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Qualitative properties of solutions to some semilinear and quasilinear elliptic PDEs

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Qualitative properties of solutions to some semilinear and quasilinear elliptic PDEs

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Abstract. We study qualitative properties of solutions of some semilinear and quasilinear elliptic equations. Symmetry and monotonicity remain an important topic in modern theory of nonlinear partial differential equations. The moving planes method is one of the most important technique that have been used in recent years to establish some qualitative properties of positive solutions of nonlinear elliptic equations as symmetry and monotonicity; in particular, it goes back to the seminal papers of A. D. Alexandrov [1] and J. Serrin [111]. In this technique maximum and comparison principles play a crucial role. Lots of well-known results about classical and more recent version of maximum and comparison principles and of the Höpf's Lemma will be presented in Chapter 1. In Chapter 2 (see also [24]), we consider positive solutions to semilinear elliptic problems with singular nonlinearity and we provide a Höpf type boundary lemma via a suitable scaling argument that allows to deal with the lack of regularity of the solutions up to the boundary. In Chapter 3 (see also [52]), we consider the quasilinear version of the problem studied in Chapter 2, that is more involved, due to the nonlinear nature of the operator. In Chapter 4 (see also [50]), we consider positive singular solutions to semilinear elliptic problems with possibly singular nonlinearity and we deduce symmetry and monotonicity properties of the solutions via the moving planes procedure in bounded or unbounded domains. In Chapter 5 (see also [51]), we consider singular solutions to quasilinear elliptic equations under zero Dirichlet boundary condition. Under suitable assumptions on the nonlinearity we deduce symmetry and monotonicity properties of positive solutions via an improved moving planes procedure, only in bounded domains. In Chapter 6 (see also [48]), we investigate qualitative properties of positive singular solutions of some elliptic systems in bounded or unbounded domains, i.e. we prove a generalization of the results obtained in Chapter 4. In Chapter 7 (see also [49]), we prove the validity of Gibbons' conjecture for a quasilinear elliptic equation. The result holds for every $(2N+2)/(N+2) < p < 2$ and for a very general class of nonlinearity f.

Riassunto. Il metodo dello spostamento degli iperpiani di A.D. Alexandrov e J.B. Serrin è lo strumento più importante utilizzato per studiare le proprietà qualitative di soluzioni di equazioni alle derivate parziali (EDP) di tipo ellittico non lineari, come simmetria e monotonia. Il Capitolo 1 tratta i principi del massimo, i principi di confronto e il lemma di Höpf che svolgono un ruolo cruciale nel metodo del moving planes. In questo capitolo, inoltre, è presente anche lo stato dell'arte nei problemi semilineari e quasilineari. Nel Capitolo 2 consideriamo le soluzioni positive di EDP ellittiche semilineari con non linearità singolari. In questo contesto, usando un argomento di "riscalamento", dimostriamo un nuovo lemma di Höpf al bordo, eludendo la perdita di regolarità di soluzioni vicine al bordo. Il Capitolo 3 tratta della versione quasilineare del problema studiato nel Capitolo 2. Dopo aver ottenuto un lemma di Höpf per questo tipo di equazione, dimostreremo la simmetria e la monotonia delle soluzioni positive nel semispazio e nei domini limitati e convessi. Nel Capitolo 4, utilizzando il metodo del moving planes, dimostriamo la simmetria e la monotonia di soluzioni positive di EDP semilineari (possibilmente singolari) in domini limitati e illimitati. Il caso quasilineare, che è molto di più delicato e tecnico, è trattato nel Capitolo 5. Il capitolo 6 è dedicato allo studio delle proprietà qualitative di soluzioni positive e singolari di alcuni sistemi ellittici cooperativi. Verrà dimostrato che i risultati ottenuti nel Capitolo 4 restano veri in questo contesto. Nell'ultimo capitolo (Capitolo 7) dimostriamo la versione quasilineare della congettura di Gibbons.

Contents

Introduction

We study qualitative properties of some semilinear and quasilinear elliptic equations. In particular we deal with weak solutions to

$$-\Delta_p u = f(u) \qquad \text{in } \Omega, \tag{0.0.1}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geq 2$. If $u \in C^2(\Omega)$, we define the **p-Laplace operator** as follows:

$$\begin{aligned}\Delta_p u &= \operatorname{div}(|\nabla u|^{p-2}\nabla u) \\ &= |\nabla u|^{p-4}\left[|\nabla u|^2\Delta u + (p-2)\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\right],\end{aligned} \tag{0.0.2}$$

where (0.0.2) is defined in the set $\{x \in \Omega : \nabla u(x) \neq 0\}$ for every $1 < p < 2$, and in the whole domain Ω for every $2 \leq p < +\infty$. As we will see in the sequel, (0.0.2) can not be used in this form and hence p-Laplace operator has to be understood in the weak sense. The hypothesis on the nonlinearity will be always specified in all the chapters, but the reader could think that f is a locally Lipschitz continuous function. We have to remark that the p-Laplace operator reduces to the classical Laplacian when $p = 2$, i.e.

$$\Delta_2 u = \operatorname{div}(\nabla u) = \Delta u = \sum_{i=1}^{N}\frac{\partial^2 u}{\partial x_i^2}.$$

In this case, sometimes, we can consider classical solutions for equation (0.0.1). When $p \neq 2$ the situation is completely different and it is well known that, since the p-Laplace operator is singular or degenerate elliptic (respectively if $1 < p < 2$ or $p > 2$), solutions of (0.0.1) are generally of class $\mathcal{C}^{1,\alpha}$, with $\alpha < 1$ (see [**46, 122**]) and have to be considered only in the weak sense. More precisely, we say that $u \in W^{1,p}(\Omega)$ solves (0.0.1) if and only if

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx = \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{0.0.3}$$

We obtain (0.0.3) by applying divergence theorem to the following:

$$\int_\Omega -\operatorname{div}(|\nabla u|^{p-2}\nabla u)\varphi\, dx = \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Now we consider the following problem with Dirichlet boundary conditions:

$$\begin{cases} -\Delta_p u = f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{0.0.4}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, with $N \geq 2$ as before, and f is assumed to be locally Lipschitz continuous. A solution u to (0.0.4)

can be defined e.g. assuming that $u \in W_0^{1,p}(\Omega)$ in the weak distributional sense. This is also the space where it is natural to prove the existence of the solutions under suitable assumptions.
Now it is important to observe that, in the weak formulation (0.0.3), the test function φ belongs to $C_c^\infty(\Omega)$, but by density arguments it is possible to show that also $\varphi \in W_0^{1,p}(\Omega)$ it is enough. In fact by the definition of $W_0^{1,p}(\Omega)$, for every $\varphi \in W_0^{1,p}(\Omega)$ there exists $\{\varphi_n\} \in C_c^\infty(\Omega)$ such that

$$\varphi_n \longrightarrow \varphi \quad \text{in} \quad W_0^{1,p}(\Omega),$$

as $n \to +\infty$. Hence taking φ_n as test functions in (0.0.3), for every n, we have

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi_n)\, dx = \int_\Omega f(u)\varphi_n\, dx \qquad \forall \varphi_n \in C_c^\infty(\Omega).$$

We want to show

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi_n)\, dx \longrightarrow \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx,$$

as n goes to $+\infty$. Subtracting the left hand side of (0.0.3) and the left hand side of (0.0.3) with φ_n as test function, we obtain

$$\begin{aligned}
&\int_\Omega \left||\nabla u|^{p-2}(\nabla u, \nabla(\varphi_n - \varphi))\right|\, dx \\
&\leq \int_\Omega |\nabla u|^{p-1}|\nabla(\varphi_n - \varphi)|\, dx \\
&\leq \left(\int_\Omega |\nabla u|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla(\varphi_n - \varphi)|^p\right)^{\frac{1}{p}} \overset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

where in the last line we used Hölder inequality and the Dominated Convergence Theorem.
Let us denote $\varrho := |\nabla u|^{p-2}$. In the set of critical points

$$\mathcal{Z}_u := \{x \in \Omega \mid \nabla u(x) = 0\}, \tag{0.0.5}$$

the equation is degenerate for $p > 2$ (i.e. $\varrho \equiv 0$) and singular for $1 < p < 2$ (i.e. $\varrho \equiv +\infty$). If $0 < \varrho(x) \leq C$ all the classical results hold (see e.g [70]), hence $u \in C^2(\Omega \setminus \mathcal{Z}_u)$. If $\varrho \approx 0$ or $\varrho \approx +\infty$ classical results are not true. In particular, in Chapter 1, we will prove the following result on the regularity of the second derivatives of solutions to problem (0.0.4):

Proposition 0.0.1 ([37]). *Let $u \in C^1(\overline{\Omega})$, $u > 0$, be a weak solution to problem (0.0.4). Assume that f is locally Lipschitz continuous. Assume that Ω is a bounded and smooth domain of $\mathbb{R}^N$. Then*

$$\int_{\Omega\setminus\{u_i=0\}} \frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{|\nabla u_i|^2}{|u_i|^\beta}\, dx \leq C, \tag{0.0.6}$$

where $0 \leq \beta < 1$, $\gamma < N-2$ ($\gamma = 0$ if $N = 2$), $1 < p < +\infty$ and the positive constant C does not depend on y. In particular, we have

$$\int_{\Omega\setminus\{\nabla u=0\}} \frac{|\nabla u|^{p-2-\beta}\|D^2 u\|^2}{|y-x|^\gamma}\, dx \leq \tilde{C}, \tag{0.0.7}$$

for a positive constant $\tilde{C}$ not depending on y.

Thanks to the previous result, it is possible to show the following summability property of $|\nabla u|$, whose proof can be found in Chapter 1:

THEOREM 0.0.2 ([37]). *Let $u \in C^1(\overline{\Omega})$, $u > 0$, be a weak solution of (0.0.4) and assume, furthermore, that $f(s) > 0$ for any $s > 0$. Then, there exists a positive constant C, independent of y, such that*

$$\int_\Omega \frac{1}{|\nabla u|^{(p-1)r}} \frac{1}{|x-y|^\gamma} \, dx \leq C \tag{0.0.8}$$

where $0 < r < 1$ and $\gamma < N - 2$ for $N \geq 3$ ($\gamma = 0$ if $N = 2$). In particular the critical set $\mathcal{Z}_u$ has zero Lebesgue measure.

We want to remark that, due to the nonlinear nature of the p-Laplace operator, several results obtained in this manuscript will be also presented and showed for $p = 2$ since the techniques used in this case are completely different from the ones used when $p \neq 2$.

0.1. Qualitative properties of solutions and the moving planes method

Qualitative properties of solutions to elliptic equations can be interpreted, in an extremely broad sense, to include every property of solutions. In this section we are going to focus on geometric properties of the solutions. Boundary conditions play an important role in the qualitative behaviour of solutions. Qualitative properties of solutions are closely related to the existence of solution to elliptic PDEs; in fact, it seems obvious that existence of solutions provides the basis for the study of qualitative properties. On the other hand, searching for solutions with particular properties could provide clues for existence. Systematic studies of qualitative properties of solutions to general nonlinear elliptic equations or systems go back to the 19th century. It should be noted, however, that earlier works in this direction on linear elliptic equations, such as symmetrization or nodal properties of eigenfunctions, have had their consequences in nonlinear equations. Symmetry and monotoncity remain an important topic in modern theory of nonlinear partial differential equations.

The moving planes method is the most important technique that have been used in recent years to establish some qualitative properties of positive solutions of nonlinear elliptic equations like symmetry and monotonicity. For instance, it is used to prove monotonicity in, say, the x_1-direction of scalar solutions of nonlinear second order elliptic equations in domains Ω in $\mathbb{R}^N$. The essential ingredient is the maximum principle, that in the semilinear case it is equivalent to the comparison principle. This method compares values of the solution of the equation at two different points.

The moving planes method goes back to A. D. Alexandrov [1], in his study of mean surfaces of constant curvature, and to J. Serrin [111] that introduced the technique in the context of elliptic PDEs, in the study of overdetermined problems. After some years, B. Gidas, W. N. Ni and L. Nirenberg, in [68], adapted this method to prove monotonicity of positive solutions vanishing on $\partial\Omega$ and, as a corollary, symmetry; in [69] the authors extended these techniques to equations in all $\mathbb{R}^N$. We refer also the reader to some other relevant papers [9, 8, 10, 12, 21, 22, 33, 35, 36, 59, 60, 61, 110, 112, 113]. In all of these papers the maximum principle plays, as we said, the crucial role, but the papers had to rely on many forms of the maximum principle. These included also the Höpf's Lemma at the boundary. The classical version of maximum and comparison principles and of the Höpf's Lemma will be presented in Chapter 1.

Now, we want to use the moving planes method in order to state the typical results that it is possible to show with this technique, in a very simple framework; to do this, let us consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{0.1.1}$$

where Ω is a bounded Lipschitz domain of $\mathbb{R}^N$, with $N \geq 2$ and f is a locally Lipschitz continuous function.

We need to fix some notations. For a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\} \tag{0.1.2}$$

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_n) \tag{0.1.3}$$

which is the reflection through the hyperplane

$$T_\lambda := \{x_1 = \lambda\}. \tag{0.1.4}$$

Also let

$$a = \inf_{x \in \Omega} x_1. \tag{0.1.5}$$

Finally we set

$$u_\lambda(x) = u(x_\lambda)\,. \tag{0.1.6}$$

We observe that, since problem (0.1.1) is invariant up to translations and rotations, u_λ defined in (0.1.6) is also a solution to (0.1.1).

Let us now state the main result

THEOREM 0.1.1 ([12, 68]). *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to (0.1.1). Assume that f is a locally Lipschitz contiunous function and that Ω is convex in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Then it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$, with*

$$u_{x_1} > 0 \qquad \text{in} \quad \Omega \cap \{x_1 < 0\}\,.$$

PROOF. Let us define

$$\Lambda_0 = \{a < \lambda < 0 : u \leq u_t \ \text{ in } \ \Omega_t \ \text{ for all } t \in (a, \lambda]\}.$$

The aim of the moving planes procedure is to show that $\Lambda_0 \neq \emptyset$ and $\sup \Lambda_0 = 0$; once we have this, we obtain automatically monotonicity for the solution u and after that, performing the moving planes in the opposite direction, we obtain also the symmetry for u. To start with the moving planes method, we have to prove that $\Lambda_0 \neq \emptyset$.

Step 1: Take $a < \lambda < a + \sigma$ with $\sigma > 0$ small. In particular, we first assume that $\sigma > 0$ is sufficiently small so that $|\Omega_\lambda| < \delta$, for some small

$\delta > 0$. Noticing that $u \equiv u_\lambda$ on T_λ and $u \le u_\lambda$ on $\partial\Omega_\lambda \setminus T_\lambda$ by the Dirichlet datum, i.e. $u \le u_\lambda$ on $\partial\Omega_\lambda$, it follows, by the weak comparison principle in small domains (see Theorem 1.2.1), that $u \le u_\lambda$ in Ω_λ, for $|\Omega_\lambda| < \delta$ with $\delta = \delta(\|u\|_\infty)$, hence $\Lambda_0 \ne \emptyset$ (see Figure 1).

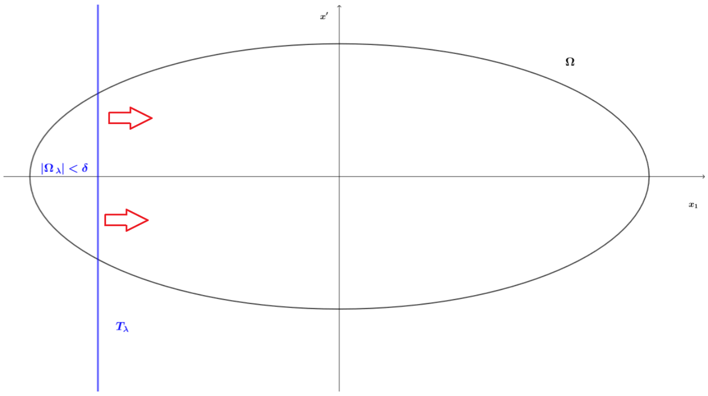

FIGURE 1. Step 1 in the moving planes method.

Step 2: Now we can set
$$\lambda_0 = \sup \Lambda_0.$$
As remarked above, to prove our result we have to show that $\lambda_0 = 0$. To do this we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \le u_{\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu}$ for any $0 < \nu < \bar{\nu}$ for some small $\bar{\nu} > 0$. By continuity we know that $u \le u_{\lambda_0}$ in Ω_{λ_0}. By the strong comparison principle, noticing that $u < u_{\lambda_0}$ on $\partial\Omega_{\lambda_0}$, we deduce that $u < u_{\lambda_0}$ in Ω_{λ_0}. Therefore, given a compact set $K \subset \Omega_{\lambda_0}$, by uniform continuity we can ensure that $u < u_{\lambda_0+\nu}$ in K for any $0 < \nu < \bar{\nu}$ for $\bar{\nu} > 0$ small. So by construction it results that $u \le u_{\lambda_0+\nu}$ on $\partial(\Omega_{\lambda_0+\nu} \setminus K)$ for any $0 < \nu < \bar{\nu}$ for $\bar{\nu} > 0$ small. For K large and $\bar{\nu}$ small by the weak comparison principle in small domains (see Theorem 1.2.1) we have $|\Omega_{\lambda_0+\nu} \setminus K|$ is small enough and therefore $u \le u_{\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu} \setminus K$ and so $u \le u_{\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu}$. But this is a contradiction with the definition of λ_0. Then $\lambda_0 = 0$ (see Figure 2).
Step 3: Since the moving planes procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ is implicit in the moving planes procedure. This provides $u_{x_1} \ge 0$ in $\{x_1 \ge 0\}$. Then $u_{x_1} > 0$ by the strong maximum principle.

□

As a consequence we have:

COROLLARY 0.1.2 ([12]). *Under the assumption of Theorem 0.1.1 if* $\Omega = B_R(0)$ *for any* $R > 0$, *then* u *is radially symmetric and monotone decreasing about the origin.*

We presented the classical version of the moving planes method for semilinear elliptic equation. As we said before, in the case $p = 2$ several results have been obtained starting by the celebrated paper of B. Gidas, W. N. Ni and L. Nirenberg [68]. This paper had a big impact not only in virtue of

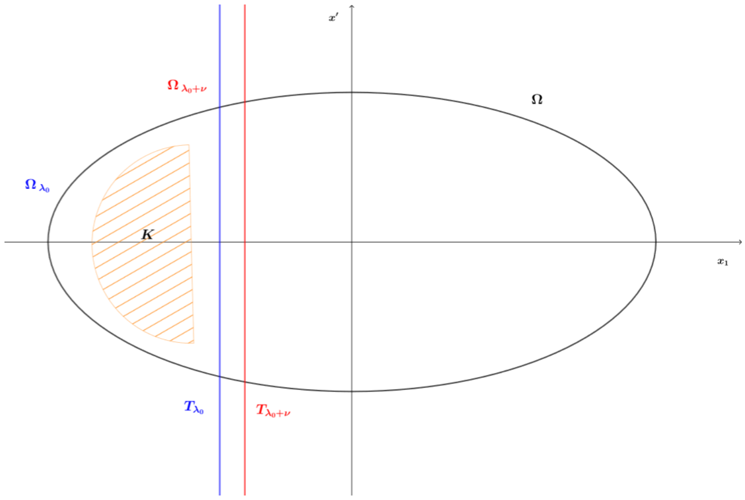

FIGURE 2. Step 2 in the moving planes method.

the several monotonicity and symmetry results that it contains, but also because it brought to attention the moving planes method which, since then, has been largely used in many different problems.

The situation is completely different when $p \neq 2$ and there are less results about monotonicity and symmetry of solutions to quasilinear elliptic problem. Let us consider

$$\begin{cases} -\Delta_p u = f(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{0.1.7}$$

where Ω is a bounded Lipschitz domain of $\mathbb{R}^N$, with $N \geq 2$, $1 < p < +\infty$ and f is a locally Lipschitz continuous function.

In this case, as remarked before, the solutions can only be considered in a weak sense. Anyway, this is not a difficulty because the moving planes method can be adapted to weak solutions of strictly elliptic problems in divergence form (see [**32, 38**]). The real difficulty with problem (0.1.7), is that the p-Laplacian operator is degenerate in the critical points of the solutions, so that comparison principles, which could substitute the maximum principles in order to use the moving planes method when the operator is not linear, are not available in the same form as for $p = 2$. Actually, counterexamples both to the validity of comparison principles and to symmetry results are available (see [**73**] and [**18**]).

A first step towards extending the moving planes method to solutions of problems involving the p-Laplacian operator has been done by L. Damascelli in [**33**]. In this paper the author mainly proves some weak and strong comparison principles for solutions of differential inequalities involving the p-Laplacian. Using these principles he adapts the moving planes method to solutions of (0.1.7) getting some monotonicity and symmetry results in the case $1 < p < 2$. Although the comparison principles of [**33**] are quite powerful in this situation, the symmetry result is not complete and relies on the assumption that the set of the critical points of u does not disconnect the caps which are constructed by the moving plane method. This extra assumption on the critical set, for $1 < p < 2$, was removed in [35]. Hence, when $p > 2$, the results contained in [33] are not enough in order to adapt the moving planes method. Some years later, L. Damascelli and B. Sciunzi in [37, 36] proved general versions of the weak comparison principle (see Theorem 1.2.5) and of the strong comparison principle (see Theorem 1.5.3) for solution to (0.1.7), that it was sufficient to extend the technique to every p.

The analogous result of Theorem 0.1.1, in the quasilinear setting, is given by the following:

THEOREM 0.1.3 ([36]). *Let $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a weak solution to (0.1.7), with $1 < p < +\infty$. Assume that f is a locally Lipschitz continuous function such that $f(s) > 0$ for $s > 0$ and that Ω is convex in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Then it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$, with*

$$u_{x_1} > 0 \qquad \text{in} \quad \Omega \cap \{x_1 < 0\}\,.$$

Moreover, if $\Omega = B_R(0)$, then u is radially symmetric and radially decreasing.

The proof of Theorem 0.1.3 is similar to the semilinear case (see Theorem 0.1.1), but the classical maximum principles, that it is equivalent to the comparison principle in the semilinear case, are replaced by comparison principles by L. Damascelli and B. Sciunzi [36, 37] (see Theorem 1.2.5 and 1.5.3) and the classical Poincaré inequality is replaced by a weighted Poincaré inequality (see Theorem 1.1.4).

The moving planes method is a very powerful technique that also can be adapted for quasilinear elliptic equations in unbounded domains. In the case of unbounded domains the main examples, arising from many applications, are provided by the whole space $\mathbb{R}^N$ and by the half-space $\mathbb{R}^N_+$. For the case of the whole space with $p = 2$, where radial symmetry of the solutions is expected, we refer to [21, 68, 69]. We refer the readers to [8, 9, 10, 34, 38, 40, 56] for results concerning monotonicity of the solutions in half-spaces, in the non-degenerate case.

The case of p-Laplace equations in unbounded domains is really harder to study. Let us only say that, the use of weighted Sobolev spaces is necessary in the case $p > 2$ and it requires the use of a weighted Poincaré type inequality with weight $\varrho = |\nabla u|^{p-2}$ (see Section 1.1). The latter involves constants that may blow up when the solution approaches zero that may happen also for positive solutions in unbounded domains. Hence, the lack of compactness plays an important role. Moreover, we have to remark that for the p-Laplace operator the Kelvin transform does not work (see [87]).

When considering the case of the half-space $\mathbb{R}^N_+$, the application of the moving planes technique is much more delicate, since weak comparison principles in small domains have to be substituted by weak comparison principles in narrow unbounded domains (see Theorem 1.3.1, Theorem 1.3.3 and Theorem 1.3.4). Also the strong comparison principle does not apply in a simple way as in the case when bounded domains are considered. In the semilinear case $p = 2$ many arguments exploited in the literature are very much related to the linear and nondegenerate nature of the operator, so that it is not possible to extend these arguments to the case of equations involving nonlinear degenerate operators.

Considering the p-Laplace operator and problems in half-spaces, first results have been obtained in the singular case $1 < p < 2$ in [58, 59], where positive locally Lipschitz continuous nonlinearities are considered. A partial answer in the more difficult degenerate case $p > 2$ was obtained in [61], where *power-like nonlinearities* are considered under the restriction $2 < p < 3$. Some years later, the restriction $2 < p < 3$ was removed in [60] and, moreover, the authors considered a larger class of nonlinearities (in particular positive nonlinearities that are superlinear at zero).

In the case of the entire space $\mathbb{R}^N$, for $p = 2$, the application of the moving planes method is quite involved, since it is needed the behaviour of the solution at infinity. In [21], L. Caffarelli, B. Gidas and J. Spruck proved, thanks to the moving planes method and to the use of Kelvin transform, that any positive solution of (0.1.1) with $f(t) \approx t^{\frac{N+2}{N-2}}$, is radially symmetric and monotone decreasing about some point of $\mathbb{R}^N$. We refer also to the seminal paper of B. Gidas, W. M. Ni and L. Nirenberg [69] for results concerning symmetry and monotonicity of solutions in $\mathbb{R}^N$, but with extra-assumption on the behaviour of solutions at infinity.

The situation for $p \neq 2$ and $\Omega = \mathbb{R}^N$ is much more complicated; the operator is not linear and, as before, one needs of comparison principle in unbounded domains (that are not equivalent to maximum principle). A first result regarding qualitative properties of solutions for quasilinear elliptic equations in the entire space is due to J. Serrin and H. Zou [112]. In this paper the authors need of an extra assumption on the decay of the solution at infinity and on the critical set. The nonlinear version of the result obtained by L. Caffarelli, B. Gidas and J. Spruck in [21], i.e. when $f(t) \approx t^{\frac{N(p-1)+p}{N-p}}$, was not so easy to obtain since the Kelvin transform for $p \neq 2$ does not work (see e.g. [87]) and also because it is not possible to start with the moving planes procedure without any a priori assumption on the decay of the solutions at infinity. This problem was solved by B. Sciunzi in [107]; the argument is based on some a priori estimates proved by J. Vetois [128], on a lower bound for the decay rate of $|\nabla u|$, the moving planes technique, Hardy's inequality and a weighted Poincaré-type inequality.

To the best of our knowledge all the symmetry results presented in this section for equations involving the p-Laplace operator in $\mathbb{R}^N$ or in $\mathbb{R}^N_+$, with $p \neq 2$, treated the case of positive nonlinearity. In Chapter 7 it will be purposed a nice variant of the moving planes method that works for a special class of changing sign nonlinearities and will be very helpful in the solution of the quasilinear version of Gibbons' conjecture for $(2N+2)/(N+2) < p < 2$.

0.2. Höpf's boundary lemma for singular elliptic equations

Starting from the seminal paper [31], singular semilinear elliptic equations have been studied from many point of view. We just quoted here the papers [4, 16, 17, 25, 26, 27, 28, 67, 75, 79, 82, 83, 95, 116] which are somehow related to the results contained in this thesis. A crucial topic in the study of singular semilinear elliptic equations is the study of the behaviour of the solutions near the boundary, namely where the solutions actually exhibit a lack of regularity. In particular, the fact that solutions are not $\mathcal{C}^1$ up to the boundary prevents the validity of the Höpf boundary lemma, see [15, 76, 103]. We address this issue and provide a generalized version of the Höpf boundary lemma, in Chapter 2 (see also [24]) for semilinear singular elliptic equations. In particular let us consider the following problem:

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} + f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2.1}$$

where $\gamma > 1$, Ω is a $\mathcal{C}^{2,\alpha}$ bounded domain of $\mathbb{R}^N$ with $0 < \alpha < 1$, $N \geq 1$ and $f : \Omega \to \mathbb{R}$ locally Lipschitz continuous.

It is well known that generally solutions to problem (0.2.1) are not smooth up to the boundary. It was in fact proved in [82] that solutions are not in $H^1_0(\Omega)$ at least when $\gamma > 3$. Therefore, having in mind the natural regularity behaviour of the solutions (see [31]) we let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The equation is well defined in the interior of the domain in the classical sense and its weak distributional formulation is

$$\int_\Omega (\nabla u, \nabla\varphi)\, dx = \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega). \tag{0.2.2}$$

Now, let us define the notion of inward pointing normal

Definition 0.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $\mathcal{C}^{2,\alpha}$ domain. Let $I_\delta(\partial\Omega)$ be a neighborhood of $\partial\Omega$ with the unique nearest point property* (see e.g. [66]). *Hence for every $x \in I_\delta(\partial\Omega)$ there exists a unique point $\hat{x} \in \partial\Omega$ such that $|x - \hat{x}| = dist(x, \partial\Omega)$. We define the* inward-pointing normal *as*

$$\eta(x) := \frac{x - \hat{x}}{|x - \hat{x}|}. \tag{0.2.3}$$

Having in mind these notations, we are now ready to state the main result of Chapter 2 (see also [24]):

Theorem 0.2.2 (Höpf type boundary lemma, [24]). *Let $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution of problem* (0.2.1). *Then there exists a neighborhood $I_\delta(\partial\Omega)$ of $\partial\Omega$ such that*

$$\partial_{\nu(x)} u > 0 \qquad \forall\, x \in I_\delta(\partial\Omega) \tag{0.2.4}$$

provided that $(\nu(x), \eta(x)) > 0$ uniformly with respect to $x \in I_\delta(\partial\Omega)$, namely provided that $(\nu(x), \eta(x)) \geq \beta > 0$ for some $\beta > 0$ for every $x \in I_\delta(\partial\Omega)$.

The proof of this result is based on a scaling argument near the boundary, which leads to the study of a limiting problem in the half-space (see problem (2.0.5)) and obeys to suitable a priori estimates. Moreover, for this limiting problem, we provide a classification result that is crucial for our technique, and may also have an independent interest (see Theorem 2.0.3).

The technique of E. Höpf [76] (see also [70]) has been already developed and improved also in the quasilinear setting. We refer the readers to [103] and Chapter 1 (see also [127]).

It was a natural question to understand if it holds an analogous result to Theorem 0.2.3 for problem (0.2.1) in the quasilinear setting. Hence, let us consider:

$$
\begin{cases} -\Delta_p u = \dfrac{1}{u^\gamma} + f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2.5}
$$

where $p > 1$, $\gamma > 1$, Ω is a $\mathcal{C}^{2,\alpha}$ bounded domain of $\mathbb{R}^N$ with $N \geq 1$ and $f : \Omega \to \mathbb{R}$ locally Lipschitz continuous.

Since the p-Laplace operator is degenerate or singular, a solution $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ of problem (0.2.5) has to be understood in the weak sense:

$$
\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx = \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{0.2.6}
$$

In collaboration with B. Sciunzi (see [52]), we obtained the following:

Theorem 0.2.3 (Höpf type boundary lemma, [52]). *Let $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution to (0.2.5). Then, for any $\beta > 0$, there exists a neighborhood $I_\delta(\partial\Omega)$ of $\partial\Omega$, such that*

$$
\partial_{\nu(x)} u > 0 \qquad \forall\, x \in I_\delta(\partial\Omega) \tag{0.2.7}
$$

whenever $\nu(x) \in \mathbb{R}^N$ with $\|\nu(x)\| = 1$ and $(\nu(x), \eta(x)) \geq \beta$.

The proof of this result will be presented in Chapter 3. Nevertheless the proof of Theorem 0.2.3, namely the proof of the Höpf lemma in the case when it appears the singular term $u^{-\gamma}$, cannot be carried out in the standard way mainly because the solutions are not of class $\mathcal{C}^1$ up to the boundary. More precisely the proofs in [70, 76, 103, 127] has the common feature of basing on the comparison of the solution with subsolutions that have a known behaviour on the boundary. This approach, with some difficulty to take into account, can be exploited also in the singular case since $t^{-\gamma}$ has the right monotonicity behaviour. This actually leads to control the behaviour of the solution near the boundary with a comparison based on the distance function. This is, in fact, also behind Theorem 3.3.2 that gives a Lazer and Mckenna type result [82]. Although some of the underlying ideas in our approach have a common flavour with the ones exploited in [24] and in Chapter 2, the proofs that we exploit in Chapter 3 are new and adapted to the degenerate nonlinear nature of the p-laplacian.

0.3. Qualitative properties of singular solutions to some elliptic problems

In Chapter 4 we study of the following singular semilinear elliptic problem:

$$(0.3.1)\qquad \begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega\setminus\Gamma \\ u > 0 & \text{in} \quad \Omega\setminus\Gamma \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$ which is convex in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. The solution has a possible singularity on the critical set $\Gamma \subset \Omega$ and thus is understood in the following sense: $u \in H^1_{loc}(\overline{\Omega}\setminus\Gamma) \cap C(\overline{\Omega}\setminus\Gamma)$ and

$$(0.3.2)\qquad \int_\Omega (\nabla u, \nabla\varphi)\, dx = \int_\Omega f(x,u)\varphi\, dx \qquad \forall\varphi \in C^1_c(\Omega\setminus\Gamma)\,.$$

The source term $f(x,u)$ is assumed to satisfy

(I_f) We say that f fulfills the condition (I_f) if $f : \overline{\Omega}\setminus\Gamma\times(0,+\infty) \to \mathbb{R}$ is a continuous function such that for $0 < t \leq s \leq M$ and for any compact set $K \subset \overline{\Omega}\setminus\Gamma$, it holds

$$f(x,s) - f(x,t) \leq C(K,M)(s-t) \qquad \text{for any} \quad x \in K\,,$$

where $C(K,M)$ is a positive constant depending on K and M. Furthermore $f(\cdot,s)$ is non-decreasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$ and symmetric with respect to the hyperplane $\{x_1 = 0\}$.

In particular, this allows us to consider equations involving Hardy-Leray type potentials, see [122].

Now we state the first main result of Chapter 4:

Theorem 0.3.1 ([50]). *Let Ω be a convex domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and let $u \in H^1_{loc}(\Omega\setminus\Gamma) \cap C(\overline{\Omega}\setminus\Gamma)$ be a solution to (0.3.1). Assume that f fulfills (I_f). Assume also that Γ is a point if $N = 2$ while Γ is closed and such that*

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0,$$

if $N \geq 3$. Then, if $\Gamma \subset \{x_1 = 0\}$, it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$. Furthermore

$$u_{x_1} > 0 \qquad \text{in} \quad \Omega \cap \{x_1 < 0\}\,.$$

We want to remark that in the work of B. Sciunzi [110], the author has been considered the singular set Γ contained in a smooth $(N-2)$-dimensional sub-manifold of the hyperplane $\{x_1 = 0\}$ if $N > 2$, while it is a point in dimension two. With the same technique, developed in [110], more general problems could be considered, e.g. cases when the critical set has zero capacity. It is also clear that, if Γ is not contained in any symmetry hyperplane of the domain, then with our technique it could be possible in any case to carry out the moving planes procedure until the hyperplane touch the critical set. This is optimal somehow, since it is implicit in the moving planes technique the monotonicity of the solution and solutions in general change their slope near the critical set. This actually shows that the moving planes procedure cannot go beyond the critical set. It is also worth emphasizing that Theorem 0.3.1 for problem (0.3.1) is still true if the Laplace operator Δu is replaced by $div(A(x)\nabla u)$ for some positive definite matrix $A(x) = A(x_2, \ldots, x_n)$ with bounded continuous coefficients. In this case all the proofs can be repeated verbatim and no Hölder's continuous regularity of the coefficients is required also in dimension two.

First results regarding the applicability of the moving planes procedure to the case of singular solutions go back to [22] (see also [121]) where the case when the singular set is a single point is considered. We follow and improve here the technique in [110], where the case of a smooth $(N-2)$-dimensional singular set was considered in the case of locally Lipschitz continuous nonlinearity. Let us mention that the technique introduced in [110] also works in the nonlocal context (see [91]).
On the other hand, in the case $\Gamma = \emptyset$, symmetry and monotonicity properties of solutions to semilinear elliptic problems involving singular nonlinearities, have been studied in [**25**, **26**]. Also in this direction our result is new and more general. In fact, while in [**25**, **26**] it is necessary to restrict the attention to problems of the form (0.2.1), here we only need to consider nonlinearities that are locally Lipschitz continuous from above. Actually, all the nonlinearities of the form

$$f(x,s) := a_1(x_1)f_1(s) + f_2(s)\,,$$

where f_1 is a decreasing continuous function in $(0,+\infty)$ and non-negative, $f_2(\cdot)$ is locally Lipschitz continuous in $[0,+\infty)$ and $a_1 \in C^0(\mathbb{R})$, a_1 is non-negative, even and non-decreasing for $x_1 < 0$, satisfy our assumptions.

The technique, as shown in [110], can be applied to study singular solutions to the following Sobolev critical equation in $\mathbb{R}^N, N \geq 3$,

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in} \quad \mathbb{R}^N \setminus \Gamma \\ u > 0 & \text{in} \quad \mathbb{R}^N \setminus \Gamma. \end{cases} \tag{0.3.3}$$

In [110] it was considered the case of a closed critical set Γ contained in a compact smooth submanifold of dimension $d \leq N-2$ and a summability property of the solution at infinity was imposed (see also [121] for the special case in which the singular set Γ is reduced to a single point). In Chapter 4 we remove both these restrictions and prove the following:

Theorem 0.3.2 ([50]). *Let $N \geq 3$ and let $u \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ be a solution to (0.3.3). Assume that the solution u has a non-removable*[1] *singularity in the singular set Γ, where Γ is a closed and proper subset of $\{x_1 = 0\}$ such that*

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0.$$

[1]Here we mean that the solution u does not admit a smooth extension all over the whole space. Namely it is not possible to find $\tilde{u} \in H^1_{loc}(\mathbb{R}^N)$ with $u \equiv \tilde{u}$ in $\mathbb{R}^N \setminus \Gamma$.

Then, u is symmetric with respect to the hyperplane $\{x_1 = 0\}$.
The same conclusion is true if the hyperplane $\{x_1 = 0\}$ is replaced by any affine hyperplane.

The results obtained in Chapter 4 in the semilinear case, in particular the once involving bounded domains can be extended in a non trivial way to the case of quasilinear elliptic equations; this is the main topic of Chapter 5. Now, let us consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \setminus \Gamma \\ u > 0 & \text{in } \Omega \setminus \Gamma \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.3.4}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and $p > 1$. The solution u has a possible singularity on the critical set Γ and in fact we shall only assume that u is of class C^1 far from the critical set. Therefore the equation is understood as in the following:

DEFINITION 0.3.3. *We say that $u \in C^1(\overline{\Omega} \setminus \Gamma)$ is a solution to (0.3.4) if $u = 0$ on $\partial\Omega$ and*

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx = \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^1_c(\Omega \setminus \Gamma)\,. \tag{0.3.5}$$

The purpose of Chapter 5 (see also [51]) is to investigate symmetry and monotonicity properties of the solutions when the domain is assumed to have symmetry properties. This issue is well understood in the semilinear case $p = 2$ as explained before.

The moving planes procedure for quasilinear elliptic problems, as remarked in Section 0.1, has been adapted when $\Gamma = \emptyset$. However, the techniques developed in Chapter 4 and described above cannot be applied straightforwardly manly for two reasons. First of all the technique developed in Chapter 4 (see also [50, 110]), that works the case $p = 2$, is strongly based on a homogeneity argument that fails for $p \neq 2$. Furthermore, since the gradient of the solution may blows up near the critical set, then the equation may exhibit both a degenerate and a singular nature at the same time. This causes, in particular, that it is no longer true that the case $1 < p < 2$ allows to get stronger results in a easier way, as it is in the case $\Gamma = \emptyset$.

Now we list all the assumptions on the singular set Γ and on the nonlinearity f in the different cases $1 < p < 2$ and $p > 2$:

(A^1_f). *For $1 < p < 2$ we assume that f is locally Lipschitz continuous so that, for any $0 \leq t, s \leq M$, there exists a positive constant $K_f = K_f(M)$ such that*

$$|f(s) - f(t)| \leq K_f |s - t|.$$

Moreover $f(s) > 0$ for $s > 0$ and

$$\lim_{t \to +\infty} \frac{f(t)}{t^q} = l \in (0, +\infty).$$

for some $q \in \mathbb{R}$ such that $p-1 < q < p^*-1$, where $p^* = Np/(N-p)$.

(A_f^2). *For $p \geq 2$ we only assume that f is locally Lipschitz continuous so that, for $0 \leq t, s \leq M$ there exists a positive constant $K_f = K_f(M)$ such that*

$$|f(s) - f(t)| \leq K_f |s - t|.$$

Furthermore $f(s) > 0$ for $s > 0$.

(A_Γ^1). *For $1 < p < 2$ and $N = 2$ we assume that $\Gamma = \{0\}$, while for $1 < p < 2$ and $N > 2$ we assume that $\Gamma \subseteq \mathcal{M}$ for some compact C^2 submanifold $\mathcal{M}$ of dimension $m \leq N - k$, with $k \geq \frac{N}{2}$.*

(A_Γ^2). *For $2 < p < N$ and $N \geq 2$, we assume that Γ closed and such that*

$$\mathrm{Cap}_p(\Gamma) = 0.$$

We prefer to start the presentation of our results, that we prove in Chapter 5, with the case $p > 2$. We have the following:

THEOREM 0.3.4 ([51]). *Let $p > 2$ and let $u \in C^1(\overline{\Omega} \setminus \Gamma)$ be a solution to (0.3.4) and assume that f is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$, namely assume (A_f^2). If Ω is convex and symmetric with respect to the x_1-direction, Γ is closed with $\mathrm{Cap}_p(\Gamma) = 0$, namely let us assume (A_Γ^2), and*

$$\Gamma \subset \{x \in \Omega \,:\, x_1 = 0\},$$

then it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.

Although the technique that we are going to develop in the proof of Theorem 0.3.4 works for any $p > 2$, the result is stated for $2 < p \leq N$ since there are no sets of zero p-capacity when $p > N$.

Surprisingly the case $1 < p < 2$ presents more difficulties related to the fact that, as already remarked, the operator may degenerate near the critical set even if $p < 2$. We will therefore need an accurate analysis on the behaviour of the gradient of the solution near Γ. We carry out such analysis exploiting the results of [100] (therefore we shall require a growth assumption on the nonlinearity) and a blow up argument. The result is the following:

THEOREM 0.3.5 ([51]). *Let $1 < p < 2$ and let $u \in C^1(\overline{\Omega} \setminus \Gamma)$ be a solution to (0.3.4) and assume that f is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$ and has subcritical growth, namely let us assume (A_f^1). Assume that Γ is closed and that $\Gamma = \{0\}$ for $N = 2$, while $\Gamma \subseteq \mathcal{M}$ for some compact C^2 submanifold $\mathcal{M}$ of dimension $m \leq N - k$, with $k \geq \frac{N}{2}$ for $N > 2$, see (A_Γ^1). Then, if Ω is convex and symmetric with respect to the x_1-direction and*

$$\Gamma \subset \{x \in \Omega \,:\, x_1 = 0\},$$

it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.

The aim of Chapter 6 is to generalize all the results obtained in Chapter 4 to the case of semilinear cooperative elliptic systems. In particular, we investigate symmetry and monotonicity properties of singular solutions to some semilinear elliptic systems in such a way to find a generalization of the results presented in Chapter 4. All the results presented here, about systems, are contained in Chapter 6 and in [48]; when [48] was completed, we learned that the case of bounded domains was also considered in [14] (see also [13]), obtaining similar results. In the first part we consider the following semilinear elliptic system

$$\begin{cases} -\Delta u_i = f_i(u_1,\dots,u_m) & \text{in} \quad \Omega\setminus\Gamma \\ u_i > 0 & \text{in} \quad \Omega\setminus\Gamma \\ u_i = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{0.3.6}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$ and $i = 1, ..., m$ ($m \geq 2$). The technique which is mostly used in this chapter is the moving planes method. For simplicity of exposition we assume directly in all the Chapter 6 that Ω is a convex domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$. The solution has a possible singularity on the critical set $\Gamma \subset \Omega$. When $m = 1$ system (6.0.2) reduces to a scalar equations that was already studied in [50, 110] and in Chapter 4. The moving planes procedure for semilinear elliptic system has been firstly adapted by Troy in [123] where he considered the cooperative system (0.3.6) with $\Gamma = \emptyset$ (see also [41, 42, 106]). This technique was also adapted in the case of cooperative semilinear systems in half spaces by Dancer in [39] and in the whole space by Busca and Sirakov in [19]. For the case of quasilinear elliptic system in bounded domains we suggest [92].

Moreover, motivated by [83], through all the chapter, we assume that the following hypotheses (denoted by (S_{f_i}) in the sequel) hold:

(S_{f_i}) (i) $f_i : \mathbb{R}^m_+ \to \mathbb{R}$ are assumed to be $\mathcal{C}^1$ functions for every $i = 1, ..., m$.

(ii) The functions f_i $(1 \leq i \leq m)$ are assumed to satisfy the monotonicity (also known as *cooperative*) conditions

$$\frac{\partial f_i}{\partial t_j}(t_1, ..., t_j, ..., t_m) \geq 0 \quad \text{for} \quad i \neq j,\ 1 \leq i, j \leq m.$$

Since we want to consider singular solutions, the natural assumption is

$$u_i \in H^1_{loc}(\Omega\setminus\Gamma) \cap C(\overline{\Omega}\setminus\Gamma) \quad \forall i = 1, ..., m$$

and thus the system is understood in the weak sense:

$$\int_\Omega (\nabla u_i, \nabla \varphi_i)\, dx = \int_\Omega f_i(u_1, u_2, ..., u_m)\varphi_i\, dx \qquad \forall \varphi_i \in C^1_c(\Omega\setminus\Gamma) \tag{0.3.7}$$

for every $i = 1, ..., m$.

Under the previous assumptions we can prove the following result:

Theorem 0.3.6 ([14, 48]). *Let Ω be a convex domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and let $(u_1, ..., u_m)$ be a solution to (0.3.6), where $u_i \in H^1_{loc}(\Omega\setminus\Gamma) \cap C(\overline{\Omega}\setminus\Gamma)$ for every $i = 1, ..., m$. Assume that each f_i fulfills (S_{f_i}). Assume also that Γ is a point if $N = 2$ while Γ is closed and such that*

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0,$$

if $N \geq 3$. Then, if $\Gamma \subset \{x_1 = 0\}$, it follows that u_i is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$, for every $i = 1, ..., m$. Furthermore

$$\partial_{x_1} u_i > 0 \qquad \text{in} \quad \Omega \cap \{x_1 < 0\},$$

for every $i = 1, ..., m$.

The technique developed in the case of bounded domains (see [50, 51, 110], and [91] for the nonlocal setting) is very powerful and can be adapted to some cooperative systems in $\mathbb{R}^N$ involving critical nonlinearity. Papers on existence or qualitative properties of solutions to systems with critical growth in $\mathbb{R}^N$ are very few, due to the lack of compactness given by the Talenti bubbles and the difficulties arising from the lack of good variational methods. We refer the reader to [19, 30, 71, 72, 74, 99] for this kind of systems. Our aim is to study qualitative properties of singular solutions to the following $m \times m$ system of equations

$$\begin{cases} -\Delta u_i = \displaystyle\sum_{j=1}^{m} a_{ij} u_j^{2^*-1} & \text{in} \quad \mathbb{R}^N \setminus \Gamma, \\ u_i > 0 & \text{in} \quad \mathbb{R}^N \setminus \Gamma, \end{cases} \tag{0.3.8}$$

where $i = 1, ..., m$, $m \geq 2$, $N \geq 3$ and the matrix $A := (a_{ij})_{i,j=1,...,m}$ is symmetric and such that

$$\sum_{j=1}^{m} a_{ij} = 1 \text{ for every } i = 1, ..., m. \tag{0.3.9}$$

These kind of systems, with $\Gamma = \emptyset$, was studied by Mitidieri in [89, 90] considering the case $m = 2$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and it is known in the literature as nonlinearity belonging to the *critical hyperbola*.

If $m = 1$, then (0.3.8) reduces to the classical critical Sobolev equation

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in} \quad \mathbb{R}^N \setminus \Gamma \\ u > 0 & \text{in} \quad \mathbb{R}^N \setminus \Gamma, \end{cases} \tag{0.3.10}$$

that can be found in [50, 110]. If Γ reduces to a single point we find the result contained in [121], while if $\Gamma = \emptyset$ then system (0.3.10) reduces to the classical Sobolev equation (see [21]). For existence results of radial and nonradial solutions for (0.3.8), we refer to some interesting papers [71, 72]. We want to remark that in [71, 72] the authors treat the general case of a matrix A in which its entries a_{ij} are not necessarily positive and this fact implies that it is not possible to apply the maximum principle. As remarked above the natural assumption is

$$u_i \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma) \quad \forall i = 1, ..., m$$

and, thus, the system is understood in the following sense:

$$\int_{\mathbb{R}^N} (\nabla u_i, \nabla \varphi_i)\, dx = \sum_{j=1}^{m} a_{ij} \int_{\mathbb{R}^N} u_j^{2^*-1} \varphi_i \, dx \qquad \forall \varphi_i \in C^1_c(\mathbb{R}^N \setminus \Gamma) \tag{0.3.11}$$

for every $i = 1, ..., m$.

What we are going to show in Chapter 6 is also the following result:

THEOREM 0.3.7 ([48]). *Let* $N \geq 3$ *and let* $(u_1, ..., u_m)$ *be a solution to* (0.3.8), *where* $u_i \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ *for every* $i = 1, ..., m$. *Assume that the matrix* $A = (a_{ij})_{i,j=1,...,m}$, *defined above, is symmetric,* $a_{ij} \geq 0$ *for every* $i, j = 1, ..., m$ *and it satisfies* (0.3.9). *Moreover at least one of* u_i *has a non-removable*[2] *singularity in the singular set* Γ, *where* Γ *is a closed and proper subset of* $\{x_1 = 0\}$ *such that*

$$\operatorname*{Cap_2}_{\mathbb{R}^N}(\Gamma) = 0.$$

Then, all u_i *are symmetric with respect to the hyperplane* $\{x_1 = 0\}$. *The same conclusion is true if* $\{x_1 = 0\}$ *is replaced by any affine hyperplane. If at least one of* u_i *has only a non-removable singularity at the origin for every* $i = 1, ..., m$, *then each* u_i *is radially symmetric about the origin and radially decreasing.*

Another interesting elliptic system involving Sobolev critical exponents is the following one:

$$\begin{cases} -\Delta u = u^{2^*-1} + \dfrac{\alpha}{2^*} u^{\alpha-1} v^{\beta} & \text{in } \mathbb{R}^N \setminus \Gamma \\ -\Delta v = v^{2^*-1} + \dfrac{\beta}{2^*} u^{\alpha} v^{\beta-1} & \text{in } \mathbb{R}^N \setminus \Gamma \\ u, v > 0 & \text{in } \mathbb{R}^N \setminus \Gamma, \end{cases} \tag{0.3.12}$$

where $\alpha, \beta > 1$, $\alpha + \beta = 2^* := \frac{2N}{N-2}$ $(N \geq 3)$

The solutions to (0.3.12) are solitary waves for a system of coupled Gross–Pitaevskii equations. This type of systems arises, e.g., in the Hartree–Fock theory for double condensates, that is, Bose-Einstein condensates of two different hyperfine states which overlap in space. Existence results for this kind of system are very complicated and the existence of nontrivial solutions is deeply related to the parameter α, β and N. This kind of systems (0.3.12) with $\Gamma = \emptyset$ was studied in [**2**, **5**, **6**, **99**, **115**, **118**]. In particular, in [**99**] the authors show a uniqueness result for least energy solutions under suitable assumptions on the parameters α, β and N, while, in [**30**] the authors study also the competitive setting, showing that the system admits infinitely many fully nontrivial solutions, which are not conformally equivalent. Motivated by their physical applications, weakly coupled elliptic systems have received much attention in recent years, and there are many results for the cubic case where $\Gamma = \emptyset$, $\alpha = \beta = 2$ and 2^* is replaced by 4 in low dimensions $N = 3, 4$, see e.g. [**2**, **5**, **6**, **84**, **85**, **117**, **118**]. Since our technique does not work when $1 < \alpha < 2$ or $1 < \beta < 2$, here we study the case $\alpha, \beta \geq 2$ and $N = 3$ or $N = 4$, since we are assuming that $\alpha + \beta = 2^*$.

The last results that it is going to be proved in Chapter 6 is given by the following:

[2] Here we mean that the solution $(u_1, ..., u_m)$ does not admit a smooth extension all over the whole space. Namely it is not possible to find $\tilde{u}_i \in H^1_{loc}(\mathbb{R}^N)$ with $u_i \equiv \tilde{u}_i$ in $\mathbb{R}^N \setminus \Gamma$, for some $i = 1, ..., m$.

THEOREM 0.3.8 ([48]). *Let $N = 3$ or $N = 4$ and let $(u, v) \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma) \times H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ be a solution to (0.3.12). Assume that the solution (u, v) has a non-removable[3] singularity in the singular set Γ, where Γ is a closed and proper subset of $\{x_1 = 0\}$ such that*

$$\operatorname*{Cap_2}_{\mathbb{R}^N}(\Gamma) = 0.$$

Moreover let us assume that $\alpha, \beta \geq 2$ and that holds $\alpha + \beta = 2^*$. Then, u and v are symmetric with respect to the hyperplane $\{x_1 = 0\}$. The same conclusion is true if $\{x_1 = 0\}$ is replaced by any affine hyperplane. If at least one between u and v has only a non-removable singularity at the origin, then (u, v) is radially symmetric about the origin and radially decreasing.

0.4. The Gibbons' conjecture for quasilinear elliptic equations

Chapter 7 concerns the study of the qualitative properties of the following quasilinear elliptic equation

$$-\Delta_p u = f(u) \quad \text{in } \mathbb{R}^N, \tag{0.4.1}$$

where we denote a generic point belonging to $\mathbb{R}^N$ by (x', y) with $x' = (x_1, x_2, \ldots, x_{N-1})$ and $y = x_N$, $p > 1$ and $N > 1$.

We summarize the assumptions on the nonlinearity f, denoted by (G_f), in the following:

(G_f): The nonlinearity $f(\cdot)$ belongs to $C^1([-1, 1])$, $f(-1) = 0$, $f(1) = 0$, $f'_+(-1) < 0$, $f'_-(1) < 0$ and the set
$$\mathcal{N}_f := \{t \in [-1, 1] \mid f(t) = 0\}$$
is finite.

The setting of our assumptions allows us to include Allen-Cahn type nonlinearities and, in fact, this chapter is motivated by some questions arising from the following problem

$$-\Delta u = u(1 - u^2) \quad \text{in } \mathbb{R}^N, \tag{0.4.2}$$

see [65]. G.W. Gibbons [29] formulated the following:

CONJECTURE [29]. – Assume $N > 1$ and consider a bounded solution u of (0.4.2) in $C^2(\mathbb{R}^N)$, such that

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1,$$

uniformly with respect to x'. Then, is it true that

$$u(x) = \tanh\left(\frac{x_N - \alpha}{\sqrt{2}}\right),$$

for some $\alpha \in \mathbb{R}$?

This conjecture is also known as the weaker version of the famous De Giorgi's conjecture [45]. We refer to [55] for a complete history on the argument.

[3]As above, we mean that the solution (u, v) does not admit a smooth extension all over the whole space. Namely it is not possible to find $(\tilde{u}, \tilde{v}) \in H^1_{loc}(\mathbb{R}^N) \times H^1_{loc}(\mathbb{R}^N)$ with $u \equiv \tilde{u}$ or $v \equiv \tilde{v}$ in $\mathbb{R}^N \setminus \Gamma$.

The Gibbons' conjecture in the semilinear case $p = 2$ is by now well understood (see [11, 54, 55, 63]). Here we address the quasilinear case for the class of nonlinearities f fulfilling (G_f). To the best of our knowledge this is the first result in this framework. This is motivated by the fact that, unlike the semilinear case (0.4.2), working with the singular operator $-\Delta_p(\cdot)$ we have to take into account that the nonlinearity f change sign and that all the techniques involved in the study of the problem (0.4.1) are not standard since we work in the whole $\mathbb{R}^N$.
Our proofs are based on the technique of the moving planes and on the use of maximum and comparison principles for the $-\Delta_p(\cdot)$ operator, which are much more involved since we have carefully take into account the critical set $\mathcal{Z}_{\nabla u}$ (see (7.2.5)) where the gradient of the solution u vanishes. Moreover, when we consider the case of unbounded domain as $\mathbb{R}^N$, the application of the moving planes technique is much more delicate since weak comparison principles in small domains have to be substituted by weak comparison principles in unbounded domains. Actually, the strong comparison principle does not apply simply as in the case when bounded domains are considered because of the lack of compactness. When we work with the Laplacian operator, i.e. the case $p = 2$, many arguments exploited in the literature are very much related to the linear and nondegenerate nature of the operator. In Chapter 7 we cannot take advantage of all the classical techniques used in the semilinear case and, thus, we need to recover these arguments in the case of equations involving nonlinear degenerate/singular operators.
The main result of Chapter 7 is given by the following:

Theorem 0.4.1 ([49]). *Let $N > 1$, $(2N+2)/(N+2) < p < 2$ and $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ be a solution of (0.4.1), such that*

$$|u| \leq 1$$

and

$$\lim_{y\to+\infty} u(x', y) = 1 \quad \text{and} \quad \lim_{y\to-\infty} u(x', y) = -1, \tag{0.4.3}$$

uniformly with respect to $x' \in \mathbb{R}^{N-1}$. If f fulfills (G_f), then u depends only on y and

$$\partial_y u > 0 \quad \text{in } \mathbb{R}^N. \tag{0.4.4}$$

To get our main result, we first recover a weak comparison principle in a suitable half-space and then we exploit it to start the moving planes procedure. The application of the moving planes method is not standard since we have to recover compactness using some translation arguments, (since we work on $\mathbb{R}^N$) and, not least, we have to take into account the fact that the nonlinearity f change sign which produces peculiar difficulties in the case $p \neq 2$, already in the case of bounded domain. Finally, we get the monotonicity in all the directions of the the upper hemi-sphere $\mathbb{S}^{N-1}_+ := \{\nu \in \mathbb{S}^{N-1}_+ \mid (\nu, e_N)\}$ that will give us the desired 1-dimensional symmetry.

1

Preliminaries

The goal of this chapter is to resume some well known results about weak and strong comparison and maximum principles involving some semilinear and quasilinear elliptic equations.

In particular we study regularity and qualitative properties of positive weak solutions of the following elliptic problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in}\,\Omega \\ u > 0 & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases} \tag{1.0.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N > 2$ and $1 < p < +\infty$. The hypothesis on the nonlinearity f will be always specified. Anyway the reader may think that $f(\cdot)$ is a locally Lipschitz continuous function and most of our results will hold in this case. In general solutions to p-Laplace equations are of class $C^{1,\alpha}$ (see e.g. [**46, 122**]).

In all the thesis we further use the following inequalities, whose proof can be found for example in [**33**]: $\forall \eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$ there exists positive constants C_1, C_2, C_3, C_4 depending on p such that

$$\begin{aligned} &[|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'][\eta - \eta'] \geq C_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2, \\ &\qquad ||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \leq C_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|, \\ &[|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'][\eta - \eta'] \geq C_3|\eta - \eta'|^p \qquad \text{if} \quad p \geq 2, \\ &\qquad ||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \leq C_4|\eta - \eta'|^{p-1} \quad \text{if} \quad 1 < p \leq 2. \end{aligned} \tag{1.0.2}$$

1.1. Regularity of the solutions

In this section we deal with the regularity of the solutions to problem (1.0.1). All the results contained in this section can be found in a paper by L. Damascelli and B. Sciunzi [**36**], where the authors essentially proved the weak maximum principle for quasilinear elliptic equations, i.e. for problem (1.0.1), via the summability properties of the solutions and thanks to the moving planes method of Alexandrov-Serrin, when $p > 2$ and $f(s) > 0$ for $s > 0$ (the case $1 < p \leq 2$ was well known). In all the thesis we will say that $u \in C^1(\Omega)$ is a weak solution to problem (1.0.1) if it satisfies the following equation

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx = \int_\Omega f(u)\varphi\, dx, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.1.1}$$

We define, as usual, the critical set $\mathcal{Z}_u$ of u by setting

$$\mathcal{Z}_u = \{x \in \Omega : \nabla u(x) = 0\} \tag{1.1.2}$$

Note that the importance of the critical set $\mathcal{Z}_u$ is due to the fact that it is exactly the set where our operator is degenerate. By Höpf boundary lemma (cf. [103] or Section 1.4), it follows that

$$\mathcal{Z}_u \cap \partial\Omega = \emptyset, \tag{1.1.3}$$

if $f(0) \geq 0$. We point out that, by standard regularity results, $u \in C^2(\Omega \setminus \mathcal{Z}_u)$. For functions $\varphi \in C_c^\infty(\Omega \setminus \mathcal{Z}_u)$, let us consider the test function $\varphi_i = \partial_{x_i}\varphi$ and denote also $u_i = \partial_{x_i} u$, for all $i = 1, \dots, N$. With this choice in (1.1.1), integrating by parts, we get

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2}(\nabla u_i, \nabla\varphi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla u_i)(\nabla u, \nabla\varphi)\, dx \\ - \int_\Omega f'(u) u_i \varphi = 0, \end{aligned} \tag{1.1.4}$$

that is, in such a way, we have defined the linearized operator $L_u(u_i, \varphi)$ at a fixed solution u of (1.1.1). Then we can write equation (1.1.4) as

$$L_u(u_i, \varphi) = 0, \qquad \forall \varphi \in C_c^\infty(\Omega \setminus \mathcal{Z}_u). \tag{1.1.5}$$

1.1.1. Summability of second derivatives.

The aim of this subsection is to show some summability results on the second derivatives of solutions to (1.0.1). The point of view of considering $|\nabla u|^{p-2}$ as a weight is particularly useful when studying qualitative properties of a fixed solution. In the following, we repeatedly use Young's inequality in the form $ab \leq \delta a^2 + C(\delta) b^2$ for all $a, b \in \mathbb{R}$ and $\delta > 0$. We can now state and prove the following:

Proposition 1.1.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution to problem (1.0.1). Assume that f is locally Lipschitz continuous. Assume that Ω is a bounded and smooth domain of $\mathbb{R}^N$. Then*

$$\int_{\Omega\setminus\{u_i=0\}} \frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{|\nabla u_i|^2}{|u_i|^\beta}\, dx \leq C, \tag{1.1.6}$$

where $0 \leq \beta < 1$, $\gamma < N-2$ ($\gamma = 0$ if $N = 2$), $1 < p < +\infty$ and the positive constant C does not depend on y. In particular, we have

$$\int_{\Omega\setminus\{\nabla u=0\}} \frac{|\nabla u|^{p-2-\beta}\|D^2 u\|^2}{|y-x|^\gamma}\, dx \leq \tilde{C}, \tag{1.1.7}$$

for a positive constant $\tilde{C}$ not depending on y.

Proof. For all $\varepsilon > 0$, let us define the smooth function $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ by setting

$$T_\varepsilon(t) = \begin{cases} t & \text{if } |t| \geq 2\varepsilon, \\ 2t - 2\varepsilon & \text{if } \varepsilon \leq t \leq 2\varepsilon, \\ 2t + 2\varepsilon & \text{if } -2\varepsilon \leq t \leq -\varepsilon, \\ 0 & \text{if } |t| \leq \varepsilon. \end{cases} \tag{1.1.8}$$

To prove (1.1.6) we choose $E \subset\subset \Omega$ such that

$$\mathcal{Z}_u \cap (\Omega \setminus E) = \emptyset.$$

Since u is C^2 in $\Omega \setminus E$, then we may reduce to prove that

$$\int_{E\setminus\{u_i=0\}} \frac{|\nabla u|^{p-2}}{|y-x|^\gamma} \frac{|\nabla u_i|^2}{|u_i|^\beta}\, dx \leq C.$$

Let us consider the cut-off function $\psi \in C_c^\infty(\Omega)$, such that the support of ψ is compactly contained in Ω, $\psi \geq 0$ in Ω and $\psi \equiv 1$ in E. Hence we set

$$\varphi_{\varepsilon,y}(x) = \frac{T_\varepsilon(u_i(x))}{|u_i(x)|^\beta} \frac{\psi(x)}{|y-x|^\gamma} \tag{1.1.9}$$

where $0 \leq \beta < 1$, $\gamma < N-2$ ($\gamma = 0$ for $N = 2$). Since $\varphi_{\varepsilon,y}$ vanishes in a neighborhood of each critical point, it follows that $\varphi_{\varepsilon,y} \in C_c^2(\Omega \setminus \mathcal{Z}_u)$ and hence we can use it as a test function in (1.1.4), getting the following result:

$$\begin{aligned}
&\int_\Omega \frac{1}{|y-x|^\gamma} \frac{|\nabla u|^{p-2}}{|u_i|^\beta} \left(T_\varepsilon'(u_i) - \beta \frac{T_\varepsilon(u_i)}{u_i} \right) \psi |\nabla u_i|^2\, dx \\
&+ \int_\Omega (p-2) \frac{1}{|y-x|^\gamma} \frac{|\nabla u|^{p-4}}{|u_i|^\beta} \left(T_\varepsilon'(u_i) - \beta \frac{T_\varepsilon(u_i)}{u_i} \right) \psi (\nabla u, \nabla u_i)^2 dx \\
&+ \int_\Omega \frac{1}{|y-x|^\gamma} |\nabla u|^{p-2} \frac{T_\varepsilon(u_i)}{|u_i|^\beta} (\nabla u_i, \nabla \psi)\, dx \\
&+ \int_\Omega (p-2) \frac{1}{|y-x|^\gamma} |\nabla u|^{p-4} \frac{T_\varepsilon(u_i)}{|u_i|^\beta} (\nabla u, \nabla u_i)(\nabla u, \nabla \psi)\, dx \\
&+ \int_\Omega |\nabla u|^{p-2} \frac{T_\varepsilon(u_i)}{|u_i|^\beta} \psi \left(\nabla u_i, \nabla_x \left(\frac{1}{|y-x|^\gamma} \right) \right) dx \\
&+ \int_\Omega (p-2) |\nabla u|^{p-4} \frac{T_\varepsilon(u_i)}{|u_i|^\beta} \psi (\nabla u, \nabla u_i) \left(\nabla u, \nabla_x \left(\frac{1}{|y-x|^\gamma} \right) \right) dx \\
&= \int_\Omega f'(u) u_i \frac{T_\varepsilon(u_i)}{|u_i|^\beta} \frac{\psi}{|y-x|^\gamma}\, dx.
\end{aligned}$$

Let us denote each term of the previous equation in a useful way for the sequel, that is

$$\begin{aligned}
A_1 &= \int_\Omega \frac{1}{|y-x|^\gamma} \frac{|\nabla u|^{p-2}}{|u_i|^\beta} \left(T_\varepsilon'(u_i) - \beta \frac{T_\varepsilon(u_i)}{u_i} \right) \psi |\nabla u_i|^2\, dx; \\
A_2 &= \int_\Omega (p-2) \frac{1}{|y-x|^\gamma} \frac{|\nabla u|^{p-4}}{|u_i|^\beta} \left(T_\varepsilon'(u_i) - \beta \frac{T_\varepsilon(u_i)}{u_i} \right) \psi (\nabla u, \nabla u_i)^2 dx; \\
A_3 &= \int_\Omega \frac{1}{|y-x|^\gamma} |\nabla u|^{p-2} \frac{T_\varepsilon(u_i)}{|u_i|^\beta} (\nabla u_i, \nabla \psi)\, dx;
\end{aligned}$$

$$
\begin{aligned}
A_4 &= \int_\Omega (p-2)\frac{1}{|y-x|^\gamma}|\nabla u|^{p-4}\frac{T_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u,\nabla u_i)(\nabla u,\nabla\psi)\,dx;\\
A_5 &= \int_\Omega |\nabla u|^{p-2}\frac{T_\varepsilon(u_i)}{|u_i|^\beta}\psi\left(\nabla u_i,\nabla_x\left(\frac{1}{|y-x|^\gamma}\right)\right)dx;\\
A_6 &= \int_\Omega (p-2)|\nabla u|^{p-4}\frac{T_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u,\nabla u_i)\left(\nabla u,\nabla_x\left(\frac{1}{|y-x|^\gamma}\right)\right)dx;\\
M &= \int_\Omega f'(u)u_i\frac{T_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}\,dx.
\end{aligned}
$$

Then we have rearranged the equation as

$$
\sum_{i=1}^{6} A_i = M. \tag{1.1.10}
$$

Notice that, since $0 \le \beta < 1$, for all $t \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$
T'_\varepsilon(t) - \frac{\beta T_\varepsilon(t)}{t} \ge 0, \qquad \lim_{\varepsilon\to 0}\left(T'_\varepsilon(t) - \frac{\beta T_\varepsilon(t)}{t}\right) = 1-\beta.
$$

From now on, we will denote

$$
\tilde{T}_\varepsilon(t) = T'_\varepsilon(t) - \beta\frac{T_\varepsilon(t)}{t}, \qquad \text{for all } t\in\mathbb{R} \text{ and } \varepsilon>0.
$$

From equation (1.1.10) one has

$$
A_1 + A_2 \le \sum_{i=3}^{6}|A_i| + |M|.
$$

We shall distinguish the proof into two cases.
Case I: $\mathbf{p \ge 2}$. This implies $A_2 \ge 0$, and hence

$$
A_1 \le A_1 + A_2 \le \sum_{i=3}^{6}|A_i| + |M|. \tag{1.1.11}
$$

Case II: $\mathbf{1 < p < 2}$. By Schwarz inequality, of course, it follows

$$
|\nabla u|^{p-4}(\nabla u,\nabla u_i)^2 \le |\nabla u|^{p-2}|\nabla u_i|^2.
$$

In turn, since $1<p<2$, this implies

$$
(p-2)\frac{\tilde{T}_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi|\nabla u|^{p-4}(\nabla u,\nabla u_i)^2}{|y-x|^\gamma} \ge (p-2)\frac{\tilde{T}_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi|\nabla u|^{p-2}|\nabla u_i|^2}{|y-x|^\gamma},
$$

so that $(p-2)A_1 \le A_2$, yielding

$$
A_1 \le \frac{1}{p-1}(A_1+A_2) \le \frac{1}{p-1}\left[\sum_{i=3}^{6}|A_i| + |M|\right]. \tag{1.1.12}
$$

In both cases, in view of (1.1.11) and (1.1.12), we want to estimate the terms in the sum

$$
\sum_{i=3}^{6}|A_i| + |M|. \tag{1.1.13}
$$

Let us start by estimating the terms A_i in the sum (1.1.13). Concerning A_3, we have

$$|A_3| \leq \int_\Omega \frac{1}{|y-x|^\gamma}|\nabla u|^{p-2}\frac{|T_\varepsilon(u_i)|}{|u_i|^\beta}|\nabla u_i||\nabla\psi|dx \leq C_3,$$

where

$$\frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^{\beta-1}}\frac{|T_\varepsilon(u_i)|}{|u_i|}|\nabla u_i||\nabla\psi| \in L^\infty(\Omega),$$

since $|\nabla u_i|$ is bounded in a neighborhood of the boundary by Höpf Lemma, $\gamma - 2 < N$, $0 \leq \beta < 1$ and the constant C_3 is independent of y. For the same reason, we also have

$$|A_4| \leq \int_\Omega \frac{|p-2|}{|y-x|^\gamma}|\nabla u|^{p-2}\frac{|T_\varepsilon(u_i)|}{|u_i|^\beta}|\nabla u_i||\nabla\psi|\,dx \leq C_4,$$

for some positive constants C_4 independent of y.
Furthermore, for a positive constant C_5 independent of y, we have

$$\begin{aligned}
|A_5| &\leq \int_\Omega |\nabla u|^{p-2}\frac{|T_\varepsilon(u_i)|}{|u_i|^\beta}\psi|\nabla u_i|\left|\nabla_x\left(\frac{1}{|y-x|^\gamma}\right)\right|\,dx\\
&\leq C_5\int_\Omega |\nabla u|^{p-2}\frac{|T_\varepsilon(u_i)|}{|u_i|^\beta}\psi|\nabla u_i|\frac{1}{|y-x|^{\gamma+1}}dx\\
&\leq C_5\delta\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{|T_\varepsilon(u_i)|}{|u_i|}|\nabla u_i|^2\,dx\\
&+ C(\delta)\int_\Omega |\nabla u|^{p-1}\frac{|T_\varepsilon(u_i)|}{|u_i|}\frac{1}{|y-x|^{\gamma+2}}\,dx\\
&\leq C_5\delta\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{|T_\varepsilon(u_i)|}{|u_i|}|\nabla u_i|^2\,dx + K_5(\delta)
\end{aligned}$$

where we used Young's inequality, $\gamma - 2 < N$ and $0 \leq \beta < 1$. In a similar way,

$$\begin{aligned}
|A_6| &\leq \int_\Omega |p-2||\nabla u|^{p-2}\frac{|T_\varepsilon(u_i)|}{|u_i|^\beta}\psi|\nabla u_i|\left|\nabla_x\left(\frac{1}{|y-x|^\gamma}\right)\right|\,dx\\
&\leq C_6\delta\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{T_\varepsilon(u_i)}{u_i}|\nabla u_i|^2dx + K_6(\delta)
\end{aligned}$$

and

$$|M| \leq \int_\Omega |f'(u)|\frac{|T_\varepsilon(u_i)|}{|u_i|^{\beta-1}}\frac{\psi}{|y-x|^\gamma}\,dx \leq C_M,$$

where the last inequality holds true since f is locally Lipschitz continuous and where C_6 and C_M are constants independent of y. Then, by these estimates above and by equations (1.1.11), (1.1.12) and (1.1.13) we write

$$\begin{aligned}
A_1 &\leq \mathcal{D}\left[\sum_{i=3}^{6}|A_i| + |M|\right]\\
&\leq \mathcal{S}\delta A_1 + \mathcal{M}\delta\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{T_\varepsilon(u_i)}{u_i}|\nabla u_i|^2dx + \mathcal{C}_\delta,
\end{aligned} \tag{1.1.14}$$

where we have set

$$\mathcal{D}=\max\left\{1,\frac{1}{p-1}\right\},\quad \mathcal{S}=\mathcal{D}\frac{C_3}{\eta},\quad \mathcal{M}=\mathcal{D}\max\left\{C_5,C_6\right\}$$
$$\mathcal{C}_\delta=\mathcal{D}\max\left\{K_5(\delta),K_6(\delta),C_3,C_4,C_M\right\}.$$

Then by (1.1.14) one has

$$\begin{aligned}&(1-\mathcal{S}\delta)\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\left(T'_\varepsilon(u_i)-\beta\frac{T_\varepsilon(u_i)}{u_i}\right)\psi|\nabla u_i|^2\,dx\\&\le \mathcal{M}\delta\int_\Omega\frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{T_\varepsilon(u_i)}{u_i}|\nabla u_i|^2dx+\mathcal{C}_\delta,\end{aligned}$$

namely

(1.1.15)
$$(1-\mathcal{S}\delta)\int_\Omega\frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\left[T'_\varepsilon(u_i)-\left(\beta+\frac{\mathcal{M}\delta}{(1-\mathcal{S}\delta)}\right)\frac{T_\varepsilon(u_i)}{u_i}\right]\psi|\nabla u_i|^2\,dx \le \mathcal{C}_\delta.$$

Let us choose $\delta>0$ such that

$$\begin{cases}1-\mathcal{S}\delta>0,\\ 1-\left(\beta+\frac{\mathcal{M}\delta}{1-\mathcal{S}\delta}\right)>0.\end{cases}\tag{1.1.16}$$

Therefore, since as $\varepsilon\to0$

$$\left[T'_\varepsilon(u_i)-\left(\beta+\frac{\mathcal{M}\delta}{(1-\mathcal{S}\delta)}\right)\frac{T_\varepsilon(u_i)}{u_i}\right]\to\left(1-\beta-\frac{\mathcal{M}\delta}{(1-\mathcal{S}\delta)}\right)>0\ \text{in}\ \{u_i\neq0\},$$

by Fatou's Lemma we get

$$\int_{\Omega\setminus\{u_i=0\}}\frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{|\nabla u_i|^2}{|u_i|^\beta}\psi dx\le C.\tag{1.1.17}$$

The proof is now complete, in view of the choose of the cut-off function ψ. □

1.1.2. Gradients summability.

In this subsection we show the gradient summability of a solution u of problem (1.0.1). We have the following:

THEOREM 1.1.2. *Let u be a weak solution of (1.0.1) and assume, furthermore, that $f(s)>0$ for any $s>0$. Then, there exists a positive constant C, independent of y, such that*

$$\int_\Omega\frac{1}{|\nabla u|^{(p-1)r}}\frac{1}{|x-y|^\gamma}\,dx\le C\tag{1.1.18}$$

where $0<r<1$ and $\gamma<N-2$ for $N\ge3$ ($\gamma=0$ if $N=2$). In particular the critical set $\mathcal{Z}_u$ has zero Lebesgue measure.

PROOF. Let E be a set with $E\subset\subset\Omega$ and $(\Omega\setminus E)\cap\mathcal{Z}_u=\emptyset$. Recall that $\mathcal{Z}_u=\{\nabla u=0\}$ and $\mathcal{Z}_u\cap\partial\Omega=\emptyset$, in view of Höpf boundary lemma (see [103] or Section 1.4). It is easy to see that, to prove the result, we may reduce to show that

$$\int_E\frac{1}{|\nabla u|^{(p-1)r}}\frac{1}{|x-y|^\gamma}\,dx\le C.\tag{1.1.19}$$

To achieve this, let us consider the function

$$\Psi(x) = \Psi_{\varepsilon,y}(x) = \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^{\gamma}} \varphi, \tag{1.1.20}$$

where $0 < r < 1$ and $\gamma < N-2$ for $N \geq 3$ ($\gamma = 0$ if $N = 2$). We also assume that φ is a positive $C_c^\infty(\Omega)$ cut-off function such that $\varphi \equiv 1$ in E. Using Ψ as test function in (1.1.1), since $f(u) \geq \sigma$ for some $\sigma > 0$ in the support of Ψ, we get

$$\begin{aligned}
\sigma \int_\Omega \Psi \, dx \leq \int_\Omega f(u)\Psi \, dx &= \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla \Psi) \, dx \\
&\leq \int_\Omega |\nabla u|^{p-2} |(\nabla u, \nabla |\nabla u|)| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \frac{1}{|x-y|^\gamma} \varphi \, dx \\
&\quad + \int_\Omega |\nabla u|^{p-2} \left| \left(\nabla u, \nabla_x \frac{1}{|x-y|^\gamma} \right) \right| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \varphi \, dx \\
&\quad + \int_\Omega |\nabla u|^{p-2} |(\nabla u, \nabla \varphi)| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^\gamma} \, dx.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_\Omega \Psi \, dx \leq C \Bigg[\int_\Omega |\nabla u|^{p-1} \|D^2 u\| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \frac{1}{|x-y|^\gamma} \varphi \, dx \\
+ \int_\Omega \frac{|\nabla u|^{p-1}}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^{\gamma+1}} \varphi \, dx \\
+ \int_\Omega \frac{|\nabla u|^{p-1}}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{|\nabla \varphi|}{|x-y|^\gamma} \, dx \Bigg].
\end{aligned}$$

Then, denoting by C_i, suitable positive constants independent of y and by C_δ a positive constant depending on δ, we obtain

$$\begin{aligned}
\int_\Omega \Psi \, dx &\leq C_1 \int_\Omega |\nabla u|^{p-1} \|D^2 u\| \cdot \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi \, dx \\
&\quad + C_2 \int_\Omega \frac{1}{|x-y|^{\gamma+1}} \, dx + C_3 \int_\Omega \frac{1}{|x-y|^\gamma} \, dx \\
&\leq C_1 \int_\Omega |\nabla u|^{p-1} \|D^2 u\| \cdot \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi \, dx + C_4 \\
&\leq C_5 \delta \int_\Omega \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi \, dx \\
&\quad + C_\delta \int_\Omega |\nabla u|^{(p-2)-(p(r-1)+2-r)} \|D^2 u\|^2 \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi \, dx + C_6 \\
&\leq C_5 \delta \int_\Omega \Psi \, dx + C_\delta.
\end{aligned} \tag{1.1.21}$$

Here we have we used that $u \in C^1(\Omega\ )$, $\gamma < N-2$ and we have exploited the regularity result of Proposition 1.1.1, with $\beta := p(r-1) + 2 - r$. Then, by (1.1.21), fixing δ sufficiently small, such that $1 - C_5\delta > 0$, one concludes

$$\int_\Omega \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^\gamma} \varphi \, dx \leq K, \tag{1.1.22}$$

for some positive constant K independent of y. Taking the limit for ε going to zero, the assertion immediately follows by Fatou's Lemma. Moreover, as a consequence, we have that

$$\mathcal{L}(\mathcal{Z}_u) = 0.$$

□

Theorem 1.1.2 provides in fact the right summability of the weight $\varrho(x) = |\nabla u(x)|^{p-2}$ in order to obtain a weighted Poincaré inequality.

1.1.3. Weighted Sobolev and Poincaré inequality.

In this paragraph we shall prove some results about weighted Sobolev and Poincaré inequality, that are essential tools in the proof of the weak comparison principle in the case $p > 2$. Let us start stating the following:

Condition (PE) We say that $u(x)$ satisfies the Condition (PE) in Ω, if

$$|u(x)| \leq \hat{C} \int_{\Omega} \frac{|\nabla u(y)|}{|x-y|^{N-1}} dy. \tag{1.1.23}$$

This condition is the same introduced in a paper by A. Farina, L. Montoro and B. Sciunzi [61]. This generally follows by potential estimates, see [70, Lemma 7.14, Lemma 7.16], that gives

$$u(x) = \hat{C} \int_{\Omega} \frac{(x_i - y_i)\frac{\partial u}{\partial x_i}(y)}{|x-y|^N} dy \quad \text{a.e. in } \Omega,$$

with

(i) $\hat{C} = \frac{1}{N\omega_N}$ if $u \in W_0^{1,1}(\Omega)$,
where ω_N is the volume of the unit ball in $\mathbb{R}^N$;

(ii) $\hat{C} = \frac{d^N}{N|S|}$ if $u \in W^{1,1}(\Omega)$ with $\displaystyle\int_S u = 0$ and Ω convex,
where $d = \operatorname{diam} \Omega$ and S any measurable subset of Ω.

Moreover let $\mu \in (0, 1]$, we define

$$V_\mu[f, \Omega](x) = \int_{\Omega} \frac{f(y)}{|x-y|^{N(1-\mu)}} dy. \tag{1.1.24}$$

It is well known that (see [70, pag.159])

$$V_\mu[1, \Omega](x) \leq \mu^{-1} \omega_N^{1-\mu} |\Omega|^\mu. \tag{1.1.25}$$

Let us state the following:

Lemma 1.1.1. *Let us consider $\tilde{\Omega} \subset \Omega$ and $V_\mu[f, \Omega](x)$ as in (1.1.24). Then for any $1 \leq q \leq \infty$ one has*

$$\|V_\mu[f, \tilde{\Omega}](x)\|_{L^q(\Omega)} \leq \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_N^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_{L^m(\tilde{\Omega})}, \tag{1.1.26}$$

where $0 \leq \delta = \dfrac{1}{m} - \dfrac{1}{q} < \mu$.

Proof. The proof follows by [70, Lemma 7.12].

□

Recall that, if $\varrho \in L^1(\Omega)$, $1 \le q < \infty$, the space $H^{1,q}_\varrho(\Omega)$ is defined as the completion of $C^\infty(\overline{\Omega})$ under the norm

$$\|v\|_{H^{1,q}_\varrho} = \|v\|_{L^q(\Omega)} + \|\nabla v\|_{L^q(\Omega,\varrho)} \tag{1.1.27}$$

where

$$\|\nabla v\|^q_{L^q(\Omega,\varrho)} = \int_\Omega |\nabla v|^q \varrho \, dx.$$

We also recall that $H^{1,q}_{0,\varrho}$ is defined as the completion of $C^\infty_c(\overline{\Omega})$ under the norm

$$\|v\|_{H^{1,q}_{0,\varrho}} = \|\nabla v\|_{L^q(\Omega,\varrho)}. \tag{1.1.28}$$

THEOREM 1.1.3 (Weighted Sobolev inequality). *Let ϱ a weight function such that*

$$\int_\Omega \frac{1}{\varrho^t |x-y|^\gamma} \le C^*, \tag{1.1.29}$$

with $t = \frac{p-1}{p-2} r$, $\frac{p-2}{p-1} < r < 1$, $\gamma < N-2$ ($\gamma = 0$ if $N = 2$). Assume, in the case $N \ge 3$, without no loss of generality that

$$\gamma > N - 2t,$$

which[1] *implies $Nt - 2N + 2t + \gamma > 0$. Then, for any $w \in H^{1,2}_{0,\varrho}(\Omega)$, there exists a constant C_{s_ϱ} such that*

$$\|w\|_{L^q(\Omega)} \le C_{s_\varrho} \|\nabla w\|_{L^2(\Omega,\varrho)} = C_{s_\varrho} \left(\int_\Omega \varrho |\nabla w|^2 \right)^{\frac{1}{2}}, \tag{1.1.30}$$

for any $1 \le q < 2^*(t)$ where

$$\frac{1}{2^*(t)} = \frac{1}{2} - \frac{1}{N} + \frac{1}{t}\left(\frac{1}{2} - \frac{\gamma}{2N}\right). \tag{1.1.31}$$

with

$$C_{s_\varrho} = \hat{C}(C^*)^{\frac{1}{2t}} (C_M)^{\frac{1}{(2t)'}}, \tag{1.1.32}$$

where $\hat{C}$ is as in (1.1.23), C^* is as in (1.1.29) and in the statement of Theorem 1.1.2, and

$$C_M = \left(\frac{1-\delta}{\frac{\alpha}{N} - \delta} \right)^{1-\delta} \omega_n^{1-\frac{\alpha}{N}} |\Omega|^{\frac{\alpha}{N} - \delta}.$$

REMARK 1.1.2. Note that the largest value of $2^*(t)$ is obtained at the limiting case $t \approx \frac{p-1}{p-2}$, and $\gamma \approx (N-2)$, $\gamma = 0$ for $N = 2$. We have therefore that (1.1.30) holds for any $q < \tilde{2}^*$ where

$$\frac{1}{\tilde{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1} \cdot \frac{1}{N},$$

Moreover one has $\tilde{2}^* > 2$.

[1]Note that the condition $\gamma > N - 2t$ holds true for $r \approx 1$ and $\gamma \approx N-2$ that we may assume with no loose of generality.

Proof. Without loss of generality we assume w belonging to $C^1(\Omega)$ or $C^1_0(\Omega)$ depending on the case (i) or (ii) of **Condition (PE)**. Hence (1.1.23) implies

$$|w(x)| \leq \hat{C}\int_\Omega \frac{|\nabla w(y)|}{|x-y|^{N-1}}dy. \tag{1.1.33}$$

Then

$$\begin{aligned}|w(x)| &\leq \hat{C}\int_\Omega \frac{|\nabla w(y)|}{|x-y|^{N-1}}\,dy\\ &\leq \hat{C}\int_\Omega \frac{1}{\varrho^{\frac{1}{2}}|x-y|^{\frac{\gamma}{2t}}}\,\frac{|\nabla w(y)|\varrho^{\frac{1}{2}}}{|x-y|^{N-1-\frac{\gamma}{2t}}}\,dy\\ &\leq \hat{C}\left(\int_\Omega \frac{1}{\varrho^t|x-y|^\gamma}\,dy\right)^{\frac{1}{2t}}\left(\int_\Omega \frac{\left(|\nabla w(y)|\varrho^{\frac{1}{2}}\right)^{(2t)'}}{|x-y|^{(N-1-\frac{\gamma}{2t})(2t)'}}\,dy\right)^{\frac{1}{(2t)'}},\end{aligned}$$

where in the last inequality we used Hölder inequality with $\frac{1}{2t}+\frac{1}{(2t)'}=1$. Hence

$$|w(x)| \leq \hat{C}(C^*)^{\frac{1}{2t}}\left(\int_\Omega \frac{\left(|\nabla w(y)|\varrho^{\frac{1}{2}}\right)^{(2t)'}}{|x-y|^{(N-1-\frac{\gamma}{2t})(2t)'}}\,dy\right)^{\frac{1}{(2t)'}}. \tag{1.1.34}$$

We point out that

$$\left(|\nabla w|\varrho^{\frac{1}{2}}\right)^{(2t)'} \in L^{\frac{2}{(2t)'}}(\Omega). \tag{1.1.35}$$

From (1.1.34), by using equation (1.1.24) with $\mu = 1-\frac{1}{N}(N-1-\frac{\gamma}{2t})(2t)'$, we obtain

$$|w(x)| \leq \hat{C}(C^*)^{\frac{1}{2t}}\left(V_\mu\left[\left(|\nabla w(y)|\varrho^{\frac{1}{2}}\right)^{(2t)'},\Omega\right](x)\right)^{\frac{1}{(2t)'}}. \tag{1.1.36}$$

Moreover we remark that the assumption $\gamma > N-2t$ implies $\mu > 0$.
We shall use now Lemma 1.1.1 setting

$$\frac{1}{m} = \frac{(2t)'}{2},$$

see (1.1.35). In order to apply (1.1.26), since by assumption $Nt-2N+2t+\gamma > 0$, a direct calculation shows that it is possible to find a $q>1$ such that

$$\frac{1}{m}-\frac{1}{q} < \mu.$$

From (1.1.34) we have

(1.1.37)

$$\left(\int_\Omega |w(x)|^{q(2t)'}\,dx\right)^{\frac{1}{q(2t)'}} \leq \hat{C}(C^*)^{\frac{1}{2t}}\left\|V_\mu\left[\left(|\nabla w(y)|\varrho^{\frac{1}{2}}\right)^{(2t)'},\Omega\right](x)\right\|^{\frac{1}{(2t)'}}_{L^q(\Omega)}.$$

From (1.1.37), by using Lemma 1.1.1 we get

(1.1.38)
$$\left(\int_\Omega |w(x)|^{q(2t)'}\right)^{\frac{1}{q(2t)'}} \leq \hat{C}(C^*)^{\frac{1}{2t}}\left(\left(\frac{1-\delta}{\frac{\alpha}{N}-\delta}\right)^{1-\delta}\omega_n^{1-\frac{\alpha}{N}}|\Omega|^{\frac{\alpha}{N}-\delta}\right)^{\frac{1}{(2t)'}}\left(\int_\Omega \varrho|\nabla w|^2\right)^{\frac{1}{2}},$$

that gives (1.1.31) with $q(2t)' = 2^*(t)$ and (1.1.32) with

$$C_M = \left(\frac{1-\delta}{\frac{\alpha}{N}-\delta}\right)^{1-\delta}\omega_n^{1-\frac{\alpha}{N}}|\Omega|^{\frac{\alpha}{N}-\delta}.$$

□

As a natural consequence of the weighted Sobolev inequality, we obtain the following:

Corollary 1.1.4 (Weighted Poincaré inequality). *Let w be as in one of the following cases*

(i) $w \in H^{1,2}_{0,\varrho}(\Omega)$,

(ii) $w \in H^{1,2}_{\varrho}(\Omega)$ *such that* $\int_\Omega w = 0$ *and* Ω *convex.*

Then, if the weight ϱ fulfill (1.1.29), then

$$\int_\Omega w^2 \leq C_p(\Omega)\hat{C}^2(C^*)^{\frac{1}{t}}(C_M)^{\frac{2}{(2t)'}}\int_\Omega \varrho|\nabla w|^2,$$

where $\hat{C}, C^*, C_M$ are as in Theorem 1.1.3 and with $C_p(\Omega) \to 0$ if $|\Omega| \to 0$. In particular, given any $0 < \vartheta < 1$, we can assume that

$$C_p(\Omega) \leq C\,|\Omega|^{\frac{2\,\vartheta}{(p-1)N}}. \tag{1.1.39}$$

Proof. Choose $2 < q < \tilde{2}^*$. By Holder inequality we get:

$$\int_\Omega w^2 \leq \left(\int_\Omega w^q\right)^{\frac{2}{q}}|\Omega|^{\frac{q-2}{q}}, \tag{1.1.40}$$

and then using Theorem 1.1.3 one has

$$\int_\Omega w^2 \leq C_p(\Omega)\hat{C}^2(C^*)^{\frac{1}{t}}(C_M)^{\frac{2}{(2t)'}}\int_\Omega \varrho|\nabla w|^2.$$

By (1.1.40) and direct computation it follows (1.1.39).

□

1.2. Weak comparison principles in bounded domains

As remarked at the beginning of this chapter, our aim is to show some results about comparison principles involving semilinear an quasilinear elliptic equations in bounded and unbounded domains. In fact, the aim of this section is to show some results about weak comparison principles in small domains. To do this, let us consider the following elliptic problem

$$\begin{cases} -\Delta_p u \le f(u) & \text{in } \Omega \\ -\Delta_p v \ge f(v) & \text{in } \Omega, \end{cases} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is a bounded domain and $1 < p < +\infty$. In all the section we assume that f is a locally Lipschitz continuous function on $[0, +\infty)$. As remarked in the introduction, we point out that it is well known that, if $p = 2$, we have classical sub and super-solution $u, v \in C^2(\overline{\Omega})$, while if $p \neq 2$ problem (1.2.1) has to be understood in the weak distributional meaning, since it is well known by regularity theory that $u, v \in C^{1,\alpha}(\overline{\Omega})$, see e.g. [**46, 122**].

We just point out the fact that if $p = 2$ the situation is well known in the literature, since, maximum principles and comparison principles are equivalent. Inspired by classical results, that can be found for example in [**70**], we exploit the following well known result whose proof can be found in a paper by A. Farina and B. Sciunzi (see [**62**]):

Theorem 1.2.1 (Weak comparison principle in small domains, $p = 2$). *Let us assume that $p = 2$ and $u, v \in C^2(\overline{\Omega})$ satisfying (1.2.1). Then there exists $\vartheta = \vartheta(\Omega, u, v, f) > 0$ such that, if for any domain $\Omega' \subset \Omega$ with $u \le v$ on $\partial\Omega'$ and $|\Omega'| \le \vartheta$ (where $|\cdot|$ denotes the Lebesgue measure of a set) it follows that*

$$u \le v \qquad in\, \Omega'.$$

Proof. Let us consider the weak formulation of problem (1.2.1) and get

$$\int_{\Omega'} (\nabla u, \nabla \varphi)\, dx - \int_{\Omega'} f(u)\varphi\, dx \le \int_{\Omega'} (\nabla v, \nabla \varphi)\, dx - \int_{\Omega'} f(v)\varphi\, dx \tag{1.2.2}$$

for every test function $\varphi \in C_c^\infty(\Omega')$, $\varphi \ge 0$ in Ω'. Taking $(u-v)^+ \in H_0^1(\Omega')$ as test function in (1.2.2), we obtain

$$\begin{aligned} \int_{\Omega'} |\nabla (u-v)^+|^2\, dx &\le \int_{\Omega'} \frac{f(u) - f(v)}{u - v} [(u-v)^+]^2\, dx \\ &\le C(\Omega, u, v, f) \int_{\Omega'} [(u-v)^+]^2\, dx \end{aligned} \tag{1.2.3}$$

where the positive constant $C(\Omega, u, v, f)$ can be determined exploiting the fact that u, v are bounded on Ω and f is locally Lipschitz continuous on $[0, \infty)$. An application of Poincaré inequality gives

$$\int_{\Omega'} |\nabla (u-v)^+|^2\, dx \le C(\Omega, u, v, f)(C_N(|\Omega'|))^{\frac{2}{N}} \int_{\Omega'} |\nabla (u-v)^+|^2\, dx \tag{1.2.4}$$

where $C_N > 0$ is a constant depending only on the Euclidean dimension N. For $|\Omega'|$ sufficiently small such that $C(\Omega, u, v, f)(C_N(|\Omega'|))^{\frac{2}{N}} < 1$, we get that $(u-v)^+ \equiv 0$ and the thesis.

□

The situation is completely different when $p \neq 2$. Since the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is nonlinear, there is a deep difference between comparison and maximum principles. We have to remark that the singular case, i.e. $1 < p < 2$, is easier than the degenerate one, since when $p > 2$ we need to apply weighted Sobolev and Poincaré inequalities proved in Section 1.1, see Theorem 1.1.3 and Corollary 1.1.4 The weak comparison principle for the quasilinear elliptic problem (1.2.1) appeared, in a very general version, in a paper by L. Damascelli [33] when $1 < p < 2$. First of all let us recall the Poincaré inequality whose proof with $A = \Omega$ and $B = \emptyset$ can be found in [70]. Now we prove the variant proposed in [33].

LEMMA 1.2.2 (Poincaré inequality). *Let us assume that Ω is an open bounded set such that $\Omega = A \cup B$ with A, B measurable set of Ω. If $u \in W_0^{1,q}(\Omega)$ with $1 < q < +\infty$, then*

$$\|u\|_{L^\infty(\Omega)} \le \omega_N^{-\frac{1}{N}} |\Omega|^{\frac{1}{Nq}} \left[|A|^{\frac{1}{Nq'}} \|\nabla u\|_{L^q(A)} + |B|^{\frac{1}{Nq'}} \|\nabla u\|_{L^q(B)} \right], \tag{1.2.5}$$

where $q' = \frac{q}{q-1}$.

PROOF. Let us define $h(x, y) := |x - y|^{1-N}$ and let us assume that C is a measurable set of Ω. If $R > 0$ is such that $|C| = |B(x, R)|$ we have

$$\begin{aligned} \int_C h(x,y)\,dy &= \int_{C\cap B(x,R)} h(x,y)\,dy + \int_{C\setminus B(x,R)} h(x,y)\,dy \\ &\le \int_{C\cap B(x,R)} h(x,y)\,dy + \int_{B(x,R)\setminus C} h(x,y)\,dy \\ &= \int_{B(x,R)} h(x,y)\,dy = N\omega_N R = N\omega_N \left(\frac{|C|}{\omega_N}\right)^{\frac{1}{N}}, \end{aligned} \tag{1.2.6}$$

where ω_N is the misure of the N-dimensional ball $B(0, 1)$. If $f \in L^q(C)$ by Fubini's Theorem for a.e. $x \in \Omega$ we have that $f(y)(h(x, y))^{\frac{1}{q}} \in L^p(C)$. Let us recall the definition of potential given in (1.1.24) with $\mu = \frac{1}{N}$

$$V_{\frac{1}{N}}[f, C](x) := \int_C f(y)h(x, y)\,dy.$$

Then by (1.2.6) and Hölder inequality it follows that

$$\begin{aligned} |V_{\frac{1}{N}}[f,C](x)| &\le \int_C |f| h^{\frac{1}{q}} h^{\frac{1}{q'}}\,dy \\ &\le \left(\int_C |f(y)|^q h(x,y)\,dy\right)^{\frac{1}{q}} \left(\int_C h(x,y)\,dy\right)^{\frac{1}{q'}} \\ &\le \left[N\omega_N \left(\frac{|C|}{\omega_N}\right)^{\frac{1}{N}}\right]^{\frac{1}{q'}} \left(\int_C |f(y)|^q h(x,y)\,dy\right)^{\frac{1}{q}}. \end{aligned} \tag{1.2.7}$$

Taking the q power and integrating in x over Ω we obtain, using again Fubini's Theorem and (1.2.7) with $C = \Omega$ and the role of x and y interchanged,

$$\|V_{\frac{1}{N}}[f, C]\|_{L^q(\Omega)} \le N\omega_N \left(\frac{|C|}{\omega_N}\right)^{\frac{1}{Nq'}} \left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{Nq}} \|f\|_{L^q(C)}. \tag{1.2.8}$$

Now, if $u \in C_c^\infty(\Omega)$ then we have the representation (see Lemma 7.14 [70])

$$u(x) = \frac{1}{N\omega_N} \int_\Omega |x - y|^{-N} (\nabla u, x - y)\,dy$$

so that if $\Omega = A \cup B$ we have that

$$u(x) \leq \frac{1}{N\omega_N}\left[V_{\frac{1}{N}}[A, |\nabla u|](x) + V_{\frac{1}{N}}[B, |\nabla u|](x)\right].$$

From (1.2.8) we obtain (1.2.5) for $u \in C_c^\infty(\Omega)$ and the general case follows by density argument.

□

Let us put, if u, v are functions in $W^{1,\infty}(\Omega)$ and $A \subseteq \Omega$

$$M_A = M_A(u, v) = \sup_A(|\nabla u| + |\nabla v|).$$

REMARK 1.2.3. *A function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous if and only if for all $R > 0$ there exists $C_1(R), C_2(R) \geq 0$ such that*

(1) *$f_1(s) = f(s) - C_1 s$ is non-increasing in $[-R, R]$.*
(2) *$f_2(s) = f(s) - C_2 s$ is non-decreasing in $[-R, R]$.*

In view of Remark 1.2.3 we prove (see also [33]) the following version of the weak comparison principles in the singular case:

THEOREM 1.2.4 (Weak comparison principle, $1 < p < 2$). *Let Ω be bounded and $u, v \in W^{1,\infty}(\Omega)$ weakly satisfy*

$$-\Delta_p u + g(x, u) - \Lambda u \leq -\Delta_p v + g(x, v) - \Lambda v \quad \text{in } \Omega \tag{1.2.9}$$

where $\Lambda \geq 0$ and $g \in C(\overline{\Omega} \times \mathbb{R})$ is such that for each $x \in \Omega$ $g(x, s)$ is non-decreasing in s for $|s| \leq \max\{\|u\|_\infty, \|v\|_\infty\}$. Let $\Omega' \subseteq \Omega$ be open and suppose $u \leq v$ on $\partial\Omega'$.

(a) *If $\Lambda = 0$ then $u \leq v$ in Ω', for all $p > 1$.*
(b) *If $1 < p < 2$ there exist $\delta, M > 0$, depending only on $p, \Lambda, \gamma, \Gamma, |\Omega|$ and M_Ω, such that the following holds:*

$$\text{if } \Omega' = A_1 \cup A_2,\ |A_1 \cap A_2| = 0,\ |A_1| < \delta \text{ and } M_{A_2} < M,$$
$$\text{then } u \leq v \text{ in } \Omega'.$$

PROOF. (a) Let us assume that $\Lambda = 0$. We pass to the weak formulation of (1.2.9)

$$\int_{\Omega'} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\varphi)\, dx + \int_{\Omega'} [g(x, u) - g(x, v)]\varphi\, dx \leq 0, \quad \forall \varphi \in C_c^\infty(\Omega'). \tag{1.2.10}$$

Since by our assumptions $u \leq v$ on $\partial\Omega'$ it follows that $\varphi = (u - v)^+ \in W_0^{1,p}(\Omega')$. Using φ as test function in (1.2.10), the fact that g is nondecreasing and also by (1.0.2) we deduce that

$$C_1 \int_{\Omega'} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u - v)|^2\, dx \leq \int_{\Omega'} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u - v)^+)\, dx \leq 0. \tag{1.2.11}$$

Hence by (1.2.11) it follows that $(u - v)^+ \equiv 0$ in Ω' and this gives the thesis.

(b) As we did above by (1.2.9), we pass to the weak formulation

(1.2.12)
$$\int_{\Omega'} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\varphi)\,dx + \int_{\Omega'} [g(x,u) - g(x,v)]\varphi\,dx - \Lambda\int_{\Omega'} (u-v)\varphi\,dx \leq 0, \qquad \forall\varphi\in C_c^\infty(\Omega').$$

Since by our assumptions $u \leq v$ on $\partial\Omega'$ it follows that $\varphi = (u-v)^+ \in W_0^{1,p}(\Omega')$. Using φ as test function in (1.2.10), the fact that g is nondecreasing and also by (1.0.2) we deduce that

$$\begin{aligned} &C_1\int_{\Omega'} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx \\ &\leq \int_{\Omega'} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+)\,dx \\ &\leq \Lambda\int_{\Omega'} [(u-v)^+]^2\varphi\,dx \end{aligned} \tag{1.2.13}$$

Let us recall that $\Omega' = A_1 \cup A_2$ and $1 < p < 2$, hence we rewrite (1.2.13)

(1.2.14)
$$\begin{aligned} &C_1 M_{\Omega'}^{p-2}\int_{A_1} |\nabla(u-v)^+|^2\,dx \;+\; C_1 M_{A_2}^{p-2}\int_{A_2} |\nabla(u-v)^+|^2\,dx \\ &\qquad\leq C_1\int_{\Omega'} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx \\ &\qquad\leq \Lambda\int_{\Omega'} [(u-v)^+]^2\,dx \end{aligned}$$

Now, by Lemma 1.2.2 with $q = q' = 2$ we have that

(1.2.15)
$$\begin{aligned} &C_1 M_{\Omega'}^{p-2}\int_{A_1} |\nabla(u-v)^+|^2\,dx \;+\; C_1 M_{A_2}^{p-2}\int_{A_2} |\nabla(u-v)^+|^2\,dx \\ &\leq 2\Lambda\omega_N^{-\frac{2}{N}}|\Omega'|^{\frac{1}{N}}\left[|A_1|^{\frac{1}{N}}\int_{A_1} |\nabla(u-v)^+|^2\,dx \;+|\Omega'|^{\frac{1}{N}}\int_{A_2} |\nabla(u-v)^+|^2\,dx\right]. \end{aligned}$$

Now, we infer that if $|A_1|$ and M_{A_2} are small we must have

$$\int_{A_i} |\nabla(u-v)^+|^2\,dx = 0,$$

for $i = 1, 2$, so that $(u-v)^+ \equiv 0$ in Ω' and hence the thesis.

□

As remarked at the beginning of this section, using regularity results and weighted Sobolev and Poincaré inequalities proved in Section 1.1, an application of previous results is given by the following:

THEOREM 1.2.5 (Weak comparison principle, $p > 2$). *Let $\tilde\Omega$ be a bounded smooth domain such that $\tilde\Omega \subseteq \Omega$. Assume that u, v are solutions to the* *problem* (1.0.1) *and assume that* $u \le v$ *on* $\partial\tilde{\Omega}$. *Then there exists a positive constant* $\vartheta = \vartheta(\Omega, u, v, f)$ *such that, assuming*

$$|\tilde{\Omega}| \le \vartheta$$

then it holds

$$u \le v \qquad in\ \tilde{\Omega}.$$

Proof. Let us recall the weak formulations for the solutions u and v to problem (1.0.1)

$$\int_{\tilde{\Omega}} |\nabla u|^{p-2}(\nabla u\,,\,\nabla\varphi)\,dx = \int_{\tilde{\Omega}} f(u)\varphi\,dx, \qquad \forall\varphi\in C^\infty_c(\tilde{\Omega}), \tag{1.2.16}$$

$$\int_{\tilde{\Omega}} |\nabla v|^{p-2}(\nabla v\,,\,\nabla\varphi)\,dx = \int_{\tilde{\Omega}} f(v)\varphi\,dx, \qquad \forall\varphi\in C^\infty_c(\tilde{\Omega}). \tag{1.2.17}$$

Then we assume by contradiction that the assertion is false, and consider

$$(u-v)^+ = \max\{u-v, 0\},$$

that, consequently, is not identically equal to zero. Let us also set $\Omega^+ \equiv \operatorname{supp}(u-v)^+ \cap \tilde{\Omega}$. Since by assumption $u \le v$ on $\partial\tilde{\Omega}$, it follows that $(u-v)^+ \in W^{1,p}_0(\tilde{\Omega})$. We can therefore choose it as admissible test function in (1.2.16) and (1.2.17). Whence, subtracting the two, we get

$$\begin{aligned}&\int_{\Omega^+} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\,,\,\nabla(u-v))\,dx\\ &= \int_{\Omega^+} (f(u)-f(v))(u-v)\,dx.\end{aligned} \tag{1.2.18}$$

By (1.0.2), it follows that

$$\begin{aligned}C_1 &\int_{\Omega^+} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx\\ &\le \int_{\Omega^+} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v))\,dx,\end{aligned}$$

so that

$$\begin{aligned}C_1 &\int_{\Omega^+} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx\\ &\le \int_{\Omega^+} \left|\frac{f(u)-f(v)}{u-v}\right| |u-v|^2\,dx\end{aligned} \tag{1.2.19}$$

Let us now evaluate the term on the right hand side of the above inequality. By the Lipschitz continuity of f, it follows

$$\begin{aligned}\int_{\Omega^+} \left|\frac{f(u)-f(v)}{u-v}\right| |u-v|\,dx &\le C\int_{\Omega^+} |u-v|^2\,dx\\ &\le C\,C_P(|\Omega^+|)\int_{\Omega^+} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx\end{aligned}$$

Concluding, exploiting the above estimates, we get

$$\begin{aligned}&\int_{\Omega^+} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx\\ &\le \tilde{C}\,C_P(|\Omega^+|)\int_{\Omega^+} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx\end{aligned}$$

which gives a contradiction for $\tilde{C}\,C_P(|\Omega^+|)) < 1$. Now, since $|\tilde{\Omega}| \leq \vartheta$ by assumption, it follows that if ϑ is sufficiently small, then we may assume that $C_P(|\Omega^+|)$ is also small, and that $\tilde{C}\,C_P(|\Omega^+|)) < \frac{1}{2}$. Hence we have a contradiction, and shows that actually $(u-v)^+ = 0$ and the thesis.

□

1.3. Weak comparison principles in narrow unbounded domains

In this section we show some well known results about weak comparison principles in narrow unbounded domains involving the p-Laplacian, that are very useful tools in the proof of monotonicity results in the half-space. It is well known that the first result for $p = 2$ is due to H. Berestycki, L. Caffarelli and L. Nirenberg and can be found in the seminal papers [8, 9, 10]. We point out that the technique used in these papers works only in the semilinear case, since the authors need to construct an explicit solution to the problem. In a series of papers by A. Farina, L. Montoro and B. Sciunzi [59, 60, 61] and also in a paper by A. Farina, L. Montoro, G. Riey and B. Sciunzi [58] the authors used a new technique to prove monotonicity results in the half-space that works also in the case of quasilinear elliptic equations involving the p-Laplace operator. The singular case $1 < p \leq 2$ (see [58, 59]), is simpler than the degenerate one $p > 2$ (see [60, 61]), since we have in force the classical Poincaré inequality (presented in the previous section), that, used in a tricky way, it is one of the main tool in the proof of the comparison principle in narrow strips. In the case $p > 2$, that we are going to consider at the end of this section, the use of weighted Sobolev spaces is naturally associated to the study of qualitative properties of the solutions, as discussed in Section 1.1. This issue is more delicate in unbounded domains. Let us only say that the use of weighted Sobolev spaces is necessary in the case $p > 2$ and it requires the use of a weighted Poincaré type inequality with weight $\varrho = |\nabla u|^{p-2}$ (see [37] and also Section 1.1). The latter involves constants that may blow up when the solution approaches zero that may happen also for positive solutions in unbounded domains. Namely once again the lack of compactness plays an important role. In the same spirit of the papers cited above, we start this section showing a result about the weak comparison principle in strips when $1 < p \leq 2$, whose proof, based on an iterative argument, is also new in the semilinear case $p = 2$ and can be found in [59].

Theorem 1.3.1 ([59]). *We suppose $N \geq 2$, $1 < p \leq 2$, $\lambda > 0$ and assume that f is locally Lipschitz continuous. Set*

$$\Sigma_{\lambda_{y_0}} := \mathbb{R}^{N-1} \times \left[y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2} \right], \quad y_0 \geq \frac{\lambda}{2}.$$

Consider respectively $u, v \in C^{1,\alpha}_{loc}(\Sigma_{\lambda_{y_0}})$ a sub and super-solution to the following quasilinear elliptic equation

$$-\Delta_p w = f(w) \ \text{in} \ \Sigma_{\lambda_{y_0}} \tag{1.3.1}$$

with $u, \nabla u, v, \nabla v \in L^\infty(\Sigma_{\lambda_{y_0}})$. If $u \leq v$ on $\partial\Sigma_{\lambda_{y_0}}$, then there exists

$$\lambda_0 = \lambda_0(N, p, \|\nabla u\|_\infty, \|\nabla v\|_\infty, \|u\|_\infty, \|v\|_\infty, f) > 0$$

such that if, $0 < \lambda < \lambda_0$, *it follows that*

$$u \le v \qquad \text{in } \Sigma_{\lambda_{y_0}}.$$

If u and v are not assumed to be bounded, the same conclusion holds, if we assume that the nonlinearity f is globally *Lipschitz continuous.*

We start proving a lemma that will be useful in the proof of Theorem 1.3.1:

Lemma 1.3.2 ([59]). *Let $\vartheta > 0$ and $\gamma > 0$ such that $\vartheta < 2^{-\gamma}$. Moreover let $R_0 > 0$, $c > 0$ and*

$$\mathcal{L} : (R_0, +\infty) \to \mathbb{R}$$

a non-negative and non-decreasing function such that

$$\begin{cases} \mathcal{L}(R) \le \vartheta \mathcal{L}(2R) + g(R) & \forall R > R_0, \\ \mathcal{L}(R) \le CR^{\gamma} & \forall R > R_0, \end{cases} \tag{1.3.2}$$

where $g : (R_0, +\infty) \to \mathbb{R}^+$ is such that

$$\lim_{R \to +\infty} g(R) = 0.$$

Then

$$\mathcal{L}(R) = 0.$$

Proof. It is sufficient to prove that

$$l := \lim_{R \to +\infty} \mathcal{L}(R) = 0.$$

By contradiction suppose that $l \neq 0$ and choose ϑ_1 such that $\vartheta < \vartheta_1 < 2^{-\gamma}$. This implies the exixtence of $R_1 = R_1(\vartheta_1) \ge R_0$ such that

$$(\vartheta - \vartheta_1)\mathcal{L}(2R) + g(R) < 0 \quad \forall R \ge R_1,$$

and then

$$\mathcal{L}(R) \le \vartheta_1 \mathcal{L}(2R) \quad \forall R \ge R_1. \tag{1.3.3}$$

By (1.3.3) we have: $\forall \bar{l} \in \mathbb{N}^*, \ \ \forall R \ge R_1$

$$\begin{aligned} \mathcal{L}(R) &\le \vartheta_1^{\bar{l}} \mathcal{L}(2^{\bar{l}} R) \\ &\le C \vartheta_1^{\bar{l}} (2^{\bar{l}} R)^{\gamma} \\ &= C (2^{\gamma} \vartheta_1)^{\bar{l}} R^{\gamma}, \end{aligned} \tag{1.3.4}$$

where we have used that $\mathcal{L}(R) \le CR^{\gamma}$ for $R > R_0$, by (1.3.2).
Since $0 < \vartheta_1 < 2^{-\gamma}$, by (1.3.4) we obtain

$$\mathcal{L}(R) \le \lim_{\bar{l} \to +\infty} C(2^{\gamma}\vartheta_1)^{\bar{l}} R^{\gamma} = 0 \quad \forall R \ge R_1,$$

getting the contradiction.

□

Proof of Theorem 1.3.1. We therefore assume that $N \ge 2$, $1 < p \le 2$, $\lambda > 0$ and that f is locally Lipschitz continuous. We consider $u, v \in C^{1,\alpha}_{loc}$ with $u, \nabla u, v, \nabla v \in L^{\infty}(\Sigma_{\lambda_{y_0}})$ such that u, v weakly solve (1.3.1).
We want to show that there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then

$$u \le v \qquad \text{in } \Sigma_{\lambda_{y_0}}.$$

We carry out the proof in the case $u, v \in L^\infty(\Sigma_{\lambda_{y_0}})$.The same proof works when u and v may be not bounded, but f is *globally* Lipschitz continuous.

First of all we remark that $(u-v)^+ \in L^\infty(\Sigma_{\lambda_{y_0}})$ since we assumed u, v to be bounded in $\Sigma_{\lambda_{y_0}}$.
Let us now define

$$\Psi = [(u-v)^+]^\alpha \varphi^2, \tag{1.3.5}$$

where $\alpha > 1$, and $\varphi(x', y) = \varphi(x') \in C_c^\infty(\mathbb{R}^{N-1})$, $\varphi \geq 0$ such that

$$\begin{cases} \varphi \equiv 1, & \text{in } B'(0,R) \subset \mathbb{R}^{N-1}, \\ \varphi \equiv 0, & \text{in } \mathbb{R}^{N-1} \setminus B'(0,2R), \\ |\nabla \varphi| \leq \frac{C}{R}, & \text{in } B'(0,2R) \setminus B'(0,R) \subset \mathbb{R}^{N-1}. \end{cases} \tag{1.3.6}$$

We note that $\Psi \in W_0^{1,p}(\Sigma_{\lambda_{y_0}})$ by (1.3.6) and since $u \leq v$ on $\partial \Sigma_{\lambda_{y_0}}$.
Let us define the cylinder

$$\mathcal{C}(R) := \left\{ \Sigma_{\lambda_{y_0}} \cap \overline{\{B'(0,R) \times \mathbb{R}\}} \right\}.$$

Then using Ψ as test function in both equations of problem (1.3.1) and substracting we get

$$\begin{aligned} &\alpha \int_{\mathcal{C}(2R)} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+\right) [(u-v)^+]^{\alpha-1}\varphi^2 \, dx \\ &+ \int_{\mathcal{C}(2R)} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi^2\right) [(u-v)^+]^\alpha \, dx \\ &= \int_{\mathcal{C}(2R)} (f(u) - f(v))[(u-v)^+]^\alpha \varphi^2 \, dx. \end{aligned}$$

Taking into account (1.0.2) and the fact that $p \leq 2$, we have

(1.3.7)

$$\begin{aligned} &\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla(u-v)^+|^2 [(u-v)^+]^{\alpha-1}\varphi^2 \, dx \\ &\leq \alpha \int_{\mathcal{C}(2R)} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+\right) [(u-v)^+]^{\alpha-1}\varphi^2 \, dx \\ &= - \int_{\mathcal{C}(2R)} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi^2\right) [(u-v)^+]^\alpha \, dx \\ &\quad + \int_{\mathcal{C}(2R)} (f(u) - f(v))[(u-v)^+]^\alpha \varphi^2 \, dx \\ &\leq \int_{\mathcal{C}(2R)} \left| \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi^2\right) \right| [(u-v)^+]^\alpha \, dx \\ &\quad + \int_{\mathcal{C}(2R)} |(f(u) - f(v))| \, [(u-v)^+]^\alpha \varphi^2 \, dx \\ &\leq C_2 \int_{\mathcal{C}(2R)} |\nabla(u-v)|^{p-1} |\nabla \varphi^2| [(u-v)^+]^\alpha \, dx \\ &\quad + \int_{\mathcal{C}(2R)} \left| \frac{(f(u) - f(v))}{(u-v)} \right| [(u-v)^+]^{\alpha+1} \varphi^2 \, dx. \end{aligned}$$

Then, since $u, v \in C^{1,\alpha}_{loc}$ have bounded gradient by assumption, one has

$$\begin{aligned}(1.3.8)\qquad &\alpha c_1 \int_{\mathcal{C}(2R)} |\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\varphi^2\,dx\\ &\le c_2\int_{\mathcal{C}(2R)}[(u-v)^+]^\alpha|\nabla\varphi^2|\,dx + \int_{\mathcal{C}(2R)} L_f[(u-v)^+]^{\alpha+1}\varphi^2\,dx\\ &:= c_2 I_1 + L_f I_2,\end{aligned}$$

where

$$c_1 = (\|\nabla u\|_\infty + \|\nabla v\|_\infty)^{p-2}C_1,$$

$$c_2 = (\|\nabla u\|_\infty + \|\nabla v\|_\infty)^{p-1}C_2.$$

L_f is the Lipschitz constant of f in the interval

$$[-\max\{\|u\|_\infty, \|v\|_\infty\}, \max\{\|u\|_\infty, \|v\|_\infty\}].$$

We now evaluate the term

$$I_1 = \int_{\mathcal{C}(2R)}[(u-v)^+]^\alpha|\nabla\varphi^2|\,dx.$$

(1.3.9)

$$\begin{aligned}I_1 &\le 2\int_{\mathcal{C}(2R)}[(u-v)^+]^\alpha\varphi|\nabla\varphi|\,dx = 2\int_{\mathcal{C}(2R)}[(u-v)^+]^\alpha\varphi|\nabla\varphi|^{\frac12}|\nabla\varphi|^{\frac12}\,dx\\ &\le 2\int_{\mathcal{C}(2R)}\frac{[(u-v)^+]^{\alpha+1}\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}}{\frac{\alpha+1}{\alpha}}\,dx + 2\int_{\mathcal{C}(2R)}\frac{|\nabla\varphi|^{\frac{\alpha+1}{2}}}{\alpha+1}\,dx\\ &\le 2\int_{\mathbb{R}^{N-1}}\left(\int_{y_0-\frac{\lambda}{2}}^{y_0+\frac{\lambda}{2}}\left([(u-v)^+]^{\frac{\alpha+1}{2}}\right)^2dy\right)\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}\,dx'\\ &\quad + 2\int_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}\,dx\\ &\le C_p^2(\lambda)\frac{(\alpha+1)^2}{2}\int_{\mathcal{C}(2R)}[(u-v)^+]^{\alpha-1}|\partial_y(u-v)^+|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}\,dx\\ &\quad + 2\int_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}\,dx\\ &\le C_p^2(\lambda)\frac{(\alpha+1)^2}{2}\int_{\mathcal{C}(2R)}[(u-v)^+]^{\alpha-1}|\nabla(u-v)^+|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}\,dx\\ &\quad + 2\int_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}\,dx.\end{aligned}$$

In (1.3.9) we used Young's inequality with conjugate exponents $\left(\dfrac{\alpha+1}{\alpha}, \alpha+1\right)$, a Poincaré inequality in the set $\left[y_0-\dfrac{\lambda}{2}, y_0+\dfrac{\lambda}{2}\right]$, denoting with C_p the associated constant and the fact that $\varphi = \varphi(x')$.
We now evaluate the term

$$I_2 = \int_{\mathcal{C}(2R)}[(u-v)^+]^{\alpha+1}\varphi^2\,dx.$$

(1.3.10)

$$\begin{aligned}
I_2 &= \int_{\mathcal{C}(2R)} \left([(u-v)^+]^{\frac{\alpha+1}{2}}\right)^2 \varphi^2\, dx \\
&= \int_{\mathbb{R}^{N-1}} \left(\int_{y_0-\frac{\lambda}{2}}^{y_0+\frac{\lambda}{2}} \left([(u-v)^+]^{\frac{\alpha+1}{2}}\right)^2 dy \right) (\varphi(x'))^2 dx' \\
&\le C_p^2(\lambda) \int_{\mathbb{R}^{N-1}} \left(\int_{y_0-\frac{\lambda}{2}}^{y_0+\frac{\lambda}{2}} \left(\frac{\alpha+1}{2}\right)^2 [(u-v)^+]^{\alpha-1} |\partial_y (u-v)^+|^2 dy \right) \varphi^2 dx' \\
&= C_p^2(\lambda) \left(\frac{\alpha+1}{2}\right)^2 \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha-1} |\nabla (u-v)^+|^2 \varphi^2\, dx
\end{aligned}$$

Now we are going to choose the constants $\alpha > 1$ and $\lambda > 0$ in such a way

$$L_f C_p^2(\lambda) \left(\frac{\alpha+1}{2}\right)^2 < \frac{\alpha c_1}{2} \tag{1.3.11}$$

so that from (1.3.8) we have

$$\begin{aligned}
&\alpha \frac{c_1}{2} \int_{\mathcal{C}(2R)} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1} \varphi^2\, dx \\
&\le c_2 \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha} |\nabla \varphi^2|\, dx = c_2 I_1 .
\end{aligned} \tag{1.3.12}$$

From (1.3.23) one has that

$$\begin{aligned}
&\alpha \frac{c_1}{2} \int_{\mathcal{C}(R)} |\nabla (u-v)^+|^2 (u-v)^{\alpha-1}\, dx \\
&\le \alpha \frac{c_1}{2} \int_{\mathcal{C}(2R)} |\nabla (u-v)^+|^2 (u-v)^{\alpha-1} \varphi^2\, dx \le c_2 I_1 .
\end{aligned}$$

Consequently we obtain

(1.3.13)

$$\begin{aligned}
&\int_{\mathcal{C}(R)} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1}\, dx \\
&\le \frac{c_2}{\alpha c_1} C_p^2(\lambda)(\alpha+1)^2 \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha-1} |\nabla (u-v)^+|^2 \varphi^{\frac{\alpha+1}{\alpha}} |\nabla \varphi|^{\frac{\alpha+1}{2\alpha}}\, dx \\
&+ 4 \frac{c_2}{\alpha c_1} \int_{\mathcal{C}(2R)} |\nabla \varphi|^{\frac{\alpha+1}{2}}\, dx .
\end{aligned}$$

From (1.3.13), setting $\alpha = 2N+1$, one has

(1.3.14)

$$\begin{aligned}
&\int_{\mathcal{C}(R)} |\nabla (u-v)^+|^2 (u-v)^{\alpha-1}\, dx \\
&\le \vartheta \int_{\mathcal{C}(2R)} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1}\, dx + 4 \frac{c_2}{\alpha c_1} C\lambda R^{N-1} R^{-(N+1)} \\
&= \vartheta \int_{\mathcal{C}(2R)} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1}\, dx + c_3 R^{-2},
\end{aligned}$$

where

$$c_3 = 4 \frac{c_2}{\alpha c_1} C\lambda \in \mathbb{R}^+,$$

$$\frac{c_2}{\alpha c_1} C_p^2(\lambda)(2N+2)^2 = \vartheta < 2^{-N}.$$

In particular to do this, recalling that $C_p^2(\lambda) \simeq \lambda^2$, $\lambda > 0$ will be taken such that

$$\frac{c_2}{\alpha c_1} C_p^2(\lambda)(\alpha+1)^2 < 2^{-N}. \tag{1.3.15}$$

Let us set

$$\mathcal{L}(R) = \int_{\mathcal{C}(R)} |\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\, dx,$$

and

$$g(R) = c_3 R^{-2}.$$

Then one has

$$\begin{cases} \mathcal{L}(R) \le \vartheta \mathcal{L}(2R) + g(R) & \forall R > 0, \\ \mathcal{L}(R) \le CR^N & \forall R > 0, \end{cases}$$

and from Lemma 1.3.2 with $\gamma = N$, since we assumed $\vartheta < 2^{-N}$, we get $\mathcal{L}(R) = 0$ and consequently the thesis.

□

We point out the fact that monotonicity results in the half-space are generally based on weak comparison principles in narrow strips as it was done in a series of papers, some of them mentioned before, by H. Beresticky, L. Caffarelli and L. Nirenberg [**8**, **9**, **10**], E. N. Dancer [**39**], L. Damascelli and F. Gladiali [**34**] and by A. Farina, L. Montoro and B. Sciunzi, see [**56**, **58**, **59**, **60**, **61**]. In our case, the presence of the therm $|\nabla u|^{p-2}$ gives rise to a phenomenon that was first pointed out in [**33**, **35**], in the case of bounded domains. Namely, it is possible to prove monotonicity results via a weak comparison principle in domains that can be decomposed into two parts. A narrow part (w.r.t. the Lebesgue measure of the section) and a part where the gradient of the solution is small.

We have the following:

Theorem 1.3.3 ([**58**]). *Let $1 < p < 2$, $N \ge 2$ and let us assume that f is a locally Lipschitz continuous function. Fix $\lambda_0 > 0$ and $M_0 > 0$. Consider $\lambda \in (0, \lambda_0]$, $\tau, \varepsilon > 0$ and set*

$$\Sigma_{\lambda_{y_0}} := \mathbb{R}^{N-1} \times \left(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\right), \quad y_0 \ge \frac{\lambda}{2}. \tag{1.3.16}$$

Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma}_{\lambda_{y_0}})$ such that $\|u\|_\infty + \|\nabla u\|_\infty \le M_0$, $\|v\|_\infty + \|\nabla v\|_\infty \le M_0$ and

$$\begin{cases} -\Delta_p u \le f(u), & \text{in } \Sigma_{\lambda_{y_0}}, \\ -\Delta_p v \ge f(v), & \text{in } \Sigma_{\lambda_{y_0}}, \\ u \le v, & \text{on } \partial \mathcal{S}_\tau^\varepsilon, \end{cases} \tag{1.3.17}$$

where the open set $\mathcal{S}_\tau^\varepsilon \subseteq \Sigma_{\lambda_{y_0}}$ is such that

$$\mathcal{S}_\tau^\varepsilon = \bigcup_{x' \in \mathbb{R}^{N-1}} I_{x'}^{\tau,\varepsilon},$$

and the open set $I_{x'}^{\tau,\varepsilon} \subseteq \{x'\} \times \big(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\big)$ *has the form*

$$I_{x'}^{\tau,\varepsilon} = A_{x'}^{\tau} \cup B_{x'}^{\varepsilon} \qquad \text{with} \quad |A_{x'}^{\tau} \cap B_{x'}^{\varepsilon}| = 0 \tag{1.3.18}$$

and, for x' *fixed,* $A_{x'}^{\tau}$, $B_{x'}^{\varepsilon} \subset \big(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\big)$ *are measurable sets such that*

$$|A_{x'}^{\tau}| \leq \tau \quad \text{and} \quad B_{x'}^{\varepsilon} \subseteq \{y \in \mathbb{R} : |\nabla u(x', y)| < \varepsilon, |\nabla v(x', y)| < \varepsilon\}.$$

Then there exist

$$\tau_0 = \tau_0(N, p, \lambda_0, M_0) > 0$$

and

$$\varepsilon_0 = \varepsilon_0(N, p, \lambda_0, M_0) > 0$$

such that, if $0 < \tau < \tau_0$ *and* $0 < \varepsilon < \varepsilon_0$*, it follows that*

$$u \leq v \qquad \text{in} \quad \mathcal{S}_{\tau}^{\varepsilon}.$$

If the functions f *is assumed to be* globally *Lipschitz continuous on* $\mathbb{R}^N_+ \times \mathbb{R}$*, the same conclusion holds true without any assumption on the boundedness of* u *and* v*.*

Moreover, as a consequence of the previous theorem we have:

Theorem 1.3.4 ([58]). *Let* $1 < p < 2$, $N \geq 2$ *and let us assume that* f *is a locally Lipschitz continuous function. Consider* $\lambda > 0$ *and set*

$$\Sigma_{\lambda_{y_0}} := \mathbb{R}^{N-1} \times \left(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\right), \quad y_0 \geq \frac{\lambda}{2}.$$

Fix $M_0 > 0$ *and let* $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma}_{\lambda_{y_0}})$ *such that* $\|u\|_\infty + \|\nabla u\|_\infty \leq M_0$, $\|v\|_\infty + \|\nabla v\|_\infty \leq M_0$ *and*

$$\begin{cases} -\Delta_p u \leq f(u), & \text{in } \Sigma_{\lambda_{y_0}}, \\ -\Delta_p v \geq f(v), & \text{in } \Sigma_{\lambda_{y_0}}, \\ u \leq v, & \text{on } \partial\mathcal{S}, \end{cases} \tag{1.3.19}$$

where $\mathcal{S} \subseteq \Sigma_{\lambda_{y_0}}$ *is an open subset.*
Then there exists

$$\overline{\lambda} = \overline{\lambda}(N, p, M_0) > 0$$

such that, if $0 < \lambda < \overline{\lambda}$*, it follows that*

$$u \leq v \qquad \text{in } \mathcal{S}.$$

If the functions f *is assumed to be* globally *Lipschitz continuous on* $\mathbb{R}^N_+ \times \mathbb{R}$*, the same conclusion holds true without any assumption on the boundedness of* u *and* v*.*

We provide now the proof of a generalized version of the Poincaré inequality in one dimension.

Lemma 1.3.5 (Poincaré type inequality). *Let* I *be an open bounded subset of* $\mathbb{R}$ *and assume that* $I = A \cup B$ *with* $|A \cap B| = 0$, A *and* B *measurable subsets of* I*. Let* $\varrho : I \to \mathbb{R} \cup \{\infty\}$ *be measurable and such that*

$$\inf_{t \in I} \varrho(t) > 0\,.$$

Then for any $w \in H_0^1(I)$ such that $\int_I \varrho(t)|\partial_t w|^2(t)dt$ is finite, the following inequality holds:

(1.3.20)
$$\int_I w^2(t)dt \le 2|I| \max\left\{|A|\sup_{t\in A}\frac{1}{\varrho(t)},\ |B|\sup_{t\in B}\frac{1}{\varrho(t)}\right\}\int_I \varrho(t)|\partial_t w|^2(t)dt.$$

Proof. Since w belongs to $H_0^1(I)$, there exists $a \in \overline{I}$ such that $w(x) = \int_a^x \partial_t w(t)dt$. Thus we have:

(1.3.21)
$$\begin{aligned}|w(x)| &\le \int_a^x |\partial_t w(t)|dt \le \int_I |\partial_t w(t)|dt = \int_A |\partial_t w(t)|dt + \int_B |\partial_t w(t)|dt\\ &\le |A|^{1/2}\Big(\int_A |\partial_t w|^2(t)dt\Big)^{1/2} + |B|^{1/2}\Big(\int_B |\partial_t w|^2(t)dt\Big)^{1/2}\\ &\le \left(|A|\sup_{t\in A}\frac{1}{\varrho(t)}\right)^{1/2}\left(\int_A \varrho(t)|\partial_t w|^2(t)dt\right)^{1/2}\\ &\quad + \left(|B|\sup_{t\in B}\frac{1}{\varrho(t)}\right)^{1/2}\left(\int_B \varrho(t)|\partial_t w|^2(t)dt\right)^{1/2}.\end{aligned}$$

Finally, using (1.3.21) we obtain:

$$\begin{aligned}&\int_I w^2(t)dt \le |I|\sup_{t\in I} w^2(t)\\ &\le 2|I|\left(|A|\sup_{t\in A}\frac{1}{\varrho(t)}\int_A \varrho\,|\partial_t w|^2(t)dt + |B|\sup_{t\in B}\frac{1}{\varrho(t)}\int_B \varrho\,|\partial_t w|^2(t)dt\right),\end{aligned}$$

from which the thesis immediately follows.

□

Proof of Theorem 1.3.3: In the proof we denote by $\|\cdot\|_\infty$, the L^∞ norm in $\Sigma_{\lambda_{y_0}}$. We remark that $(u-v)^+$ belongs to $L^\infty(\Sigma_{\lambda_{y_0}})$ since u and v are bounded in $\Sigma_{\lambda_{y_0}}$.
For $\alpha > 1$ we define

(1.3.22)
$$\psi = [(u-v)^+]^\alpha \varphi^2,$$

where $\varphi(x', y) = \varphi(x') \in C_c^\infty(\mathbb{R}^{N-1})$ is such that

(1.3.23)
$$\begin{cases}\varphi \ge 0, & \text{in } \mathbb{R}^N_+\\ \varphi \equiv 1, & \text{in } B'(0,R) \subset \mathbb{R}^{N-1},\\ \varphi \equiv 0, & \text{in } \mathbb{R}^{N-1}\setminus B'(0,2R),\\ |\nabla\varphi| \le \frac{C}{R}, & \text{in } B'(0,2R)\setminus B'(0,R) \subset \mathbb{R}^{N-1},\end{cases}$$

where $B'(0,R) = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$, $R > 1$ and C is a positive constant.

Let $\mathcal{C}(R)$ be defined as

$$\mathcal{C}(R) := \left\{\mathcal{S}_\tau^\varepsilon \cap \{B'(0,R)\times\mathbb{R}\}\right\}.$$

The assumptions in (1.3.23) and the inequality $u \leq v$ on $\partial\mathcal{S}_\tau^\varepsilon$ imply that $\psi \in W_0^{1,p}(\mathcal{C}(2R))$. This allows us to use ψ as test function in both equations of problem (1.3.17) and to get (by subtracting):

(1.3.24)
$$\int_{\mathcal{C}(2R)} \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\psi\big)\, dx \leq \int_{\mathcal{C}(2R)} (f(u) - f(v))\psi\, dx\,,$$

Using (1.3.22) we obtain:

$$\begin{aligned} &\alpha \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+)[(u-v)^+]^{\alpha-1}\varphi^2\, dx \\ &\leq \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\varphi^2)[(u-v)^+]^{\alpha}\, dx \\ &\quad + L_f \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha+1}\varphi^2\, dx. \end{aligned} \tag{1.3.25}$$

Recalling (1.0.2), $|\nabla u|$ and $|\nabla v|$ are bounded and $\alpha > 1$, from (1.3.25) we obtain

$$\begin{aligned} &C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\varphi^2\, dx \\ &\leq C_2 \int_{\mathcal{C}(2R)} |\nabla(u-v)^+|^{p-1}|\nabla\varphi^2|[(u-v)^+]^{\alpha}\, dx \\ &\quad + L_f \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha+1}\varphi^2\, dx \\ &\leq C_2(2M_0)^{p-1} \int_{\mathcal{C}(2R)} |\nabla\varphi^2|[(u-v)^+]^{\alpha}\, dx \\ &\quad + L_f \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha+1}\varphi^2\, dx. \end{aligned} \tag{1.3.26}$$

Let us define

$$c_1 := \frac{C_2(2M_0)^{p-1}}{C_1}, \tag{1.3.27}$$

$$c_2 := \frac{L_f}{C_1}, \tag{1.3.28}$$

$$I_1 := \int_{\mathcal{C}(2R)} |\nabla\varphi^2|[(u-v)^+]^{\alpha}\, dx\,, \quad I_2 := \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha+1}\varphi^2\, dx$$

and note that both c_1 and c_2 depend only on p and M_0, in particular they are independent of $\alpha > 1$.

Thus, with the definitions above, we now rewrite (1.3.26) as follows: for every $\alpha > 1$,

$$\begin{aligned} &\int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\varphi^2\, dx \\ &\leq c_1 I_1 + c_2 I_2. \end{aligned} \tag{1.3.29}$$

We also observe that

$$\int_{\mathbb{R}^{N-1}}\Big(\int_{I^{\tau,\varepsilon}_{x'}}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\Big)dy\varphi^2(x')dx'$$
$$=\int_{\mathcal{C}(2R)}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\varphi^2\,dx<+\infty$$

since φ depends only on x' and the right-hand-side of (1.3.29) is finite. Hence, for almost every $x'\in\mathbb{R}^{N-1}$ we have that

$$\int_{I^{\tau,\varepsilon}_{x'}}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}dy<+\infty, \tag{1.3.30}$$

which also entails: for almost every $x'\in\mathbb{R}^{N-1}$

$$\int_{I^{\tau,\varepsilon}_{x'}}(|\nabla u|+|\nabla v|)^{p-2}|\partial_y(u-v)^+|^2[(u-v)^+]^{\alpha-1}dy<+\infty. \tag{1.3.31}$$

Estimate for I_1. Let us recall the decomposition stated in (1.3.18) which gives

$$\mathcal{S}^\varepsilon_\tau=\bigcup_{x'\in\mathbb{R}^{n-1}}I^{\tau,\varepsilon}_{x'}\qquad\text{with}\qquad I^{\tau,\varepsilon}_{x'}=A^\tau_{x'}\cup B^\varepsilon_{x'}.$$

We set

$$\varrho_{x'}(t)=(|\nabla u(x',t)|+|\nabla v(x',t)|)^{p-2}$$

in order to apply Lemma 1.3.5 in each $I^{\tau,\varepsilon}_{x'}$, for which (1.3.31) holds true, with $\varrho(t):=\varrho_{x'}(t)$, $A:=A^\tau_{x'}$, $B:=B^\varepsilon_{x'}$ and $w(t)=[(u-v)^+(x',t)]^{\frac{\alpha+1}{2}}$. Note that the constant in (1.3.20) in this case is given by:

$$C_{\tau,\varepsilon}(x')=2\lambda\max\left\{|A^\tau_{x'}|\sup_{t\in A^\tau_{x'}}\frac{1}{\varrho_{x'}(t)},\,|B^\varepsilon_{x'}|\sup_{t\in B^\varepsilon_{x'}}\frac{1}{\varrho_{x'}(t)}\right\}.$$

Therefore, for almost every $x'\in\mathbb{R}^{N-1}$, we have

$$C_{\tau,\varepsilon}(x')\le C_{\tau,\varepsilon}:=2\lambda_0\max\left\{\tau(2M_0)^{2-p}\,,\ \lambda_0(2\varepsilon)^{2-p}\right\}, \tag{1.3.32}$$

so that, since $1<p<2$, $C_{\tau,\varepsilon}$ can be chosen arbitrary small, for τ and ε sufficiently small. Now, recalling that φ depends only on x' and using Young's inequality with conjugate exponents $\frac{\alpha+1}{\alpha}$ and $\alpha+1$, we get:

(1.3.33)
$$I_1\le 2\int_{\mathcal{C}(2R)}[(u-v)^+]^\alpha\varphi|\nabla\varphi|\,dx=2\int_{\mathcal{C}(2R)}[(u-v)^+]^\alpha\varphi|\nabla\varphi|^{\frac12}|\nabla\varphi|^{\frac12}\,dx$$
$$\le 2\int_{\mathcal{C}(2R)}\frac{[(u-v)^+]^{\alpha+1}\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}}{\frac{\alpha+1}{\alpha}}\,dx+2\int_{\mathcal{C}(2R)}\frac{|\nabla\varphi|^{\frac{\alpha+1}{2}}}{\alpha+1}\,dx$$
$$\le 2\int_{\mathbb{R}^{N-1}}\left(\int_{I^{\tau,\varepsilon}_{x'}}\left([(u-v)^+]^{\frac{\alpha+1}{2}}\right)^2dy\right)\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}\,dx'$$
$$+2\int_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}\,dx$$

and the application of Lemma 1.3.5 yields

(1.3.34)

$$\begin{aligned} I_1 \leq C_{\tau,\varepsilon}\frac{(\alpha+1)^2}{2} \\ \int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}[(u-v)^+]^{\alpha-1}|\partial_y(u-v)^+|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}\,dx \\ +2\int_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}\,dx \\ \leq C_{\tau,\varepsilon}\frac{(\alpha+1)^2}{2} \\ \int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}[(u-v)^+]^{\alpha-1}|\nabla(u-v)^+|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}\,dx \\ +2\int_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}\,dx\,, \end{aligned}$$

where $C_{\tau,\varepsilon}$ has been defined in (1.3.32).

Estimate for I_2. We use the same notations as in the evaluation of I_1 and we get:

(1.3.35)

$$\begin{aligned} I_2 &= \int_{\mathcal{C}(2R)}\left([(u-v)^+]^{\frac{\alpha+1}{2}}\right)^2\varphi^2\,dx \\ &= \int_{\mathbb{R}^{N-1}}\left(\int_{I_{x'}^{\tau,\varepsilon}}\left([(u-v)^+]^{\frac{\alpha+1}{2}}\right)^2dy\right)(\varphi(x'))^2dx' \\ &\leq C_{\tau,\varepsilon}\left(\frac{\alpha+1}{2}\right)^2 \\ &\int_{\mathbb{R}^{N-1}}\left(\int_{I_{x'}^{\tau,\varepsilon}}(|\nabla u|+|\nabla v|)^{p-2}[(u-v)^+]^{\alpha-1}|\partial_y(u-v)^+|^2dy\right)\varphi^2dx' \\ &\leq C_{\tau,\varepsilon}\left(\frac{\alpha+1}{2}\right)^2\int_{\mathcal{C}(2R)}(|\nabla u|+|\nabla v|)^{p-2}[(u-v)^+]^{\alpha-1}|\nabla(u-v)^+|^2\varphi^2\,dx. \end{aligned}$$

Let us fix

$$\alpha = 2N+1 > 1. \tag{1.3.36}$$

Recalling that $C_{\tau,\varepsilon}$ tends to 0, as both τ and ε go to zero, we can take $\tau>0$ and $\varepsilon>0$ small enough, such that

$$c_2C_{\tau,\varepsilon}\left(\frac{\alpha+1}{2}\right)^2<\frac{1}{2},\qquad c_1C_{\tau,\varepsilon}(\alpha+1)^2<2^{-N} \tag{1.3.37}$$

so that from (1.3.29) we have

$$\int_{\mathcal{C}(2R)}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}\varphi^2\,dx \leq 2c_1I_1. \tag{1.3.38}$$

By (1.3.23) we infer that

$$\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 (u-v)^{\alpha-1}\, dx \tag{1.3.39}$$
$$\leq \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 (u-v)^{\alpha-1} \varphi^2\, dx \leq 2c_1 I_1$$

and, using (1.3.34), we obtain

(1.3.40)

$$\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1}\, dx$$
$$\leq c_1 C_{\tau,\varepsilon} (\alpha+1)^2$$
$$\int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} [(u-v)^+]^{\alpha-1} |\nabla (u-v)^+|^2 \varphi^{\frac{\alpha+1}{\alpha}} |\nabla \varphi|^{\frac{\alpha+1}{2\alpha}}\, dx$$
$$+ 4c_1 \int_{\mathcal{C}(2R)} |\nabla \varphi|^{\frac{\alpha+1}{2}}\, dx.$$

Recalling (1.3.36) one has:

(1.3.41)

$$\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 (u-v)^{\alpha-1}\, dx$$
$$\leq \vartheta \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1}\, dx + \hat{C} R^{-2},$$

where

$$\vartheta = c_1 C_{\tau,\varepsilon} (\alpha+1)^2,$$
$$\hat{C} = 4c_1 \lambda C^{\frac{\alpha+1}{2}} > 0$$

exploiting also (1.3.23). Notice that, in view of (1.3.37), we also have that $\vartheta < 2^{-N}$. In order to apply Lemma 1.3.2 we set

$$\mathcal{L}(R) = \int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1}\, dx,$$

and

$$g(R) = \hat{C} R^{-2}.$$

Then from (1.3.41) we have:

$$\begin{cases} \mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) + g(R) & \forall R > 0, \\ \mathcal{L}(R) \leq C R^N & \forall R > 0. \end{cases}$$

Applying Lemma 1.3.2 with $\beta = N$, we get $\mathcal{L}(R) = 0$ and consequently the thesis.

□

Proof of Theorem 1.3.4.
The desired result is obtained with the same proof of that of Theorem 1.3.3 with the following slight (but necessary) modifications. Replace $\mathcal{S}_\tau^\varepsilon$ by $\mathcal{S}$, set $\varepsilon = \tau = \lambda$, $B_{x'}^\varepsilon = \emptyset$, $I_{x'}^{\tau,\varepsilon} = A_{x'}^\tau = \mathcal{S} \cap \{x'\} \times \left(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\right)$ and observe that (1.3.32) becomes

$$C_\lambda(x') \leq C_\lambda := 2\lambda^2 (2M_0)^{2-p}, \tag{1.3.42}$$

and that (1.3.37) becomes

$$c_2 C_\lambda \left(\frac{\alpha+1}{2}\right)^2 < \frac{1}{2}, \qquad c_1 C_\lambda (\alpha+1)^2 < 2^{-N}. \tag{1.3.43}$$

The conclusion the follows by taking λ small enough in the latter one.

□

Now we want to present a weak comparison principle in unbounded strips that has a natural application in the study of monotonicity properties of solutions in the half-space of the following degenerate quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \mathbb{R}^N_+ \\ u(x', y) \geq 0 & \text{in } \mathbb{R}^N_+ \\ u(x', 0) = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{1.3.44}$$

where $N \geq 2$, $p > 2$ and $f(\cdot)$ satisfies:

(E_f) the nonlinearity f is *positive* i.e. $f(t) > 0$ for $t > 0$, locally Lipschitz continuous in $\mathbb{R}^+ \cup \{0\}$ and

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = f_0 \in \mathbb{R}^+ \cup \{0\}.$$

We state the following:

THEOREM 1.3.6. *Let $p > 2$ and let $u, v \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ be two positive weak solutions to (1.3.44) with $|\nabla u|, |\nabla v| \in L^\infty(\mathbb{R}^N_+)$. For $\lambda > 0$ fixed such that $0 \leq \alpha < \beta \leq \lambda$, let $\Sigma_{(\alpha,\beta)} := \mathbb{R}^{N-1} \times (\alpha, \beta)$, $\Sigma_\beta = \Sigma_{(0,\beta)}$ and assume that*

$$u \leq v \qquad \text{on} \quad \partial\Sigma_{(\alpha,\beta)}\,. \tag{1.3.45}$$

Assume furthermore that, setting

$$\mathcal{I}^+_{(\lambda)} = \Big\{(x', \lambda) \,:\, x' \in \mathcal{P}\big(Supp\,(u-v)^+\big)\Big\},$$

it holds that

$$u(x) \geq \gamma > 0 \quad \text{on} \quad \mathcal{I}^+_{(\lambda)}. \tag{1.3.46}$$

Then, for $\Lambda > 0$ fixed such that

$$\Lambda \geq 2\lambda + 1\,,$$

it follows that there exists $h_0 = h_0(f, p, \gamma, N, \|\nabla u\|_{L^\infty(\Sigma_\Lambda)}, \|\nabla v\|_{L^\infty(\Sigma_\Lambda)})$ such that if $\beta - \alpha \leq h_0$ then we have

$$u \leq v \quad \text{in} \quad \Sigma_{(\alpha,\beta)}.$$

PROOF. We remark that $(u-v)^+ \in L^\infty(\Sigma_{(\alpha,\beta)})$ since we assumed $|\nabla u|$ and $|\nabla v|$ are bounded. We put

$$\mathcal{C}_{(\alpha,\beta)}(R) = \mathcal{C}(R) = \Sigma_{(\alpha,\beta)} \cap \{B'(0,R) \times \mathbb{R}\}. \tag{1.3.47}$$

Let us now define

$$\Psi = (u-v)^+ \varphi_R^2, \tag{1.3.48}$$

where $\varphi_R(x',y)=\varphi_R(x')\in C_c^\infty(\mathbb{R}^{N-1})$, $\varphi_R\geq 0$ such that

$$
\begin{cases}
\varphi_R\equiv 1, & \text{in } B^{'}(0,R)\subset\mathbb{R}^{N-1},\\
\varphi_R\equiv 0, & \text{in } \mathbb{R}^{N-1}\setminus B^{'}(0,2R),\\
|\nabla\varphi_R|\leq \frac{C}{R}, & \text{in } B^{'}(0,2R)\setminus B^{'}(0,R)\subset\mathbb{R}^{N-1},
\end{cases}
\tag{1.3.49}
$$

where $B^{'}(0,R)$ denotes the ball in $\mathbb{R}^{N-1}$ with center 0 and radius $R>0$. From now on, for the sake of simplicity, we set $\varphi_R(x',y):=\varphi(x',y)$. By (1.3.49) and by the fact that $u\leq v$ on $\partial\Sigma_\lambda^\beta$ (see (1.3.45)), it follows that

$$\Psi\in W_0^{1,p}(\mathcal{C}_{(\alpha,\beta)}(2R)).$$

Since u is a solution to problem (1.3.44), then it follows that u,v are solutions to

$$
\begin{cases}
-\Delta_p u=f(u) & \text{in } \Sigma_{(\alpha,\beta)},\\
-\Delta_p v=f(v) & \text{in } \Sigma_{(\alpha,\beta)},\\
\quad u\leq v & \text{on } \partial\Sigma_{(\alpha,\beta)}.
\end{cases}
\tag{1.3.50}
$$

Then using Ψ as test function in both equations of problem (1.3.17) and substracting we get

$$
\begin{aligned}
&\int_{\mathcal{C}(2R)}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla(u-v)^+\big)\varphi^2\,dx\\
&+\int_{\mathcal{C}(2R)}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla\varphi^2\big)(u-v)^+\,dx\\
&=\int_{\mathcal{C}(2R)}\big(f(u)-f(v)\big)(u-v)^+\varphi^2\,dx,
\end{aligned}
\tag{1.3.51}
$$

where $\mathcal{C}(\cdot)$ denotes the cylinder defined in (1.3.47). By (1.0.2) and the fact that $p\geq 2$, from (1.3.51) we deduce that

$$
\begin{aligned}
&\dot{C}\int_{\mathcal{C}(2R)}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\varphi^2\,dx\\
&\leq\int_{\mathcal{C}(2R)}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla(u-v)^+\big)\varphi^2\,dx\\
&=-\int_{\mathcal{C}(2R)}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla\varphi^2\big)(u-v)^+\,dx\\
&\quad+\int_{\mathcal{C}(2R)}\big(f(u)-f(v)\big)(u-v)^+\varphi^2\,dx\\
&\leq\int_{\mathcal{C}(2R)}\big|\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla\varphi^2\big)\big|\,(u-v)^+\,dx\\
&\quad+\int_{\mathcal{C}(2R)}\big(f(u)-f(v)\big)(u-v)^+\varphi^2\,dx\\
&\leq\check{C}\int_{\mathcal{C}(2R)}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+||\nabla\varphi^2|(u-v)^+\,dx\\
&\quad+\int_{\mathcal{C}(2R)}\big(f(u)-f(v)\big)(u-v)^+\varphi^2\,dx,
\end{aligned}
\tag{1.3.52}
$$

where in the last line we used Schwarz inequality and (1.0.2). Setting

$$I_1 := \check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+| |\nabla \varphi^2| (u-v)^+ \, dx \tag{1.3.53}$$

and

$$I_2 := \int_{\mathcal{C}(2R)} \big(f(u) - f(v)\big)(u-v)^+ \varphi^2 \, dx, \tag{1.3.54}$$

(1.3.52) becomes

$$\dot{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 \varphi^2 \, dx \le I_1 + I_2. \tag{1.3.55}$$

In order to estimate the terms I_1 and I_2 in (1.3.55) we will exploit the weighted Poincaré type inequality (see Section 1.1, Corollary 1.1.4) and a covering argument that goes back to [61]. Let us consider the hypercubes Q_i of $\mathbb{R}^N$ defined by

$$Q_i = Q_i' \times [\alpha, \beta],$$

where $Q_i' \subset \mathbb{R}^{N-1}$ are hypercubes of $\mathbb{R}^{N-1}$, with edge $\beta - \alpha$ and such that

$$\bigcup_i Q_i' = \mathbb{R}^{N-1}.$$

Moreover we assume that $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and

$$\bigcup_{i=1}^{\overline{N}} \overline{Q_i} \supset \mathcal{C}(2R). \tag{1.3.56}$$

It follows as well, that each set Q_i has diameter

$$\mathrm{diam}(Q_i) = d_Q = \sqrt{N}(\beta - \alpha), \qquad i = 1, \cdots, \overline{N}. \tag{1.3.57}$$

The covering in (1.3.56) will allow us to use in each Q_i the weighted Poincaré type inequality and to take advantage of the constant C_p in Corollary 1.1.4, that turns to be not depending on the index i of (1.3.56). Later we will recollect the estimates.

Let us define

$$w(x) := \begin{cases} \Big(u - v\Big)^+ (x', y) & \text{if } (x', y) \in \overline{Q}_i; \\ -\Big(u - v\Big)^+ (x', 2\beta - y) & \text{if } (x', y) \in \overline{Q}_i^r, \end{cases} \tag{1.3.58}$$

where $(x', y) \in \overline{Q}_i^r$ iff $(x', 2\beta - y) \in \overline{Q}_i$. We claim that

$$\int_{Q_i} w^2 \, dx \le C_p(Q_i) \int_{Q_i} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \tag{1.3.59}$$

where $C_p(Q_i)$ is given by Theorem 1.1.4 and has the property that it goes to zero if the diameter of Q_i goes to zero. Actually, since $p \ge 2$, we will deduce (1.3.59) by

$$\int_{Q_i} w^2 \, dx \le C_p(Q_i) \int_{Q_i} |\nabla v|^{p-2} |\nabla w|^2 \, dx. \tag{1.3.60}$$

The fact that Corollary 1.1.4 can be applied to deduce (1.3.60) is somehow technical and we describe the procedure here below.

We have $\int_{Q_i \cup Q_i^r} w(x)\, dx = 0$ and therefore, see [70, Lemma 7.14, Lemma 7.16], it follows that

$$w(x) = \hat{C} \int_{Q_i \cup Q_i^r} \frac{(x_i - z_i)\frac{\partial w}{\partial x_i}(z)}{|x-z|^N}\, dz \qquad \text{a.e. } x \in Q_i \cup Q_i^r,$$

where $\hat{C} = \hat{C}(d_Q, N)$, is a positive constant. Arguing as in the proof of Lemma 1.2.2, then for almost every $x \in Q_i$ we have

$$\begin{aligned} |w(x)| &\leq \hat{C} \int_{Q_i \cup Q_i^r} \frac{|\nabla w(z)|}{|x-z|^{N-1}}\, dz \\ &= \hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x-z|^{N-1}} dz + \hat{C} \int_{Q_i^r} \frac{|\nabla w(z)|}{|x-z|^{N-1}}\, dz \\ &\leq 2\hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x-z|^{N-1}}\, dz\,, \end{aligned}$$

where in the last line we used the following standard changing of variables

$$(z^t)' = z' \quad \text{and} \quad z_N^t = 2\beta - z_N,$$

the fact that for $x \in Q_i$, it holds that $(|x-z|)\Big|_{z \in Q_i} \leq (|x - z^t|)\Big|_{z \in Q_i}$ and that, by (1.3.58) it holds that $|\nabla w(z)| = |\nabla w(z^t)|$.

Hence (1.1.33) holds and, in order to prove (1.3.60), we need to show that (1.1.29) holds with

$$\varrho := |\nabla u|^{p-2}\,.$$

Note now that, if w vanishes identically in Q_i, then there is nothing to prove. If not it is easy to see that by our assumptions (see (1.3.46)) and by the classical Harnack inequality, it follows that there exists $\bar{\gamma} > 0$ such that

$$u \geq \bar{\gamma} > 0 \qquad \text{in} \quad \tilde{Q}_i' \times [\lambda/2\,, 4\lambda] \tag{1.3.61}$$

where

$$\tilde{Q}_i' := \{x \in \mathbb{R}^{N-1} \;:\; dist(x, Q_i') < 1\}\,.$$

Let us consider $Q_i^{\mathcal{R}_\lambda}$ obtained by the reflection of Q_i with respect to the hyperplane $T_\lambda = \{(x', y) \in \mathbb{R}^N \;:\; y = \lambda\}$. Since $Q_i^{\mathcal{R}_\lambda}$ is bounded away from the boundary $\mathbb{R}^N$, namely

$$\text{dist}\,(Q_i^{\mathcal{R}_\lambda}\,, \{y = 0\}) \geq \lambda > 0,$$

thanks to (1.3.61) and since a sufficient condition to the summability of ϱ holds, more precisely we can apply Proposition 2.4 [60], we obtain that

$$\int_{Q_i^{\mathcal{R}_\lambda}} \frac{1}{|\nabla u|^{p-2}} \frac{1}{|x-y|^\gamma}\, dy \leq C_1^*(\beta_1, \beta_2) \qquad \text{for any} \quad x \in Q_i^{\mathcal{R}_\lambda},$$

where

$$\beta_1 = \min_{t\in[\bar\gamma, \|u\|_{L^\infty(\Sigma_\Lambda)}]} f(t) \qquad \text{and} \qquad \beta_2 = \lambda.$$

We deduce the same for v:

$$\int_{Q_i} \frac{1}{|\nabla v|^{p-2}} \frac{1}{|x-y|^\gamma}\, dy \le C_1^*(\beta_1, \beta_2) \qquad \text{for any} \quad x \in Q_i,$$

so that we can exploit Corollary 1.1.4 to deduce (1.3.60) and consequently (1.3.59).

Let us now estimate the right hand side of (1.3.55). Recalling (1.3.53) we get

$$\begin{aligned}
I_1 =& 2\check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+| \varphi |\nabla \varphi| (u-v)^+\, dx \\
=& 2\check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}+\frac{p-2}{2}} |\nabla (u-v)^+|\, \varphi |\nabla \varphi| (u-v)^+\, dx \\
\le& \delta' \check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 \varphi^2\, dx \\
&+ \frac{\check{C}}{\delta'} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 [(u-v)^+]^2\, dx,
\end{aligned}$$

where in the last inequality we used weighted Young's inequality, with δ' to be chosen later. Hence

$$I_1 \le I_1^a + I_1^b, \tag{1.3.62}$$

where

$$\begin{aligned}
I_1^a &:= \delta' \check{C} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 \varphi^2\, dx, \\
I_1^b &:= \frac{\check{C}}{\delta'} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 [(u-v)^+]^2\, dx.
\end{aligned} \tag{1.3.63}$$

Using the covering in (1.3.56), the properties of the cut-off function in (1.3.48) and the fact that $|\nabla u|$ and $|\nabla v|$ are bounded, by (1.3.59) we deduce that

(1.3.64)

$$\begin{aligned}
I_1^b \le& \sum_{i=1}^{\overline{N}} \frac{C}{\delta' R^2} \int_{\mathcal{C}(2R)\cap Q_i} [(u-v)^+]^2\, dx \\
\le& \max_i C_P(Q_i) \sum_{i=1}^{\overline{N}} \frac{C}{\delta' R^2} \int_{\mathcal{C}(2R)\cap Q_i} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2\, dx \\
\le& C_P^* \frac{C}{\delta' R^2} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2\, dx
\end{aligned}$$

where $C_P^* = \max_i C_P(Q_i)$ and $C = C(p, \|\nabla u\|_{L^\infty(\Sigma_\Lambda)})$.

Now we estimate the term I_2 in (1.3.55). Being f locally Lipschitz continuous form (1.3.54), arguing as in (1.3.64), we get that

$$\begin{aligned} I_2 &\leq \int_{\mathcal{C}(2R)} \frac{f(u)-f(v)}{u-v}[(u-v)^+]^2\,dx \\ &\leq C_P^* \cdot C \int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx, \end{aligned}$$

where C_P^* is as in (1.3.64) and $C = C(f,\lambda,\|\nabla u\|_{L^\infty(\Sigma_\Lambda)})$. Actually the constant C will depend on the Lipschitz constant of f in the interval

$$\left[0, \max\{\|u\|_{L^\infty(\Sigma_\Lambda)}, \|v\|_{L^\infty(\Sigma_\Lambda)}\}\right].$$

By (1.3.55), (1.3.62), (1.3.63) and (1.3.64), up to redefining the constants, we obtain

(1.3.65)

$$\begin{aligned} C\int_{\mathcal{C}(2R)} &(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\varphi^2\,dx \\ &\leq \delta' \int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx \\ &\quad + \frac{C_P^*}{R}\int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx \\ &\quad + C_P^*\int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx. \end{aligned}$$

Let us choose δ' small in (1.3.65) such that $C-\delta' > C/2$ and fix $R>1$. Then we obtain

$$\begin{aligned} &\int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\varphi^2\,dx \\ &\leq 4\frac{C_P^*}{C}\int_{\mathcal{C}(2R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx. \end{aligned} \tag{1.3.66}$$

To conclude we set now

$$\mathcal{L}(R) := \int_{\mathcal{C}(R)} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)^+|^2\,dx. \tag{1.3.67}$$

We can fix $h_0 = h_0(f,p,\gamma,\lambda,N,\|\nabla u\|_{L^\infty(\Sigma_\Lambda)},\|\nabla v\|_{L^\infty(\Sigma_\Lambda)})$ positive, such that if

$$\beta - \alpha \leq h_0,$$

(recall that $C_P^* \to 0$ in this case since $\operatorname{diam}(Q_i)\to 0$, see (1.3.57)) then

$$\vartheta := 4\frac{C_P^*}{C} < 2^{-N}.$$

Then, by (1.3.66) and (1.3.67), we have

$$\begin{cases} \mathcal{L}(R) \leq \vartheta\mathcal{L}(2R) & \forall R>1, \\ \mathcal{L}(R) \leq CR^N & \forall R>1. \end{cases}$$

From Lemma 1.3.2 with $\nu = N$ and $\vartheta < 2^{-N}$, we get

$$\mathcal{L}(R) \equiv 0$$

and consequently that $(u-v)^+ \equiv 0$.

□

Finally, the last result that we present a weak comparison principle that works in unbounded strips (we do not need to consider narrow parts). This results works for a general class of changing-sign nonlinearities and to the best of our knowledge this results is new since it is the first time that, in the quasilinear case, these nonlinearities are considered in unbounded domains. This results will be crucial in the proof of Gibbons conjecture in the quasilinear case that will be presented in Chapter 7 and it is also contained in a paper in collaboration with A. Farina, L. Montoro and B. Sciunzi [49]. Let us consider the following quailinear elliptic problem:

$$\begin{cases} -\Delta_p u \leq f(u) & \text{in } \Sigma_{(a,b)} \\ -\Delta_p v \geq f(v) & \text{in } \Sigma_{(a,b)} \\ u \leq v & \text{on } \partial\Sigma_{(a,b)}, \end{cases} \tag{1.3.68}$$

ordered on the boundary of some half-space $\Sigma_{(a,b)}$ of $\mathbb{R}^N$, with $p > 1$ and $N < 1$. More precisely

$$\Sigma_{(a,b)} := \mathbb{R}^{N-1} \times (a,b),$$

where either $a = -\infty$ and $b \in \mathbb{R}$, or $a \in \mathbb{R}$ and $b = +\infty$.

We summarize the assumptions on the nonlinearity f (denoted by (G_f) in the following) as follows:

(G_f): The nonlinearity $f(\cdot)$ belongs to $C^1([-1,1])$, $f(-1) = 0$, $f(1) = 0$, $f'_+(-1) < 0$, $f'_-(1) < 0$ and the set

$$\mathcal{N}_f := \{t \in [-1,1] \mid f(t) = 0\}$$

is finite. We provide the following

Theorem 1.3.7 ([49]). *Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma_{(a,b)}})$ satisfying problem (1.3.68), $N \geq 1$, $p > 1$, where $\Sigma_{(a,b)}$ is some half-space of $\mathbb{R}^N$ and f fulfils (G_f). Moreover, let us assume that*

$$|\nabla u|, |\nabla v| \in L^\infty(\Sigma_{(a,b)}),$$

for some δ sufficiently small

$$-1 \leq u \leq -1 + \delta \quad \text{in } \Sigma_{\lambda_{y_0}}$$

and for some $L > 0$

$$f'(t) < -L \quad \text{in } [-1, -1+\delta]. \tag{1.3.69}$$

Then

$$u \leq v \quad \text{in } \Sigma_{(a,b)}. \tag{1.3.70}$$

The same result is true if

$$1 - \delta \leq v \leq 1 \quad \text{in } \Sigma_{(a,b)} \quad \text{and} \quad f'(t) < -L \quad \text{in } [1-\delta, 1].$$

Proof. We prove the result in the case $-1 \leq u \leq -1+\delta$.
We distinguish two cases:
Case 1: $1 < p < 2$**.** We set

$$\psi := w^{\alpha}\varphi_R^{\alpha+1}, \tag{1.3.71}$$

where $\alpha > 1$, $R > 0$ large, $w := (u-v)^+$ and φ_R is a standard cutoff function such that $0 \leq \varphi_R \leq 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R}, with $|\nabla\varphi_R| \leq 2/R$ in $B_{2R} \setminus B_R$. Let us define $\mathcal{C}(2R) := \Sigma_{(a,b)} \cap B_{2R} \cap \operatorname{supp}(\omega)$. First of all we notice that $\psi \in W_0^{1,p}(\mathcal{C}(2R))$. By density arguments we can take ψ as test function in the weak formulation of (1.3.68), so that, subtracting we obtain

$$\begin{aligned}
&\alpha \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w) w^{\alpha-1}\varphi_R^{\alpha+1}\, dx \\
&\leq -(\alpha+1)\int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\varphi_R) w^{\alpha}\varphi_R^{\alpha+1}\, dx \\
&\quad + \int_{\mathcal{C}(2R)} [f(u) - f(v)] w^{\alpha}\varphi_R^{\alpha+1}\, dx\,.
\end{aligned} \tag{1.3.72}$$

From (1.3.72), using (1.0.2) and noticing that f is decreasing in $[-1, -1+\delta]$, we obtain

$$\begin{aligned}
&\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1}\varphi_R^{\alpha+1}\, dx \\
&\leq \alpha \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w) w^{\alpha-1}\varphi_R^{\alpha+1}\, dx \\
&\leq -(\alpha+1)\int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\varphi_R) w^{\alpha}\varphi_R^{\alpha}\, dx \\
&\quad + \int_{\mathcal{C}(2R)} f'(\xi)(u-v)^+ w^{\alpha}\varphi_R^{\alpha+1}\, dx \\
&\leq (\alpha+1)C_3 \int_{\mathcal{C}(2R)} |\nabla w|^{p-1}|\nabla\varphi_R| w^{\alpha}\varphi_R^{\alpha}\, dx - L\int_{\mathcal{C}(2R)} (u-v)^+ w^{\alpha}\varphi_R^{\alpha+1}\, dx,
\end{aligned} \tag{1.3.73}$$

where ξ is some point that belongs to (v, u). Hence, recalling also that $|\nabla u|, |\nabla v| \in L^{\infty}(\Sigma_{(a,b)})$, we deduce

$$\begin{aligned}
&\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1}\varphi_R^{\alpha+1}\, dx \\
&\leq (\alpha+1)C_3 \int_{\mathcal{C}(2R)} |\nabla w|^{p-1}|\nabla\varphi_R| w^{\alpha}\varphi_R^{\alpha}\, dx - L\int_{\mathcal{C}(2R)} w^{\alpha+1}\varphi_R^{\alpha+1}\, dx \\
&\leq (\alpha+1)C \int_{\mathcal{C}(2R)} |\nabla\varphi_R| w^{\alpha}\varphi_R^{\alpha}\, dx - L\int_{\mathcal{C}(2R)} w^{\alpha+1}\varphi_R^{\alpha+1}\, dx
\end{aligned} \tag{1.3.74}$$

where $C = C(p, \|\nabla u\|_{L^\infty(\Sigma_{(a,b)})}, \|\nabla v\|_{L^\infty(\Sigma_{(a,b)})})$. Exploiting the weighted Young's inequality with exponents $\alpha+1$ and $(\alpha+1)/\alpha$ in (1.3.74), we obtain

$$\begin{aligned}
&\alpha C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1} \varphi_R^{\alpha+1}\, dx \\
&\leq \frac{C}{\sigma^{\alpha+1}} \int_{\mathcal{C}(2R)} |\nabla \varphi_R|^{\alpha+1}\, dx + \alpha C \sigma^{\frac{\alpha+1}{\alpha}} \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \\
&\quad - L \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \\
&\leq \frac{C}{\sigma^{\alpha+1}} \int_{\mathcal{C}(2R)} |\nabla \varphi_R|^{\alpha+1}\, dx + \left(\alpha C \sigma^{\frac{\alpha+1}{\alpha}} - L\right) \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx \\
&\leq \frac{2^{\alpha+1} C}{\sigma^{\alpha+1} R^{\alpha-(N-1)}} + \left(\alpha C \sigma^{\frac{\alpha+1}{\alpha}} - L\right) \int_{\mathcal{C}(2R)} w^{\alpha+1} \varphi_R^{\alpha+1}\, dx.
\end{aligned}$$

Now taking $\alpha > N-1$, if we choose

$$\sigma = \sigma(p, \alpha, L, N, \|\nabla u\|_{L^\infty(\Sigma_{(a,b)})}, \|\nabla v\|_{L^\infty(\Sigma_{(a,b)})}) > 0$$

sufficiently small so that

$$\alpha C \sigma^{\frac{\alpha+1}{\alpha}} - L < 0,$$

we obtain

$$\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1}\, dx \leq \frac{\tilde{C}}{\alpha \sigma^{\alpha+1} R^{\alpha-(N-1)}}. \tag{1.3.75}$$

Passing to the limit in (1.3.75) for $R \to +\infty$, by Fatou's Lemma we have

$$\int_{\Sigma_{\lambda_{y_0}}} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 w^{\alpha-1}\, dx \leq 0.$$

This implies that $u \leq v$ in $\Sigma_{(a,b)}$.
Case 2: $p \geq 2$. We set

$$\psi := w \varphi_R^2, \tag{1.3.76}$$

where $R > 0$, $w := (u-v)^+$ and φ_R is the standard cutoff function defined above. First of all we notice that $\psi \in W_0^{1,p}(B_{2R})$. Let us define $\mathcal{C}(2R) := \Sigma_{(a,b)} \cap B_{2R} \cap \mathrm{supp}(\omega)$. By density arguments we can take ψ as test function in the weak formulation of (1.3.68), so that, subtracting we obtain

$$\begin{aligned}
&\int_{\mathcal{C}(2R)} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla w) \varphi_R^2\, dx \\
&\leq -2 \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \varphi_R) w \varphi_R\, dx \\
&\quad + \int_{\mathcal{C}(2R)} [f(u) - f(v)] w \varphi_R^2\, dx\,.
\end{aligned} \tag{1.3.77}$$

From (1.3.77), using (1.0.2) and that $f'(u) \le -L$ in $[-1, -1+\delta]$, we obtain

(1.3.78)
$$
\begin{aligned}
&C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w|^2 \varphi_R^2 \, dx \\
&\le \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w)\varphi_R^2 \, dx \\
&\le -2 \int_{\mathcal{C}(2R)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla \varphi_R) w \varphi_R \, dx \\
&\quad + \int_{\mathcal{C}(2R)} f'(\xi)(u-v)^+ w \varphi_R^2 \, dx \\
&\le 2C_2 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w| \; |\nabla \varphi_R| w \varphi_R \, dx \\
&\quad - L \int_{\mathcal{C}(2R)} (u-v)^+ w \varphi_R^2 \, dx,
\end{aligned}
$$

where ξ is some point tha belongs to (v, u). Using in (1.3.78) the weighted Young's inequality (and the fact that $|\nabla u|, |\nabla v| \in L^\infty(\Sigma_{(a,b)})$), we obtain

(1.3.79)
$$
\begin{aligned}
&C_1 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w|^2 \varphi_R^2 \, dx \\
&\quad \le 2C_2 \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} \, |\nabla w| \, (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} \, |\nabla \varphi_R| w \varphi_R \, dx \\
&\qquad - L \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx \\
&\quad \le C_2 \sigma \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w|^2 \, dx \\
&\qquad + \frac{C_2}{\sigma} \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla \varphi_R|^2 w^2 \varphi_R^2 \, dx \\
&\qquad - L \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx. \\
&\quad \le C_2 \sigma \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w|^2 \, dx \\
&\qquad + \left(\frac{C}{\sigma R^2} - L \right) \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx,
\end{aligned}
$$

where $C = C(p, \|\nabla u\|_{L^\infty(\Sigma_{(a,b)})}, \|\nabla v\|_{L^\infty(\Sigma_{(a,b)})})$ is a positive constant. Hence, up to redefine the constants, we have

(1.3.80)
$$
\begin{aligned}
\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w|^2 \, dx &\le C\sigma \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} \, |\nabla w|^2 \, dx \\
&\quad + \frac{1}{C_1} \left(\frac{C}{\sigma R^2} - L \right) \int_{\mathcal{C}(2R)} w^2 \varphi_R^2 \, dx.
\end{aligned}
$$

Now we set

$$\mathcal{L}(R) := \int_{\mathcal{C}(R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx.$$

By our assumption,$|\nabla u|, |\nabla v| \in L^\infty(\Sigma_{(a,b)})$, it follows that $\mathcal{L}(R) \le \dot{C}R^N$ for every $R > 0$ and for some $\dot{C} = \dot{C}(p, \|\nabla u\|_{L^\infty(\Sigma_{(a,b)})}, \|\nabla v\|_{L^\infty(\Sigma_{(a,b)})})$. Moreover, in equation (1.3.80), we take $\sigma = \sigma(p, N, \|\nabla u\|_{L^\infty(\Sigma_{(a,b)})}, \|\nabla v\|_{L^\infty(\Sigma_{(a,b)})}) > 0$ sufficiently small so that $C\sigma < 1/2^N$. Finally we fix $R_0 > 0$ such that

$$\frac{C}{\sigma R^2} - L < 0$$

for every $R \ge R_0$. Therefore by (1.3.80) we deduce that

$$\begin{cases} \mathcal{L}(R) \le \vartheta \mathcal{L}(2R) & \forall R \ge R_0 \\ \mathcal{L}(R) \le \dot{C}R^N & \forall R \ge R_0, \end{cases} \tag{1.3.81}$$

where $\vartheta := C\sigma < 1/2^N$. By applying Lemma 1.3.2 it follows that $\mathcal{L}(R) = 0$ for all $R \ge R_0$. Hence $u \le v$ in $\Sigma_{(a,b)}$.

□

1.4. The Höpf boundary lemma and the strong maximum principle

The aim of this section is to present two classical results: the Höpf boundary lemma and the strong comparison principle for quasilinear elliptic equations. It is well know that the Höpf boundary lemma always implies the strong maximum principle. Here, borrowing the ideas of J. L. Vazquez contained in the celebrated paper [127] we would like to present this well known results for the following quasilinear elliptic problem: Let us consider the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + \beta(u) = f(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.1}$$

where $1 < p < +\infty$, Ω is any connected domain of $\mathbb{R}^N$, $N \ge 2$, $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function with $\beta(0) = 0$ and $f \ge 0$ a.e. in Ω. As already mentioned, we now state the results by J. L. Vazquez in the celebrated paper [127], but we have to remark that similar results for quasilinear elliptic equations were obtained also by P. Pucci and J. Serrin, that considered a more general class of operators and of nonlinearity (see e.g. [103]). Here we prove the case $p = 2$ (as it was done by J.L. Vazquez in [127]) and we give some ideas for the quasilinear case.

Theorem 1.4.1 ([127]). *Let $p = 2$ and $u \in L^1_{loc}(\Omega)$ be such that is a solution to (1.4.1) such that $\Delta u \in L^1_{loc}(\Omega)$ in the sense of distribution in Ω and $\Delta u \le \beta(u)$ in $\{x \in \Omega \mid 0 < u(x) < a\}$, where a is a positive constant and $\beta : [0, a] \to \mathbb{R}$ is a continuous nondecreasing function with $\beta(0) = 0$. Under the assumption that $\beta(S) = 0$ for some $S > 0$ or*

$$\int_0^{\frac{a}{2}} \frac{1}{\sqrt{\beta(S)S}} \, dS = \infty \tag{1.4.2}$$

if $\beta(S) > 0$ *for* $S > 0$, *then either* $u \equiv 0$ *a.e. in* Ω *or* u *is strictly positive in* Ω *in the sense that for every compact set* $k \subset \Omega$ *there exists a constant* $\gamma = \gamma(K) > 0$ *such that*

$$u \geq \gamma \quad a.e. \ in \ \Omega. \tag{1.4.3}$$

In particular if u *vanishes a.e. in a set of positive measure it must vanish a.e. in* Ω.

In other words what we are going to prove is that, for a suitable class of solutions of (1.4.1) for $p = 2$, the strong maximum principle holds if and only if either $\beta(S) = 0$ for some $S > 0$ or $\beta(S) > 0$ for $S > 0$ and

$$\int_0^1 \frac{1}{\sqrt{j(S)}}\, dS = \infty, \tag{1.4.4}$$

where $j(S) = \displaystyle\int_0^s \beta(t)\, dt$. We notice that for every $S > 0$ since β is monotone nondecreasing then also j it is and we have

$$\frac{S}{2}\beta\left(\frac{S}{2}\right) \leq j\left(\frac{S}{2}\right) \leq j(S) \leq \beta(S).$$

Hence condition (1.4.4) is equivalent to the following one

$$\int_0^1 \frac{1}{\sqrt{\beta(S)}}\, dS = \infty. \tag{1.4.5}$$

Remark 1.4.1. We want to observe that if $\beta(s) = s^q$ with $q > 0$ it follows that, by simple computations, condition (1.4.5) holds if and only if $q \geq 1$. When (1.4.5) does not hold, in particular for this kind of nonlinearity when $0 < q < 1$, it follows that there exist the so called **dead core solutions**, for more details we refer to the seminal paper [127].

In the proof of Theorem 1.4.1 we need of the following technical lemma that will be useful to build a radial solution of problem (1.4.1), in order to compare it with other solutions of problem (1.4.1) with $p = 2$:

Lemma 1.4.2. *For all* $k_1, k_2, r_1, v_1 > 0$ *and* $\beta : \mathbb{R} \to \mathbb{R}$ *continuous nondecresing function with* $\beta(0) = 0$, *there exists a unique* $v = v(r, k_1, k_2, r_1, v_1)$ *defined in* $[0, r_1]$ *of class* $\mathcal{C}^2$ *that is a solution of the following nonlinear two-point boundary value problem*

$$\begin{cases} v'' = k_1 v' + k_2 \beta(v) & 0 < r < r_1 \\ v(0) = 0 \quad v(r_1) = v_1 \end{cases} \tag{1.4.6}$$

and $v, v', v'' \geq 0$. *Moreover if* β *satisfies* (1.4.5) *then* $v'(0) = 0$ *and* $v_1 > v > 0$ *in* $(0, r_1)$.

Proof. For the existence and the uniqueness of the solution v to problem we refer to the works [7, 78]. The fact that $v(r) \geq 0$ for $0 \leq r \leq r_1$ follows from comparison arguments between sub and super-solutions of (1.4.6) as exploited in [7, 78]. Now we observe that by multiplying both sides of the ordinary differntial equation of (1.4.6) for $e^{-k_1 r}$ we obtain:

$$\left(e^{-k_1 r} v'(r)\right)' = k_2 e^{-k_1 r} \beta(v(r));$$

from this it follows that $e^{-k_1 r}v'(r)$ is a nondecreasing function, hence by simple computations $v'(r) \geq 0$ for $0 \leq r \leq r_1$. Since $v''(r) = k_1 v'(r) + k_2\beta(v(r))$ for $0 \leq r \leq r_1$ then also $v''(r) \geq 0$ for $0 \leq r \leq r_1$.

Now let us consider r_0 the largest r for which $v(r) = 0$. Necessarily $0 \leq r_0 < r_1$, $v : [r_0, r_1] \to [0, v_1]$ is bijective and

$$\int_{r_0}^{r_1} \frac{v'(r)}{\sqrt{j(v(r))}}\, dr = \int_0^1 \frac{1}{\sqrt{\beta(S)}}\, dS = \infty. \tag{1.4.7}$$

Now, if $w = (v')^2$ we have

$$k_2 j'(v) = k_2\beta(v)v' = (v'' - k_1 v')v'$$

and so

$$2k_2 e^{-2k_1 r} j'(v) = \left(e^{-2k_1 r} w\right).$$

Since $v'(r_0) = 0$, by integrating the previous equation between r_0 and r we have

$$\begin{aligned} 2k_2 e^{-2k_1 r_0} j(v(r)) &= 2k_2 e^{-2k_1 r_0} \int_{r_0}^{r} j'(v(r))\, dr \\ &\geq \int_{r_0}^{r} 2k_2 e^{-2k_1 r} j'(v(r))\, dr = e^{-2k_1 r_1} w(r) \end{aligned}$$

and hence we have

$$\int_{r_0}^{r_1} \frac{v'(r)}{\sqrt{j(v(r))}}\, dr \leq \sqrt{2k_2} e^{k_1(r_1 - r_0)}(r_1 - r_0) < +\infty,$$

but this gives a contradiction with our assumption (1.4.5). Hence $v'(r_0) > 0$ and this implies $r_0 = 0$. It follows that $v'(0) > 0$ and $v'(r) > 0$ for $o < r < r_1$. □

First Proof of Theorem 1.4.1. Let us prove the theorem in the case $u \in C^1(\Omega)$. Let us assume that u vanishes somewhere in Ω but it is not identically zero. Hence we can choose a point $x_0 \in \Omega$ and a ball $B = B_R(x_1)$ such that $x_0 \in \partial B$ and $u(x_0) = 0$ and $0 < u(x) < a$ for each $x \in B$. It is sufficient to take $x_1 \in \Omega$ such that $u(x_1) > 0$ and for $\varepsilon > 0$ sufficiently small $d(x_1, N) < \varepsilon$ and $d(x, \partial\Omega)$ with $N = \{x \in \Omega \mid u(x) = 0\}$ and $R = \sup_{r>0}\{B_r(x_1) \subset \Omega \setminus N\}$. Taking $G = \left\{x \in \mathbb{R}^N \mid \frac{R}{2} < |x - x_1 < R\right\}$, $u > 0$ in G, $v_1 = \inf_u \left\{|x - x_1| = \frac{R}{2}\right\}$ and using Lemma 1.4.2 we can construct the function

$$\hat{u}(x) := v\left(R - |x - x_1|, k_1, 1, \frac{R}{2}, v_1\right) \tag{1.4.8}$$

in the annulus G defined above. Now by Lemma 1.4.2 we have that

$$\Delta\hat{u} = k_1|\nabla\hat{u}| + \beta(\hat{u}).$$

Moreover, since $\hat{u}$ is radial, by taking $k_1 \geq \frac{2(N-1)}{R}\hat{u}$ we have

$$\Delta\hat{u} \geq \beta(\hat{u}).$$

Now by Kato inequality we have

$$\Delta(\hat{u} - u)^+ \geq \operatorname{sign}(\hat{u} - u)\Delta(\hat{u} - u) = \operatorname{sign}(\hat{u} - u)(\beta(u) - \beta(\hat{u})) \geq 0.$$

Then $u \geq \hat{u}$ and since $v'(0) > 0$ it follows that

$$\liminf_{h\to 0} \frac{u(x_0 + h(x_1 - x_0)) - u(x_0)}{h} > 0$$

against the assumption that $u \in C^1(\Omega)$, which implies that $\nabla u(x_0) = 0$.

If $u \notin C^1(\Omega)$, assume that $u \not\equiv 0$. Then there must exists a nall $B_R(\bar{x}) \subset \Omega$, $R > 0$ such that the trace g of $\hat{u}$ on the sphere $S_R(\bar{x})$ is not zero a.e. We have to show that there exists a unique solution $v \in C^1(\Omega)$ to the following semilinear problem

$$\begin{cases} -\Delta v + \beta(v) = 0 & \text{in } B_R(\bar{x}) \\ v = \min(g, a/2) & \text{on } S_R(\bar{x}). \end{cases} \tag{1.4.9}$$

This is a well known result in the literature and we refer the reader to the celebrated papers [**81**, **97**]. After that, the proof follows by the one developed in the case of $\mathcal{C}^1$ solutions.

□

Now we are ready to prove the following

THEOREM 1.4.2 (Höpf boundary lemma). *Let Ω, β, p and u as in Theorem 1.4.1 and let x_0 be a point on $\partial\Omega$ satisying the interior sphere condition. Let B one such sphere and ν the corresponding interior normal at x_0. Then there exists $\gamma > 0$ such that*

$$\operatorname{ess}\liminf_{x\to x_0} \frac{u(x)}{(x - x_0, \nu)} \geq \gamma \quad x \in B. \tag{1.4.10}$$

In particular if $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ we have

$$\frac{\partial u}{\partial \nu}(x_0) \geq \gamma. \tag{1.4.11}$$

PROOF. Now we take the annulus G corresponding to the ball $B_R(x_1)$ that occurs in the definition of interior sphere condition at x_0. Since G touches $\partial\Omega$ we replace x_1 by $x_1^\varepsilon = x_1 + \varepsilon\nu$ for a small $\varepsilon > 0$ and keep R fixed. If ε is sufficiently small the new annulus G_ε is such that $\overline{G_\varepsilon} \subset \Omega$. Arguing as in Theorem 1.4.1 we have

$$u(x) \geq \hat{u}(x - \varepsilon)$$

a.e. in G. Passing to the limit for $\varepsilon \to 0$ and remembering that $v'(0) > 0$ we obtain (1.4.10) with $\gamma = v'(0)$.

□

Theorems 1.4.1 and 1.4.2 hold also in the quasilinear case, when the classical Laplace operator is replaced by the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and conditions (1.4.4) and (1.4.5) are replaced respectively by

$$\int_0^1 \frac{1}{(j(S))^{\frac{1}{p}}}\, dS = \infty, \tag{1.4.12}$$

and

$$\int_0^1 \frac{1}{(\beta(S))^{\frac{1}{p}}}\, dS = \infty. \tag{1.4.13}$$

In the case of quasilinear elliptic equations it will be different the ODE analysis and also comparison arguments. All this details are not contained in the work by J. L. Vazquez and so we refer the reader to the book of P. Pucci and J. Serrin [103] for completeness.

We want to conclude this section saying that it is always possible to prove that the strong comparison principle follows by Höpf boundary lemma. This fact is quite natural in the case of semilinear equation, but for quasilinear operator could be very useful in the proof of qualitative properties of solutions, since it is well known that maximum principles and comparison principles are not equivalent. Here we state another proof of Theorem 1.4.1 having in force the Höpf lemma:

Second Proof of Theorem 1.4.1. Let us assume that $u \in C^1(\Omega)$ is a solution to (1.4.1). Arguing by contradiction, let us assume that u vanishes somewhere in Ω but it is not identically zero. Hence we can choose a point $x_0 \in \Omega$ and a ball $B = B_R(x_1) \subset \Omega$ such that $x_0 \in \partial B$, $u(x_0) = 0$ and $0 < u(x) < a$ for each $x \in B$. We observe that

$$\begin{cases} -\Delta u \leq \beta(u) & \text{in } B \\ u(x_0) = 0 & x_0 \in \partial B. \end{cases}$$

Let us note that ∂B satisfies the interior sphere condition at x_0, hence by the Höpf boundary lemma, i.e. Theorem 1.4.2, we have that

$$\frac{\partial u}{\partial \nu}(x_0) = (\nabla u(x_0), \nu) > 0$$

where ν is the interior normal at x_0. But, since $u(x_0) = 0$, $u \geq 0$ in Ω and $u \in C^1(\Omega)$ it follows that x_0 is a minimum point for u and this also implies $\nabla u(x_0) = 0$. This fact gives a contradiction with Höpf boundary lemma. □

1.5. Strong comparison principles for $p \neq 2$

The aim of this section is to recall two important result: the strong comparison principle for quasilinear elliptic equations and the strong maximum principle for linearized equations. Both these principles are remarkable consequences of Harnack type inequalities which give informations about the critical set $\mathcal{Z}_u$ of solutions to the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5.1}$$

where Ω is any domain of $\mathbb{R}^N$, $N \geq 1$, $1 < p < +\infty$, f is positive and locally Lipschitz continuous. As remarked above, the main tool in the proof of strong comparison principles are results regarding Harnack type inequalities:

Theorem 1.5.1 (Harnack Comparison Inequality). *Let $p > (2N+2)/(N+2)$ and let $u, v \in C^1_{loc}(\Omega)$ with u or v weak solution to (1.5.1) in Ω. Suppose that $\overline{B(x, 6\delta)} \subset \Omega' \subset \Omega$ for some $\delta > 0$ and that*

$$u \leq v \quad \text{in} \quad B(x, 6\delta).$$

Then there exists $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega')}, \|\nabla u\|_{L^\infty(\Omega')}, \|\nabla v\|_{L^\infty(\Omega')}) > 0$ *such that*

$$\sup_{B(x,\delta)} (v - u) \leq C \inf_{B(x,2\delta)} (v - u). \tag{1.5.2}$$

The iterative technique that is used to prove Theorem 1.5.1 is due to J. K. Moser [93] and was first used to prove Hölder continuity properties of solutions of some strictly elliptic linear operators (this problem had been previously studied by E. De Giorgi [43] and J. Nash [94] in their famous papers).
In [124] and in [126] N. S. Trudinger considers the case of degenerate operators which satisfy some a-priori assumptions on the matrix of the coefficients (see [124]). The works of N. S. Trudinger stemmed originally from the paper of J. K. Moser, but it make no use of (a variant of) the famous John-Nirenberg Lemma (see [93]), exploiting in the proof only weighted Sobolev inequalities and a clever use of test-functions techniques. For the proof of this result we refer the reader to the work of L. Damascelli and B. Sciunzi [36].

Remark 1.5.2. *As in Remark 1.2.3, note that a function* $f : I \longrightarrow \mathbb{R}$ *is locally Lipschitz continuous in the interval* I *if and only if, for each compact subinterval* $[a, b] \subset I$*, there exist two positive costants* C_1 *and* C_2 *such that*

i) $f_1(s) = f(s) - C_1 s$ *is nonincreasing in* $[a, b]$.
ii) $f_2(s) = f(s) + C_2 s$ *is nondecreasing in* $[a, b]$.

Therefore we get that, if

$$-\Delta_p u - f(u) \leqslant -\Delta_p v - f(v) \qquad u \leqslant v \quad in \quad B(x, 5\delta) \tag{1.5.3}$$

then

$$-\Delta_p u + \Lambda u \leqslant -\Delta_p v + \Lambda v \qquad u \leqslant v \quad in \quad B(x, 5\delta) \tag{1.5.4}$$

for $\Lambda \in \mathbb{R}$ *sufficiently large, and the previous result works also in this case.*

This implies in turn the following

Theorem 1.5.3 (Strong Comparison Principle). *Let* $u, v \in C^1(\overline{\Omega})$ *where* Ω *is a bounded smooth connected domain of* $\mathbb{R}^N$ *with* $\frac{2N+2}{N+2} < p < 2$ *or* $p > 2$. *Suppose that either* u *or* v *is a weak solution of* (1.5.1). *Assume*

$$-\Delta_p u + \Lambda u \leqslant -\Delta_p v + \Lambda v \qquad u \leqslant v \quad in \quad \Omega \tag{1.5.5}$$

where $\Lambda \in \mathbb{R}$. *Then* $u \equiv v$ *in* Ω *unless*

$$u < v \qquad in \qquad \Omega \tag{1.5.6}$$

The same result holds (see Remark 1.5.2) if u *and* v *are weak solutions of* (1.5.1) *or more generally if*

$$-\Delta_p u - f(u) \leqslant -\Delta_p v - f(v) \qquad u \leqslant v \quad in \quad \Omega \tag{1.5.7}$$

with u *or* v *weakly solving* (1.5.1).

Proof. Let us define

$$K_{uv} = \{x \in \Omega \,|\, u(x) = v(x)\} \tag{1.5.8}$$

By the continuity of u and v we have that K_{uv} is closed in Ω. Since, by Theorem 1.5.1, for any $x \in K_{uv}$ there exists a ball $B(x)$ centered in x all contained in K_{uv}, then K_{uv} is also open in Ω and the thesis follows, since Ω is connected.

$\square$

Theorem 1.5.3 improves previous similar results. In particular we refer to [70] for the case of strictly elliptic operators or for the case of degenerate operators with $f = 0$ (see also [47]).

In a similar way to the case of solutions to problem (1.5.1), we want to prove a Strong Maximum Principle for the linearized equation. We recall that any derivative $u_i := u_{x_i}$, $1 \leq i \leq N$, satisfies the linearized equations of (1.5.1), i.e.

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2}(\nabla u_i, \nabla\varphi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla u_i)(\nabla u, \nabla \varphi)\,dx \\ - \int_\Omega f'(u)u_i\varphi = 0, \qquad \forall \varphi \in C_c^\infty(\Omega). \end{aligned} \tag{1.5.9}$$

Let us now state the result on the Harnack type inequality for (1.5.9):

Theorem 1.5.4 (Harnack Inequality for the Linearized Operator). *Let $u_i \in H_\varrho^{1,2}(\Omega) \cap L^\infty(\Omega)$ be a nonnegative weak solution of* (1.5.9) *in a bounded smooth domain Ω of $\mathbb{R}^N$, $N \geqslant 2$, with f positive and such that is a continuous function which is locally Lipschitz continuous in $(0, +\infty)$ and $p > 2$. Suppose that $\overline{B(x, 5\delta)} \subset \Omega$. Let us put*

$$\frac{1}{\overline{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

(consequently $\overline{2}^* > 2$ for $m > 2$) and let 2^* be any real number such that $2 < 2^* < \overline{2}^*$. Then for every $0 < s < \chi$, $\chi \equiv \frac{2^*}{2}$, there exists $C > 0$ such that

$$\sup_{B(x,\delta)} u_i \leq C \inf_{B(x,2\delta)} u_i \tag{1.5.10}$$

where C is a constant depending on x, s, N, u, m, f.
If $\frac{2N+2}{N+2} < p < 2$ the same result holds with χ replaced by $\chi' \equiv \frac{2^\sharp}{s^\sharp}$ where $2^\sharp$ is the classical Sobolev exponent ($2^\sharp = \frac{2N}{N-2}$), $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ and $s < \frac{p-1}{p-m}$.

We prove now a remarkable consequence of weak Harnack inequality which give information about the critical set $\mathcal{Z}_u$ of solutions of (1.5.1). This is particularly interesting since $\mathcal{Z}_u$ is also the set of point where the operator is degenerate elliptic.

Theorem 1.5.5 (Strong Maximum Principle for the Linearized Operator). *Let $u_i \in H_\varrho^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ be a weak solution of* (1.5.9) *in a bounded smooth domain Ω of $\mathbb{R}^N$, $N \geq 2$ with $\frac{2N+2}{N+2} < p < 2$ or $p > 2$ where f is positive and locally Lipschitz continuous in $(0, +\infty)$. Then, for any connected domain $\Omega' \subset \Omega$ with $u_i \geq 0$ in Ω', we have $u_i \equiv 0$ in Ω' or $u_i > 0$ in Ω'.*

Proof. Let us define $K_{u_i} = \{x \in \Omega' \,|\, u_i(x) = 0\}$. By the continuity of v, then K_{u_i} is closed in Ω'. Moreover by Theorem 1.5.4 K_{u_i} is also open in Ω' and the thesis follows.

$\square$

2

The Höpf boundary lemma for singular semilinear elliptic equations

In this chapter we deal with positive weak solutions to the singular semilinear elliptic problem:

$$
\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} + f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.0.1}
$$

where $\gamma > 1$, Ω is a $\mathcal{C}^{2,\alpha}$ bounded domain of $\mathbb{R}^N$ with $0 < \alpha < 1$, $N \geq 1$ and $f : \Omega \to \mathbb{R}$ locally Lipschitz continuous.

As remarked in the introduction, it is well known that generally solutions to problem (2.0.1) are not smooth up to the boundary. It was in fact proved in [82] that solutions are not in $H_0^1(\Omega)$ at least when $\gamma > 3$. Therefore, having in mind the natural regularity behaviour of the solutions (see [31]) we let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The equation is well defined in the interior of the domain in the classical meaning and its weak distributional formulations is

$$
\int_\Omega (\nabla u, \nabla \varphi)\, dx = \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{2.0.2}
$$

Now, let us define the notion of inward pointing normal

Definition 2.0.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $\mathcal{C}^{2,\alpha}$ domain. Let $I_\delta(\partial\Omega)$ be a neighborhood of $\partial\Omega$ with the unique nearest point property* (see e.g. [66]). *Hence for every $x \in I_\delta(\partial\Omega)$ there exists a unique point $\hat{x} \in \partial\Omega$ such that $|x - \hat{x}| = dist(x, \partial\Omega)$. We define the* inward-pointing normal *as*

$$
\eta(x) := \frac{x - \hat{x}}{|x - \hat{x}|}. \tag{2.0.3}
$$

Having in mind these notations, we are now ready to state the main result of this chapter:

Theorem 2.0.2 (Höpf type boundary lemma). *Let $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution of problem* (2.0.1). *Then there exists a neighborhood $I_\delta(\partial\Omega)$ of $\partial\Omega$ such that*

$$
\partial_{\nu(x)} u > 0 \qquad \forall\, x \in I_\delta(\partial\Omega) \tag{2.0.4}
$$

provided that $(\nu(x), \eta(x)) > 0$ uniformly with respect to $x \in I_\delta(\partial\Omega)$, namely provided that $(\nu(x), \eta(x)) \geq \beta > 0$ for some $\beta > 0$ for every $x \in I_\delta(\partial\Omega)$.

In cases when solutions are not smooth up to the boundary, the Höpf lemma is generally replaced by comparison of the solutions with respect to the distance function. Exploiting also such a kind of arguments, our Theorem 2.0.2 provides an information on the sign of the inward derivatives as soon as we look at the solution in the interior of the domain.

Actually we exploit a scaling argument near the boundary which leads to the study of a limiting problem in the half space:

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} & \text{in } \mathbb{R}^N_+ \\ u > 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{2.0.5}$$

where $\gamma > 1$, $N \geq 1$, $\mathbb{R}^N_+ := \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_N > 0\}$ and $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$. As above problem (2.0.5) has to be understood in the weak distributional meaning with test functions with compact support in $\mathbb{R}^N_+$, that is

$$\int_{\mathbb{R}^N_+} (\nabla u, \nabla \varphi)\, dx = \int_{\mathbb{R}^N_+} \frac{\varphi}{u^\gamma}\, dx \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^N_+). \tag{2.0.6}$$

Our scaling argument leads to the study of a limiting profile which is a solution to (2.0.5) and obeys to suitable a priori estimates. The following classification result is therefore crucial for our technique, and may also have an independent interest:

Theorem 2.0.3. *Let $\gamma > 1$. Let $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ be a solution to problem (2.0.5) such that*

$$|u(x)| \leq C x_N^t \quad \forall x \in \mathbb{R}^N_+ \tag{2.0.7}$$

where $t := \dfrac{2}{1+\gamma}$. Then

$$u(x) = u(x_N) = M x_N^t \tag{2.0.8}$$

where $M := \left(\dfrac{(\gamma+1)^2}{2(\gamma-1)}\right)^{\frac{t}{2}}$.

We will prove Theorem 2.0.3 in Section 2.1 together with useful preliminary results. Then in Section 2.2 we exploit Theorem 2.0.3 and a scaling argument to prove Theorem 2.0.2.

2.1. Classification results for singular semilinear elliptic problems in the half-space

Here we introduce some notations and preliminary results. We say that u is a weak *subsolution* of problem (2.0.1) if

$$\int_\Omega (\nabla u, \nabla \varphi)\, dx \leq \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \geq 0. \tag{2.1.1}$$

Similarly, we say that u is a weak *supersolution* of problem (2.0.1) if

$$\int_\Omega (\nabla u, \nabla \varphi)\, dx \geq \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \geq 0. \tag{2.1.2}$$

We shall prove a weak maximum principle in unbounded domain, borrowing some ideas from [9] (see also [15]).

THEOREM 2.1.1. *Let* $\gamma > 1$, $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ *be a weak subsolution of problem* (2.0.5), *as in* (2.1.1), *and* $v \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ *be a weak supersolution of problem* (2.0.5), *as in* (2.1.2). *Let us assume that there exists a constant* $K > 0$ *such that*

$$|u(x)| + |v(x)| \leq K x_N^t \quad \forall x \in \mathbb{R}^N_+ \tag{2.1.3}$$

with $t := \dfrac{2}{1+\gamma}$ *and*

$$u \leq v \quad \text{on } \partial\mathbb{R}^N_+. \tag{2.1.4}$$

Then

$$u \leq v \quad \text{in } \mathbb{R}^N_+. \tag{2.1.5}$$

PROOF. We set

$$w = u - v. \tag{2.1.6}$$

In the weak meaning, we have

$$-\Delta w \leq c(x) w \quad \text{in } \mathbb{R}^N_+, \tag{2.1.7}$$

where

$$c(x) = \left[\frac{1}{u^\gamma} - \frac{1}{v^\gamma}\right] \frac{1}{u-v} \leq 0 \quad \text{and} \quad c(x) \in C(\mathbb{R}^N_+).$$

Now passing to spherical coordinates $x = \varrho\xi$, with $\varrho > 0$ and $\xi \in S^{N-1}$, we obtain:

$$-\Delta w = -\sum_{i=1}^{N} w_{x_i x_i} = -w_{\varrho\varrho} - \frac{N-1}{\varrho} w_\varrho - \frac{1}{\varrho^2}\Delta_S w \leq c(\cdot) w,$$

where Δ_S is the Laplace-Beltrami operator on the sphere S^{N-1}. Now we take an infinite open connected cone $\mathcal{C}$ such that its closure is disjoint from $\overline{\mathbb{R}^N_+}$. Hence we consider the following eigenvalue problem

$$\begin{cases} -\Delta_S \psi = \lambda \psi & \text{in } G \\ \psi = 0 & \text{on } \partial G, \end{cases} \tag{2.1.8}$$

where $G = S^{N-1} \setminus \overline{\mathcal{C}}$ and $\lambda > 0$.

It is well known (see for example [77, 114]) that the eigenvalues of the Laplace-Beltrami operator $-\Delta_S$ on the (N-1)-sphere S^{N-1} are $\mu_k = k(k+N-2)$ where $k \in \mathbb{N}$. Now we fix $\alpha > 0$ such that $\lambda_1 := \alpha(\alpha+N-2)$ is the principal eigenvalue of the problem (2.1.8) and ψ_1 is the corresponding eigenfunction. Since $G \subset S^{N-1}$ it follows that $\alpha(\alpha+N-2) = \lambda_1 \geq \mu_1 = N-1$. As a consequence of this fact we have that $\alpha \geq 1$. Using, as before, spherical coordinates $x = \varrho\xi$, let us define the following

$$g(x) = g(\varrho, \xi) := \varrho^\alpha \psi_1(\xi), \quad \xi \in G. \tag{2.1.9}$$

Then g is an harmonic function, hence

$$\Delta g + c(x) g \leq 0.$$

Since $\mathbb{R}^N_+$ lies outside the cone $\mathcal{C}$, the function g is strictly positive on $\mathbb{R}^N_+$. Hence we can consider the function

$$\sigma := \frac{w}{g}\,.$$

By the definition of σ, we have

$$\nabla\sigma = \frac{1}{g}\nabla w - \frac{w}{g^2}\nabla g,$$

and

$$\Delta\sigma = \frac{1}{g}\Delta w - \frac{2}{g^2}\nabla g\cdot\nabla w - \frac{w}{g^2}\Delta g + 2\frac{w}{g^3}|\nabla g|^2.$$

Finally it follows that

$$L\,\sigma := -\left(\Delta\sigma + \frac{2}{g}\nabla\sigma\cdot\nabla g + \frac{\Delta g + c(x)g}{g}\sigma\right) \leq 0 \quad \text{in}\,\mathbb{R}^N_+. \tag{2.1.10}$$

Moreover $\sigma \leq 0$ on $\partial\mathbb{R}^N_+$. Noticing that $g \leq \beta\varrho^\alpha$ and by the growth hypothesis (2.1.3) we have

$$|\sigma| = \frac{|w|}{g} \leq \frac{|u|+|v|}{g} \leq \frac{K(x_N)^{\frac{2}{\gamma+1}}}{\beta\varrho^\alpha} \leq \frac{K}{\beta}\varrho^{\frac{2}{\gamma+1}-\alpha}.$$

Recalling that $\dfrac{2}{\gamma+1} < 1$ and $\alpha \geq 1$, it follows that $\dfrac{2}{\gamma+1} - \alpha < 0$. Hence we have

$$\limsup_{|x|\to+\infty}\sigma(x) \leq 0.$$

By the *weak maximum principle* (see e.g. [70] or Theorem 1.2.1) it follows now that $\sigma \leq 0$ in $\mathbb{R}^N_+$. Since g is strictly positive by construction, it follows that

$$w \leq 0 \quad \text{in}\,\mathbb{R}^N_+.$$

□

Proof of Theorem 2.0.3. Let

$$M = \left(\frac{(\gamma+1)^2}{2(\gamma-1)}\right)^{\frac{1}{\gamma+1}} \quad \text{and} \quad t = \frac{2}{\gamma+1}.$$

Setting

$$u(x) = Mx_N^t,$$

a simple computation shows that:

$$-\Delta u = -\frac{\partial^2 u}{\partial x_N^2} = -Mt(t-1)x_N^{t-2} = \frac{1}{(Mx_N^t)^\gamma} = \frac{1}{u^\gamma} \quad \text{in}\,\mathbb{R}^N_+\,. \tag{2.1.11}$$

The uniqueness of the solution follows by Theorem 2.1.1.

□

2.2. Local estimates for the solutions and proof of the Höpf boundary lemma

Relying on technique of [82] we prove the following local estimates for the solution u of problem (2.0.1):

THEOREM 2.2.1. *Let $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution to problem (2.0.1) and let ϕ_1 denote the first eigenfunction corresponding to the first eigenvalue λ_1 of the problem*

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega \\ \phi > 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.2.1}$$

Then there exist two positive constants m_1, m_2 and there exists $\delta > 0$ sufficiently small such that

$$m_1\phi_1(x)^{\frac{2}{1+\gamma}} \leq u(x) \leq m_2\phi_1(x)^{\frac{2}{1+\gamma}} \quad \forall\, x \in \overline{I_\delta(\partial\Omega)}. \tag{2.2.2}$$

PROOF. We rewrite the equation of problem (2.0.1) as

$$-\Delta u = \frac{1}{u^\gamma} + f(u) = \frac{p(x)}{u^\gamma} \qquad \text{in } \Omega \tag{2.2.3}$$

where $p(x) := 1 + u^\gamma f(u(x))$ and we fix $\delta_0 > 0$ sufficiently small so that, for every $0 < \delta < \delta_0$ we have that

$$p(x) > 0 \qquad \forall\, x \in I_\delta(\partial\Omega).$$

Arguing as in [82], we consider the principal eigenfunction ϕ_1 of problem (2.2.1). It is well known that $\phi_1 \in C^2(\overline{\Omega})$ and, by Höpf boundary lemma

$$\nabla\phi_1(x) \neq 0 \qquad \forall\, x \in \partial\Omega.$$

Let us consider $t := \dfrac{2}{1+\gamma}$ and $\Psi(x) := s\phi_1(x)^t$ with $s > 0$. The function Ψ satisfies the following equation

$$-\Delta\Psi(x) = \frac{g(x,s)}{\Psi(x)^\gamma} \qquad x \in I_\delta(\partial\Omega)$$

where

$$g(x,s) := s^{1+\gamma}[t(1-t)|\nabla\phi_1(x)|^2 + t\lambda_1\phi_1(x)^2]. \tag{2.2.4}$$

Since $0 < t < 1$, by the definition of g in (2.2.4) we can choose two positive constants s_1 and s_2 such that $0 < s_1 < s_2$ and

$$g(x,s_1) < p(x) < g(x,s_2) \qquad \forall\, x \in \overline{I_\delta(\partial\Omega)}. \tag{2.2.5}$$

Hence, setting $u_1 := s_1\phi_1(x)^t$ and $u_2 := s_2\phi_1(x)^t$, we have that

$$-\Delta u_1 < \frac{p(x)}{u_1^\gamma} \qquad \text{in } I_\delta(\partial\Omega) \tag{2.2.6}$$

in the distributional meaning of (2.1.1), and

$$-\Delta u_2 > \frac{p(x)}{u_2^\gamma} \qquad \text{in } I_\delta(\partial\Omega) \tag{2.2.7}$$

in the distributional meaning of (2.1.2).

Now we consider $u_\beta := \beta u$ and observe that u_β satisfies the following equation

$$-\Delta u_\beta(x) = \beta(-\Delta u(x)) = \beta\frac{p(x)}{u(x)^\gamma} = \beta^{\gamma+1}\frac{p(x)}{u_\beta(x)^\gamma}.$$

By taking $\beta_1 > 0$ sufficiently large it follows that u_{β_1} and u_1 satisfy the following problem:

$$\begin{cases} -\Delta u_{\beta_1} \geq \dfrac{p(x)}{u_{\beta_1}^\gamma} & \text{in } I_\delta(\partial\Omega) \\ -\Delta u_1 < \dfrac{p(x)}{u_1^\gamma} & \text{in } I_\delta(\partial\Omega) \\ u_{\beta_1} \geq u_1 & \text{on } \partial I_\delta(\partial\Omega). \end{cases} \tag{2.2.8}$$

Here we note that the boundary datum of problem (2.2.8) is fulfilled for β_1 sufficiently large. Now, we claim that

$$u_{\beta_1} = \beta_1 u(x) \geq u_1(x) > 0 \qquad \text{in } I_\delta(\partial\Omega). \tag{2.2.9}$$

If this is not the case, then there would exists an x_0 in $\overline{I_\delta(\partial\Omega)}$ such that $0 < u_{\beta_1}(x_0) < u_1(x_0)$ and the minimum of $u_{\beta_1} - u_1$ on $\overline{I_\delta(\partial\Omega)}$ should be assumed at x_0. But according to the argument above, this would imply that

$$\Delta\left(u_{\beta_1} - u_1\right)(x_0) < p(x_0)\left[\frac{1}{u_1(x_0)^\gamma} - \frac{1}{u_{\beta_1}(x_0)^\gamma}\right] < 0,$$

which is impossible by the *maximum principle* (see e.g. [70]). This would provide a contradiction showing that (2.2.9) holds.

Similarly, choosing $\beta_2 > 0$ sufficiently small it follows that u_{β_2} and u_2 satisfy the following problem:

$$\begin{cases} -\Delta u_{\beta_2} \leq \dfrac{p(x)}{u_{\beta_2}^\gamma} & \text{in } I_\delta(\partial\Omega) \\ -\Delta u_2 > \dfrac{p(x)}{u_2^\gamma} & \text{in } I_\delta(\partial\Omega) \\ u_{\beta_2} \leq u_2 & \text{on } \partial I_\delta(\partial\Omega). \end{cases} \tag{2.2.10}$$

Repeating verbatim all the arguments above, it follows that

$$u_{\beta_2} = \beta_2 u(x) \leq u_2(x) \qquad \text{in } I_\delta(\partial\Omega). \tag{2.2.11}$$

Hence, taking $m_1 := \dfrac{s_1}{\beta_1}$ and $m_2 := \dfrac{s_2}{\beta_2}$, we have (2.2.2) and the thesis is proved.

□

Proof of Theorem 2.0.2. Since the domain is of class $C^{2,\alpha}$ we may and do reduce to work in a neighborhood of the boundary $I_\delta(\partial\Omega)$ where the *unique nearest point property* holds (see e.g. [66]). Arguing by contradiction, let us assume that there exists a sequence of points $\{x_n\}$ in $I_\delta(\partial\Omega)$, such that $x_n \longrightarrow x_0 \in \partial\Omega$, as $n \to +\infty$, and

$$\partial_{\nu(x_n)} u(x_n) \leq 0, \;\text{ with } (\nu(x_n), \eta(x_n)) \geq \beta > 0. \tag{2.2.12}$$

Without loss of generality, we can assume that $x_0 = 0 \in \partial\Omega$ and $\eta(x_n) = e_N$. This follows by the fact that the Laplace operator is invariant under isometries. More precisely, for each $n \in \mathbb{N}$, we can consider an isometry

$T_n : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with the above mentioned properties just composing a translation and a rotation of the axes. This procedure generates a new sequence of points $\{y_n\}$, where $y_n := T_n x_N$, such that every $y_n \in \text{span}\langle e_N \rangle$ and $y_n \longrightarrow 0$ as $n \to +\infty$. Setting $u_n(y) := u(T_n^{-1}(y))$, it follows that

$$-\Delta u_n = \frac{1}{u_n^\gamma} + f(u_n) \qquad \text{in } \Omega_n = T_n(\Omega). \tag{2.2.13}$$

Now we set

$$w_n(y) := \frac{u_n(\delta_n y)}{M_n} \tag{2.2.14}$$

where $\delta_n := \text{dist}(x_n, \partial\Omega) = \text{dist}(T_n x_n, 0)$ and $M_n := u_n(\delta_n e_N) = u(x_n)$. It follows that $\delta_n \to 0$ as $n \to +\infty$ and

- w_n is defined in Ω_n^*, where $\Omega_n^* := \dfrac{\Omega_n}{\delta_n}$.
- $w_n(e_N) = 1$.
- $M_n \to 0$, as $n \to +\infty$.

Moreover w_n satisfies

$$\begin{aligned}
-\Delta w_n &= \frac{1}{M_n}\left[-\Delta(u_n(\delta_n y))\right] \\
&= \frac{\delta_n^2}{M_n}\left(\frac{1}{u_n(\delta_n y)^\gamma} + f(u_n(\delta_n y))\right) \\
&= \frac{\delta_n^2}{M_n^{\gamma+1}}\left(\frac{M_n^\gamma}{u_n(\delta_n y)^\gamma} + M_n^\gamma f(u_n(\delta_n y))\right) \\
&= \frac{\delta_n^2}{M_n^{\gamma+1}}\left(\frac{1}{w_n(y)^\gamma} + M_n^\gamma f(u_n(\delta_n y))\right) \qquad \text{in } \Omega_n^*.
\end{aligned} \tag{2.2.15}$$

Here it is important to observe that the term $\dfrac{\delta_n^2}{M_n^{\gamma+1}}$ is bounded in a neighborhood $I_\delta(\partial\Omega_n^*)$; this is a consequence of the Theorem 2.2.1.

In the following we shall deduce a limiting problem with a limiting solution that will be denoted by u_∞. The reader should keep in mind that f is bounded, the term $M_n^\gamma f(u_n(\delta_n y))$ vanishes and the limiting equation is therefore:

$$-\Delta w_\infty = \frac{\tilde{C}}{w_\infty^\gamma} \qquad \text{in } \mathbb{R}^N_+. \tag{2.2.16}$$

Let us provide the details needed to pass to the limit. We have that:

- $w_n \overset{C^{2,\alpha}}{\longrightarrow} w_\infty$, as $n \to +\infty$, in any compact set K of $\mathbb{R}^N_+$.
- $w_\infty \in C^{2,\alpha}(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$.
- $\Omega_n^* := \dfrac{\Omega_n}{\delta_n} \overset{C^2}{\longrightarrow} \mathbb{R}^N_+$, as $n \to +\infty$.

To prove this let us consider a compact set K in $I_{\delta_n}(\partial\Omega_n^*)$ such that $\text{dist}(K, \partial\Omega_n^*) \geq \overline{C} > 0$ for every $n \in \mathbb{N}$, for some tubular neighborhoods $I_{\delta_n}(\partial\Omega_n^*)$ such that Theorem 2.2.1 holds.

Claim 1. We claim that $w_n(y) > 0$ for all $y \in K$ and for all $n \in \mathbb{N}$.
Let $y \in K$. Hence, by Theorem 2.2.1

$$w_n(y) := \frac{u_n(\delta_n y)}{M_n} \geq L \frac{(\text{dist}(\delta_n y, \partial\Omega_n^*))^{\frac{2}{\gamma+1}}}{M_n}.$$

In particular, by the fact that $\text{dist}(\delta_n y, \partial\Omega_n^*) \geq \overline{C}\delta_n$, it follows that

$$w_n(y) \geq L \frac{(\overline{C}\delta_n)^{\frac{2}{\gamma+1}}}{M_n} \geq C(K, \gamma, m_1, m_2) > 0 \tag{2.2.17}$$

for every $n \in \mathbb{N}$.

Claim 2. We claim that $w_n \overset{C^{2,\alpha}}{\longrightarrow} w_\infty$, as $n \to +\infty$, in any compact set K of $\mathbb{R}^N_+$.
Since $\text{dist}(y, \partial\Omega_n^*) \leq C$ for every $y \in K$, by Theorem 2.2.1 it follows that

$$\begin{aligned} w_n(y) = \frac{u_n(\delta_n)}{M_n} &\leq m_2 \frac{[\text{dist}(\delta_n y, \partial\Omega_n)]^{\frac{2}{\gamma+1}}}{M_n} \\ &= L m_2 \frac{\delta_n^{\frac{2}{\gamma+1}} [\text{dist}(y, \partial\Omega_n^*)]^{\frac{2}{\gamma+1}}}{M_n} \\ &\leq L m_2 C^{\frac{2}{\gamma+1}} \frac{\delta_n^{\frac{2}{\gamma+1}}}{M_n} \\ &\leq L m_2 C^{\frac{2}{\gamma+1}} C(K, m_1). \end{aligned} \tag{2.2.18}$$

Hence

$$\|w_n\|_{L^\infty(K)} \leq C_1$$

for any compact set K of $\mathbb{R}^N_+$. By applying regularity theory, see e.g. [46], there exists a compact set $K' \subset K$ such that

$$\|w_n\|_{C^{1,\alpha}(K')} \leq C_2, \tag{2.2.19}$$

where C_2 is a positive constant depending only upon N, M and $\text{dist}(K, \partial\Omega_n)$. By standard elliptic estimates (see e.g. [70], Theorem 6.6, pp 98.) it follows that

$$\|w_n\|_{C^{2,\alpha}(K')} \leq C_3, \tag{2.2.20}$$

where C_3 is a positive constant depending only upon N, K', $\|u\|_{C^{1,\alpha}(K')}$ and $\|h_n\|_{C^{0,\alpha}(K')}$. Therefore, by Ascoli-Arzelà Theorem, the sequence $\{w_n\}$ admits a subsequence that we call $\{w_n\}$ such that converges on the compact set $K' \subset \mathbb{R}^N_+$.

Now we consider an increasing sequence of compact sets $\{K_n\}$ of $\mathbb{R}^N_+$, i.e.

$$K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots \subset \mathbb{R}^N_+.$$

Our aim is to use a diagonal procedure to construct the limit function. We note that there exists a subsequence $\{w_n^{(1)}\}$ of $\{w_n\}$ such that

$$w_n^{(1)} \overset{C^2}{\longrightarrow} w_1 \quad \text{in } K_1.$$

In the same way there exists a subsequence $\{w_n^{(2)}\}$ of $\{w_n^{(1)}\}$ such that

$$w_n^{(2)} \xrightarrow{C^2} w_2 \quad \text{in } K_2$$

and

$$w_2 = w_1 \quad \text{in } K_1.$$

In general we consider the compact set K_m, so that there exists a subsequence $\{w_n^{(m)}\}$ of $\{w_n^{(m-1)}\}$ such that

$$w_n^{(m)} \xrightarrow{C^2} w_m \quad \text{in } K_m$$

and

$$w_m = w_{m-1} \quad \text{in } K_{m-1}.$$

Finally we found the limit function and it is such that

$$w_\infty = w_m \quad \text{in } K_m, \tag{2.2.21}$$

for every $m \in \mathbb{N}$. Hence w_∞ is a solution of the limit equation (2.2.16).

Claim 3. We claim that $\dfrac{\Omega_n}{\delta_n} \xrightarrow{C^2} \mathbb{R}^N_+$, as $n \to +\infty$.

Since the domain Ω is $C^{2,\alpha}$, then there exists $g \in C^2(\mathbb{R}^{N-1})$ such that $\Omega \cap B_R(0) := \{x = (x', x_N) \in \Omega \cap B_R(0) \mid x_N > g(x_1, ..., x_{N-1}) = g(x')\}$ for some $R > 0$. Arguing as above, we have to consider for every $n \in \mathbb{N}$ the function $g_n(y) := g(T_n^{-1}(y))$ and the domain $\Omega_n := T_n(\Omega)$. Without loss of generality we can assume that $g_n(0) = 0$ and $\nabla_{\mathbb{R}^{N-1}} g_n(0) = 0$. Moreover, by hypothesis $\|g_n\|_{C^2} = \|g\|_{C^2} \le C$, where C is a positive constant. We note that $z \in \dfrac{\Omega_n}{\delta_n}$ if and only if $\delta_n z \in \Omega_n$.

Now, noticing that $\dfrac{\Omega_n}{\delta_n} \cap B_R(0) := \left\{x_N > \dfrac{g_n(\delta_n x')}{\delta_n}\right\}$, we want to show that

$$\frac{g_n(\delta_n x')}{\delta_n} \xrightarrow{C^0_{loc}} 0, \tag{2.2.22}$$

as n goes to $+\infty$. Let us consider the second order Taylor approximation of the function g_n centered at the point $x' = 0$ in a compact set $K \subset \mathbb{R}^{N-1}$, with $0 \in K$:

$$g_n(p) := g_n(0) + \langle \nabla g_n(0), p\rangle + \frac{1}{2}\langle D^2 g_n(\xi p)p, p\rangle, \tag{2.2.23}$$

where $\xi \in (0,1)$ for every $n \in \mathbb{N}$ and for every $p \in K$.

Noticing that $g_n(0) = 0$ and $\nabla g_n(0) = 0$ and recalling that g_n is a C^2 function, it follows that

$$\left|\frac{g_n(\delta_n x)}{\delta_n}\right| = \frac{1}{2\delta_n}|\langle D^2 g_n(\xi\delta_n x)\delta_n x, \delta_n x\rangle| \le \frac{C}{2}\delta_n \|x\|^2 \le C_K \delta_n, \tag{2.2.24}$$

where C_K is a positive constant depending only by the compact set K. Hence by (2.2.24) it follows that (2.2.22) holds and so we have the convergence of the domain.

It remains to verify the Dirichlet datum for the limiting profile u_∞. More precisely we have to show that $w_\infty = 0$ on $\partial\mathbb{R}^N_+$. By Theorem 2.2.1 it follows that

$$
\begin{aligned}
w_n(y) = \frac{u_n(\delta_n)}{M_n} &\leq m_2 \frac{[(\phi_1)_n(\delta_n y)]^{\frac{2}{\gamma+1}}}{M_n} \\
&\leq L m_2 \frac{[\mathrm{dist}(\delta_n y, \partial\Omega_n)]^{\frac{2}{\gamma+1}}}{M_n} \\
&= L m_2 \frac{\delta_n^{\frac{2}{\gamma+1}} [\mathrm{dist}(y, \partial\Omega_n^*)]^{\frac{2}{\gamma+1}}}{M_n} \\
&\leq C(K, L, m_2, m_1) [\mathrm{dist}(y, \partial\Omega_n^*)]^{\frac{2}{\gamma+1}} \quad \text{in } \Omega_n^*.
\end{aligned} \tag{2.2.25}
$$

Since $\Omega_n^* \to \mathbb{R}^N_+$, as n goes to $+\infty$, by (2.2.25) and (2.2.17), passing to the limit we have that

$$
0 \leq w_\infty(y) \leq C(K, L, m_2, m_1) \left[\mathrm{dist}(y, \partial\mathbb{R}^N_+)\right]^{\frac{2}{\gamma+1}} . \tag{2.2.26}
$$

Hence for any $y \in \partial\mathbb{R}^N_+$ we have that $w_\infty(y) = 0$. Moreover, it is important to observe that in (2.2.26), w_∞ satisfies the growth hypothesis (2.1.3) of Theorem 2.1.1.

So we can pass to the limit in (2.2.15) to obtain equation (2.2.16). But, by Theorem 2.0.3, we have that w_∞ is the unique solution of problem (2.0.5), hence

$$
w_\infty(x) = w_\infty(x_N) = \left(\frac{\tilde{C}}{2} \frac{(\gamma+1)^2}{\gamma - 1} \right)^{\frac{1}{\gamma+1}} (x_N)^{\frac{2}{\gamma+1}}. \tag{2.2.27}
$$

Then, by (2.2.27), it follows that

$$
\partial_{\nu(x)} w_\infty(x) > 0 \quad \forall x \in \mathbb{R}^N_+
$$

for every $\nu \in \mathbb{R}^N$ such that $(\nu, \eta) > 0$ and this provides a contradiction with (2.2.12). Hence we have the thesis (2.0.4) and the result is proved.

□

3

The Höpf boundary lemma for quasilinear elliptic problems involving singular nonlinearities and applications

In this chapter, we deal with positive weak solutions to the singular quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u = \dfrac{1}{u^\gamma} + f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.0.1}$$

where $p > 1$, $\gamma > 1$, Ω is a $\mathcal{C}^{2,\alpha}$ bounded domain of $\mathbb{R}^N$ with $N \geq 1$ and $f : \Omega \to \mathbb{R}$ locally Lipschitz continuous. A key point to have in mind in the study of semilinear or quasilinear problems involving singular nonlinearities is the fact that the source term loses regularity at zero, namely the problem is singular near the boundary, as pointed out in the previous chapter. As a first consequence, solutions are not smooth up to the boundary (see [82]) and the gradient generally blows up near the boundary in such a way that $u \notin W_0^{1,p}(\Omega)$. Therefore, here and in all the chapter, we mean that $u \in C^{1,\alpha}(\Omega)$ is a solution to (3.0.1) in the weak distributional meaning according to Definition 3.1.1. Existence and uniqueness results regarding problem (3.0.1) can be found e.g. in [**4**, **16**, **17**, **25**, **26**, **28**, **67**, **95**, **96**].

In this setting we prove a general version of a Höpf type boundary lemma regarding the sign of the derivatives of the solution near the boundary and in the interior of the domain, as we have done in the previous chapter in the semilinear context. To state our result we need some notation; thus we shall denote with $I_\delta(\partial\Omega)$ a neighborhood of the boundary with the *unique nearest point property* (see e.g. [**66**]). We have to recall Definition 2.0.1 of inward pointing normal defined by:

$$\eta(x) := \frac{x - \hat{x}}{|x - \hat{x}|}. \tag{3.0.2}$$

With this notation we have the following:

Theorem 3.0.1 (Höpf type boundary lemma). *Let $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution to (3.0.1). Then, for any $\beta > 0$, there exists a neighborhood $I_\delta(\partial\Omega)$ of $\partial\Omega$, such that*

$$\partial_{\nu(x)} u > 0 \qquad \forall\, x \in I_\delta(\partial\Omega) \tag{3.0.3}$$

whenever $\nu(x) \in \mathbb{R}^N$ with $\|\nu(x)\| = 1$ and $(\nu(x), \eta(x)) \geq \beta$.

We are mainly concerned with the study of the sign of the derivatives near the boundary. Such a control is generally deduced a posteriori, by contradiction, assuming that the solution is $\mathcal{C}^1$ up to the boundary. In our setting this is not a natural assumption and we develop a different technique that in any case exploits very basic arguments of common use. In fact we carry out a scaling argument near the boundary that leads to a limiting problem in the half space.

$$\begin{cases} -\Delta_p u = \dfrac{1}{u^\gamma} & \text{in } \mathbb{R}^N_+ \\ u > 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{3.0.4}$$

where $p > 1$, $\gamma > 1$, $N \geq 1$, $\mathbb{R}^N_+ := \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_N > 0\}$ and $u \in C^{1,\alpha}(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$.

Our scaling argument leads in fact to the study of a limiting profile which is a solution to (3.0.4) and obeys to suitable decay assumptions. It is therefore crucial for our technique, and may also have an independent interest, the following classification result:

THEOREM 3.0.2. *Let $\gamma > 1$ and let $u \in C^{1,\alpha}(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ be a solution to problem (3.0.4) such that*

$$\underline{c} x_N^\beta \leq u(x) \leq \overline{C} x_N^\beta \qquad \text{with} \quad \beta := \frac{p}{\gamma + p - 1} \tag{3.0.5}$$

and $\underline{c}, \overline{C} \in \mathbb{R}^+$. Then
(3.0.6)

$$u(x) = u(x_N) = M x_N^\beta \qquad \text{with} \quad M := \left[\frac{(\gamma + p - 1)^p}{p^{p-1}(p-1)(\gamma - 1)}\right]^{\frac{1}{\gamma+p-1}}.$$

The Höpf boundary lemma is a fundamental tool in many applications. We exploit it here to develop the *moving planes method* (see [1, 12, 68, 111]) for problem (3.0.1) obtaining the following:

THEOREM 3.0.3. *Let Ω be a bounded smooth domain of $\mathbb{R}^N$ which is strictly convex in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution of problem (3.0.1) with $f(s) > 0$ for $s > 0$ ($f(0) \geq 0$). Then it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.*
In particular if the domain is a ball, then the solution is radial and radially decreasing.

For the reader's convenience we sketch the proofs here below.

- In Section 3.1 we prove 1D-symmetry result in half spaces for problem (3.0.4), see Theorem 3.1.2. Mainly we develop a comparison principle to compare the solution u and it's translation $u_\tau := u(x - \tau e_1)$. Even if the source term is decreasing, a quite technical approach is needed because the operator is nonlinear and we are reduced to work in unbounded domains. The 1D-symmetry result obtained leads us to the study of a one dimensional problem in $\mathbb{R}^+$. We carry out this analysis proving a uniqueness result (see Proposition 3.2.2) that provides, as a corollary, the proof of Theorem 3.0.2.

- In Section 3.2 we prove Lemma 3.2.1, that is a very useful tool in the ODE analysis. Moreover we run through again, in the quasilinear setting, the technique of [82] to provide asymptotic estimates for the solutions near the boundary in terms of the distance function, see Theorem 3.3.2. Finally we prove Lemma 3.3.1 that is a weak comparison principle in bounded domain that we used in the proof of Theorem 3.3.2.
- Section 3.3 is the core of the chapter. We prove here Theorem 3.0.1 developing the scaling argument that leads to the problem in the half-space. To this aim we strongly exploit the asymptotic estimates deduced in Theorem 3.3.2. The proof follows by contradiction thanks to the classification result Theorem 3.0.2.
- Finally, in Section 3.4, we apply our Höpf type boundary lemma to prove the symmetry and monotonicity result stated in Theorem 3.0.3. The proof is based on the joint use of the *moving planes method* and the monotonicity information near the boundary provided by Theorem 3.0.1 that allow to avoid the region where the problem is singular.

3.1. One dimensional symmetry in the half-space

The aim of this section is to show the first part of Theorem 3.0.2, in particular we are going to prove that each solution u to problem (3.0.4) satisfying (3.2.2) is one-dimensional. Solutions to p-Laplace equations are generally of class $\mathcal{C}^{1,\alpha}$, see [46, 122]. Therefore a solution to (3.0.1) has to be understood in the weak distributional meaning taking into account the singular nonlinearity. We state the following:

Definition 3.1.1. *We say that $u \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$, $u > 0$ in Ω, is a weak solution to problem* (3.0.1) *if*

(3.1.1)
$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx = \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega).$$

We say that $u \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$, $u > 0$ in Ω, is a weak subsolution *of problem* (3.0.1) *if*

(3.1.2)
$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx \le \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0.$$

Similarly, we say that $u \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$, $u > 0$ in Ω, is a weak supersolution *of problem* (3.0.1) *if*

(3.1.3)
$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi)\, dx \ge \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0.$$

Theorem 3.1.2. *Let $\gamma > 1$ and let $u \in C^{1,\alpha}(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ be a solution to problem* (3.0.4) *such that*

$$\underline{c}x_N^\beta \le u(x) \le \overline{C}x_N^\beta \quad \forall x \in \mathbb{R}^N_+ \tag{3.1.4}$$

with $\beta := \dfrac{p}{\gamma + p - 1}$. *Then*

$$u_{x_i} \equiv 0 \tag{3.1.5}$$

for every $i = 1, \ldots, N-1$. *Namely,* $u(x) = u(x_N)$.

PROOF OF THEOREM 3.1.2. We start with a gradient estimate showing that

$$|\nabla u(x)| \leq \frac{C_b}{x_N^{\frac{\gamma-1}{\gamma+p-1}}} \, . \tag{3.1.6}$$

To prove this fact we use the notation $x = (x_1, ..., x_N) = (x', x_N) \in \mathbb{R}^N$ and, with no loss of generality we consider a point $P_c := (0', x_N^c)$. Setting

$$w(x) := \frac{u(x_N^c \cdot x)}{(x_N^c)^\beta}$$

it follows that

$$-\Delta_p w = \frac{1}{w^\gamma} \qquad \text{in } \mathbb{R}^N_+ \, . \tag{3.1.7}$$

We restrict our attention to the problem

$$\begin{cases} -\Delta_p w = \dfrac{1}{w^\gamma} & \text{in } B_{\frac{1}{2}}(0', 1) \\ w > 0 & \text{in } B_{\frac{1}{2}}(0', 1) \end{cases} \tag{3.1.8}$$

so that, by (3.1.4), it follows that w is bounded and $\frac{1}{w^\gamma} \in L^\infty(B_{\frac{1}{2}}(0', 1))$. Therefore, by standard $C^{1,\alpha}$ estimates [**46, 122**], we deduce that

$$\|w\|_{C^1(B_{\frac{1}{4}}(0',1))} \leq C_b.$$

Scaling back we get (3.1.6).

Arguing by contradiction, without loss of generality, we assume that there exists $P_0 \in \mathbb{R}^N$ such that $u_{x_1}(P_0) > 0$. Hence there exists $\delta > 0$ sufficiently small such that $u_{x_1}(x) > 0$ for all $x \in B_\delta(P_0)$. Now we define

$$u_\tau(x) := u(x - \tau e_1) \tag{3.1.9}$$

where $0 < \tau < \delta$. Hence by the Mean Value Theorem it follows that

$$u(P_0) - u_\tau(P_0) = u_{x_1}(\xi)\tau > \hat{C}\tau > 0 \tag{3.1.10}$$

where $\xi \in \{tP_0 + (1-t)(P_0 - \tau e_1), t \in [0,1]\}$. Moreover, there exists $k > 0$ sufficiently large such that, by the Mean Value Theorem and (3.1.6), we have

$$|u - u_\tau| \leq \frac{\check{C}\tau}{x_N^{\frac{\gamma-1}{\gamma+p-1}}} \quad \text{in } \mathbb{R}^N_+ \cap \{x_N \geq k\}. \tag{3.1.11}$$

Now we set

$$S := \sup_{x \in \mathbb{R}^N_+} (u - u_\tau) > 0. \tag{3.1.12}$$

We also note that $S < +\infty$ by (3.1.4) and (3.1.11). Let us consider

$$w_{\tau,\varepsilon}(x) := [u - u_\tau - (S - \varepsilon)]^+ \tag{3.1.13}$$

for every $\varepsilon > 0$ small enough. We notice that, by (3.1.4) and (3.1.11),

$$\operatorname{supp}(w_{\tau,\varepsilon}) \subset\subset \{\hat{k} \leq x_N \leq \hat{K}\} \tag{3.1.14}$$

for some $\hat{k}, \hat{K} > 0$. We consider a standard cutoff function $\varphi_R := \varphi_R(x')$ such that $\varphi_R = 1$ in $B'_R(0)$, $\varphi_R = 0$ in $(B'_{2R}(0))^c$ and $|\nabla\varphi_R| \leq \frac{2}{R}$ in $B'_{2R}(0) \setminus B'_R(0)$, where $B'_R(0)$ denotes the $(N-1)$-dimensional ball of center 0 and radius R.

We distinguish two cases:

Case 1: $1 < p < 2$. We set

$$\psi := w^{\alpha}_{\tau,\varepsilon}\varphi^2_R \tag{3.1.15}$$

where $\alpha > 0$, $w_{\tau,\varepsilon}$ is defined in (3.1.13) and φ_R is the cutoff function defined here above. First of all we notice that ψ belongs to $W^{1,p}_0(\mathbb{R}^N_+)$. By density argument we can take ψ as test function in the weak formulation of problem (3.0.4), see Definition 3.1.1, so that, subtracting the equation for u and u_τ, we obtain

$$\begin{aligned}
&\alpha \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla w_{\tau,\varepsilon}) w^{\alpha-1}_{\tau,\varepsilon}\varphi^2_R \, dx \\
&= -2 \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla\varphi_R) w^{\alpha}_{\tau,\varepsilon}\varphi_R \, dx \\
&\quad + \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} \left(\frac{1}{u^\gamma} - \frac{1}{u^\gamma_\tau}\right) w^{\alpha}_{\tau,\varepsilon}\varphi^2_R \, dx \, .
\end{aligned} \tag{3.1.16}$$

From (3.1.16), using (1.0.2) and the Mean Value Theorem, we obtain

$$\begin{aligned}
&\alpha C_1 \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w^{\alpha-1}_{\tau,\varepsilon}\varphi^2_R \, dx \\
&\leq \alpha \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla w_{\tau,\varepsilon}) w^{\alpha-1}_{\tau,\varepsilon}\varphi^2_R \, dx \\
&= -2 \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla\varphi_R) w^{\alpha}_{\tau,\varepsilon}\varphi_R \, dx \\
&\quad + \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} \left(\frac{1}{u^\gamma} - \frac{1}{u^\gamma_\tau}\right) w^{\alpha}_{\tau,\varepsilon}\varphi^2_R \, dx \\
&\leq 2C_4 \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} |\nabla(u - u_\tau)|^{p-1} |\nabla\varphi_R| w^{\alpha}_{\tau,\varepsilon}\varphi_R \, dx \\
&\quad - \gamma \int_{\mathbb{R}^N_+\cap\operatorname{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w^{\alpha}_{\tau,\varepsilon}\varphi^2_R \, dx
\end{aligned} \tag{3.1.17}$$

where ξ belongs to $\overrightarrow{(u,u_\tau)}$. Hence, recalling also (3.1.6), we deduce that

$$\begin{aligned}
&\alpha C_1\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\varphi_R^2\,dx\\
&\le 2C_4\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}|\nabla(u-u_\tau)|^{p-1}|\nabla\varphi_R|w_{\tau,\varepsilon}^{\alpha}\varphi_R\,dx\\
&\quad-\gamma\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}\frac{1}{\xi^{\gamma+1}}w_{\tau,\varepsilon}^{\alpha+1}\varphi_R^2\,dx\\
&\le \check{C}\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}|\nabla\varphi_R|w_{\tau,\varepsilon}^{\alpha}\,dx
\end{aligned}\tag{3.1.18}$$

where $\check{C}:=2C_4\|\nabla(u-u_\tau)\varphi_R\|^{p-1}_{L^\infty(\mathbb{R}^N_+\cap\mathrm{supp}(\psi))}$. Exploiting the weighted Young's inequality with exponents $\left(\dfrac{\alpha+1}{\alpha},\ \alpha+1\right)$ we obtain

$$\begin{aligned}
&\alpha C_1\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\varphi_R^2\,dx\\
&\le \check{C}\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}|\nabla\varphi_R|w_{\tau,\varepsilon}^{\alpha}\,dx\\
&\le \frac{\check{C}}{\sigma^{\alpha+1}(\alpha+1)}\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}|\nabla\varphi_R|^{\alpha+1}\,dx\\
&\quad+\frac{\check{C}\alpha}{\alpha+1}\sigma^{\frac{\alpha}{\alpha+1}}\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}w_{\tau,\varepsilon}^{\alpha+1}\,dx\\
&\le \frac{\dot{C}}{R^{\alpha-(N-2)}}+\frac{\check{C}\alpha}{\alpha+1}\sigma^{\frac{\alpha}{\alpha+1}}\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}\left[w_{\tau,\varepsilon}^{\frac{\alpha+1}{2}}\right]^2\,dx.
\end{aligned}\tag{3.1.19}$$

From (3.1.19) and exploiting the Poincaré inequality in the x_N-direction it follows that

$$\begin{aligned}
&\alpha C_1\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\varphi_R^2\,dx\\
&\le \frac{\dot{C}}{R^{\alpha-(N-2)}}+\frac{\check{C}\alpha}{\alpha+1}\sigma^{\frac{\alpha}{\alpha+1}}\int_{B'_{2R}(0)}\left(\int_{\{y\le k\}}\left[w_{\tau,\varepsilon}^{\frac{\alpha+1}{2}}\right]^2\,dy\right)\,dx'\\
&\le \frac{\dot{C}}{R^{\alpha-(N-2)}}+\frac{\check{C}\alpha}{\alpha+1}\sigma^{\frac{\alpha}{\alpha+1}}C_P^2(k)\left(\frac{\alpha+1}{2}\right)^2\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}|\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\,dx\\
&\le \frac{\check{C}\alpha}{\alpha+1}\sigma^{\frac{\alpha}{\alpha+1}}C_P^2(k)\left(\frac{\alpha+1}{2}\right)^2\\
&\qquad\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}(|\nabla u|+|\nabla u_\tau|)^{(2-p)+(p-2)}\,|\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\,dx\\
&\quad+\frac{\dot{C}}{R^{\alpha-(N-2)}}
\end{aligned}\tag{3.1.20}$$

where C_P is the Poincaré constant. Let us point out that, by (3.1.4), (3.1.14) and standard regularity theory [46, 122], it follows that

$$|\nabla u| + |\nabla u_\tau| \leq C \qquad \text{in } \operatorname{supp}(w_{\tau,\varepsilon}) \subset\subset \{\hat{k} \leq x_N \leq \hat{K}\}\,. \tag{3.1.21}$$

Hence we have

$$\begin{aligned} &\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla u_\tau|)^{p-2}\, |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\, dx \\ &\leq \vartheta \int_{\mathcal{C}(2R)} (|\nabla u| + |\nabla u_\tau|)^{p-2}\, |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\, dx \,+ \frac{\dot{C}}{R^{\alpha-(N+2)}} \end{aligned} \tag{3.1.22}$$

where $\mathcal{C}(R) := \left(\mathbb{R}^N_+ \cap (B'_R(0) \times \mathbb{R})\right)$ and $\vartheta := \frac{\check{C}\alpha}{\alpha+1} \sigma^{\frac{\alpha}{\alpha+1}} C_P^2(k) \left(\frac{\alpha+1}{2}\right)^2 \|(|\nabla u| + |\nabla u_\tau|)^{2-p}\|_\infty$. We set

$$g(R) := \frac{\dot{C}}{R^{\alpha-(N+2)}}$$

and

$$\mathcal{L}(R) := \int_{\mathcal{C}(R)} (|\nabla u| + |\nabla u_\tau|)^{p-2}\, |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1}\, dx$$

so that

$$\mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) + g(R)\,.$$

Now we fix α sufficiently large so that $g(R) \to 0$ as $R \to +\infty$ and, consequently, we take σ small enough so that $\vartheta < 2^{(\alpha-\gamma)\beta+1}$. This allows to exploit Lemma 2.1 of [59]: it follows that

$$\mathcal{L}(R) = 0$$

for any $R > 0$. This proves that actually $w_{\tau,\varepsilon}$ is constant and therefore $w_{\tau,\varepsilon} = 0$ since it vanishes near the boundary. This is a contradiction with (3.1.10) thus proving the result in the case $1 < p < 2$.

Case 2: $p \geq 2$**.** We set

$$\psi := w_{\tau,\varepsilon}\varphi_R^2 \tag{3.1.23}$$

with $w_{\tau,\varepsilon}$ and φ_R defined as in the previous case $1 < p < 2$. Arguing exactly as in the case $1 < p < 2$ we arrive to

$$\begin{aligned} &\int_{\mathbb{R}^N_+ \cap \operatorname{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla w_{\tau,\varepsilon})\varphi_R^2\, dx \\ &= -\,2 \int_{\mathbb{R}^N_+ \cap \operatorname{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla \varphi_R) w_{\tau,\varepsilon}\varphi_R\, dx \\ &\quad + \int_{\mathbb{R}^N_+ \cap \operatorname{supp}(\psi)} \left(\frac{1}{u^\gamma} - \frac{1}{u_\tau^\gamma}\right) w_{\tau,\varepsilon}\varphi_R^2\, dx \end{aligned} \tag{3.1.24}$$

From (3.1.24), using (1.0.2) and the Mean Value Theorem, we deduce that

$$
\begin{aligned}
&C_1 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 \varphi_R^2 \, dx \\
&\le \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla w_{\tau,\varepsilon})\varphi_R^2 \, dx \\
&= -2 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla \varphi_R) w_{\tau,\varepsilon}\varphi_R \, dx \\
&\quad + \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} \left(\frac{1}{u^\gamma} - \frac{1}{u_\tau^\gamma} \right) w_{\tau,\varepsilon}\varphi_R^2 \, dx \\
&\le 2C_2 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}| w_{\tau,\varepsilon} |\nabla \varphi_R| \varphi_R \, dx \\
&\quad - \gamma \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon}\varphi_R^2 \, dx
\end{aligned}
\tag{3.1.25}
$$

where ξ belongs to $\overrightarrow{(u, u_\tau)}$. Exploiting the Young's inequality to the right hand side we have

$$
\begin{aligned}
&C_1 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 \varphi_R^2 \, dx \\
&\le 2C_2 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}| \, |\nabla \varphi_R| w_{\tau,\varepsilon}\varphi_R \, dx \\
&\quad - \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon}\varphi_R^2 \, dx \\
&\le \sigma C_2 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 \, dx \\
&\quad + \frac{C_2}{\sigma} \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla \varphi_R|^2 w_{\tau,\varepsilon}^2 \varphi_R^2 \, dx \\
&\quad - \gamma \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon}\varphi_R^2 \, dx
\end{aligned}
\tag{3.1.26}
$$

As above we shall exploit the fact that $|\nabla u|$ and $|\nabla u_\tau|$ are uniformly bounded in $\mathbb{R}^N_+ \cap \mathrm{supp}(\psi)$, see (3.1.21). Therefore we get

$$
\begin{aligned}
&C_1 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 \varphi_R^2 \, dx \\
&\le \sigma C_2 \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 \, dx \\
&\quad + \left(\frac{\check{C}}{\sigma R^2} - \dot{C} \right) \int_{\mathbb{R}^N_+\cap \mathrm{supp}(\psi)} w_{\tau,\varepsilon}^2 \varphi_R^2 \, dx
\end{aligned}
\tag{3.1.27}
$$

where $\check{C}$ e $\dot{C}$ are positive constants. By taking $R_0 > 0$ sufficiently large it follows that $\dfrac{\check{C}}{\sigma R^2} - \dot{C} < 0$ for every $R \ge R_0$. Hence we have

(3.1.28)
$$\begin{aligned}&\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2\varphi_R^2\,dx\\&\leq\frac{\sigma C_2}{C_1}\int_{\mathbb{R}^N_+\cap\mathrm{supp}(\psi)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2\,dx.\end{aligned}$$

As above, for $\mathcal{C}(R):=\big(\mathbb{R}^N_+\cap(B'_R(0)\times\mathbb{R})\big)$, we set

$$\mathcal{L}(R):=\int_{\mathcal{C}(R)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2\,dx$$

so that

$$\mathcal{L}(R)\leq\vartheta\mathcal{L}(2R)\,.$$

where $\vartheta:=\frac{\sigma C_2}{C_1}>0$ is sufficiently small when $\sigma>0$ is sufficiently small. Applying again Lemma 2.1 of [59] it follows that

$$\int_{\mathcal{C}(R)}(|\nabla u|+|\nabla u_\tau|)^{p-2}\,|\nabla w_{\tau,\varepsilon}|^2\,dx=0$$

for any $R\geq R_0$. This provides a contradiction exactly as in the case $1<p<2$ so that the thesis follows also in the case $p\geq 2$.

□

3.2. ODE analysis and classification result

The aim of this section is to show the second part of Theorem 3.0.2, in particular we are going to prove that each one dimensional solution u to problem (3.0.4) satisfying (3.2.2) is given by (3.0.6). The one dimensional symmetry result proved in Theorem 3.1.2 leads to the study of the one dimensional problem:

(3.2.1)
$$\begin{cases}-\big(|u'|^{p-2}u'\big)'=\dfrac{1}{u^\gamma} & t\in\mathbb{R}^+\\ u>0 & t\in\mathbb{R}^+\\ u(0)=0\end{cases}$$

where $\gamma>1$ and $u\in C^{1,\alpha}(\mathbb{R}^+)\cap C(\mathbb{R}^+\cup\{0\})$. As a consequence, we expect uniqueness for such a problem, since the source term is decreasing. By the way the proof is not straightforward since the source term is decreasing but singular at zero. Now we are ready to prove the following lemma:

Lemma 3.2.1. *Let $u\in C^{1,\alpha}(\mathbb{R}^+)\cap C(\mathbb{R}^+\cup\{0\})$ be a solution to (3.2.1). Assume that there exists a positive constant C_u such that*

(3.2.2)
$$\frac{t^\beta}{C_u}\leq u(t)\leq C_u t^\beta$$

for t sufficiently large and $\beta:=\frac{p}{\gamma+p-1}$. Then there exists a positive constant C'_u such that

(3.2.3)
$$\frac{t^{\beta-1}}{C'_u}\leq u'(t)\leq C'_u t^{\beta-1}$$

for t large enough.

Proof. We first claim that $u'(t) \geq 0$ for every $t > 0$. To prove this fact we argue by contradiction and assume that there exist $t_0 \geq 0$ such that $u'(t_0) < 0$. Setting

$$w(t) := |u'(t)|^{p-2}u'(t)$$

it follows by the equation in (3.2.1) that w is a strictly decreasing function. Therefore $u'(t) \leq -C := u'(t_0) < 0$ for every $t \geq t_0$ and

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\,ds \leq u(t_0) - \int_{t_0}^{t} C\,ds = -Ct + Ct_0 + u(t_0). \tag{3.2.4}$$

This would force u to be negative for t large in contradiction with with the fact that u is positive by assumption. Therefore we deduce that $u'(t), w(t) \geq 0$ for t sufficiently large. Recalling that w is a strictly decreasing function, we deduce that actually $u'(t), w(t) > 0$. Furthermore $w(t) \to M \geq 0$ as t goes to $+\infty$. I is easy to show that $M = 0$. If $M > 0$ in fact, arguing as in (3.2.4), we would have

$$u(t) \geq Mt + c$$

for t sufficiently large. This gives a contradiction with our initial assumption (3.2.2), hence $M = 0$.

Let us now set

$$h(t) := \frac{t^{-\beta\gamma+1}}{\beta\gamma - 1}.$$

By Cauchy's Theorem we have that for t large enough and $k > t$ fixed there exists $\xi_t \in (t, t+k)$ such that

$$\frac{w(t) - w(t+k)}{h(t) - h(t+k)} = \frac{w'(\xi_t)}{h'(\xi_t)}. \tag{3.2.5}$$

Letting $k \to +\infty$ in (3.2.5) we obtain

$$\frac{w(t)}{h(t)} = -\frac{([u']^{p-1})'(\xi_t)}{(\xi_t)^{-\beta\gamma}} = \frac{t^{\beta\gamma}}{u^{\gamma}}. \tag{3.2.6}$$

for t large enough. By (3.2.2) and (3.2.6) we deduce that $\frac{w(t)}{h(t)}$ is bounded at infinity, thus proving (3.2.3).

□

Now we are ready to prove our uniqueness result:

Proposition 3.2.2. *Problem (3.2.1) admits a unique solution $u \in C^{1,\alpha}(\mathbb{R}^+) \cap C(\mathbb{R}^+ \cup \{0\})$ satisfying (3.2.2) given by*

$$u(t) = Mt^{\beta} \tag{3.2.7}$$

where $M := \left[\dfrac{(\gamma+p-1)^p}{p^{p-1}(p-1)(\gamma-1)}\right]^{\frac{1}{\gamma+p-1}}$ *and* $\beta := \dfrac{p}{\gamma+p-1}$.

Proof. Arguing by contradiction we assume that there exist two positive solution $u, v \in C^{1,\alpha}(\mathbb{R}^+) \cap C(\mathbb{R}^+ \cup \{0\})$ to problem (3.2.1) such that $u \not\equiv v$. Let us consider the cutoff function $\varphi_R \in C_c^{\infty}(\mathbb{R})$, $R > 0$, such that $\varphi_R(t) = 1$ if $t \in [-R, R]$, $\varphi_R(t) = 0$ if $t \in (-\infty, -2R) \cup (2R, +\infty)$ and $|\varphi'(t)| < \frac{2}{R}$ for every $t \in (-2R, -R) \cup (R, 2R)$. For $\varepsilon > 0$ (small) we set

$$w_{\varepsilon} = (u - v - \varepsilon)^+ \qquad \text{and} \qquad \psi := [(u - v - \varepsilon)^+]^{\alpha}\varphi_R^2$$

with $\alpha > 0$ (large). Passing through the weak formulation of problem (3.2.1) for u and v, subtracting and using standard elliptic estimates and (1.0.2) we obtain

(3.2.8)
$$\begin{aligned}
&\alpha C_1 \int_0^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\varphi_R^2\,dt\\
&\leq \alpha \int_0^{2R} (|u'|^{p-2}u' - |v'|^{p-2}v', w_\varepsilon')w_\varepsilon^{\alpha-1}\varphi_R^2\,dt\\
&= -2\int_R^{2R} (|u'|^{p-2}u' - |v'|^{p-2}v', \varphi_R')w_\varepsilon^{\alpha}\varphi_R\,dt + \int_0^{2R}\left(\frac{1}{u^\gamma}-\frac{1}{v^\gamma}\right) w_\varepsilon^{\alpha}\varphi_R^2\,dt\\
&\leq 2C_2 \int_R^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'||w_\varepsilon^{\alpha}||\varphi_R'|\varphi_R\,dt - \gamma\int_R^{2R}\frac{1}{\xi^{\gamma+1}}(u-v)w_\varepsilon^{\alpha}\varphi_R^2\,dt
\end{aligned}$$

with $\xi \in \overrightarrow{(u,v)}$. Exploiting the weighted Young's inequality in the right hand side we have

$$\begin{aligned}
&\alpha C_1 \int_0^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\varphi_R^2\,dt\\
&\leq LC_2 \int_R^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\,dt\\
&\quad+\frac{C_2}{LR^2}\int_R^{2R} \left(|u'|+|v'|\right)^{p-2} w_\varepsilon^{\alpha+1}\varphi_R^2\,dt\\
&\quad-\gamma\int_R^{2R}\frac{1}{\xi^{\gamma+1}}w_\varepsilon^{\alpha+1}\varphi_R^2\,dt
\end{aligned} \tag{3.2.9}$$

By Lemma 3.2.1 it follows that

$$\begin{aligned}
&\alpha C_1 \int_0^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\varphi_R^2\,dt\\
&\leq LC_2 \int_R^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\,dt\\
&\quad+\frac{\check{C}}{L}\int_R^{2R} t^{(\beta-1)(p-2)-2} w_\varepsilon^{\alpha+1}\varphi_R^2\,dt\\
&\quad-\dot{C}\int_R^{2R} t^{-\beta(\gamma+1)} w_\varepsilon^{\alpha+1}\varphi_R^2\,dt\\
&\leq LC_2 \int_R^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\,dt\\
&\quad+\left(\frac{\check{C}}{L}-\hat{C}\right)\frac{1}{R^{\beta(\gamma+1)}}\int_R^{2R} w_\varepsilon^{\alpha+1}\varphi_R^2\,dt
\end{aligned} \tag{3.2.10}$$

where we also used the fact that $t/2 \leq R \leq t$ when $t \in [R, 2R]$. Now we fix L sufficiently large such that $\frac{\check{C}}{L} - \hat{C} \leq 0$ so that

(3.2.11)
$$\int_0^{R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\,dt \leq \frac{LC_2}{\alpha C_1}\int_0^{2R} \left(|u'|+|v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\,dt.$$

Hence we define

$$\mathcal{L}(R) := \int_0^R \left(|u'| + |v'|\right)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1}\, dt.$$

By Lemma 3.2.1 we deduce that $\mathcal{L}(\cdot)$ has polynomial growth, namely

$$\mathcal{L}(R) \le CR^{(\beta-1)(p-2)} R^{2(\beta-1)} R^{\beta(\alpha-1)} \int_0^R dt = CR^{(\beta-1)p+\beta(\alpha-1)+1} = CR^{\sigma}$$

with $\sigma := (\alpha-\gamma)\beta + 1$. We take $\alpha > 0$ sufficiently large so that $\sigma > 0$ and $\frac{\vartheta C_2}{\alpha C_1} < 2^{-\sigma}$ so that Lemma 2.1 of [59] apply and shows that

$$\mathcal{L}(R) = 0.$$

From this it follows that $u \le v + \varepsilon$ for every $\varepsilon > 0$, hence $u \le v$. Arguing in the same way it follows that $u \ge v$ and this proves the uniqueness result. To conclude the proof it is now sufficient to check that the function defined in (3.2.7) solves the problem.

□

Proof of Theorem 3.0.2. Once that Theorem 3.1.2 is in force, the proof of Theorem 3.0.2 is a consequence of Proposition 3.2.2.

□

3.3. Asymptotic analysis near the boundary and proof of the Höpf boundary lemma

We start this section considering the auxiliary problem:

$$\begin{cases} -\Delta_p u = \dfrac{p(x)}{u^\gamma} & \text{in } \mathcal{D} \\ u > 0 & \text{in } \mathcal{D} \end{cases} \tag{3.3.1}$$

where $\mathcal{D}$ is a bounded smooth domain of $\mathbb{R}^N$, where $p \in L^\infty(\mathcal{D})$ and $p(x) \ge c > 0$ a.e. in $\mathcal{D}$, $\gamma > 1$ and $u \in W^{1,p}_{loc}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$. For this kind of problems, generally, the weak comparison principle holds true. This is manly due to the monotonicity properties of the source term. In spite of this remark, the proof is not straightforward when considering sub/super solutions that are not smooth up the boundary. Therefore we provide here below a self contained proof of a comparison principle that we shall exploit later on.

Lemma 3.3.1. *Let $u \in W^{1,p}_{loc}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ be a subsolution of problem (3.3.1) in the sense of (3.1.2) and let $v \in W^{1,p}_{loc}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ be a supersolution of problem (3.3.1) in the sense of (3.1.3). Then, if $u \le v$ on $\partial\mathcal{D}$ it follows that $u \le v$ in $\mathcal{D}$.*

Proof. Let us set:

$$w_\varepsilon := (u - v - \varepsilon)^+ \tag{3.3.2}$$

where $\varepsilon > 0$. We notice that w_ε is suitable as test function since $\operatorname{supp}(w_\varepsilon) \subset\subset \mathcal{D}$ and $u, v \in W^{1,p}_{loc}(\mathcal{D})$. Hence $w_\varepsilon \in W^{1,p}_0(\mathcal{D})$ and, by density arguments, we can plug w_ε as test function in (3.1.2) and (3.1.3) and by subtracting we obtain

$$\begin{aligned}(3.3.3)\qquad &\int_{\mathcal{D}\cap \mathrm{supp}(w_\varepsilon)} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w_\varepsilon\right)\, dx \\ &\leq \int_{\mathcal{D}\cap \mathrm{supp}(w_\varepsilon)} p(x)\left(\frac{1}{u^\gamma}-\frac{1}{v^\gamma}\right) w_\varepsilon\, dx.\end{aligned}$$

Taking into account the fact that $u-v \geq u-v-\varepsilon$, the fact that $p(\cdot)$ is positive and $u^{-\gamma}$ is decreasing, it follows that

$$(3.3.4)\qquad \int_{\mathcal{D}\cap \mathrm{supp}(w_\varepsilon)} \left(|\nabla u| + |\nabla v|\right)^{p-2} |\nabla w_\varepsilon|^2\, dx \leq 0.$$

By Fatou's Lemma, as ε tends to zero, we deduce that

$$\int_{\mathcal{D}} \left(|\nabla u| + |\nabla v|\right)^{p-2} |\nabla (u-v)^+|^2\, dx \leq 0$$

showing that $(u-v)^+$ is constant, and therefore zero by the boundary data. Thus we deduce that $u \leq v$ in $\mathcal{D}$ proving the thesis.

□

We exploit now Lemma 3.3.1 to study the boundary behaviour of the solutions to (3.0.1). The proof is actually the one in [82]. Since we could not find an appropriate reference for the estimates that we need, we repeat the argument. We denote with ϕ_1 the first (positive) eigenfunction of the p-laplacian in Ω. Namely

$$(3.3.5)\qquad \begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} & \text{in } \Omega\\ \phi_1 = 0 & \text{on } \partial\Omega.\end{cases}$$

Having in mind Lemma 3.3.1 we can prove a similar result to the one in [82], but in the quasilinear setting.

THEOREM 3.3.2. *Let $u \in C^{1,\alpha}_{loc}(\Omega)\cap C(\overline{\Omega})$ be a positive solution to* (3.0.1)*. Then there exist two positive constants m_1, m_2 and there exists $\delta > 0$ sufficiently small such that*

$$(3.3.6)\qquad m_1\phi_1(x)^{\frac{p}{\gamma+p-1}} \leq u(x) \leq m_2\phi_1(x)^{\frac{p}{\gamma+p-1}} \quad \forall\, x \in I_\delta(\partial\Omega).$$

PROOF. We rewrite the equation in (3.0.1) as

$$(3.3.7)\qquad -\Delta_p u = \frac{1}{u^\gamma} + f(u) = \frac{p(x)}{u^\gamma} \qquad \text{in } \Omega$$

where $p(x) := 1 + u(x)^\gamma f(u(x))$. In the following we assume that δ is small enough so that

$$p(x) > 0 \qquad \forall\, x \in I_\delta(\partial\Omega).$$

Arguing as in [82], we exploit the principal eigenfunction ϕ_1 of problem (3.3.5) and the fact that $\phi_1 \in C^{1,\alpha}(\overline{\Omega})$ (see e.g. [3, 86, 80]) and

$$\nabla\phi_1(x) \neq 0 \qquad \forall\, x \in \partial\Omega.$$

For $t := \dfrac{p}{\gamma+p-1}$ we set $\Psi := s\,\phi_1^t$, $s > 0$. It is easy to see that

$$-\Delta_p \Psi = \frac{g(x,s)}{\Psi^\gamma} \qquad \text{in } I_\delta(\partial\Omega)$$

where

$$g(x,s) := s^{\gamma+p-1}t^{p-1}\left[\frac{(\gamma-1)(p-1)}{\gamma+p-1}|\nabla\phi_1(x)|^p + \lambda_1\phi_1(x)^p\right]. \tag{3.3.8}$$

Since $0 < t < 1$, we can choose two positive constants s_1 and s_2 such that $0 < s_1 < s_2$ and

$$g(x,s_1) < p(x) < g(x,s_2) \qquad \forall\ x \in I_\delta(\partial\Omega). \tag{3.3.9}$$

Hence, setting $u_1 := s_1\phi_1^t$ and $u_2 := s_2\phi_1^t$, we have that

$$-\Delta_p u_1 < \frac{p(x)}{u_1^\gamma} \qquad \text{in } I_\delta(\partial\Omega) \tag{3.3.10}$$

and

$$-\Delta_p u_2 > \frac{p(x)}{u_2^\gamma} \qquad \text{in } I_\delta(\partial\Omega) \tag{3.3.11}$$

In order to control the datum on the boundary of $I_\delta(\partial\Omega)$ (in the interior of the domain), we need to switch from u to $u_\beta := \beta u$ observing that

$$-\Delta_p u_\beta \;=\; \beta^{\gamma+p-1}\frac{p(x)}{u_\beta^\gamma}.$$

For $\beta_1 > 0$ large it follows that u_{β_1} and u_1 satisfy the following problem:

$$\begin{cases} -\Delta_p u_{\beta_1} \geq \dfrac{p(x)}{u_{\beta_1}^\gamma} & \text{in } I_\delta(\partial\Omega) \\ -\Delta_p u_1 < \dfrac{p(x)}{u_1^\gamma} & \text{in } I_\delta(\partial\Omega) \\ u_{\beta_1} \geq u_1 & \text{on } \partial I_\delta(\partial\Omega). \end{cases} \tag{3.3.12}$$

By Lemma 3.3.1 it follows now that

$$u_{\beta_1} = \beta_1 u \geq u_1 \qquad \text{in } I_\delta(\partial\Omega). \tag{3.3.13}$$

Similarly, for $\beta_2 > 0$ small, it follows that u_{β_2} and u_2 satisfy the problem:

$$\begin{cases} -\Delta_p u_{\beta_2} \leq \dfrac{p(x)}{u_{\beta_2}^\gamma} & \text{in } I_\delta(\partial\Omega) \\ -\Delta_p u_2 > \dfrac{p(x)}{u_2^\gamma} & \text{in } I_\delta(\partial\Omega) \\ u_{\beta_2} \leq u_2 & \text{on } \partial I_\delta(\partial\Omega). \end{cases} \tag{3.3.14}$$

By Lemma 3.3.1 it follows that

$$u_{\beta_2} = \beta_2 u \leq u_2 \qquad \text{in } I_\delta(\partial\Omega). \tag{3.3.15}$$

Hence the thesis is proved with $m_1 := \dfrac{s_1}{\beta_1}$ and $m_2 := \dfrac{s_2}{\beta_2}$.

□

We are now ready to prove Theorem 3.0.1 exploiting the previous preliminary results.

Proof of Theorem 3.0.1. Since the domain is of class $C^{2,\alpha}$ we may and do reduce to work in a neighborhood of the boundary $I_{\bar{\delta}}(\partial\Omega)$ where the *unique nearest point property* holds (see e.g. [66]). Arguing by contradiction, let us assume that there exists a sequence of points $\{x_n\}$ in $I_{\bar{\delta}}(\partial\Omega)$, such that $x_n \longrightarrow x_0 \in \partial\Omega$, as $n \to +\infty$, and

$$\partial_{\nu(x_n)} u(x_n) \leq 0, \text{ with } (\nu(x_n), \eta(x_n)) \geq \beta > 0. \tag{3.3.16}$$

Without loss of generality, we can assume that $x_0 = 0 \in \partial\Omega$ and $\eta(x_n) = e_N$. This follows by the fact that the p-Laplace operator is invariant under isometries. More precisely, for each $n \in \mathbb{N}$, we can consider an isometry $T_n : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with the above mentioned properties just composing a translation and a rotation of the axes. This procedure generates a new sequence of points $\{y_n\}$, where $y_n := T_n x_n$, such that every $y_n \in \operatorname{span}\langle e_N \rangle$ and $y_n \longrightarrow 0$ as $n \to +\infty$. Setting $u_n(y) := u(T_n^{-1}(y))$, it follows that

$$-\Delta_p u_n = \frac{1}{u_n^\gamma} + f(u_n) \qquad \text{in } \Omega_n = T_n(\Omega). \tag{3.3.17}$$

Now we set

$$w_n(y) := \frac{u_n(\delta_n y)}{M_n} \tag{3.3.18}$$

where $\delta_n := \operatorname{dist}(x_n, \partial\Omega) = \operatorname{dist}(T_n x_N, 0)$ and $M_n := u_n(\delta_n e_N) = u(x_n)$. It follows that $\delta_n \to 0$ as $n \to +\infty$ and

- w_n is defined in $\Omega_n^* := \dfrac{\Omega_n}{\delta_n}$.
- $w_n(e_N) = 1$.
- $M_n \to 0$, as $n \to +\infty$.

It is easy to see that w_n weakly satisfies

$$-\Delta_p w_n = \frac{\delta_n^p}{M_n^{\gamma+p-1}} \left(\frac{1}{w_n(y)^\gamma} + M_n^\gamma f(u_n(\delta_n y)) \right) \qquad \text{in } \Omega_n^*. \tag{3.3.19}$$

The key idea of the proof is to argue by contradiction exploiting a limiting profile, that we shall denote by u_∞, which is a solution to a limiting problem in a half space. The contradiction will then follows applying the classification result in Theorem 3.0.2. Here below we develop this argument and we suggest to the reader to keep in mind that f is bounded, the term $M_n^\gamma f(u_n(\delta_n y))$ will vanish since M_n goes to zero and $\dfrac{\delta_n^p}{M_n^{\gamma+p-1}}$ is bounded as a consequence of Theorem 3.3.2. Therefore the expected limiting equation is:

$$-\Delta_p w_\infty = \frac{\tilde{C}}{w_\infty^\gamma} \qquad \text{in } \mathbb{R}^N_+. \tag{3.3.20}$$

Let us provide the details needed to pass to the limit. We claim that:

- $w_n \overset{C^{1,\alpha}}{\longrightarrow} w_\infty$, as $n \to +\infty$, in any compact set K of $\mathbb{R}^N_+$.
- $w_\infty \in C^{1,\alpha}(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$.

- $w_\infty = 0$ on $\partial\mathbb{R}^N_+$.

To prove this let us consider a compact set $K \subset \mathbb{R}^N_+$. For $n \in \mathbb{N}$ large we can assume that $K \subset I_{\bar{\delta}/\delta_n}(\partial\Omega_n^*)$ so that Theorem 3.3.2 can be exploited.

Claim 1. We claim that $w_n(y) > 0$ for all $y \in K$ and for $n \in \mathbb{N}$ large. Let $y \in K$. Hence, by Theorem 3.3.2

$$w_n(y) := \frac{u_n(\delta_n y)}{M_n} \geq L \frac{(\mathrm{dist}(\delta_n y, \partial\Omega_n))^{\frac{p}{\gamma+p-1}}}{M_n}.$$

In particular, by the fact that $\mathrm{dist}(\delta_n y, \partial\Omega_n) \geq \overline{C}\delta_n$, it follows that

$$w_n(y) \geq L \frac{(\overline{C}\delta_n)^{\frac{p}{\gamma+p-1}}}{M_n} \geq C(K, \gamma, m_1) > 0\,. \tag{3.3.21}$$

Claim 2. We claim that $w_n \overset{C^{1,\alpha}}{\longrightarrow} w_\infty$, as $n \to +\infty$, in any compact set K of $\mathbb{R}^N_+$.
Since $\mathrm{dist}(y, \partial\Omega_n^*) \leq C$ for every $y \in K$, by Theorem 3.3.2 it follows that

$$\begin{aligned} w_n(y) = \frac{u_n(\delta_n\, y)}{M_n} &\leq L m_2 \frac{[\mathrm{dist}(\delta_n y, \partial\Omega_n)]^{\frac{p}{\gamma+p-1}}}{M_n} \\ &= L m_2 \frac{\delta_n^{\frac{p}{\gamma+p-1}}\,[\mathrm{dist}(y, \partial\Omega_n^*)]^{\frac{p}{\gamma+p-1}}}{M_n} \\ &\leq L m_2 C^{\frac{p}{\gamma+p-1}} \frac{\delta_n^{\frac{p}{\gamma+p-1}}}{M_n} \\ &\leq L C^{\frac{p}{\gamma+p-1}} C(K, m_2). \end{aligned} \tag{3.3.22}$$

Hence

$$\|w_n\|_{L^\infty(K)} \leq C(K)$$

for any compact set K of $\mathbb{R}^N_+$. By standard regularity theory (see e.g. [70]) it follows that w_n is uniformly bounded in $C^{1,\alpha}(K')$ for any compact set $K' \subset K$. Therefore, by Ascoli's Theorem, we can pass to the limit in any compact set and with $C^{1,\alpha'}$ convergence. Exploiting a standard diagonal process, in the same spirit of the previous chapter, we can therefore define the limiting function w_∞ that turns out to be a solution to (3.3.20) in the half space. The fact that $\dfrac{\Omega_n}{\delta_n}$ leads to the limiting domain $\mathbb{R}^N_+$ as $n \to +\infty$ follows by standard arguments that we discussed in the previous chapter.

It remains to verify the Dirichlet datum for the limiting profile w_∞. More precisely we have to show that $w_\infty = 0$ on $\partial\mathbb{R}^N_+$. By Theorem 3.3.2 it follows that

$$\begin{aligned} w_n(y) = \frac{u_n(\delta_n\, y)}{M_n} &\leq L m_2 \frac{[\mathrm{dist}(\delta_n y, \partial\Omega_n)]^{\frac{p}{\gamma+p-1}}}{M_n} \\ &= L m_2 \frac{\delta_n^{\frac{p}{\gamma+p-1}}\,[\mathrm{dist}(y, \partial\Omega_n^*)]^{\frac{p}{\gamma+p-1}}}{M_n} \\ &\leq C(K, L, m_2, m_1)\,[\mathrm{dist}(y, \partial\Omega_n^*)]^{\frac{p}{\gamma+p-1}} \quad \text{in } \Omega_n^*. \end{aligned} \tag{3.3.23}$$

Since $\Omega_n^* \to \mathbb{R}^N_+$, as n goes to $+\infty$, by (3.3.23) and (3.3.21), passing to the limit we have that

$$0 \le w_\infty(y) \le C(K, L, m_2, m_1) \left[\mathrm{dist}(y, \partial\mathbb{R}^N_+)\right]^{\frac{p}{\gamma+p-1}} . \tag{3.3.24}$$

In a similar fashion, and exploiting again Theorem 3.3.2, we also deduce that

$$w_\infty(y) \ge C(K, L, m_2, m_1) \left[\mathrm{dist}(y, \partial\mathbb{R}^N_+)\right]^{\frac{p}{\gamma+p-1}} . \tag{3.3.25}$$

By (3.3.24) it follows that $w_\infty(y) = 0$ as claimed. Furthermore, collecting (3.3.24) and (3.3.25), we deduce that w_∞ has the right asymptotic behaviour needed to apply Theorem 3.0.2, see (3.0.5). This shows that that w_∞ is the unique solution to (3.3.20) given by

$$w_\infty(x) = w_\infty(x_N) = \left[\frac{\tilde{C}}{p^{p-1}} \frac{(\gamma+p-1)^p}{(p-1)(\gamma-1)}\right]^{\frac{1}{\gamma+p-1}} (x_N)^{\frac{p}{\gamma+p-1}}. \tag{3.3.26}$$

On the other hand, passing to the limit in (3.3.16), it would follows that

$$\partial_{\bar{\nu}} w_\infty(e_N) \le 0$$

for some $\bar{\nu} \in \mathbb{R}^N$ with $(\bar{\nu}, e_N) > 0$. Clearly this is a contradiction with (3.3.26) thus proving the result.

□

Now using the Theorem 3.0.1 we want to prove the symmetry result.

3.4. Symmetry and monotonicity result

In this section we prove our symmetry (and monotonicity) result. Actually we provide the details needed for the application of the *moving planes method*. For the semilinear case see [24, 25, 26], in the quasilinear setting we use the technique developed in [37].

We start with some notation: for a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\} \tag{3.4.1}$$

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \dots, x_N) \tag{3.4.2}$$

which is the reflection through the hyperplane $T_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}$. Also let

$$a = \inf_{x\in\Omega} x_1. \tag{3.4.3}$$

Now we set

$$u_\lambda(x) = u(x_\lambda) . \tag{3.4.4}$$

Finally we define

$$\Lambda_0 = \{a < \lambda < 0 : u \le u_t \text{ in } \Omega_t \text{ for all } t \in (a, \lambda]\} .$$

In the following the critical set of u

$$\mathcal{Z}_u := \{\nabla u = 0\}$$

will play a crucial role. Let us first note that, as a consequence of Theorem 3.0.1, we know that

$$\mathcal{Z}_u \subset\subset \Omega\,.$$

This fact allows to exploit the results of [37] since the solution is positive in the interior of the domain (and the nonlinearity is no more singular there). Therefore we conclude that

$$|\mathcal{Z}_u| = 0 \qquad \text{and} \qquad \Omega \setminus \mathcal{Z}_u \quad \text{is connected.}$$

Proof of Theorem 3.0.3. The proof follows via the *moving planes technique*. We start showing that:

$$\Lambda_0 \neq \emptyset\,.$$

To prove this, let us consider $\lambda > a$ with $\lambda - a$ small. By Theorem 3.0.1 it follows that

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in} \quad \Omega_\lambda \cup R_\lambda(\Omega_\lambda),$$

and this immediately proves that $u < u_\lambda$ in Ω_λ.
Now we define

$$\lambda_0 := \sup \Lambda_0.$$

We shall show that $u \leq u_\lambda$ in Ω_λ for every $\lambda \in (a, 0]$, namely that:

$$\lambda_0 = 0\,.$$

To prove this, we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \leq u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau}$ for any $0 < \tau < \bar{\tau}$ for some $\bar{\tau} > 0$ (small). By continuity we know that $u \leq u_{\lambda_0}$ in Ω_{λ_0}. The *strong comparison principle* (see e.g. [103, 127] or Chapter 1) holds true in $\Omega_{\lambda_0} \setminus Z_u$, providing that

$$u < u_{\lambda_0} \qquad \text{in} \quad \Omega_{\lambda_0} \setminus Z_u\,.$$

Note in fact that, in each connected component $\mathcal{C}$ of $\Omega_{\lambda_0} \setminus Z_u$, the *strong comparison principle* implies that $u < u_{\lambda_0}$ in $\mathcal{C}$ unless $u \equiv u_{\lambda_0}$ in $\mathcal{C}$. Actually the latter case is not possible. In fact, if $\partial\mathcal{C} \cap \partial\Omega \neq \emptyset$ this is not possible in view of the zero Dirichlet baundary datum since u is positive in the interior of the domain. If else $\partial\mathcal{C} \cap \partial\Omega = \emptyset$ then we should have a *local symmetry region* causing $\Omega \setminus Z_u$ to be not connected, against what we already remarked above.

Therefore, given a compact set $K \subset \Omega_{\lambda_0} \setminus Z_u$, by uniform continuity we can ensure that $u < u_{\lambda_0+\tau}$ in K for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. Moreover, by Theorem 3.0.1 and taking into account the zero Dirichlet boundary datum, it is easy to show that, for some $\delta > 0$, we have that

$$u < u_{\lambda_0+\tau} \quad \text{in} \quad I_\delta(\partial\Omega) \cap \Omega_{\lambda_0+\tau} \tag{3.4.5}$$

for any $0 < \tau < \bar{\tau}$. This is quite standard once that Theorem 3.0.1 is in force. The hardest part is the study in the region near $\partial\Omega \cap T_{\lambda_0+\tau}$. Here we exploit the monotonicity properties of the solutions proved in Theorem 3.0.1 that works once we note that $(e_1, \eta(x)) > 0$ in a neighborhood of $\partial\Omega \cap T_{\lambda_0+\tau}$ since the domain is smooth and strictly convex.

Now we define

$$w_{\lambda_0+\tau} := (u - u_{\lambda_0+\tau})^+$$

for any $0 < \tau < \bar{\tau}$. We already showed in (3.4.5) that $\mathrm{supp}(w_{\lambda_0+\tau}) \subset\subset \Omega_{\lambda_0+\tau}$. Moreover $w_{\lambda_0+\tau} = 0$ in K by construction.

For any $\tau > 0$ fixed, we can choose $\bar{\tau}$ small and K large so that

$$|\Omega_{\lambda_0+\tau} \setminus K| < \tau\,.$$

Here we are also exploiting the fact that the critical set Z_u has zero Lebesgue measure (see [37]).

In particular we take τ sufficiently small so that the *weak comparison principle in small domains* (see [37]) works, showing that

$$w_{\lambda_0+\tau} = 0 \quad \text{in} \quad \Omega_{\lambda_0+\tau}$$

for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. But this is in contradiction with the definition of λ_0. Hence $\lambda_0 = 0$.

The desired symmetry (and monotonicity) result follows now performing the procedure in the same way but in the opposite direction.

□

4

Qualitative properties of singular solutions to semilinear elliptic problems

The aim of this chapter is to investigate symmetry and monotonicity properties of singular solutions to semilinear elliptic equations. We address the issue of problems involving singular nonlinearity. More precisely let us consider the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega \setminus \Gamma \\ u > 0 & \text{in} \quad \Omega \setminus \Gamma \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.0.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$. Our results will be obtained by means of the moving planes technique, see [**1**, **12**, **68**, **111**]. Such a technique can be performed in general domains providing partial monotonicity results near the boundary and symmetry when the domain is convex and symmetric. For semplicity of exposition we assume directly in all the chapter that Ω is a convex domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$. The solution has a possible singularity on the critical set $\Gamma \subset \Omega$. Furthermore in all the chapter the nonlinearity f will be assumed to be uniformly locally Lipschitz continuous from above far from the singular set. More precisely we recall the following:

Definition 4.0.1 (I_f). *We say that f fulfills the condition (I_f) if $f : \overline{\Omega} \setminus \Gamma \times (0, +\infty) \to \mathbb{R}$ is a continuous function such that for $0 < t \leq s \leq M$ and for any compact set $K \subset \overline{\Omega} \setminus \Gamma$, it holds*

$$f(x,s) - f(x,t) \leq C(K,M)(s-t) \qquad \textit{for any} \quad x \in K\,,$$

where $C(K,M)$ is a positive constant depending on K and M. Furthermore $f(\cdot, s)$ is non-decreasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$ and symmetric with respect to the hyperplane $\{x_1 = 0\}$.

A typical example is provided by positive solutions to

$$-\Delta u = \frac{1}{u^\alpha} + g(u) \quad \text{in} \quad \Omega \setminus \Gamma \tag{4.0.2}$$

where $\alpha > 0$ and g is locally Lipschitz continuous. Such a problem, in the case $\Gamma = \emptyset$, as been widely investigated in the literature. We refer the readers to the pioneering work [**31**] and to [**17**, **23**, **25**, **27**, **28**, **82**, **95**, **120**]. In particular, by [**82**], it is known that solutions generally have no H^1-regularity up to the boundary. Therefore, having this example in mind, the natural assumption in this chapter is

$$u \in H^1_{loc}(\Omega \setminus \Gamma) \cap C(\overline{\Omega} \setminus \Gamma)$$

and thus the equation is understood in the following sense:

$$\int_\Omega (\nabla u, \nabla \varphi)\, dx = \int_\Omega f(x,u)\varphi\, dx \qquad \forall \varphi \in C^1_c(\Omega \setminus \Gamma)\,. \tag{4.0.3}$$

Remark 4.0.2. *Note that, by the assumption* (I_f)*, the right hand side of* (4.0.1) *is locally bounded. Therefore, by standard elliptic regularity theory, it follows that*

$$u \in C^{1,\alpha}_{loc}(\Omega \setminus \Gamma),$$

where $0 < \alpha < 1$*.*

Let us now state our main result

Theorem 4.0.3. *Let* Ω *be a convex domain which is symmetric with respect to the hyperplane* $\{x_1 = 0\}$ *and let* $u \in H^1_{loc}(\Omega \setminus \Gamma) \cap C(\overline{\Omega} \setminus \Gamma)$ *be a solution to* (4.0.1)*. Assume that* f *fulfills* (I_f) *(see Definition 4.0.1). Assume also that* Γ *is a point if* $N = 2$ *while* Γ *is closed and such that*

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0,$$

if $N \geq 3$*. Then, if* $\Gamma \subset \{x_1 = 0\}$*, it follows that* u *is symmetric with respect to the hyperplane* $\{x_1 = 0\}$ *and increasing in the* x_1*-direction in* $\Omega \cap \{x_1 < 0\}$*. Furthermore*

$$u_{x_1} > 0 \qquad in \quad \Omega \cap \{x_1 < 0\}\,.$$

Remark 4.0.4. *Theorem 4.0.3 is proved for convex domains. It will be clear from the proofs that this is only used to prove that* $\partial\Omega \cap \{x_1 = \lambda\}$ *is discrete in dimension two while* $\partial\Omega \cap \{x_1 = \lambda\}$ *has zero capacity for* $N \geq 3$*. Therefore the result holds true more generally once that such an information is available. In all this cases we could assume that* Ω *is convex only in the* x_1*-direction.*

The technique, as shown in [110] and as discussed in the Introduction, can be applied to study singular solutions to the following Sobolev critical equation in $\mathbb{R}^N, N \geq 3$,

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in} \quad \mathbb{R}^N \setminus \Gamma \\ u > 0 & \text{in} \quad \mathbb{R}^N \setminus \Gamma. \end{cases} \tag{4.0.4}$$

In [110] it was considered the case of a closed critical set Γ contained in a compact smooth submanifold of dimension $d \leq N-2$ and a summability property of the solution at infinity was imposed (see also [121] for the special case in which the singular set Γ is reduced to a single point). Here we remove both these restrictions and we prove the following:

Theorem 4.0.5. *Let* $N \geq 3$ *and let* $u \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ *be a solution to* (4.0.4)*. Assume that the solution* u *has a non-removable*[1] *singularity in the singular set* Γ*, where* Γ *is a closed and proper subset of* $\{x_1 = 0\}$ *such that*

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0.$$

[1]Here we mean that the solution u does not admit a smooth extension all over the whole space. Namely it is not possible to find $\tilde{u} \in H^1_{loc}(\mathbb{R}^N)$ with $u \equiv \tilde{u}$ in $\mathbb{R}^N \setminus \Gamma$.

Then, u is symmetric with respect to the hyperplane $\{x_1 = 0\}$.
The same conclusion is true if the hyperplane $\{x_1 = 0\}$ is replaced by any affine hyperplane.

Some interesting consequences of the previous result are contained in the following

COROLLARY 4.0.6. *Let $N \geq 3$ and let $u \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ be a solution to (4.0.4) with a non-removable singularity in the singular set Γ.*
(i) If $\Gamma = \{x_0\}$, then u is radially symmetric with respect to x_0.
(ii) If $\Gamma = \{x_0, x_1\}$, then u has cylindrical symmetry with respect to the axis passing through x_0 and x_1.
More generally we have :
(iii) assume $1 \leq k \leq N-2$ and suppose that Γ is a closed subset of an affine $k-$dimensional subspace of $\mathbb{R}^N$. Then, up to isometry, the solution u has the form $u(x) = u(x_1, ..., x_k, |x^{'}|)$, where $x^{'} := (x_{k+1}, ...x_N)$ and $|x^{'}| := \sqrt{x^2_{k+1} + ... + x^2_n}$.

The following example shows that Theorem 4.0.5 and item (iii) of Corollary 4.0.6 are sharp for $N \geq 5$ and also that singular solutions exhibiting un unbounded critical set Γ exist.

For $N \geq 5$ and $1 \leq k < \frac{N-2}{2}$, k integer, we set $p = p(N) = \frac{N+2}{N-2} > 1$ and $A = A(N,k) = [(\frac{N}{2} - k - 1)\frac{N}{2}]^{\frac{N-2}{4}} > 0$. Then, the function $v(r) = Ar^{-\frac{2}{p(N)-1}}$ is a singular positive radial solution of $-\Delta v = v^{p(N)}$ in $\mathbb{R}^{N-k} \setminus \{0^{'}\}$, which is smooth in $\mathbb{R}^{N-k} \setminus \{0^{'}\}$. Hence $u = u(x_1, ..., x_N) := v(|x^{'}|)$ is a singular solution to (4.0.4) in $\mathbb{R}^N \setminus \Gamma$, with Γ given by the $k-$dimensional subspace $\{x_1 = ... = x_k = 0\} \subset \mathbb{R}^N$, moreover $u \in C^\infty(\mathbb{R}^N \setminus \Gamma)$.

4.1. Notations and preliminary results

For a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\} \tag{4.1.1}$$

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_N) \tag{4.1.2}$$

which is the reflection through the hyperplane $T_\lambda := \{x_1 = \lambda\}$. Also let

$$a = \inf_{x \in \Omega} x_1. \tag{4.1.3}$$

Since Γ is compact and of zero capacity, u is defined a.e. on Ω and Lebesgue measurable on Ω. Therefore the function

$$u_\lambda := u \circ R_\lambda \tag{4.1.4}$$

is Lebesgue measurable on $R_\lambda(\Omega)$. Similarly, ∇u and ∇u_λ are Lebesgue measurable on Ω and $R_\lambda(\Omega)$ respectively.

It is easy to see that, if $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0$, then $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(R_\lambda(\Gamma)) = 0$. Another consequence of our assumptions is that $\underset{B^\lambda_\epsilon}{\mathrm{Cap}_2}(R_\lambda(\Gamma)) = 0$ for any open neighborhood $\mathcal{B}^\lambda_\epsilon$ of $R_\lambda(\Gamma)$. Indeed, recalling that Γ is a point if $N = 2$ while Γ

is closed with $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0$ if $N \geq 3$ by assumption, it follows that

$$\underset{\mathcal{B}^\lambda_\epsilon}{\mathrm{Cap}_2}(R_\lambda(\Gamma)) := \inf \left\{ \int_{\mathcal{B}^\lambda_\epsilon} |\nabla \varphi|^2 dx < +\infty \ : \ \varphi \geq 1 \text{ in } \mathcal{B}^\lambda_\delta, \ \varphi \in \ C^\infty_c(\mathcal{B}^\lambda_\epsilon) \right\} = 0,$$

for some neighborhood $\mathcal{B}^\lambda_\delta \subset \mathcal{B}^\lambda_\varepsilon$ of $R_\lambda(\Gamma)$. From this, it follows that there exists $\varphi_\varepsilon \in \ C^\infty_c(\mathcal{B}^\lambda_\epsilon)$ such that $\varphi_\varepsilon \geq 1$ in $\mathcal{B}^\lambda_\delta$ and $\displaystyle\int_{\mathcal{B}^\lambda_\epsilon} |\nabla \varphi_\varepsilon|^2 dx < \varepsilon$.

Now we construct a function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N, [0,1])$ (see Figure 1) that $\psi_\varepsilon = 1$ outside $\mathcal{B}^\lambda_\varepsilon$, $\psi_\varepsilon = 0$ in $\mathcal{B}^\lambda_\delta$ and

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 dx = \int_{\mathcal{B}^\lambda_\epsilon} |\nabla \psi_\varepsilon|^2 dx < 4\varepsilon.$$

To this end we consider the following Lipschitz continuous function

$$T_1(s) = \begin{cases} 1 & \text{if} \quad s \leq 0 \\ -2s + 1 & \text{if} \quad 0 \leq s \leq \frac{1}{2} \\ 0 & \text{if} \quad s \geq \frac{1}{2} \end{cases}$$

and we set

$$\psi_\varepsilon := T_1 \circ \varphi_\varepsilon \tag{4.1.5}$$

where we have extended φ_ε by zero outside $\mathcal{B}^\lambda_\varepsilon$. Clearly $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N), 0 \leq \psi_\varepsilon \leq 1$ and

$$\int_{\mathcal{B}^\lambda_\epsilon} |\nabla \psi_\varepsilon|^2 dx \leq 4 \int_{\mathcal{B}^\lambda_\epsilon} |\nabla \varphi_\varepsilon|^2 dx < 4\varepsilon.$$

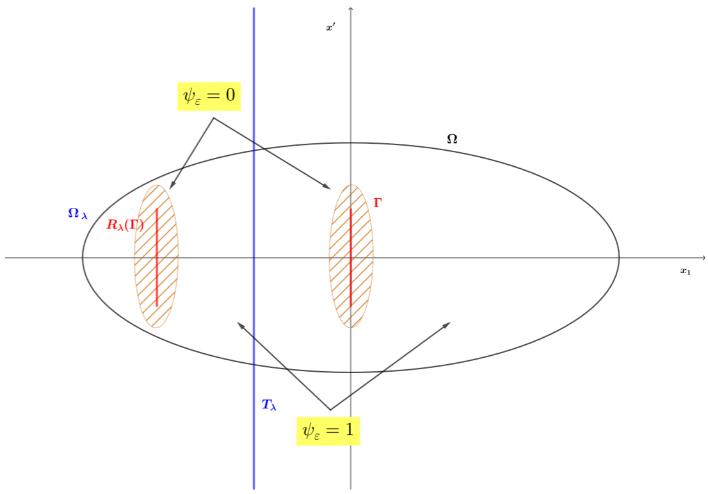

FIGURE 1. The cutoff function ψ_ε.

Now we set $\gamma_\lambda := \partial\Omega \cap T_\lambda$. Recalling that Ω is convex, it is easy to deduce that γ_λ is made of two points in dimension two. If else $N \geq 3$ then it follows that γ_λ is a smooth manifold of dimension $N - 2$. Note in fact that locally $\partial\Omega$ is the zero level set of a smooth function $g(\cdot)$ whose gradient is not parallel to the x_1-direction since Ω is convex. Then it is sufficient to observe that locally $\partial\Omega \cap T_\lambda \equiv \{g(\lambda, x') = 0\}$ and use the *implicit function theorem* exploiting the fact that $\nabla_{x'} g(\lambda, x') \neq 0$. This implies that $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\gamma_\lambda) = 0$, see e.g. [53]. So, as before, $\underset{\mathcal{I}_\tau^\lambda}{\mathrm{Cap}_2}(\gamma_\lambda) = 0$ for any open neighborhood of γ_λ and then there exists $\varphi_\tau \in C_c^\infty(\mathcal{I}_\tau^\lambda)$ such that $\varphi_\tau \geq 1$ in a neighborhood $\mathcal{I}_\sigma^\lambda$ with $\gamma_\lambda \subset \mathcal{I}_\sigma^\lambda \subset \mathcal{I}_\tau^\lambda$. As above, we set

$$\phi_\tau := T_1 \circ \varphi_\tau \tag{4.1.6}$$

where we have extended φ_τ by zero outside $\mathcal{I}_\tau^\lambda$. Then, $\phi_\tau \in C^{0,1}(\mathbb{R}^N), 0 \leq \phi_\tau \leq 1, \phi_\tau = 1$ outside $\mathcal{I}_\tau^\lambda, \phi_\tau = 0$ in $\mathcal{I}_\sigma^\lambda$ (see Figure 2) and

$$\int_{\mathbb{R}^N} |\nabla \phi_\tau|^2 dx = \int_{\mathcal{I}_\tau^\lambda} |\nabla \phi_\tau|^2 dx \leq 4 \int_{\mathcal{I}_\tau^\lambda} |\nabla \varphi_\tau|^2 dx < 4\tau.$$

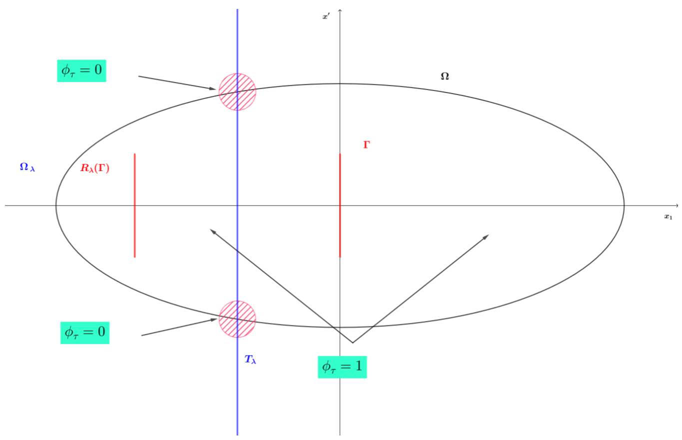

FIGURE 2. The cutoff function ϕ_τ.

4.2. Symmetry and monotonicity result in bounded domains

In the following we will exploit the fact that u_λ is a solution to:

$$\int_{R_\lambda(\Omega)} (\nabla u_\lambda, \nabla \varphi)\, dx = \int_{R_\lambda(\Omega)} f(x_\lambda, u_\lambda)\varphi\, dx \qquad \forall \varphi \in C_c^1(R_\lambda(\Omega) \setminus R_\lambda(\Gamma)) \tag{4.2.1}$$

and we also observe that, for any $a < \lambda < 0$, the function $w_\lambda := u - u_\lambda$ satisfies $0 \leq w_\lambda^+ \leq u$ a.e. on Ω_λ and so $w_\lambda^+ \in L^2(\Omega_\lambda)$, since $u \in C^0(\overline{\Omega_\lambda})$. To proceed further, we need the following two results

LEMMA 4.2.1. *Let $\lambda \in (a, 0)$ be such that $R_\lambda(\Gamma) \cap \overline{\Omega} = \emptyset$ and consider the function*

$$\varphi := \begin{cases} w_\lambda^+ \phi_\tau^2 & in \quad \Omega_\lambda, \\ 0 & in \quad \mathbb{R}^N \setminus \Omega_\lambda, \end{cases}$$

where ϕ_τ *is as in* (4.1.6)*. Then,* $\varphi \in C^{0,1}_c(\Omega) \cap C^{0,1}_c(R_\lambda(\Omega))$*,* φ *has compact support contained in* $(\Omega \setminus \Gamma) \cap (R_\lambda(\Omega) \setminus R_\lambda(\Gamma)) \cap \{x_N \leq \lambda\}$ *and*

(4.2.2)
$$\nabla\varphi = \phi_\tau^2(\nabla w_\lambda \chi_{supp(w_\lambda^+)\cap supp(\varphi)}) + 2\phi_\tau(w_\lambda^+ \chi_{supp(\varphi)})\nabla\phi_\tau \quad a.e.\ on\ \ \Omega \cup R_\lambda(\Omega).$$

If $\lambda \in (a, 0)$ *is such that* $R_\lambda(\Gamma) \cap \overline{\Omega} \neq \emptyset$*, the same conclusions hold true for the function*

$$\varphi := \begin{cases} w_\lambda^+ \psi_\varepsilon^2 \phi_\tau^2 & in \quad \Omega_\lambda, \\ 0 & in \quad \mathbb{R}^N \setminus \Omega_\lambda, \end{cases}$$

where ψ_ε *is defined as in* (4.1.5) *and* ϕ_τ *as in* (4.1.6)*. Furthermore, a.e. on* $\Omega \cup R_\lambda(\Omega)$*,*

(4.2.3)
$$\nabla\varphi = \psi_\varepsilon^2\phi_\tau^2(\nabla w_\lambda \chi_{supp(w_\lambda^+)\cap supp(\varphi)}) + 2(w_\lambda^+ \chi_{supp(\varphi)})(\psi_\varepsilon^2\phi_\tau\nabla\phi_\tau + \psi_\varepsilon\phi_\tau^2\nabla\psi_\varepsilon).$$

In particular, $\varphi \in C^{0,1}(\overline{\Omega_\lambda})$*,* $\varphi_{|\partial\Omega_\lambda} = 0$ *and so* $\varphi \in H^1_0(\Omega_\lambda)$*.*

Proof. Let us consider the case when $\lambda \in (a, 0)$ is such that $R_\lambda(\Gamma) \cap \overline{\Omega} \neq \emptyset$ (the other case being similar and easier). We first prove that for every $x \in \Omega$ there is an open ball B_x centered at x, such that $\overline{B_x} \subset \Omega$ and $\varphi \in C^{0,1}(\overline{B_x})$, and then that there exists $\eta > 0$ such that $supp(\varphi)$ is contained in the compact set $\{x \in \Omega : dist(x, \partial\Omega) \geq \eta\} \cap \{x_N \leq \lambda\} \cap (\mathbb{R}^N \setminus V) \subset (\Omega \setminus \Gamma) \cap (R_\lambda(\Omega) \setminus R_\lambda(\Gamma))$, where V is any open set contained in the neighborhood $\mathcal{B}^\lambda_\delta$ appearing in the construction of ψ_ε.

If $x \in \Omega \cap \{x_N > \lambda\}$ then $\varphi \equiv 0$ in an open neighbourhood of x and so $\varphi \in C^{0,1}(\overline{B_x})$ for a suitable ball B_x. If $x \in \Omega \cap T_\lambda$ then we can find a small open ball $B_x \subset \Omega$ such that $B_x \cap (\partial\Omega \cup R_\lambda(\Gamma)) = \emptyset$. Therefore, both u and u_λ belong to $C^1(\overline{B_x} \cap \{x_N \leq \lambda\})$ and so, $\varphi \in C^{0,1}(\overline{B_x} \cap \{x_N \leq \lambda\})$, thanks to the Lipschitz character of ϕ_τ and ψ_ε. On the other hand we also have that $\varphi \equiv 0$ on $\overline{B_x} \cap T_\lambda$, by definition of w_λ. Thus $\varphi \in C^{0,1}(\overline{B_x})$ and we are done also in this case. If $x \in R_\lambda(\Gamma) \cap \Omega$ then $\varphi \equiv 0$ in an open neighbourhood of x by definition of ψ_ε and so $\varphi \in C^{0,1}(\overline{B_x})$ for a suitable ball B_x. Finally, if $x \in \Omega_\lambda \setminus R_\lambda(\Gamma)$ then, as before, we can find a small open ball B_x such that $\overline{B_x} \subset \Omega_\lambda \setminus R_\lambda(\Gamma)$. In this case, both u and u_λ belong to $C^1(\overline{B_x})$. This yields $w_\lambda \in C^{0,1}(\overline{B})$ and so is φ, again thanks to the Lipschitz character of ϕ_τ and ψ_ε.

To prove the second part of the claim we observe that $\varphi \equiv 0$ on $\Omega \setminus \Omega_\lambda$ and that, for any point x of the compact set $(\partial\Omega) \cap \{x_N \leq \lambda\}$ there is a small open ball B_x, centered at x, such that $\varphi = 0$ on $B_x \cap \Omega$. The latter clearly holds for any point of γ_λ, by definition of ϕ_τ, and for any point of $\partial\Omega \cap R_\lambda(\Gamma)$, by definition of ψ_ε. It is also true for any $x \in (\partial\Omega) \cap \{x_N < \lambda\}$, since $u - u_\lambda$ is well-defined, continuous and negative on the set $[(\partial\Omega) \cap \{x_N < \lambda\}] \setminus R_\lambda(\Gamma)$. The arguments above immediately yield that $\varphi \in C^{0,1}_c(\Omega)$ and the formula (4.2.3). A similar argument also shows that $\varphi \in C^{0,1}_c(R_\lambda(\Omega))$.

To compute $\nabla\varphi$ we also took into consideration the Remark 4.0.2.

□

Lemma 4.2.2. *Under the assumptions of Theorem 4.0.3, let* $a < \lambda < 0$. *Then* $w_\lambda^+ \in H_0^1(\Omega_\lambda)$ *and*

$$\int_{\Omega_\lambda} |\nabla w_\lambda^+|^2\, dx \le c(f, |\Omega|, \|u\|_{L^\infty(\Omega_\lambda)}),$$

where $|\Omega|$ *denotes the* $n-$*dimensional Lebesgue measure of* Ω.

Proof. We first prove that $\nabla w_\lambda \chi_{supp(w_\lambda^+)} \in L^2(\Omega_\lambda)$ and then that the distributional gradient of w_λ^+ is given by $\nabla w_\lambda \chi_{supp(w_\lambda^+)}$. We do this only for the case in which λ is such that $R_\lambda(\Gamma) \cap \overline{\Omega} \neq \emptyset$, the other case being similar and easier. For ψ_ε as in (4.1.5) and ϕ_τ as in (4.1.6), we consider the function φ defined in Lemma 4.2.1. In view of the properties of φ, stated in Lemma 4.2.1, and a standard density argument, we can use φ as test function in (4.0.3) and (4.2.1) so that, subtracting, we get

$$\begin{aligned}
\int_{\Omega_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \phi_\tau^2\, dx =& -2 \int_{\Omega_\lambda} \nabla w_\lambda \nabla \psi_\varepsilon w_\lambda^+ \psi_\varepsilon \phi_\tau^2\, dx \\
& -2 \int_{\Omega_\lambda} \nabla w_\lambda \nabla \phi_\tau w_\lambda^+ \psi_\varepsilon^2 \phi_\tau\, dx \\
& + \int_{\Omega_\lambda} (f(x,u) - f(x_\lambda, u_\lambda))\, w_\lambda^+ \psi_\varepsilon^2 \phi_\tau^2\, dx \\
\le& -2 \int_{\Omega_\lambda} \nabla w_\lambda \nabla \psi_\varepsilon w_\lambda^+ \psi_\varepsilon \phi_\tau^2\, dx \\
& -2 \int_{\Omega_\lambda} \nabla w_\lambda \nabla \phi_\tau w_\lambda^+ \psi_\varepsilon^2 \phi_\tau\, dx \\
& + \int_{\Omega_\lambda} (f(x,u) - f(x, u_\lambda)) w_\lambda^+ \psi_\varepsilon^2 \phi_\tau^2\, dx.
\end{aligned}$$

Here we also used the monotonicity properties of $f(\cdot, s)$, see (I_f). Exploiting Young's inequality we get that

(4.2.4)

$$\begin{aligned}
\int_{\Omega_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \phi_\tau^2\, dx \le& \frac{1}{4} \int_{\Omega_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \phi_\tau^2\, dx \\
& + 4 \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 (w_\lambda^+)^2 \phi_\tau^2\, dx \\
& + \frac{1}{4} \int_{\Omega_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \phi_\tau^2\, dx \\
& + 4 \int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_\lambda^+)^2 \psi_\varepsilon^2\, dx \\
& + \int_{\Omega_\lambda} (f(x,u) - f(x, u_\lambda)) w_\lambda^+ \psi_\varepsilon^2 \phi_\tau^2\, dx.
\end{aligned}$$

Now we observe that the last integral is actually computed on the set $\{x \in \Omega_\lambda \setminus R_\lambda(\Gamma) : u(x) > u_\lambda(x) > 0\} \subset \overline{\Omega_\lambda} \subset \overline{\Omega} \setminus \Gamma$ and so, we can apply condition (I_f) with the compact set $K = \overline{\Omega_\lambda}$ and $M = \|u\|_{L^\infty(\Omega_\lambda)}$. We get therefore that

(4.2.5)
$$\int_{\Omega_\lambda} (f(x,u) - f(x,u_\lambda)) w_\lambda^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx \le c(f, \|u\|_{L^\infty(\Omega_\lambda)}) \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx$$

and so, from (4.2.4), we infer that

(4.2.6)
$$\begin{aligned}\int_{\Omega_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx \le & 8 \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 (w_\lambda^+)^2 \phi_\tau^2 \, dx \\ & + 8 \int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_\lambda^+)^2 \psi_\varepsilon^2 \, dx \\ & + 2c(f, \|u\|_{L^\infty(\Omega_\lambda)}) \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx.\end{aligned}$$

Taking into account the properties of ψ_ε and ϕ_τ, we see that

$$\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 \, dx = \int_{\Omega_\lambda \cap (\mathcal{B}_\varepsilon^\lambda \setminus \mathcal{B}_\delta^\lambda)} |\nabla \psi_\varepsilon|^2 \, dx < 4\varepsilon, \tag{4.2.7}$$

$$\int_{\Omega_\lambda} |\nabla \phi_\tau|^2 \, dx = \int_{\Omega_\lambda \cap (\mathcal{I}_\tau^\lambda \setminus \mathcal{I}_\sigma^\lambda)} |\nabla \phi_\tau|^2 \, dx < 4\tau, \tag{4.2.8}$$

which combined with $0 \le w_\lambda^+ \le u$, immediately lead to

$$\begin{aligned}\int_{\Omega_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx \le & 32(\varepsilon + \tau) \|u\|^2_{L^\infty(\Omega_\lambda)} \\ & + 2c(f, \|u\|_{L^\infty(\Omega_\lambda)}) \|u\|^2_{L^\infty(\Omega_\lambda)} |\Omega| \, .\end{aligned}$$

By Fatou's Lemma, as ε and τ tend to zero, we deduce that $\nabla w_\lambda \chi_{supp(w_\lambda^+)} \in L^2(\Omega_\lambda)$. To conclude we note that $\varphi \to w_\lambda^+$ in $Ł^2(\Omega)$, as ε and τ tend to zero, by definition of φ. Also, $\nabla \varphi \to \nabla w_\lambda \chi_{supp(w_\lambda^+)}$ in $L^2(\Omega_\lambda)$, by (4.2.3). Therefore, $\nabla w_\lambda \chi_{supp(w_\lambda^+)}$ is the distributional gradient of ∇w_λ^+ and w_λ^+ in $H^1_0(\Omega_\lambda)$, since $\varphi \in H^1_0(\Omega_\lambda)$ again by Lemma 4.2.1. Which concludes the proof.

□

Proof of Theorem 4.0.3. We define

$$\Lambda_0 = \{a < \lambda < 0 : u \le u_t \text{ in } \Omega_t \setminus R_t(\Gamma) \text{ for all } t \in (a, \lambda]\}$$

and to start with the moving planes procedure, we have to prove that

Step 1 : $\Lambda_0 \neq \emptyset$ (See Figure 3). Fix a $\lambda_0 \in (a, 0)$ such that $R_{\lambda_0}(\Gamma) \subset \Omega^c$, then for every $a < \lambda < \lambda_0$, we also have that $R_\lambda(\Gamma) \subset \Omega^c$. For any λ in this set we consider, on the domain Ω, the function $\varphi := w_\lambda^+ \phi_\tau^2 \chi_{\Omega_\lambda}$, where ϕ_τ is as in (4.1.6) and we proceed as in the proof of Lemma 4.2.2. That is, by Lemma 4.2.1 and a density argument, we can use φ as test function in (4.0.3) and (4.2.1) so that, subtracting, we get

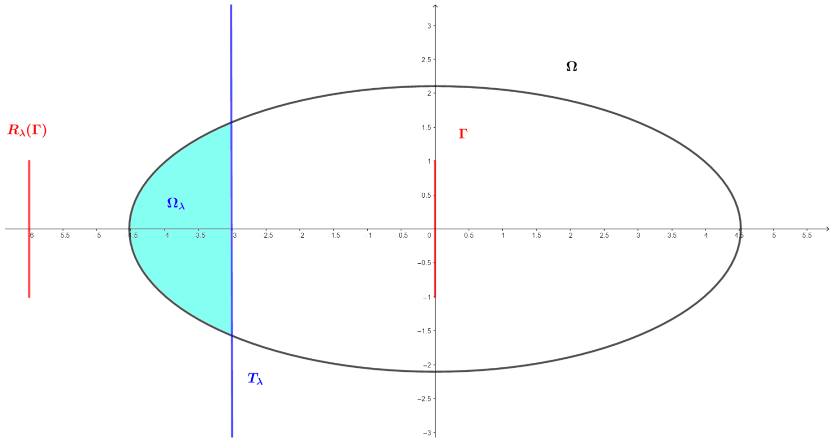

FIGURE 3. Step 1 of the moving planes method: $\Lambda_0 \neq \emptyset$.

$$\begin{aligned}\int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \phi_\tau^2 \, dx = &- 2\int_{\Omega_\lambda} \nabla w_\lambda^+ \nabla \phi_\tau w_\lambda^+ \phi_\tau \, dx \\ &+ \int_{\Omega_\lambda} (f(x,u) - f(x_\lambda, u_\lambda))\, w_\lambda^+ \phi_\tau^2 \, dx \\ \leq &- 2\int_{\Omega_\lambda} \nabla w_\lambda^+ \nabla \phi_\tau w_\lambda^+ \phi_\tau \, dx \\ &+ \int_{\Omega_\lambda} (f(x,u) - f(x, u_\lambda)) w_\lambda^+ \phi_\tau^2 \, dx.\end{aligned}$$

Exploiting Young's inequality and the assumption (I_f), with $K = \overline{\Omega}_{\lambda_0}$ and $M = \|u\|^2_{L^\infty(\Omega_{\lambda_0})}$, we then get that

$$\begin{aligned}\int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \phi_\tau^2 \, dx \leq &\frac{1}{2}\int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \phi_\tau^2 \, dx + 2\int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_\lambda^+)^2 dx \\ &+ c(f, \|u\|_{L^\infty(\Omega_{\lambda_0})}) \int_{\Omega_\lambda} (w_\lambda^+)^2 \phi_\tau^2 \, dx.\end{aligned}$$

Taking into account the properties of ϕ_τ, we see that

(4.2.9)
$$\int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_\lambda^+)^2 dx \leq \|u\|^2_{L^\infty(\Omega_\lambda)} \int_{\Omega_\lambda \cap (\mathcal{I}_\tau^\lambda \setminus \mathcal{I}_\sigma^\lambda)} |\nabla \phi_\tau|^2 \, dx \leq 4\|u\|^2_{L^\infty(\Omega_\lambda)} \cdot \tau.$$

We therefore deduce that

$$\int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \phi_\tau^2 \, dx \leq 16\|u\|_{L^\infty(\Omega_\lambda)} \cdot \tau + 2c(f, \|u\|_{L^\infty(\Omega_{\lambda_0})}) \int_{\Omega_\lambda} (w_\lambda^+)^2 \phi_\tau^2 \, dx.$$

By Fatou's Lemma, as τ tend to, zero we have

$$\begin{aligned}\int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \, dx &\leq 2c(f, \|u\|_{L^\infty(\Omega_{\lambda_0})}) \int_{\Omega_\lambda} (w_\lambda^+)^2 \, dx \\ &\leq 2c(f, \|u\|_{L^\infty(\Omega_{\lambda_0})}) c_p^2(\Omega_\lambda) \int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \, dx,\end{aligned} \tag{4.2.10}$$

where $c_p(\cdot)$ is the Poincaré constant (in the Poincaré inequality in $H^1_0(\Omega_\lambda)$). Since $c_p^2(\Omega_\lambda) \to 0$ as $\lambda \to a$, we can find $\lambda_1 \in (a, \lambda_0)$, such that

$$\forall \lambda \in (a, \lambda_1) \qquad 2c(f, \|u\|_{L^\infty(\Omega_{\lambda_0})})c_p^2(\Omega_\lambda) < \frac{1}{2}\,,$$

so that by (4.2.10), we deduce that

$$\forall \lambda \in (a, \lambda_1) \qquad \int_{\Omega_\lambda} |\nabla w_\lambda^+|^2 \, dx \leq 0,$$

proving that $u \leq u_\lambda$ in $\Omega_\lambda \setminus R_\lambda(\Gamma)$ for λ close to a, which implies the desired conclusion $\Lambda_0 \neq \emptyset$.
Now we can set

$$\lambda_0 = \sup \Lambda_0.$$

Step 2: here we show that $\lambda_0 = 0$ (see Figure 4). To this end we assume

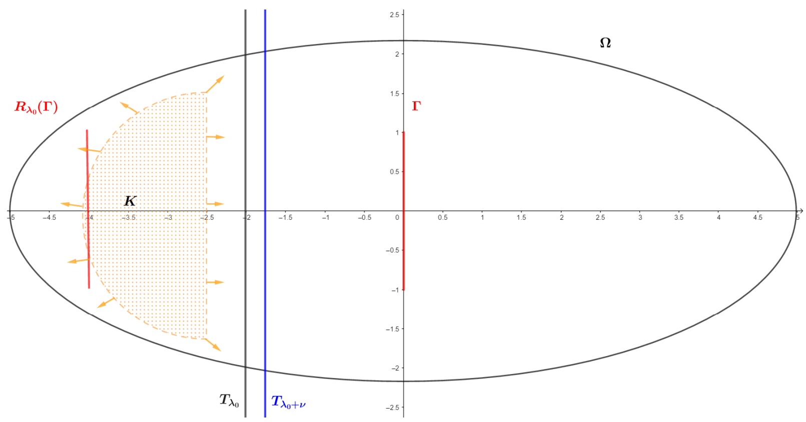

FIGURE 4. Step 2 of the moving planes method: $\lambda_0 = 0$.

that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \leq u_{\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu} \setminus R_{\lambda_0+\nu}(\Gamma)$ for any $0 < \nu < \bar{\nu}$ for some small $\bar{\nu} > 0$. By continuity we know that $u \leq u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. Since Ω is convex in the x_1-direction and the set $R_{\lambda_0}(\Gamma)$ lies in the hyperplane of equation $\{\, x_1 = -2\lambda_0 \,\}$, we see that $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ is open and connected. Therefore, by the strong maximum principle we deduce that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ (here we have also used that $u, u_{\lambda_0} \in C^1(\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma))$ by Remark 4.0.2, as well as the assumption (I_f)).

Now, note that for $K \subset \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$, there is $\nu = \nu(K, \lambda_0) > 0$, sufficiently small, such that $K \subset \Omega_\lambda \setminus R_\lambda(\Gamma)$ for every $\lambda \in [\lambda_0, \lambda_0 + \nu]$. Consequently u and u_λ are well defined on K for every $\lambda \in [\lambda_0, \lambda_0+\nu]$. Hence, by the uniform continuity of the function $g(x, \lambda) := u(x) - u(2\lambda - x_1, x^{'})$ on the compact set $K \times [\lambda_0, \lambda_0 + \nu]$ we can ensure that $K \subset \Omega_{\lambda_0+\nu} \setminus R_{\lambda_0+\nu}(\Gamma)$ and $u < u_{\lambda_0+\nu}$ in K for any $0 \leq \nu < \bar{\nu}$, for some $\bar{\nu} = \bar{\nu}(K, \lambda_0) > 0$ small. Clearly we can also assume that $\bar{\nu} < \frac{|\lambda_0|}{4}$.

Let us consider ψ_ε constructed in such a way that it vanishes in a neighborhood of $R_{\lambda_0+\nu}(\Gamma)$ and ϕ_τ constructed in such a way it vanishes in a neighborhood of $\gamma_{\lambda_0+\nu} = \partial\Omega \cap T_{\lambda_0+\nu}$. As swown in the proof of lemma 4.2.2, the functions

$$\varphi := \begin{cases} w^+_{\lambda_0+\nu}\psi^2_\varepsilon\phi^2_\tau & \text{in} \quad \Omega_{\lambda_0+\nu} \\ 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega_{\lambda_0+\nu} \end{cases}$$

are such that $\varphi \to w^+_{\lambda_0+\nu}$ in $H^1_0(\Omega_{\lambda_0+\nu})$, as ε and τ tend to zero. Moreover, $\varphi \in C^{0,1}(\overline{\Omega_{\lambda_0+\nu}})$ and $\varphi_{|\partial\Omega_{\lambda_0+\nu}} = 0$, by Lemma 4.2.1, and $\varphi = 0$ on an open neighborhood of K, by the above argument. Therefore, $\varphi \in H^1_0(\Omega_{\lambda_0+\nu} \setminus K)$ and thus, also $w^+_{\lambda_0+\nu}$ belongs to $H^1_0(\Omega_{\lambda_0+\nu} \setminus K)$. We also note that $\nabla w^+_{\lambda_0+\nu} = 0$ on an open neighborhood of K.
Now we argue as in Lemma 4.2.2 and we plug φ as test function in (4.0.3) and (4.2.1) so that, by subtracting, we get

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}} |\nabla w^+_{\lambda_0+\nu}|^2\psi^2_\varepsilon\phi^2_\tau\,dx \leq &- 2\int_{\Omega_{\lambda_0+\nu}} \nabla w_{\lambda_0+\nu}\nabla\psi_\varepsilon w^+_{\lambda_0+\nu}\psi_\varepsilon\phi^2_\tau\,dx \\ &- 2\int_{\Omega_{\lambda_0+\nu}} \nabla w_{\lambda_0+\nu}\phi_\tau w^+_{\lambda_0+\nu}\psi^2_\varepsilon\phi_\tau\,dx \\ &+ \int_{\Omega_{\lambda_0+\nu}} (f(x,u) - f(x,u_\lambda))w^+_{\lambda_0+\nu}\psi^2_\varepsilon\phi^2_\tau\,dx\end{aligned}$$

where we also use the monotonicity of $f(\cdot, s)$ in the x_1-direction. Therefore, taking into account the properties of $w^+_{\lambda_0+\nu}$ and $\nabla w^+_{\lambda_0+\nu}$ we also have

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{\lambda_0+\nu}|^2\psi^2_\varepsilon\phi^2_\tau\,dx \leq &-2\int_{\Omega_{\lambda_0+\nu}\setminus K} \nabla w^+_{\lambda_0+\nu}\nabla\psi_\varepsilon w^+_{\lambda_0+\nu}\psi_\varepsilon\phi^2_\tau\,dx \\ &- 2\int_{\Omega_{\lambda_0+\nu}\setminus K} \nabla w^+_{\lambda_0+\nu}\nabla\phi_\tau w^+_{\lambda_0+\nu}\psi^2_\varepsilon\phi_\tau\,dx \\ &+ \int_{\Omega_{\lambda_0+\nu}\setminus K} (f(x,u) - f(x,u_\lambda))w^+_{\lambda_0+\nu}\psi^2_\varepsilon\phi^2_\tau\,dx.\end{aligned}$$

Furthermore, since f is locally uniformly Lipschitz continuous from above, we deduce that

(4.2.11)
$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{\lambda_0+\nu}|^2\psi^2_\varepsilon\phi^2_\tau\,dx \leq &\, 2\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{\lambda_0+\nu}||\nabla\psi_\varepsilon|w^+_{\lambda_0+\nu}\psi_\varepsilon\phi^2_\tau\,dx \\ &+ 2\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{\lambda_0+\nu}||\nabla\phi_\tau|w^+_{\lambda_0+\nu}\psi^2_\varepsilon\phi_\tau\,dx \\ &+ c(f, \|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})\int_{\Omega_{\lambda_0+\nu}\setminus K} (w^+_{\lambda_0+\nu})^2\psi^2_\varepsilon\phi^2_\tau\,dx.\end{aligned}$$

Now, as in the proof of Lemma 4.2.2, we use Young's inequality to deduce that

(4.2.12)

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\psi_\varepsilon^2\phi_\tau^2\,dx \le 8\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla\psi_\varepsilon|^2(w^+_{\lambda_0+\nu})^2\phi_\tau^2\,dx\\ +8\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla\phi_\tau|^2(w^+_{\lambda_0+\nu})^2\psi_\varepsilon^2\,dx\\ +2c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})\int_{\Omega_{\lambda_0+\nu}\setminus K}(w^+_\lambda)^2\psi_\varepsilon^2\phi_\tau^2\,dx,\end{aligned}$$

which in turns yields

(4.2.13)

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\psi_\varepsilon^2\phi_\tau^2\,dx \le 32\|u\|^2_{L^\infty(\Omega_{\lambda_0+\bar\nu})}(\epsilon+\tau)\\ +2c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})\int_{\Omega_{\lambda_0+\nu}\setminus K}(w^+_\lambda)^2\psi_\varepsilon^2\phi_\tau^2\,dx.\end{aligned}$$

Passing to the limit, as $(\epsilon,\tau)\to(0,0)$, in the latter we get

(4.2.14)

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\,dx &\le 2c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})\int_{\Omega_{\lambda_0+\nu}\setminus K}(w^+_{\lambda_0+\nu})^2\,dx\\ &\le 2c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})c_p^2(\Omega_{\lambda_0+\nu}\setminus K)\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\,dx\,,\end{aligned}$$

where $c_p(\cdot)$ is the Poincaré constant (in the Poincaré inequality in $H^1_0(\Omega_{\lambda_0+\nu}\setminus K)$). Now we recall that $c_p^2(\Omega_{\lambda_0+\nu}\setminus K)\le Q(n)|\Omega_{\lambda_0+\nu}\setminus K|^{\frac{2}{N}}$, where $Q=Q(n)$ is a positive constant depending only on the dimension n, and therefore, by summarizing, we have proved that for every compact set $K\subset\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ there is a small $\bar\nu=\bar\nu(K,\lambda_0)\in(0,\frac{|\lambda_0|}{4})$ such that for every $0\le\nu<\bar\nu$ we have

(4.2.15)

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\,dx \le& 2c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})\\ & Q(n)|\Omega_{\lambda_0+\nu}\setminus K|^{\frac{2}{N}}\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\,dx.\end{aligned}$$

Now we first fix a compact $K\subset\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ such that

$$|\Omega_{\lambda_0}\setminus K|^{\frac{2}{N}}<[20c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})Q(n)]^{-1},$$

this is possible since $|R_{\lambda_0}(\Gamma)|=0$ by the assumption on Γ, and then we take $\bar\nu_0<\bar\nu$ such that for every $0\le\nu<\bar\nu_0$ we have $|\Omega_{\lambda_0+\nu}\setminus\Omega_{\lambda_0}|^{\frac{2}{N}}<[20c(f,\|u\|_{L^\infty(\Omega_{\lambda_0+\frac{|\lambda_0|}{4}})})Q(n)]^{-1}$. Inserting those informations into (4.2.15) we immediately get that

$$(4.2.16) \qquad \int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{\lambda_0+\nu}|^2\,dx < \frac{1}{2}\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{\lambda_0+\nu}|^2\,dx$$

and so $\nabla w^+_{\lambda_0+\nu}$ on $\Omega_{\lambda_0+\nu}\setminus K$ for every $0 \le \nu < \bar\nu_0$. On the other hand, we recall that $\nabla w^+_{\lambda_0+\nu}$ on an open neighbourhood of K for every $0 \le \nu < \bar\nu$, thus $\nabla w^+_{\lambda_0+\nu}$ on $\Omega_{\lambda_0+\nu}$ for every $0 \le \nu < \bar\nu_0$. The latter proves that $u \le u_{\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu}\setminus R_{\lambda_0+\nu}(\Gamma)$ for every $0 < \nu < \bar\nu_0$. Such a contradiction shows that

$$\lambda_0 = 0\,.$$

Step 3: conclusion. Since the moving planes procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ is implicit in the moving planes procedure. Since u has C^1 regularity, see Remark 4.0.2, the fact that u_{x_1} is positive for $x_1 < 0$ follows by the maximum principle, the Höpf lemma and the assumption (I_f).

□

4.3. Symmetry and monotonicity results in $\mathbb{R}^N$ involving critical nonlinearities

In this section we prove Theorem 4.0.5. We first note that, thanks to a well-known result of Brezis and Kato [20] and standard elliptic estimates (see also [119]), the solution u is smooth in $\mathbb{R}^N\setminus\Gamma$. Furthermore we observe that it is enough to prove the theorem for the special case in which *the origin does not belong to* Γ. Indeed, if the result is true in this special case, then we can apply it to the function $u_z(x) := u(x+z)$, where $z \in \{x_1 = 0\}\setminus\Gamma \neq \emptyset$, which satisfies the equation (4.0.4) with Γ replaced by $-z+\Gamma$ (note that $-z+\Gamma$ is a closed and proper subset of $\{x_1 = 0\}$ with $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(-z+\Gamma) = 0$ and such that the origin does not belong to it).
Under this assumption, we consider the map $K : \mathbb{R}^N\setminus\{0\} \longrightarrow \mathbb{R}^N\setminus\{0\}$ defined by $K = K(x) := \frac{x}{|x|^2}$. Given u solution to (4.0.4), its Kelvin transform is given by

$$(4.3.1) \qquad v(x) := \frac{1}{|x|^{N-2}}\,u\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^N\setminus\{\Gamma^*\cup\{0\}\},$$

where $\Gamma^* = K(\Gamma)$. It follows that v weakly satisfies (4.0.4) in $\mathbb{R}^N\setminus\{\Gamma^*\cup\{0\}\}$ and that $\Gamma^* \subset \{x_1 = 0\}$ since, by assumption, $\Gamma \subset \{x_1 = 0\}$. Furthermore, we also have that Γ^* is bounded (not necessarily closed) since we assumed that $0 \notin \Gamma$.

To proceed further we need the following lemmata

Lemma 4.3.1. *Let $F : \mathbb{R}^N\setminus\{0\} \longrightarrow \mathbb{R}^N\setminus\{0\}$ be a C^1-diffeomorphism and let A be a bounded open set of $\mathbb{R}^N\setminus\{0\}$. If $C \subset A$ is a compact set such that*

$$(4.3.2) \qquad \underset{A}{\mathrm{Cap}_2}(C) = 0,$$

then

$$\operatorname*{Cap_2}_{F(A)}(F(C)) = 0. \tag{4.3.3}$$

PROOF. By hypothesis (4.3.2) and by definition of 2-capacity, for every $\varepsilon > 0$ let $\varphi_\varepsilon \in C_c^\infty(A)$ such that

(i) $\int_A |\nabla \varphi_\varepsilon|^2 \, dx < \varepsilon$

(ii) $\varphi_\varepsilon \geq 1$ in a neighborhood $\mathcal{B}_\varepsilon$ of C.

Let $\psi_\varepsilon := \varphi_\varepsilon \circ G$, where $G := F^{-1}$. By definition of ψ_ε, we immediately have that $\psi_\varepsilon \geq 1$ in a neighborhood $\mathcal{B}_\varepsilon'$ of the compact set $F(C)$. Moreover

$$\begin{aligned}
&\int_{F(A)} |\nabla \psi_\varepsilon(y)|^2 \, dy \\
&\qquad = \int_{F(A)} |JG(y_1, ..., y_n) \cdot \nabla \varphi_\varepsilon(G_1(y_1), ..., G_N(y_n))|^2 \, dy_1 \cdots dy_n \\
&\qquad \leq \int_{F(A)} \|JG\|_{\infty, \overline{F(A)}} |\nabla \varphi_\varepsilon(G_1(y_1), ..., G_N(y_n))|^2 \, dy_1 \cdots dy_n \\
&\qquad \leq C(F, A) \int_{F(A)} |\nabla \varphi_\varepsilon(G_1(y_1), ..., G_N(y_n))|^2 \, dy_1 \cdots dy_n \\
&\qquad = C(F, A) \int_A |\nabla \varphi_\varepsilon(x_1, ..., x_N)|^2 |\det(JF(x_1, ..., x_N))| dx_1 \cdots dx_N \\
&\qquad \leq \tilde{C}(F, A) \int_A |\nabla \varphi_\varepsilon|^2 \, dx < \tilde{C}(F, A)\varepsilon.
\end{aligned}$$

Since $\tilde{C}(F, A)$ is independent of ε, the desired conclusion follows at once.

□

LEMMA 4.3.2. *Let Γ be a closed subset of $\mathbb{R}^N$, with $N \geq 3$. Also suppose that $0 \notin \Gamma$ and*

$$\operatorname*{Cap_2}_{\mathbb{R}^N}(\Gamma) = 0. \tag{4.3.4}$$

Then

$$\operatorname*{Cap_2}_{\mathbb{R}^N}(\Gamma^*) = 0. \tag{4.3.5}$$

PROOF. Since 0 belongs to the open set $\mathbb{R}^N \setminus \Gamma$, there exists $r_0 \in (0, 1)$ such that $B_{r_0}(0) \cap \Gamma = \emptyset$. Therefore, $\Gamma = \bigcup_{m=1}^{+\infty} \left[\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right]$ and so

$$\operatorname*{Cap_2}_{\mathbb{R}^N} \left[\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right] = 0, \quad \forall m \in \mathbb{N},$$

since (4.3.4) is in force. The latter and $N \geq 3$ imply that

$$\operatorname*{Cap_2}_{A_m} \left[\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right] = 0, \quad \forall m \in \mathbb{N},$$

where $A_m := B_{m+1}(0) \setminus \overline{B_{\frac{r_0}{2}}(0))}$ is an open and bounded set for every $m \geq 1$. An application of lemma 4.3.1 with $F = K$, the inversion $x \to \frac{x}{|x|^2}$, $A = A_m$

and $C = \Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))$ yields

$$\underset{K(A_m)}{\mathrm{Cap}_2}\, K\left(\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right) = 0, \quad \forall m \in \mathbb{N}$$

and so

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2} K\left(\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right) = 0, \quad \forall m \in \mathbb{N}.$$

But

$$\begin{aligned}\Gamma^* = K(\Gamma) = K\left(\bigcup_{m=1}^{+\infty}\left[\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right]\right)\\ = \bigcup_{m=1}^{+\infty} K\left(\Gamma \cap (\overline{B_m(0)} \setminus B_{r_0}(0))\right)\end{aligned}$$

and the 2-capacity is an exterior measure (see e.g. [53]), so the desired conclusion (4.3.5) follows.

□

Let us now fix some notations. We set

$$\Sigma_\lambda = \{x \in \mathbb{R}^N \,:\, x_1 < \lambda\}\,. \tag{4.3.6}$$

As above $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_N)$ is the reflection of x through the hyperplane $T_\lambda = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_1 = \lambda\}$. Finally we consider the Kelvin transform v of u defined in (4.3.1) and we set

$$w_\lambda(x) = v(x) - v_\lambda(x) = v(x) - v(x_\lambda). \tag{4.3.7}$$

Note that v weakly solves

$$\int_{\mathbb{R}^N} (\nabla v, \nabla\varphi)\, dx = \int_{\mathbb{R}^N} v^{2^*-1}\varphi\, dx \qquad \forall \varphi \in C^1_c(\mathbb{R}^N \setminus \Gamma^* \cup \{0\})\,. \tag{4.3.8}$$

and v_λ weakly solves

(4.3.9)
$$\int_{\mathbb{R}^N} (\nabla v_\lambda, \nabla\varphi)\, dx = \int_{\mathbb{R}^N} v_\lambda^{2^*-1}\varphi\, dx \qquad \forall \varphi \in C^1_c(\mathbb{R}^N \setminus R_\lambda(\Gamma^* \cup \{0\}))\,.$$

The properties of the Kelvin transform, the fact that $0 \notin \Gamma$ and the regularity of u imply that $|v(x)| \leq C|x|^{2-N}$ for every $x \in \mathbb{R}^N$ such that $|x| \geq R$, where C and R are positive constants (depending on u). In particular, for every $\lambda < 0$, we have

$$v \in L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap C^0(\overline{\Sigma_\lambda})\,. \tag{4.3.10}$$

LEMMA 4.3.3. *Under the assumption of Theorem 4.0.5, for every $\lambda < 0$, we have that $w_\lambda^+ \in L^{2^*}(\Sigma_\lambda), \nabla w_\lambda^+ \in L^2(\Sigma_\lambda)$ and*

$$\|w_\lambda^+\|^2_{L^{2^*}(\Sigma_\lambda)} \leq C_S^2 \int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2\, dx \leq 2C_S^2 \frac{N+2}{N-2}\|v\|^{2^*}_{L^{2^*}(\Sigma_\lambda)}, \tag{4.3.11}$$

where C_S denotes the best constant in Sobolev embedding.

Proof. We immediately see that $w_\lambda^+ \in L^{2^*}(\Sigma_\lambda)$, since $0 \le w_\lambda^+ \le v \in L^{2^*}(\Sigma_\lambda)$. The rest of the proof follows the lines of the one of lemma 4.2.2. Arguing as in section 2, for every $\varepsilon > 0$, we can find a function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N, [0,1])$ (see Figure 1) such that

$$\int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 < 4\varepsilon$$

and $\psi_\varepsilon = 0$ in an open neighborhood $\mathcal{B}_\varepsilon$ of $R_\lambda(\{\Gamma^* \cup \{0\}\})$, with $\mathcal{B}_\varepsilon \subset \Sigma_\lambda$.

Fix $R_0 > 0$ such that $R_\lambda(\{\Gamma^* \cup \{0\}) \subset B_{R_0}$ and, for every $R > R_0$, let φ_R be a standard cut off function such that $0 \le \varphi_R \le 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R} with $|\nabla \varphi_R| \le 2/R$ (see Figure 5) and consider

$$\varphi := \begin{cases} w_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 & \text{in} \quad \Sigma_\lambda, \\ 0 & \text{in} \quad \mathbb{R}^N \setminus \Sigma_\lambda. \end{cases}$$

Now, as in Lemma 4.2.1 we see that $\varphi \in C_c^{0,1}(\mathbb{R}^N)$ with $supp(\varphi)$ contained

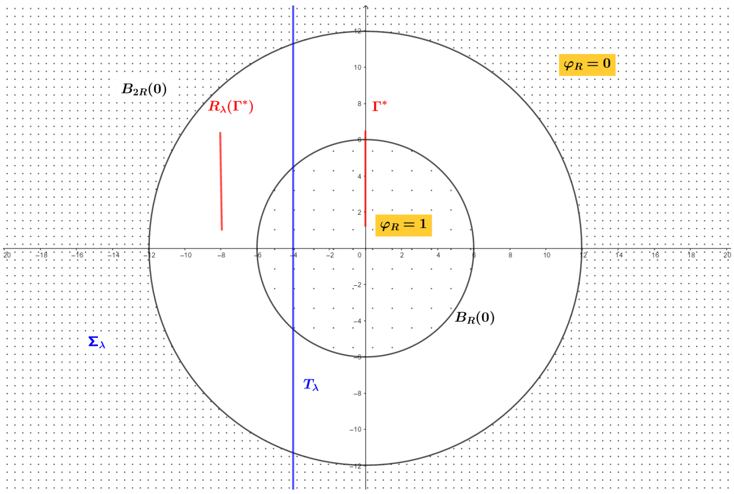

Figure 5. The cutoff function φ_R.

in $\overline{\Sigma_\lambda \cap B_{2R}} \setminus R_\lambda(\{\Gamma^* \cup \{0\}\})$ and

(4.3.12)
$$\nabla \varphi = \psi_\varepsilon^2 \varphi_R^2 (\nabla w_\lambda \chi_{supp(w_\lambda^+) \cap supp(\varphi)}) + 2(w_\lambda^+ \chi_{supp(\varphi)})(\psi_\varepsilon^2 \varphi_R \nabla \varphi_R + \psi_\varepsilon \varphi_R^2 \nabla \psi_\varepsilon).$$

Therefore, by a standard density argument, we can use φ as test function in (4.3.8) and in (4.3.9) so that, subtracting we get

$$\begin{aligned} \int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx = & -2 \int_{\Sigma_\lambda} \nabla w_\lambda \nabla \psi_\varepsilon w_\lambda^+ \psi_\varepsilon \varphi_R^2 \, dx \\ & -2 \int_{\Sigma_\lambda} \nabla w_\lambda \nabla \varphi_R w_\lambda^+ \varphi_R \psi_\varepsilon^2 \, dx \\ & + \int_{\Sigma_\lambda} (v^{2^*-1} - v_\lambda^{2^*-1}) w_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\ =: & \; I_1 + I_2 + I_3 \, . \end{aligned} \tag{4.3.13}$$

Exploiting also Young's inequality and recalling that $0 \leq w_\lambda^+ \leq v$, we get that

$$
\begin{aligned}
|I_1| &\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 4\int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 (w_\lambda^+)^2 \varphi_R^2 \, dx \\
&\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 16\varepsilon \|v\|^2_{L^\infty(\Sigma_\lambda)}.
\end{aligned} \tag{4.3.14}
$$

Furthermore we have that

(4.3.15)

$$
\begin{aligned}
|I_2| \leq & \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + 4\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} |\nabla \varphi_R|^2 (w_\lambda^+)^2 \psi_\varepsilon^2 \, dx \\
\leq & \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + 4\left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} |\nabla \varphi_R|^n \, dx\right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} v^{2^*} \, dx\right)^{\frac{N-2}{N}} \\
\leq & \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \, + \, C(N) \left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} v^{2^*} \, dx\right)^{\frac{N-2}{N}}
\end{aligned}
$$

where $C(N)$ is a positive constant depending only on the dimension N.

Let us now estimate I_3. Since $v(x), v_\lambda(x) > 0$, by the convexity of $t \to t^{2^*-1}$, for $t > 0$, we obtain

$$
v^{2^*-1}(x) - v_\lambda^{2^*-1}(x) \leq \frac{N+2}{N-2} v_\lambda^{2^*-2}(x)(v(x) - v_\lambda(x)),
$$

for every $x \in \Sigma_\lambda$. Thus, by making use of the monotonicity of $t \to t^{2^*-2}$, for $t > 0$ and the definition of w_λ^+ we get

$$
(v^{2^*-1} - v_\lambda^{2^*-1}) w_\lambda^+ \leq \frac{N+2}{N-2} v_\lambda^{2^*-2}(v - v_\lambda) w_\lambda^+ \leq \frac{N+2}{N-2} v^{2^*-2}(w_\lambda^+)^2.
$$

Therefore

(4.3.16)

$$
\begin{aligned}
I_3 &\leq \frac{N+2}{N-2}\int_{\Sigma_\lambda} v^{2^*-2}(w_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\
&\leq \frac{N+2}{N-2}\int_{\Sigma_\lambda} v^{2^*-2} v^2 dx \, = \frac{N+2}{N-2}\int_{\Sigma_\lambda} v^{2^*} \, dx = \frac{N+2}{N-2}\|v\|^{2^*}_{L^{2^*}(\Sigma_\lambda)}
\end{aligned}
$$

where we also used that $0 \leq w_\lambda^+ \leq v$. Taking into account the estimates on I_1, I_2 and I_3, by (4.3.13) we deduce that

$$\begin{aligned}\int_{\Sigma_\lambda} |\nabla w_\lambda \chi_{supp(w_\lambda^+)}|^2 \varphi_\varepsilon^2 \varphi_R^2 \, dx \leq & 32\varepsilon \|v\|^2_{L^\infty(\Sigma_\lambda)} \\ & + 2C(N) \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} v^{2^*} \, dx \right)^{\frac{N-2}{N}} \\ & + 2\frac{N+2}{N-2} \|v\|^{2^*}_{L^{2^*}(\Sigma_\lambda)}. \end{aligned} \tag{4.3.17}$$

By Fatou's Lemma, as ε tends to zero and R tends to infinity, we deduce that $\nabla w_\lambda \chi_{supp(w_\lambda^+)} \in L^2(\Sigma_\lambda)$. We also note that $\varphi \to w_\lambda^+$ in $L^{2^*}(\Sigma_\lambda)$, by definition of φ, and that $\nabla \varphi \to \nabla w_\lambda \chi_{supp(w_\lambda^+)}$ in $L^2(\Sigma_\lambda)$, by (4.3.12) and the fact that $w_\lambda^+ \in L^{2^*}(\Sigma_\lambda)$. Therefore, $\nabla w_\lambda \chi_{supp(w_\lambda^+)}$ is the distributional gradient of ∇w_λ^+ and so ∇w_λ^+ in $L^2(\Sigma_\lambda)$ with (taking limit in (4.3.17))

$$\int_{\Sigma_\lambda} |\nabla w_\lambda|^2 \, dx \leq 2\frac{N+2}{N-2} \|v\|^{2^*}_{L^{2^*}(\Sigma_\lambda)}. \tag{4.3.18}$$

Since $\varphi \in C_c^{0,1}(\mathbb{R}^N)$ we also have

$$\left(\int_{\Sigma_\lambda} \varphi^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C_S^2 \int_{\Sigma_\lambda} |\nabla \varphi|^2 \, dx \tag{4.3.19}$$

where C_S denotes de best constant in Sobolev embedding. Thus, passing to the limit in (4.3.19) and using the above convergence results, we get the desired conclusion (4.3.11).

□

Proof of Theorem 4.0.5. We can now complete the proof of Theorem 4.0.5. As for the proof of Theorem 4.0.3, we split the proof into three steps and we start with
Step 1: there exists $M > 1$ such that $v \leq v_\lambda$ in $\Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\})$, for all $\lambda < -M$.

Arguing as in the proof of Lemma 4.3.3 and using the same notations and the same construction for ψ_ε, φ_R and φ, we get

$$\begin{aligned}\int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2 \varphi_\varepsilon^2 \varphi_R^2 \, dx = & - 2\int_{\Sigma_\lambda} \nabla w_\lambda^+ \nabla \varphi_\varepsilon w_\lambda^+ \varphi_\varepsilon \varphi_R^2 \, dx \\ & - 2\int_{\Sigma_\lambda} \nabla w_\lambda^+ \nabla \varphi_R w_\lambda^+ \varphi_R \varphi_\varepsilon^2 \, dx \\ & + \int_{\Sigma_\lambda} (v^{2^*-1} - v_\lambda^{2^*-1}) w_\lambda^+ \varphi_\varepsilon^2 \varphi_R^2 \, dx \\ =: & \, I_1 + I_2 + I_3 \,, \end{aligned} \tag{4.3.20}$$

where I_1, I_2 and I_3 can be estimated exactly as in (4.3.14), (4.3.15) and (4.3.16). The latter yield

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2 \varphi_\varepsilon^2 \varphi_R^2 \, dx \leq & 32\varepsilon \|v\|^2_{L^\infty(\Sigma_\lambda)} \\
& + 2C(N) \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} v^{2^*} \, dx \right)^{\frac{N-2}{N}} \\
& + 2\frac{N+2}{N-2} \int_{\Sigma_\lambda} v^{2^*-2} (w_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx.
\end{aligned} \tag{4.3.21}
$$

Taking the limit in the latter, as ε tends to zero and R tends to infinity, leads to

$$
\int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2 \, dx \leq 2\frac{N+2}{N-2} \int_{\Sigma_\lambda} v^{2^*-2} (w_\lambda^+)^2 \, dx < +\infty \tag{4.3.22}
$$

which combined with Lemma 4.3.3 gives

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2 \, dx & \leq 2\frac{N+2}{N-2} \int_{\Sigma_\lambda} v^{2^*-2} (w_\lambda^+)^2 dx \\
& \leq 2\frac{N+2}{N-2} \left(\int_{\Sigma_\lambda} v^{2^*} \, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda} (w_\lambda^+)^{2^*} \, dx \right)^{\frac{2}{2^*}} \\
& \leq 2\frac{N+2}{N-2} C_S^2 \left(\int_{\Sigma_\lambda} v^{2^*} \, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2 \, dx \right).
\end{aligned} \tag{4.3.23}
$$

Recalling that $v \in L^{2^*}(\Sigma_\lambda)$, we deduce the existence of $M > 1$ such that

$$
2\frac{N+2}{N-2} C_S^2 \left(\int_{\Sigma_\lambda} v^{2^*} \, dx \right)^{\frac{2}{N}} < 1
$$

for every $\lambda < -M$. The latter and (4.3.23) lead to

$$
\int_{\Sigma_\lambda} |\nabla w_\lambda^+|^2 \, dx = 0 \, .
$$

This implies that $w_\lambda^+ = 0$ by Lemma 4.3.3 and the claim is proved.

To proceed further we define

$$
\Lambda_0 = \{\lambda < 0 : v \leq v_t \text{ in } \Sigma_t \setminus R_t(\Gamma^* \cup \{0\}) \text{ for all } t \in (-\infty, \lambda]\}
$$

and

$$
\lambda_0 = \sup \Lambda_0.
$$

Step 2: we have that $\lambda_0 = 0$. We argue by contradiction and suppose that $\lambda_0 < 0$. By continuity we know that $v \leq v_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$. By the strong maximum principle we deduce that $v < v_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$. Indeed, $v = v_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$) is not possible if $\lambda_0 < 0$, since in this case v would be singular somewhere on $R_{\lambda_0}(\Gamma^* \cup \{0\})$. Now, for some $\bar{\tau} > 0$, that will be fixed later on, and for any $0 < \tau < \bar{\tau}$ we show that $v \leq v_{\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ obtaining a contradiction with the definition of λ_0 and proving thus the claim. To this end we are going to show that, for every $\delta > 0$ there are $\bar{\tau}(\delta, \lambda_0) > 0$ and a compact set K (depending on δ and λ_0) such that

$$K \subset \Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\}), \qquad \int_{\Sigma_\lambda \setminus K} v^{2^*}\, dx < \delta, \qquad \forall\, \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}].$$

To see this, we note that for every every $\delta > 0$ there are $\tau_1(\delta, \lambda_0) > 0$ and a compact set K (depending on δ and λ_0) such that $\displaystyle\int_{\Sigma_{\lambda_0} \setminus K} v^{2^*}\, dx < \frac{\delta}{2}$ and $K \subset \Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\})$ for every $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$. Consequently u and u_λ are well defined on K for every $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$. Hence, by the uniform continuity of the function $g(x, \lambda) := u(x) - u(2\lambda - x_1, x')$ on the compact set $K \times [\lambda_0, \lambda_0 + \tau_1]$ we can ensure that $K \subset \Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ and $u < u_{\lambda_0+\tau}$ in K for any $0 \leq \tau < \tau_2$, for some $\tau_2 = \tau(\delta, \lambda_0) \in (0, \tau_1)$. Clearly we can also assume that $\tau_2 < \frac{|\lambda_0|}{4}$. Finally, since $v^{2^*} \in L^1(\Sigma_{\lambda_0 + \frac{|\lambda_0|}{4}})$ and $\displaystyle\int_{\Sigma_{\lambda_0} \setminus K} v^{2^*}\, dx < \frac{\delta}{2}$, we obtain the existence of $\bar{\tau} \in (0, \tau_2)$ such that $\displaystyle\int_{\Sigma_\lambda \setminus K} v^{2^*}\, dx < \delta$ for all $\lambda \in [\lambda_0, \lambda_0 + \bar{\tau}]$.

Now we repeat verbatim the arguments used in the proof of Lemma 4.3.3 but using the test function

$$\varphi := \begin{cases} w^+_{\lambda_0+\tau} \psi^2_\varepsilon \varphi^2_R & \text{in} \quad \Sigma_{\lambda_0+\tau} \\ 0 & \text{in} \quad \mathbb{R}^N \setminus \Sigma_{\lambda_0+\tau}. \end{cases}$$

Thus we recover the first inequality in (4.3.23), which immediately gives, for any $0 \leq \tau < \bar{\tau}$

(4.3.24)

$$\begin{aligned} \int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla w^+_{\lambda_0+\tau}|^2\, dx &\leq 2\frac{N+2}{N-2} \int_{\Sigma_{\lambda_0+\tau} \setminus K} v^{2^*-2} (w^+_{\lambda_0+\tau})^2\, dx \\ &\leq 2\frac{N+2}{N-2} \left(\int_{\Sigma_{\lambda_0+\tau} \setminus K} v^{2^*}\, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_{\lambda_0+\tau} \setminus K} (w^+_{\lambda_0+\tau})^{2^*}\, dx \right)^{\frac{2}{2^*}} \\ &\leq 2\frac{N+2}{N-2} C_S^2 \left(\int_{\Sigma_{\lambda_0+\tau} \setminus K} v^{2^*}\, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla w^+_{\lambda_0+\tau}|^2\, dx \right) \end{aligned}$$

since $w^+_{\lambda_0+\tau}$ and $\nabla w^+_{\lambda_0+\tau}$ are zero in a neighbourhood of K, by the above construction. Now we fix $\delta < \frac{1}{2}\left[2\frac{N+2}{N-2} C_S^2\right]^{-\frac{N}{2}}$ and we observe that with this choice we have

$$2\frac{N+2}{N-2} C_S^2 \left(\int_{\Sigma_{\lambda_0+\tau} \setminus K} v^{2^*}\, dx \right)^{\frac{2}{N}} < \frac{1}{2}, \qquad \forall\, 0 \leq \tau < \bar{\tau}$$

which plugged into (4.3.24) implies that $\displaystyle\int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla w^+_{\lambda_0+\tau}|^2\, dx = 0$ for every $0 \leq \tau < \bar{\tau}$. Hence $\displaystyle\int_{\Sigma_{\lambda_0+\tau}} |\nabla w^+_{\lambda_0+\tau}|^2\, dx = 0$ for every $0 \leq \tau < \bar{\tau}$, since

$\nabla w^+_{\lambda_0+\tau}$ is zero in a neighborhood of K. The latter and Lemma 4.3.3 imply that $w^+_{\lambda_0+\tau} = 0$ on $\Sigma_{\lambda_0+\tau}$ for every $0 \leq \tau < \bar{\tau}$ and thus $v \leq v_{\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ for every $0 \leq \tau < \bar{\tau}$. Which proves the claim of Step 2.

Step 3: conclusion. The symmetry of the Kelvin transform v follows now performing the moving planes method in the opposite direction. The fact that that v is symmetric w.r.t. the hyperplane $\{x_1 = 0\}$ implies the symmetry of the solution u w.r.t. the hyperplane $\{x_1 = 0\}$. The last claim then follows by the invariance of the considered problem with respect to isometries (translations and rotations).

□

Proof of Corollary 4.0.6. The function $v(x) = u(x + x_0)$ satisfies the assumptions of Theorem 4.0.5 with $\Gamma = \{0\}$. An application of Theorem 4.0.5 yields that v is symmetric with respect to every hyperplane through the origin and so the original solution u must be radially symmetric with respect to x_0. This proves item (i). Since item (ii) is a special case of item (iii) with $k = 1$, we need only to prove item (iii). To this end we observe that, up to an isometry, we can suppose that the affine $k-$dimensional subspace is $\{x_{k+1} = ... = x_N = 0\}$. Therefore, we can apply Theorem 4.0.5 to get that u is symmetric with respect to each hyperplane of $\mathbb{R}^N$ containing $\{x_{k+1} = ... = x_N = 0\}$; i.e., u is invariant with respect to every rotation of $\mathbb{R}^N$ which leaves invariant the set $\{x_{k+1} = ... = x_N = 0\}$. Note that we can apply Theorem 4.0.3 since any affine $k-$dimensional subspace of $\mathbb{R}^N$, with $1 \leq k \leq N-2$, has zero 2-capacity in $\mathbb{R}^N$ (and so $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0$).

□

5

Monotonicity and symmetry of singular solutions to quasilinear problems

In this chapter we consider the problem

$$
\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \setminus \Gamma \\ u > 0 & \text{in } \Omega \setminus \Gamma \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.0.1}
$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and $p > 1$. The solution u has a possible singularity on the critical set Γ and in fact we shall only assume that u is of class $\mathcal{C}^1$ far from the critical set. Therefore the equation is understood as in the following

DEFINITION 5.0.1. *We say that $u \in C^1(\overline{\Omega} \setminus \Gamma)$ is a solution to (5.0.1) if $u = 0$ on $\partial\Omega$ and*

$$
\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\,dx = \int_\Omega f(u)\varphi\,dx \qquad \forall\varphi \in C^1_c(\Omega \setminus \Gamma)\,. \tag{5.0.2}
$$

The purpose of this chapter is to investigate symmetry and monotonicity properties of the solutions when the domain is assumed to have symmetry properties. This issue is well understood in the semilinear case $p = 2$ when $\Gamma = \emptyset$. The symmetry of the solutions in this case can be deduced by the celebrated *moving planes method*, see [**1**, **12**, **68**, **111**]. In [**50**, **110**] and in Chapter 4 the moving planes procedure has been adapted to the case when the singular set has zero capacity, in the semilinear setting $p = 2$.

As remarked in the introduction, here we extend the result obtained in Chapter 4 in bounded domains to singular solutions of problem (5.0.1). We prefer to start the presentation of our results with the case $p > 2$. We have the following:

THEOREM 5.0.2. *Let $p > 2$ and let $u \in C^1(\overline{\Omega} \setminus \Gamma)$ be a solution to (5.0.1) and assume that f is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$, namely assume (A^2_f). If Ω is convex and symmetric with respect to the x_1-direction, Γ is closed with $\mathrm{Cap}_p(\Gamma) = 0$, namely let us assume (A^2_Γ), and*

$$
\Gamma \subset \{x \in \Omega \,:\, x_1 = 0\},
$$

then it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.

Although the technique that we will develop to prove Theorem 5.0.2 works for any $p > 2$, the result is stated for $2 < p \leq N$ since there are no sets of zero p-capacity when $p > N$.

Surprisingly, as we explained in the introduction, the case $1 < p < 2$ presents more difficulties related to the fact that the operator may degenerate near the critical set even if $p < 2$. We will therefore need an accurate analysis on the behaviour of the gradient of the solution near Γ. We carry out such analysis exploiting the results of [100] (therefore we shall require a growth assumption on the nonlinearity) and a blow up argument. The result is the following:

THEOREM 5.0.3. *Let $1 < p < 2$ and let $u \in C^1(\overline{\Omega} \setminus \Gamma)$ be a solution to (5.0.1) and assume that f is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$ and has subcritical growth, namely let us assume (A_f^1). Assume that Γ is closed and that $\Gamma = \{0\}$ for $N = 2$, while $\Gamma \subseteq \mathcal{M}$ for some compact C^2 submanifold $\mathcal{M}$ of dimension $m \leq N - k$, with $k \geq \frac{n}{2}$ for $N > 2$, see (A_Γ^1). Then, if Ω is convex and symmetric with respect to the x_1-direction and*

$$\Gamma \subset \{x \in \Omega \,:\, x_1 = 0\},$$

it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.

5.1. Notations and technical results

Notation. Generic fixed and numerical constants will be denoted by C (with subscript in some case) and they will be allowed to vary within a single line or formula. By $|A|$ we will denote the Lebesgue measure of a measurable set A.

As we did in the previous chapter, we fix some standard notations in the moving planes method. For a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\} \tag{5.1.1}$$

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_N) \tag{5.1.2}$$

which is the reflection through the hyperplane $T_\lambda := \{x \in \mathbb{R}^N \,:\, x_1 = \lambda\}$. Also let

$$a = \inf_{x \in \Omega} x_1. \tag{5.1.3}$$

Finally we set

$$u_\lambda(x) = u(x_\lambda)\,. \tag{5.1.4}$$

We recall also the definition of p-capacity of a compact set $\mathcal{A} \subset \mathbb{R}^N$. For $1 \leq p \leq N$ we define $\mathrm{Cap}_p(\mathcal{A})$ as

(5.1.5)
$$\mathrm{Cap}_p(\mathcal{A}) := \inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^p dx < +\infty \;:\; \varphi \in C_c^\infty(\mathbb{R}^N) \;\text{ and }\; \varphi \geq \chi_{\mathcal{A}}\right\},$$

where $\chi_{\mathcal{S}}$ denotes the characteristic function of a set $\mathcal{S}$. By the invariance under reflections of (5.1.5), it follows that

$$\mathrm{Cap}_p(\Gamma) = \mathrm{Cap}_p(R_\lambda(\Gamma)). \tag{5.1.6}$$

Moreover it can be shown that, if $\text{Cap}_p(R_\lambda(\Gamma)) = 0$, then we have that

$$\text{Cap}_p^D(R_\lambda(\Gamma)) = 0, \tag{5.1.7}$$

where $D \subset \mathbb{R}^N$ denotes a bounded subset and with $\text{Cap}_p^D(\mathcal{A})$ ($\mathcal{A} \subset D$ a compact set of $\mathbb{R}^N$) we mean

$$\text{Cap}_p^D(\mathcal{A}) := \inf\left\{ \int_D |\nabla\varphi|^p dx < +\infty \ : \ \varphi \in C_c^\infty(D) \ \text{ and } \ \varphi \geq \chi_{\mathcal{A}} \right\}.$$

Let $\varepsilon > 0$ small and let $\mathcal{B}_\epsilon^\lambda$ be a ε-neighborhood of $R_\lambda(\Gamma)$. From (5.1.6) and (5.1.7) it follows that there exists $\varphi_\varepsilon \in C_c^\infty(\mathcal{B}_\epsilon^\lambda)$ such that $\varphi_\varepsilon \geq 1$ on $\chi_{R_\lambda(\Gamma)}$ and

$$\int_{\mathcal{B}_\epsilon^\lambda} |\nabla\varphi_\varepsilon|^p dx < \varepsilon.$$

To carry on our analysis we need to construct a function $\psi_\varepsilon \in W^{1,p}(\Omega)$ (see Figure 1) such that $\psi_\varepsilon = 1$ in $\Omega \setminus \mathcal{B}_\varepsilon^\lambda$, $\psi_\varepsilon = 0$ in a δ_ε-neighborhood $\mathcal{B}_{\delta_\varepsilon}^\lambda$ of $R_\lambda(\Gamma)$ (with $\delta_\varepsilon < \varepsilon$) and such that

$$\int_{\mathcal{B}_\epsilon^\lambda} |\nabla\psi_\varepsilon|^p dx \leq C\varepsilon, \tag{5.1.8}$$

for some positive constant C that does not depend on ε. To construct such a test function we consider the real functions $T : \mathbb{R} \to \mathbb{R}_0^+$ and $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$T(s) := \max\{0; \min\{s; 1\}\},\ s \in \mathbb{R} \quad \text{and} \quad g(s) := \max\{0; -2s + 1\},\ s \in \mathbb{R}_0^+. \tag{5.1.9}$$

Finally we set

$$\psi_\varepsilon(x) := g(T(\varphi_\varepsilon(x))). \tag{5.1.10}$$

By the definitions (5.1.9), it follows that ψ_ε satisfies (5.1.8).

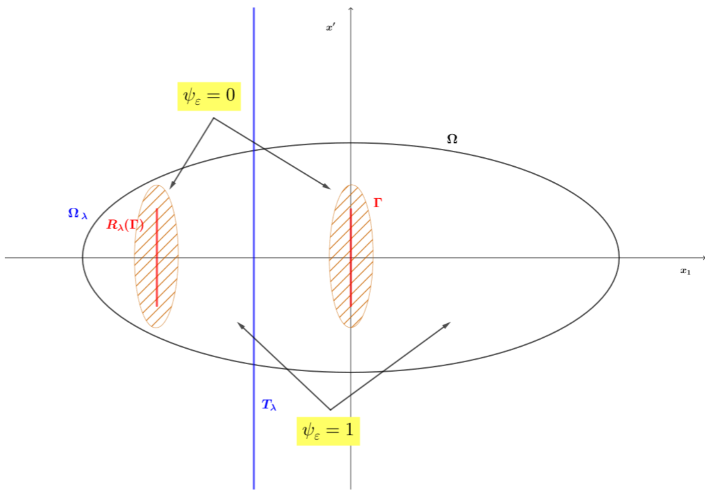

FIGURE 1. The cutoff function ψ_ε.

To simplify the presentation we summarize the assumptions of the main results as follows:

(A_f^1). *For $1 < p < 2$ we assume that f is locally Lipschitz continuous so that, for any $0 \leq t, s \leq M$, there exists a positive constant $K_f = K_f(M)$ such that*

$$|f(s) - f(t)| \leq K_f |s - t|.$$

Moreover $f(s) > 0$ for $s > 0$ and

$$\lim_{t \to +\infty} \frac{f(t)}{t^q} = l \in (0, +\infty).$$

for some $q \in \mathbb{R}$ such that $p - 1 < q < p^* - 1$, where $p^* = Np/(N - p)$.

(A_f^2). *For $p \geq 2$ we only assume that f is locally Lipscitz continuous so that, for $0 \leq t, s \leq M$ there exists a positive constant $K_f = K_f(M)$ such that*

$$|f(s) - f(t)| \leq K_f |s - t|.$$

Furthermore $f(s) > 0$ for $s > 0$.

(A_Γ^1). *For $1 < p < 2$ and $N = 2$ we assume that $\Gamma = \{0\}$, while for $1 < p < 2$ and $N > 2$ we assume that $\Gamma \subseteq \mathcal{M}$ for some compact C^2 submanifold $\mathcal{M}$ of dimension $m \leq N - k$, with $k \geq \frac{N}{2}$.*

(A_Γ^2). *For $2 < p < N$ and $N \geq 2$, we assume that Γ closed and such that*

$$\mathrm{Cap}_p(\Gamma) = 0.$$

Remark 5.1.1. *We want just to remark that in the case $1 < p < 2$ and $N > 2$ if $\Gamma \subseteq \mathcal{M}$ for some compact C^2 submanifold $\mathcal{M}$ of dimension $m \leq N - k$ then $\mathrm{Cap}_p(\Gamma) = 0$. In this case we consider $\mathcal{B}_\varepsilon$ a tubular neighborhood of radius ε of $\mathcal{M}$, i.e.*

$$\mathcal{B}_\varepsilon := \{x \in \Omega \ : \ dist(x, \mathcal{M}) < \varepsilon\},$$

with $\varepsilon > 0$ sufficiently small so that $\mathcal{M}$ has the unique nearest point property in the neighborhood of $\mathcal{M}$ of radius ε. We may and do also assume that Fermi coordinates are well defined in such neighborhood, see e.g. [98]. Therefore, using the defintion (5.1.5) above, it can be shown that $\mathrm{Cap}_p(\Gamma) =$ 0.

In the following we will exploit the fact that u_λ (in the sense of Definition 5.0.1) is a solution to

(5.1.11)
$$\int_{R_\lambda(\Omega)} |\nabla u_\lambda|^{p-2} (\nabla u_\lambda, \nabla \varphi)\, dx = \int_{R_\lambda(\Omega)} f(u_\lambda) \varphi\, dx \quad \forall \varphi \in C_c^1(R_\lambda(\Omega) \backslash R_\lambda(\Gamma)).$$

We set

(5.1.12)
$$w_\lambda(x) := (u - u_\lambda)(x), \quad x \in \overline{\Omega}_\lambda \setminus R_\lambda(\Gamma).$$

Lemma 5.1.2. *Let $p > 1$ and let u and u_λ be solutions to (5.0.2) and (5.1.11) respectively and let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Let us assume $\Gamma \subset \Omega$ closed and such that*

$$\mathrm{Cap}_p(\Gamma) = 0.$$

Let a be defined as in (5.1.3) and $a < \lambda < 0$. Then

$$\int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \, dx \le C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) \, .$$

PROOF. In all the proof, according to our assumptions, we assume that $0 \le t, s \le M$, there exists a positive constant $K_f = K_f(M)$ such that

$$|f(s) - f(t)| \le K_f |s - t|.$$

For ψ_ε defined as in (5.1.10), we consider

$$\varphi_\varepsilon := w_\lambda^+ \psi_\varepsilon^p \chi_{\Omega_\lambda}.$$

By standard arguments, since $w_\lambda^+ \le \|u\|_{L^\infty(\Omega_\lambda)}$ (recall that in particular $u \in C(\overline{\Omega} \setminus \Gamma)$) and by construction $0 \le \psi_\varepsilon \le 1$, we have that $\varphi_\varepsilon \in W_0^{1,p}(\Omega_\lambda)$. By a density argument we use φ_ε as test function in (5.0.2) and (5.1.11). Subtracting we get

$$\begin{aligned} &\int_{\Omega_\lambda} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w_\lambda^+) \psi_\varepsilon^p \, dx \\ &+ p \int_{\Omega_\lambda} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \psi_\varepsilon) \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\ &= \int_{\Omega_\lambda} (f(u) - f(u_\lambda)) w_\lambda^+ \psi_\varepsilon^p \, dx \end{aligned} \tag{5.1.13}$$

Now it is useful to split the set Ω_λ as the union of two disjoint subsets A_λ and B_λ such that $\Omega_\lambda = A_\lambda \cup B_\lambda$. In particular, for $\dot{C} > 1$ that will be fixed large, we set

$$A_\lambda = \{x \in \Omega_\lambda \ : \ |\nabla u_\lambda(x)| < \dot{C} |\nabla u(x)|\}$$

and

$$B_\lambda = \{x \in \Omega_\lambda \ : \ |\nabla u_\lambda(x)| \ge \dot{C} |\nabla u(x)|\}.$$

Then it follows that

- By the definition of A_λ it follows that there exists $\hat{C}$ such that

$$|\nabla u| + |\nabla u_\lambda| < \hat{C} |\nabla u|. \tag{5.1.14}$$

- By the definition of the set B_λ and standard triangular inequalities, we can deduce the existence of a positive constant $\check{C}$ such that

$$\frac{1}{\check{C}} |\nabla u_\lambda| \le |\nabla u_\lambda| - |\nabla u| \le |\nabla w_\lambda| \le |\nabla u_\lambda| + |\nabla u| \le \check{C} |\nabla u_\lambda|. \tag{5.1.15}$$

We distinguish two cases:

Case 1: $1 < p < 2$. From (5.1.13), using (1.0.2) and (A_f^1) we have

(5.1.16)
$$\begin{aligned}
C_1 \int_{\Omega_\lambda} &(|\nabla u|+|\nabla u_\lambda|)^{p-2}|\nabla w_\lambda^+|^2\psi_\varepsilon^p \, dx \\
&\leq \int_{\Omega_\lambda} (|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla w_\lambda^+)\psi_\varepsilon^p \, dx \\
&\leq p \int_{\Omega_\lambda} \left||\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right| \, |\nabla \psi_\varepsilon|\psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
&\quad + \int_{\Omega_\lambda} \frac{f(u)-f(u_\lambda)}{u-u_\lambda}(w_\lambda^+)^2\psi_\varepsilon^p \, dx \\
&\leq pC_4 \int_{\Omega_\lambda} |\nabla w_\lambda^+|^{p-1} \, |\nabla \psi_\varepsilon|\psi_\varepsilon^{p-1} w_\lambda^+ \, dx + K_f \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^p \, dx \\
&\leq C\Big(I_1 + I_2\Big) + C \int_{\Omega_\lambda} \psi_\varepsilon^p \, dx,
\end{aligned}$$

where

$$I_1 := \int_{A_\lambda} |\nabla w_\lambda^+|^{p-1} \, |\nabla \psi_\varepsilon|\psi_\varepsilon^{p-1} w_\lambda^+ \, dx$$

and

$$I_2 := \int_{B_\lambda} |\nabla w_\lambda^+|^{p-1} \, |\nabla \psi_\varepsilon|\psi_\varepsilon^{p-1} w_\lambda^+ \, dx,$$

and $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$ is a positive constant.

Step 1: Evaluation of I_1. Using Young's inequality and (5.1.14), we have

(5.1.17)
$$\begin{aligned}
I_1 = \int_{A_\lambda} &|\nabla w_\lambda^+|^{p-1}|\nabla \psi_\varepsilon|\psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
&\leq \left(\int_{A_\lambda} |\nabla w_\lambda^+|^p \psi_\varepsilon^p \, dx\right)^{\frac{p-1}{p}} \left(\int_{A_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^p \psi_\varepsilon^p \, dx\right)^{\frac{p-1}{p}} \left(\int_{A_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx\right)^{\frac{1}{p}} \\
&\leq \left(\hat{C} \int_{A_\lambda} |\nabla u|^p \psi_\varepsilon^p \, dx\right)^{\frac{p-1}{p}} \left(\int_{A_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx\right)^{\frac{1}{p}} \\
&\leq C \left(\int_{\Omega_\lambda} |\nabla u|^p \, dx\right)^{\frac{p-1}{p}} \left(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx\right)^{\frac{1}{p}},
\end{aligned}$$

where $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$ is a positive constant.

Step 2: Evaluation of I_2. Using the weighted Young's inequality and (5.1.15) we get

$$
\begin{aligned}
I_2 =& \int_{B_\lambda} |\nabla w_\lambda^+|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
\leq& \delta \int_{B_\lambda} |\nabla w_\lambda^+|^p \psi_\varepsilon^p \, dx \\
&+ \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \\
\leq& \delta \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} \left(|\nabla u| + |\nabla u_\lambda|\right)^2 \psi_\varepsilon^p \, dx \\
&+ \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \\
\leq& \delta \check{C}^2 \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla u_\lambda|^2 \psi_\varepsilon^p \, dx \\
&+ \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \\
\leq& \delta \check{C}^4 \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&+ \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \\
\leq& \delta C \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&+ \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx,
\end{aligned} \tag{5.1.18}
$$

where $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$ is a positive constant. Finally, using (5.1.16), (5.1.17) and (5.1.18), we obtain

$$
\begin{aligned}
&\int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\leq \delta C \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\quad + C \left(\int_{\Omega_\lambda} |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx \right)^{\frac{1}{p}} \\
&\quad + \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx + C \int_{\Omega_\lambda} \psi_\varepsilon^p \, dx,
\end{aligned} \tag{5.1.19}
$$

for some positive constant $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$.

Case 2: $p \geq 2$**.** From (5.1.13), using (1.0.2) and (A_f^2) we have

$$
\begin{aligned}
& C_1 \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
& \leq \int_{\Omega_\lambda} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w_\lambda^+) \psi_\varepsilon^p \, dx \\
& = -p \int_{\Omega_\lambda} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \psi_\varepsilon) \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
& \quad + \int_{\Omega_\lambda} (f(u) - f(u_\lambda)) w_\lambda^+ \psi_\varepsilon^p \, dx \\
& \leq p \int_{\Omega_\lambda} \left| |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda \right| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
& \quad + \int_{\Omega_\lambda} \frac{f(u) - f(u_\lambda)}{u - u_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^p \, dx \\
& \leq p C_2 \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
& \quad + K_f \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^p \, dx \\
& = p C_2 \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
& \quad + p C_2 \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
& \quad + K_f \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^p \, dx \\
& \leq C \Big(I_1 + I_2 \Big) + C \int_{\Omega_\lambda} \psi_\varepsilon^p \, dx,
\end{aligned}
\tag{5.1.20}
$$

where

$$
I_1 := \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx
$$

and

$$
I_2 := \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx,
$$

and $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$ is a positive constant.

Step 1: Evaluation of I_1. Using the weighted Young's inequality we have

$$
\begin{aligned}
I_1 & = \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
& \leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
& \quad + \frac{1}{\delta} \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \psi_\varepsilon|^2 \psi_\varepsilon^{p-2} (w_\lambda^+)^2 \, dx.
\end{aligned}
$$

Using (5.1.14) and Hölder inequality, we obtain

$$
\begin{aligned}
I_1 &\leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\quad + \frac{\hat{C}^{p-2}}{\delta} \int_{A_\lambda} |\nabla u|^{p-2} |\nabla \psi_\varepsilon|^2 \psi_\varepsilon^{p-2} (w_\lambda^+)^2 \, dx \\
&\leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\quad + \frac{C}{\delta} \left(\int_{A_\lambda} |\nabla u|^p \psi_\varepsilon^p \, dx \right)^{\frac{p-2}{p}} \left(\int_{A_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \right)^{\frac{2}{p}} \\
&\leq \delta \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\quad + \frac{C}{\delta} \left(\int_{\Omega_\lambda} |\nabla u|^p \, dx \right)^{\frac{p-2}{p}} \left(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx \right)^{\frac{2}{p}},
\end{aligned} \tag{5.1.21}
$$

with $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$ is a positive constant.

Step 2: Evaluation of I_2. By the weighted Young's inequality

$$
\begin{aligned}
I_2 &:= \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \psi_\varepsilon^{p-1} w_\lambda^+ \, dx \\
&\leq \delta \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p(p-2)}{p-1}} |\nabla w_\lambda^+|^{\frac{p}{p-1}} \psi_\varepsilon^p \, dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \\
&= \delta \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p(p-2)}{p-1}} |\nabla w_\lambda^+|^2 |\nabla w_\lambda^+|^{\frac{p}{p-1}-2} \psi_\varepsilon^p \, dx \\
&\quad + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx.
\end{aligned}
$$

Using (5.1.15) and noticing that

$$
\frac{p}{(p-1)} - 2 < 0,
$$

we obtain the following estimate

$$
\begin{aligned}
I_2 &\leq \delta \check{C}^{\frac{(p-2)(p+1)}{p-1}} \int_{B_\lambda} |\nabla u_\lambda|^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p \, dx \\
&\leq \delta C \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx + \frac{C}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p \, dx \\
&\leq \delta C \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx,
\end{aligned} \tag{5.1.22}
$$

with $C = C(p, \|u\|_{L^\infty(\Omega_\lambda)})$. In the second line of (5.1.22) we exploited the fact that, since $p \geq 2$ then

$$
|\nabla u_\lambda|^{p-2} \leq (|\nabla u| + |\nabla u_\lambda|)^{p-2}.
$$

Collecting (5.1.20), (5.1.21) and (5.1.22) we deduce that

$$\begin{aligned}
&\int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\quad \leq \delta C \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^p \, dx \\
&\quad + \frac{C}{\delta} \left(\int_{\Omega_\lambda} |\nabla u|^p \, dx \right)^{\frac{p-2}{p}} \left(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx \right)^{\frac{2}{p}} \\
&\quad + \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, dx + C \int_{\Omega_\lambda} \psi_\varepsilon^p \, dx,
\end{aligned} \tag{5.1.23}$$

for some positive constant $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$.

For δ small, from (5.1.19) and (5.1.23), using (5.1.8) and the fact that for $\lambda < 0$ the solution $u \in W^{1,p}(\Omega_\lambda)$, letting $\varepsilon \to 0$ by Fatou's Lemma we obtain

$$\int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 \, dx \leq C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}),$$

concluding the proof.

□

5.2. Symmetry and monotonicity results in the singular and in the degenerate case

We recall the fact that u_λ (in the sense of Definition 5.0.1) is a solution to

$$\int_{R_\lambda(\Omega)} |\nabla u_\lambda|^{p-2} (\nabla u_\lambda, \nabla \varphi) \, dx = \int_{R_\lambda(\Omega)} f(u_\lambda) \varphi \, dx \quad \forall \varphi \in C_c^1(R_\lambda(\Omega) \backslash R_\lambda(\Gamma)) \, . \tag{5.2.1}$$

We set

$$w_\lambda(x) := (u - u_\lambda)(x), \quad x \in \overline{\Omega} \setminus (\Gamma \cup R_\lambda(\Gamma)).$$

Since in the following we will exploit weighted Sobolev inequalities, it is convenient to set weight

$$\hat{\varrho} := |\nabla u|^{p-2} \qquad \frac{1}{\hat{\varrho}} := |\nabla u|^{2-p} \, . \tag{5.2.2}$$

We have the following

Lemma 5.2.1. *Let $1 < p < 2$. Under the same assumption of Theorem 5.0.3, define*

$$\Omega_\lambda^+ := \Omega_\lambda \cap supp\,(w_\lambda^+).$$

Then

$$|\nabla u|^{2-p} \in L^t(R_\lambda(\Omega_\lambda^+)), \tag{5.2.3}$$

for some $t > \frac{N}{2}$.

Proof. By definition of Ω_λ^+ we have

$$\|u\|_{L^\infty(R_\lambda(\Omega_\lambda^+))} = \|u_\lambda\|_{L^\infty(\Omega_\lambda^+)} \leq \|u\|_{L^\infty(\Omega_\lambda^+)} \leq C(\lambda, \|u\|_{L^\infty(\Omega_\lambda)}).$$

Taking $x_0 \in R_\lambda(\Omega_\lambda^+) \setminus \Gamma$, we set:

(5.2.4) $$g(x) := u(dx + x_0) \quad \text{in } B_{\frac{1}{2}}(0),$$

where $d := \mathrm{dist}(x_0, \Gamma)$. Since u is a solution (in the sense of Definition 5.0.1) to (5.0.1), we deduce that for any $\varphi \in C_c^1(B_{1/2}(0))$

(5.2.5)
$$\begin{aligned}
&\int_{B_{\frac{1}{2}}(0)} |\nabla g|^{p-2}(\nabla g, \nabla\varphi)\, dx \\
&= d^{p-1} \int_{B_{\frac{1}{2}}(0)} |\nabla u(dx + x_0)|^{p-2}(\nabla u(dx + x_0), \nabla\varphi)\, dx \\
&= d^{p-n} \int_{B_{\frac{d}{2}}(x_0)} |\nabla u(x)|^{p-2} \left(\nabla u(x), \nabla\left(\varphi\left(\frac{x - x_0}{d}\right)\right)\right) dx \\
&= d^{p-n} \int_{B_{\frac{d}{2}}(x_0)} f(u(x))\varphi\left(\frac{x - x_0}{d}\right) dx \\
&= d^{p} \int_{B_{\frac{1}{2}}(0)} f(u(dx + x_0))\varphi(x)\, dx \\
&= \int_{B_{\frac{1}{2}}(0)} c(x)(g(x))^{p-1}\varphi(x)\, dx,
\end{aligned}$$

with

(5.2.6) $$c(x) := d^p \frac{f(u(dx + x_0))}{u^{p-1}(dx + x_0)}.$$

From (5.2.5) we deduce that in distributional sense

$$-\Delta_p g = c(x) g^{p-1} \quad \text{in } B_{\frac{1}{2}}(0).$$

On the other hand u as well (in distributional sense) is a positive solution to $-\Delta_p u = f(u)$ in $B_d(x_0)$. Therefore using [100, Theorem 3.1] we have

(5.2.7) $$0 < u(x) \leq C(1 + d^{-\frac{p}{q+1-p}}),$$

where $C = C(f, n, p) > 0$. By (5.2.6), using (A_f^1) we have

(5.2.8) $$c(x) = Cd^p(1 + u^{q+1-p}),$$

with $C = C(l, p, K_f)$ is a positive constant. Finally, collecting (5.2.7) and (5.2.8) we deduce

$$c(x) \leq Cd^p(1 + d^{-p}) \leq C,$$

with $C = C(f, l, N, p, q, K_f, \Omega)$. Hence $c(x) \in L^\infty(B_{1/2}(0))$. By [103, Theorem 7.2.1], recalling (5.2.4), for every $x \in B_{1/8}(0)$ it follows

(5.2.9)
$$\begin{aligned}
g(x) &\leq \sup_{x \in B_{\frac{1}{4}}(0)} g(x) \\
&\leq C_H \inf_{x \in B_{\frac{1}{4}}(0)} g(x) \leq C_H g(0) \leq C
\end{aligned}$$

where $C = C(f, l, N, p, q, K_f, \Omega)$ is a positive constant. Hence g is bounded in $B_{1/8}(0)$ and as consequence, see e.g. [46, 122]

$$g \in C^{1,\alpha}(B_{\frac{1}{16}}(0)).$$

Then there exists a positive constant $C = C(N, p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)})$ such that

$$|\nabla g(x)| \leq C \quad \forall x \in B_{\frac{1}{16}}(0).$$

By (5.2.4) it follows

$$d|\nabla u(dx + x_0)| \leq C \quad \forall x \in B_{\frac{1}{16}}(0),$$

namely

$$|\nabla u| \leq \frac{C}{d} \quad \text{in } B_{\frac{d}{16}}(x_0). \tag{5.2.10}$$

Using (A^1_Γ), we can consider $\mathcal{B}_\varepsilon$ a tubular neighborhood of radius ε of $\mathcal{M}$, i.e.

$$\mathcal{B}_\varepsilon := \{x \in \Omega \ : \ \mathrm{dist}(x, \mathcal{M}) < \varepsilon\}.$$

We now exploit an integration in Fermi coordinates,see e.g. [98]. We indicate a point of $\mathcal{B}_\varepsilon$ via the coordinate $(\sigma, x)'$ where σ is the variable describing the manifold $\mathcal{M}$ and $x' \in \mathbb{R}^k$ is the Euclidean variable on the normal section. For σ fixed, D_σ will stand for the normal section at σ. By (5.2.10), and passing to polar coordinates we obtain

(5.2.11)
$$\begin{aligned}
\int_{R_\lambda(\Omega_\lambda^+)} \left(|\nabla u|^{2-p}\right)^t dx &= \int_{R_\lambda(\Omega_\lambda^+)\setminus\mathcal{B}_\varepsilon} \left(|\nabla u|^{2-p}\right)^t dx + \int_{\mathcal{B}_\varepsilon} \left(|\nabla u|^{2-p}\right)^t dx \\
&\leq C + C\int_{\mathcal{M}} d\sigma \int_{D_\sigma} \left(|\nabla u|^{2-p}\right)^t dx' \\
&\leq C + C\int_{\mathcal{M}} d\sigma \int_0^\varepsilon \frac{1}{r^{(2-p)t-(k-1)}}\, dr \\
&= C(N, p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) + CE_1,
\end{aligned}$$

with

$$E_1 := \int_{\mathcal{M}} d\sigma \int_0^\varepsilon \frac{1}{r^{(2-p)t-(k-1)}}\, dr < +\infty, \tag{5.2.12}$$

if $t < k/(2-p)$, recalling that $1 < p < 2$. Hence, since $k \geq N/2$, inequality (5.2.12) holds for some

$$t \in \left(\frac{N}{2}, \frac{k}{2-p}\right),$$

being $2k > N(2-p)$ under our assumption. □

Let us now set

$$\mathcal{Z}_\lambda := \{x \in \Omega_\lambda \setminus R_\lambda(\Gamma) \mid \nabla u(x) = \nabla u_\lambda(x) = 0\}.$$

We have the following

Lemma 5.2.2. *Let u be a solution to (5.0.2) with $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that $f(s) > 0$ for $s > 0$. Let $a < \lambda < 0$. If $C_\lambda \subset \Omega_\lambda \setminus (R_\lambda(\Gamma) \cup \mathcal{Z}_\lambda)$ is a connected component of $\Omega_\lambda \setminus (R_\lambda(\Gamma) \cup \mathcal{Z}_\lambda)$ and $u = u_\lambda$ in C_λ, then*

$$C_\lambda = \emptyset.$$

Proof. Let

$$C := C_\lambda \cup R_\lambda(C_\lambda).$$

Arguing by contradiction we assume $C \neq \emptyset$. Now for $\varepsilon > 0$, we define $h_\varepsilon(t) : \mathbb{R}^+_0 \to \mathbb{R}$ as

$$h_\varepsilon(t) = \begin{cases} \frac{G_\varepsilon(t)}{t} & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases}$$

where $G_\varepsilon(t) = (2t - 2\varepsilon)\chi_{[\varepsilon\,,2\varepsilon]}(t) + t\chi_{[2\varepsilon\,,\infty)}(t)$ for $t \geqslant 0$. Moreover we consider the cut-off function ψ_ε on the set $\Gamma \cup R_\lambda(\Gamma)$ defined in a similar way as in (5.1.10). Hence we define the test function

$$\varphi_\varepsilon := h_\varepsilon(|\nabla u|)\psi_\varepsilon^2 \chi_C.$$

We point out that the $\operatorname{supp} \varphi_\varepsilon \subset C$ and therefore we can use it as test function in (5.0.2). We obtain

$$\begin{aligned} 0 < \int_C f(u) h_\varepsilon(|\nabla u|)\psi_\varepsilon^2\, dx = & \int_C |\nabla u|^{p-2}(\nabla u, \nabla|\nabla u|) h'_\varepsilon(|\nabla u|)\psi_\varepsilon^2\, dx \\ & + 2\int_C |\nabla u|^{p-2}(\nabla u, \nabla \psi_\varepsilon) h_\varepsilon(|\nabla u|)\psi_\varepsilon\, dx. \end{aligned}$$

Using Schwarz inequality, observing that

$$h_\varepsilon(t) \leq 1 \quad \text{and} \quad h'_\varepsilon(t) \leq 2/\varepsilon,$$

we obtain

(5.2.13)

$$\begin{aligned} 0 &< \int_C f(u) \frac{G_\varepsilon(|\nabla u|)}{|\nabla u|} \psi_\varepsilon^2\, dx \\ &\leq 2\int_{C\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} |\nabla u|^{p-2}\|D^2 u\|\psi_\varepsilon^2 \frac{|\nabla u|}{\varepsilon}\, dx + 2\int_C |\nabla u|^{p-1}|\nabla\psi_\varepsilon|\psi_\varepsilon\, dx \\ &\leq 4\int_{C\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} |\nabla u|^{p-2}\|D^2 u\|\psi_\varepsilon^2\, dx + 2\int_C |\nabla u|^{p-1}|\nabla\psi_\varepsilon|\psi_\varepsilon\, dx \\ &\leq 4\int_C |\nabla u|^{p-2}\|D^2 u\|\psi_\varepsilon^2\chi_{A_\varepsilon}\, dx + 2\left(\int_C |\nabla u|^p\, dx\right)^{\frac{p-1}{p}} \left(\int_C |\nabla\psi_\varepsilon|^p\, dx\right)^{\frac{1}{p}}, \end{aligned}$$

where $A_\varepsilon := C \cap \{\varepsilon < |\nabla u| < 2\varepsilon\}$. Now we note that by the definition of the region C and because $u = u_\lambda$ in C_λ, then the solution u is bounded and $C^{1,\alpha}$ by classical regularity results. Moreover

$$|\nabla u|^{p-2}\|D^2 u\|\psi_\varepsilon^2\chi_{A_\varepsilon} \leq |\nabla u|^{p-2}\|D^2 u\|$$

and $|\nabla u|^{p-2}\|D^2 u\| \in L^1(\mathcal{C})$ by [37] (see also [88, Lemma 5] for details). It is important to note that the regularity of the solution in $R_\lambda(C_\lambda)$ is induced by symmetry by the regularity in C_λ. Noticing that $|\nabla u|^{p-2}\|D^2 u\|\psi_\varepsilon^2\chi_{A_\varepsilon} \to 0$

as ε goes to 0, then letting $\varepsilon \to 0$ in (5.2.13), by Dominated Convergence Theorem and (5.1.8) it follows

$$0 < \int_C f(u)\, dx \le 0,$$

and this gives a contradiction. Hence $C = \emptyset$.

□

Proof of Theorem 5.0.3. Since the singular set Γ is contained in the hyperplane $\{x_1 = 0\}$, then the moving planes procedure can be started in the standard way (see e.g [35, 36, 37]) and, for $a < \lambda < a + \sigma$ with $\sigma > 0$ small, we have that $w_\lambda \le 0$ in Ω_λ (see (5.1.12)) by the weak comparison principle in small domains. Note that the crucial point here is that w_λ has a singularity at Γ and at $R_\lambda(\Gamma)$. For λ close to a the singularity does not play a role. To proceed further we define

$$\Lambda_0 = \{a < \lambda < 0 : u \le u_t \text{ in } \Omega_t \setminus R_t(\Gamma) \text{ for all } t \in (a, \lambda]\}$$

and $\lambda_0 = \sup \Lambda_0$, since we proved above that Λ_0 is not empty. To prove our result we have to show that $\lambda_0 = 0$. To do this we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \le u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma)$ for any $0 < \tau < \bar\tau$ for some small $\bar\tau > 0$. We remark that $|\mathcal{Z}_{\lambda_0}| = 0$, see [37]. Let us take $A_{\lambda_0} \subset \Omega_{\lambda_0}$ be an open set such that $\mathcal{Z}_{\lambda_0} \cap \Omega_{\lambda_0} \subset A_{\lambda_0} \subset\subset \Omega$. Such set exists by Höpf lemma (see Chapter 1). Moreover note that, since $|\mathcal{Z}_{\lambda_0}| = 0$, we can take A_{λ_0} of arbitrarily small measure. By continuity we know that $u \le u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. We can exploit the strong comparison principle, see e.g. [103, Theorem 2.5.2] or Chapter 1, to get that, in any connected component of $\Omega_{\lambda_0} \setminus \mathcal{Z}_{\lambda_0}$, we have

$$u < u_{\lambda_0} \qquad \text{or} \qquad u \equiv u_{\lambda_0}.$$

The case $u \equiv u_{\lambda_0}$ in some connected component C_{λ_0} of $\Omega_{\lambda_0} \setminus \mathcal{Z}_{\lambda_0}$ is not possible, since by symmetry, it would imply the existence of a local symmetry phenomenon and consequently that $\Omega \setminus \mathcal{Z}_{\lambda_0}$ would be not connected, in spite of what we proved in Lemma 5.2.2. Hence we deduce that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. Therefore, given a compact set $K \subset \Omega_{\lambda_0} \setminus (R_{\lambda_0}(\Gamma) \cup A_{\lambda_0})$, by uniform continuity we can ensure that $u < u_{\lambda_0+\tau}$ in K for any $0 < \tau < \bar\tau$ for some small $\bar\tau > 0$. Note that to do this we implicitly assume, with no loss of generality, that $R_{\lambda_0}(\Gamma)$ remains bounded away from K. Arguing in a similar way as in Lemma 5.1.2, we consider

$$\varphi_\varepsilon := w^+_{\lambda_0+\tau} \psi^p_\varepsilon \chi_{\Omega_{\lambda_0+\tau}}. \tag{5.2.14}$$

By density arguments as above, we plug φ_ε as test function in (5.0.2) and (5.2.1) so that, subtracting, we get

(5.2.15)
$$\begin{aligned}&\int_{\Omega_{\lambda_0+\tau}\setminus K}(|\nabla u|^{p-2}\nabla u-|\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau},\nabla w^+_{\lambda_0+\tau})\psi^p_\varepsilon\,dx\\&+p\int_{\Omega_{\lambda_0+\tau}\setminus K}(|\nabla u|^{p-2}\nabla u-|\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau},\nabla\psi_\varepsilon)\psi^{p-1}_\varepsilon w^+_{\lambda_0+\tau}\,dx\\&=\int_{\Omega_{\lambda_0+\tau}\setminus K}(f(u)-f(u_\lambda))w^+_{\lambda_0+\tau}\psi^p_\varepsilon\,dx.\end{aligned}$$

Now we split the set $\Omega_{\lambda_0+\tau}\setminus K$ as the union of two disjoint subsets $A_{\lambda_0+\tau}$ and $B_{\lambda_0+\tau}$ such that $\Omega_{\lambda_0+\tau}\setminus K=A_{\lambda_0+\tau}\cup B_{\lambda_0+\tau}$. In particular, for $\dot C>1$, we set

$$A_{\lambda_0+\tau}=\{x\in\Omega_{\lambda_0+\tau}\setminus K\;:\;|\nabla u_{\lambda_0+\tau}(x)|<\dot C|\nabla u(x)|\}$$

and

$$B_{\lambda_0+\tau}=\{x\in\Omega_{\lambda_0+\tau}\setminus K\;:\;|\nabla u_{\lambda_0+\tau}(x)|\geq\dot C|\nabla u(x)|\}.$$

From (5.2.15), using (1.0.2) and (A^1_f), repeating verbatim arguments in (5.1.16), (5.1.17) and in (5.1.18) we have

$$\begin{aligned}&\int_{\Omega_{\lambda_0+\tau}\setminus K}(|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\psi^p_\varepsilon\,dx\\&\leq\delta C\int_{\Omega_{\lambda_0+\tau}\setminus K}(|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\psi^p_\varepsilon\,dx\\&\quad+C\left(\int_{\Omega_\lambda}|\nabla u|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^p\,dx\right)^{\frac1p}\\&\quad+\frac{C}{\delta}\int_{\Omega_{\lambda_0+\tau}\setminus K}|\nabla\psi_\varepsilon|^p\,dx+K_f\int_{\Omega_{\lambda_0+\tau}\setminus K}(w^+_{\lambda_0+\tau})^2\psi^p_\varepsilon\,dx,\end{aligned}$$

for some positive constant $C=C(p,\lambda,\|u\|_{L^\infty(\Omega_{\lambda+\bar\tau})})$. Taking $\delta>0$ sufficiently small and using (A^1_Γ), as we did above passing to the limit for $\varepsilon\to0$ we obtain

(5.2.16)
$$\int_{\Omega_{\lambda_0+\tau}\setminus K}(|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\,dx\leq CK_f\int_{\Omega_{\lambda_0+\tau}\setminus K}(w^+_{\lambda_0+\tau})^2\,dx.$$

Now we set

$$\varrho:=\left(1+|\nabla u|^2+|\nabla u_\lambda|^2\right)^{\frac{p-2}{2}}$$

in order to exploit the weighted Sobolev inequality from [124] (see also Chapter 1). The results of [124] apply if $\varrho\in L^1(\Omega_\lambda)$ and

$$\frac{1}{\varrho}\in L^t(\Omega_\lambda),$$

for some $t > n/2$. In particular, $H^1_{0,\varrho}(\Omega')$ (see [**37, 124**]) coincides with the closure of $C^\infty_c(\Omega')$ with respect to the norm

$$\|w\|_\varrho := \|\nabla w\|_{L^2(\Omega',\varrho)} := \left(\int_{\Omega'} \varrho|\nabla w|^2\,dx\right)^{\frac{1}{2}}$$

and it holds that

$$\|w\|_{L^{2^*_\varrho}(\Omega')} \le C_S \|\nabla w\|_{L^2(\Omega',\varrho)} \qquad \text{for any } w \in H^1_{0,\varrho}(\Omega'),$$

where

$$\frac{1}{2^*_\varrho} := \frac{1}{2}\left(1+\frac{1}{t}\right) - \frac{1}{n}.$$

Note that

$$(5.2.17) \qquad \left(1+|\nabla u|^2+|\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}} \le K_1 + K_2|\nabla u_{\lambda_0+\tau}|^{2-p},$$

in $\Omega^+_{\lambda_0+\tau} := \Omega_{\lambda_0+\tau} \cap \operatorname{supp}(w^+_{\lambda_0+\tau})$, where K_1 and K_2 are positive constants depending only on p and on $\|u\|_{C^1(\bar\Omega_{\lambda_0+\bar\tau})}$. By Lemma 5.2.1 and (5.2.17), we deduce that

$$\frac{1}{\varrho} := \left(1+|\nabla u|^2+|\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}} \in L^t(\Omega_{\lambda_0+\tau}),$$

for some $t > n/2$ and this allows us to use the above mentioned results of [**124**]. We shall use the fact that

$$(5.2.18) \qquad \begin{aligned}(|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{2-p} &\le 2^{\frac{2-p}{2}}\left(|\nabla u|^2+|\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}}\\ &\le 2^{\frac{2-p}{2}}\left(1+|\nabla u|^2+|\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}}.\end{aligned}$$

In particular, by (5.2.18), Hölder inequality and weighted Sobolev inequality, in (5.2.16), we obtain

(5.2.19)

$$\begin{aligned}\int_{\Omega_{\lambda_0+\tau}\setminus K} \varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx &\le 2^{\frac{2-p}{2}}\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\,dx\\ &\le 2^{\frac{2-p}{2}} C K_f \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2\,dx\\ &\le 2^{\frac{2-p}{2}} C K_f |\Omega_{\lambda_0+\tau}\setminus K|^{\frac{1}{(\frac{2}{2^*_\varrho})'}}\left(\int_{\Omega_{\lambda_0+\tau}\setminus K}(w^+_{\lambda_0+\tau})^{2^*_\varrho}\,dx\right)^{\frac{2}{2^*_\varrho}}\\ &\le 2^{\frac{2-p}{2}} C K_f C_p(|\Omega_{\lambda_0+\tau}\setminus K|)\int_{\Omega_{\lambda_0+\tau}\setminus K}\varrho|\nabla w^+_{\lambda_0+\tau}|^2 dx,\end{aligned}$$

where $C_p(\cdot)$ tends to zero if the measure of the domain tends to zero. For $\bar\tau$ small and K large, we may assume that

$$2^{\frac{2-p}{2}} C K_f C_p(|\Omega_{\lambda_0+\tau}\setminus K|) < \frac{1}{2}$$

so that by (5.2.19), we deduce that

$$\int_{\Omega_{\lambda_0+\tau}} \varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx = \int_{\Omega_{\lambda_0+\tau}\setminus K} \varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx \le 0,$$

proving that $u \leq u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma)$ for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. Such a contradiction shows that

$$\lambda_0 = 0.$$

Since the moving planes procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ is implicit in the moving planes procedure.

□

Proof of Theorem 5.0.2. Arguing verbatim as in the previous case up to (5.2.14), we consider

$$\varphi_\varepsilon := w^+_{\lambda_0+\tau}\psi^p_\varepsilon \chi_{\Omega_{\lambda_0+\tau}}$$

and by a density arguments, we plug it as test function in (5.0.2) and (5.1.11). Subtracting, we get

(5.2.20)

$$\begin{aligned}
&\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla w^+_{\lambda_0+\tau})\psi^p_\varepsilon \, dx \\
&+ p\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla \psi_\varepsilon)\psi^{p-1}_\varepsilon w^+_{\lambda_0+\tau} \, dx \\
&= \int_{\Omega_{\lambda_0+\tau}\setminus K} (f(u) - f(u_\lambda))w^+_{\lambda_0+\tau}\psi^p_\varepsilon \, dx.
\end{aligned}$$

Using the split

$$\begin{aligned}
A_{\lambda_0+\tau} &= \{x \in \Omega_{\lambda_0+\tau} \setminus K \;:\; |\nabla u_{\lambda_0+\tau}(x)| < \dot{C}|\nabla u(x)|\}, \\
B_{\lambda_0+\tau} &= \{x \in \Omega_{\lambda_0+\tau} \setminus K \;:\; |\nabla u_{\lambda_0+\tau}(x)| \geq \dot{C}|\nabla u(x)|\},
\end{aligned}$$

from (5.2.20), using (1.0.2),(A^2_f) and arguing as in Lemma 5.1.2, we obtain

$$\begin{aligned}
&\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\psi^p_\varepsilon \, dx \\
&\leq \delta C \int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\psi^p_\varepsilon \, dx \\
&\quad + \frac{C}{\delta}\left(\int_{\Omega_{\lambda_0+\tau}\setminus K} |\nabla u|^p \, dx\right)^{\frac{p-2}{p}} \left(\int_{\Omega_{\lambda_0+\tau}\setminus K} |\nabla \psi_\varepsilon|^p \, dx\right)^{\frac{2}{p}} \\
&\quad + K_f \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2\psi^p_\varepsilon \, dx,
\end{aligned}$$

for some positive constant $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_{\lambda+\bar{\tau}})})$. As we did above passing to the limit for $\varepsilon \to 0$, by Fatou's Lemma we obtain

(5.2.21)

$$\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2 \, dx \leq CK_f \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2 \, dx.$$

In this case we have $|\nabla u|^{p-2} \leq (|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{p-2}$ since $p > 2$. Then we set $\varrho := |\nabla u|^{p-2}$ and we see that ϱ is bounded in $\Omega_{\lambda_0+\tau}$, hence $\varrho \in L^1(\Omega_{\lambda_0+\tau})$.

By applying the weighted Poincaré inequality to (5.2.21), see [37, Theorem 1.2], we deduce that

(5.2.22)

$$\begin{aligned}\int_{\Omega_{\lambda_0+\tau}\setminus K} \varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx &\le \int_{\Omega_{\lambda_0+\tau}\setminus K}(|\nabla u|+|\nabla u_{\lambda_0+\tau}|)^{p-2}|\nabla w^+_{\lambda_0+\tau}|^2\,dx\\ &\le CK_f\int_{\Omega_{\lambda_0+\tau}\setminus K}(w^+_{\lambda_0+\tau})^2\,dx\\ &\le CK_fC_p(|\Omega_{\lambda_0+\tau}\setminus K|)\int_{\Omega_{\lambda_0+\tau}\setminus K}\varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx\end{aligned}$$

where $C_p(\cdot)$ tends to zero if the measure of the domain tends to zero. For $\bar\tau$ small and K large, we may assume that

$$CK_fC_p(|\Omega_{\lambda_0+\tau}\setminus K|)<\frac{1}{2}$$

so that by (5.2.22), we deduce that

$$\int_{\Omega_{\lambda_0+\tau}}\varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx=\int_{\Omega_{\lambda_0+\tau}\setminus K}\varrho|\nabla w^+_{\lambda_0+\tau}|^2\,dx\le 0,$$

proving that $u\le u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau}\setminus R_{\lambda_0+\tau}(\Gamma)$ for any $0<\tau<\bar\tau$ for some small $\bar\tau>0$. Such a contradiction shows that

$$\lambda_0=0.$$

Since the moving planes procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1<0\}$ is implicit in the moving planes procedure.

□

6

Symmetry and monotonicity properties of singular solutions to some cooperative semilinear elliptic systems involving critical nonlinearities

The aim of this chapter is to investigate symmetry and monotonicity properties of singular solutions to some semilinear elliptic systems in such a way to find a generalization of the results presented in Chapter 4. In the first part we consider the following semilinear elliptic system

$$\begin{cases} -\Delta u_i = f_i(u_1,\dots,u_m) & \text{in} \quad \Omega\setminus\Gamma \\ u_i > 0 & \text{in} \quad \Omega\setminus\Gamma \\ u_i = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{6.0.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$ and $i = 1,...,m$ $(m \geq 2)$.

Motivated by [83], through all the chapter, we assume that the following hypotheses (denoted by (S_{f_i}) in the sequel) hold:

(S_{f_i}) (i) $f_i : \mathbb{R}^m_+ \to \mathbb{R}$ are assumed to be $\mathcal{C}^1$ functions for every $i = 1,...,m$.

(ii) The functions f_i $(1 \leq i \leq m)$ are assumed to satisfy the monotonicity (also known as *cooperative*) conditions

$$\frac{\partial f_i}{\partial t_j}(t_1,...,t_j,...,t_m) \geq 0 \quad \text{for} \quad i \neq j,\ 1 \leq i,j \leq m.$$

In this chapter the case of singular nonlinearities for systems is not included, while it was considered in the case of scalar equations, see [50] or chapter 4; about these problems we have also to mention the pioneering work of Crandall, Rabinowitz and Tartar [31] and also [17, 24, 52, 82, 120] for the scalar case. It would be interesting to consider in future projects a more general class of nonlinearities. Since we want to consider singular solutions, the natural assumption is

$$u_i \in H^1_{loc}(\Omega\setminus\Gamma) \cap C(\overline{\Omega}\setminus\Gamma) \quad \forall i = 1,...,m$$

and thus the system is understood in the following sense:

$$\int_\Omega (\nabla u_i, \nabla\varphi_i)\,dx = \int_\Omega f_i(u_1,u_2,...,u_m)\varphi_i\,dx \qquad \forall\varphi_i \in C^1_c(\Omega\setminus\Gamma) \tag{6.0.2}$$

for every $i = 1,...,m$.

REMARK 6.0.1. *Note that, by the assumption (S_{f_i}), the right hand side in the system (6.0.2) is locally bounded. Therefore, by standard elliptic regularity theory, it follows that*

$$u_i \in C^{1,\alpha}_{loc}(\Omega\setminus\Gamma),$$

where $0 < \alpha < 1$. We remark that, in 1968, E. De Giorgi provided a counterexample showing that the scalar case is special and the regularity theory does not work in general for elliptic systems, see [44], but in the case of equations involving Laplace operator, Schauder theory is still applicable.

Under the previous assumptions we can prove the following result:

THEOREM 6.0.2. *Let Ω be a convex domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and let $(u_1, ..., u_m)$ be a solution to (6.0.1), where $u_i \in H^1_{loc}(\Omega \setminus \Gamma) \cap C(\overline{\Omega} \setminus \Gamma)$ for every $i = 1, ..., m$. Assume that each f_i fulfills (S_{f_i}). Assume also that Γ is a point if $N = 2$ while Γ is closed and such that*

$$\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0,$$

if $N \geq 3$. Then, if $\Gamma \subset \{x_1 = 0\}$, it follows that u_i is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$, for every $i = 1, ..., m$. Furthermore

$$\partial_{x_1} u_i > 0 \qquad in \quad \Omega \cap \{x_1 < 0\},$$

for every $i = 1, ..., m$.

The technique developed in the case of bounded domains, see [**50, 51, 110**] (see also [**91**] for the nonlocal setting) is very powerful and can be adapted to some cooperative systems in $\mathbb{R}^N$ involving critical nonlinearity. Our aim is to study qualitative properties of singular solutions to the following $m \times m$ system of equations

$$\begin{cases} -\Delta u_i = \displaystyle\sum_{j=1}^{m} a_{ij} u_j^{2^*-1} & \text{in} \quad \mathbb{R}^N \setminus \Gamma, \\ u_i > 0 & \text{in} \quad \mathbb{R}^N \setminus \Gamma, \end{cases} \tag{6.0.3}$$

where $i = 1, ..., m$, $m \geq 2$, $N \geq 3$ and the matrix $A := (a_{ij})_{i,j=1,...,m}$ is symmetric and such that

$$\sum_{j=1}^{m} a_{ij} = 1 \text{ for every } i = 1, ..., m. \tag{6.0.4}$$

This kind of system, with $\Gamma = \emptyset$, was studied by Mitidieri in [**89, 90**] considering the case $m = 2$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and it is known in the literature as nonlinearity belonging to the *critical hyperbola*.

As remarked above the natural assumption is

$$u_i \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma) \quad \forall i = 1, ..., m$$

and thus the system is understood in the following sense:

$$\int_{\mathbb{R}^N} (\nabla u_i, \nabla \varphi_i)\, dx = \sum_{j=1}^{m} a_{ij} \int_{\mathbb{R}^N} u_j^{2^*-1} \varphi_i \, dx \qquad \forall \varphi_i \in C^1_c(\mathbb{R}^N \setminus \Gamma) \tag{6.0.5}$$

for every $i = 1, ..., m$.

What we are going to show is the following result:

THEOREM 6.0.3. *Let $N \geq 3$ and let $(u_1, ..., u_m)$ be a solution to (6.0.3), where $u_i \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ for every $i = 1, ..., m$. Assume that the matrix $A = (a_{ij})_{i,j=1,...,m}$, defined above, is symmetric, $a_{ij} \geq 0$ for every $i, j = 1, ..., m$ and it satisfies (6.0.4). Moreover at least one of u_i has a non-removable*[1] *singularity in the singular set Γ, where Γ is a closed and proper subset of $\{x_1 = 0\}$ such that*

$$\operatorname*{Cap_2}_{\mathbb{R}^N}(\Gamma) = 0.$$

Then, all u_i are symmetric with respect to the hyperplane $\{x_1 = 0\}$. The same conclusion is true if $\{x_1 = 0\}$ is replaced by any affine hyperplane. If at least one of u_i has only a non-removable singularity at the origin for every $i = 1, ..., m$, then each u_i is radially symmetric about the origin and radially decreasing.

Another interesting elliptic system involving Sobolev critical exponents is the following one:

$$\begin{cases} -\Delta u = u^{2^*-1} + \dfrac{\alpha}{2^*} u^{\alpha-1} v^{\beta} & \text{in} \quad \mathbb{R}^N \setminus \Gamma \\ -\Delta v = v^{2^*-1} + \dfrac{\beta}{2^*} u^{\alpha} v^{\beta-1} & \text{in} \quad \mathbb{R}^N \setminus \Gamma \\ u, v > 0 & \text{in} \quad \mathbb{R}^N \setminus \Gamma, \end{cases} \tag{6.0.6}$$

where $\alpha, \beta > 1$, $\alpha + \beta = 2^* := \frac{2N}{N-2}$ $(N \geq 3)$

The solutions to (6.0.6) are solitary waves for a system of coupled Gross–Pitaevskii equations.

We prove the following:

THEOREM 6.0.4. *Let $N = 3$ or $N = 4$ and let $(u, v) \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma) \times H^1_{loc}(\mathbb{R}^N \setminus \Gamma)$ be a solution to (6.0.6). Assume that the solution (u, v) has a non-removable*[2] *singularity in the singular set Γ, where Γ is a closed and proper subset of $\{x_1 = 0\}$ such that*

$$\operatorname*{Cap_2}_{\mathbb{R}^N}(\Gamma) = 0.$$

Moreover let us assume that $\alpha, \beta \geq 2$ and that holds $\alpha + \beta = 2^*$. Then, u and v are symmetric with respect to the hyperplane $\{x_1 = 0\}$. The same conclusion is true if $\{x_1 = 0\}$ is replaced by any affine hyperplane. If at least one between u and v have only a non-removable singularity at the origin, then (u, v) is radially symmetric about the origin and radially decreasing.

6.1. Notations

We need to fix some notations and since they are similar to the ones introduced in the previous chapters, we recall someone of them for the reader's

[1] Here we mean that the solution $(u_1, ..., u_m)$ does not admit a smooth extension all over the whole space. Namely it is not possible to find $\tilde{u}_i \in H^1_{loc}(\mathbb{R}^N)$ with $u_i \equiv \tilde{u}_i$ in $\mathbb{R}^N \setminus \Gamma$, for some $i = 1, ..., m$.

[2] As above, we mean that the solution (u, v) does not admit a smooth extension all over the whole space. Namely it is not possible to find $(\tilde{u}, \tilde{v}) \in H^1_{loc}(\mathbb{R}^N) \times H^1_{loc}(\mathbb{R}^N)$ with $u \equiv \tilde{u}$ or $v \equiv \tilde{v}$ in $\mathbb{R}^N \setminus \Gamma$.

convenience. For a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\} \tag{6.1.1}$$

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \dots, x_N) \tag{6.1.2}$$

which is the reflection through the hyperplane $T_\lambda := \{x_1 = \lambda\}$. Also let

$$a = \inf_{x\in\Omega} x_1. \tag{6.1.3}$$

We define the functions

$$u_{i,\lambda} := u_i \circ R_\lambda \tag{6.1.4}$$

and we recall that they are Lebesgue measurable on $R_\lambda(\Omega)$. Similarly, ∇u_i and $\nabla u_{i,\lambda}$ are Lebesgue measurable on Ω and $R_\lambda(\Omega)$ respectively.

Recalling that Γ is a point if $N = 2$ while Γ is closed with $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\Gamma) = 0$ if $N \geq 3$ by assumption, it follows that

$$\underset{\mathcal{B}^\lambda_\epsilon}{\mathrm{Cap}_2}(R_\lambda(\Gamma)) := \inf\left\{\int_{\mathcal{B}^\lambda_\epsilon} |\nabla\varphi|^2 dx < +\infty \ : \ \varphi \geq 1 \text{ in } \mathcal{B}^\lambda_\delta,\ \varphi \in\ C^\infty_c(\mathcal{B}^\lambda_\epsilon)\right\} = 0,$$

for some neighborhood $\mathcal{B}^\lambda_\delta \subset \mathcal{B}^\lambda_\varepsilon$ of $R_\lambda(\Gamma)$. From this, it follows that there exists $\varphi_\varepsilon \in\ C^\infty_c(\mathcal{B}^\lambda_\epsilon)$ such that $\varphi_\varepsilon \geq 1$ in $\mathcal{B}^\lambda_\delta$ and $\int_{\mathcal{B}^\lambda_\epsilon} |\nabla\varphi_\varepsilon|^2 dx < \varepsilon$.

Now we construct a function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N, [0,1])$ such that $\psi_\varepsilon = 1$ outside $\mathcal{B}^\lambda_\varepsilon$, $\psi_\varepsilon = 0$ in $\mathcal{B}^\lambda_\delta$ and

$$\int_{\mathbb{R}^N} |\nabla\psi_\varepsilon|^2 dx = \int_{\mathcal{B}^\lambda_\epsilon} |\nabla\psi_\varepsilon|^2 dx < 4\varepsilon.$$

To this end we consider the following Lipschitz continuous function

$$T_1(s) = \begin{cases} 1 & \text{if} \quad s \leq 0 \\ -2s+1 & \text{if} \quad 0 \leq s \leq \frac{1}{2} \\ 0 & \text{if} \quad s \geq \frac{1}{2} \end{cases}$$

and we set

$$\psi_\varepsilon := T_1 \circ \varphi_\varepsilon \tag{6.1.5}$$

where we have extended φ_ε by zero outside $\mathcal{B}^\lambda_\varepsilon$ (see Figure 1). Clearly $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N)$, $0 \leq \psi_\varepsilon \leq 1$ and

$$\int_{\mathcal{B}^\lambda_\epsilon} |\nabla\psi_\varepsilon|^2 dx \leq 4 \int_{\mathcal{B}^\lambda_\epsilon} |\nabla\varphi_\varepsilon|^2 dx < 4\varepsilon.$$

Now we set $\gamma_\lambda := \partial\Omega \cap T_\lambda$. Hence $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(\gamma_\lambda) = 0$, see e.g. [53] and Chapter 4. So, as in Chapter 4, $\underset{\mathcal{I}^\lambda_\tau}{\mathrm{Cap}_2}(\gamma_\lambda) = 0$ for any open neighborhood of γ_λ and then there exists $\varphi_\tau \in C^\infty_c(\mathcal{I}^\lambda_\tau)$ such that $\varphi_\tau \geq 1$ in a neighborhood $\mathcal{I}^\lambda_\sigma$ with $\gamma_\lambda \subset \mathcal{I}^\lambda_\sigma \subset \mathcal{I}^\lambda_\tau$. As above, we set

$$\phi_\tau := T_1 \circ \varphi_\tau \tag{6.1.6}$$

where we have extended φ_τ by zero outside $\mathcal{I}^\lambda_\tau$ (see Figure 2). Then, $\phi_\tau \in C^{0,1}(\mathbb{R}^N), 0 \leq \phi_\tau \leq 1, \phi_\tau = 1$ outside $\mathcal{I}^\lambda_\tau, \phi_\tau = 0$ in $\mathcal{I}^\lambda_\sigma$ and

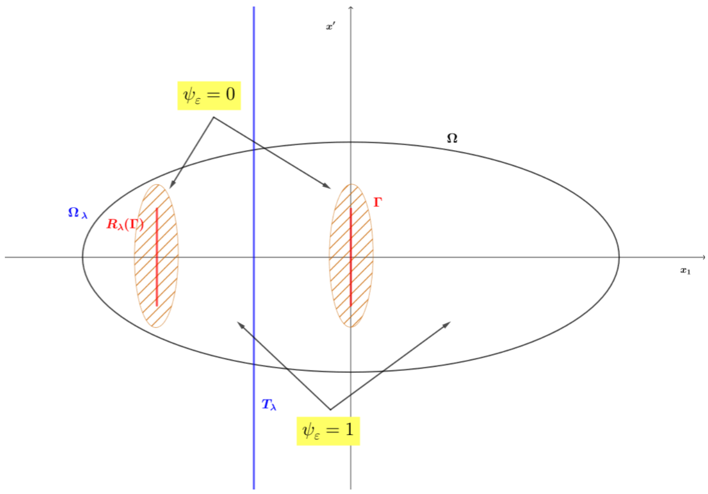

FIGURE 1. The cutoff function ψ_ε.

$$\int_{\mathbb{R}^N} |\nabla \phi_\tau|^2 dx = \int_{\mathcal{I}_\tau^\lambda} |\nabla \phi_\tau|^2 dx \leq 4 \int_{\mathcal{I}_\tau^\lambda} |\nabla \varphi_\tau|^2 dx < 4\tau.$$

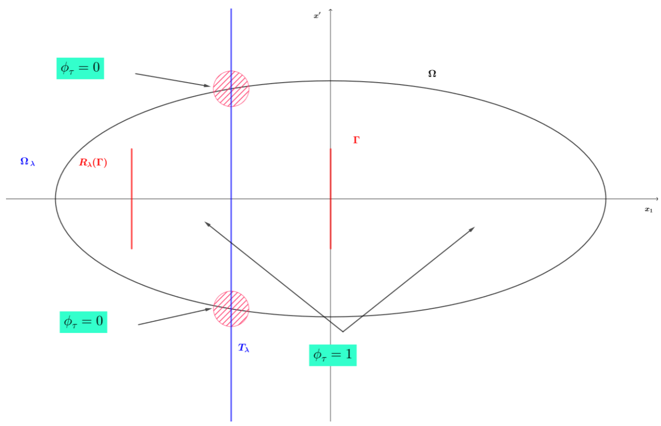

FIGURE 2. The cutoff function ϕ_τ.

6.2. Symmetry and monotonicity results in bounded domains

Let us set

$$w_{i,\lambda}^+ = (u_i - u_{i,\lambda})^+$$

where $i = 1, ..., m$. We will prove the result by showing that, actually, it holds $w_{i,\lambda}^+ \equiv 0$ for $i = 1, ..., m$. To prove this, we have to perform the moving planes method.

In the following we will exploit the fact that $(u_{1,\lambda}, ..., u_{1,\lambda})$ is a solution to

(6.2.1)

$$\int_{\Omega_\lambda} (\nabla u_{i,\lambda}, \nabla \varphi_i)\, dx = \int_{\Omega_\lambda} f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})\varphi_i\, dx \quad \forall \varphi_i \in C_c^1(\Omega_\lambda \backslash R_\lambda(\Gamma))$$

for every $i = 1, ..., m$, where $\Omega_\lambda := R_\lambda(\Omega)$.

Now we are ready to prove an essential tool that we will use to start the moving planes procedure.

Lemma 6.2.1. *Under the assumptions of Theorem 6.0.2, let $a < \lambda < 0$. Then $w^+_{i,\lambda} \in H^1_0(\Omega_\lambda)$ for every $i = 1, ..., m$ and*

$$\sum_{i=1}^m \int_{\Omega_\lambda} |\nabla w^+_{i,\lambda}|^2 \, dx \le \frac{m}{2} \sum_{i=1}^m (1 + C_i^2) \|u_i\|^2_{L^\infty(\Omega_\lambda)} |\Omega| \,. \tag{6.2.2}$$

where $|\Omega|$ denotes the $N-$dimensional Lebesgue measure of Ω and C_i is a positive constant only depending on f_i.

Proof. For ψ_ε as in (6.1.5) and ϕ_τ as in (6.1.6), we consider the functions

$$\varphi_i := \begin{cases} w^+_{i,\lambda} \psi_\varepsilon^2 \phi_\tau^2 & \text{in} \quad \Omega_\lambda, \\ 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega_\lambda, \end{cases}$$

In view of the properties of φ_i, stated in Lemma 4.2.1, and a standard density argument, we can use φ_i as test function in (6.0.2) and (6.2.1) so that, subtracting, we get

(6.2.3)
$$\begin{aligned} \int_{\Omega_\lambda} |\nabla w^+_{i,\lambda}|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx &= -2 \int_{\Omega_\lambda} (\nabla w^+_{i,\lambda}, \nabla \psi_\varepsilon) w^+_{i,\lambda} \psi_\varepsilon \phi_\tau^2 \, dx \\ &\quad - 2 \int_{\Omega_\lambda} (\nabla w^+_{i,\lambda}, \nabla \phi_\tau w^+_{i,\lambda}) \psi_\varepsilon^2 \phi_\tau \, dx \\ &\quad + \int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w^+_{i,\lambda} \psi_\varepsilon^2 \phi_\tau^2 \, dx. \end{aligned}$$

Exploiting Young's inequality in the right hand side of (6.2.3), we get that

(6.2.4)
$$\begin{aligned} \int_{\Omega_\lambda} |\nabla w^+_{i,\lambda}|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx &\le \frac{1}{4} \int_{\Omega_\lambda} |\nabla w^+_{i,\lambda}|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx + 4 \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 (w^+_{i,\lambda})^2 \phi_\tau^2 \, dx \\ &\quad + \frac{1}{4} \int_{\Omega_\lambda} |\nabla w^+_{i,\lambda}|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx + 4 \int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w^+_{i,\lambda})^2 \psi_\varepsilon^2 \, dx \\ &\quad + \int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w^+_{i,\lambda} \psi_\varepsilon^2 \phi_\tau^2 \, dx. \end{aligned}$$

The last term of the right hand side of (6.2.4) can be rewritten as follows

(6.2.5)
$$\begin{aligned}
&\int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx \\
&= \int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_2, ..., u_m) \\
&\qquad + f_i(u_{1,\lambda}, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx \\
&= \int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_2, ..., u_m) \\
&\qquad + f_i(u_{1,\lambda}, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, u_3, ..., u_m) \\
&\qquad + f_i(u_{1,\lambda}, u_{2,\lambda}, u_3, ..., u_m) - \cdots - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx.
\end{aligned}$$

Using the fact that f_i are $\mathcal{C}^1$ functions $(S_{f_i})-(i)$ and they satisfy $(S_{f_i})-(ii)$, by (6.2.5) we have

$$\begin{aligned}
&\int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx \\
&\qquad \leq \sum_{j=1}^m C_j(f_j) \int_{\Omega_\lambda} w_{j,\lambda}^+ w_{i,\lambda}^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx.
\end{aligned} \tag{6.2.6}$$

Now compiling all the previous estimates and exploiting Young's inequality in the right hand side of 6.2.6 we obtain

(6.2.7)
$$\begin{aligned}
\int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx &\leq 8 \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 (w_{i,\lambda}^+)^2 \phi_\tau^2 \, dx + 8 \int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_{i,\lambda}^+)^2 \psi_\varepsilon^2 \, dx \\
&\quad + m \int_{\Omega_\lambda} (w_{i,\lambda}^+)^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx + \sum_{j=1}^m C_j^2 \int_{\Omega_\lambda} (w_{j,\lambda}^+)^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx.
\end{aligned}$$

Adding all the equations of (6.0.3) we get

$$\begin{aligned}
\sum_{i=1}^m \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx &\leq 8 \sum_{i=1}^m \Big(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 (w_{i,\lambda}^+)^2 \phi_\tau^2 \, dx \\
&\qquad + \int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_{i,\lambda}^+)^2 \psi_\varepsilon^2 \, dx \Big) \\
&\quad + \frac{m}{2} \sum_{i=1}^m (1 + C_i^2) \int_{\Omega_\lambda} (w_{i,\lambda}^+)^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx.
\end{aligned} \tag{6.2.8}$$

Taking into account the properties of ψ_ε and ϕ_τ, we see that

$$\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^2 \, dx = \int_{\Omega_\lambda \cap (\mathcal{B}_\varepsilon^\lambda \setminus \mathcal{B}_\delta^\lambda)} |\nabla \psi_\varepsilon|^2 \, dx < 4\varepsilon, \tag{6.2.9}$$

$$\int_{\Omega_\lambda} |\nabla \phi_\tau|^2 \, dx = \int_{\Omega_\lambda \cap (\mathcal{I}_\tau^\lambda \setminus \mathcal{I}_\sigma^\lambda)} |\nabla \phi_\tau|^2 \, dx < 4\tau, \tag{6.2.10}$$

which combined with $0 \le w_{i,\lambda}^+ \le u_i$, for every $i = 1, ..., m$, immediately lead to

$$\sum_{i=1}^{m} \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx \le 32(\varepsilon + \tau) \sum_{i=1}^{m} \|u_i\|^2_{L^\infty(\Omega_\lambda)} + \frac{m}{2} \sum_{i=1}^{m} (1 + C_i^2) \|u_i\|^2_{L^\infty(\Omega_\lambda)} |\Omega| \, .$$

By Fatou's Lemma, as ε and τ tend to zero, we have (6.2.2). To conclude we note that $\varphi_i \to w_{i,\lambda}^+$ in $L^2(\Omega)$, as ε and τ tend to zero, by definition of φ_i for every $i = 1, ..., m$. Also, $\nabla\varphi \to \nabla w_{i,\lambda}^+$ in $L^2(\Omega_\lambda)$, by (4.2.3). Therefore, $w_{i,\lambda}^+$ in $H_0^1(\Omega_\lambda)$, since $\varphi_i \in H_0^1(\Omega_\lambda)$ again by Lemma 4.2.1, for every $i = 1, ..., m$, which concludes the proof.

□

Proof of Theorem 6.0.2. We define

$$\Lambda_0 = \{a < \lambda < 0 : u_i \le u_{i,t} \text{ in } \Omega_t \setminus R_t(\Gamma) \text{ for all } t \in (a, \lambda] \text{ and i=1,...,m}\}$$

and to start with the moving planes procedure, we have to prove that

Step 1 : $\Lambda_0 \neq \emptyset$. Fix $\lambda_0 \in (a, 0)$ such that $R_{\lambda_0}(\Gamma) \subset \Omega^c$, then for every $a < \lambda < \lambda_0$, we also have that $R_\lambda(\Gamma) \subset \Omega^c$. For any λ in this set we consider, on the domain Ω, the function $\varphi_i := w_{i,\lambda}^+ \phi_\tau^2 \chi_{\Omega_\lambda}$, where ϕ_τ is as in (6.1.6) and we proceed as in the proof of Lemma 6.2.1. That is, by Lemma 4.2.1 and a density argument, we can use φ_i as test function in (6.0.2) and (6.2.1) so that, subtracting, we get

(6.2.11)

$$\int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \phi_\tau^2 \, dx = -2 \int_{\Omega_\lambda} \nabla w_{i,\lambda}^+ \nabla \phi_\tau w_{i,\lambda}^+ \phi_\tau \, dx + \int_{\Omega_\lambda} [f_i(u_1, u_2, ..., u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, ..., u_{m,\lambda})] w_{i,\lambda}^+ \phi_\tau^2 \, dx.$$

Exploiting Young's inequality and the assumption (S_{f_i}), then we get that

$$\int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \phi_\tau^2 \, dx \le \frac{1}{2} \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \phi_\tau^2 \, dx + 2 \int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_{i,\lambda}^+)^2 dx + \sum_{i=1}^{m} C_j \int_{\Omega_\lambda} w_{j,\lambda}^+ w_{i,\lambda}^+ \phi_\tau^2 \, dx.$$

Taking into account the properties of ϕ_τ, we see that

(6.2.12)

$$\int_{\Omega_\lambda} |\nabla \phi_\tau|^2 (w_{i,\lambda}^+)^2 dx \le \|u\|^2_{L^\infty(\Omega_\lambda)} \int_{\Omega_\lambda \cap (\mathcal{I}_\tau^\lambda \setminus \mathcal{I}_\sigma^\lambda)} |\nabla \phi_\tau|^2 \, dx \le 4 \|u\|^2_{L^\infty(\Omega_\lambda)} \cdot \tau.$$

We therefore deduce that

$$\sum_{i=1}^{m} \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \phi_\tau^2 \, dx \le 16\tau \sum_{i=1}^{m} \|u_i\|_{L^\infty(\Omega_\lambda)} + \frac{m}{2} \sum_{i=1}^{m} (1 + C_i^2) \int_{\Omega_\lambda} (w_{i,\lambda}^+)^2 \phi_\tau^2 \, dx.$$

By Fatou's Lemma, as τ tends to zero, we have

$$\sum_{i=1}^{m}\int_{\Omega_\lambda}|\nabla w_{i,\lambda}^+|^2\,dx \le \frac{m}{2}\sum_{i=1}^{m}(1+C_i^2)\int_{\Omega_\lambda}(w_{i,\lambda}^+)^2\,dx \tag{6.2.13}$$
$$\le \frac{m}{2}\sum_{i=1}^{m}(1+C_i^2)C_{i,p}^2(\Omega_\lambda)\int_{\Omega_\lambda}|\nabla w_{i,\lambda}^+|^2\,dx,$$

where $C_{i,p}(\cdot)$ is the Poincaré constant (in the Poincaré inequality in $H_0^1(\Omega_\lambda)$). Since $C_{i,p}(\Omega_\lambda)\to 0$ as $\lambda\to a$, we can find $\lambda_1\in(a,\lambda_0)$, such that

$$C_{i,p}(\Omega_\lambda)<\frac{1}{\sqrt{m(1+C_i^2)}}\qquad \forall\lambda\in(a,\lambda_1)\text{ and for every } i=1,...,m,$$

so that by (6.2.13), we deduce that

$$\int_{\Omega_\lambda}|\nabla w_{i,\lambda}^+|^2\,dx\le 0\qquad \forall\lambda\in(a,\lambda_1)\text{ and for every } i=1,...,m,$$

proving that $u_i\le u_{i,\lambda}$ in $\Omega_\lambda\setminus R_\lambda(\Gamma)$ for λ close to a, which implies the desired conclusion $\Lambda_0\neq\emptyset$.

Now we can set

$$\lambda_0=\sup\Lambda_0.$$

Step 2: here we show that $\lambda_0=0$. To this end we assume that $\lambda_0<0$ and we reach a contradiction by proving that $u_i\le u_{i,\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu}\setminus R_{\lambda_0+\nu}(\Gamma)$ for any $0<\nu<\bar\nu$ for some small $\bar\nu>0$ and for every $i=1,...,m$. By continuity we know that $u_i\le u_{i,\lambda_0}$ in $\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ for every $i=1,...,m$. Since Ω is convex in the x_1-direction and the set $R_{\lambda_0}(\Gamma)$ lies in the hyperplane of equation $\{x_1=-2\lambda_0\}$, we see that $\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ is open and connected. Moreover, using $(S_{f_i})-(ii)$ we have that

$$\begin{aligned}-\Delta(u_i-u_{i,\lambda_0})&=f(u_1,...,u_m)-f(u_{1,\lambda_0},...,u_{m,\lambda_0})\\&=[f(u_1,...,u_m)-f(u_{1,\lambda_0},...,u_m)]+\cdots\\&\cdots+[f(u_{1,\lambda_0},...,u_m)-f(u_{1,\lambda_0},...,u_{m,\lambda_0})]\le 0.\end{aligned}$$

Therefore, by the strong maximum principle we deduce that $u_i<u_{i,\lambda_0}$ in $\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ and for every $i=1,...,m$.

Now, note that for $K\subset\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$, there is $\nu=\nu(K,\lambda_0)>0$, sufficiently small, such that $K\subset\Omega_\lambda\setminus R_\lambda(\Gamma)$ for every $\lambda\in[\lambda_0,\lambda_0+\nu]$. Consequently u_i and $u_{i,\lambda}$ are well defined on K for every $\lambda\in[\lambda_0,\lambda_0+\nu]$ and for every $i=1,...,m$. Hence, by the uniform continuity of the functions $g_i(x,\lambda):=u_i(x)-u_i(2\lambda-x_1,x')$ on the compact set $K\times[\lambda_0,\lambda_0+\nu]$ we can ensure that $K\subset\Omega_{\lambda_0+\nu}\setminus R_{\lambda_0+\nu}(\Gamma)$ and $u_i<u_{i,\lambda_0+\nu}$ in K for any $0\le\nu<\bar\nu$, for some $\bar\nu=\bar\nu(K,\lambda_0)>0$ small. Clearly we can also assume that $\bar\nu<\frac{|\lambda_0|}{4}$.

Let us consider ψ_ε constructed in such a way that it vanishes in a neighborhood of $R_{\lambda_0+\nu}(\Gamma)$ and ϕ_τ constructed in such a way it vanishes in a neighborhood of $\gamma_{\lambda_0+\nu}=\partial\Omega\cap T_{\lambda_0+\nu}$. As shown in the proof of Lemma 6.2.1, the functions

$$\varphi_i:=\begin{cases}w_{i,\lambda_0+\nu}^+\psi_\varepsilon^2\phi_\tau^2 & \text{in}\quad \Omega_{\lambda_0+\nu}\\ 0 & \text{in}\quad \mathbb{R}^N\setminus\Omega_{\lambda_0+\nu}\end{cases}$$

are such that $\varphi_i \to w^+_{i,\lambda_0+\nu}$ in $H^1_0(\Omega_{\lambda_0+\nu})$, as ε and τ tend to zero. Moreover, $\varphi_i \in C^{0,1}(\overline{\Omega_{\lambda_0+\nu}})$ and $\varphi_{i|_{\partial\Omega_{\lambda_0+\nu}}} = 0$, by Lemma 4.2.1, and $\varphi_i = 0$ on an open neighborhood of K, by the above argument. Therefore, $\varphi_i \in H^1_0(\Omega_{\lambda_0+\nu} \setminus K)$ and thus, also $w^+_{i,\lambda_0+\nu}$ belongs to $H^1_0(\Omega_{\lambda_0+\nu} \setminus K)$. We also note that $\nabla w^+_{i,\lambda_0+\nu} = 0$ on an open neighborhood of K.
Now we argue as in Lemma 6.2.1 and we plug φ_i as test function in (6.0.2) and (6.2.1) so that, by subtracting, we get

(6.2.14)
$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}} |\nabla w^+_{i,\lambda_0+\nu}|^2 \psi^2_\varepsilon \phi^2_\tau \, dx = &-2 \int_{\Omega_{\lambda_0+\nu}} (\nabla w^+_{i,\lambda_0+\nu}, \nabla \psi_\varepsilon) w^+_{i,\lambda_0+\nu} \psi_\varepsilon \phi^2_\tau \, dx \\ &- 2 \int_{\Omega_{\lambda_0+\nu}} (\nabla w^+_{i,\lambda_0+\nu}, \nabla \phi_\tau) w^+_{i,\lambda_0+\nu} \psi^2_\varepsilon \phi_\tau \, dx \\ &+ \int_{\Omega_{\lambda_0+\nu}} [f_i(u_1, u_2, ..., u_m) \\ &\quad - f_i(u_{1,\lambda_0+\nu}, u_{2,\lambda_0+\nu}, ..., u_{m,\lambda_0+\nu})] \, w^+_{i,\lambda_0+\nu} \psi^2_\varepsilon \phi^2_\tau \, dx.\end{aligned}$$

Therefore, taking into account the properties of $w^+_{\lambda_0+\nu}$ and $\nabla w^+_{\lambda_0+\nu}$ we also have

$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}|^2 \psi^2_\varepsilon \phi^2_\tau \, dx \le &-2 \int_{\Omega_{\lambda_0+\nu}\setminus K} (\nabla w^+_{i,\lambda_0+\nu}, \nabla \psi_\varepsilon) w^+_{i,\lambda_0+\nu} \psi_\varepsilon \phi^2_\tau \, dx \\ &- 2 \int_{\Omega_{\lambda_0+\nu}\setminus K} (\nabla w^+_{i,\lambda_0+\nu}, \nabla \phi_\tau) w^+_{i,\lambda_0+\nu} \psi^2_\varepsilon \phi_\tau \, dx \\ &+ \int_{\Omega_{\lambda_0+\nu}\setminus K} [f_i(u_1, u_2, ..., u_m) - \\ &\quad - f_i(u_{1,\lambda_0+\nu}, u_{2,\lambda_0+\nu}, ..., u_{m,\lambda_0+\nu})] w^+_{i,\lambda_0+\nu} \psi^2_\varepsilon \phi^2_\tau \, dx.\end{aligned}$$

Furthermore, since f_i are $\mathcal{C}^1$ functions, we deduce that

(6.2.15)
$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}|^2 \psi^2_\varepsilon \phi^2_\tau \, dx \le &\, 2 \int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}||\nabla \psi_\varepsilon| w^+_{i,\lambda_0+\nu} \psi_\varepsilon \phi^2_\tau \, dx \\ &+ 2 \int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}||\nabla \phi_\tau| w^+_{i,\lambda_0+\nu} \psi^2_\varepsilon \phi_\tau \, dx \\ &+ \sum_{j=1}^m C_j(f_i) \int_{\Omega_{\lambda_0+\nu}\setminus K} w^+_{j,\lambda_0+\nu} w^+_{i,\lambda_0+\nu} \psi^2_\varepsilon \phi^2_\tau \, dx.\end{aligned}$$

Now, as in the proof of Lemma 6.2.1, we use Young's inequality to deduce that

(6.2.16)
$$\begin{aligned}\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{i,\lambda_0+\nu}|^2\psi^2_\varepsilon\phi^2_\tau\,dx &\le 8\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla\psi_\varepsilon|^2(w^+_{i,\lambda_0+\nu})^2\phi^2_\tau\,dx\\ &+8\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla\phi_\tau|^2(w^+_{i,\lambda_0+\nu})^2\psi^2_\varepsilon\,dx\\ &+\sum_{j=1}^m C_j\int_{\Omega_{\lambda_0+\nu}\setminus K}w^+_{j,\lambda_0+\nu}w^+_{i,\lambda_0+\nu}\psi^2_\varepsilon\phi^2_\tau\,dx,\end{aligned}$$

which in turns yields

(6.2.17)
$$\begin{aligned}\sum_{i=1}^m\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{i,\lambda_0+\nu}|^2\psi^2_\varepsilon\phi^2_\tau\,dx &\le 32(\epsilon+\tau)\sum_{i=1}^m\|u\|^2_{L^\infty(\Omega_{\lambda_0+\bar\nu})}\\ &+\frac{m}{2}\sum_{i=1}^m(1+C_i^2)\int_{\Omega_{\lambda_0+\nu}\setminus K}(w^+_{i,\lambda_0+\nu})^2\psi^2_\varepsilon\phi^2_\tau\,dx.\end{aligned}$$

Passing to the limit, as $(\epsilon,\tau)\to(0,0)$, in the latter we get

(6.2.18)
$$\begin{aligned}\sum_{i=1}^m\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{i,\lambda_0+\nu}|^2\,dx &\le \frac{m}{2}\sum_{i=1}^m(1+C_i^2)\int_{\Omega_{\lambda_0+\nu}\setminus K}(w^+_{i,\lambda_0+\nu})^2\,dx.\\ \le \frac{m}{2}\sum_{i=1}^m(1+C_i^2)C^2_{i,p}(\Omega_{\lambda_0+\nu}\setminus K)&\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{i,\lambda_0+\nu}|^2\,dx\,,\end{aligned}$$

where $C_{i,p}(\cdot)$ are the Poincaré constants (in the Poincaré inequalities in $H^1_0(\Omega_{\lambda_0+\nu}\setminus K)$). Now we recall that $C^2_{i,p}(\Omega_{\lambda_0+\nu}\setminus K)\le Q(n)|\Omega_{\lambda_0+\nu}\setminus K|^{\frac{2}{N}}$ for every $i=1,...,m$, where $Q=Q(n)$ is a positive constant depending only on the dimension N, and therefore, by summarizing, we have proved that for every compact set $K\subset\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ there is a small $\bar\nu=\bar\nu(K,\lambda_0)\in(0,\frac{|\lambda_0|}{4})$ such that for every $0\le\nu<\bar\nu$ we have

$$\begin{aligned}\sum_{i=1}^m\int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{i,\lambda_0+\nu}|^2\,dx \le \frac{m}{2}\sum_{i=1}^m(1+C_i^2)Q(n)|\Omega_{\lambda_0+\nu}\setminus K|^{\frac{2}{N}}\\ \int_{\Omega_{\lambda_0+\nu}\setminus K}|\nabla w^+_{\lambda_0+\nu}|^2\,dx.\end{aligned} \tag{6.2.19}$$

Now we first fix a compact $K\subset\Omega_{\lambda_0}\setminus R_{\lambda_0}(\Gamma)$ such that

$$|\Omega_{\lambda_0}\setminus K|^{\frac{2}{N}}<[m(1+C_i^2)Q(n)]^{-1}\text{ for every }i=1,...,m,$$

this is possible since $|R_{\lambda_0}(\Gamma)|=0$ by the assumption on Γ, and then we take $\bar\nu_0<\bar\nu$ such that for every $0\le\nu<\bar\nu_0$ we have $|\Omega_{\lambda_0+\nu}\setminus\Omega_{\lambda_0}|^{\frac{2}{N}}<[4m(1+C_i^2)Q(n)]^{-1}$. Inserting those informations into (6.2.19) we immediately get that

(6.2.20)

$$\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}|^2\,dx < \frac{1}{2}\int_{\Omega_{\lambda_0+\nu}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}|^2\,dx \quad \text{for every } i = 1,...,m$$

and so $\nabla w^+_{i,\lambda_0+\nu} = 0$ on $\Omega_{\lambda_0+\nu} \setminus K$ for every $0 \le \nu < \bar{\nu}_0$ and $i = 1,...,m$. On the other hand, we recall that $\nabla w^+_{i,\lambda_0+\nu} = 0$ on an open neighborhood of K for every $0 \le \nu < \bar{\nu}$ and $i = 1,...,m$, thus $\nabla w^+_{i,\lambda_0+\nu} = 0$ on $\Omega_{\lambda_0+\nu}$ for every $0 \le \nu < \bar{\nu}_0$ and $i = 1,...,m$. The latter proves that $u_i \le u_{i,\lambda_0+\nu}$ in $\Omega_{\lambda_0+\nu} \setminus R_{\lambda_0+\nu}(\Gamma)$ for every $0 < \nu < \bar{\nu}_0$ and $i = 1,...,m$. Such a contradiction shows that

$$\lambda_0 = 0\,.$$

Step 3: conclusion. Since the moving planes procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ is implicit in the moving planes procedure. Since u has C^1 regularity, the fact that $\partial_{x_1} u_i$ is positive for $x_1 < 0$ follows by the maximum principle, the Höpf lemma and the assumption (S_{f_i}).

□

6.3. Moving plane method for systems involving the critical hyperbola in the whole space

Proof of Theorem 6.0.3. We first note that, thanks to a well-known result of Brezis and Kato [20] and standard elliptic estimates (see also [119]), the solution $(u_1, ..., u_m)$ to (6.0.3) is smooth in $\mathbb{R}^N \setminus \Gamma$. Furthermore we observe that it is enough to prove the theorem for the special case in which *the origin does not belong to* Γ. Indeed, if the result is true in this special case, then we can apply it to the functions $u^{(i)}_z(x) := u_i(x + z)$ for every $i = 1,...,m$, where $z \in \{x_1 = 0\} \setminus \Gamma \neq \emptyset$, which satisfies the system (6.0.3) with Γ replaced by $-z+\Gamma$ (note that $-z+\Gamma$ is a closed and proper subset of $\{x_1 = 0\}$ with $\underset{\mathbb{R}^N}{\mathrm{Cap}_2}(-z + \Gamma) = 0$ and such that the origin does not belong to it).

Under this assumption, we consider the map $K : \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}^N \setminus \{0\}$ defined by $K(x) := \frac{x}{|x|^2}$. Given $(u_1, ..., u_m)$ solution to (6.0.3), the Kelvin transform of u_i is given by

$$\hat{u}_i(x) := \frac{1}{|x|^{N-2}} u_i\left(\frac{x}{|x|^2}\right) \qquad x \in \mathbb{R}^N \setminus \{\Gamma^* \cup \{0\}\}, \tag{6.3.1}$$

where $\Gamma^* = K(\Gamma)$ and $i = 1,...,m$. It follows that $(\hat{u}_1, ..., \hat{u}_m)$ weakly satisfies (6.0.3) in $\mathbb{R}^N \setminus \{\Gamma^* \cup \{0\}\}$ and that $\Gamma^* \subset \{x_1 = 0\}$ since, by assumption, $\Gamma \subset \{x_1 = 0\}$. Furthermore, we also have that Γ^* is bounded (not necessarily closed) since we assumed that $0 \notin \Gamma$.

Let us now fix some notations. We set

$$\Sigma_\lambda = \{x \in \mathbb{R}^N \,:\, x_1 < \lambda\}\,. \tag{6.3.2}$$

As above $x_\lambda = (2\lambda - x_1, x_2, \dots, x_N)$ is the reflection of x through the hyperplane $T_\lambda = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_1 = \lambda\}$. Finally we consider the Kelvin transform $(\hat{u}_1, ..., \hat{u}_m)$ of $(u_1, ..., u_m)$ defined in (6.3.1) and we set

$$w_{i,\lambda}^+ = (\hat{u}_i - \hat{u}_{i,\lambda})^+ \tag{6.3.3}$$

where $i = 1, ..., m$. Note that $(\hat{u}_1, ..., \hat{u}_m)$ weakly solves

(6.3.4)
$$\int_{\mathbb{R}^N} (\nabla \hat{u}_i, \nabla \varphi_i)\, dx = \sum_{j=1}^m a_{ij} \int_{\mathbb{R}^N} \hat{u}_j^{2^*-1} \varphi_i\, dx \qquad \forall \varphi_i \in C_c^1(\mathbb{R}^N \setminus \Gamma^* \cup \{0\}) .$$

and $(\hat{u}_{1,\lambda}, ..., \hat{u}_{m,\lambda})$ weakly solves

(6.3.5)
$$\int_{\mathbb{R}^N} (\nabla \hat{u}_{i,\lambda}, \nabla \varphi_i)\, dx = \sum_{j=1}^m a_{ij} \int_{\mathbb{R}^N} \hat{u}_{j,\lambda}^{2^*-1} \varphi_i\, dx \qquad \forall \varphi_i \in C_c^1(\mathbb{R}^N \setminus R_\lambda(\Gamma^* \cup \{0\})) .$$

where $i = 1, ..., m$. The properties of the Kelvin transform, the fact that $0 \notin \Gamma$ and the regularity of u_i imply that $|\hat{u}_i(x)| \le C|x|^{2-N}$ for every $x \in \mathbb{R}^N$ and $i = 1, ..., m$ such that $|x| \ge R$, where C and R are positive constants (depending on u_i). In particular, for every $\lambda < 0$, we have

$$\hat{u}_i \in L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap C^0(\overline{\Sigma_\lambda}) \tag{6.3.6}$$

for every $i = 1, ..., m$. We will prove the result by showing that, actually, it holds $\hat{w}_{i,\lambda}^+ \equiv 0$ for every $i = 1, ..., m$. To prove this, we have to perform the moving planes method.

Lemma 6.3.1. *Under the assumption of Theorem 6.0.3, for every $\lambda < 0$, we have that $\hat{w}_{i,\lambda}^+ \in L^{2^*}(\Sigma_\lambda), \nabla \hat{w}_{i,\lambda}^+ \in L^2(\Sigma_\lambda)$ and*

$$\begin{aligned} \sum_{i=1}^m \|w_{i,\lambda}^+\|_{L^{2^*}(\Sigma_\lambda)}^2 &\le \sum_{i=1}^m C_{i,S}^2 \int_{\Sigma_\lambda} |\nabla \hat{w}_{i,\lambda}^+|^2\, dx \\ &\le 2\frac{N+2}{N-2} \sum_{i,j=1}^m a_{ij} C_{i,S}^2 \|\hat{u}_j\|_{L^{2^*}(\Sigma_\lambda)}^{2^*-1} \|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}. \end{aligned} \tag{6.3.7}$$

where $C_{i,S}$ are the best constants in Sobolev embeddings.

Proof. We immediately see that $w_{i,\lambda}^+ \in L^{2^*}(\Sigma_\lambda)$, since $0 \le w_{i,\lambda}^+ \le \hat{u}_i \in L^{2^*}(\Sigma_\lambda)$ for every $i = 1, ..., m$. The rest of the proof follows the lines of the one of Lemma 6.2.1. Arguing as in section 2, for every $\varepsilon > 0$, we can find a function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N, [0,1])$ such that

$$\int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 < 4\varepsilon$$

and $\psi_\varepsilon = 0$ in an open neighborhood $\mathcal{B}_\varepsilon$ of $R_\lambda(\{\Gamma^* \cup \{0\}\})$, with $\mathcal{B}_\varepsilon \subset \Sigma_\lambda$.

Fix $R_0 > 0$ such that $R_\lambda(\{\Gamma^* \cup \{0\}) \subset B_{R_0}$ and, for every $R > R_0$, let φ_R be a standard cut off function such that $0 \le \varphi_R \le 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R} with $|\nabla \varphi_R| \le 2/R$, and consider

$$\varphi_i := \begin{cases} w_{i,\lambda}^+ \psi_\varepsilon^2 \varphi_R^2 & \text{in } \Sigma_\lambda, \\ 0 & \text{in } \mathbb{R}^N \setminus \Sigma_\lambda \end{cases}$$

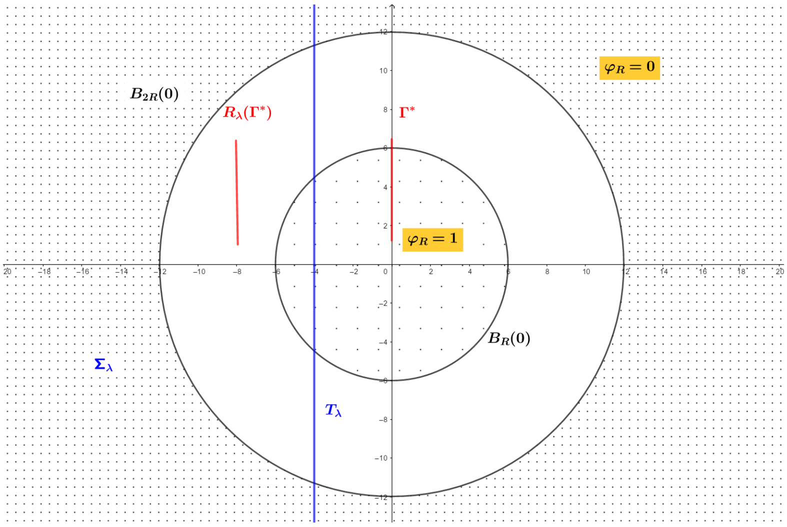

FIGURE 3. The cutoff function φ_R.

for every $i = 1, ..., m$.

Now, as in Lemma 4.2.1 we see that $\varphi_i \in C_c^{0,1}(\mathbb{R}^N)$ with $supp(\varphi_i)$ contained in $\overline{\Sigma_\lambda \cap B_{2R}} \setminus R_\lambda(\{\Gamma^* \cup \{0\}\})$ and

$$\nabla \varphi_i = \psi_\varepsilon^2 \varphi_R^2 \nabla w_{i,\lambda}^+ + 2 w_{i,\lambda}^+ (\psi_\varepsilon^2 \varphi_R \nabla \varphi_R + \psi_\varepsilon \varphi_R^2 \nabla \psi_\varepsilon). \tag{6.3.8}$$

Therefore, by a standard density argument, we can use φ_i as test functions respectively in (6.3.4) and in (6.3.5) so that, subtracting we get

$$\begin{aligned}\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx &= -2 \int_{\Sigma_\lambda} (\nabla w_{i,\lambda}^+, \nabla \psi_\varepsilon) w_{i,\lambda}^+ \psi_\varepsilon \varphi_R^2 \, dx \\ &\quad - 2 \int_{\Sigma_\lambda} (\nabla w_{i,\lambda}^+, \nabla \varphi_R) w_{i,\lambda}^+ \varphi_R \psi_\varepsilon^2 \, dx \\ &\quad + \sum_{i=1}^m a_{ij} \int_{\Sigma_\lambda} (\hat{u}_j^{2^*-1} - \hat{u}_{j,\lambda}^{2^*-1}) w_{i,\lambda}^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\ &=: \, I_1 + I_2 + I_3 \, .\end{aligned} \tag{6.3.9}$$

Exploiting also Young's inequality and recalling that $0 \leq w_{i,\lambda}^+ \leq \hat{u}_i$, we get that

$$\begin{aligned}|I_1| &\leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 4 \int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 (w_{i,\lambda}^+)^2 \varphi_R^2 \, dx \\ &\leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 16 \varepsilon \|\hat{u}_i\|_{L^\infty(\Sigma_\lambda)}^2 .\end{aligned} \tag{6.3.10}$$

Furthermore we have that

(6.3.11)

$$
\begin{aligned}
|I_2| &\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 4\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} |\nabla \varphi_R|^2 (w_{i,\lambda}^+)^2 \psi_\varepsilon^2 \, dx \\
&\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\
&\quad + 4\left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} |\nabla \varphi_R|^n \, dx\right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} \hat{u}_i^{2^*} \, dx\right)^{\frac{N-2}{N}} \\
&\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \, + \, C(N)\left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} \hat{u}_i^{2^*} \, dx\right)^{\frac{N-2}{N}}
\end{aligned}
$$

where $C(N)$ is a positive constant depending only on the dimension N. Let us now estimate I_3. Since $\hat{u}_i(x), \hat{u}_{i,\lambda}(x) > 0$, by the convexity of $t \to t^{2^*-1}$, for $t > 0$, we obtain

$$
\hat{u}_i^{2^*-1}(x) - \hat{u}_{i,\lambda}^{2^*-1}(x) \leq \frac{N+2}{N-2}\hat{u}_{i,\lambda}^{2^*-2}(x)(\hat{u}_i(x) - \hat{u}_{i,\lambda}(x)),
$$

for every $x \in \Sigma_\lambda$ and $i = 1, ..., m$. Thus, by making use of the monotonicity of $t \to t^{2^*-2}$, for $t > 0$ and the definition of $w_{i,\lambda}^+$ we get

$$
(\hat{u}_i^{2^*-1} - \hat{u}_{i,\lambda}^{2^*-1}) w_{i,\lambda}^+ \leq \frac{N+2}{N-2}\hat{u}_{i,\lambda}^{2^*-2}(\hat{u}_i - \hat{u}_{i,\lambda}) w_{i,\lambda}^+ \leq \frac{N+2}{N-2}\hat{u}_i^{2^*-2}(w_{i,\lambda}^+)^2
$$

for every $i = 1, ..., m$. Therefore

(6.3.12)

$$
\begin{aligned}
|I_3| &\leq \frac{N+2}{N-2}\sum_{j=1}^m a_{ij}\int_{\Sigma_\lambda} \hat{u}_j^{2^*-2} w_{j,\lambda}^+ w_{i,\lambda}^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
&\leq \frac{N+2}{N-2}\sum_{j=1}^m a_{ij}\int_{\Sigma_\lambda} \hat{u}_j^{2^*-2}\hat{u}_j\hat{u}_i dx \, = \frac{N+2}{N-2}\sum_{j=1}^m a_{ij}\int_{\Sigma_\lambda} \hat{u}_j^{2^*-1} u_i \, dx \\
&= \frac{N+2}{N-2}\left(a_{ii}\|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}^{2^*} + \sum_{\substack{j=1 \\ j\neq i}}^m a_{ij}\int_{\Sigma_\lambda} \hat{u}_j^{2^*-1} u_i \, dx\right) \\
&\leq \frac{N+2}{N-2}\left(a_{ii}\|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}^{2^*} + \sum_{\substack{j=1 \\ j\neq i}}^m a_{ij}\left(\int_{\Sigma_\lambda} \hat{u}_j^{2^*}\right)^{\frac{N+2}{2N}}\left(\int_{\Sigma_\lambda} u_i^{2^*}\right)^{\frac{1}{2^*}} dx\right) \\
&= \frac{N+2}{N-2}\sum_{j=1}^m a_{ij}\|\hat{u}_j\|_{L^{2^*}(\Sigma_\lambda)}^{2^*-1}\|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}
\end{aligned}
$$

where we also used that $0 \leq w_{i,\lambda}^+ \leq \hat{u}_i$ for every $i = 1, ..., m$ and Hölder inequality.

Taking into account the estimates on I_1, I_2 and I_3, by (6.3.9) we deduce that

(6.3.13)

$$\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \le 32\varepsilon \|\hat{u}_i\|^2_{L^\infty(\Sigma_\lambda)} + 2C(N) \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{u}_i^{2^*} \, dx \right)^{\frac{N-2}{N}} + 2\frac{N+2}{N-2} \sum_{j=1}^m a_{ij} \|\hat{u}_j\|^{2^*-1}_{L^{2^*}(\Sigma_\lambda)} \|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}$$

which in turns yields

(6.3.14)

$$\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \le 32\varepsilon \sum_{i=1}^m \|\hat{u}_i\|^2_{L^\infty(\Sigma_\lambda)} + 2C(N) \sum_{i=1}^m \|\hat{u}_i\|^2_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} + 2\frac{N+2}{N-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \|\hat{u}_j\|^{2^*-1}_{L^{2^*}(\Sigma_\lambda)} \|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}.$$

By Fatou's Lemma, as ε tends to zero and R tends to infinity, we deduce that $\nabla w_{i,\lambda}^+ \in L^2(\Sigma_\lambda)$ for every $i = 1, ..., m$. We also note that $\varphi_i \to w_{i,\lambda}^+$ in $L^{2^*}(\Sigma_\lambda)$, by definition of φ_i, and that $\nabla \varphi_i \to \nabla w_{i,\lambda}$ in $L^2(\Sigma_\lambda)$, by (6.3.8) and the fact that $w_{i,\lambda}^+ \in L^{2^*}(\Sigma_\lambda)$ for every $i = 1, ..., m$. Therefore by (6.3.14) we have

$$\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \, dx \le 2\frac{N+2}{N-2} \sum_{i,j=1}^m a_{ij} \|\hat{u}_j\|^{2^*-1}_{L^{2^*}(\Sigma_\lambda)} \|\hat{u}_i\|_{L^{2^*}(\Sigma_\lambda)}. \tag{6.3.15}$$

Since $\varphi_i \in C_c^{0,1}(\mathbb{R}^N)$ we also have

$$\left(\int_{\Sigma_\lambda} \varphi_i^{2^*} \, dx \right)^{\frac{2}{2^*}} \le C_{i,S}^2 \int_{\Sigma_\lambda} |\nabla \varphi_i|^2 \, dx \tag{6.3.16}$$

where $C_{i,S}$ are the best constants in Sobolev embeddings. Thus, passing to the limit in (6.3.16) and using the above convergence results, we get the desired conclusion (6.3.7).

□

We can now complete the proof of Theorem 6.0.3. As for the proof of Theorem 6.0.2, we split the proof into three steps and we start with
Step 1: there exists $M > 1$ such that $\hat{u}_i \le \hat{u}_{i,\lambda}$ in $\Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\})$, for all $\lambda < -M$ and $i = 1, ..., m$.

Arguing as in the proof of Lemma 6.3.1 and using the same notations and the same construction for ψ_ε, φ_R and φ_i, we get

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2\, dx = &-2\int_{\Sigma_\lambda} \nabla w_{i,\lambda}^+ \nabla \psi_\varepsilon w_{i,\lambda}^+ \psi_\varepsilon \varphi_R^2\, dx \\
&- 2\int_{\Sigma_\lambda} \nabla w_{i,\lambda}^+ \nabla \varphi_R w_{i,\lambda}^+ \varphi_R \psi_\varepsilon^2\, dx \\
&+ \sum_{i=1}^m a_{ij} \int_{\Sigma_\lambda} (\hat{u}_j^{2^*-1} - \hat{u}_{j,\lambda}^{2^*-1}) w_{i,\lambda}^+ \psi_\varepsilon^2 \varphi_R^2\, dx \\
=:&\ I_1 + I_2 + I_3\,.
\end{aligned} \tag{6.3.17}
$$

where I_1, I_2 and I_3 can be estimated exactly as in (6.3.10), (6.3.11) and (6.3.12). The latter yield

(6.3.18)

$$
\begin{aligned}
\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \varphi_R^2\, dx \le 32\varepsilon \sum_{i=1}^m \|\hat{u}_i\|^2_{L^\infty(\Sigma_\lambda)} + 2C(N)\sum_{i=1}^m \|\hat{u}_i\|^2_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} \\
+ 2\frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij} \int_{\Sigma_\lambda} \hat{u}_j^{2^*-2} w_{j,\lambda}^+ w_{i,\lambda}^+ \psi_\varepsilon^2 \varphi_R^2\, dx.
\end{aligned}
$$

Taking the limit in the latter, as ε tends to zero and R tends to infinity, leads to

$$
\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2\, dx \le 2\frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij} \int_{\Sigma_\lambda} \hat{u}_j^{2^*-2} w_{j,\lambda}^+ w_{i,\lambda}^+\, dx < +\infty \tag{6.3.19}
$$

which combined with Lemma 6.3.1 gives

(6.3.20)

$$
\begin{aligned}
&\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \le 2\frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij} \int_{\Sigma_\lambda} \hat{u}_j^{2^*-2} w_{j,\lambda}^+ w_{i,\lambda}^+\, dx \\
&\le \frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij} \left(\int_{\Sigma_\lambda} \hat{u}_j^{2^*-2} (w_{j,\lambda}^+)^2\, dx + \int_{\Sigma_\lambda} \hat{u}_j^{2^*-2} (w_{i,\lambda}^+)^2\, dx \right) \\
&\le \frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij} \left[\left(\int_{\Sigma_\lambda} \hat{u}_j^{2^*}\, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda} (w_{j,\lambda}^+)^{2^*}\, dx \right)^{\frac{2}{2^*}} \right. \\
&\qquad\qquad \left. + \left(\int_{\Sigma_\lambda} \hat{u}_j^{2^*}\, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda} (w_{i,\lambda}^+)^{2^*}\, dx \right)^{\frac{2}{2^*}} \right] \\
&\le \frac{N+2}{N-2}\sum_{i=1}^m \sum_{j=1}^m a_{ij} \|\hat{u}_j\|^{2^*-2}_{L^{2^*}(\Sigma_\lambda)} \left(C_{j,S}^2 \int_{\Sigma_\lambda} |\nabla w_{j,\lambda}^+|^2\, dx + C_{i,S}^2 \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2\, dx \right) \\
&= \frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij} \left(2\delta_{ij} C_{i,S}^2 \|\hat{u}_i\|^{2^*-2}_{L^{2^*}(\Sigma_\lambda)} \right. \\
&\qquad\qquad \left. + (1-\delta_{ij}) C_{j,S}^2 \|\hat{u}_j\|^{2^*-2}_{L^{2^*}(\Sigma_\lambda)} \right) \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2\, dx
\end{aligned}
$$

Recalling that $\hat{u}_i, \hat{u}_j \in L^{2^*}(\Sigma_\lambda)$ for every $i,j = 1,...,m$, we deduce the existence of $M > 1$ such that

$$\frac{N+2}{N-2}\sum_{j=1}^{m} a_{ij}\left(2\delta_{ij}C^2_{i,S}\|\hat{u}_i\|^{2^*-2}_{L^{2^*}(\Sigma_\lambda)} + (1-\delta_{ij})C^2_{j,S}\|\hat{u}_j\|^{2^*-2}_{L^{2^*}(\Sigma_\lambda)}\right) < 1$$

for every $\lambda < -M$ and $i = 1,....,m$. The latter and (6.4.27) lead to

$$\int_{\Sigma_\lambda} |\nabla w^+_{i,\lambda}|^2\, dx = 0\,.$$

This implies that for every $i = 1,...,m$ we have $w^+_{i,\lambda} = 0$ by Lemma 6.3.1 and the claim is proved.

To proceed further we define

$\Lambda_0 := \{\lambda < 0 : \hat{u}_i \leq \hat{u}_{i,t}$ in $\Sigma_t \setminus R_t(\Gamma^* \cup \{0\})$ for all $t \in (a,\lambda]$ and i=1,...,m$\}$

and

$$\lambda_0 := \sup\, \Lambda_0.$$

Step 2: we have that $\lambda_0 = 0$. We argue by contradiction and suppose that $\lambda_0 < 0$. By continuity we know that $\hat{u}_i \leq \hat{u}_{i,\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$ for every $i = 1,...,m$. By the strong maximum principle we deduce that $\hat{u}_i < \hat{u}_{i,\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$ for every $i = 1,...,m$. Indeed, $\hat{u}_i = \hat{u}_{i,\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\}))$ is not possible if $\lambda_0 < 0$, since in this case each $\hat{u}_i$ would be singular somewhere on $R_{\lambda_0}(\Gamma^* \cup \{0\})$. Now, for some $\bar{\tau} > 0$, that will be fixed later on, and for any $0 < \tau < \bar{\tau}$ we show that $\hat{u}_i \leq \hat{u}_{i,\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ obtaining a contradiction with the definition of λ_0 and proving thus the claim. To this end we are going to show that, for every $\delta > 0$ there are $\bar{\tau}(\delta, \lambda_0) > 0$ and a compact set K (depending on δ and λ_0) such that

$$K \subset \Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\}), \qquad \int_{\Sigma_\lambda \setminus K} \hat{u}_i^{2^*}\, dx < \delta,\ \forall\, \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}] \text{ and } i = 1,...,m.$$

To see this, we note that for every every $\delta > 0$ there are $\tau_1(\delta, \lambda_0) > 0$ and a compact set K (depending on δ and λ_0) such that $\displaystyle\int_{\Sigma_{\lambda_0} \setminus K} \hat{u}_i^{2^*}\, dx < \frac{\delta}{2}$ for every $i = 1,...,m$ and $K \subset \Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\})$ for every $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$. Consequently $\hat{u}_i$ and $\hat{u}_{i,\lambda}$ are well defined on K for every $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$. Hence, by the uniform continuity of the functions $g_i(x,\lambda) := \hat{u}_i(x) - \hat{u}_i(2\lambda - x_1, x')$ on the compact set $K \times [\lambda_0, \lambda_0 + \tau_1]$ we can ensure that $K \subset \Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ and $\hat{u}_i < \hat{u}_{i,\lambda_0+\tau}$ in K for any $0 \leq \tau < \tau_2$, for some $\tau_2 = \tau(\delta, \lambda_0) \in (0, \tau_1)$. Clearly we can also assume that $\tau_2 < \frac{|\lambda_0|}{4}$. Finally, since $\hat{u}_i^{2^*} \in L^1(\Sigma_{\lambda_0 + \frac{|\lambda_0|}{4}})$ and $\displaystyle\int_{\Sigma_{\lambda_0} \setminus K} \hat{u}_i^{2^*}\, dx < \frac{\delta}{2}$ for each $i = 1,...,m$, we obtain the existence of $\bar{\tau} \in (0, \tau_2)$ such that $\displaystyle\int_{\Sigma_\lambda \setminus K} \hat{u}_i^{2^*}\, dx < \delta$ for all $\lambda \in [\lambda_0, \lambda_0 + \bar{\tau}]$ and $i = 1,...,m$.

Now we repeat verbatim the arguments used in the proof of Lemma 6.3.1 but using the test functions

$$\varphi_i := \begin{cases} w^+_{i,\lambda_0+\tau}\psi^2_\varepsilon\varphi^2_R & \text{in} \quad \Sigma_{\lambda_0+\tau} \\ 0 & \text{in} \quad \mathbb{R}^N \setminus \Sigma_{\lambda_0+\tau}. \end{cases}$$

Thus we recover the last inequality in (6.3.20), which immediately gives, for any $0 \le \tau < \bar\tau$

(6.3.21)

$$\begin{aligned} &\sum_{i=1}^m \int_{\Sigma_{\lambda_0+\tau}\setminus K} |\nabla w^+_{i,\lambda_0+\nu}|^2 \\ &\le \frac{N+2}{N-2}\sum_{i,j=1}^m a_{ij}\Big[2\delta_{ij}C^2_{i,S}\|\hat u_i\|^{2^*-2}_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)} \\ &\qquad +(1-\delta_{ij})C^2_{j,S}\|\hat u_j\|^{2^*-2}_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}\Big]\int_{\Sigma_{\lambda_0+\tau}\setminus K}|\nabla w^+_{i,\lambda}|^2\,dx \end{aligned}$$

since $w^+_{i,\lambda_0+\tau}$ and $\nabla w^+_{i,\lambda_0+\tau}$ are zero in a neighborhood of K, by the above construction for every $i = 1,...,m$. Now we fix $\delta > 0$ such that for every $i = 1,...,m$ we have

$$\frac{N+2}{N-2}\sum_{j=1}^m a_{ij}\Big[2\delta_{ij}C^2_{i,S}\|\hat u_i\|^{2^*-2}_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)} + (1-\delta_{ij})C^2_{j,S}\|\hat u_j\|^{2^*-2}_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}\Big] < \frac{1}{2},$$

for all $0 \le \tau < \bar\tau$, which plugged into (6.3.21) implies that

$$\int_{\Sigma_{\lambda_0+\tau}\setminus K}|\nabla w^+_{i,\lambda_0+\tau}|^2\,dx = 0$$

for every $0 \le \tau < \bar\tau$ and $i = 1,...,m$. Hence $\int_{\Sigma_{\lambda_0+\tau}}|\nabla w^+_{i,\lambda_0+\tau}|^2\,dx = 0$ for every $0 \le \tau < \bar\tau$, since $\nabla w^+_{i,\lambda_0+\tau}$ are zero in a neighborhood of K. The latter and Lemma 6.3.1 imply that $w^+_{i,\lambda_0+\tau} = 0$ on $\Sigma_{\lambda_0+\tau}$ for every $0 \le \tau < \bar\tau$ and $i = 1,...,m$, thus $\hat u_i \le \hat u_{i,\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau}\setminus R_{\lambda_0+\tau}(\Gamma^*\cup\{0\})$ for every $0 \le \tau < \bar\tau$ and $i = 1,...,m$. Which proves the claim of Step 2.

Step 3: conclusion. The symmetry of the Kelvin transform $(\hat u_1,...,\hat u_m)$ follows now performing the moving planes method in the opposite direction. The fact that every $\hat u_i$ is symmetric w.r.t. the hyperplane $\{x_1 = 0\}$ implies the symmetry of the solution $(u_1,...,u_m)$ w.r.t. the hyperplane $\{x_1 = 0\}$. The last claim then follows by the invariance of the considered problem with respect to isometries (translations and rotations).

□

6.4. Moving plane method for a cooperative Gross-Pitaevskii type system in low dimension

PROOF OF THEOREM 6.0.4. As we observed in the proof of Theorem 6.0.3, thanks to a well-known result of Brezis and Kato [20] and standard elliptic estimates (see also [119]), the solution (u,v) is smooth in $\mathbb{R}^N \setminus \Gamma$.

Furthermore we recall that it is enough to prove the theorem for the special case in which *the origin does not belong to* Γ.
Under this assumption, we consider the map $K : \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}^N \setminus \{0\}$ defined by $K = K(x) := \frac{x}{|x|^2}$. Given (u, v) solution to (6.0.6), its Kelvin transform is given by
(6.4.1)

$$(\hat{u}(x), \hat{v}(x)) := \left(\frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right), \frac{1}{|x|^{N-2}} v\left(\frac{x}{|x|^2}\right)\right) \quad x \in \mathbb{R}^N \setminus \{\Gamma^* \cup \{0\}\},$$

where $\Gamma^* = K(\Gamma)$. It follows that $(\hat{u}, \hat{v})$ weakly satisfies (6.0.6) in $\mathbb{R}^N \setminus \{\Gamma^* \cup \{0\}\}$ and that $\Gamma^* \subset \{x_1 = 0\}$ since, by assumption, $\Gamma \subset \{x_1 = 0\}$. Furthermore, we also have that Γ^* is bounded (not necessarily closed) since we assumed that $0 \notin \Gamma$.
Let us now fix some notations. We set

$$\Sigma_\lambda = \{x \in \mathbb{R}^N \, : \, x_1 < \lambda\} \,. \tag{6.4.2}$$

As above $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_N)$ is the reflection of x through the hyperplane $T_\lambda = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_1 = \lambda\}$. Finally we consider the Kelvin transform $(\hat{u}, \hat{v})$ of (u, v) defined in (6.4.1) and we set

$$\begin{aligned} \xi_\lambda(x) &= \hat{u}(x) - \hat{u}_\lambda(x) = \hat{u}(x) - \hat{u}(x_\lambda), \\ \zeta_\lambda(x) &= \hat{v}(x) - \hat{v}_\lambda(x) = \hat{v}(x) - \hat{v}(x_\lambda). \end{aligned} \tag{6.4.3}$$

Note that $(\hat{u}, \hat{v})$ weakly solves

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla \hat{u}, \nabla \varphi) \, dx \; &= \int_{\mathbb{R}^N} \hat{u}^{2^*-1} \varphi \, dx + \frac{\alpha}{2^*} \int_{\mathbb{R}^N} \hat{u}^{\alpha-1} \hat{v}^\beta \varphi \, dx \,, \\ \int_{\mathbb{R}^N} (\nabla \hat{v}, \nabla \psi) \, dx \; &= \int_{\mathbb{R}^N} \hat{v}^{2^*-1} \psi \, dx + \frac{\beta}{2^*} \int_{\mathbb{R}^N} \hat{u}^{\alpha} \hat{v}^{\beta-1} \psi \, dx \,, \end{aligned} \tag{6.4.4}$$

for all $\varphi \in C^1_c(\mathbb{R}^N \setminus \Gamma^* \cup \{0\})$ and $(\hat{u}_\lambda, \hat{v}_\lambda)$ weakly solves

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla \hat{u}_\lambda, \nabla \varphi) \, dx \; &= \int_{\mathbb{R}^N} \hat{u}_\lambda^{2^*-1} \varphi \, dx \; + \frac{\alpha}{2^*} \int_{\mathbb{R}^N} \hat{u}_\lambda^{\alpha-1} \hat{v}_\lambda^\beta \varphi \, dx \,, \\ \int_{\mathbb{R}^N} (\nabla \hat{v}_\lambda, \nabla \psi) \, dx \; &= \int_{\mathbb{R}^N} \hat{v}_\lambda^{2^*-1} \psi \, dx + \frac{\beta}{2^*} \int_{\mathbb{R}^N} \hat{u}_\lambda^{\alpha} \hat{v}_\lambda^{\beta-1} \psi \, dx \,, \end{aligned} \tag{6.4.5}$$

for all $\varphi \in C^1_c(\mathbb{R}^N \setminus \Gamma^* \cup \{0\})$. The properties of the Kelvin transform, the fact that $0 \notin \Gamma$ and the regularity of u, v imply that $|\hat{u}(x)| \le C_u |x|^{2-N}$ and $|\hat{v}(x)| \le C_v |x|^{2-N}$ and for every $x \in \mathbb{R}^N$ such that $|x| \ge R$, where C_u, C_v and R are positive constants (depending on u and v). In particular, for every $\lambda < 0$, we have

$$\hat{u}, \hat{v} \in L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap C^0(\overline{\Sigma_\lambda}) \,. \tag{6.4.6}$$

Lemma 6.4.1. *Under the assumption of Theorem 6.0.3, for every* $\lambda < 0$, *we have that* $\xi_\lambda^+, \zeta_\lambda^+ \in L^{2^*}(\Sigma_\lambda), \nabla \xi_\lambda^+, \nabla \zeta_\lambda^+ \in L^2(\Sigma_\lambda)$ *and*

$$\begin{aligned} &\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + \int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \, dx \\ &\qquad \le 2 \frac{N+2}{N-2} \left[(1+\alpha) \|\hat{u}\|^{2^*}_{L^{2^*}(\Sigma_\lambda)} + (1+\beta) \|\hat{v}\|^{2^*}_{L^{2^*}(\Sigma_\lambda)} \right]. \end{aligned} \tag{6.4.7}$$

PROOF. We immediately see that $\xi_\lambda^+, \zeta_\lambda^+ \in L^{2^*}(\Sigma_\lambda)$, since $0 \leq \xi_\lambda^+ \leq \hat{u} \in L^{2^*}(\Sigma_\lambda)$ and $0 \leq \zeta_\lambda^+ \leq \hat{v} \in L^{2^*}(\Sigma_\lambda)$. The rest of the proof follows the lines of the one of Lemma 6.2.1. Arguing as in section 2, for every $\varepsilon > 0$, we can find a function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N, [0,1])$ such that

$$\int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 < 4\varepsilon$$

and $\psi_\varepsilon = 0$ in an open neighborhood $\mathcal{B}_\varepsilon$ of $R_\lambda(\{\Gamma^* \cup \{0\}\})$, with $\mathcal{B}_\varepsilon \subset \Sigma_\lambda$.

Fix $R_0 > 0$ such that $R_\lambda(\{\Gamma^* \cup \{0\}) \subset B_{R_0}$ and, for every $R > R_0$, let φ_R (see Figure 4) be a standard cut off function such that $0 \leq \varphi_R \leq 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R} with $|\nabla \varphi_R| \leq 2/R$, and consider

$$\varphi := \begin{cases} \xi_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 & \text{in } \Sigma_\lambda, \\ 0 & \text{in } \mathbb{R}^N \setminus \Sigma_\lambda \end{cases} \quad \text{and} \quad \psi := \begin{cases} \zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 & \text{in } \Sigma_\lambda, \\ 0 & \text{in } \mathbb{R}^N \setminus \Sigma_\lambda. \end{cases}$$

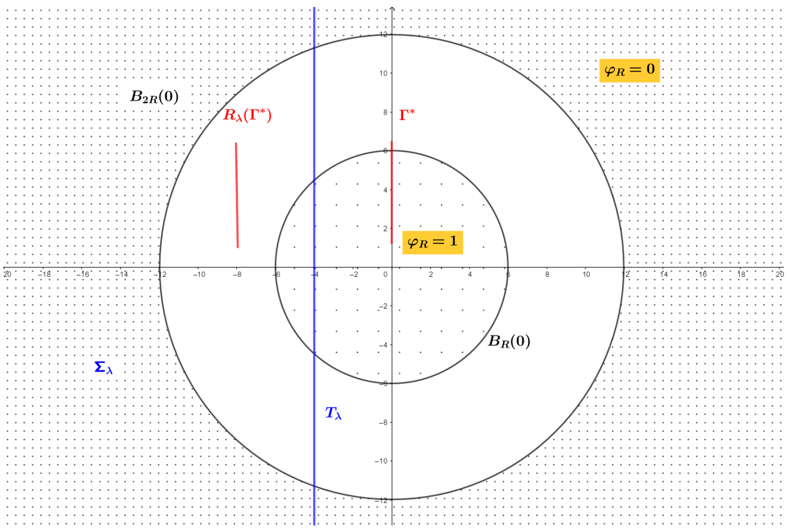

FIGURE 4. The cutoff function φ_R.

Now, as in Lemma 4.2.1 we see that $\varphi, \psi \in C_c^{0,1}(\mathbb{R}^N)$ with $supp(\varphi)$ and $supp(\psi)$ contained in $\overline{\Sigma_\lambda \cap B_{2R}} \setminus R_\lambda(\{\Gamma^* \cup \{0\}\})$ and

$$\nabla \varphi = \psi_\varepsilon^2 \varphi_R^2 \nabla \xi_\lambda^+ + 2\xi_\lambda^+ (\psi_\varepsilon^2 \varphi_R \nabla \varphi_R + \psi_\varepsilon \varphi_R^2 \nabla \psi_\varepsilon). \tag{6.4.8}$$

$$\nabla \psi = \psi_\varepsilon^2 \varphi_R^2 \nabla \zeta_\lambda^+ + 2\zeta_\lambda^+ (\psi_\varepsilon^2 \varphi_R \nabla \varphi_R + \psi_\varepsilon \varphi_R^2 \nabla \psi_\varepsilon). \tag{6.4.9}$$

Therefore, by a standard density argument, we can use φ and ψ as test functions respectively in (6.4.4) and in (6.4.5) so that, subtracting we get

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx = & -2 \int_{\Sigma_\lambda} (\nabla \xi_\lambda^+, \nabla \psi_\varepsilon) \xi_\lambda^+ \psi_\varepsilon \varphi_R^2 \, dx \\
& - 2 \int_{\Sigma_\lambda} (\nabla \xi_\lambda^+, \nabla \varphi_R) \xi_\lambda^+ \varphi_R \psi_\varepsilon^2 \, dx \\
& + \int_{\Sigma_\lambda} (\hat{u}^{2^*-1} - \hat{u}_\lambda^{2^*-1}) \xi_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + \frac{\alpha}{2^*} \int_{\Sigma_\lambda} (\hat{u}^{\alpha-1} \hat{v}^\beta - \hat{u}_\lambda^{\alpha-1} \hat{v}_\lambda^\beta) \xi_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
=: & \; I_1 + I_2 + I_3 + I_4 \, .
\end{aligned} \tag{6.4.10}
$$

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx = & -2 \int_{\Sigma_\lambda} (\nabla \zeta_\lambda^+, \nabla \psi_\varepsilon) \zeta_\lambda^+ \psi_\varepsilon \varphi_R^2 \, dx \\
& - 2 \int_{\Sigma_\lambda} (\nabla \zeta_\lambda^+, \nabla \varphi_R) \zeta_\lambda^+ \varphi_R \psi_\varepsilon^2 \, dx \\
& + \int_{\Sigma_\lambda} (\hat{v}^{2^*-1} - \hat{v}_\lambda^{2^*-1}) \zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + \frac{\beta}{2^*} \int_{\Sigma_\lambda} (\hat{u}^{\alpha} \hat{v}^{\beta-1} - \hat{u}_\lambda^{\alpha} \hat{v}_\lambda^{\beta-1}) \zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
=: & \; E_1 + E_2 + E_3 + E_4 \, .
\end{aligned} \tag{6.4.11}
$$

Exploiting also Young's inequality and recalling that $0 \le \xi_\lambda^+ \le \hat{u}$ and $0 \le \zeta_\lambda^+ \le \hat{v}$, we get that

$$
\begin{aligned}
|I_1| & \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 4 \int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 (\xi_\lambda^+)^2 \varphi_R^2 \, dx \\
& \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 16 \varepsilon \|\hat{u}\|^2_{L^\infty(\Sigma_\lambda)} .
\end{aligned} \tag{6.4.12}
$$

$$
\begin{aligned}
|E_1| & \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 4 \int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 (\zeta_\lambda^+)^2 \varphi_R^2 \, dx \\
& \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 16 \varepsilon \|\hat{v}\|^2_{L^\infty(\Sigma_\lambda)} .
\end{aligned} \tag{6.4.13}
$$

Furthermore we have that

$$
\begin{aligned}
|I_2| & \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + 4 \int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} |\nabla \varphi_R|^2 (\xi_\lambda^+)^2 \psi_\varepsilon^2 \, dx \\
& \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + 4 \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} |\nabla \varphi_R|^n \, dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{u}^{2^*} \, dx \right)^{\frac{N-2}{N}} \\
& \le \frac{1}{4} \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \; + \; c(N) \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{u}^{2^*} \, dx \right)^{\frac{N-2}{N}} .
\end{aligned} \tag{6.4.14}
$$

(6.4.15)

$$
\begin{aligned}
|E_2| &\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla\zeta_\lambda^+|^2\psi_\varepsilon^2\varphi_R^2\,dx + 4\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} |\nabla\varphi_R|^2(\zeta_\lambda^+)^2\psi_\varepsilon^2\,dx\\
&\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla\zeta_\lambda^+|^2\psi_\varepsilon^2\varphi_R^2\,dx\\
&\quad + 4\left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} |\nabla\varphi_R|^N\,dx\right)^{\frac{2}{N}}\left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} \hat{v}^{2^*}\,dx\right)^{\frac{N-2}{N}}\\
&\leq \frac{1}{4}\int_{\Sigma_\lambda} |\nabla\zeta_\lambda^+|^2\psi_\varepsilon^2\varphi_R^2\,dx \,+\, c(N)\left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} \hat{v}^{2^*}\,dx\right)^{\frac{N-2}{N}}
\end{aligned}
$$

where $c(N)$ is a positive constant depending only on the dimension N.

Let us now estimate I_3 and E_3. Since $\hat{u}(x), \hat{u}_\lambda(x), \hat{v}(x), \hat{v}_\lambda(x) > 0$, by the convexity of $t \to t^{2^*-1}$, for $t > 0$, we obtain

$$\hat{u}^{2^*-1}(x) - \hat{u}_\lambda^{2^*-1}(x) \leq \frac{N+2}{N-2}\hat{u}_\lambda^{2^*-2}(x)(\hat{u}(x) - \hat{u}_\lambda(x))$$

and

$$\hat{v}^{2^*-1}(x) - \hat{v}_\lambda^{2^*-1}(x) \leq \frac{N+2}{N-2}\hat{v}_\lambda^{2^*-2}(x)(\hat{v}(x) - \hat{v}_\lambda(x)),$$

for every $x \in \Sigma_\lambda$. Thus, by making use of the monotonicity of $t \to t^{2^*-2}$, for $t > 0$ and the definition of ξ_λ^+ and ζ_λ^+ we get

$$(\hat{u}^{2^*-1} - \hat{u}_\lambda^{2^*-1})\xi_\lambda^+ \leq \frac{N+2}{N-2}\hat{u}_\lambda^{2^*-2}(\hat{u} - \hat{u}_\lambda)\xi_\lambda^+ \leq \frac{N+2}{N-2}\hat{u}^{2^*-2}(\xi_\lambda^+)^2$$

and

$$(\hat{v}^{2^*-1} - \hat{v}_\lambda^{2^*-1})\zeta_\lambda^+ \leq \frac{N+2}{N-2}\hat{v}_\lambda^{2^*-2}(\hat{v} - \hat{v}_\lambda)\zeta_\lambda^+ \leq \frac{N+2}{N-2}\hat{v}^{2^*-2}(\zeta_\lambda^+)^2.$$

Therefore

(6.4.16)

$$
\begin{aligned}
|I_3| &\leq \frac{N+2}{N-2}\int_{\Sigma_\lambda} \hat{u}^{2^*-2}(\xi_\lambda^+)^2\psi_\varepsilon^2\varphi_R^2\,dx\\
&\leq \frac{N+2}{N-2}\int_{\Sigma_\lambda} \hat{u}^{2^*-2}\hat{u}^2 dx \;=\; \frac{N+2}{N-2}\int_{\Sigma_\lambda} \hat{u}^{2^*}\,dx = \frac{N+2}{N-2}\|\hat{u}\|_{L^{2^*}(\Sigma_\lambda)}^{2^*}
\end{aligned}
$$

(6.4.17)

$$
\begin{aligned}
|E_3| &\leq \frac{N+2}{N-2}\int_{\Sigma_\lambda} \hat{v}^{2^*-2}(\zeta_\lambda^+)^2\psi_\varepsilon^2\varphi_R^2\,dx\\
&\leq \frac{N+2}{N-2}\int_{\Sigma_\lambda} \hat{v}^{2^*-2}\hat{v}^2 dx \;=\; \frac{N+2}{N-2}\int_{\Sigma_\lambda} \hat{v}^{2^*}\,dx = \frac{N+2}{N-2}\|\hat{v}\|_{L^{2^*}(\Sigma_\lambda)}^{2^*}
\end{aligned}
$$

where we also used that $0 \leq \xi_\lambda^+ \leq \hat{u}$ and $0 \leq \zeta_\lambda^+ \leq \hat{v}$.

Finally we have to estimate I_4 and E_4. Since $\hat{u}(x), \hat{u}_\lambda(x), \hat{v}(x), \hat{v}_\lambda(x) > 0$, by the convexity of the functions $t \to t^\alpha, t \to t^{\alpha-1}, t \to t^\beta, t \to t^{\beta-1}$ for $t > 0$, we obtain

$$\begin{aligned}
\hat{u}^{\alpha}(x)-\hat{u}_{\lambda}^{\alpha}(x) &\leq \alpha \hat{u}_{\lambda}^{\alpha-1}(x)(\hat{u}(x)-\hat{u}_{\lambda}(x)),\\
\hat{u}^{\alpha-1}(x)-\hat{u}_{\lambda}^{\alpha-1}(x) &\leq (\alpha-1)\hat{u}_{\lambda}^{\alpha-2}(x)(\hat{u}(x)-\hat{u}_{\lambda}(x)),\\
\hat{v}^{\beta}(x)-\hat{v}_{\lambda}^{\beta}(x) &\leq \beta \hat{v}_{\lambda}^{\beta-1}(x)(\hat{v}(x)-\hat{v}_{\lambda}(x)),\\
\hat{v}^{\beta-1}(x)-\hat{v}_{\lambda}^{\beta-1}(x) &\leq (\beta-1)\hat{v}_{\lambda}^{\beta-2}(x)(\hat{v}(x)-\hat{v}_{\lambda}(x)),
\end{aligned}$$

for every $x \in \Sigma_\lambda$. By the monotonicity of $t \to t^{\alpha}, t \to t^{\alpha-1}, t \to t^{\beta}, t \to t^{\beta-1}$ for $t > 0$ and the definition of ξ_λ^+ and ζ_λ^+ we get

$$\begin{aligned}
(\hat{u}^{\alpha}(x)-\hat{u}_{\lambda}^{\alpha}(x))\xi_\lambda^+ &\leq \alpha \hat{u}_{\lambda}^{\alpha-2}(\hat{u}-\hat{u}_{\lambda})\xi_\lambda^+ \leq \alpha \hat{u}^{\alpha-2}(\xi_\lambda^+)^2,\\
(\hat{u}^{\alpha-1}(x)-\hat{u}_{\lambda}^{\alpha-1}(x))\xi_\lambda^+ &\leq (\alpha-1)\hat{u}_{\lambda}^{\alpha-2}(\hat{u}-\hat{u}_{\lambda})\xi_\lambda^+ \leq (\alpha-1)\hat{u}^{\alpha-2}(\xi_\lambda^+)^2,\\
(\hat{v}^{\beta}-\hat{v}_{\lambda}^{\beta})\zeta_\lambda^+ &\leq \beta \hat{v}_{\lambda}^{\beta-2}(\hat{v}-\hat{v}_{\lambda})\zeta_\lambda^+ \leq \beta \hat{v}^{\beta-2}(\zeta_\lambda^+)^2,\\
(\hat{v}^{\beta-1}-\hat{v}_{\lambda}^{\beta-1})\zeta_\lambda^+ &\leq (\beta-1)\hat{v}_{\lambda}^{\beta-2}(\hat{v}-\hat{v}_{\lambda})\zeta_\lambda^+ \leq (\beta-1)\hat{v}^{\beta-2}(\zeta_\lambda^+)^2.
\end{aligned}$$

Now, having in mind all these estimates, we need a fine analysis in view of the cooperativity of the system. Since $\alpha+\beta = 2^* = \frac{2N}{N-2}$ and $\alpha, \beta \geq 2$ we have to split

(6.4.18)

$$\begin{aligned}
|I_4| &\leq \frac{\alpha}{2^*}\int_{\Sigma_\lambda} |\hat{u}^{\alpha-1}\hat{v}^{\beta}-\hat{u}^{\alpha-1}\hat{v}_{\lambda}^{\beta}|\xi_\lambda^+\psi_\varepsilon^2\varphi_R^2\,dx\\
&\quad+\frac{\alpha}{2^*}\int_{\Sigma_\lambda} |\hat{u}^{\alpha-1}\hat{v}_{\lambda}^{\beta}-\hat{u}_{\lambda}^{\alpha-1}\hat{v}_{\lambda}^{\beta}|\xi_\lambda^+\psi_\varepsilon^2\varphi_R^2\,dx\\
&\leq \frac{\alpha\beta}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha-1}\hat{v}_{\lambda}^{\beta-1}\xi_\lambda^+\zeta_\lambda^+\psi_\varepsilon^2\varphi_R^2\,dx+\frac{\alpha(\alpha-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}_{\lambda}^{\alpha-2}\hat{v}_{\lambda}^{\beta}(\xi_\lambda^+)^2\psi_\varepsilon^2\varphi_R^2\,dx\\
&\leq \frac{\alpha\beta}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha-1}\hat{v}^{\beta-1}\hat{u}\hat{v}\psi_\varepsilon^2\varphi_R^2\,dx+\frac{\alpha(\alpha-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha-2}\hat{v}^{\beta}\hat{u}^2\psi_\varepsilon^2\varphi_R^2\,dx\\
&\leq \frac{\alpha\beta}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha}\hat{v}^{\beta}\,dx+\frac{\alpha(\alpha-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha}\hat{v}^{\beta}\,dx\\
&= \frac{\alpha(2^*-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha}\hat{v}^{\beta}\,dx
\end{aligned}$$

(6.4.19)

$$
\begin{aligned}
|E_4| \leq & \frac{\beta}{2^*}\int_{\Sigma_\lambda} |\hat{u}^\alpha \hat{v}^{\beta-1} - \hat{u}^\alpha \hat{v}_\lambda^{\beta-1}|\zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + \frac{\beta}{2^*}\int_{\Sigma_\lambda} |\hat{u}^\alpha \hat{v}_\lambda^{\beta-1} - \hat{u}_\lambda^\alpha \hat{v}_\lambda^{\beta-1}|\zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
\leq & \frac{\alpha\beta}{2^*}\int_{\Sigma_\lambda} \hat{u}_\lambda^{\alpha-1}\hat{v}^{\beta-1}\xi_\lambda^+\zeta_\lambda^+\psi_\varepsilon^2\varphi_R^2\, dx + \frac{\beta(\beta-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}_\lambda^\alpha \hat{v}_\lambda^{\beta-2}(\zeta_\lambda^+)^2\psi_\varepsilon^2\varphi_R^2\, dx \\
\leq & \frac{\alpha\beta}{2^*}\int_{\Sigma_\lambda} \hat{u}^{\alpha-1}\hat{v}^{\beta-1}\hat{u}\hat{v}\psi_\varepsilon^2\varphi_R^2\, dx + \frac{\beta(\beta-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}^\alpha \hat{v}^{\beta-2}\hat{v}^2\psi_\varepsilon^2\varphi_R^2\, dx \\
\leq & \frac{\alpha\beta}{2^*}\int_{\Sigma_\lambda} \hat{u}^\alpha\hat{v}^\beta\, dx + \frac{\alpha(\alpha-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}^\alpha\hat{v}^\beta\, dx \\
= & \frac{\beta(2^*-1)}{2^*}\int_{\Sigma_\lambda} \hat{u}^\alpha\hat{v}^\beta\, dx
\end{aligned}
$$

Hence, by applying Hölder inequality with exponents $\left(\frac{\alpha}{2^*}, \frac{\beta}{2^*}\right)$ it follows that

(6.4.20)
$$|I_4| + |E_4| \leq (2^*-1)\int_{\Sigma_\lambda} \hat{u}^\alpha\hat{v}^\beta\, dx \leq (2^*-1)\|\hat{u}\|^\alpha_{L^{2^*}(\Sigma_\lambda)}\|\hat{v}\|^\beta_{L^{2^*}(\Sigma_\lambda)}$$

Taking into account the estimates on I_1, I_2, I_3, I_4, E_1, E_2, E_3 and E_4, by adding (6.4.10) and (6.4.11), we deduce that

(6.4.21)

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla\xi_\lambda^+|^2\psi_\varepsilon^2\varphi_R^2\, dx + \int_{\Sigma_\lambda} |\nabla\zeta_\lambda^+|^2\psi_\varepsilon^2\varphi_R^2\, dx \leq & 32\varepsilon\left(\|\hat{u}\|^2_{L^\infty(\Sigma_\lambda)} + \|\hat{v}\|^2_{L^\infty(\Sigma_\lambda)}\right) \\
& + 2C(N)\left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} \hat{u}^{2^*}\, dx\right)^{\frac{N-2}{N}} \\
& + 2C(N)\left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} \hat{v}^{2^*}\, dx\right)^{\frac{N-2}{N}} \\
& + 2\frac{N+2}{N-2}\left(\|\hat{u}\|^{2^*}_{L^{2^*}(\Sigma_\lambda)} + \|\hat{v}\|^{2^*}_{L^{2^*}(\Sigma_\lambda)}\right) \\
& + 2(2^*-1)\|\hat{u}\|^\alpha_{L^{2^*}(\Sigma_\lambda)}\|\hat{v}\|^\beta_{L^{2^*}(\Sigma_\lambda)}
\end{aligned}
$$

By Fatou's Lemma, as ε tends to zero and R tends to infinity, we deduce that $\nabla\xi_\lambda^+, \nabla\zeta_\lambda^+ \in L^2(\Sigma_\lambda)$. We also note that $\varphi \to \xi_\lambda^+$ and $\psi \to \zeta_\lambda^+$ in $L^{2^*}(\Sigma_\lambda)$, by definition of φ and ψ, and that $\nabla\varphi \to \nabla\xi_\lambda^+$ and $\nabla\psi \to \nabla\zeta_\lambda^+$ in $L^2(\Sigma_\lambda)$, by (6.4.8), (6.4.9) and the fact that $\xi_\lambda^+, \zeta_\lambda^+ \in L^{2^*}(\Sigma_\lambda)$. Therefore

(6.4.22)

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla\xi_\lambda^+|^2\, dx + \int_{\Sigma_\lambda} |\nabla\zeta_\lambda^+|^2\, dx \leq & 2\frac{N+2}{N-2}\left(\|\hat{u}\|^{2^*}_{L^{2^*}(\Sigma_\lambda)} + \|\hat{v}\|^{2^*}_{L^{2^*}(\Sigma_\lambda)}\right) \\
& + 2(2^*-1)\|\hat{u}\|^\alpha_{L^{2^*}(\Sigma_\lambda)}\|\hat{v}\|^\beta_{L^{2^*}(\Sigma_\lambda)}
\end{aligned}
$$

Exploiting Young's inequality in the right hand side of (6.4.22), with conjugate exponents $\left(\frac{\alpha}{2^*}, \frac{\beta}{2^*}\right)$, we obtain (6.4.7).

□

We can now complete the proof of Theorem 6.0.4. As for the proof of Theorem 6.0.2 and Theorem 6.0.3, we split the proof into three steps and we start with

Step 1: there exists $M > 1$ such that $\hat{u} \leq \hat{u}_\lambda$ and $\hat{v} \leq \hat{v}_\lambda$ in $\Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\})$, for all $\lambda < -M$.

Arguing as in the proof of Lemma 6.4.1 and using the same notations and the same construction for ψ_ε, φ_R, φ and ψ, we get

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx = & - 2 \int_{\Sigma_\lambda} (\nabla \xi_\lambda^+, \nabla \psi_\varepsilon) \xi_\lambda^+ \psi_\varepsilon \varphi_R^2 \, dx \\
& - 2 \int_{\Sigma_\lambda} (\nabla \xi_\lambda^+, \nabla \varphi_R) \xi_\lambda^+ \varphi_R \psi_\varepsilon^2 \, dx \\
& + \int_{\Sigma_\lambda} (\hat{u}^{2^*-1} - \hat{u}_\lambda^{2^*-1}) \xi_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + \frac{\alpha}{2^*} \int_{\Sigma_\lambda} (\hat{u}^{\alpha-1} \hat{v}^\beta - \hat{u}_\lambda^{\alpha-1} \hat{v}_\lambda^\beta) \xi_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
=: & \; I_1 + I_2 + I_3 + I_4 \, .
\end{aligned}
\tag{6.4.23}
$$

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \psi_\varepsilon^2 \varphi_R^2 \, dx = & - 2 \int_{\Sigma_\lambda} (\nabla \zeta_\lambda^+, \nabla \psi_\varepsilon) \zeta_\lambda^+ \psi_\varepsilon \varphi_R^2 \, dx \\
& - 2 \int_{\Sigma_\lambda} (\nabla \zeta_\lambda^+, \nabla \varphi_R) \zeta_\lambda^+ \varphi_R \psi_\varepsilon^2 \, dx \\
& + \int_{\Sigma_\lambda} (\hat{v}^{2^*-1} - \hat{v}_\lambda^{2^*-1}) \zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
& + \frac{\beta}{2^*} \int_{\Sigma_\lambda} (\hat{u}^{\alpha} \hat{v}^{\beta-1} - \hat{u}_\lambda^{\alpha} \hat{v}_\lambda^{\beta-1}) \zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
=: & \; E_1 + E_2 + E_3 + E_4 \, .
\end{aligned}
\tag{6.4.24}
$$

where I_1, E_1, I_2, E_2, I_3, E_3, I_4 and E_4 can be estimated exactly as in (6.4.12), (6.4.13), (6.4.14), (6.4.15), (6.4.16), (6.4.17), (6.4.18) and (6.4.19).

The latter yield

(6.4.25)

$$\begin{aligned}\int_{\Sigma_\lambda} \left(|\nabla \xi_\lambda^+|^2 + |\nabla \zeta_\lambda^+|^2\right) \psi_\varepsilon^2 \varphi_R^2 \, dx \leq & 32\varepsilon \left(\|\hat{u}\|^2_{L^\infty(\Sigma_\lambda)} + \|\hat{v}\|^2_{L^\infty(\Sigma_\lambda)} \right) \\ & + 2C(N) \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{u}^{2^*} \, dx \right)^{\frac{2}{2^*}} \\ & + 2C(N) \left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{v}^{2^*} \, dx \right)^{\frac{2}{2^*}} \\ & + 2\frac{N+2}{N-2} \int_{\Sigma_\lambda} \hat{u}^{2^*-2} (\xi_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\ & + 2\frac{N+2}{N-2} \int_{\Sigma_\lambda} \hat{v}^{2^*-2} (\zeta_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\ & + 4\frac{\alpha\beta}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha-1} \hat{v}^{\beta-1} \xi_\lambda^+ \zeta_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\ & + \frac{\alpha(\alpha-1)}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha-2} \hat{v}^{\beta} (\xi_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \\ & + \frac{\beta(\beta-1)}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha} \hat{v}^{\beta-2} (\zeta_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx \, .\end{aligned}$$

Passing to the limit in the latter, as ε tends to zero and R tends to infinity, we obtain

$$\begin{aligned}\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + \int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \, dx & \\ \leq & 2\frac{N+2}{N-2} \left(\int_{\Sigma_\lambda} \hat{u}^{2^*-2} (\xi_\lambda^+)^2 \, dx + \int_{\Sigma_\lambda} \hat{v}^{2^*-2} (\zeta_\lambda^+)^2 \, dx \right) \\ & + 4\frac{\alpha\beta}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha-1} \hat{v}^{\beta-1} \xi_\lambda^+ \zeta_\lambda^+ \, dx \\ & + \frac{\alpha(\alpha-1)}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha-2} \hat{v}^{\beta} (\xi_\lambda^+)^2 \, dx \\ & + \frac{\beta(\beta-1)}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha} \hat{v}^{\beta-2} (\zeta_\lambda^+)^2 \, dx \; < +\infty\end{aligned} \tag{6.4.26}$$

which combined with Young's inequality gives

$$\begin{aligned}\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + \int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \, dx & \\ \leq & 2\frac{N+2}{N-2} \left(\int_{\Sigma_\lambda} \hat{u}^{2^*-2} (\xi_\lambda^+)^2 \, dx + \int_{\Sigma_\lambda} \hat{v}^{2^*-2} (\zeta_\lambda^+)^2 \, dx \right) \\ & + \frac{\alpha(2^*+\beta-1)}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha-2} \hat{v}^{\beta} (\xi_\lambda^+)^2 \, dx \\ & + \frac{\beta(2^*+\beta-1)}{2^*} \int_{\Sigma_\lambda} \hat{u}^{\alpha} \hat{v}^{\beta-2} (\zeta_\lambda^+)^2 \, dx \\ = & : A_1 + A_2 + A_3.\end{aligned} \tag{6.4.27}$$

Exploiting Hölder inequality with conjugate exponents $\left(\dfrac{2^*}{2^*-2}, \dfrac{2^*}{2}\right)$ we obtain

$$|A_1| \le 2\frac{N+2}{N-2}\left[\left(\int_{\Sigma_\lambda}\hat u^{2^*}\,dx\right)^{\frac{2}{N}}\left(\int_{\Sigma_\lambda}(\xi_\lambda^+)^{2^*}\,dx\right)^{\frac{2}{2^*}} + \left(\int_{\Sigma_\lambda}\hat v^{2^*}\,dx\right)^{\frac{2}{N}}\left(\int_{\Sigma_\lambda}(\zeta_\lambda^+)^{2^*}\,dx\right)^{\frac{2}{2^*}}\right]. \tag{6.4.28}$$

Exploiting Hölder inequality with conjugate exponents $\left(\dfrac{2^*}{\alpha-2}, \dfrac{2^*}{\beta}, \dfrac{2^*}{2}\right)$ (we note that if $\alpha = 2$ we have $\beta = 2$ and the conjugate exponents would be $(\frac{2^*}{2}, \frac{2^*}{2})$) we obtain

$$|A_2| \le \frac{\alpha(2^*+\beta-1)}{2^*}\left(\int_{\Sigma_\lambda}\hat u^{2^*}\,dx\right)^{\frac{\alpha-2}{2^*}}\left(\int_{\Sigma_\lambda}\hat v^{2^*}\,dx\right)^{\frac{\beta}{2^*}}\left(\int_{\Sigma_\lambda}(\xi_\lambda^+)^{2^*}\,dx\right)^{\frac{2}{2^*}}. \tag{6.4.29}$$

Exploiting Hölder inequality with conjugate exponents $\left(\dfrac{2^*}{\alpha}, \dfrac{2^*}{\beta-2}, \dfrac{2^*}{2}\right)$ (we note that if $\beta = 2$ we have $\alpha = 2$ and the conjugate exponents would be $(\frac{2^*}{2}, \frac{2^*}{2})$)we obtain

$$|A_3| \le \frac{\beta(2^*+\alpha-1)}{2^*}\left(\int_{\Sigma_\lambda}\hat u^{2^*}\,dx\right)^{\frac{\alpha}{2^*}}\left(\int_{\Sigma_\lambda}\hat v^{2^*}\,dx\right)^{\frac{\beta-2}{2^*}}\left(\int_{\Sigma_\lambda}(\zeta_\lambda^+)^{2^*}\,dx\right)^{\frac{2}{2^*}}. \tag{6.4.30}$$

Combining (6.4.28), (6.4.29) and (6.4.30) and applying Sobolev inequality

$$\int_{\Sigma_\lambda}|\nabla\xi_\lambda^+|^2\,dx + \int_{\Sigma_\lambda}|\nabla\zeta_\lambda^+|^2\,dx \le C_1\int_{\Sigma_\lambda}|\nabla\xi_\lambda^+|^2\,dx + C_2\int_{\Sigma_\lambda}|\nabla\zeta_\lambda^+|^2\,dx, \tag{6.4.31}$$

where $C_1 := \left[2\frac{N+2}{N-2}\|\hat u\|^{\beta}_{L^{2^*}(\Sigma_\lambda)} + \frac{\alpha(2^*+\beta-1)}{2^*}\|\hat v\|^{\beta}_{L^{2^*}(\Sigma_\lambda)}\right] C^2_{u,S}\|\hat u\|^{\alpha-2}_{L^{2^*}(\Sigma_\lambda)}$, $C_2 := \left[2\frac{N+2}{N-2}\|\hat v\|^{\alpha}_{L^{2^*}(\Sigma_\lambda)} + \frac{\beta(2^*+\beta-1)}{2^*}\|\hat u\|^{\alpha}_{L^{2^*}(\Sigma_\lambda)}\right] C^2_{v,S}\|\hat v\|^{\beta-2}_{L^{2^*}(\Sigma_\lambda)}$, $C_{u,S}$ and $C_{v,S}$ are the Sobolev constants. Recalling that $\hat u, \hat v \in L^{2^*}(\Sigma_\lambda)$, we deduce the existence of $M > 1$ such that

$$C_1 := \left[2\frac{N+2}{N-2}\|\hat u\|^{\beta}_{L^{2^*}(\Sigma_\lambda)} + \frac{\alpha(2^*+\beta-1)}{2^*}\|\hat v\|^{\beta}_{L^{2^*}(\Sigma_\lambda)}\right] C^2_{u,S}\|\hat u\|^{\alpha-2}_{L^{2^*}(\Sigma_\lambda)} < 1$$

and

$$C_2 := \left[2\frac{N+2}{N-2}\|\hat v\|^{\alpha}_{L^{2^*}(\Sigma_\lambda)} + \frac{\beta(2^*+\beta-1)}{2^*}\|\hat u\|^{\alpha}_{L^{2^*}(\Sigma_\lambda)}\right] C^2_{v,S}\|\hat v\|^{\beta-2}_{L^{2^*}(\Sigma_\lambda)} < 1$$

for every $\lambda < -M$. The latter and (6.4.27) lead to

$$\int_{\Sigma_\lambda}|\nabla\xi_\lambda^+|^2\,dx = 0 \quad \text{and} \quad \int_{\Sigma_\lambda}|\nabla\zeta_\lambda^+|^2\,dx = 0.$$

This implies that $\xi_\lambda^+ = \zeta_\lambda^+ = 0$ by Lemma 6.4.1 and the claim is proved.

To proceed further we define

$$\Lambda_0 = \{\lambda < 0 : \hat{u} \leq \hat{u}_t \text{ and } \hat{v} \leq \hat{v}_t \text{ in } \Sigma_t \setminus R_t(\Gamma^* \cup \{0\}) \text{ for all } t \in (-\infty, \lambda]\}$$

and

$$\lambda_0 = \sup \Lambda_0.$$

Step 2: we have that $\lambda_0 = 0$. We argue by contradiction and suppose that $\lambda_0 < 0$. By continuity we know that $\hat{u} \leq \hat{u}_{\lambda_0}$ and $\hat{v} \leq \hat{v}_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$. By the strong maximum principle we deduce that $\hat{u} < \hat{u}_{\lambda_0}$ and $\hat{v} < \hat{v}_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\})$. Indeed, $\hat{u} = \hat{u}_{\lambda_0}$ and $\hat{v} = \hat{v}_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\}))$ is not possible if $\lambda_0 < 0$, since in this case $\hat{u}$ and $\hat{v}$ would be singular somewhere on $R_{\lambda_0}(\Gamma^* \cup \{0\})$. Now, for some $\bar{\tau} > 0$, that will be fixed later on, and for any $0 < \tau < \bar{\tau}$ we show that $\hat{u} \leq \hat{u}_{\lambda_0+\tau}$ and $\hat{v} \leq \hat{v}_{\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ obtaining a contradiction with the definition of λ_0 and proving thus the claim. To this end we recall that, repeating verbatim the argument used in the roof of Theorem 6.0.3, it is possible to prove that for every $\delta > 0$ there are $\bar{\tau}(\delta, \lambda_0) > 0$ and a compact set K (depending on δ and λ_0) such that

$$K \subset \Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\}), \int_{\Sigma_\lambda \setminus K} \hat{u}^{2^*} < \delta \text{ and } \int_{\Sigma_\lambda \setminus K} \hat{v}^{2^*} < \delta, \ \forall \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}].$$

Now we repeat verbatim the arguments used in the proof of Lemma 6.4.1 but using the test function

$$\varphi := \begin{cases} \xi^+_{\lambda_0+\tau} \psi_\varepsilon^2 \varphi_R^2 & \text{in } \Sigma_{\lambda_0+\tau} \\ 0 & \text{in } \mathbb{R}^N \setminus \Sigma_{\lambda_0+\tau}. \end{cases}$$

and

$$\psi := \begin{cases} \zeta^+_{\lambda_0+\tau} \psi_\varepsilon^2 \varphi_R^2 & \text{in } \Sigma_{\lambda_0+\tau} \\ 0 & \text{in } \mathbb{R}^N \setminus \Sigma_{\lambda_0+\tau}. \end{cases}$$

Thus we recover the first inequality in (6.4.27), and repeating verbatim the arguments used in (6.4.28), (6.4.29) and (6.4.30) which immediately gives, for any $0 \leq \tau < \bar{\tau}$

$$\begin{aligned} &\int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla \xi^+_{\lambda_0+\tau}|^2 \, dx + \int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla \zeta^+_{\lambda_0+\tau}|^2 \, dx \\ &\qquad \leq C_1 C_{u,S}^2 \|\hat{u}\|^{\alpha-2}_{L^{2^*}(\Sigma_{\lambda_0+\tau} \setminus K)} \int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla \xi^+_{\lambda_0+\tau}|^2 \, dx \\ &\qquad\quad + C_2 C_{v,S}^2 \|\hat{v}\|^{\beta-2}_{L^{2^*}(\Sigma_{\lambda_0+\tau} \setminus K)} \int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla \zeta^+_{\lambda_0+\tau}|^2 \, dx, \end{aligned} \tag{6.4.32}$$

where

$$C_1 := 2\frac{N+2}{N-2} \|\hat{u}\|^{\beta}_{L^{2^*}(\Sigma_{\lambda_0+\tau} \setminus K)} + \frac{\alpha(2^* + \beta - 1)}{2^*} \|\hat{v}\|^{\beta}_{L^{2^*}(\Sigma_{\lambda_0+\tau} \setminus K)},$$

$$C_2 := 2\frac{N+2}{N-2} \|\hat{v}\|^{\alpha}_{L^{2^*}(\Sigma_{\lambda_0+\tau} \setminus K)} + \frac{\beta(2^* + \beta - 1)}{2^*} \|\hat{u}\|^{\alpha}_{L^{2^*}(\Sigma_{\lambda_0+\tau} \setminus K)},$$

$C_{u,S}$ and $C_{v,S}$ are the Sobolev constants. Now we fix

$$\delta < \min\left\{C_1 C_{u,S}^2 \|\hat{u}\|_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}^{\alpha-2}, C_2 C_{v,S}^2 \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}^{\beta-2}\right\}$$

and we observe that with this choice we have

$$C_1 C_{u,S}^2 \|\hat{u}\|_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}^{\alpha-2} < 1 \qquad \text{and} \qquad C_2 C_{v,S}^2 \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}^{\beta-2} < 1,$$

for all $0 \leq \tau < \bar{\tau}$, which plugged into (6.4.32) implies that

$$\int_{\Sigma_{\lambda_0+\tau}\setminus K} |\nabla \xi_{\lambda_0+\tau}^+|^2\, dx = \int_{\Sigma_{\lambda_0+\tau}\setminus K} |\nabla \zeta_{\lambda_0+\tau}^+|^2\, dx = 0$$

for every $0 \leq \tau < \bar{\tau}$. Hence

$$\int_{\Sigma_{\lambda_0+\tau}} |\nabla \xi_{\lambda_0+\tau}^+|^2\, dx = \int_{\Sigma_{\lambda_0+\tau}} |\nabla \zeta_{\lambda_0+\tau}^+|^2\, dx = 0$$

for every $0 \leq \tau < \bar{\tau}$, since $\nabla \xi_{\lambda_0+\tau}^+$ and $\nabla \zeta_{\lambda_0+\tau}^+$ are zero in a neighbourhood of K. The latter and Lemma 6.4.1 imply that $\xi_{\lambda_0+\tau}^+ = 0$ and $\zeta_{\lambda_0+\tau}^+ = 0$ on $\Sigma_{\lambda_0+\tau}$ for every $0 \leq \tau < \bar{\tau}$ and thus $\hat{u} \leq \hat{u}_{\lambda_0+\tau}$ and $\hat{v} \leq \hat{v}_{\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ for every $0 \leq \tau < \bar{\tau}$. Which proves the claim of Step 2.

Step 3: conclusion. The symmetry of the Kelvin transform v follows now performing the moving planes method in the opposite direction. The fact that $\hat{u}$ and $\hat{v}$ are symmetric w.r.t. the hyperplane $\{x_1 = 0\}$ implies the symmetry of the solution (u, v) w.r.t. the hyperplane $\{x_1 = 0\}$. The last claim then follows by the invariance of the considered problem with respect to isometries (translations and rotations).

□

7

Gibbons' Conjecture for equations involving the p-Laplacian

This Chapter concerns the study of the qualitative properties of the following quasilinear elliptic equation

$$-\Delta_p u = f(u) \quad \text{in } \mathbb{R}^N, \tag{7.0.1}$$

where we denote a generic point belonging to $\mathbb{R}^N$ by (x', y) with $x' = (x_1, x_2, \ldots, x_{N-1})$ and $y = x_N$, $p > 1$ and $N > 1$. As well known, see [46, 122], the solutions of equations involving the p-Laplace operator are generally of class $C^{1,\alpha}$. Therefore the equation (7.0.1) has to be understood in the weak sense, see Definition 7.1.6 below. We summarize the assumptions on the nonlinearity f (denoted by (G_f) in the following) as follows:

(G_f): The nonlinearity $f(\cdot)$ belongs to $C^1([-1,1])$, $f(-1) = 0$, $f(1) = 0$, $f'_+(-1) < 0$, $f'_-(1) < 0$ and the set

$$\mathcal{N}_f := \{t \in [-1,1] \mid f(t) = 0\}$$

is finite.

As remarked in the Introduction, the setting of our assumptions allows us to include Allen-Cahn type nonlinearities and in fact this chapter is motivated by some questions arising from the following problem

$$-\Delta u = u(1-u^2) \quad \text{in } \mathbb{R}^N, \tag{7.0.2}$$

see [65]. G.W. Gibbons [29] formulated the following

Gibbons' conjecture [29]. – Assume $N > 1$ and consider a bounded solution u of (7.0.2) in $C^2(\mathbb{R}^N)$, such that

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1,$$

uniformly with respect to x'. Then, is it true that

$$u(x) = \tanh\left(\frac{x_N - \alpha}{\sqrt{2}}\right),$$

for some $\alpha \in \mathbb{R}$?

This conjecture is also known as the weaker version of the famous De Giorgi's conjecture [45]. We refer to [55] for a complete history on the argument. Our main result is the following

Theorem 7.0.1. *Let* $N > 1$, $(2N+2)/(N+2) < p < 2$ *and* $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ *be a solution of* (7.0.1), *such that*

$$|u| \leq 1$$

and

$$\lim_{y\to+\infty} u(x',y) = 1 \quad \text{and} \quad \lim_{y\to-\infty} u(x',y) = -1, \tag{7.0.3}$$

uniformly with respect to $x' \in \mathbb{R}^{N-1}$. *If* f *fulfills* (G_f), *then* u *depends only on* y *and*

$$\partial_y u > 0 \quad \text{in } \mathbb{R}^N. \tag{7.0.4}$$

Remark 7.0.2. *We want to point out that, by the strong maximum principle* [127], *see also Theorem 7.1.1, applied to* (7.0.1), *we deduce that a solution to* (7.0.1) *under the assumptions of Theorem 7.0.1, actually satisfies* $|u| < 1$ *otherwise* $u \equiv \pm 1$ *in all* $\mathbb{R}^N$. *We will use this information several times throughout the chapter. Moreover by classical regularity results* [46], *since* $\|f(u)\|_{L^\infty(\mathbb{R}^N)} \leq C$, *with* C *a positive constant that does not depend on* u, *we also deduce that*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

To get our main result, we first recover a weak comparison principle in a suitable half-space and then we exploit it to start the moving planes procedure. The application of the moving planes method is not standard since we have to recover compactness using some translation arguments, (since we work on $\mathbb{R}^N$) and, not least, we have to take into account the fact that the nonlinearity f change sign which produces peculiar difficulties in the case $p \neq 2$, already in the case of bounded domain. Finally we get the monotonicity in all the directions of the the upper hemi-sphere $\mathbb{S}^{N-1}_+ := \{\nu \in \mathbb{S}^{N-1}_+ \mid (\nu, e_N)\}$ that will give us the desired 1-dimensional symmetry. This chapter is organized as follows: In Section 7.1 we recall some results about strong maximum and comparison principles just for the reader's convenience, already presented in Chapter 1. In Section 7.2 we prove the monotonicity of the solution in the x_N-direction, exploiting the moving planes procedure. In Section 7.3 we prove the 1-dimensional symmetry and finally we prove our main result.

7.1. Preliminary results

The aim of this section is to recall, for the reader's convenience, some well known results about strong comparison principles and strong maximum principles for quasilinear elliptic equations that will be used several times in the proof of our main theorem. Lots of this results are proved in Chapter 1. Let us consider the following quasilinear elliptic equation

$$-\Delta_p w = f(w) \qquad \text{in } \Omega \tag{7.1.1}$$

where Ω is any domain of $\mathbb{R}^N$ and f is a locally Lipschitz continuous function. Any solution w to (7.1.1) has to be understood in the weak distributional sense (see Definition 7.1.6) and generally is of class $C^{1,\alpha}$.

The first result that we recall the classical strong maximum principle due to J. L. Vazquez [127] (see also P. Pucci and J. Serrin book [103] and Theorem 1.4.1):

Theorem 7.1.1 (Strong Maximum Principle and Höpf's Lemma, [103, 127]). *Let $u \in C^1(\Omega)$ be a non-negative weak solution to*

$$-\Delta_p u + cu^q = g \geq 0 \quad in \quad \Omega$$

with $1 < p < +\infty$, $q \geq p-1$, $c \geq 0$ and $g \in L^\infty_{loc}(\Omega)$. If $u \neq 0$, then $u > 0$ in Ω. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that $u \in C^1(\Omega) \cup \{x_0\}$ and $u(x_0) = 0$ we have that $\partial_\nu u > 0$ for any inward directional derivative (this means that if y approaches x_0 in a ball $B \subseteq \Omega$ that has x_0 on its boundary, then $\lim_{y\to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).

It is very simple to guess that in the quasilinear case, maximum and comparison principles are not equivalent; for this reason we need also to recall the classical version of the strong comparison principle for quasilinear elliptic equations:

Theorem 7.1.2 (Classical Strong Comparison Principle, [33, 103]). *Let $u, v \in C^1(\Omega)$ be two solutions to (7.1.1) such that $u \leq v$ in Ω, with $1 < p < +\infty$ and let $\mathcal{Z} = \{x \in \Omega \mid |\nabla u(x)| + |\nabla v(x)| \neq 0\}$. If $x_0 \in \Omega \setminus \mathcal{Z}$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus \mathcal{Z}$ containing x_0.*

For the proof of this result we suggest [33, 103]. The main feature of Theorem 7.1.2 is that holds far from the critical set. Now we present a result which holds, under stronger assumptions, all over the critical set and generalizes Theorem 1.5.3:

Theorem 7.1.3 (Strong Comparison Principle, [36]). *Let $u, v \in C^1(\overline{\Omega})$ be two solutions to (7.1.1), where Ω is a bounded smooth connected domain of $\mathbb{R}^N$ and $\frac{2N+2}{N+2} < p < +\infty$. Assume that at least one of the following two conditions (f_u),(f_v) holds:*

(f_u): *either*

$$f(u(x)) > 0 \quad in \quad \overline{\Omega} \tag{7.1.2}$$

or

$$f(u(x)) < 0 \quad in \quad \overline{\Omega}; \tag{7.1.3}$$

(f_v): *either*

$$f(v(x)) > 0 \quad in \quad \overline{\Omega} \tag{7.1.4}$$

or

$$f(v(x)) < 0 \quad in \quad \overline{\Omega}. \tag{7.1.5}$$

Moreover, if

$$u \leq v \quad in \quad \Omega. \tag{7.1.6}$$

Then $u \equiv v$ in Ω unless

$$u < v \quad in \quad \Omega. \tag{7.1.7}$$

PROOF. The proof of this result follows by the same arguments in [36, 61, 108, 109]. Note in fact that under the assumption (f_u) or (f_v), it follows that $|\nabla u|^{-1}$ or $|\nabla v|^{-1}$ has the summability properties exposed by Theorem 3.1 in [109] (see also Theorem 1.1.2). Then the weighted Sobolev inequality is in force, see e.g. Theorem 8 in [61] (see also Theorem 1.1.3).

Now, it is sufficient to note that the Harnack comparison inequality given by Corollary 3.2 in [36] holds true (see also Theorem 1.5.2), since the proof it is only based on the weighted Sobolev inequality.

Finally it is standard to see that the Strong Comparison Principle follows by the weak comparison Harnack inequality, see Theorem 1.4 in [36] (see also Theorem 1.5.3).

□

Let us now recall that the linearized operator at a fixed solution w of (7.1.1) (see also equation (1.1.4)), $L_w(v,\varphi)$, is well defined, for every v and φ in the weighted Sobolev space $H^{1,2}_\varrho(\Omega)$ with $\varrho = |\nabla w|^{p-2}$, by

(7.1.8)

$$L_w(v,\varphi) \equiv \int_\Omega |\nabla w|^{p-2}(\nabla v, \nabla\varphi) + (p-2)\int_\Omega |\nabla w|^{p-4}(\nabla w, \nabla v)(\nabla w, \nabla\varphi)\, dx \\ - \int_\Omega f'(w)v\varphi\, dx, \qquad \forall \varphi \in C^\infty_c(\Omega).$$

Moreover $v \in H^{1,2}_\varrho(\Omega)$ is a weak solution of the linearized operator if

$$L_w(v,\varphi) = 0. \tag{7.1.9}$$

As in the case of equation (7.1.1), also for the linearized equation (7.1.9) a classical version of the strong maximum principle holds:

THEOREM 7.1.4 (Classical Strong Maximum Principle for the Linearized Operator, [103]). *Let $u \in C^1(\overline{\Omega})$ be a solution to problem (7.1.1), with $1 < p < +\infty$. Let $\eta \in \mathbb{S}^{N-1}$ and let us assume that for any connected domain $\Omega' \subset \Omega \setminus \mathcal{Z}$*

$$\partial_\eta u \geq 0 \quad in \quad \Omega'. \tag{7.1.10}$$

Then $\partial_\eta u \equiv 0$ in Ω' unless

$$\partial_\eta u > 0 \quad in \quad \Omega'. \tag{7.1.11}$$

Let us recall now a more general result which holds all over the critical set $\mathcal{Z}$ and that generalizes Theorem 1.5.5 of Chapter 1:

THEOREM 7.1.5 (Strong Maximum Principle for the Linearized Operator, [36]). *Let $u \in C^1(\Omega)$ be a solution to problem (7.1.1), with $\frac{2N+2}{N+2} < p < +\infty$. Assume that either*

$$f(u(x)) > 0 \quad in \quad \overline{\Omega} \tag{7.1.12}$$

or

$$f(u(x)) < 0 \quad in \quad \overline{\Omega}. \tag{7.1.13}$$

If $\partial_\eta \geq 0$ in Ω, for some $\eta \in \mathbb{S}^{N-1}$, then either $\partial_\eta u \equiv 0$ in Ω or $\partial_\eta u > 0$ in Ω.

A solution to (7.0.1) has to be understood in the weak distributional sense. We start giving the following

Definition 7.1.6. *Let $\Omega \subseteq \mathbb{R}^N$ an open set. We say that $u \in C^{1,\alpha}_{loc}(\Omega)$ is a* weak subsolution *to*

$$-\Delta_p u = f(u) \quad in\ \Omega, \tag{7.1.14}$$

if

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\, dx \le \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0. \tag{7.1.15}$$

Similarly, we say that $u \in C^{1,\alpha}_{loc}(\Omega)$ is a weak supersolution *to* (7.0.1) *if*

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\, dx \ge \int_\Omega f(u)\varphi\, dx \qquad \forall \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0. \tag{7.1.16}$$

Finally, we say that $u \in C^{1,\alpha}_{loc}(\Omega)$ is a weak solution *of equation* (7.1.14)*, if* (7.1.15) *and* (7.1.16) *hold.*

Moreover we need to recall the weak comparison principle (see Theorem 1.3.7) between a subsolution and a supersolution to (7.0.1) ordered on the boundary of some open half-space Σ of $\mathbb{R}^N$, whose proof is included in Chapter 1.

Theorem 7.1.7. *Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma})$, $N > 1$, $p > 1$, $a \in \mathbb{R}$ such that*

$$\begin{cases} -\Delta_p u \le f(u) & in\ \Sigma \\ -\Delta_p v \ge f(v) & in\ \Sigma \\ u \le v & on\ \partial\Sigma, \end{cases} \tag{7.1.17}$$

where Σ is some open half-space of $\mathbb{R}^N$ and $f \in C^1(\mathbb{R})$. Moreover, let us assume that

$$|\nabla u|, |\nabla v| \in L^\infty(\Sigma),$$

for some δ sufficiently small

$$-1 \le u \le -1 + \delta \quad in\ \Sigma := (-\infty, a)$$

and for some $L > 0$

$$f'(t) < -L \quad in\ [-1, -1+\delta]. \tag{7.1.18}$$

Then

$$u \le v \quad in\ \Sigma. \tag{7.1.19}$$

The same result is true if

$$1 - \delta \le v \le 1 \quad in\ \Sigma := (a, +\infty) \quad and \quad f'(t) < -L \quad in\ [1-\delta, 1].$$

Let us recall another weak comparison principle in narrow domains that will be an essential tool in the proof of Theorem 7.0.1, whose proof is also included in Chapter 1 (see Theorem 1.3.3).

THEOREM 7.1.8 ([58]). *Let* $1 < p < 2$ *and* $N > 1$. *Fix* $\lambda_0 > 0$ *and* $L_0 > 0$. *Consider* $a, b \in \mathbb{R}$, *with* $a < b$, $\tau, \epsilon > 0$ *and set*

$$\Sigma_{(a,b)} := \left\{\mathbb{R}^{N-1} \times (a,b)\right\}.$$

Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma}_{(a,b)})$ *such that* $\|u\|_\infty + \|\nabla u\|_\infty \le L_0$, $\|v\|_\infty + \|\nabla v\|_\infty \le L_0$, f *fulfills* (h_f) *and*

$$\begin{cases} -\Delta_p u \le f(u) & \text{in } \Sigma_{(a,b)} \\ -\Delta_p v \ge f(v) & \text{in } \Sigma_{(a,b)} \\ u \le v & \text{on } \partial\mathcal{S}_{(\tau,\epsilon)}, \end{cases} \tag{7.1.20}$$

where the open set $\mathcal{S}_{(\tau,\epsilon)} \subseteq \Sigma_{(a,b)}$ *is such that*

$$\mathcal{S}_{(\tau,\epsilon)} = \bigcup_{x' \in \mathbb{R}^{N-1}} I^{\tau,\epsilon}_{x'},$$

and the open set $I^{\tau,\epsilon}_{x'} \subseteq \{x'\} \times (a,b)$ *has the form*

$$I^{\tau,\epsilon}_{x'} = A^\tau_{x'} \cup B^\epsilon_{x'}, \text{ with } |A^\tau_{x'} \cap B^\epsilon_{x'}| = \emptyset$$

and, for x' *fixed,* $A^\tau_{x'}, B^\epsilon_{x'} \subset (a,b)$ *are measurable sets such that*

$$|A^\tau_{x'}| \le \tau \quad \text{and} \quad B^\epsilon_{x'} \subseteq \{x_N \in \mathbb{R} \mid |\nabla u(x', x_N)| < \epsilon, \ |\nabla v(x', x_N)| < \epsilon\}.$$

Then there exist

$$\tau_0 = \tau_0(N, p, a, b, L_0) > 0$$

and

$$\epsilon_0 = \epsilon_0(N, p, a, b, L_0) > 0$$

such that, if $0 < \tau < \tau_0$ *and* $0 < \epsilon < \epsilon_0$, *it follows that*

$$u \le v \quad \text{in } \mathcal{S}_{(\tau,\epsilon)}.$$

7.2. Monotonicity with respect to x_N

The purpose of this section consists in showing that all the non-trivial solutions u to (7.0.1) that satisfies (7.0.3) are increasing in the x_N direction. Since in our problem the right hand side depends only on u, it is possible to define the following set

$$\mathcal{Z}_{f(u)} := \{x \in \mathbb{R}^N \mid u(x) \in \mathcal{N}_f\}.$$

Without any apriori assumption on the behaviour of ∇u, the set $\mathcal{Z}_{f(u)}$ may be very wild, see Figure 1.

We start proving a lemma that we will use repeatedly in the sequel of the work.

Let us define the upper hemisphere

$$\mathbb{S}^{N-1}_+ := \{\nu \in \mathbb{S}^{N-1} \mid (\nu, e_N) > 0\}. \tag{7.2.1}$$

LEMMA 7.2.1. *Let* $\mathcal{U}$ *a connected component of* $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$, $\eta \in \mathbb{S}^{N-1}_+$ *and let us assume that* $\partial_\eta u \ge 0$ *in* $\mathcal{U}$. *Then*

$$\partial_\eta u > 0 \quad \text{in } \mathcal{U}.$$

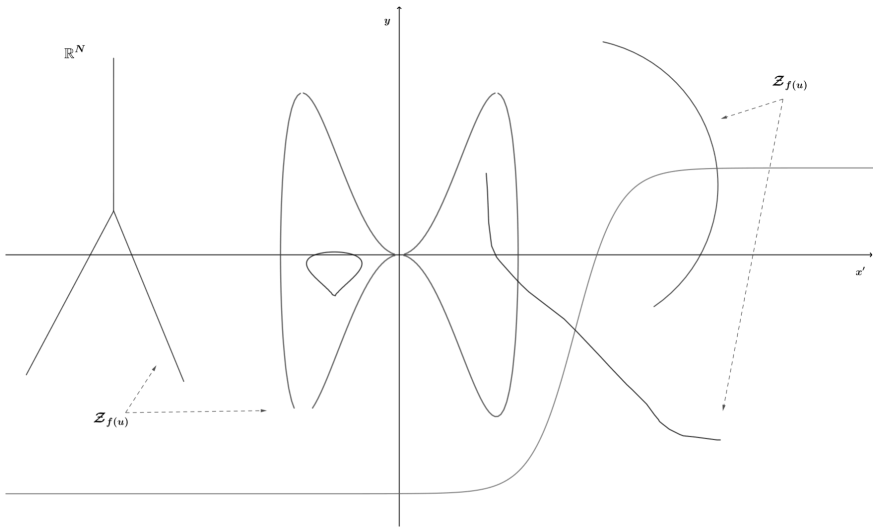

FIGURE 1. The set $\mathcal{Z}_{f(u)}$

PROOF. Using Theorem 7.1.5 we deduce that either $\partial_\eta u > 0$ in $\mathcal{U}$ or $\partial_\eta u \equiv 0$ in $\mathcal{U}$. Let us suppose that $\partial_\eta u \equiv 0$ in $\mathcal{U}$. Let $P_0 \in \mathcal{U}$ and let us define

$$r(t) = P_0 + t\eta \quad t \in \mathbb{R}.$$

Let us set

$$t_0 = \inf \Big\{ t \in \mathbb{R} \, : \, r(\vartheta) \in \mathcal{U}, \, \forall \vartheta \in (t, 0] \Big\}. \tag{7.2.2}$$

We note that the infimum in (7.2.2) is well defined, since by definition the connected component $\mathcal{U}$ is an open set. We deduce that either

$$t_0 = -\infty \quad \text{or} \quad t_0 > -\infty.$$

In the case $t_0 = -\infty$, we deduce that $u(P_0) = -1$. Indeed u should be constant (recall that $\partial_\eta u \equiv 0$ in $\mathcal{U}$) on $r(t)$ for $t \in (-\infty, 0]$ and (7.0.3) holds. This would be a contradiction, see Remark 7.0.2.

In the case $t_0 > -\infty$, we deduce that $r(t_0) \in \mathcal{Z}_{f(u)}$ and therefore $f(u(r(t_0))) = f(u(P_0 + t_0\eta)) = 0$. But u should be constant on $r(t)$ for $t_0 \leq t \leq 0$, implying $f(u(P_0)) = f(u(P_0 + t_0\eta)) = 0$, namely $P_0 \in \mathcal{Z}_{f(u)}$ against the assumption.

□

PROPOSITION 7.2.2. *Under the assumptions of Theorem 7.0.1, we have that*

$$\partial_{x_N} u > 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{Z}_{f(u)}. \tag{7.2.3}$$

The proof is based on a nontrivial modification of the moving planes method. Let us recall some notations. We define the half-space Σ_λ and the hyperplane T_λ by

$$\Sigma_\lambda := \{x \in \mathbb{R}^N \mid x_N < \lambda\}, \qquad T_\lambda := \partial\Sigma_\lambda = \{x \in \mathbb{R}^N \mid x_N = \lambda\} \tag{7.2.4}$$

and the reflected function $u_\lambda(x)$ by

$$u_\lambda(x) = u_\lambda(x', x_N) := u(x', 2\lambda - x_N) \quad \text{in } \mathbb{R}^N.$$

We also define the critical set $\mathcal{Z}_{\nabla u}$ by

$$\mathcal{Z}_{\nabla u} := \{x \in \mathbb{R}^N \mid \nabla u(x) = 0\}. \tag{7.2.5}$$

The first step in the proof of the monotonicity is to get a property concerning the local symmetry regions of the solution, namely any $C \subseteq \Sigma_\lambda$ such that $u \equiv u_\lambda$ in C.

Having in mind these notations we are able to prove the following:

PROPOSITION 7.2.3. *Under the assumption of Theorem 7.0.1, let us assume that u is a solution to (7.0.1) satisfying (7.0.3), such that*

(i) *u is monotone non-decreasing in Σ_λ*

and

(ii) *$u \leq u_\lambda$ in Σ_λ.*

Then $u < u_\lambda$ in $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$.

PROOF. By (7.0.3), given $0 < \delta_0 < 1$ there exists $M_0 = M_0(\delta_0) > 0$, with $\lambda > -M_0$, such that $u(x) = u(x', x_N) < -1 + \delta_0$ in $\{x_N < -M_0\}$ and $u_\lambda(x) = u(x', 2\lambda - x_N) > 1 - \delta_0$ in $\{x_N < -M_0\}$. We fix δ_0 sufficiently small such that $f'(u) < -L$ in $\{x_N < -M_0\}$, for some $L > 0$. Arguing by contradiction, let us assume that there exists $P_0 = (x_0', x_{N,0}) \in \Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$ such that $u(P_0) = u_\lambda(P_0)$. Let $\mathcal{U}_0$ the connected component of $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$ containing P_0. By Theorem 7.1.3, since $u(P_0) = u_\lambda(P_0)$, we deduce that $\mathcal{U}_0$ is a local symmetry region, i.e. $u \equiv u_\lambda$ in $\mathcal{U}_0$.

We notice that, by construction, $u < u_\lambda$ in Σ_{-M_0}, since $u(x) < -1 + \delta_0$ and $u_\lambda(x) = u(x', 2\lambda - x_N) > 1 - \delta_0$ in Σ_{-M_0}. Since $\mathcal{U}_0$ is an open set of $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$ (and also of $\mathbb{R}^N$) there exists $\varrho_0 = \varrho_0(P_0) > 0$ such that

$$B_{\varrho_0}(P_0) \subset \mathcal{U}_0. \tag{7.2.6}$$

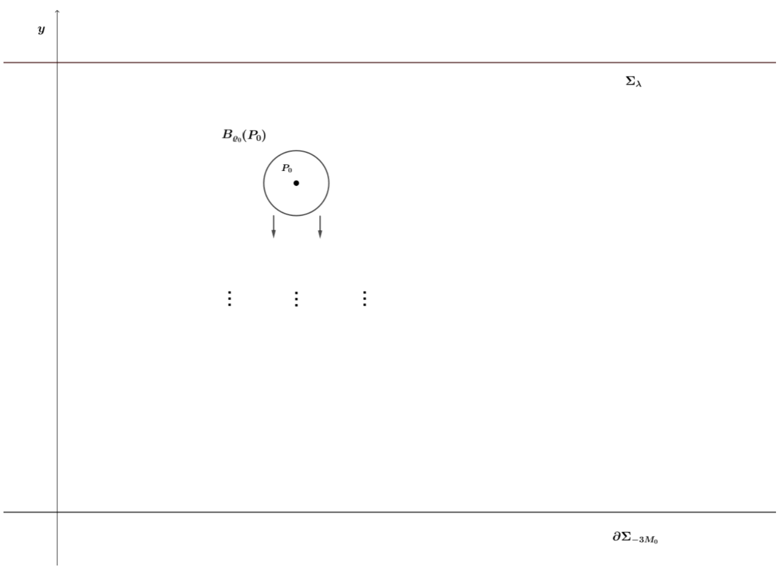

FIGURE 2. The slided ball $B_{\varrho_0}(P_0)$

We can slide B_{ϱ_0} in $\mathcal{U}_0$, towards to $-\infty$ in the y-direction and keeping its centre on the line $\{x' = x'_0\}$ (see Figure 2), until it touches for the first time $\partial\mathcal{U}_0$ at some point $z_0 \in \mathcal{Z}_{f(u)}$. In Figure 3, we show some possible examples of *first contact point* with the set $\mathcal{Z}_{f(u)}$.

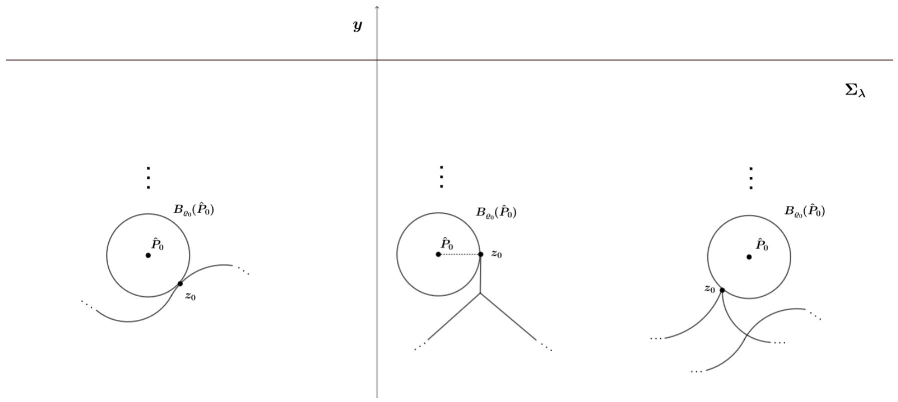

FIGURE 3. The first contact point z_0

Now we consider the function

$$w_0(x) := u(x) - u(z_0)$$

and we observe that $w_0(x) \neq 0$ for every $x \in B_{\varrho 0}(\hat{P}_0)$, where $\hat{P}_0$ is the new centre of the slided ball. In fact, if this is not the case there would exist a point $\bar{z} \in B_{\varrho_0}(\hat{P}_0)$ such that $w_0(\bar{z}) = 0$, but this is in contradiction with the fact that $\mathcal{U}_0 \cap \mathcal{Z}_{f(u)} = \emptyset$. We have to distinguish two cases. Since $p < 2$ and f is locally Lipschitz, we have that

Case 1: If $w_0(x) > 0$ in $B_{\varrho 0}(\hat{P}_0)$, then

$$\begin{cases} \Delta_p w_0 \leq C w_0^{p-1} & \text{in } B_{\varrho 0}(\hat{P}_0) \\ w_0 > 0 & \text{in } B_{\varrho 0}(\hat{P}_0) \\ w(z_0) = 0 & z_0 \in \partial B_{\varrho 0}(\hat{P}_0), \end{cases}$$

where C is a positive constant.

Case 2: If $w_0(x) < 0$ in $B_{\varrho 0}(\hat{P}_0)$, setting $v_0 = -w_0$ we have

$$\begin{cases} \Delta_p v_0 \leq C v_0^{p-1} & \text{in } B_{\varrho 0}(\hat{P}_0) \\ v_0 > 0 & \text{in } B_{\varrho 0}(\hat{P}_0) \\ v_0(z_0) = 0 & z_0 \in \partial B_{\varrho 0}(\hat{P}_0), \end{cases}$$

where C is a positive constant.

In both cases, by the Höpf boundary lemma (see e.g. [**103**, **127**]), it follows that $|\nabla w(z_0)| = |\nabla u(z_0)| \neq 0$.

Using the Implicit Function Theorem we deduce that the set $\{u = u(z_0)\}$ is a smooth manifold near z_0. Now we want to prove that

$$u_{x_N}(z_0) > 0$$

and actually that the set $\{u = u(z_0)\}$ is a graph in the y-direction near the point z_0. By our assumption we know that $u_{x_N}(z_0) := u_y(z_0) \geq 0$. According to [36, 37] and (7.1.8), the linearized operator of (7.0.1) is well defined

$$L_u(u_y, \varphi) \equiv$$

$$\int_{\Sigma_\lambda} [|\nabla u|^{p-2}(\nabla u_y, \nabla\varphi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_y)(\nabla u, \nabla\varphi)]\, dx + \\ - \int_{\Sigma_\lambda} f'(u) u_y \varphi\, dx \tag{7.2.7}$$

for every $\varphi \in C^1_c(\Sigma_\lambda)$. Moreover u_y satisfies the linearized equation (7.1.9), i.e.

$$L_u(u_y, \varphi) = 0 \quad \forall \varphi \in C^1_c(\Sigma_\lambda). \tag{7.2.8}$$

Let us set $z_0 = (z'_0, y_0)$. We have two possibilities: $u_y(z_0) = 0$ or $u_y(z_0) > 0$.

Claim: We show that the case $u_y(z_0) = 0$ is not possible.

If $u_y(z_0) = 0$, then $u_y(x) \equiv 0$ in all $B_{\hat\varrho}(z_0)$ for some positive $\hat\varrho$; to prove this we use the fact that $|\nabla u(z_0)| \neq 0$, $u \in C^{1,\alpha}$ and that Theorem 7.1.4 holds.

By construction of z_0 there exists $\varepsilon_1 > 0$ such that every point $z \in \mathcal{S}_1 := \{(z'_0, t) \in \mathcal{U}_0 : y_0 < t < y_0 + \varepsilon_1\}$ has the following properties:

(1) $z \in \overline{\mathcal{U}_0}$, since the ball is sliding along the segment $\mathcal{S}_1$;
(2) $z \notin \partial\mathcal{U}_0$, since z_0 is the first contact point with $\partial\mathcal{U}_0$.

In particular, for every $z \in \mathcal{S}_1$ we have

$$z \in \overline{\mathcal{U}_0} \setminus \partial\mathcal{U}_0 = \mathcal{U}_0. \tag{7.2.9}$$

Since $|\nabla u(z_0)| \neq 0$ and $u \in C^{1,\alpha}$, by Theorem 7.1.4 it follows that there exists $0 < \varepsilon_2 < \varepsilon_1$ such that

$$u_y(x) = 0 \quad \forall x \in B_{\varepsilon_2}(z_0).$$

Let us consider $\mathcal{S}_2 := \{(z'_0, t) \in \mathcal{U}_0 : y_0 < t < y_0 + \varepsilon_2\}$; by definition $\mathcal{S}_2 \subset \mathcal{S}_1$ and every point of $\mathcal{S}_2$ belongs also to $\mathcal{Z}_{f(u)}$, since $u(z) = u(z_0)$ for every $z \in \mathcal{S}_2$ and $z_0 \in \mathcal{Z}_{f(u)}$ by our assumptions. But this gives a contradiction with (7.2.9).

From what we have seen above, we have $|\nabla u(z_0)| \neq 0$ and hence there exists a ball $B_r(z_0)$ where $|\nabla u(x)| \neq 0$ for every $x \in B_r(z_0)$. By Theorem 7.1.2 it follows that $u \equiv u_\lambda$ in $B_r(z_0)$ namely $u \equiv u_\lambda$ in a neighborhood of the point $z_0 \in \partial\mathcal{U}_0$. Since $u_y(z_0) > 0$ and $\mathcal{N}_f$ is finite

$$B_r(z_0) \cap \Big((\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}) \setminus \mathcal{U}_0 \Big) \neq \emptyset$$

and $u_y(x) > 0$ in $B_r(z_0)$, as consequence, the set $\{u = u(z_0)\}$ is a graph in the y-direction in a neighborhood of the point z_0. Now we have to distinguish two cases:

Case 1: $u(z_0) = \min \Big[\mathcal{N}_f \setminus \{-1\} \Big]$.

Define the sets

$$\mathcal{C}_1 := \Big\{ x \in \mathbb{R}^N \,:\, x' \in (B_r(z_0) \cap \{y := y_0\}) \quad \text{and} \quad u(x) < u(z_0) \Big\}$$

$$\mathcal{C}_2 := B_r(z_0) \cup \big((B_r(z_0) \cap \{y := y_0\}) \times (-\infty, y_0) \big)$$

and

$$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2.$$

We observe that $\mathcal{C}$ is an open unbounded path-connected set (actually a deformed cylinder), see Figure 4. Since $f(u(z_0))$ has the right sign, by Theorem 7.1.3 it follows that $u \equiv u_\lambda$ in $\mathcal{C}$ and this in contradiction with the uniform limit conditions (7.0.3).

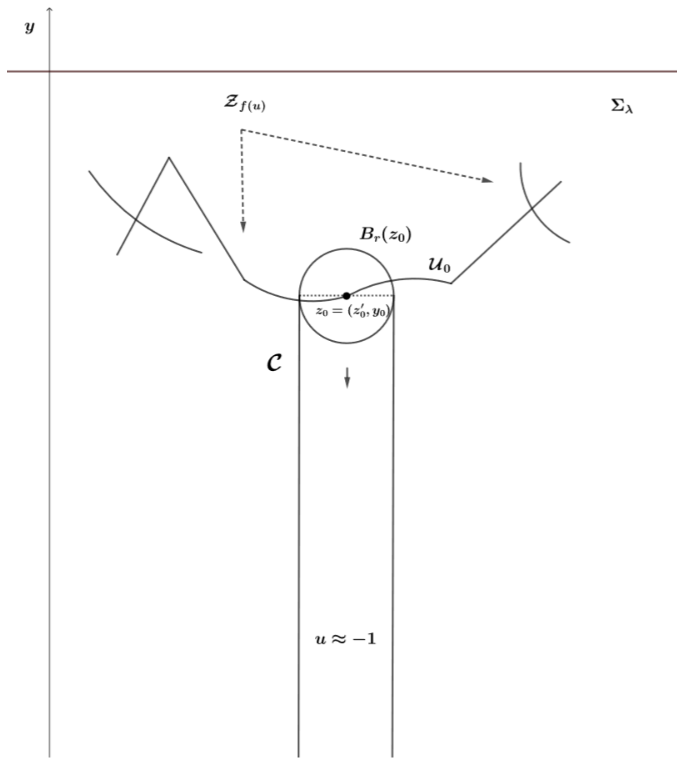

FIGURE 4. Case 1: $u(z_0) = \min \Big[\mathcal{N}_f \setminus \{-1\} \Big]$

Case 2: $u(z_0) > \min \Big[\mathcal{N}_f \setminus \{-1\} \Big]$.

In this case the open ball $B_r(z_0)$ must intersect another connected component (i.e. $\not\equiv \mathcal{U}_0$) of $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$, such that $u \equiv u_\lambda$ in a such component, see Figure 5. Here we used the fact that near the (new) first contact point, the corresponding level set is a graph in the y-direction. Now, it is clear that repeating a finite number of times the argument leading to the existence of the touching point z_0, we can find a touching point z_m such that

$$u(z_m) = \min \Big[\mathcal{N}_f \setminus \{-1\} \Big].$$

The contradiction then follows exactly as in Case 1.

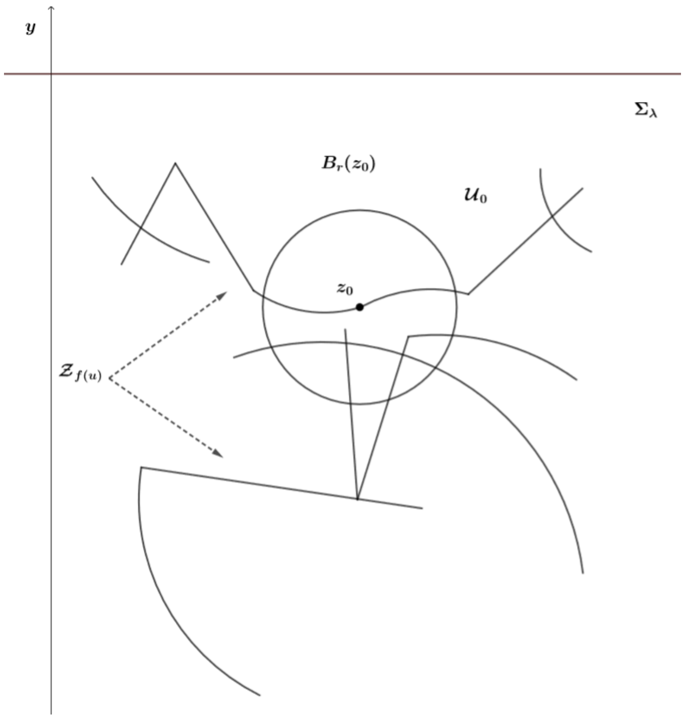

FIGURE 5. Case 2: $u(z_0) > \min\left[\mathcal{N}_f \setminus \{-1\}\right]$

Hence $u < u_\lambda$ in $\Sigma_\lambda \setminus \mathcal{Z}_{f(u)}$.

□

To prove Proposition 7.2.2 we need of the following result:

LEMMA 7.2.4. *Under the assumption of Theorem 7.0.1, let u be a solution to (7.0.1). Then there exist $M_0 = M_0(p, f, N, \|\nabla u\|_{L^\infty(\mathbb{R}^N)}) > 0$ sufficiently large such that for every $M \geq M_0$ there exits a constant $C^* = C^*(M) > 0$ such that*

$$|\nabla u| \geq \partial_{x_N} u \geq C^* > 0 \quad in\ \{-M-1 < x_N < -M+1\}. \tag{7.2.10}$$

PROOF. Performing the moving planes procedure, using (7.0.3) and (G_f), by the Proposition 7.1.7 with $v = u_\lambda$ and $\Sigma = \Sigma_\lambda$, we infer that there exists a constant $M_0 = M_0(p, f, N, \|\nabla u\|_{L^\infty(\mathbb{R}^N)}) > 0$ such that $\partial_{x_N} u \geq 0$ in $\{x_N < -M_0 + 1\}$. Now we can assume

$$\mathcal{Z}_{f(u)} \cap \{x_N < -M_0 + 1\} = \emptyset,$$

then by Theorem 7.1.5 it follows that $\partial_{x_N} u > 0$ in $\{x_N < -M_0 + 1\}$, since the case $\partial_{x_N} u = 0$ would imply a contradiction, i.e. $u(x) = -1$ in $\{x_N < -M_0 + 1\}$. We observe that in particular it holds $|\nabla u| \geq \partial_{x_N} u > 0$ in $\{-M_0 - 1 < x_N < -M_0 + 1\}$. We want to prove that for all $M \geq M_0$, there exists $C^* = C^*(M) > 0$ such that $\partial_{x_N} u \geq C^* > 0$ in $\{-M-1 < x_N < -M+1\}$.

Arguing by contradiction let us assume that there exists a sequence of point $P_n = (x'_n, x_{N,n})$, with $-M-1 < x_{N,n} < -M+1$ for every $n \in \mathbb{N}$, such that $\partial_{x_N} u(P_n) \to 0$ as $n \to +\infty$ in $\{-M-1 < x_N < -M+1\}$. Up to subsequences, let us assume that

$$x_{N,n} \to \bar{x}_N \text{ with } -M-1 \leq \bar{x}_N \leq -M+1.$$

Let us now define

$$\tilde{u}_n(x', x_N) := u(x' + x'_n, x_N)$$

so that $\|\tilde{u}_n\|_\infty = \|u\|_\infty \leq 1$. By standard regularity theory, see [46, 122], we have that

$$\|\tilde{u}_n\|_{C^{1,\alpha}_{loc}(\mathbb{R}^N)} \leq C$$

for some $0 < \alpha < 1$. By Ascoli's Theorem we have

$$\tilde{u}_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u}$$

up to subsequences, for $\alpha' < \alpha$. By construction $\partial_{x_N} \tilde{u} \geq 0$ and $\partial_{x_N} \tilde{u}(0, \bar{x}_N) = 0$, hence by Theorem 7.1.4 it follows that $\partial_{x_N} \tilde{u} = 0$ in $\{-M-1 < x_N < -M+1\}$ and therefore $\partial_{x_N} \tilde{u} = 0$ in all $\{(x', x_n) : x_N < -M+1\}$ by Theorem 7.1.5, since $\mathcal{Z}_{f(u)} \cap \{x_N < -M_0 + 1\} = \emptyset$. This gives a contradiction (by Theorem 7.1.4) with the fact that $\lim_{x_N \to -\infty} u(x', x_N) = -1$ (this implies that $\lim_{x_N \to -\infty} \tilde{u}(x', x_N) = -1$), see Remark 7.0.2. □

With the notation introduced above, we set

$$\Lambda := \{\lambda \in \mathbb{R} \mid u \leq u_t \text{ in } \Sigma_t \ \forall t < \lambda\}. \tag{7.2.11}$$

Note that, by Proposition 7.1.7 (with $v = u_t$), it follows that $\Lambda \neq \emptyset$, hence we can define

$$\bar{\lambda} := \sup \Lambda. \tag{7.2.12}$$

Moreover it is important to say that by the continuity of u and u_λ, it follows that

$$u \leq u_{\bar{\lambda}} \quad \text{in } \Sigma_{\bar{\lambda}}.$$

The proof of the fact that $u(x', x_N)$ is monotone increasing in the x_N-direction in the entire space $\mathbb{R}^N$ is done once show that $\bar{\lambda} = +\infty$. To do this we assume by contradiction that $\bar{\lambda} < +\infty$, and we prove a crucial result, which allows us to localize the support of $(u - u_{\bar{\lambda}})^+$. This localization, that we are going to obtain, will be useful to apply the weak comparison principle given by Proposition 7.1.7 and Theorem 7.1.8.

Proposition 7.2.5. *Under the assumption of Theorem 7.0.1, let u be a solution to (7.0.1). Assume that $\bar{\lambda} < +\infty$ (see (7.2.12)) and set*

$$W_\varepsilon := (u - u_{\bar{\lambda}+\varepsilon})\chi_{\{x_N \leq \bar{\lambda}+\varepsilon\}}.$$

Let $M, \kappa > 0$ be such that $M > 2|\bar{\lambda}|$. Then for all $\mu \in (0, (\bar{\lambda} + M)/2)$ there exists $\bar{\varepsilon} > 0$ such that for every $0 < \varepsilon < \bar{\varepsilon}$

$$\operatorname{supp} W^+_\varepsilon \subset \{x_N \leq -M\} \cup \{\bar{\lambda} - \mu \leq x_N \leq \bar{\lambda} + \varepsilon\} \cup \{|\nabla u| \leq \kappa\}. \tag{7.2.13}$$

Proof. Assume by contradiction that (7.2.13) is false, so that there exists $\mu > 0$ in such a way that, given any $\bar{\varepsilon} > 0$, we find $0 < \varepsilon \leq \bar{\varepsilon}$ so that there exists a corresponding $x_\varepsilon = (x'_\varepsilon, x_{N,\varepsilon})$ such that

$$u(x'_\varepsilon, x_{N,\varepsilon}) \geq u_{\bar{\lambda}+\varepsilon}(x'_\varepsilon, x_{N,\varepsilon}),$$

with $x_\varepsilon = (x'_\varepsilon, x_{N,\varepsilon})$ belonging to the set

$$\{(x', x_N) \in \mathbb{R}^N \,:\, M < x_{N,\varepsilon} < \bar\lambda - \mu\}$$

and such that $|\nabla u(x_\varepsilon)| \geq \kappa$.

Taking $\bar\varepsilon = 1/n$, then there exists $\varepsilon_n \leq 1/n$ going to zero, and a corresponding sequence

$$x_n = (x'_n, x_{N,n}) = (x'_{\varepsilon_n}, x_{N,\varepsilon_n})$$

such that

$$u(x'_n, x_{N,n}) \geq u_{\bar\lambda+\varepsilon_n}(x'_n, x_{N,n})$$

with $-M < x_{N,n} < \bar\lambda - \mu$. Up to subsequences, let us assume that

$$x_{N,n} \to \bar x_N \text{ with } -M \leq \bar x_N \leq \bar\lambda - \mu.$$

Let us define

$$\tilde u_n(x', x_N) := u(x' + x'_n, x_N)$$

so that $\|\tilde u_n\|_\infty = \|u\|_\infty \leq 1$. By standard regularity theory, see [46, 122], we have that

$$\|\tilde u_n\|_{C^{1,\alpha}_{loc}(\mathbb{R}^N)} \leq C$$

for some $0 < \alpha < 1$. By Ascoli's Theorem we have

$$\tilde u_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde u$$

up to subsequences, for $\alpha' < \alpha$. By construction it follows that

- $\tilde u \leq \tilde u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$;
- $\tilde u(0, \bar x_N) = \tilde u_{\bar\lambda}(0, \bar x_N)$;
- $|\nabla \tilde u(0, \bar x_N)| \geq \kappa$.

Since $|\nabla \tilde u(0, \bar x_N)| \geq \kappa$ there exists $\varrho > 0$ and a ball $B_\varrho(0, \bar x_N) \subset \Sigma_{\bar\lambda}$ such that $|\nabla u(x)| \neq 0$ for every $x \in B_\varrho(0, \bar x_N)$. Now, if $\tilde u(0, \bar x_N) \in \mathcal{Z}_{f(u)}$, since $\tilde u$ is non constant in $B_\varrho(0, \bar x_N)$, there exists $P_0 \in B_\varrho(0, \bar x_N)$ such that $u(P_0) \notin \mathcal{Z}_{f(u)}$. By Theorem 7.1.2 it follows that

$$\tilde u \equiv \tilde u_{\bar\lambda} \quad \text{in } B_\varrho(0, \bar x_N). \tag{7.2.14}$$

On the other hand, by Proposition 7.2.3 it follows that

$$\tilde u < \tilde u_{\bar\lambda} \qquad \text{in } \Sigma_{\bar\lambda} \setminus \mathcal{Z}_{f(u)}.$$

This gives a contradiction with (7.2.14). Hence we have (7.2.13).

□

Proof of Proposition 7.2.2. Let us assume by contradiction that $\bar\lambda < +\infty$, see (7.2.12). Let $\hat M > 0$ be such that Proposition 7.1.7 and Lemma 7.2.4 apply. Let $C^* = C^*(\hat M)$ be the constant given in Lemma 7.2.4. By Proposition 7.2.5 (choose $M = 4\hat M + 1$ there, redefining $\hat M$ if necessary) we have that

$$supp\, W^+_\varepsilon \subset \{x_N \leq -4\hat M - 1\} \cup \{-4\hat M + 1 \leq x_N \leq \bar\lambda + \varepsilon\}, \tag{7.2.15}$$

where $W_\varepsilon := (u - u_{\bar\lambda+\varepsilon})\chi_{\{x_N \leq \bar\lambda+\varepsilon\}}$. In particular, to get (7.2.15), we choose κ in Proposition 7.2.5 such that $2\kappa = C^*$. Then we deduce that

$$u \leq u_{\bar\lambda+\varepsilon} \text{ in } \{(x, x_N) \in \mathbb{R}^N \,:\, -4\hat M - 1 < x_N < -4\hat M + 1\}. \tag{7.2.16}$$

Using (7.2.16), we can apply Proposition 7.1.7 in $\{x_N < -4\hat{M} - 1\}$ and therefore, together Lemma 7.2.4 and Proposition 7.2.5, we actually deduce

$$supp\, W_\varepsilon^+ \subset \{-4\hat{M} + 1 \leq x_N \leq \bar{\lambda} + \varepsilon\}.$$

In particular, if we look to (7.2.13), we deduce that $supp\, W_\varepsilon^+$ must belong to the set

$$A := \big\{\{\bar{\lambda} - \mu \leq x_N \leq \bar{\lambda} + \varepsilon\} \cup \{|\nabla u| \leq \kappa\}\big\} \cap \big\{x_N \geq -4\hat{M} + 1\big\}.$$

We now apply Theorem 7.1.8 in the set A. Let us choose (in Theorem 7.1.8)

$$L_0 = 1 + \|\nabla u\|_{L^\infty(\mathbb{R}^N)}$$

and take $\tau_0 = \tau_0(p, \bar{\lambda}, \hat{M}, N, L_0) > 0$ and $\epsilon_0 = \epsilon_0(p, \bar{\lambda}, \hat{M}, N, L_0) > 0$ as in Theorem 7.1.8. Let μ, ε in Proposition 7.2.5 such that $2(\mu + \varepsilon) < \tau_0$ and let us redefine κ eventually such that $\kappa := \min\{C^*/2, \epsilon_0\}$. We finally apply Theorem 7.1.8 concluding that actually $W_\varepsilon^+ = 0$ in the set A. This gives a contradiction, in view of the definition (7.2.12) of $\bar{\lambda}$. Consequently we deduce that $\bar{\lambda} = +\infty$. This implies the monotonicity of u, that is $\partial_{x_N} u \geq 0$ in $\mathbb{R}^N$. By Theorem 7.1.5, it follows that

$$\partial_{x_N} u > 0 \quad \text{in}\, \mathbb{R}^N \setminus \mathcal{Z}_{f(u)},$$

since by Lemma 7.2.1, the case $\partial_{x_N} u \equiv 0$ in some connected component, say $\mathcal{U}$, of $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$ can not hold.

□

7.3. 1-Dimensional Symmetry

In this section we pass from the monotonicity in x_N to the monotonicity in all the directions of the upper hemisphere $\mathbb{S}_+^{N-1}$ defined in (7.2.1). We refer to [55] for the case of the Laplacian operator, where in the proof the linearity of the operator was crucial. Here we have to take into account the singular nature and the nonlinearity of the operator p-Laplacian.

LEMMA 7.3.1. *Under the same assumption of Theorem 7.0.1, given $\varrho > 0$ and $k > 0$, we define*

$$\Sigma_k^\varrho := \{x \in \mathbb{R}^N \mid -k < x_N < k\} \cap \{|\nabla u| > \varrho\}.$$

Assume $\eta \in \mathbb{S}_+^{N-1}$ and suppose that

$$\partial_\eta u \geq 0 \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \partial_\eta u > 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{Z}_{f(u)}. \tag{7.3.1}$$

Then, there exists an open neighbourhood $\mathcal{O}_\eta$ of η in $\mathbb{S}_+^{N-1}$, such that

$$\partial_\nu u = (\nabla u, \nu) > 0 \quad \text{in } \Sigma_k^\varrho, \tag{7.3.2}$$

for every $\nu \in \mathcal{O}_\eta$.

PROOF. Arguing by contradiction let us assume that there exist two sequences $\{P_m\} \in \mathbb{R}^N$ and $\{\nu_m\} \in \mathbb{S}_+^{N-1}$ such that, for every $m \in \mathbb{N}$ we have that $P_m = (x'_m, x_{N,m}) \in \Sigma_k^\varrho$, $|(\nu_m, \eta) - 1| < 1/m$ and $\partial_{\nu_m} u(P_m) \leq 0$. Since $-k < x_{N,m} < k$ for every $m \in \mathbb{N}$, then up to subsequences $x_{N,m} \to \bar{x}_N$. Now, let us define

$$\tilde{u}_m(x', x_N) := u(x' + x'_m, x_N)$$

so that $\|\tilde{u}_m\|_\infty = \|u\|_\infty \leq 1$. By standard regularity theory, see [46, 122], we have that

$$\|\tilde{u}_m\|_{C^{1,\alpha}_{loc}(\mathbb{R}^N)} \leq C.$$

By Ascoli's Theorem, via a standard diagonal process, we have, up to subsequences

$$\tilde{u}_m \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u},$$

for some $0 < \alpha' < \alpha$.

By uniform convergence and (7.3.1) it follows that

$$\partial_\eta \tilde{u}(0, \bar{x}_N) = 0 \quad \text{and} \quad |\nabla \tilde{u}(0, \bar{x}_N)| \geq \varrho.$$

- If $P_0 := (0, \bar{x}_N) \in \mathcal{Z}_{f(\tilde{u})}$, since $|\nabla \tilde{u}(0, \bar{x}_N)| \geq \varrho$, then there exists a ball $B_r(P_0)$ such that $|\nabla \tilde{u}(x)| \neq 0$ for every $x \in B_r(P_0)$. By Theorem 7.1.4, applied having in mind that $|\nabla \tilde{u}(x)| \neq 0$ in $B_r(P_0)$, it follows that $\partial_\eta \tilde{u}(x) = 0$ for every $x \in B_r(P_0)$. In particular $\partial_\eta \tilde{u}(x) = 0$ for every $x \in B_r(P_0) \cap \left(\Sigma_k^\varrho \setminus \mathcal{Z}_{f(\tilde{u})}\right)$, hence by Theorem 7.1.5 we deduce that $\partial_\eta \tilde{u} \equiv 0$ in the connected component $\mathcal{U}$ of $\Sigma_k^\varrho \setminus \mathcal{Z}_{f(\tilde{u})}$ containing $B_r(P_0)$ (possibly redefining r), but this is in contradiction with Lemma 7.2.1.
- If $P_0 \in \Sigma_k^\varrho \setminus \mathcal{Z}_{f(\tilde{u})}$ by Theorem 7.1.5 it follows that $\partial_\eta \tilde{u} > 0$ in the connected component of $\mathbb{R}^N \setminus \mathcal{Z}_{f(\tilde{u})}$ containing the point P_0. Indeed the case $\partial_\eta \tilde{u} \equiv 0$ in the connected component of $\mathbb{R}^N \setminus \mathcal{Z}_{f(\tilde{u})}$ containing P_0 can not hold since Lemma 7.2.1.

Hence we deduce (7.3.2). □

Having in mind the previous lemma, now we are able to prove the monotonicity in a small cone of direction around η in the entire space.

Proposition 7.3.2. *Under the assumption of Theorem 7.0.1, assume $\eta \in \mathbb{S}^{N-1}_+$ such that $\partial_\eta u > 0$ in $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$. Then, there exists an open neighbourhood $\mathcal{O}_\eta$ of η in $\mathbb{S}^{N-1}_+$, such that*

$$\partial_\nu u = u_\nu \geq 0 \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \partial_\nu u = u_\nu > 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{Z}_{f(u)}, \tag{7.3.3}$$

for every $\nu \in \mathcal{O}_\eta$.

Proof. We fix $\tilde{\delta} > 0$ and let $k = k(\tilde{\delta}) > 0$ be such that $u < -1 + \tilde{\delta}$ in $\{x_N < -k\}$, $u > 1 - \tilde{\delta}$ in $\{x_N > k\}$ and (7.1.18) holds in $\{|x_N| > k\}$. By Lemma 7.3.1 it follows that for all $\varrho > 0$ one has

$$\operatorname{supp}(u_\nu^-) \subseteq \Big(\{|x_N| \geq k\} \cup (\{-k < x_N < k\} \cap \{|\nabla u| \leq \varrho\})\Big).$$

For simplicity of exposition we set

$$A := \{|x_N| \geq k\} \quad \text{and} \quad D := \big(\{-k < x_N < k\} \cap \{|\nabla u| \leq \varrho\}\big).$$

Our claim is to show that $u_\nu^- = 0$ in $A \cup D$. In order to do this we split the proof in two part.

Step 1. We show that $u_\nu^- = 0$ in A.
We set

$$\varphi := (u_\nu^-)^\alpha \varphi_R^2 \chi_{\mathcal{A}(2R)} \tag{7.3.4}$$

where $\alpha > 1$, $R > 0$ large, $\mathcal{A}(2R) := A \cap B_{2R}$ and φ_R is a standard cutoff function such that $0 \le \varphi_R \le 1$ on $\mathbb{R}^N$, $\varphi_R = 1$ in B_R, $\varphi_R = 0$ outside B_{2R}, with $|\nabla \varphi_R| \le 2/R$ in $B_{2R} \setminus B_R$. First of all we notice that φ belongs to $W_0^{1,p}(\mathcal{A}(2R))$. To see this, use the definition of φ_R and note that by Lemma 7.2.4 and Lemma 7.3.1, it follows that $u_\nu^- = 0$ on the hyperplanes $|x_N| = k$, namely on ∂A.

According to [36, 37], the linearized operator is well defined

$$
\begin{aligned}
&L_u(u_\nu, \varphi) \equiv \\
&\int_{\mathbb{R}^N} [|\nabla u|^{p-2}(\nabla u_\nu, \nabla \varphi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_\nu)(\nabla u, \nabla \varphi)]\, dx + \\
&\qquad - \int_{\mathbb{R}^N} f'(u) u_\nu \varphi \, dx
\end{aligned}
\tag{7.3.5}
$$

for every $\varphi \in C_c^1(\mathbb{R}^N)$. Moreover it satisfies the following equation

$$
L_u(u_\nu, \varphi) = 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^N).
\tag{7.3.6}
$$

Taking φ defined in (7.3.4) in the previous equation, we obtain

$$
\begin{aligned}
&\alpha \int_{\mathcal{A}(2R)} |\nabla u|^{p-2} (\nabla u_\nu, \nabla u_\nu^-)(u_\nu^-)^{\alpha-1} \varphi_R^2 \\
&+ 2 \int_{\mathcal{A}(2R)} |\nabla u|^{p-2} (\nabla u_\nu, \nabla \varphi_R)(u_\nu^-)^{\alpha} \varphi_R \\
&+ \alpha(p-2) \int_{\mathcal{A}(2R)} |\nabla u|^{p-4} (\nabla u, \nabla u_\nu)(\nabla u, \nabla u_\nu^-)(u_\nu^-)^{\alpha-1} \varphi_R^2 \, dx \\
&+ 2(p-2) \int_{\mathcal{A}(2R)} |\nabla u|^{p-4} (\nabla u, \nabla u_\nu)(\nabla u, \nabla \varphi_R)(u_\nu^-)^{\alpha} \varphi_R \, dx \\
&= \int_{\mathcal{A}(2R)} f'(u) (u_\nu^-)^{\alpha+1} \varphi_R^2 \, dx
\end{aligned}
\tag{7.3.7}
$$

Making some computations we obtain

$$\begin{aligned}
&\alpha\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&=-2\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}(\nabla u_\nu^-,\nabla\varphi_R)(u_\nu^-)^{\alpha}\varphi_R\,dx\\
&+\alpha(2-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-4}(\nabla u,\nabla u_\nu^-)^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&+2(2-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-4}(\nabla u,\nabla u_\nu^-)(\nabla u,\nabla\varphi_R)(u_\nu^-)^{\alpha}\varphi_R\,dx\\
&+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx\\
&\le\alpha(2-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&+2(3-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|\;|\nabla\varphi_R|(u_\nu^-)^{\alpha}\varphi_R\,dx\\
&+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx.
\end{aligned}\tag{7.3.8}$$

Now it is possible to rewrite (7.3.8) as follows

$$\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&\le 2(3-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|\;|\nabla\varphi_R|(u_\nu^-)^{\alpha}\varphi_R\,dx\\
&+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx.
\end{aligned}\tag{7.3.9}$$

Exploiting the weighted Young inequality we obtain

$$\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&\le 2(3-p)\int_{\mathcal{A}(2R)}|\nabla u|^{\frac{p-2}{2}}|\nabla u_\nu^-|\;(u_\nu^-)^{\frac{\alpha-1}{2}}|\nabla u|^{\frac{p-2}{2}}|\nabla\varphi_R|(u_\nu^-)^{\frac{\alpha+1}{2}}\varphi_R\,dx\\
&+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx\\
&\le\sigma(3-p)\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&+\frac{3-p}{\sigma}\int_{\mathcal{A}(2R)}|\nabla u|^{p-2}|\nabla\varphi_R|^2(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx\\
&+\int_{\mathcal{A}(2R)}f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx.
\end{aligned}\tag{7.3.10}$$

Since $u_\nu = (\nabla u, \nu)$, where $\|\nu\| = 1$, we have

$$
\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&+\frac{3-p}{\sigma}\int_{\mathcal{A}(2R)} |\nabla u|^{p-1}|\nabla \varphi_R|^2(u_\nu^-)^{\alpha}\varphi_R^2\,dx\\
&+\int_{\mathcal{A}(2R)} f'(u)(u_\nu^-)^{\alpha+1}\varphi_R^2\,dx\\
&\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&+\hat{C}\int_{\mathcal{A}(2R)} |\nabla \varphi_R|(u_\nu^-)^{\alpha}\varphi_R^2|\nabla\varphi_R|\,dx\\
&-L\int_{\mathcal{A}(2R)} (u_\nu^-)^{\alpha+1}\varphi_R^2\,dx,
\end{aligned}
\tag{7.3.11}
$$

where we used (7.1.18) and where $\hat{C} := 3-p/\sigma\|\nabla u\|_\infty^{p-1}$. Exploiting the Young inequality with exponents $(\alpha+1)/\alpha$ and $\alpha+1$ we obtain

$$
\begin{aligned}
&\alpha(p-1)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\varphi_R^2\,dx\\
&\le \sigma(3-p)\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&+\frac{\hat{C}}{\alpha+1}\int_{\mathcal{A}(2R)} |\nabla\varphi_R|^{\alpha+1}\,dx\\
&+\frac{\hat{C}(\alpha+1)}{\alpha}\int_{\mathcal{A}(2R)} |\nabla\varphi_R|^{\frac{\alpha+1}{\alpha}}(u_\nu^-)^{\alpha+1}\varphi_R^{2\frac{\alpha+1}{\alpha}}\,dx\\
&-L\int_{\mathcal{A}(2R)} (u_\nu^-)^{\alpha+1}\varphi_R^2\,dx,
\end{aligned}
\tag{7.3.12}
$$

Since $|\nabla\varphi_R| \le 2/R$ in $B_{2R}\setminus B_R$, $0 \le \varphi_R \le 1$ in $\mathbb{R}^N$ and $\varphi_R = 1$ in B_R, we obtain

$$
\begin{aligned}
&\int_{\mathcal{A}(R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&\le \vartheta\int_{\mathcal{A}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\alpha-1}\,dx\\
&+\frac{1}{\alpha(p-1)}\left(\frac{\hat{C}(\alpha+1)}{\alpha R^{\frac{\alpha+1}{\alpha}}}-L\right)\int_{\mathcal{A}(2R)} (u_\nu^-)^{\alpha+1}\varphi_R^2\,dx,\\
&+\frac{\bar{C}}{R^{\alpha-(N-1)}},
\end{aligned}
\tag{7.3.13}
$$

where $\vartheta := \sigma(3-p)/\alpha(p-1)$ and $\bar{C} := 2\hat{C}/\alpha(\alpha+1)(p-1)$. Now we fix $\alpha > 0$ such that $\alpha > N-1$, $\sigma > 0$ sufficiently small such that $\vartheta < 2^{-N}$ and finally $R_0 > 0$ such that $\hat{C}(\alpha+1)/\alpha R^{\frac{\alpha+1}{\alpha}} - L < 0$. Having in mind all these fixed parameters let us define

$$\mathcal{L}(R) := \int_{\mathcal{A}(R)} |\nabla u|^{p-2} |\nabla u_\nu^-|^2 (u_\nu^-)^{\alpha-1}\, dx.$$

It is easy to see that $\mathcal{L}(R) \le CR^N$. By (7.3.13) we deduce that holds

$$\mathcal{L}(R) \le \vartheta \mathcal{L}(2R) + \frac{\bar{C}}{R^{\alpha-(N-1)}}$$

for every $R \ge R_0$. By applying Lemma 2.1 in [59] it follows that $\mathcal{L}(R) = 0$ for all $R \ge R_0$. Hence passing to the limit we obtain that $u_\nu^- = 0$ in A.

Step 2. $u_\nu^- = 0$ in D.

Let us denote by B' the $(N-1)$ dimensional ball in $\mathbb{R}^{N-1}$ and $\psi_R(x', x_N) = \psi_R(x') \in C_c^\infty(\mathbb{R}^{N-1})$ is a standard cutoff function such that

$$\begin{cases} \psi_R \equiv 1, & \text{in } B'(0,R) \subset \mathbb{R}^{N-1}, \\ \psi_R \equiv 0, & \text{in } \mathbb{R}^{N-1} \setminus B'(0,2R), \\ |\nabla \psi_R| \le \frac{2}{R}, & \text{in } B'(0,2R) \setminus B'(0,R) \subset \mathbb{R}^{N-1}. \end{cases} \tag{7.3.14}$$

Let us define the cylinder

$$\mathcal{C}(R) := \left\{ (x', x_N) \in \mathbb{R}^N \,:\, \{x \in \mathbb{R}^N \mid -k < x_N < k\} \cap \overline{\{B'(0,R) \times \mathbb{R}\}} \right\}.$$

We set

$$\psi := (u_\nu^-)^\beta \psi_R^2 \chi_{\mathcal{C}(2R)} \tag{7.3.15}$$

where $\beta > 1$. First of all we notice that ψ belongs to $W_0^{1,p}(\mathcal{C}(2R))$ by (7.3.14) and since $u_\nu^- = 0$ on ∂A (as above, see Lemma 7.2.4 and Lemma 7.3.1). Recalling (7.3.5) we have also in this case that

$$L_u(u_\nu, \psi) = 0 \quad \forall \psi \in C_c^1(\mathbb{R}^N). \tag{7.3.16}$$

Taking ψ defined in (7.3.15) in the previous equation, we obtain

$$\begin{aligned} &\beta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} (\nabla u_\nu, \nabla u_\nu^-)(u_\nu^-)^{\beta-1} \psi_R^2\, dx \\ &+ 2 \int_{\mathcal{C}(2R)} |\nabla u|^{p-2} (\nabla u_\nu, \nabla \psi_R)(u_\nu^-)^{\beta} \psi_R\, dx \\ &+ \beta(p-2) \int_{\mathcal{C}(2R)} |\nabla u|^{p-4} (\nabla u, \nabla u_\nu)(\nabla u, \nabla u_\nu^-)(u_\nu^-)^{\beta-1} \psi_R^2\, dx \\ &+ 2(p-2) \int_{\mathcal{C}(2R)} |\nabla u|^{p-4} (\nabla u, \nabla u_\nu)(\nabla u, \nabla \varphi_R)(u_\nu^-)^{\beta} \psi_R\, dx \\ &= \int_{\mathcal{C}(2R)} f'(u)(u_\nu^-)^{\beta+1} \psi_R^2\, dx. \end{aligned} \tag{7.3.17}$$

Repeating verbatim the same argument of (7.3.8), (7.3.9) and (7.3.10), starting by (7.3.17) we obtain

$$\begin{aligned}
&\beta(p-1)\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\leq \sigma(3-p)\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx\\
&+\frac{(3-p)}{\sigma}\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla\psi_R|^2(u_\nu^-)^{\beta+1}\psi_R^2\,dx\\
&+\int_{\mathcal{C}(2R)}f'(u)(u_\nu^-)^{\beta+1}\psi_R^2\,dx.
\end{aligned}\tag{7.3.18}$$

Since $u_\nu = (\nabla u, \nu)$ and $|\nabla u| \leq \varrho$ in $\mathcal{C}(2R)$ we have

$$\begin{aligned}
&\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\leq \vartheta\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx\\
&+\hat{C}\varrho^{p-1}\int_{\mathcal{C}(2R)}|\nabla\psi_R|^2(u_\nu^-)^{\beta}\psi_R^2\,dx\\
&+C_k\int_{\mathcal{C}(2R)}(u_\nu^-)^{\beta+1}\psi_R^2\,dx.
\end{aligned}\tag{7.3.19}$$

where $\vartheta := \sigma(3-p)/\beta(p-1)$, $\hat{C} := (3-p)/\sigma\beta(p-1)$ and

$$\tilde{C} := \|f'\|_{L^\infty((-1,1))}\beta(p-1).$$

Exploiting the Young inequality with exponents $(\beta+1)/\beta$ and $\beta+1$ we obtain

$$\begin{aligned}
&\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2(u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\leq \vartheta\int_{\mathcal{C}(2R)}|\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx\\
&+\frac{\hat{C}\varrho^{p-1}}{\beta+1}\int_{\mathcal{C}(2R)}|\nabla\psi_R|^{\beta+1}\,dx\\
&+\frac{\hat{C}\varrho^{p-1}(\beta+1)}{\beta}\int_{\mathcal{C}(2R)}|\nabla\psi_R|^{\frac{\beta+1}{\beta}}(u_\nu^-)^{\beta+1}\psi_R^{2\frac{\beta+1}{\beta}}\,dx\\
&+\tilde{C}\int_{\mathcal{C}(2R)}(u_\nu^-)^{\beta+1}\psi_R^2\,dx.
\end{aligned}\tag{7.3.20}$$

Since $|\nabla\psi_R| \leq 2/R$ in $B'_{2R} \setminus B'_R$, $0 \leq \psi_R \leq 1$ in $\mathbb{R}^N$ and $\psi_R = 1$ in B'_R, we obtain

(7.3.21)
$$
\begin{aligned}
&\int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\quad\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx + \bar{C}_R \int_{\mathcal{C}(2R)} (u_\nu^-)^{\beta+1}\psi_R^2\,dx\\
&\qquad + \frac{2^{\beta+1}\hat{C}\varrho^{p-1}}{(\beta+1)R^{\beta-(N-2)}}\\
&\quad\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx\\
&\qquad + \bar{C}_R \int_{B'(0,2R)} \left(\int_{-k}^{k} \left[(u_\nu^-)^{\frac{\beta+1}{2}}\right]^2 dx_N\right) \psi_R^2(x')\,dx'\\
&\qquad + \frac{2^{\beta+1}\hat{C}\varrho^{p-1}}{(\beta+1)R^{\beta-(N-2)}}\\
&\quad\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx\\
&\qquad + \bar{C}_R C_p(k)^2 \frac{(\beta+1)^2}{4} \int_{\mathcal{C}(2R)} |\partial_{x_N} u_\nu^-|^2 (u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\qquad + \frac{2^{\beta+1}\hat{C}\varrho^{p-1}}{(\beta+1)R^{\beta-(N-2)}},
\end{aligned}
$$

with $\bar{C}_R := \hat{C}\varrho^{p-1}(\beta+1)/\beta R^{\frac{\beta+1}{\beta}} + \tilde{C}$. We point out that in (7.3.21) we used a Poincaré inequality in the set $[-k,k]$ (denoting with C_p the associated constant) together with the fact that $\psi_R = \psi_R(x')$. By (7.3.21) we obtain

(7.3.22)
$$
\begin{aligned}
&\int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\quad\leq \vartheta \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2\ (u_\nu^-)^{\beta-1}\,dx\\
&\qquad + \bar{C}_R C_p(k)^2 \frac{(\beta+1)^2}{4} \varrho^{2-p} \int_{\mathcal{C}(2R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1}\psi_R^2\,dx\\
&\qquad + \frac{2^{\beta+1}\hat{C}\varrho^{p-1}}{(\beta+1)R^{\beta-(N-2)}}.
\end{aligned}
$$

Finally we choose $\beta > 0$ such that $\beta > N-2$, $\vartheta > 0$ sufficiently small such that $\vartheta < 2^{-N+1}$ and $\varrho > 0$ sufficently small such that

$$\bar{C}_R C_p(k)^2 \frac{(\beta+1)^2}{2} \varrho^{2-p} < 1.$$

Having in mind all these fixed parameters let us define

$$\mathcal{L}(R) := \int_{\mathcal{C}(R)} |\nabla u|^{p-2}|\nabla u_\nu^-|^2 (u_\nu^-)^{\beta-1}\,dx.$$

It is easy to see that $\mathcal{L}(R) \leq CR^{N-1}$. By (7.3.22) (up to a redefining of the constant involved) we deduce that

$$\mathcal{L}(R) \leq \vartheta\mathcal{L}(2R) + \frac{C}{R^{\beta-(N-2)}} \tag{7.3.23}$$

holds for every $R > 0$. By applying Lemma 2.1 in [59] it follows that $\mathcal{L}(R) = 0$ for all $R > 0$. Since $p < 2$, passing to the limit in (7.3.23), we deduce that for a.e. $x \in D$

$$\text{either } u_\nu^-(x) = 0, \qquad \text{or } |\nabla u_\nu^-(x)| = 0. \tag{7.3.24}$$

This actually implies that $u_\nu^-(x) = 0$ in D. Indeed let us suppose that would exist a point $P \in D$ such that $u_\nu^-(P) \neq 0$. Let us consider the connected component $\mathcal{U}$ of $D \setminus \{x \in D \; : \; u_\nu^-(x) = 0\}$ containing P. By the continuity of u_ν^-, it follows that $u_\nu^- = 0$ on the boundary $\partial\mathcal{U}$. On the other hand u_ν^- must be constant in $\mathcal{U}$ (since by (7.3.24) $|\nabla u_\nu^-| = 0$ there) .This is a contradiction.

By this two step we deduce that $u_\nu \geq 0$ in $\mathbb{R}^N$. Finally by Lemma 7.2.1 we get (7.3.3).

□

Proof of Theorem 7.0.1. Using Proposition 7.2.2 we get that the solution is monotone increasing in the y-direction and this implies that $\partial_y u \geq 0$ in $\mathbb{R}^N$. In particular we have $\partial_y u > 0$ in $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$ by (7.2.3). By Proposition 7.3.2, actually we obtain that the solution is increasing in a cone of directions close to the y-direction. This allows us to show that in fact, for $i = 1, 2, \cdots, N-1$, $\partial_{x_i} u = 0$ in $\mathbb{R}^N$, just exploiting the arguments in [55, 63, 64]. We provide the details for the sake completeness. Let Ω be the set of the directions $\eta \in \mathbb{S}_+^{N-1}$ for which there exists an open neighborhood $\mathcal{O}_\eta \subset \mathbb{S}_+^{N-1}$ such that

$$\partial_\nu u = u_\nu \geq 0 \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \partial_\nu u = u_\nu > 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{Z}_{f(u)},$$

for every $\nu \in \mathcal{O}_\eta$. The set Ω is non-empty, since $e_N \in \Omega$, and it is also open by Proposition 7.3.2. Now we want to show that it is also closed. Let $\bar\eta \in \mathbb{S}_+^{N-1}$ and let us consider the sequence $\{\eta_n\}$ in Ω such that $\eta_n \to \bar\eta$ as $n \to +\infty$ in the topology of $\mathbb{S}_+^{N-1}$. Since by our assumptions $\partial_{\eta_n} u \geq 0$ in $\mathbb{R}^N$, passing to the limit we obtain that $\partial_{\bar\eta} u \geq 0$ in $\mathbb{R}^N$. By Lemma 7.2.1 it follows that $\partial_{\bar\eta} u > 0$ in $\mathbb{R}^N \setminus \mathcal{Z}_{f(u)}$. By Proposition 7.3.2 there exists an open neighborhood $\mathcal{O}_{\bar\eta}$ such that (7.3.3) is true for every $\nu \in \mathcal{O}_{\bar\eta}$; hence $\bar\eta \in \Omega$ and this implies that Ω is also closed. Now, since $\mathbb{S}_+^{N-1}$ is a path-connected set, we have that $\Omega = \mathbb{S}_+^{N-1}$. Then there exists $v \in C^{1,\alpha}_{loc}(\mathbb{R})$ such that $u(x', y) = v(y)$. Now, let us assume that there exists $b \in \mathcal{Z}_{f(u)}$ such that $u'(b) = 0$. Then the level set $\{u = u(b)\}$ is a closed interval, i.e. there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ such that

$$\{u = u(b)\} = [\alpha, \beta].$$

By Höpf's Lemma we have $u'(\beta) > 0$, but this implies that $\{u = u(b)\} = \{b\}$ and so $u'(b) > 0$, that is in contradiction with our initial assumption. Hence we deduce that $\partial_y u > 0$ in $\mathbb{R}^N$, concluding the proof.

□

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