$

Since $D^2 = 36-28-2 = 6$ and $p_a(D) = \frac{5\cdot 4}{2} - 7 = 3$, then $h^1(\mathcal{O}_D(D)) = 0$ (see [20], Example $IV.1.3.4$). Since $h^1(\mathcal{O}_{\overline{X'}}) = 0$ by definition, then $h^1(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0$ from the exact sequence (43). Consequently $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 5$ from the exact sequence (42).

Since $\overline{J}$ is rational, then $h^0(\mathcal{O}_{\overline{J}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 5$ and $h^1(\mathcal{O}_{\overline{J}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0$, so, from the exact sequence (41), we have that $h^0(\mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 10$ and $h^1(\mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0$.

Finally, from the exact sequence (40), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 11$. The claim is proved for x_{11} and x_{12} distinct.

◇ Let x_{11} and x_{12} be infinitely near. We can consider the following exact sequences:

$$0 \to \mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to$$

$$\to \mathcal{O}_{\overline{X'}}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to \mathcal{O}_{\mathbb{P}^1} \to 0; \tag{44}$$

$$0 \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to \mathcal{O}_{\overline{X'}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to$$

$$\to \mathcal{O}_{\overline{J}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to 0 \tag{45}$$

$$0 \to \mathcal{O}_{\overline{X'}}(2\sum_{i=1}^{3} E_i - E_{11} - 2E_{12}) \to \mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to$$

$$\to \mathcal{O}_{\overline{J}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) \to 0. \tag{46}$$

Exactly as the previous case, we can prove that $h^0(\mathcal{O}_{\overline{X'}}(2\sum_{i=1}^{3} E_i - E_{11} - 2E_{12})) = h^2(\mathcal{O}_{\overline{X'}}(2\sum_{i=1}^{3} E_i - E_{11} - 2E_{12})) = 0$ and $h^1(2\sum_{i=1}^{3} E_i - E_{11} - 2E_{12})) = 4$.

Since $\overline{J} \cdot (6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}) = 8$, then $h^0(\mathcal{O}_{\overline{J}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12})) = 9$. Moreover we have that $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12})) \geq \binom{8}{2} - 7 \cdot 3 - 2 = 5$. To show that equality holds, it is sufficient to prove that $h^1(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12})) = 0$.

Similarly to before, the generic curve D of $|6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12}|$ is irreducible by Bertini's Theorem.

Let us consider the exact sequence

$$0 \to \mathcal{O}_{\overline{X'}} \to \mathcal{O}_{\overline{X'}}(D) \to \mathcal{O}_D(D) \to 0. \tag{47}$$

Since $D^2 = 36 - 28 - 1 - 4 + 2 + 2 = 7$ and $p_a(D) = p_g(D) = \frac{5 \cdot 4}{2} - 7 = 3$, then $h^1(\mathcal{O}_D(D)) = 0$ (see [20], Example $IV.1.3.4$). Since $h^1(\mathcal{O}_{\overline{X'}}) = 0$ by definition, then $h^1(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12})) = 0$ from the exact sequence (47). Consequently $h^0(\mathcal{O}_{\overline{X'}}(6l - 2\sum_{i=4}^{10} E_i - E_{11} - 2E_{12})) = 5$ from the exact sequence (46).

With the same computations as before, using the exact sequences (45) and (44), we obtain that $h^0(\mathcal{O}_{\overline{X'}}(18l - 4\sum_{i=1}^{3} E_i - 6\sum_{i=4}^{10} E_i - E_{11} - 2E_{12})) = 11$. The claim is proved for x_{11} and x_{12} infinitely near.

□

We have found a new example of rational surface $X \subset \mathbb{P}^{12}$ of degree $\deg(X) = \frac{C'^2}{\deg i_{L'}} = 24$ with Prym-canonical hyperplane sections and only one singularity, a quartic rational singularity.

We can list some possible rational surface X with Prym-canonical hyperplane sections in terms of numerical values.

Lemma 3.6. *The following list contains linear systems L' on $X' = Bl_{\{x_1,\dots,x_{10}\}}(\mathbb{P}^2)$ whose image $X = i_{L'}(X')$ satisfies the conclusions of Lemma 3.4 and* $\dim L' \geq 4$, *so they are candidates to be surfaces with Prym-canonical hyperplane sections.*

1. $L' = |6ml - m\sum_{i=1}^{2} E_i - 2m\sum_{i=3}^{10} E_i|$, *for* $m \geq 2$, $m \in \mathbb{N}$;
2. $L' = |7ml - 2m\sum_{i=1}^{9} E_i - 3mE_{10}|$, *for* $m \geq 2$, $m \in \mathbb{N}$;
3. $L' = |9ml - 2m\sum_{i=1}^{3} E_i - 3m\sum_{i=4}^{10} E_i|$, *for* $m \geq 2$, $m \in \mathbb{N}$;
4. $L' = |9ml - mE_1 - 2mE_2 - 3m\sum_{i=3}^{10} E_i|$, *for* $m \geq 2$, $m \in \mathbb{N}$;
5. $L' = |10ml - 3m\sum_{i=1}^{10} E_i|$, *for* $m \geq 1$, $m \in \mathbb{N}$;
6. $L' = |10ml - 2mE_1 - 3m\sum_{i=2}^{9} E_i - 4mE_{10}|$, *for* $m \geq 1$, $m \in \mathbb{N}$.

Remark 3.7. *To conclude that the surfaces $X = i_{L'}(X')$ of the previous Lemma are surfaces with Prym-canonical hyperplane sections, it remains to show that $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$ while $\mathcal{O}_{C'}(-K_{X'}) \ncong \mathcal{O}_{C'}$ and, moreover, L' has the expected dimension, it is base-point free and it defines an embedding outside a finite number of contracted curves, that determine the isolated singularities of X. However, if we only know that L' defines a birational map, then the image X could have non-isolated singularities.*

Proof of Lemma 3.6. If $X'' = \mathbb{P}^2$, then $-2K_{X''} \sim 6l$, in particular we assume that $-2K_{X''}$ is an irreducible sextic with 10 nodes $\{x_1, ..., x_{10}\}$. Let L'' be a linear system with base points $\{x_1, ..., x_{10}\}$ of multiplicity $r_i > 0$, for $i = 1, ..., 10$, and let $X' = Bl_{\{x_1,...,x_{10}\}}(\mathbb{P}^2)$ be the blowing up of X'' along the base points of L''. As seen in Remark 3.3, the anticanonical divisor $-K_{X'}$ is not effective. Clearly the generic element of L' is $C' \sim bl - \sum_{i=1}^{10} r_i E_i$, for $b \in \mathbb{N}_{>0}$.

We consider all the linear systems L' on X' of curves C' of degree b passing through $\{x_1, ..., x_{10}\}$ with multiplicity $r_i > 0$, for $i = 1, .., 10$, such that $3b = \sum_{i=1}^{10} r_i$ and $\dim(L') \geq 4$. Thus, by Lemma 3.4 and by definition of surface with Prym-canonical hyperplane sections, we expect the image $X = i_{L'}(X')$ to be a possible surface with Prym-canonical hyperplane sections. As in Example 3.2, the linear system L' contracts $-2K_{X'}$ to a quadruple rational singularity.

We observe that, if $b \leq 3$, then $3b < 10$, so the variable b never satisfies the condition $3b = \sum_{i=1}^{10} r_i$, for $r_i > 0$, for $i = 1, .., 10$.

- If $\mathbf{b} = \mathbf{4}$, then the relation $3b = \sum_{i=1}^{10} r_i$ is satisfied either if

$$\begin{cases} r_i = 1 & for \quad \text{i=1,...,8} \\ r_9 = r_{10} = 2 \end{cases}$$

  or if

$$\begin{cases} r_i = 1 & for \quad \text{i=1,...,9} \\ r_{10} = 3, \end{cases}$$

  up to renaming the multiplicities r_i. It is clear that all the positive multiples satisfy the same relation, in the sense that $3mb = \sum_{i=1}^{10} mr_i$, for any $m \in \mathbb{N}_{>0}$.

  If $C' \sim 4ml - m\sum_{i=1}^{8} E_i - 2mE_9 - 2mE_{10}$, for $m > 0$, then we can observe that

$$h^0(\mathcal{O}_{X'}(C')) \geq \frac{(4m+2)(4m+1)}{2} - 8\frac{m(m+1)}{2} - 2\frac{2m(2m+1)}{2} =$$
$$= 8m^2 + 6m + 1 - 4m^2 - 4m - 4m^2 - 2m = 1.$$

  Since we do not know if $\dim |C'| \geq 4$, then we exclude this case.

  Similarly, if $C' \sim 4ml - m\sum_{i=1}^{9} E_i - 3mE_{10}$, we observe that

$$h^0(\mathcal{O}_{X'}(C')) \geq \frac{(4m+2)(4m+1)}{2} - 9\frac{m(m+1)}{2} - \frac{3m(3m+1)}{2} =$$

$$= 8m^2 + 6m + 1 - \frac{9}{2}m^2 - \frac{9}{2}m - \frac{9}{2}m^2 - \frac{3}{2}m = -m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. We exclude this case for the same reason as before.

- If $\mathbf{b = 5}$, then the relation $3b = \sum_{i=1}^{10} r_i$ is satisfied if

$$\begin{cases} r_i = 1 & for \;\; \text{i=1,...,5} \\ r_j = 2 & for \;\; \text{j=6,...,10,} \end{cases}$$

up to renaming the terms. All the positive multiples of $C' \sim 5l - \sum_{i=1}^{5} E_i - 2\sum_{j=6}^{10} E_j$ satisfy the same relation. We can observe that

$$h^0(\mathcal{O}_{X'}(mC')) \geq \frac{(5m+2)(5m+1)}{2} - 5\frac{m(m+1)}{2} - 5\frac{2m(2m+1)}{2} =$$
$$= \frac{25}{2}m^2 + \frac{15}{2}m + 1 - \frac{5}{2}m^2 - \frac{5}{2}m - 10m^2 - 5m = 1,$$

for any $m \in \mathbb{N}_{>0}$. Since we do not know if $\dim |mC'| \geq 4$, for $m > 0$, then we also exclude this case.

There are other $\{r_1, ..., r_{10}\}$ satisfying the relation $3b = \sum_{i=1}^{10} r_i$, with values of r_i greater than or equal to the previous ones, but it is clear that $h^0(\mathcal{O}_{X'}(C')) \leq 1$ in these other possible cases (in the previous case the value of $h^0(\mathcal{O}_{X'}(C'))$ was 1). Then we do not consider them.

- If $\mathbf{b = 6}$, the relation $3b = \sum_{i=1}^{10} r_i$ is satisfied if

$$\begin{cases} r_1 = r_2 = 1 & \\ r_i = 2 & for \;\; \text{i=2,...,10,} \end{cases}$$

up to renaming the multiplicities r_i. We also consider all the positive multiples of $C' \sim 6l - E_1 - E_2 - 2\sum_{i=2}^{10} E_i$, so

$$h^0(\mathcal{O}_{X'}(mC')) \geq \frac{(6m+2)(6m+1)}{2} - 2\frac{m(m+1)}{2} - 8\frac{2m(2m+1)}{2} =$$
$$= 18m^2 + 9m + 1 - m^2 - m - 16m^2 - 8m = m^2 + 1,$$

for any $m \in \mathbb{N}_{>0}$. We are sure that $\dim L' \geq 4$ if and only if $m \geq 2$. Hence the linear systems $L' = |mC'|$, for $m \geq 2$, are such that the image is a "possible" surface with Prym-canonical hyperplane sections.

Another set $\{r_1, ..., r_{10}\}$ that satisfies the relation $3b = \sum_{i=1}^{10} r_i$, for $b = 6$, is

$$\begin{cases} r_i = 1 & for \;\; \text{i=1,2,3} \\ r_j = 2 & for \;\; \text{j=4,...,9} \\ r_{10} = 3. & \end{cases}$$

Again, if $C' \sim 6ml - m\sum_{i=1}^{3} E_i - 2m\sum_{i=4}^{9} E_i - 3E_{10}$, for $m \in \mathbb{N}_{>0}$, then we have that

$$h^0(\mathcal{O}_{X'}(C')) \geq \frac{(6m+2)(6m+1)}{2} - 3\frac{m(m+1)}{2} - 6\frac{2m(2m+1)}{2} - \frac{3m(3m+1)}{2} =$$

$$= 18m^2 + 9m + 1 - \frac{3}{2}m^2 - \frac{3}{2}m - 12m^2 - 6m - \frac{9}{2}m^2 - \frac{3}{2}m = 1.$$

Since we do not know if $\dim |C'| \geq 4$, then we exclude this case.

We do not consider the other possible $\{r_1, ..., r_{10}\}$ that satisfy the relation $3b = \sum_{i=1}^{10} r_i$ because the associated linear systems have expected dimension less than or equal to 1.

- With the same techniques as before, if $\mathbf{b = 7}$, then $C' \sim 7ml - 2m\sum_{i=1}^{9} E_i - 3mE_{10}$ is such that

$$h^0(\mathcal{O}_{X'}(C')) \geq \frac{49}{2}m^2 + \frac{21}{2}m + 1 - 18m^2 - 9m - \frac{9}{2}m^2 - \frac{3}{2}m = 2m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. We are sure that $\dim L' \geq 4$ if and only if $m \geq 2$. So we only consider the linear systems $L' = |C'|$ with $m \geq 2$, whose image is a "possible" surface with Prym-canonical hyperplane sections.

If $C' \sim 7ml - mE_1 - 2m\sum_{i=2}^{8} E_i - 3mE_9 - 3mE_{10}$, then

$$h^0(\mathcal{O}_{X'}(C')) = \frac{49}{2}m^2 + \frac{21}{2}m + 1 - \frac{m^2}{2} - \frac{m}{2} - 14m^2 - 7m - 9m^2 - 3m = m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. We are sure that $\dim L' \geq 4$ if and only if $m \geq 2$. If we apply a quadratic transformation in the two points of multiplicity $3m$ and in one of the points of multiplicity $2m$ of C', then these linear systems are the same found in the case "$b = 6$".

If $C' \sim 7ml - mE_1 - mE_2 - 2m\sum_{i=3}^{7} E_i - 3m\sum_{i=8}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) = \frac{49}{2}m^2 + \frac{21}{2}m + 1 - m^2 - m - 10m^2 - 5m - \frac{27}{2}m^2 - \frac{9}{2}m = 1,$$

for $m \in \mathbb{N}_{>0}$. We do not know if $\dim L' \geq 4$, so we exclude this case. We also exclude the other possible $\{r_1, ..., r_{10}\}$ that satisfy the relation $3b = \sum_{i=1}^{10} r_i$, with values of r_i greater than or equal to the previous ones, because the associated linear systems have expected dimension less than or equal to 1.

- Similarly, if $\mathbf{b = 8}$, then $C' \sim 8ml - 2m\sum_{i=1}^{6} E_i - 3m\sum_{i=7}^{10} E_i$ is such that

$$h^0(\mathcal{O}_{X'}(C')) = 32m^2 + 12m + 1 - 12m^2 - 6m - 18m^2 - 6m = 2m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. We are sure that $\dim L' \geq 4$ if and only if $m \geq 2$. So we only consider the linear systems $L' = |C'|$ with $m \geq 2$. If we apply a quadratic transformation in three of the points of multiplicity $3m$ of C', then these linear systems are the same of the previous case with $b = 7$.

If $C' \sim 8ml - E_1 - 2m\sum_{i=2}^{5} E_i - 3m\sum_{i=6}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) = 32m^2 + 12m + 1 - \frac{m^2}{2} - \frac{m}{2} - 8m^2 - 4m - \frac{45}{2}m^2 - \frac{15}{2}m = m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Again we are sure that $\dim L' \geq 4$ if and only if $m \geq 2$. If we apply a quadratic transformation in three of the points of multiplicity $3m$ of C', then these linear systems are the same found in the case "$b = 7$".

If $C' \sim 8ml - E_1 - E_2 - 2E_3 - 2E_4 - 3m\sum_{i=5}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) = 32m^2 + 12m + 1 - m^2 - m - 4m^2 - 2m - 27m^2 - 9m = 1,$$

for $m \in \mathbb{N}_{>0}$. We do not know if $\dim L' \geq 4$, so we exclude this case and similarly all the other possible $\{r_1, ..., r_{10}\}$ that satisfy the relation $3b = \sum_{i=1}^{10} r_i$, with values of r_i greater than or equal to the previous ones, because the associated linear systems have expected dimension less than or equal to 1.

- If $\mathbf{b = 9}$, then $C' \sim 9ml - 2m\sum_{i=1}^{3} E_i - 3m\sum_{i=4}^{10} E_i$ is such that

$$h^0(\mathcal{O}_{X'}(C')) = \frac{81}{2}m^2 + \frac{27}{2}m + 1 - 6m^2 - 3m - \frac{63}{2}m^2 - \frac{21}{2}m = 3m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Because if $m \geq 2$ then $\dim L' \geq 4$, we only consider the linear systems $L' = |C'|$ with $m \geq 2$, whose image is a "possible" surface with Prym-canonical hyperplane sections.

If $C' \sim 9ml - mE_1 - 2mE_2 - 3m\sum_{i=3}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) = \frac{81}{2}m^2 + \frac{27}{2}m + 1 - \frac{m^2}{2} - \frac{m}{2} - 2m^2 - m - 36m^2 - 12m = 2m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Because if $m \geq 2$ then $\dim L' \geq 4$, we only consider the linear systems $L' = |C'|$ with $m \geq 2$, whose image is a "possible" surface with Prym-canonical hyperplane sections.

If $C' \sim 9ml - 2m\sum_{i=1}^{4} E_i - 3m\sum_{i=5}^{9} E_i - 4mE_{10}$, then

$$h^0(\mathcal{O}_{X'}(C')) = \frac{81}{2}m^2 + \frac{27}{2}m + 1 - 8m^2 - 4m - \frac{45}{2}m^2 - \frac{15}{2}m - 8m^2 - 2m = 2m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Applying a quadratic transformation in the point of multiplicity $4m$ and in two of the points of multiplicity $3m$ of C', we obtain the same linear systems of the case with $b = 8$.

If $C' \sim 9ml - 2m\sum_{i=1}^{5} E_i - 3m\sum_{i=6}^{8} E_i - 4m\sum_{i=9}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) = \frac{81}{2}m^2 + \frac{27}{2}m + 1 - 10m^2 - 5m - \frac{27}{2}m^2 - \frac{9}{2}m - 16m^2 - 4m = m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Applying a quadratic transformation in the two points of multiplicity $4m$ and in one of the points of multiplicity $3m$ of C', we obtain the same linear systems of the case with $b = 7$.

If $C' \sim 9ml - 2m\sum_{i=1}^{6} E_i - 3mE_7 - 4m\sum_{i=8}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) = \frac{81}{2}m^2 + \frac{27}{2}m + 1 - 12m^2 - 6m - \frac{9}{2}m^2 - \frac{3}{2}m - 24m^2 - 6m = 1,$$

for $m \in \mathbb{N}_{>0}$. We do not know if $\dim L' \geq 4$, so we exclude this case and similarly all the other possible $\{r_1, ..., r_{10}\}$ that satisfy the relation $3b = \sum_{i=1}^{10} r_i$, with values of r_i greater than or equal to the previous ones, because the associated linear systems have expected dimension less than or equal to 1.

- Finally, if $\mathbf{b = 10}$, we can consider $C' \sim 10ml - 3m\sum_{i=1}^{10} E_i$, for $m \in \mathbb{N}_{>0}$. It is easy to prove that

$$h^0(\mathcal{O}_{X'}(C')) \geq 5m^2 + 1,$$

with $5m^2 + 1 \geq 5$ for any m. Example 3.2 corresponds to this case with $m = 1$.
If $C' \sim 10ml - 2mE_1 - 3m\sum_{i=2}^{9} E_i - 4mE_{10}$, then

$$h^0(\mathcal{O}_{X'}(C')) \geq 4m^2 + 1 \geq 5$$

for any $m \in \mathbb{N}_{>0}$. So the linear systems $L' = |C'|$, for $m \geq 1$, are such that the image is a "possible" surface with Prym-canonical hyperplane sections.
If $C' \sim 10ml - 2mE_1 - 2mE_2 - 3m\sum_{i=3}^{8} E_i - 4mE_9 - 4mE_{10}$, then

$$h^0(\mathcal{O}_{X'}(C')) \geq 3m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Using a quadratic transformation centered in the two points of multiplicity $4m$ and in one of the points of multiplicity $3m$ of C', then these linear systems are referable to the cases with $b = 9$.
If $C' \sim 10ml - 2m\sum_{i=1}^{3} E_i - 3m\sum_{i=4}^{7} E_i - 4m\sum_{i=8}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) \geq 2m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Using a quadratic transformation centered in the three points of multiplicity $4m$ of C', then these linear systems are the same found in the case "$b = 8''$.
If $C' \sim 10ml - 2m\sum_{i=1}^{4} E_i - 3m\sum_{i=5}^{6} E_i - 4m\sum_{i=7}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) \geq m^2 + 1,$$

for $m \in \mathbb{N}_{>0}$. Using two quadratic transformations, one centered in three of the points of multiplicity $4m$ of C' and the other centered in the point of multiplicity $4m$ and in the two points of multiplicity $3m$ of the transformed curve, then these linear systems are the same of the previous cases with $b = 6$.
If $C' \sim 10ml - 2m\sum_{i=1}^{5} E_i - 4m\sum_{i=6}^{10} E_i$, then

$$h^0(\mathcal{O}_{X'}(C')) \geq 1,$$

for $m \in \mathbb{N}_{>0}$. We do not know if $\dim L' \geq 4$, so we exclude this case and the other possible $\{r_1, ..., r_{10}\}$ that satisfy the relation $3b = \sum_{i=1}^{10} r_i$, with values of r_i greater than or equal to the previous ones, because the associated linear systems have expected dimension less than or equal to 1.

□

Remark 3.8. *It is possible to extend the list of Lemma 3.6 continuing with the same method as before for all* $b \geq 10$*, so there are infinite possible linear systems* $L' = |bl - \sum_{i=1}^{10} r_i E_i|$ *that satisfy*

$$3b = \sum_{i=1}^{10} r_i \qquad \textit{and} \qquad \dim(L') \geq 4. \tag{48}$$

In particular, we can prove that infinite linear systems not multiple one of the other exist.

Let $L' = |bl - \sum_{i=1}^{10} r_i E_i|$ *be a linear system such that* $3b = \sum_{i=1}^{10} r_i$ *and*

$$\dim(L') = h^0(\mathcal{O}_{X'}(C')) - 1 = \binom{b+2}{2} - \sum_{i=1}^{10} \frac{r_i(r_i+1)}{2} - 1 =$$

$$= \frac{b^2+3b+2}{2} - \sum_{i=1}^{10} \frac{r_i(r_i+1)}{2} - 1 = \frac{b^2 - \sum_{i=1}^{10} r_i^2}{2} + \frac{3b - \sum_{i=1}^{10} r_i}{2} \geq 4.$$

Then it is clear that, if $k \in \mathbb{N}_{\geq 2}$*, the linear system* $\overline{L'} = |kbl - \sum_{i=1}^{10} kr_i|$ *satisfies* $3kb = k\sum_{i=1}^{10} r_i$ *and*

$$dim\overline{L'} = \binom{kb+2}{2} - \sum_{i=1}^{10} \frac{kr_i(kr_i+1)}{2} - 1 = k^2 \frac{b^2 - \sum_{i=1}^{10} r_i^2}{2} + k\frac{3b - \sum_{i=1}^{10} r_i}{2} > 4.$$

We can prove that there are other examples of linear systems of degree kb that satisfy the requests of (48) that are not multiples of linear systems of lower degree.
For example, we suppose that $\min\{kr_1, ..., kr_{10}\} = kr_1$ *and* $\max\{kr_1, ..., kr_{10}\} = kr_{10}$*. At this point, we can consider the linear system* $\widetilde{L'} = kbl - \widetilde{r_1}E_1 - \widetilde{r_{10}}E_{10} - \sum_{i=2}^{9} kr_i E_i$*, where* $\widetilde{r_1} = kr_1 + 1$ *and* $\widetilde{r_{10}} = kr_{10} - 1$*. Hence* $\widetilde{L'}$ *cannot be a multiple of a linear system that satisfies* (48) *of lower degree. It is clear that* $3kb = \sum_{i=1}^{10} kr_i = \sum_{i=2}^{9} kr_i + \widetilde{r_1} + \widetilde{r_2}$*, so one of the two properties of (48) is satisfied. Similarly we have that*

$$\dim \widetilde{L'} = \binom{kb+2}{2} - \sum_{i=2}^{9} \frac{kr_i(kr_i+1)}{2} - \frac{\widetilde{r_1}(\widetilde{r_1}+1)}{2} - \frac{\widetilde{r_{10}}(\widetilde{r_{10}}+1)}{2} - 1 =$$

$$= \dim(\overline{L'}) + \frac{kr_1(kr_1+1)}{2} + \frac{kr_{10}(kr_{10}+1)}{2} - \frac{\widetilde{r_1}(\widetilde{r_1}+1)}{2} - \frac{\widetilde{r_{10}}(\widetilde{r_{10}}+1)}{2}.$$

We have already observed that $\dim \overline{L'} > 4$*. Let us fix*

$$x := \frac{kr_1(kr_1+1)}{2} + \frac{kr_{10}(kr_{10}+1)}{2} - \frac{\widetilde{r_1}(\widetilde{r_1}+1)}{2} - \frac{\widetilde{r_{10}}(\widetilde{r_{10}}+1)}{2} =$$

$$= \frac{k^2r_1^2 + kr_1}{2} + \frac{k^2r_{10}^2 + kr_{10}}{2} - \frac{k^2r_1^2 + 3kr_1 + 2}{2} - \frac{k^2r_{10}^2 - kr_{10}}{2} =$$

$$= \frac{kr_1 + kr_{10} - 3kr_1 - 2 + kr_{10}}{2} = k(r_{10} - r_1) - 1.$$

We observe that $kr_{10} \geq kr_1$ *by the previous assumption. So, if* $kr_{10} = kr_1$*, then* $x = -1$ *and* $\dim \widetilde{L'} = \dim \overline{L'} + x \geq 4$*, instead, if* $kr_{10} > kr_1$*, then* $x \geq 2(r_{10} - r_1) - 1 \geq 1$*, hence* $\dim \widetilde{L'} > 5$*. The two properties of (48) are satisfied by* $\widetilde{L'}$*.*

We conclude that, for any degree, we can find linear systems that are not multiples of linear system of lower degree.

As seen in Remark 3.3, in order to have $-K_{X'}$ not effective, at least 10 points of $\mathbb{P}^2$ must be blown up. If we suppose that $-2K_{X'}$ is effective and irreducible, then only 10 points must be blown up, as discussed above.
Instead, if $-2K_{X''}$ is effective and reducible, we can blow up more than 10 points of $\mathbb{P}^2$ keeping $-2K_{X'}$ effective.

Remark 3.9. *We know that* $-2K_{\mathbb{P}^2} \sim 6l$ *is linearly equivalent to a sextic curve.*

We recall that a nodal plane curve irreducible and effective is such that the number of nodes δ *is at most equal to the arithmetic genus of the curve. It is clear that lines and conics cannot have nodes. Instead, an irreducible cubic* C *can have at most one node since* $p_a(C) = 1$*, an irreducible quartic* Q *can have at most* 3 *nodes since* $p_a(Q) = 3$*, an irreducible quintic* T *can have at most* 6 *nodes since* $p_a(T) = 6$ *and so on.*

If r *is the number of nodes of* $-2K_{\mathbb{P}^2}$*, then the possible reduced and reducible antibicanonical divisors of* $\mathbb{P}^2$*, whose corresponding antibicanonical divisor* $-2K_{X'}$ *of* $X' = Bl_{x_1,..,x_r}\mathbb{P}^2$ *is effective while the anticanonical divisor* $-K_{X'}$ *is not effective, are formed by:*

- *if* $r = 11$*, the possibilities are*
  - *union of a quintic with* 6 *nodes and a line;*
  - *union of a quartic with* 3 *nodes and a conic;*
  - *union of a quartic with* 2 *nodes and two lines;*
  - *union of two cubics, each one with one node;*
  - *union of a smooth cubic, a conic and a line;*
- *if* $r = 12$*, then we have*
  - *union of a quartic with* 3 *nodes and two lines;*
  - *union of a cubic with one node, a conic and a line;*
  - *union of a smooth cubic and three lines;*
  - *union of three conics;*
- *if* $r = 13$*, the possibilities are*
  - *union of a cubic with one node and three lines;*
  - *union of two conics and two lines;*

- *if $r = 14$, the only possible antibicanonical divisor of $\mathbb{P}^2$ is*
  - *union of one conic and four lines;*
- *if $r = 15$, the only possible antibicanonical divisor of $\mathbb{P}^2$ is*
  - *union of six lines.*

We can generalize Lemma 3.4.

Lemma 3.10. *Let X be a surface with Prym-canonical hyperplane sections whose minimal model is $X'' = \mathbb{P}^2$ and let X' be the minimal resolution of the singularities of X. If we also assume that $-2K_{X'}$ is effective, $-K_{X'}$ is not effective and C' is a general divisor of L', then $C' \sim bl - \sum_{i=1}^{k} r_i E_i$ is such that $3b = \sum_{i=1}^{k} r_i$, where $10 \leq k \leq 15$ and $r_i > 0$ for any i.*

Proof. Similar to the proof of Lemma 3.4 but now $-2K_{X''}$ can be reducible, so $10 \leq k \leq 15$ as seen in Remarks 3.3 and 3.9. □

Remark 3.11. *The previous Lemma is useful to list in terms of numerical values the possible surfaces with Prym-canonical hyperplane sections, as seen in Lemma 3.6 for $-2K_{X'}$ irreducible.*

4 Moduli of surfaces with Prym-canonical hyperplane sections

Let $X \subset \mathbb{P}^{g-1}$ be a surface with Prym-canonical hyperplane sections that is not a cone, for $g \geq 5$. We assume that a general hyperplane section C of X has Clifford index $\mathrm{Cliff}(C) \geq 5$, so, in particular, C is not hyperelliptic and it is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$ (see [23], Lemma 2.1).

Let $\pi : X' \to X$ be the minimal resolution of singularities of X and let $C' = \pi^* C$, as seen in the previous chapters. Let us suppose that X' is not an Enriques surface, so it is birationally equivalent to $\mathbb{P}^2$ or a ruled surface X'' over a base curve Γ of genus $q \geq 0$ (see Theorem 1.10). Let $\phi : X' \to X''$ be a relatively minimal model for X'. We have already proved that $h^0(\mathcal{O}_{X'}(-2K_{X'})) = 1$ (see Theorem 1.12), so let W' be the only effective antibicanonical divisor on X'. Since $-K_{X'}|_{C'}$ is a two-torsion divisor by hypothesis, then $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$.

In the following, we will suppose that all these assumptions are valid. Moreover, in the first phase, we will work assuming $X'' = \mathbb{P}^2$ or X'' a rational ruled surface, with $q = q(X') = q(X'') = 0$.

Lemma 4.1. *If X' is rational, we have that $h^0(\mathcal{O}_{W'}) = -K_{X'}^2$.*

Proof. We use the following exact sequence:

$$0 \to \mathcal{O}_{X'}(-W') \to \mathcal{O}_{X'} \to \mathcal{O}_{W'} \to 0. \tag{49}$$

Since W' is effective, then $h^0(\mathcal{O}_{X'}(-W')) = 0$. In addition $h^0(\mathcal{O}_{X'}) = 1$ and $h^1(\mathcal{O}_{X'}) = 0$ by hypothesis.
On the other hand, by the Riemann-Roch Theorem, we have that

$$h^1(\mathcal{O}_{X'}(-W')) = h^2(\mathcal{O}_{X'}(2K_{X'})) - \frac{1}{2}(2K_{X'})(K_{X'}) - 1 =$$

$$= h^0(\mathcal{O}_{X'}(-K_{X'})) - K_{X'}^2 - 1 = -K_{X'}^2 - 1.$$

From the long exact sequence associated with (49), we conclude that $h^0(\mathcal{O}_{W'}) = -K_{X'}^2$. □

Remark 4.2. *By Mumford's Theorem (see [25], Chapter 1), we know that $-K_{X'}^2 > 0$ since $-2K_{X'}$ is contracted by π.*

The divisor W' has one or more connected components, $D_1, ..., D_n$, so, by Proposition 1.15, $D_1, ..., D_n$ are contracted to rational singularities on X with multiplicity $\mu_1, ..., \mu_n$ respectively.

Let us suppose that W' is irreducible, so W' is contracted to a rational singularity $x \in X$ with multiplicity μ. Then, by definition, we have that $\mu = -W'^2$.

Let h be the number of blown up points in ϕ (also infinitely near). With the same notation of Proposition 1.22, we have that

$$-2K_{X'_{t+1}} = \pi^*(-2K_{X'_t}) - 2E_{t+1,1} - ... - 2E_{t+1,i(t)},$$

where $E_{t+1,i}$ is the exceptional divisor associated with the blown up point $x_{t,i}$, for $t = 1, ..., s-1$ and $i = 1, .., i(t)$. So $(-2K_{X'_{t+1}})^2 = (-2K_{X'_t})^2 - 4i(t)$. Hence, iterating this process and observing that $i(1) + ... + i(s-1) = h$ by definition, we have that $-\mu = W'^2 = (-2K_{X''})^2 - 4h$ and, in particular,

$$\mu = 4h - (2K_{X''})^2. \tag{50}$$

If $X'' = \mathbb{P}^2$, then $\mu = 4h - 36$. Since μ is strictly positive, then $h > 9$ in this case.

We claim that surfaces of the type of X as above depend on a number of moduli that has an upper bound. Before showing this, we will prove preliminary lemmas.

Lemma 4.3. *Let $X \subset \mathbb{P}^{g-1}$ be a surface with Prym-canonical hyperplane sections that is not a cone, for $g \geq 5$. Assume that a general hyperplane section C of X has* Cliff$(C) \geq 5$. *Then X is an intersection of quadrics (ideal-theoretic intersection of quadrics).*

Remark 4.4. *As already observed, a general hyperplane section C of X with* Cliff$(C) \geq$ 5 *is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$ (see [18], Theorem 1).*

Proof. As seen in point 1. of Theorem 1.10, we have that $h^1(\mathcal{O}_X(n)) = 0$ for any $n \geq 0$. We consider the exact sequence

$$0 \to \mathcal{I}_{C|X}(2) \to \mathcal{I}_{X|\mathbb{P}^{g-1}}(2) \to \mathcal{I}_{C|\mathbb{P}^{g-2}}(2) \to 0,$$

where $\mathcal{I}_{C|X}(2) \cong \mathcal{O}_X(1)$. By the previous remark, we know that $h^1(\mathcal{O}_X(1)) = 0$, hence

$$H^0(\mathcal{I}_{X|\mathbb{P}^{g-1}}(2)) \twoheadrightarrow H^0(\mathcal{I}_{C|\mathbb{P}^{g-2}}(2)). \tag{51}$$

By [23], Corollary 2.3, we have that C is an intersection of quadrics. Let $\{q_1, ..., q_p\}$ be the equations of a basis for the space of quadrics containing C. Let H be a general hyperplane section such that $X \cap H = C$. Using (51), for every hypersurface $f|_H$ of $\mathbb{P}^{g-2}$ containing C of degree $k \geq 2$, there exists a hypersurface f containing X in $\mathbb{P}^{g-1}$ of degree k such that f induces $f|_H$ on $\mathbb{P}^{g-2}$. Let $\{Q_1, ..., Q_p\}$ be the equations of a set of quadrics containing X such that $Q_i|_H = q_i$, for $i = 1, .., p$. Similarly to the techniques of [28], Corollary 7.11, let λ be the equation of a hypersurface of degree k containing X. By the previous remark, we have that $\lambda|_H = \sum_i u_i q_i$, where u_i are polynomials of degree $k-2$. So $\lambda - \sum_i U_i Q_i$, where $U_i|_H = u_i$ for any i, vanishes on X and H, hence $\lambda - \sum_i U_i Q_i = h \cdot \lambda'$, where h defines H and λ' of degree $k-1$ vanishes on X. By induction on k, the ideal of X is generated by quadrics and the claim is proved. □

Let $\rho : X \to F \subset \mathbb{P}^{g-2}$ be the projection from x on a hyperplane, where x is the rational singularity of X obtained by the contraction of W', that we assume irreducible.

Lemma 4.5. *Let X be a rational surface with Prym-canonical hyperplane sections, not a cone, and we assume that a general hyperplane section C of X has* $\mathrm{Cliff}(C) \geq 5$ *and, in particular, C is projectively normal. The projection map $\rho : X \to F$ is birational onto its image.*

Proof. Let us suppose that ρ is not birational on its image, then X has trisecant lines passing through x. Since X is an intersection of quadrics, then it contains these trisecant curves. Thus X would be a cone of vertex x but it is rational by assumption, then X would have rational hyperplane sections. This is impossible because we assume that X is a surface with Prym-canonical hyperplane sections. The claim is proved. □

We give an upper bound to h, that is the number of blown up points in ϕ.

Theorem 4.6. *Let X be a surface with Prym-canonical hyperplane sections C of genus g and let $\pi : X' \to X$ be the minimal resolution of singularities of X. Let $\phi : X' \to X''$ be a relatively minimal model for X' such that X'' is $\mathbb{P}^2$ or a ruled surface over a base curve of genus $q = 0$. If* $\mathrm{Cliff}(C) \geq 5$ *and, in particular, C is projectively normal and $W' \in |-2K_{X'}|$ is irreducible, then*

$$h \leq \frac{1}{6}g + \frac{5}{6} + K^2_{X''}.$$

More precisely, if $X'' = \mathbb{P}^2$, then $h \leq \frac{1}{6}g + \frac{59}{6}$.

Proof. By Lemma 4.5, we have that ρ is birational onto its image F.
Let us suppose that F is a ruled surface. Then X is a surface described by curves of degree d passing through x with multiplicity $d-1$. Since X is intersection of quadrics (see Lemma 4.3), then $d \leq 2$. If $d=1$, then X would have rational hyperplane sections but this is not possible because $g(C) \geq 5$. On the other hand, if $d=2$, then C would have a g_2^1 but this is not possible because $\mathrm{Cliff}(C) \geq 5$. Thus F cannot be a ruled surface.
By [21], Horowitz Formula, we have that

$$\deg(F) \geq \frac{4}{3}(g-2-2) = \frac{4}{3}g - \frac{16}{3}.$$

On the other hand, we have that

$$\deg(F) = \deg(X) - \mu = 2g - 2 - \mu.$$

By (50), we can say that $2g-2-\mu = 2g-2-4h+(2K_{X''})^2$. Combining the two results, we obtain that $\frac{4}{3}g - \frac{16}{3} \leq 2g-2-4h+(2K_{X''})^2$, from which

$$4h \leq \frac{2}{3}g + \frac{10}{3} + 4K_{X''}^2.$$

The claim is proved. □

Finally, let σ be the number of moduli of the surfaces of the type of X.

Theorem 4.7. *With the same hypothesis of Theorem 4.6, we can conclude that:*

1. *if X'' is birationally equivalent to a rational ruled surface, then*

$$\sigma \leq \begin{cases} 4K_{X''}^2 + \frac{1}{6}g - \frac{31}{6} & if\ \ 0 \leq e < 3 \\ 4K_{X''}^2 + \frac{1}{6}g - \frac{31}{6} + 2e - 5 & if\ \ 3 \leq e < 6 \\ 4K_{X''}^2 + \frac{1}{6}g - \frac{31}{6} + 3e - 10 & if\ \ e \geq 6. \end{cases}$$

2. *if $X'' = \mathbb{P}^2$, then*

$$\sigma \leq \frac{1}{6}g + \frac{173}{6}.$$

Proof. 1. Let X'' be a surface birationally equivalent to a rational ruled surface. Using Riemann-Roch's Theorem, we can compute that

$$\dim|-2K_{X''}| = h^0(\mathcal{O}_{X''}(-2K_{X''})) - 1 = h^1(\mathcal{O}_{X''}(-2K_{X''})) - h^2(\mathcal{O}_{X''}(-2K_{X''})) +$$

$$+\frac{1}{2}(-2K_{X''})(-3K_{X''}) + 1 - 1.$$

By Serre Duality, we have that $h^2(\mathcal{O}_{X''}(-2K_{X''})) = h^0(\mathcal{O}_{X''}(3K_{X''})) = 0$. By Lemma 2.5, we know that $h^1(\mathcal{O}_{X''}(-2K_{X''})) = h^1(\mathcal{O}_{\mathbb{P}^1}(4+2e)) + h^1(\mathcal{O}_{\mathbb{P}^1}(4+e)) + h^1(\mathcal{O}_{\mathbb{P}^1}(4)) + h^1(\mathcal{O}_{\mathbb{P}^1}(4-e)) + h^1(\mathcal{O}_{\mathbb{P}^1}(4-2e))$, where $e = -deg(D) \geq 0$ is an invariant of X'', as seen in the previous sections.

Using [20], Example $IV.1.3.4.$ and Riemann-Roch's Theorem, we obtain that

$$h^1(\mathcal{O}_{X''}(-2K_{X''})) = \begin{cases} 0 & if\ 0 \le e < 3 \\ 2e-5 & if\ 3 \le e < 6 \\ 3e-10 & if\ e \ge 6. \end{cases}$$

Then

$$\tau := \dim|-2K_{X''}| = \begin{cases} 3K^2_{X''} & if\ 0 \le e < 3 \\ 3K^2_{X''} + 2e - 5 & if\ 3 \le e < 6 \\ 3K_{X''} + 3e - 10 & if\ e \ge 6. \end{cases} \tag{52}$$

Since $\dim|-2K_{X'}| = 0$ (see Theorem 1.12), we have to blow up at least τ points that impose independent conditions on $|-2K_{X''}|$. Moreover, we blow up other $h-\tau$ points on the antibicanonical divisor, so that it remains effective on X'. It is obvious that $h - \tau \ge 0$ by definition of h.

Hence $\sigma \le 2\tau + (h-\tau) - \dim(\mathrm{Aut}(X''))$. Since X'' is a minimal rational ruled surface, then $\dim(\mathrm{Aut}(X'')) \ge 6$. Thus $\sigma \le \tau + h - 6$. By Theorem 4.6, we can write that

$$\sigma \le \tau + \frac{1}{6}g + \frac{5}{6} + K^2_{X''} - 6 = \tau + \frac{1}{6}g - \frac{31}{6} + K^2_{X''}.$$

The claim is proved using the inequalities contained in (52).

2. Let $X'' = \mathbb{P}^2$. Then

$$\dim|-2K_{X''}| = \dim|6l| = h^0(\mathcal{O}_{X''}(6l)) - 1 = h^1(\mathcal{O}_{X''}(6l)) - h^2(\mathcal{O}_{X''}(6l)) +$$

$$+\frac{1}{2}(6l)(9l) + 1 - 1,$$

where l is the class of a line of $\mathbb{P}^2$. By Serre Duality, we have that $h^2(\mathcal{O}_{X''}(6l)) = h^0(\mathcal{O}_{X''}(-9l)) = 0$. Moreover, by [20], Theorem $III.5.1$, we know that $h^1(\mathcal{O}_{X''}(-2K_{X''})) = 0$. Then

$$\dim|-2K_{X''}| = \frac{1}{2}(6l)(9l) + 1 - 1 = 27.$$

Since $\dim|-2K_{X'}| = 0$ (see Theorem 1.12), we have to blow up at least 27 points that impose independent conditions on $|-2K_{X''}|$. Moreover, we blow up other $h-27$ points on the antibicanonical divisor, so that it remains effective on X'. It is obvious that $h - 27 \ge 0$ by definition of h.

Hence $\sigma \le 2 \cdot 27 + (h-27) - \dim(\mathrm{Aut}(X''))$. Since $X'' = \mathbb{P}^2$, then $\dim(\mathrm{Aut}(X'')) = 8$. Thus $\sigma \le 27 + h - 8$. By Theorem 4.6, we can write that

$$\sigma \le 27 + \frac{1}{6}g + \frac{59}{6} - 8 = \frac{1}{6}g + \frac{173}{6}.$$

The claim is proved.

□

From now on, we will assume X'' a non-rational ruled surface, with $q = q(X') = q(X'') > 0$. Since X' is irregular and it has negative Kodaira dimension (see Theorem 1.10, we have already excluded X' Enriques surface in the assumptions), then X' is a ruled surface and we will denote by $a \geq 1$ the degree of the rational curves of the ruling. Also X is a ruled surface in curves of degree a.

Let σ be the number of moduli of the surfaces of the type of X and let m be the number of moduli of hyperplane sections of X. Since $X \subseteq \mathbb{P}^{g-1}$ by assumption and there are ∞^{g-1} possible hyperplane in $\mathbb{P}^{g-1}$, then $m \leq \sigma + g - 1$.

Theorem 4.8. *Let X be a surface with Prym-canonical hyperplane sections C of genus g such that C is projectively normal and let $\pi : X' \to X$ be the minimal resolution of singularities of X. Let $\phi : X' \to X''$ be a relatively minimal model for X' such that X'' is a ruled surface over a base curve of genus $q > 0$. If m is the number of moduli of hyperplane sections of X, then*

$$m \leq \begin{cases} 2g-1 & if \;\; q = 1 \\ 3g-3 & if \;\; q > 1 \;\; and \;\; a = 1 \\ 2g-2-(2a-3)(q-1) < 2g-1 & if \;\; q > 1 \;\; and \;\; a > 1. \end{cases}$$

Proof. If C is a general hyperplane section of X, then it is a covering $a : 1$ of a curve of genus q.

If $q = 1$, then the number of moduli of elliptic curves is 1 and, by Hurwitz formula, the degree of the ramification divisor is $2g - 2$. Thus $m \leq 1 + 2g - 2 = 2g - 1$.

Similarly, if $q > 1$, then the number of moduli of curves of genus q is $3q - 3$ and, by Hurwitz formula, the degree of the ramification divisor is $2g - 2 - a(2q - 2)$. Hence $m \leq 3q - 3 + 2g - 2 - a(2q - 2) = (3 - 2a)(q - 1) + (2g - 2) = 2g - 2 - (2a - 3)(q - 1)$. If $a = 1$, then it is obvious that $g = q$, so $m \leq 2g - 2 + (g - 1) = 3g - 3$.
On the other hand, if $a \neq 1$, then the quantity $1 + (2a - 3)(q - 1) > 0$. Hence $m \leq 2g - 2 - (2a - 3)(q - 1) < 2g - 1$. The claim is proved.

□

Ringraziamenti

Otto anni fa non avrei mai immaginato di arrivare a questo eccezionale traguardo ed il merito non è stato solo mio, ma anche di tutti coloro che mi sono stati vicini.

Un ringraziamento particolare va ai miei tre supervisori, alla Professoressa Concettina Galati, per i suoi preziosi consigli e suggerimenti, che ha permesso la stesura di questo lavoro, al Professor Andreas L. Knutsen, il cui sostegno è stato fondamentale e che ha reso i tre mesi in Norvegia un'esperienza non solo formativa, ma anche piacevole e stimolante, ed infine, ma non per importanza, voglio dire grazie al Professor Ciro Ciliberto, luminare della Geometria Algebrica, senza il quale la tesi non avrebbe raggiunto tali livelli.

Infine, a conclusione di questo mio ciclo di studi, non posso che ringraziare la mia famiglia, i miei amici ed i miei colleghi, che mi hanno supportata nei momenti di tensione e sempre spronata a dare il meglio di me stessa.

References

[1] Accola R.D.M., *On the Castelnuovo-Severi inequality for Riemann surfaces*, Kodai Math. J. 29 , No. 2 (2006), 299-317.

[2] Akizuki Y, *Theorems of Bertini on Linear systems*, Journal of the Mathematical Society of Japan, Vol. 3, No. 1 (1951).

[3] Arbarello E., Cornalba M., Griffiths P., Harris J., *Geometry of algebraic curves*, Springer Nature (2011).

[4] Beltrametti M., Sommese A.J., *Zero cycles and k-th order embeddings of smooth projective surfaces*, Sympos. Math. XXXII, Problems in the theory of surfaces and their classification - Cortona 1988, Academic Press, London (1991), 33-48.

[5] Catanese F., Ciliberto C., *Symmetric products of elliptic curves and surfaces of general type with* $p_g = q = 1$, J. Algebraic Geom. 2, No. 3 (1993), 389-411.

[6] Ciliberto C., Dedieu T., Galati C., Knutsen A.L., *On the locus of Prym curves where the Prym-canonical map is not an embedding*, Ark. Mat., No. 58 (2020), 71-85.

[7] Ciliberto C., Dedieu T., Galati C., Knutsen A.L., *Nonemptiness of Severi varieties on Enriques surfaces*, in preparation.

[8] Ciliberto C., Van der Geer G., *On the jacobian of a hyperplane section of a surface*, Lecture Notes in Math., Proceedings of the Trento Conference 1990, Vol. 1515 (1992), 33-40.

[9] Coble A., *The ten nodes of a rational sextic and of the Cayley symmetroid*, Amer. J. Math. 41 (1919), 1-27.

[10] Diaz S., Harris J., *Geometry of the Severi Variety*, Transactions of the American Mathematical Society 309, No.1 (1988), 1-34.

[11] Dolgachev I.V., *Classical algebraic geometry: a modern view*, Cambridge University Press, New edizione (2012).

[12] Dolgachev I.V., Zhang D., *Coble Rational surfaces*, Amer. J. Math. 123, No.1 (2001), 79-114.

[13] Epema D., *PhD thesis entitled "Surfaces with canonical hyperplane sections"*, Mathematisch Centrum, Amsterdam (1983).

[14] Epema D., *Surfaces with canonical hyperplane sections*, Nederl. Akad. Wetensch. Indag. Math. 45, No. 2 (1983), 173-184.

[15] Fuentes L., Pedreira M., *The projective theory of ruled surfaces*, Note Mat. 24, No. 1 (2005), 25-63.

[16] Fulton W. *Intersection Theory*, Springer Verlag, Berlin (1998).

[17] Garcia A., Lax F., *Rational Nodal Curves with no Smooth Weierstrass Points*, Proceedings of the American Mathematical Society, Vol. 124, No. 2 (1996), 407-413.

[18] Green M., Lazarsfeld R., *On the projective normality of complete linear series on an algebraic curve*, Inventiones mathematicae, No. 83 (1986), 73-90.

[19] Griffiths P., Harris J., *Principles of algebraic geometry*, Wiley and Sons (1994).

[20] Hartshorne R., *Algebraic Geometry*, Springer Verlag (2010).

[21] Horowitz T., *Varieties of low* $\Delta-$*genus*, Duke Math. J., Vol. 50, No. 3 (1983), 667-683.

[22] Kawamata Y., *A generalization of Kodaira-Ramanujam's vanishing Theorem*, Mathematische Annalen 261 (1982), 43-46.

[23] Lange H., Sernesi E., *Quadrics containing a Prym-canonical curve*, J. Algebraic Geometry 5 (1996), 387-399.

[24] Manetti M., *Degenerations of Algebraic Surfaces and applications to Moduli problems*, Tesi di Perfezionamento alla Scuola Normale Superiore di Pisa (1996).

[25] Mumford D., *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math. de l'I.H.E.S., Vol. 9 (1961).

[26] M. Nagata, *On rational surfaces. I. Irreducible curves of arithmetic genus 0 or 1*, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. Vol. 32, No. 3 (1960), 351-370.

[27] Reider I., *Vector Bundles of Rank* 2 *and Linear Systems on Algebraic Surfaces*, Annals of Mathematics, Vol. 127, No. 2 (1988), 309-316.

[28] Saint Donat B., *Projective models of* $K3$ *surfaces*, Amer. J. of Math. No. 96 (1974), 602-639.

[29] The Stacks Project, *https://stacks.math.columbia.edu/tag/01CF.*

[30] Tomaru T., *On Gorenstein surface singularities with fundamental genus* $p_f \geq 2$ *which satisfy some minimality conditions*, Pacific J. Math. Vol. 170, No. 1 (1995), 271-295.

[31] Umezu Y., *On normal projective surfaces with trivial dualizing sheaf*, Tokyo J. Math. Vol. 4, No. 2 (1981).

[32] Viehweg E., *Vanishing Theorems*, Journal für die reine und angewandte Mathematik, Vol. 335 (1982), 1-8.

[33] Zariski O., *Complete Linear Systems on Normal Varieties and a Generalization of a Lemma of Enriques-Severi*, Annals of Mathematics, Second Series, Vol. 55, No. 3 (1952), 552-592.