Università della Calabria

Dottorato di Ricerca in Matematica ed Informatica
XIX Ciclo
Tesi di Dottorato (S.S.D. MAT/05 Analisi Matematica)

Multiplicity of critical points for some noncoercive functionals

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A.A. 2007-2008

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Ph.D. Thesis

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November, 30 2007

Acknowledgements

First of all I wish to thanks Prof. Annamaria Canino, who introduced me to the field of critical point theory for nonsmooth functionals, for her human care, endless patience, friendly support and mathematical suggestions.

My special thanks go to Annarosa, Angela, Rosilde and Marianna which as sisters, have encouraged me during the period of my Ph.D., sharing some difficult periods and nervous days with great patience and love.

I thank to Giuseppe for his help in drafting this thesis and his precious advices.

I am in debt with Luigi and Gabriella for giving me their thesis of Ph.D.

I thank Francesco for his fruitful conversations in the time of development of this work.

I thank all members of the Department of Mathematica of Unical, Professors and personal administrative which provided a pleasant atmosphere during the period of my Ph.D.

I thank also all my friends in Department and my colleagues of Courses of Calcolo, for the wonderful period that I have spent with them in the University.

I thank my brothers, Giulio and Nazaria, for their moral support.

Finally I thank my parents, who have always encouraged me in my studies, growing me up in the respect of all ideas and souls.

I wish to dedicate this thesis to all these people and and to all those who have a dream come true, because they have not to be afraid to fight to achieve it: I realized my dream.

NOTATIONS

$\mathbb{N}, \mathbb{R}$ denote the set of natural, real numbers;

$\mathbb{R}^n$ is the usual real Euclidean space;

Ω is an open set in $\mathbb{R}^n$;

$\partial\Omega$ is the boundary of Ω;

a.e. stands for almost everywhere;

p' is the Hölder conjugate exponent of p;

$L^p(\Omega)$ is the space of u measurable with $\int_\Omega |u|^p\,dx < \infty$, with $1 \leq p < \infty$;

$L^\infty(\Omega)$ is the space of u measurable with $|u(x)| \leq C$ for a.e. $x \in \Omega$;

$\|\cdot\|_p$ and $\|\cdot\|_\infty$ norms of the spaces $L^p(\Omega)$ and L^∞;

$\nabla u(x)$ stands for $(D_{x_i}u(x), \ldots, D_{x_n}u(x))$ where $D_{x_i}u(x)$ is the i_{th} partial derivatives of u at x;

$H_0^1(\Omega), W_0^{1,p}(\Omega)$ Sobolev spaces;

$H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$

$\|\cdot\|_{1,2}$ the standard norm of the space $H_0^1(\Omega)$;

$\|\cdot\|_{-1,2}$ the standard norm of the space $H^{-1}(\Omega)$;

$\mathscr{D}'(\Omega)$ is the space of all distributions in Ω;

$C_c^k(\mathbb{R})$ $k-$ differentiable functions in $\mathbb{R}$ with compact support;

$C_0^k(\Omega)$ $k-$ differentiable functions in Ω which are 0 on $\partial\Omega$;

$2^* = \frac{2n}{n-2}$ the Sobolev embedding exponent of real number p;

$f'(u)(v)$ derivative of the functional f in u in the direction v;

$\overline{A}$ is the closure of a set A;

$|A|$ the Lebesgue measure of set A;

$\langle \cdot, \cdot \rangle$ is the scalar product in the duality $H^{-1}(\Omega), H_0^1(\Omega)$;

$d(x, y)$ is the distance of x from y;

$a_t(x, t)$ is the derivative of a respect to second variable;

Contents

Introduction

In this thesis we study the existence and the multiplicity of critical points of noncoercive functionals like

$$f(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p \qquad u \in H_0^1(\Omega),$$

with $2 < p < 2^*(1-\alpha)$. These critical points are weak solutions of boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) + \frac{1}{2}a_t(x,u)|\nabla u|^2 = g(x,u) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

In this framework the main difficulties are that the functional is not differentiable on the whole $H_0^1(\Omega)$, but only in $H_0^1(\Omega) \cap L^\infty(\Omega)$, even if $a(x,s)$ is smooth and that the associated differential operator

$$-\mathrm{div}(a(x,u)\nabla u) + \frac{1}{2}\frac{\partial a}{\partial s}(x,u)|\nabla u|^2,$$

involves a lower order term with quadratic growth in the gradient, which may not be in the dual space $H^{-1}(\Omega)$. Minimization results for integral functionals noncoercive were proved by Boccardo and Orsina. Specifically, the authors considered functionals whose model is

$$f(u) = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(b(x)+|u|)^{2\alpha}} - \int_\Omega hu, \quad u \in H_0^1(\Omega),$$

where $\alpha \in (0, 1/2), 0 < \beta_1 \leq b(x) \leq \beta_2$ and $h \in L^p(\Omega)$ for some $p \geq 1$. This functional, which is clearly well defined thanks to Sobolev embedding if $p \geq (2^*)'$, is however non coercive on $H_0^1(\Omega)$. Instead there exists a function h, and a sequence $\{u_n\}$ whose norm diverges in $H_0^1(\Omega)$ such that $f(u_n)$ tends to $-\infty$. Thus, even if f is lower semicontinuous on $H_0^1(\Omega)$ as a consequence of a result of De Giorgi, the lack of coerciveness implies that f may not attain its minimum on $H_0^1(\Omega)$ even in the case in which f is bounded from below. The structure of the functional has however enough properties in order to prove that if $h \in L^p(\Omega)$, with $p \geq [2^*(1-\alpha)]'$, then f (suitably extended) is coercive on $W_0^{1,q}(\Omega)$ for some $q < 2$ depending on α. Thus f attains its minimum on this larger space. This idea is followed also by Arcoya, Boccardo e Orsina and their results are presented in the work *Existence of critical points for some noncoercive functionals*. The authors prove the existence of a non trivial critical point in $H_0^1(\Omega) \cap L^\infty(\Omega)$ using a version of the Ambrosetti-Rabinowitz Mountain Pass Theorem for functionals which are not differentiable along every directions . In such way it shows the existence of a non trivial critical point in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Nevertheless the techniques used make it impossible to obtain a result of multiplicity of critical points. From 1992 the groups of M. Degiovanni ([12], [16]) and A. Ioffe ([20],[21]), developed independently a critical point theory for continuous and lower semicontinuous functional that generalizes the classical results. This theory is based on the notion of the weak slope, which is a generalization of the norm of the Fréchet derivative. In this thesis we apply this theory for to prove our result of multiplicity.

The advantage of this approach is that we find critical point belongs to the " energy space" $H_0^1(\Omega)$. In the specifically we consider the functional

$$f(u) = \frac{1}{2} \int_\Omega a(x, u) |\nabla u|^2 \, dx - \int_\Omega G(x, u),$$

where $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is , measurable respect to x in Ω for all $s \in \mathbb{R}$, and continuous respect to $s \in \mathbb{R}$ for a. e. $x \in \Omega$) such that

$$\frac{c_1}{(1 + |s|)^{2\alpha}} \leq a(x, s) \leq c_2 \tag{2}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, where c_1 and c_2 are positive constant, and $0 \leq \alpha < \frac{n}{2n-2}$.
we also assume that the function $s \mapsto a(x,s)$ is differentiable on $\mathbb{R}$ for almost every $x \in \Omega$, and its derivative $a_s(x,s) \equiv \frac{\partial a}{\partial s}(x,s)$ is such that

$$-2\beta a(x,s) \leq a_s(x,s)(1+|s|)sgn(s) \leq 0 \tag{3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, with $|s| \geq R$, where R and β are positive constants such that $0 < \beta < c_1$.
Suppose also that there exists a positive constant c_3 such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ with $|s| \leq R$

$$|a_s(x,s)| \leq c_3. \tag{4}$$

and that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$

$$a(x,s) = a(x,-s)\,; \tag{5}$$

let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be another Carathéodory function that satisfies the assumption

$$|g(x,s)| \leq b|s|^{p-1} + d, \tag{6}$$

for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, where b, d are two positive constant and $2 < p < 2^*(1-\alpha)$. Another we assume that there exists $\nu > 2$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ with $|s| \geq R$ we have

$$0 < \nu G(x,s) \leq s g(x,s). \tag{7}$$

and that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$

$$g(x,-s) = -g(x,s). \tag{8}$$

Under these hypotheses the functional f has infinitely critical points. Note that f is lower semicontinuous. The approach used in the thesis is based on the techniques developed by Pellacci and Squassina in *Unbounded critical points for a class of lower semicontinuous functionals* even if we have not their condition

$$\frac{\partial a}{\partial s}(x,u)u|\nabla u|^2 \geq 0$$

when $|s| \geq R$. Infact in our case we have a sign condition opposed. Performing a suitable change variable, $u = \varphi(v)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, we overcome this obstacle. A simple model of function φ is the follows

$$\varphi(s) = \{[(1-\alpha)|s| + 1]^{\frac{1}{1-\alpha}} - 1\} sgn(s).$$

Then we will show that the functional

$$\tilde{f}(v) = f(\varphi(v))$$

can be studied by means of nonsmooth critical point theory for lower semicontinuous functionals. An other difference with the method proposed by Pellacci-Squassina is that we have not a standard condition that is fundamental to prove Palais-smale condition. Fortunately the structure of the functional $\tilde{f}$ allows us to obtain this result with a brilliant technique. Then we prove that $\tilde{f}$ has infinitely critical point in $H_0^1(\Omega)$. Finally, by a new result presented in this thesis, which connect the critical points of $\tilde{f}$ and f, we will prove that for every critical point of $\tilde{f}$ there exist a critical point of f.

Let us illustrate more precisely the content of the thesis.

Chapter 1. We recall some tools of nonsmooth critical point theory which are useful in this thesis. In particular we recall Theorem 1.4.4 which will play an essential role, with the theory of Chapter 2 and Chapter 4, to obtain the multiplicity results.

Chapter 2. We describes the framework and the results concerning multiplicity of unbounded critical points for a class of lower semicontinuous functionals exposed in [25]. These results will give the abstract setting to improve the results of existence of Chapter 3.

Chapter 3. We present the problem. First of all we expose some results of Boccardo and Orsina (see [6]) on the existence and regularity of a minimum for functionals noncoercive. Another we are going to give some examples and counterexamples, in order to explain which kind of problems can arise when studying these functionals. Finally we expose the problem of Arcoya, Boccardo and Orsina (see [5]) that motivated this work .

Chapter 4. We apply the abstract results of Chapter 1 to improve the results of Chapter 3, according to the results of Chapter 2. In particular we will prove that the noncoercive functional studied in [5] has infinitely critical points in $H_0^1(\Omega)$.

Chapter 1

Recalls of nonsmooth critical point theory

1.1 The weak slope

In this section we recall some results of abstract critical point theory developed in [12], [16] and, independently, in [20],[21]. In order to show how this theory works, we will present some proofs. Moreover we give a new result that characterize the weak slope of a function composed.

Let X be a metric space endowed with the metric d. In the following $B(u,r)$ will denote the open ball of centre u and radius r.

Definition 1.1.1. *Let $f : X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|\, df \,|\,(u)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathscr{H} :\, B(u,\delta) \times [0,\delta] \to X$$

such that $\forall v \in B(u,\delta), \forall t \in [0,\delta]$

$$d(\mathscr{H}(v,t),v) \leq t \quad \text{and} \quad f(\mathscr{H}(v,t)) \leq f(v) - \sigma t. \tag{1.1}$$

The extended real number $|\, df \,|\,(u)$ is called the " weak slope" of f at u.

Example 1.1.2. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. We observe that the origin is a global minimum, but f is not differentiable at this point. However $|df|(0)$ exists and there holds $|df|(0) = 0$. In fact, let $\sigma > 0$ and let assume, by contradiction, that there exists $\delta > 0$ and a continuous mapping $\mathscr{H} : (-\delta, \delta) \times [0, \delta] \to \mathbb{R}$ such that for all $x \in (-\delta, \delta)$ and $t \in [0, \delta]$ there holds

$$|\mathscr{H}(x,t) - x| \leq t \quad \text{and} \quad |\mathscr{H}(x,t)| \leq |x| - \sigma t.$$

Then for $x = 0$ we have $|\mathscr{H}(0,t)| \leq -\sigma t$, which is absurd.

Definition 1.1.3. *Let $f : X \to \mathbb{R}$ lower semicontinuous. We define*

$$|\nabla f|(u) = \begin{cases} \limsup\limits_{v \to u} \dfrac{f(u) - f(v)}{d(u,v)} & \textit{if } u \textit{ is not a local minimum} \\ 0 & \textit{if } u \textit{ is a local minimum} \end{cases}$$

The extended real number $|\nabla f|(u)$ is called the " strong slope" of f at u.

Proposition 1.1.4. *We have that*

$$|df|(u) \leq |\nabla f|(u) \quad \textit{for every} \quad u \in X.$$

This justified the terminology " weak slope."

Proof. If $|df|(u) = 0$ the thesis follows from the fact that $|\nabla f|(u)$ is a positive real number. If $|df|(u) \neq 0$, then from definition of weak slope, we have

$$f(\mathscr{H}(u,t)) \leq f(u) - \sigma t \leq f(u) - \sigma d(\mathscr{H}(u,t), u)$$

then

$$\frac{f(u) - f(\mathscr{H}(u,t))}{d(\mathscr{H}(u,t), u)} \geq \sigma.$$

Passing to the limsup for $t \to 0$ we have

$$|\nabla f|(u) \geq \sigma,$$

and passing to the sup of σ

$$|\nabla f|(u) \geq |df|(u).$$

□

The notion of weak slope is a generalization of the norm of the derivative in the case of a smooth function. Indeed we have the following results

Proposition 1.1.5. *Let X be a Banach space, $f : X \to \mathbb{R}$ Fréchet differentiable at $u \in X$. Then $|\nabla f|(u) = \|f'(u)\|$.*

Proof. We recall that a function Fréchet differentiable it is also Gateaux differentiable. Since

$$\|f'(u)\| = \sup_{\|w\|=1} \langle f'(u), w\rangle,$$

it follows that there exists $w \in X$ with $\|w\| = 1$ such that

$$\langle f'(u), -w\rangle \geq \|f'(u)\| - \varepsilon$$

for small enough $\varepsilon > 0$.

Then, putting $v = u + tw$, we have:

$$\limsup_{v\to u} \frac{f(u) - f(v)}{\|v - u\|} \geq \lim_{t\to 0} \frac{f(u) - f(u + tw)}{t\|w\|} = \left\langle f'(u), -\frac{w}{\|w\|} \right\rangle \geq \|f'(u)\| - \varepsilon.$$

The arbitrary of ε allows us to conclude

$$\limsup_{v\to u} \frac{f(u) - f(v)}{\|v - u\|} \geq \|f'(u)\|.$$

On the other hand, writing

$$f(v) = f(u) + \langle f'(u), v - u\rangle + o(\|v - u\|)$$

we get

$$\frac{f(u) - f(v)}{\|v - u\|} = \left\langle f'(u), \frac{u - v}{\|v - u\|} \right\rangle + \varepsilon_v \leq \|f'(u)\| + \varepsilon_v$$

where ε_v goes to 0, as v tends to u and hence

$$\limsup_{v\to u} \frac{f(u) - f(v)}{\|v - u\|} \leq \|f'(u)\|.$$

□

Theorem 1.1.6. *Let X be a Banach space, $f \in C^1(X, \mathbb{R})$. Then we have $|df|(u) = \|f'(u)\|$ for every $u \in X$.*

Proof. Fix $u \in X$ and $t > 0$. From Propositions 1.1.4 and 1.1.5 we have to prove only that $|df|(u) \geq \|f'(u)\|$. Take $0 < \sigma < \|f'(u)\|$. There is a unit vector $w \in X : \langle f'(u), w\rangle > \sigma$. By continuity of f' at u, there exists $\delta > 0$ such that $\langle f'(\xi), w\rangle > \sigma \; \forall \xi \in B(u, 2\delta)$. Let us consider the map $\mathscr{H} : B(u, \delta) \times [0, \delta] \to X$, defined by $\mathscr{H}(v, t) = v - tw$. Clearly have that $\mathscr{H}$ is continuous and for all $(v, t) \in B(u, \delta) \times [0, \delta]$ there holds:

$$\|\mathscr{H}(v, t) - v\| \leq t.$$

Moreover from Lagrange's Theorem we have that there exists $\xi \in B(u, 2\delta)$ such that

$$f(v) - f(v - tw) = \langle f'(\xi), tw\rangle$$

whence

$$f(\mathscr{H}(v, t)) - f(v) = -t\langle f'(\xi), w\rangle < -t\sigma;$$

then the thesis follows. □

Theorem 1.1.7. *Let X be a Banach space, $\varphi : X \to X$ be a diffeomorphism and let $f : X \to \mathbb{R}$ be a continuous function. We define*

$$\tilde{f}(u) = f(\varphi(u)).$$

Then

$$|d\tilde{f}|(u) \geq \|\varphi'(u)\| \cdot |df|(\varphi(u)).$$

Proof. If $|df|(\varphi(u)) = 0$, it is true. Otherwise, let $0 < \sigma < |df|(\varphi(u))$ and let $\mathscr{H} : B(\varphi(u), \delta) \times [0, \delta] \to X$ such that $\forall w \in B(\varphi(u), \delta), \forall t \in [0, \delta]$

$$d(H(w, t), w) \leq t,$$

$$f(H(w, t)) \leq f(w) - \sigma t.$$

By continuity of φ it follows that fixed $\delta > 0$ there exist $\bar{\delta} > 0$ such that if $d(\bar{w}, u) < \bar{\delta}$ then $d(\varphi(\bar{w}), \varphi(u)) < \delta$. Let $\tilde{\delta} = \min\{\bar{\delta}, \lambda\delta\}$ where

$$\lambda = \sup_{\zeta \in B(\varphi(u), 2\delta)} \|(\varphi^{-1})'(\zeta)\|.$$

Consider $\tilde{\mathscr{H}} : B(u, \tilde{\delta}) \times [0, \tilde{\delta}] \to X$ defined by

$$\tilde{\mathscr{H}}(\bar{w}, t) = \varphi^{-1}\left(\mathscr{H}\left(\varphi(\bar{w}), \frac{t}{\lambda}\right)\right),$$

and let $w = \varphi(\bar{w})$. Of course $\tilde{\mathscr{H}}$ is continuous and by applying Lagrange's Theorem we get

$$d(\tilde{H}(\bar{w}, t), \bar{w}) = d\left(\varphi^{-1}\left(\mathscr{H}\left(\varphi(\bar{w}), \frac{t}{\lambda}\right)\right), \bar{w}\right)$$

$$= d\left(\varphi^{-1}\left(\mathscr{H}\left(w, \frac{t}{\lambda}\right)\right), \varphi^{-1}(w)\right) \leq \lambda\, d(\mathscr{H}(w, t/\lambda), w) \leq t.$$

Furthermore we have

$$\tilde{f}(\tilde{\mathscr{H}}(\bar{w}, t)) - \tilde{f}(\bar{w}) = f(\varphi(\varphi^{-1}(\mathscr{H}(\varphi(\bar{w}), t/\lambda)))) - f(\varphi(\bar{w}))$$

$$= f(H(w, t/\lambda)) - f(w) \leq -\sigma\frac{t}{\lambda}.$$

By definition of weak slope we have

$$|d\tilde{f}|(u) \geq \frac{\sigma}{\sup_{\zeta \in B(\varphi(u), 2\delta)} \|(\varphi^{-1})'(\zeta)\|}$$

and since δ can be take arbitrarily small

$$|d\tilde{f}|(u) \geq \frac{\sigma}{\|(\varphi^{-1})'(\varphi(u))\|} = \sigma\|\varphi'(u)\|,$$

which implies the assertion. □

1.2 The case of lower semicontinuous functionals

This section is devoted to some consideration about lower semicontinuous functionals. We refer the reader to ([10],[16]). Let X be a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We put

$$\mathrm{dom}(f) = \{u \in X : f(u) < +\infty\}.$$

Moreover we introduce the set

$$epi(f) = \{(u, \eta) \in X \times \mathbb{R} : f(u) \leq \eta\}$$

and the function $\mathscr{G}_f : epi(f) \to \mathbb{R}$ defined by

$$\mathscr{G}_f(u, \eta) := \eta. \tag{1.2}$$

The set $epi(f)$ is endowed with metric

$$d((u, \eta), (v, \mu)) = (d(u, v)^2 + (\eta - \mu)^2)^{\frac{1}{2}}.$$

Of course $\mathscr{G}_f$ is Lipschitz continuous of constant 1. According to the previous Definition 1.1.1, for every lower semicontinuous function f we can consider the metric space $epi(f)$ so that the weak slope of $\mathscr{G}_f$ is well defined. Note that $|d\mathscr{G}_f|(u, \eta) \leq 1$ for every $(u, \eta) \in epi(f)$. Therefore, we can define the weak slope of a lower semicontinuous function f by using $|d\mathscr{G}_f|(u, f(u))$. More precisely, we have the following definition.

Definition 1.2.1. *For every $u \in dom(f)$ let*

$$|df|(u) = \begin{cases} \dfrac{|d\mathscr{G}_f|(u, f(u))}{\sqrt{1 - |d\mathscr{G}_f|(u, f(u))^2}} & \text{if} \quad |d\mathscr{G}_f|(u, f(u)) < 1; \\ \\ +\infty & \text{if} \quad |d\mathscr{G}_f|(u, f(u)) = 1. \end{cases}$$

Based on the weak slope, we introduce the following fundamental notions .

Definition 1.2.2. *Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. We say that $u \in dom(f)$ is a (lower) critical point of f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of f if there exists a (lower) critical point $u \in dom(f)$ of f with $f(u) = c$.*

Definition 1.2.3. *Let X be a complete metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function and let $c \in \mathbb{R}$. We say that f satisfies the Palais-Smale condition at level c ($(PS)_c$ in short), if every sequence $\{u_n\}$ in $dom(f)$ such that*

$$|df|(u_n) \to 0,$$

$$f(u_n) \to c,$$

admits a subsequence $\{u_{n_k}\}$ converging in X.

Now, for every $\eta \in \mathbb{R}$, let us define the set

$$f^\eta = \{u \in X : f(u) < \eta\}. \tag{1.3}$$

The following result give a criterion to obtain a lower estimate of $|df|(u)$.

Proposition 1.2.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function defined on the complete metric space X, and let $u \in dom(f)$. Let us assume that there exist $\delta > 0, \eta > f(u), \sigma > 0$ and a continuous function $\mathscr{H} : B(u, \delta) \cap f^\eta \times [0, \delta] \to X$ such that*

$$d(\mathscr{H}(v,t), v) \le t, \quad \forall v \in B(u,\delta) \cap f^\eta$$

$$f(\mathscr{H}(v,t)) \le f(v) - \sigma t \quad \forall v \in B(u,\delta) \cap f^\eta.$$

Then $|df|(u) \ge \sigma$.

Proof. The case $|\mathscr{G}_f|(u, f(u)) = 1$ is trivial. Let us assume $|\mathscr{G}_f|(u, f(u)) < 1$. Let $\delta' \in [0, \delta]$ be such that $\mu \leq b$ for $(\nu, \mu) \in B((u, f(u)), \delta')$ and let us define

$$\mathscr{K} : B((u, f(u)), \delta') \times [0, \delta'] \to epi(f)$$

by

$$\mathscr{K}((\nu, \mu), t) = \left(\mathscr{H}\left(\nu, \frac{t}{\sqrt{1+\sigma^2}}\right), \mu - \frac{\sigma t}{\sqrt{1+\sigma^2}}\right).$$

Since $(d(u,\nu)^2 + (\mu - f(u))^2)^{\frac{1}{2}} < \delta'$ we have $f(u) < \mu + \delta'$. By lower semicontinuity of f it follows that fixed $\varepsilon > 0$ there exists $\tilde{\delta} > 0$ such that if $d(u,\nu) < \tilde{\delta}$ then $f(\nu) < f(u) + \varepsilon$. Putting $\delta' < \min\{\tilde{\delta}, \varepsilon\}$ we have $f(\nu) < \mu + 2\varepsilon$. Since ε can be made arbitrarily small,$f(\nu) < \mu$.

Therefore

$$f\left(\mathscr{H}\left(\nu, \frac{t}{\sqrt{1+\sigma^2}}\right)\right) \leq f(\nu) - \frac{\sigma t}{\sqrt{1+\sigma^2}} \leq \mu - \frac{\sigma t}{\sqrt{1+\sigma^2}},$$

and we have $\mathscr{K}((\nu,\mu),t) \in epi(f)$. Of course $\mathscr{K}$ is continuous and

$$d(\mathscr{K}((\nu,\mu),t),(\nu,\mu)) = \left[d\left(\mathscr{H}\left(\nu, \frac{t}{\sqrt{1+\sigma^2}}\right), \nu\right)^2 + \left(\frac{\sigma t}{\sqrt{1+\sigma^2}}\right)^2\right]^{\frac{1}{2}} \leq$$

$$\leq \left(\frac{t^2}{1+\sigma^2} + \frac{\sigma^2 t^2}{1+\sigma^2}\right)^{\frac{1}{2}} = t.$$

Furthermore we have

$$\mathscr{G}_f(\mathscr{K}((\nu,\mu),t)) = \mu - \frac{\sigma t}{\sqrt{1+\sigma^2}} = \mathscr{G}_f(\nu,\mu) - \frac{\sigma t}{\sqrt{1+\sigma^2}}.$$

It follows that

$$|d\mathscr{G}_f|(u, f(u)) \geq \frac{\sigma}{\sqrt{1+\sigma^2}},$$

which can be rewritten

$$\sigma^2 \leq \frac{(|d\mathscr{G}_f|(u,f(u)))^2}{1 - (|d\mathscr{G}_f|(u,f(u)))^2} = [|df|(u)]^2.$$

□

Now we want generalizes the Theorem 1.1.7 for the lower semicontinuous functionals.

Theorem 1.2.5. *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function and $\varphi : X \to X$ a diffeomorphism. We define*

$$\tilde{f}(u) = f(\varphi(u)).$$

Then

$$|d\tilde{f}|(u) \geq \|\varphi'(u)\| \cdot |df|(\varphi(u)). \tag{1.4}$$

Proof. If $|df|(\varphi(u)) = 0$ is obvious, otherwise, let $0 < \sigma < |d\mathscr{G}_f|(\varphi(u), f(\varphi(u)))$. Then there exists a continuous map

$$\mathscr{K} : B((\varphi(u), f(\varphi(u))), \delta) \times [0, \delta] \to epi(f)$$

such that $\forall (\nu, \mu) \in B((\varphi(u), f(\varphi(u))), \delta), \forall t \in [0, \delta]$ we have

$$d(\mathscr{K}((\nu, \mu), t), (\nu, \mu)) \leq t. \tag{1.5}$$

and

$$\mathscr{G}_f(\mathscr{K}(\nu, \mu), t)) \leq \mathscr{G}_f(\nu, \mu) - \sigma t. \tag{1.6}$$

Let us $\mathscr{K} = (\mathscr{K}_1, \mathscr{K}_2)$. From the conditions

$$d(\mathscr{K}_1(((\nu, \mu), t), \nu))^2 + (\mathscr{K}_2(((\nu, \mu), t) - \mu)^2 \leq t^2,$$

and

$$\mathscr{K}_2(((\nu, \mu), t) \leq \mu - \sigma t$$

we deduce that

$$d(\mathscr{K}_1(((\nu, \mu), t)), \nu) \leq \sqrt{1 - \sigma^2}\, t$$

By continuity of φ it follows that fixed $\delta > 0$ there exist $\bar{\delta} > 0$ such that if $d(\bar{\nu}, u) < \bar{\delta}$ then $d(\varphi(\bar{\nu}), \varphi(u)) < \delta$. Let

$$\gamma = \sup_{\zeta \in B(\varphi(u), 2\delta)} \|(\varphi^{-1})'(\zeta)\|$$

and

$$\theta = \sqrt{\gamma^2 - \sigma^2\gamma^2 + \sigma^2}.$$

We set $\tilde{\delta} = \min\{\bar{\delta}, \theta\delta\}$ and define a continuous map

$$\tilde{\mathscr{K}} : B((u, \tilde{f}(u)), \tilde{\delta}) \times [0, \tilde{\delta}] \to epi(\tilde{f})$$

by

$$\tilde{\mathscr{K}}((\bar{\nu}, \mu), t) = \left(\varphi^{-1}\left(\mathscr{K}_1\left((\varphi(\bar{\nu}), \mu), \frac{t}{\theta}\right)\right), \mu - \frac{\sigma}{\theta}t\right).$$

From (1.6) we have

$$\tilde{f}\left(\varphi^{-1}\left(\mathscr{K}_1\left((\varphi(\bar{\nu}), \mu), \frac{t}{\theta}\right)\right)\right) = f\left(\mathscr{K}_1\left((\varphi(\bar{\nu}), \mu), \frac{t}{\theta}\right)\right)$$

$$\leq \mathscr{K}_2\left((\varphi(\bar{\nu}), \mu), \frac{t}{\theta}\right) \leq \mu - \frac{\sigma}{\theta}t.$$

Hence $\tilde{\mathscr{K}}((\bar{\nu}, \mu), t) \in epi(\tilde{f})$.

Then we can apply Lagrange's Theorem and obtain

$$d(\tilde{\mathscr{K}}((\bar{\nu}, \mu), t), (\nu, \mu)) =$$

$$\sqrt{d\left(\varphi^{-1}\left(\mathscr{K}_1\left((\varphi(\bar{\nu}), \mu), \frac{t}{\theta}\right)\right), \varphi^{-1}(\nu)\right)^2 + \frac{\sigma^2 t^2}{\theta^2}}$$

$$\leq \sqrt{\gamma^2\frac{(1-\sigma^2)t^2}{\theta^2} + \frac{\sigma^2 t^2}{\theta^2}} = t$$

Moreover

$$\mathscr{G}_{\tilde{f}}(\tilde{\mathscr{K}}((\bar{\nu}, \mu), t)) = \mu - \frac{\sigma t}{\theta} = \mathscr{G}_{\tilde{f}}(\bar{\nu}, \mu) - \frac{\sigma t}{\theta},$$

hence

$$|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)) \geq \frac{\sigma}{\theta}.$$

Since δ can be take arbitrarily small we have

$$|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)) \geq \frac{\sigma}{\sqrt{\|(\varphi^{-1})'(\varphi(u))\|^2 - \sigma^2\|(\varphi^{-1})'(\varphi(u))\|^2 + \sigma^2}},$$

that is

$$|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)) \geq \frac{\|\varphi'(u)\|\sigma}{\sqrt{1-\sigma^2+\sigma^2\|\varphi'(u)\|^2}}.$$

Then we can write

$$(|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)))^2(1-\sigma^2) \geq \sigma^2\|\varphi'(u)\|^2(1-(|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)))^2),$$

hence

$$\frac{(|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)))^2}{1-(|d\mathscr{G}_{\tilde{f}}|(u, \tilde{f}(u)))^2} \geq \|\varphi'(u)\|^2 \frac{\sigma^2}{1-\sigma^2},$$

which implies the assertion. □

1.3 The equivariant case

We need also use the notion of equivariant weak slope introduced in [10]. Let X be a metric space on which a compact Lie group G acts by isometric transformations (a metric $G-$space in short) and let d be the metric in X.

Definition 1.3.1. *Let $f : X \to \mathbb{R}$ be a continuous invariant function and let $u \in X$. We denote by $|d_G f|(u)$ the supremum of the σ's in $[0, +\infty)$ such that there exist an invariant neighborhood U of $u, \delta > 0$, and a continuous map $\mathscr{H} : U \times [0, \delta] \to X$ such that $\forall v \in U, \forall t \in [0, \delta]$*

$$d(\mathscr{H}(v,t), v) \leq t,$$

$$f((\mathscr{H}(v,t)) \leq f(v) - \sigma t$$

and such that $\mathscr{H}(\cdot, t)$ is equivariant for all $t \in [0, \delta]$, that is,

$$\forall t \in [0, \delta], \forall v \in U, \forall g \in G : \qquad \mathscr{H}(gv, t) = g\mathscr{H}(v, t).$$

The extended real number $|d_G f|(u)$ is called the " invariant weak slope" of f at u.

Remark 1.3.2. It is readily seen that the function $|d_G f| : X \to [0, +\infty)$ is invariant. We also set $|df|(u) := |d_H f|(u)$, where $H = \{e\}$ is the trivial group with the obvious action.

Now we consider a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and the set $epi(f)$. The space $epi(f)$ has a natural structure of metric $G-$space, through the action

$$G \times epi(f) \to epi(f)$$

$$(g,(u,\xi)) \mapsto (gu,\xi).$$

Of course the function $\mathscr{G}_f$ as defined in (1.2) is invariant, thus we can apply Definition 1.3.1 to the function $\mathscr{G}_f$.

Definition 1.3.3. *For every $u \in dom(f)$ let*

$$|d_G f|(u) = \begin{cases} \dfrac{|d_G \mathscr{G}_f|(u,f(u))}{\sqrt{1-|d_G\mathscr{G}_f|(u,f(u))^2}} & \text{if} \quad |d_G\mathscr{G}_f|(u,f(u)) < 1; \\ \\ +\infty & \text{if} \quad |d_G\mathscr{G}_f|(u,f(u)) = 1. \end{cases}$$

Definition 1.3.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous invariant function. An orbit $\mathscr{O} \subseteq dom(f)$ is said to be critical, if $|d_G f|(u) = 0$ for every (equivalently, for some) $u \in \mathscr{O}$.*

Definition 1.3.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous invariant function and let $c \in \mathbb{R}$. We say that f satisfies the $G-$ Palais-Smale condition at level c ($G-(PS)_c$ in short), if every sequence $\{u_n\}$ in $dom(f)$ such that*

$$|d_G f|(u_n) \to 0,$$

$$f(u_n) \to c,$$

admits a subsequence $\{u_{n_k}\}$ converging in X.

In the particular case of $G = Z_2$ we have the following:

Definition 1.3.6. *Let X be a normed linear space and $f : X \to \mathbb{R} \cup \{+\infty\}$ an even lower semicontinuous function with $f(0) < +\infty$. For every $(0,\eta) \in$*

$epi(f)$ we denote with $|d_{\mathbb{Z}_2}\mathscr{G}_f|(0,\eta)$ the supremum of the numbers $\sigma \in [0,\infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2) : (B((0,\eta),\delta) \cap epi(f)) \times [0,\delta] \to epi(f)$$

satisfying

$$d(\mathscr{H}((w,\mu),t),(w,\mu)) \leq t \quad \mathscr{H}_2((w,\mu),t) \leq \mu - \sigma t,$$

$$\mathscr{H}_1((-w,\mu),t) = -\mathscr{H}_1((w,\mu),t),$$

for every $(w,\mu) \in B((0,\eta),\delta) \cap epi(f)$ and $t \in [0,\delta]$.

Remark 1.3.7. In Definition 1.1.1 if there exist $\varrho > 0$ and a continuous map $\mathscr{H}$ satisfying

$$d(\mathscr{H}(v,t),v) \leq \varrho t, \qquad f(\mathscr{H}(v,t)) \leq f(v) - \sigma t,$$

instead of (1.1), we can deduce that

$$|df|(u) \geq \frac{\sigma}{\varrho}.$$

A similar Remark applies to Definition 1.3.6.

In the following, we refer to [23]. For every $c \in \mathbb{R}$ we set

$$K_c = \{u \in \mathrm{dom}(f) : |d_G f|(u) = 0, f(u) = c\},$$

and f^c defined as (1.3).

Definition 1.3.8. *Let X be a real Banach space and let $\mathscr{E}$ denote the family of sets $A \subset X \setminus \{0\}$ such that A is closed in X and symmetric with respect to 0, that is, $x \in A$ implies $-x \in A$. For $A \in \mathscr{E}$, define the genus of A to be n (denoted by $\gamma(A) = n$) if there is a map $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property. When there does not exist a finite such n, set $\gamma(A) = \infty$. Finally set $\gamma(\emptyset) = 0$.*

We may now state the main (equivariant) Deformation Theorem.

Theorem 1.3.9. *Let X be a complete metric $G-$ space, $f : X \to \mathbb{R}$ a continuous invariant function, and $c \in \mathbb{R}$. Assume that f satisfies $G-(PS)_c$. Then, given $\bar{\varepsilon} > 0$, an invariant neighborhood U of K_c (if $K_c = \emptyset$, we allow $U = \emptyset$) and $\lambda > 0$, there exist $\varepsilon > 0$ and a map $\eta : X \times [0,1] \to X$ continuous with :*

(a) $d(\eta(u,t),u) \leq \lambda t$;

(b) $f(\eta(u,t)) \leq f(u)$;

(c) $f(u) \notin]c-\bar{\varepsilon}, c+\bar{\varepsilon}[\Rightarrow \eta(u,t) = u$;

(d) $\eta(f^{c+\varepsilon} \setminus U, 1) \subseteq f^{c-\varepsilon}$;

(e) $\eta(\cdot,t)$ *is equivariant for every* $t \in [0,1]$.

Proof. A step-by-step analysis of the proof in non-equivariant case (see [26], Theorem 2.14) shows that the same argument also works in the general case. □

Now we can prove a first version of the Ambrosetti-Rabinowitz Theorem ([1],[26],[27]) involving a lower semicontinuous functional.

Theorem 1.3.10. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous even function. Let $G = \mathbb{Z}_2$ and consider X as a $G-$ space. Assume that there exists a strictly increasing sequence $\{V_h\}$ of finite-dimensional subspaces of X with the following properties:*

(a) *there exist $\varrho > 0, \alpha > f(0)$ and a closed subspace $Z \subset X$ such that*

$$X = V_0 \oplus Z \quad \textit{and} \quad \forall u \in Z : \|u\| = \varrho \Rightarrow f(u) \geq \alpha;$$

(b) *there exists a sequence (R_h) in (ϱ, ∞) such that*

$$\forall u \in V_h : \|u\| \geq R_h \Rightarrow f(u) \leq f(0);$$

(c) *the function* $\mathscr{G}_f$ *satisfies* $(PS)_c$ *for every* $c \geq \alpha$;

(d) $|d_G\mathscr{G}_f|(0,\xi) \neq 0$ *whenever* $\xi \geq \alpha$.

Then there exists a sequence $(u_h,\xi_h) \in epi(f)$ *such that*

$$|d\mathscr{G}_f|(u_h,\xi_h) = 0 \quad \textit{and} \quad \mathscr{G}_f(u_h,\xi_h) = \xi_h \to +\infty.$$

Proof. Without loss of generality, we can assume $f(0) = 0$. Let us consider $\mathscr{G}_f : epi(f) \to \mathbb{R}$. First of all, it is easy to see that $|d_G\mathscr{G}_f|(u,\xi) = |d\mathscr{G}_f|(u,\xi)$ whenever $u \neq 0$. Therefore the function $\mathscr{G}_f$ actually satisfies $G-(PS)_c$ for every $c \geq \alpha$. Moreover, for every $c \geq \alpha$ we have $K_c \subseteq (X \setminus \{0\}) \times \{c\}$. Let $\mathscr{E}$ and γ be as in Definition 1.3.8 and let $k = \dim V_0$. Without loss of generality, we can assume that $\dim V_h = h + k$ for all $h \in \mathbb{N}$. Set $D_h = \overline{B(0,R_h)} \cap V_h$. Let

$$\Phi_h = \{\varphi \in C(D_h, epi(f)) : \varphi \quad \text{is equivariant}$$
$$\text{and} \quad \varphi(u) = (u,0) \quad \forall u \in \partial B(0,R_h) \cap V_h\},$$

$$\Gamma_j = \{\varphi\overline{(D_h \setminus Y)} : \varphi \in \Phi_h,\ h \geq j,\ Y \in \mathscr{E}, \quad \text{and} \quad \gamma(Y) \leq h - j\}.$$

Then the following facts hold:

(1) $\Gamma_j \neq \emptyset$ for all $j \in \mathbb{N}$;

(2) $\Gamma_{j+1} \subseteq \Gamma_j$;

(3) if $\Psi \in C(epi(f), epi(f))$ is equivariant and $\Psi(u,0) = (u,0) \forall u \in \partial B(0,R_h) \cap V_h$ and for all $h \geq j$, then $\Psi(B) \in \Gamma_j$ for every $B \in \Gamma_j$;

(4) if $B \in \Gamma_j, S \in \mathscr{E}$ and $\gamma(S) \leq s < j$, we have $B \setminus \overline{(S \times \mathbb{R})} \in \Gamma_{j-s}$;

(5) if $B \in \Gamma_j$ and $L = \{(u,\xi) \in epi(f) : u \in Z, \|u\| = \rho\}$, we have $B \cap L \neq \emptyset$.

The proof of (1)−(4) is essentially the same as that given in [[26], Proposition 9.18]. To prove (5), we denote by $\pi : X \times \mathbb{R} \to X$ the canonical projection of $X \times \mathbb{R}$ onto X. Then, it is readily seen that

$$B \cap L \neq \emptyset \Leftrightarrow \pi(B) \cap (\partial B(0,\rho)) \cap Z \neq \emptyset.$$

Assume that $B = \varphi\overline{(D_h \setminus Y)}$. The function $\pi \circ \varphi \in C(D_h, X)$ is odd and moreover $\pi \circ \varphi = id$ on $\partial B(0, R_h) \cap V_h$, so we can apply the argument of [[26], Proposition 9.23] to the set $(\pi\circ\varphi)\overline{(D_h \setminus Y)} = \pi(B)$. Hence (5) is proved. We now define the minimax values of $\mathscr{G}_f$, setting

$$c_j = \inf_{B\in\Gamma_j} \max_{(u,\xi)\in B} \mathscr{G}_f(u,\xi), \qquad j \in \mathbb{N}.$$

Properties (1)−(5) allow us to obtain for $\mathscr{G}_f$ the results of [[26], Propositions 9.29, 9.30, 9.33], provided that Theorem 1.3.9 is used instead of the classical Deformation Theorem [[26], Theorem A.4]. Then the thesis follows. □

1.4 The results of nonsmooth analysis

Now we recall from [16][17],[18],[25] some basic results. In order to compute $|d\mathscr{G}_f|(u,\eta)$ it will be useful the following result.

Proposition 1.4.1. *Let X be a normed linear space, $J : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous functional, $I : X \to \mathbb{R}$ a C^1 functional and let $f = J + I$. Then the following facts hold:*

(a) *for every $(u,\eta) \in epi(f)$ we have*

$$|d\mathscr{G}_f|(u,\eta) = 1 \Leftrightarrow |d\mathscr{G}_J|(u, \eta - I(u)) = 1;$$

(b) *if J and I are even, for every $\eta \geq f(0)$ we have*

$$|d_{\mathbb{Z}_2}\mathscr{G}_f|(0,\eta) = 1 \Leftrightarrow |d_{\mathbb{Z}_2}\mathscr{G}_J|(0, \eta - I(0)) = 1;$$

(c) *if $u \in dom(f)$ and $I'(u) = 0$, then*

$$|df|(u) = |dJ|(u).$$

Proof. (a) Let $0 < \sigma < |d\mathscr{G}_J|(u, \eta - I(u))$ and let

$$\mathscr{H} : (B((u, \eta - I(u)), \delta) \cap epi(J)) \times [0, \delta] \to epi(J)$$

be as in Definition 1.1.1. Since I is a C^1 functional, we can assume that I is Lipschitz continuous of constant ε in $B(u, 2\delta)$. Let $\delta' \in (0, \delta]$ be such that $(\nu, \mu - I(\nu)) \in B((u, \eta - I(u)), \delta)$ for every $(\nu, \mu) \in B((u, \eta), \delta')$ and let $\mathscr{K} : (B((u, \eta), \delta') \cap epi(f)) \times [0, \delta'] \to epi(f)$ be defined by

$$\mathscr{K}((\nu, \mu), t) = \left(\mathscr{H}_1\left((\nu, \mu - I(\nu)), \frac{t}{1+\varepsilon}\right), \mathscr{H}_2\left((\nu, \mu - I(\nu)), \frac{t}{1+\varepsilon}\right) + \right.$$

$$\left. I\left(\mathscr{H}_1\left((\nu, \mu - I(\nu)), \frac{t}{1+\varepsilon}\right)\right)\right),$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ are the component of the function $\mathscr{H}$. By the triangular inequality we get:

$$d(\mathscr{K}((\nu, \lambda + I(\nu)), (1+\varepsilon)s), (\nu, \lambda + I(\nu))) =$$

$$d(\mathscr{H}((\nu, \lambda), s), (\nu, \lambda + I(\nu) - I(\mathscr{H}_1((\nu, \lambda), s)))) \leq$$

$$d(\mathscr{H}((\nu, \lambda), s), (\nu, \lambda)) + |I(\mathscr{H}_1((\nu, \lambda), s)) - I(\nu)| \leq$$

$$s + \varepsilon s = (1+\varepsilon)s.$$

Furthermore, it is

$$\mathscr{G}_f(\mathscr{K}((\nu, \mu), t)) = \mathscr{H}_2\left((\nu, \mu - I(\nu)), \frac{t}{1+\varepsilon}\right) + I\left(\mathscr{H}_1\left((\nu, \mu - I(\nu)), \frac{t}{1+\varepsilon}\right)\right)$$

$$\leq \mu - I(\nu) - \sigma\frac{t}{1+\varepsilon} + I\left(\mathscr{H}_1\left((\nu, \mu - I(\nu)), \frac{t}{1+\varepsilon}\right)\right)$$

$$\leq \mathscr{G}_f(\nu, \mu) - \left(\frac{\sigma}{1+\varepsilon} - \varepsilon\right)t.$$

Hence

$$|d\mathscr{G}_f|(u, \eta) \geq \frac{\sigma}{1+\varepsilon} - \varepsilon$$

and, since ε can be made arbitrarily small, we obtain:

$$|d\mathscr{G}_f|(u, \eta) \geq \sigma.$$

Now, if $|d\mathscr{G}_J|(u, \eta - I(u)) = 1$, we deduce that $|d\mathscr{G}_f|(u, \eta) = 1$. The opposite implication is obtained by replacing the function J with the function I and the function I with the function $(-I)$. Assertion (c) follow by arguing as in the previous case. Assertion (b) can be reduced to (a) after observing that, since I is even it results $I'(0) = 0$. □

In [12],[16] it is shown that the following condition is fundamental in order to apply nonsmooth critical point theory to the study of lower semicontinuous functions.

$$\forall (u, \eta) \in epi(f) : f(u) < \eta \Rightarrow |d\mathscr{G}_f|(u, \eta) = 1. \tag{1.7}$$

The next Theorem gives a criterion to verify condition (1.7). The follows Proposition allows us to prove this Theorem.

Proposition 1.4.2. *Let $(u, \eta) \in epi(f)$. Assume that there exist $\varrho, \sigma, \delta, \varepsilon > 0$ and a continuous map*

$$\mathscr{H} : \{w \in B(u, \delta) : f(w) < \eta + \delta\} \to X$$

satisfying

$$d(\mathscr{H}(w, t), w) \leq \varrho t, \qquad f(\mathscr{H}(w, t)) \leq \max\{f(w) - \sigma t, \eta - \varepsilon\}$$

whenever $w \in B(u, \delta)$, $f(w) < \eta + \delta$ and $t \in [0, \delta]$. Then we have

$$|d\mathscr{G}_f|(u, \eta) \geq \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}.$$

If moreover X is a normed space, f is even, $u = 0$ and it results $\mathscr{H}(-w, t) = -\mathscr{H}(w, t)$, then we have

$$|d_{\mathbb{Z}_2}\mathscr{G}_f|(0, \eta) \geq \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}.$$

Proof. Let $\delta' \in (0, \delta]$ be such that $\delta + \sigma\delta' \leq \varepsilon$ and let

$$\mathscr{K} : (B((u, \lambda), \delta') \cap epi(f)) \times [0, \delta'] \to epi(f)$$

be defined by $\mathscr{K}((w,\mu),t) = (\mathscr{H}(w,t), \mu - \sigma t)$. If $(w,\mu) \in B((u,\eta),\delta') \cap epi(f)$ and $t \in [0,\delta']$, we have

$$\eta - \varepsilon \le \eta - \delta' - \sigma\delta' < \mu - \sigma t, \qquad f(w) - \sigma t \le \mu - \sigma t,$$

hence

$$f(\mathscr{H}(w,t)) \le \max\{f(w) - \sigma t, \eta - \varepsilon\} \le \mu - \sigma t.$$

Therefore $\mathscr{K}$ actually takes its values in $epi(f)$. Furthermore, it is

$$d(\mathscr{K}((w,\mu),t),(w,\mu)) \le t\sqrt{\varrho^2 + \sigma^2},$$

and

$$\mathscr{G}_f(\mathscr{K}((w,\mu),t)) = \mu - \sigma t = \mathscr{G}_f(w,\mu) - \sigma t.$$

Taking into account Definition 1.1.1 and Remark 1.3.7, the first assertion follows. In the symmetric case, $\mathscr{K}$ automatically satisfies the further condition required in Definition 1.3.6. □

Theorem 1.4.3. *Let $(u,\eta) \in epi(f)$ with $f(u) < \eta$. Assume that, for every $\varrho > 0$ there exist $\delta > 0$ and a continuous map*

$$\mathscr{H} : \{w \in B(u,\delta) : f(w) < \eta + \delta\} \times [0,\delta] \to X$$

satisfying

$$d(\mathscr{H}(w,t),w) \le \rho t \quad \text{and} \quad f(\mathscr{H}(w,t)) \le (1-t)f(w) + t(f(u) + \varrho)$$

whenever $w \in B(u,\delta), f(w) < \eta + \delta, t \in [0,\delta]$. Then we have $|d\mathscr{G}_f|(u,\eta) = 1$. If moreover X is a normed space, f is even, $u = 0$ and $\mathscr{H}(-w,t) = -\mathscr{H}(w,t)$, then we have $|d_{\mathbb{Z}_2}\mathscr{G}_f|(0,\eta) = 1$.

Proof. Let $\varepsilon > 0$ with $\eta - 2\varepsilon > f(u)$, let $0 < \varrho < \eta - f(u) - 2\varepsilon$ and let δ and $\mathscr{H}$ be as in the hypothesis. By reducing δ, we may also assume that

$$\delta \le 1, \qquad \delta(|\eta - 2\varepsilon| + |f(u) + \varrho|) \le \varepsilon.$$

Now consider $w \in B(u,\delta)$ with $f(w) < \eta + \delta$ and $t \in [0,\delta]$. If $f(w) \le \eta - 2\varepsilon$, we have

$$f(w) + t(f(u) - f(w) + \varrho) = (1-t)f(w) + t(f(u) + \varrho) \le$$

$$\le (1-t)(\eta - 2\varepsilon) + t(f(u) + \varrho) \le$$

$$\le \eta - 2\varepsilon + t|\eta - 2\varepsilon| + t|f(u) + \varrho| \le \eta - \varepsilon,$$

while, if $f(w) > \eta - 2\varepsilon$, we have

$$f(w) + t(f(u) - f(w) + \varrho) \le f(w) - (\eta - f(u) - 2\varepsilon - \varrho)t.$$

In any case it follows

$$f(\mathscr{H}(w,t)) \le \max\{f(w) - (\eta - f(u) - 2\varepsilon - \varrho)t, \eta - \varepsilon\}.$$

From Proposition 1.4.2 we get

$$|d\mathscr{G}_f|(u,\eta) \ge \frac{\eta - f(u) - 2\varepsilon - \varrho}{\sqrt{\varrho^2 + (\eta - f(u) - 2\varepsilon - \varrho)^2}}$$

and the first assertion follows by the arbitrariness of ϱ.
The same proof works also in the symmetric case. □

Now we prove a second version of the classical Theorem of Ambrosetti-Rabinowitz.

Theorem 1.4.4. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous even function. Let us assume that there exists a strictly increasing sequence $\{W_h\}$ of finite-dimensional subspaces of X with the following properties:*

(a) *there exist $\varrho > 0, \gamma > f(0)$ and a subspace $V \subset X$ of finite codimension such that*

$$\forall u \in V : \|u\| = \varrho \Rightarrow f(u) \ge \gamma;$$

(b) *there exists a sequence* (R_h) *in* (ϱ, ∞) *such that*

$$\forall u \in W_h : \|u\| \geq R_h \Rightarrow f(u) \leq f(0);$$

(c) f *satisfies* $(PS)_c$ *for any* $c \geq \gamma$ *and* f *satisfies (1.7);*

(d) $|d_{\mathbb{Z}_2}\mathscr{G}_f|(0, \eta) \neq 0$ *for every* $\eta > f(0)$.

Then there exist a sequence $\{u_h\}$ *of critical points of* f *such that*

$$\lim_{h \to \infty} f(u_h) = +\infty.$$

Proof. Because of assumption (c), the function $\mathscr{G}_f$ satisfies $(PS)_c$ for any $c \geq \alpha$. Then the assertion follows by Theorem 1.3.10. □

Chapter 2

Unbounded critical points for a class of lower semicontinuous functionals

In this Chapter we refer to [25], where the authors apply the abstract setting mentioned in Chapter 1 to prove a multiplicity results of unbounded critical points for a class of lower semicontinuous functionals. Specifically they considered the following quasilinear problem

$$\begin{cases} -\mathrm{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.1}$$

where $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ denote the derivatives of a function $j(x,s,\xi)$ with respect of the variables s and ξ respectively. Moreover j, j_ξ, j_s and g satisfied suitable assumptions that will be specified later. Problem (2.1) has a variational structure, given by the functional $f : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$f(u) = \int_\Omega j(x,u,\nabla u) - \int_\Omega G(x,u)$$

where $G(x,s)$ is the primitive of the function $g(x,s)$ with $G(x,0) = 0$. The main point of the paper [25] is that, under suitable assumptions, f is unbounded from below, so that we cannot look for a global minimum. We want to stress that we are dealing with integrands $j(x,s,\xi)$ which may be unbounded with respect to s. This class of functionals has also been treated in [3]. In these paper the existence of a nontrivial solution $u \in L^\infty(\Omega)$ is proved when $g(x,s) = |s|^{p-2}s$. Note that, in this case it is natural to expect solutions in $L^\infty(\Omega)$. In order to prove the existence result, in a fundamental step is to prove that every cluster point of a Palais-Smale sequence belongs to $L^\infty(\Omega)$. That is, to prove that u is bounded before knowing that it is a solution. In our case if u is in $L^\infty(\Omega)$ and $v \in C_0^\infty(\Omega)$ then $j_\xi(x,u,\nabla u)\cdot\nabla v$ and $j_s(x,u,\nabla u)v$ are in $L^1(\Omega)$. Therefore, if $g(x,s) = |s|^{p-2}s$, it would be possible to define a solution as a function $u \in L^\infty(\Omega)$ that satisfies the equation associated to (2.1) in the distributional sense. In our case we considered nonlinearities of the following types

$$g(x,s) = a(x)arctgs + |s|^{p-2}s,$$

where $a(x) \in L^{\frac{2n}{2n+2}}(\Omega)$, with $a(x) > 0$, and $2 < p < \dfrac{2n}{n-2}$. So that we can only expect to find solutions in $H_0^1(\Omega)$. For this reason, we have given a definition of solution weaker than the distributional one. Moreover, if $g(x,s)$ is odd with respect to s and if $j(x,-s,-\xi) = j(x,s,\xi)$, it would be natural to expect the existence of infinitely many solutions as in the semilinear case (see [1]). Unfortunately, we cannot apply any of the classical results of critical point theory, because $\int_\Omega j(x,u,\nabla u)$ in our case is not differentiable, hence f is not of class C^1 on $H_0^1(\Omega)$. More precisely, since $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ are not supposed to be bounded with respect to s, the terms $j_\xi(x,u,\nabla u)\cdot\nabla v$ and $j_s(x,s,\xi)v$ may not belong to $L^1(\Omega)$ even if $v \in C_0^\infty(\Omega)$. Notice that if $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ were supposed to be bounded with respect to s, f would be Gateaux derivable for every $u \in H_0^1(\Omega)$ and along any direction $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see [4],[9],[10],[24],[28] for the study of this class of functionals). On the contrary, in our case, for every $u \in H_0^1(\Omega)$, $f'(u)(v)$ does not even exist along directions $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

2.1 The main result

Let Ω be a bounded open subset of $\mathbb{R}^n (n \geq 3)$ and let $j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following regularity condition

$$\begin{cases} \text{for all } (s,\xi) \in \mathbb{R} \times \mathbb{R}^n & j(x,s,\xi) \text{ is measurable with respect to } x, \\ \text{for a.e. } x \in \Omega, & j(x,s,\xi) \text{ is of class } C^1 \text{ with respect to } (s,\xi) \end{cases}$$

Suppose also that j satisfies the following hypotheses:

$$\text{the function} \quad \{\xi \mapsto j(x,s,\xi)\} \quad \text{is strictly convex} \tag{2.2}$$

for almost every $x \in \Omega$ and every $s \in \mathbb{R}$;

moreover, we suppose that there exist a constant $\alpha_0 > 0$ and a positive increasing function $\alpha \in C(\mathbb{R})$ such that

$$\alpha_0 |\xi|^2 \leq j(x,s,\xi) \leq \alpha(|s|)|\xi|^2 \tag{2.3}$$

for almost every $x \in \Omega$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
We will also assume that

$$\lim_{|s| \to +\infty} \frac{\alpha(|s|)}{|s|^{p-2}} = 0 \tag{2.4}$$

where $2 < p < \dfrac{2n}{n-2}$.

Regarding the function $j_s(x,s,\xi)$ we suppose that there exist a positive increasing function $\beta \in C(\mathbb{R})$ and a positive constant R such that the following conditions are satisfied almost everywhere in Ω and for every $\xi \in \mathbb{R}^n$:

$$|j_s(x,s,\xi)| \leq \beta(|s|)|\xi|^2 \quad \text{for every} \quad s \in \mathbb{R}, \tag{2.5}$$

and

$$j_s(x,s,\xi)s \geq 0 \quad \text{for every} \quad s \in \mathbb{R} \quad \text{with} \quad |s| \geq R. \tag{2.6}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, that is

$$\begin{cases} \text{for all } s \in \mathbb{R} & g(x,s) \text{ is measurable with respect to } x \\ \text{for a.e. } x \in \Omega, & g(x,s) \text{ is continuous with respect to } s \end{cases}$$

Suppose also that g is a nonlinearity with subcritical growth, that is for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega)$ such that

$$|g(x,s)| \leq a_\varepsilon(x) + \varepsilon|s|^{\frac{n+2}{n-2}} \tag{2.7}$$

for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$.
Let $G(x,s) = \int_0^s g(x,t)dt$. We assume that there exist $q > 2$ and functions $a_0(x), \bar{a}(x) \in L^1(\Omega), b_0(x), \bar{b}(x) \in L^{\frac{2n}{n+2}}(\Omega)$ and $k(x) \in L^\infty(\Omega)$ with $k(x) > 0$ almost everywhere, such that

$$qG(x,s) \leq g(x,s)s + a_0(x) + b_0(x)|s|, \tag{2.8}$$

$$G(x,s) \geq k(x)|s|^q - \bar{a}(x) - \bar{b}(x)|s| \tag{2.9}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$
Finally, we suppose that there exist $R' > 0$ and $\delta > 0$ such that if $|s| \geq R'$ then

$$qj(x,s,\xi) - j_s(x,s,\xi)s - j_\xi(x,s,\xi) \cdot \xi \geq \delta|\xi|^2 \tag{2.10}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Remark 2.1.1. In the classical results of critical point theory different conditions from (2.7)-(2.9) are usually supposed. Indeed, as a growth condition on $g(x,s)$, it is assumed that

$$|g(x,s)| \leq a(x) + b|s|^{p-1}, \tag{2.11}$$

where $b \in \mathbb{R}^+, a(x) \in L^{\frac{2n}{n+2}}(\Omega)$ and p is defined as (2.4). Note that (2.11) implies (2.7). Indeed, suppose that $g(x,s)$ satisfies (2.11), then Young inequality implies that (2.7) is satisfied with $a_\varepsilon(x) = a(x) + C(b,\varepsilon)$. Moreover, as a superlinear condition, it is usually assumed that there exist $q > 2$ and $R > 0$ such that

$$0 < qG(x,s) \leq g(x,s)s \tag{2.12}$$

for every $s \in \mathbb{R}$ with $|s| \geq R$. Note that this condition is stronger than conditions (2.8) and (2.9). Indeed, suppose that $g(x,s)$ satisfies (2.12) and notice that this implies that there exists $a_0 \in L^1(\Omega)$ such that

$$qG(x,s) \leq g(x,s)s + a_0(x) \tag{2.13}$$

for every $s \in \mathbb{R}$. Indeed, if $|s| \geq R$ we have (2.12), otherwise, if $|s| < R$ by (2.7), we obtain

$$g(x,s) \leq |g(x,s)| \leq a_\varepsilon(x) + \varepsilon|s|^{\frac{n+2}{n-2}}.$$

Then, it follows that

$$G(x,s) = \int_0^s g(x,t)dt \leq \int_0^s (a_\varepsilon(x) + \varepsilon|t|^{\frac{n+2}{n-2}})\, dt$$

$$\leq \int_0^s (a_\varepsilon(x) + \varepsilon R^{\frac{n+2}{n-2}}) = R\, a_\varepsilon(x) + \varepsilon R^{\frac{2n}{n-2}} \in L^1(\Omega)$$

because Ω is a bounded set and $a_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega) \subset L^1(\Omega)$. Therefore, setting $a_0(x) = \varepsilon R^{\frac{2n}{n-2}} + R\, a_\varepsilon(x)$, it follows (2.13). Then (2.8) is satisfied with $b_0(x) \equiv 0$. Moreover, from (2.12) we deduce that for all $s \in \mathbb{R}$ with $|s| \geq R$ one has

$$G(x,s) \geq \frac{G\left(x, R\frac{s}{|s|}\right)}{R^q} \geq \gamma_0(x)|s|^q$$

where

$$\gamma_0(x) = R^{-q} \inf\{G(x,s) : |s| = R\} > 0$$

a.e. $x \in \Omega$. Therefore there exists $\bar{a}(x) \in L^1(\Omega)$ such that

$$G(x,s) \geq \gamma_0(x)|s|^q - \bar{a}(x)$$

a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ so that also (2.9) is satisfied.

In order to deal with the Euler equation of f let us define the following subspace of $H_0^1(\Omega)$ for a fixed u in $H_0^1(\Omega)$:

$$W_u = \{v \in H_0^1(\Omega) : j_\xi(x,u,\nabla u)\nabla v \in L^1(\Omega) \quad \text{and} \quad j_s(x,u,\nabla u)v \in L^1(\Omega)\}.$$

Now we give the definition of generalized solution.

Definition 2.1.2. *Let $\Lambda \in H^{-1}(\Omega)$ and assume (2.2), (2.3),(2.5). We say that u is a generalized solution of*

$$\begin{cases} -div(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = \Lambda & in \quad \Omega \\ \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{2.14}$$

if $u \in H_0^1(\Omega)$ and it results

$$j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega), \quad j_s(x,u,\nabla u)u \in L^1(\Omega),$$

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle\Lambda, v\rangle \qquad \forall v \in W_u.$$

Notice that if $u \in H_0^1(\Omega)$ is a generalized solution of problem (2.14) and $u \in L^\infty(\Omega)$, then u is a distributional solution of (2.14).
We will prove the following

Theorem 2.1.3. *Assume that conditions (2.2)-(2.10) hold and let us suppose that*

$$j(x,-s,-\xi) = j(x,s,\xi) \qquad \text{and} \qquad g(x,-s) = -g(x,s)$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$. Then there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ of generalized solutions of problem (2.14) with $f(u_n) \to +\infty$.

2.2 A fundamental theorem

Let us consider the functional $J : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$J(v) = \int_\Omega j(x, v, \nabla v). \tag{2.15}$$

From hypothesis (2.3), we have that $j(x, v, \nabla v) \geq 0$. Then by Fatou's Lemma, we immediately obtain that J is lower semicontinuous. Now we prove that J satisfies the condition (1.7) that is fundamental in order to apply all the abstract results of Chapter 1. To this aim, for every $k \geq 1$, we define the truncation $T_k : \mathbb{R} \to \mathbb{R}$ at height k, defined as

$$T_k(s) = \begin{cases} s, & \text{if} \quad |s| \leq k \\ k\,\dfrac{s}{|s|}, & \text{if} \quad |s| \geq k. \end{cases} \tag{2.16}$$

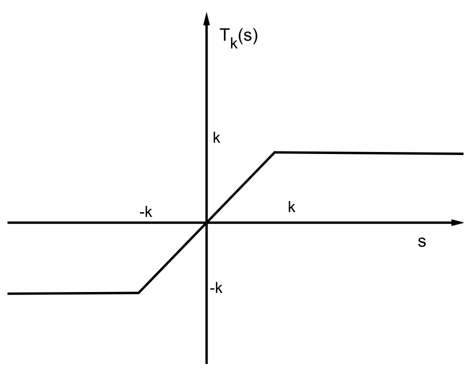

Figure 2.1: The function $T_k(s)$

We will prove the following

Theorem 2.2.1. *Assume that conditions (2.2),(2.3),(2.6) hold. Then, for every $(u, \eta) \in epi(J)$ with $J(u) < \eta$, there holds*

$$|d\mathscr{G}_J|(u, \eta) = 1.$$

Moreover, if $j(x, -s, -\xi) = j(x, s, \xi)$ is satisfied, then $\forall \eta > J(0)$ it results $|d_{\mathbb{Z}_2}\mathscr{G}_J|(0, \eta) = 1$.

Proof. Let $(u,\eta)\in epi(J)$ with $J(u)<\eta$ and let $\varrho>0$. Since $T_k(v)$ tends to v in $H_0^1(\Omega)$ as k goes to ∞, there exists $\delta\in(0,1], \delta=\delta(\rho)$, and $k\geq 1, k=k(\rho)$, such that $k\geq R$ (where R is as in (2.6)) and

$$\|T_k(v)-v\|_{1,2}<\varrho \quad \text{for every} \quad v\in B(u,\delta). \tag{2.17}$$

From (2.3) we have

$$j(x,v,\nabla T_k(v))\leq \alpha(k)|\nabla v|^2.$$

On the other hand if $v\in B(u,\delta)$ then v converges to u for a.e. $x\in\Omega$. Indeed, if we take $v_n\in H_0^1(\Omega)$ such that v_n converges to u in $H_0^1(\Omega)$ then, using the Sobolev embedding theorem, it immediately follows that v_n converges to u in $L^2(\Omega)$. Hence, up to a subsequence, v_n converges to u for a.e. $x\in\Omega$. The continuity of j with respect to ξ implies that

$$j(x,v,\nabla T_k(v))\to j(x,u,\nabla T_k(u))$$

for a.e. $x\in\Omega$. Thus, up to reducing δ, using the Lebesgue Dominated Convergence Theorem, we get the following inequality

$$\int_\Omega j(x,v,\nabla T_k(v))<\int_\Omega j(x,u,\nabla T_k(u))+\rho. \tag{2.18}$$

Now, since $j(x,u,0)=0$, and by definition of T_k one has

$$j(x,u,\nabla T_k(u))=\begin{cases} 0 & \text{if} \quad |u|\geq k \\ \\ j(x,u,\nabla u) & \text{if} \quad |u|\leq k \end{cases}.$$

This implies that

$$j(x,u,\nabla T_k(u))\leq j(x,u,\nabla u). \tag{2.19}$$

From (2.18) and (2.19) it follows

$$\int_\Omega j(x, v, \nabla T_k(v)) < \int_\Omega j(x, u, \nabla u) + \varrho \tag{2.20}$$

for each $v \in B(u, \delta)$. We now prove that, for every $t \in [0, \delta]$ and $v \in B(u, \delta)$, there holds

$$J((1-t)v + tT_k(v)) \leq (1-t)J(v) + t(J(u) + \varrho). \tag{2.21}$$

We consider the expression

$$j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, \nabla v). \tag{2.22}$$

Adding and subtracting the quantity $j(x, v, (1-t)\nabla v + t\nabla T_k(v))$, one has

$$j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, \nabla v)$$

$$= j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, (1-t)\nabla v + t\nabla T_k(v))$$
$$+ j(x, v, (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, \nabla v).$$

From (2.2) we obtain that

$$j(x, v, (1-t)\nabla v + t\nabla T_k(v)) \leq (1-t)j(x, v, \nabla v) + tj(x, v, \nabla T_k(v)) =$$

$$= t(j(x, v, \nabla T_k(v)) - j(x, v, \nabla v)) + j(x, v, \nabla v).$$

Therefore we have

$$j(x, v, (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, \nabla v)$$
$$\leq t(j(x, v, \nabla T_k(v)) - j(x, v, \nabla v)). \tag{2.23}$$

Since $j(x, s, \xi)$ is of class C^1 with respect to the variable s, there exists $\theta \in [0, 1]$ such that

$$j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, (1-t)\nabla v + t\nabla T_k(v))$$

$$= tj_s(x, v + \theta t(T_k(v) - v), (1-t)\nabla v + t\nabla T_k(v))(T_k(v) - v). \qquad (2.24)$$

We observe that

$$\begin{cases} v(x) + \theta t(T_k(v(x)) - v(x)) \geq R & \text{if} \quad v(x) \geq k \\ \\ v(x) + \theta t(T_k(v(x)) - v(x)) \leq -R & \text{if} \quad v(x) \leq -k \end{cases}$$

Indeed, if $v(x) \geq k$ then $T_k(v) = k$, and therefore:

$$\begin{aligned} v(x) + \theta t(T_k(v(x)) - v(x)) &= v(x)(1-\theta t) + \theta t \cdot T_k(v(x)) \\ &\geq k(1-\theta t) + k\theta t = k \geq R. \end{aligned}$$

Similarly, if $v(x) \leq -k$ then $T_k(v) = -k$, and we get

$$\begin{aligned} v(x) + \theta t(T_k(v(x)) - v(x)) &= v(x)(1-\theta t) + \theta t \cdot T_k(v) \\ &\leq -k(1-\theta t) - k\theta t = -k \leq -R. \end{aligned}$$

Then by (2.6) it follows that, if $|v + \theta t(T_k(v(x)) - v(x))| \geq R,$ then

$$j_s(x, v + \theta t(T_k(v) - v), (1-t)\nabla v + t\nabla T_k(v))(v + \theta t(T_k(v(x)) - v(x))) \geq 0.$$

Hence one has

$$\begin{cases} j_s(x, v + \theta t(T_k(v) - v), (1-t)\nabla v + t\nabla T_k(v)) \geq 0 & \text{if} \quad v(x) \geq k \\ \\ j_s(x, v + \theta t(T_k(v) - v), (1-t)\nabla v + t\nabla T_k(v)) \leq 0 & \text{if} \quad v(x) \leq -k \end{cases}.$$

Moreover

$$\begin{cases} T_k(v) - v \leq 0 & \text{if} \quad v(x) \geq k \\ \\ T_k(v) - v \geq 0 & \text{if} \quad v(x) \leq -k \end{cases}$$

Finally, taking into account that if $|v| \leq k$ then $T_k(v) = v$, one has

$$j_s(x, v + \theta t(T_k(v) - v), (1-t)\nabla v + t\nabla T_k(v))(T_k(v) - v) \leq 0.$$

Then from (2.24) it follows

$$\begin{aligned} &j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) - \\ &\qquad j(x, v, (1-t)\nabla v + t\nabla T_k(v)) \leq 0 \end{aligned} \tag{2.25}$$

Combining (2.23) and (2.25), we deduce that (2.22) becomes

$$\begin{aligned} j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) - j(x, v, \nabla v) \\ \leq t(j(x, v, \nabla T_k(v)) - j(x, v, \nabla v)) \end{aligned}$$

and therefore, it follows that

$$j(x, (1-t)v + tT_k(v), (1-t)\nabla v + t\nabla T_k(v)) \leq (1-t)j(x, v, \nabla v) + tj(x, v, \nabla T_k(v)).$$

Integrating both member of the previous inequality and in view of (2.20), we get (2.21). In order to apply Theorem 1.4.3 we define

$$\mathscr{H} : \{v \in B(u, \delta) : J(v) < \eta + \delta\} \times [0, \delta] \to H_0^1(\Omega)$$

by setting

$$\mathscr{H}(v, t) = (1-t)v + tT_k(v).$$

Then, taking into account (2.17) and (2.21), we have:

$$d(\mathscr{H}(v, t), v) \leq \varrho t$$

and

$$J(\mathscr{H}(v,t)) \leq (1-t)J(v) + t(J(u)+\rho).$$

for $v \in B(u,\delta)$, $J(v) < \eta + \delta$ and $t \in [0,\delta]$. The first assertion now follows from Theorem 1.4.3. Finally, since $\mathscr{H}(-v,t) = -\mathscr{H}(v,t)$, one also has $|d_{\mathbb{Z}_2}\mathscr{G}_J|(0,\eta) = 1$, whenever $j(x,-s,-\xi) = j(x,s,\xi)$. □

2.3 The variational setting

This section regards the relations between $|dJ|(u)$ and the directional derivatives of the functional J. Moreover, we will obtain some Brezis-Browder (see [8]) type results. First of all, we make a few observations.

Remark 2.3.1. Hypotheses (2.2) and the right inequality of (2.3) readily imply that there exists a positive increasing function $\bar{\alpha}(|s|)$ such that

$$|j_\xi(x,s,\xi)| \leq \bar{\alpha}(|s|)|\xi| \tag{2.26}$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Indeed, from (2.2) one has

$$j(x,s,\xi + |\xi|v) \geq j(x,s,\xi) + j_\xi(x,s,\xi) \cdot v|\xi|$$

for every $v \in \mathbb{R}^n$ such that $|v| \leq 1$. This and (2.3) yield

$$\begin{aligned} j_\xi(x,s,\xi) \cdot v|\xi| &\leq \alpha(|s|)|\xi + v|\xi||^2 - \alpha_0|\xi|^2 \\ &\leq 4\alpha(|s|)|\xi|^2. \end{aligned}$$

From the arbitrariness of v, (2.26) follows. On the other hand, if (2.26) holds, we have

$$|j(x,s,\xi)| \leq \int_0^1 |j_\xi(x,s,t\xi) \cdot \xi| dt \leq \frac{1}{2}\bar{\alpha}(|s|)|\xi|^2.$$

As a consequence, it is not restrictive to suppose that the functions in the right-hand side of (2.3) and (2.26) are the same, that is $\alpha = \bar{\alpha}$. Notice that, in particular, there holds $j_\xi(x,s,0) = 0$.

Remark 2.3.2. The hypotheses (2.2) and (2.3) imply that,

$$j_\xi(x, s, \xi) \cdot \xi \geq \alpha_0 |\xi|^2. \tag{2.27}$$

Indeed, we have

$$0 = j(x, s, 0) \geq j(x, s, \xi) + j_\xi(x, s, \xi) \cdot (0 - \xi)$$

so that inequality (2.27) follows by (2.3).

Now for every $u \in H_0^1(\Omega)$, we define the subspace

$$V_u = \{v \in H_0^1(\Omega) \cap L^\infty(\Omega) : u \in L^\infty(\{x \in \Omega : v(x) \neq 0\})\}. \tag{2.28}$$

It easy to see that V_u is a linear subspace of $H_0^1(\Omega)$.

Theorem 2.3.3. *For every $u \in H_0^1(\Omega)$ and for every $v \in H_0^1(\Omega)$ there exists a sequence $\{v_h\}$ in V_u converging to v in $H_0^1(\Omega)$ with $-v^-(x) \leq v_h(x) \leq v^+(x)$ a.e. In particular, V_u is a dense linear subspace of $H_0^1(\Omega)$.*

Proof. It is enough to treat the case $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Let$\{\vartheta_h\} \subset C_c^\infty(\mathbb{R})$ such that

$$\begin{cases} \vartheta_h(s) = 1, & \forall s \in [-h+1, h-1] \\ \vartheta_h(s) = 0, & \forall s \in \mathbb{R} \setminus [-h, h] \\ |\vartheta_h'(s)| \leq 2, & \forall s \in \mathbb{R} \end{cases}$$

We set $v_h = (\vartheta_h \circ u)v$; then one has that v_h belongs to V_u, $v_h(x)$ converges to $v(x)$, $\nabla v_h(x)$ converges to $\nabla v(x)$ and $-v^-(x) \leq v_h(x) \leq v^+(x)$ for a.e. $x \in \Omega$. Moreover, for a.e. $x \in \Omega$ we have

$$|\vartheta_h(u(x))v(x)| \leq |v(x)|,$$

$$|\vartheta_h'(u(x))\nabla u(x)v(x) + \vartheta_h(u(x)\nabla v(x)|$$

$$\leq 2|\nabla u(x)||v(x)| + |\nabla v(x)|.$$

By Lebesgue' s Theorem, v_h converges to v in $H_0^1(\Omega)$ and the thesis follows. □

Since $V_u \subset W_u \subset H_0^1(\Omega)$ and $\overline{V_u} = H_0^1(\Omega)$, then also W_u is dense in $H_0^1(\Omega)$. In the following proposition we study the conditions under which we can compute the directional derivatives of J.

Proposition 2.3.4. *Assume that conditions (2.3),(2.5),(2.26) hold. Then there exists $J'(u)(v)$ for every $u \in dom(J)$ and $v \in V_u$. Furthermore, we have*

$$j_s(x,u,\nabla u)v \in L^1(\Omega) \quad \textit{and} \quad j_\xi(x,u,\nabla u)\nabla v \in L^1(\Omega)$$

and

$$J'(u)(v) = \int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v.$$

Proof. Let $u \in \text{dom}(J)$ and $v \in V_u$. For every $t \in (0,1)$ and for a.e. $x \in \Omega$, we set

$$F(x,t) = j(x, u(x) + tv(x), \nabla u(x) + t\nabla v(x)).$$

Since $v \in V_u$ using (2.3), it follows that

$$F(x,t) \leq \alpha(|u+tv|)|\nabla u + t\nabla v|^2 \leq \alpha(\|u\|_\infty + \|v\|_\infty)(|\nabla u| + |\nabla v|)^2$$

whence $F(x,t) \in L^1(\Omega)$. Moreover, it results

$$\frac{\partial F}{\partial t}(x,t) = j_s(x, u+tv, \nabla u + t\nabla v)v + j_\xi(x, u+tv, \nabla u + t\nabla v)\cdot\nabla v.$$

From hypotheses (2.5) and (2.26) we get that for every $x \in \Omega$ with $v(x) \neq 0$, it results

$$\left|\frac{\partial F}{\partial t}(x,t)\right| \leq |v|\cdot\beta(|u+tv|)|\nabla u + t\nabla v|^2 + |\nabla v|\cdot\alpha(|u+tv|)|\nabla u + t\nabla v|$$

$$\leq \|v\|_\infty \cdot \beta(\|u\|_\infty + \|v\|_\infty)\cdot(|\nabla u| + |\nabla v|)^2$$

$$+\alpha(\|u\|_\infty + \|v\|_\infty)(|\nabla u| + |\nabla v|)|\nabla v|$$

Since the function in the right-hand side of the previous inequality belongs to $L^1(\Omega)$, the assertion follows using Lebesgue's Theorem. □

In the sequel we will often use the cut-off function $H \in C^\infty(\mathbb{R})$ given by

$$\begin{cases} H(s) = 1, & \forall s \in [-1,1] \\ H(s) = 0, & \forall s \in \mathbb{R} \setminus [-2,2] \\ |H'(s)| \le 2, & \forall s \in \mathbb{R} \end{cases} \tag{2.29}$$

Now, we can prove a fundamental inequality regarding the weak slope of J.

Proposition 2.3.5. *Assume conditions (2.3),(2.5),(2.26). Then we have*

$$|d(J-w)|(u)$$
$$\ge \sup\left\{\int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v - \langle w, v\rangle : v \in V_u, \|v\|_{1,2} \le 1\right\}$$

for every $u \in dom(J)$ and every $w \in H^{-1}(\Omega)$.

Proof. Taking into account that the weak slope of $J-w$ is a real number in $[0,\infty]$, it results that if $|d(J-w)|(u) = +\infty$ or if it holds

$$\sup\left\{\int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v - \langle w, v\rangle : v \in V_u, \|v\|_{1,2} \le 1\right\} = 0,$$

the inequality is satisfied. Otherwise, let $u \in \mathrm{dom}(J)$ and let $\eta \in \mathbb{R}^+$ be such that $J(u) < \eta$. Moreover, let us consider $\bar\sigma > 0$ and $\bar v \in V_u$ such that $\|\bar v\|_{1,2} \le 1$ and

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla \bar v + \int_\Omega j_s(x,u,\nabla u)\bar v - \langle w, \bar v\rangle < -\bar\sigma. \tag{2.30}$$

Let us set $v_k = H\left(\frac{u}{k}\right)\bar v$, where $H(s)$ is defined as in (2.29). Since $\bar v \in V_u$ we deduce that $v_k \in V_u$ for every $k \ge 1$. Moreover for a.e. $x \in \Omega$ we have

$$|v_k(x)| \le |\bar v(x)|,$$
$$|\nabla v_k(x)| = \left|H'\left(\frac{u(x)}{k}\right)\frac{\nabla u(x)}{k}\bar v(x) + H\left(\frac{u(x)}{k}\right)\nabla \bar v(x)\right|$$
$$\le 2|\nabla u(x)||\bar v(x)| + |\nabla \bar v(x)|.$$

By Lebesgue' s Theorem v_k converges to v in $H_0^1(\Omega)$. Then, let us fix $\varepsilon > 0$ there exists $k_0 \geq 1$ such that $\|v_{k_0} - \bar{v}\|_{1,2} < \frac{\varepsilon}{2}$. Hence we have:

$$\|v_{k_0}\|_{1,2} - \|\bar{v}\|_{1,2} < \frac{\varepsilon}{2},$$

that is

$$\|v_{k_0}\|_{1,2} < \|\bar{v}\|_{1,2} + \frac{\varepsilon}{2}.$$

This, recalling that $\|\bar{v}\|_{1,2} \leq 1$, implies

$$\left\| H\left(\frac{u}{k_0}\right)\bar{v} \right\|_{1,2} < 1 + \frac{\varepsilon}{2}. \tag{2.31}$$

Moreover, by Proposition 2.3.4 we can consider the directional derivatives

$$J'(u)(v_k) = \int_\Omega j_\xi(x,u,\nabla u)\nabla v_k + \int_\Omega j_s(x,u,\nabla u)v_k.$$

In addition, as k goes to infinity, we have

$$j_s(x,u(x),\nabla u(x))v_k(x) \to j_s(x,u(x),\nabla u(x))\bar{v}(x) \quad \text{for} \quad \text{a.e.} \quad x \in \Omega,$$

$$j_\xi(x,u(x),\nabla u(x))\nabla v_k(x) \to j_\xi(x,u(x),\nabla u(x))\nabla\bar{v}(x) \quad \text{for} \quad \text{a.e.} \quad x \in \Omega.$$

Moreover, we get

$$|j_s(x,u,\nabla u)v_k| \leq |j_s(x,u,\nabla u)\bar{v}|,$$

$$|j_\xi(x,u,\nabla u)\cdot\nabla v_k| \leq |j_\xi(x,u,\nabla u)||\nabla\bar{v}| + 2|\bar{v}||j_\xi(x,u,\nabla u)\cdot\nabla u|.$$

Since $v \in V_u$ and by using (2.5) and (2.26), we can apply Lebesgue's Dominated Covergence Theorem to obtain

$$\lim_{k\to\infty}\int_\Omega j_s(x,u,\nabla u)v_k = \int_\Omega j_s(x,u,\nabla u)\bar{v}$$

$$\lim_{k\to\infty}\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v_k = \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\bar{v},$$

Taking into account (2.30) it follows

$$\int_\Omega j_s(x,u,\nabla u)H\left(\frac{u}{k_0}\right)\bar{v}$$

$$+\int_\Omega j_\xi(x,u,\nabla u)\nabla\cdot\left[H\left(\frac{u}{k_0}\right)\bar{v}\right]-\left\langle w, H\left(\frac{u}{k_0}\right)\bar{v}\right\rangle<-\bar{\sigma}. \tag{2.32}$$

In order to apply Proposition 1.2.4, let us consider J^η defined as in (1.3). Now, we take $u_n \in J^\eta$ such that u_n converges to u in $H_0^1(\Omega)$ and set

$$v_n = H\left(\frac{u_n}{k_0}\right)\bar{v}.$$

We have that v_n converges to $H\left(\frac{u}{k_0}\right)\bar{v}$ in $H_0^1(\Omega)$. Hence let us fixed $\varepsilon > 0$ there exists $\delta_1 > 0$ such that, if $\|u_n - u\|_{1,2} < \delta_1$, then there holds

$$\left\|v_n - H\left(\frac{u}{k_0}\right)\bar{v}\right\|_{1,2} < \frac{\varepsilon}{2}.$$

Therefore

$$\|v_n\|_{1,2} < \left\|H\left(\frac{u}{k_0}\right)\bar{v}\right\|_{1,2} + \frac{\varepsilon}{2}$$

and by (2.31) one has $\|v_n\|_{1,2} < 1+\varepsilon$. Then we can conclude that

$$\left\|H\left(\frac{z}{k_0}\right)\bar{v}\right\|_{1,2} < 1+\varepsilon, \tag{2.33}$$

for every $z \in B(u,\delta_1)\cap J^\eta$. Moreover, note that $v_n \in V_n$, so that from Proposition 2.3.4 we deduce that we can consider $J'(u_n)(v_n)$. From (2.5) and (2.26) it follows

$$|j_s(x,u_n,\nabla u_n)v_n| \le \beta(2k_0)\|\bar{v}\|_\infty|\nabla u_n|^2,$$

$$|j_\xi(x,u_n,\nabla u_n)\nabla v_n| \le \alpha(2k_0)|\nabla u_n|\left[\frac{2}{k_0}|\nabla u_n|\|v\|_\infty + |\nabla\bar{v}|\right].$$

Then, we obtain

$$\lim_{n\to\infty}\int_\Omega j_s(x,u_n,\nabla u_n)v_n = \int_\Omega j_s(x,u,\nabla u)H\left(\frac{u}{k_0}\right)\bar{v},$$

$$\lim_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla v_n = \int_\Omega j_\xi(x,u,\nabla u)\nabla\left[H\left(\frac{u}{k_0}\right)\bar v\right],$$

which together with (2.32) immediately imply

$$\int_\Omega j_s(x,z,\nabla z)H\left(\frac{z}{k_0}\right)\bar v$$
$$+\int_\Omega j_\xi(x,z,\nabla z)\cdot\nabla\left[H\left(\frac{z}{k_0}\right)\bar v\right]-\left\langle w,H\left(\frac{z}{k_0}\right)\bar v\right\rangle<-\bar\sigma. \tag{2.34}$$

for every $z\in B(u,\delta_1)\cap J^\eta$. Now, observe that inequality (2.34) is equivalent to:

$$J'(z)\left(H\left(\frac{z}{k_0}\right)\bar v\right)-\left\langle w,H\left(\frac{z}{k_0}\right)\bar v\right\rangle<-\bar\sigma.$$

Since

$$J'(z)\left(H\left(\frac{z}{k_0}\right)\bar v\right)=\lim_{t\to 0}\left[\frac{J\left(z+\frac{t}{1+\varepsilon}H\left(\frac{z}{k_0}\right)\bar v\right)-J(z)}{t}\right]\cdot(1+\varepsilon)$$

then there exists $\delta<\delta_1$ with

$$J\left(z+\frac{t}{1+\varepsilon}H\left(\frac{z}{k_0}\right)\bar v\right)-J(z)$$
$$-\left\langle w,\frac{t}{1+\varepsilon}H\left(\frac{z}{k_0}\right)\bar v\right\rangle\le-\frac{\bar\sigma}{1+\varepsilon}t \tag{2.35}$$

for every $t\in[0,\delta]$ and $z\in B(u,\delta)\cap J^\eta$. Finally, let us define the continuous function $\mathscr{H}:B(u,\delta)\cap J^\eta\times[0,\delta]\to H_0^1(\Omega)$ given by

$$\mathscr{H}(z,t)=z+\frac{t}{1+\varepsilon}H\left(\frac{z}{k_0}\right)\bar v.$$

From (2.33) and (2.35) we deduce that $\mathscr{H}$ satisfies all the hypotheses of Proposition 1.2.4. Then, $|d(J-w)|(u)\ge\dfrac{\bar\sigma}{1+\varepsilon}$, and the conclusion follows from the arbitrariness of ε. □

The next lemma will be useful in proving two Brezis-Browder type results for J.

Lemma 2.3.6. *Assume conditions (2.2),(2.3),(2.5),(2.6) and let* $u \in dom(J)$. *Then*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u \le |dJ|(u)\|u\|_{1,2}. \tag{2.36}$$

In particular, if $|dJ|(u) < +\infty$, *there holds*

$$j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega) \quad \textit{and} \quad j_s(x,u,\nabla u)u \in L^1(\Omega)$$

Proof. First, notice that if u is such that $|dJ|(u) = +\infty$, or

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla u + \int_\Omega j_s(x,u,\nabla u)u \le 0,$$

then the conclusion trivially holds. Otherwise, let u belong to dom(J) with $|dJ|(u) < +\infty$, and $\sigma > 0$ such that

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla u + \int_\Omega j_s(x,u,\nabla u)u > \sigma\|u\|_{1,2} > 0.$$

Let $T_k(s)$ defined by (2.16), then we can to consider $k \ge 1$ such that

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla T_k(u) + \int_\Omega j_s(x,u,\nabla u)T_k(u) > \sigma\|T_k(u)\|_{1,2}.$$

Indeed, putting $\Omega_1 = \{x \in \Omega : |u(x)| \le k\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| \ge k\}$ we have

$$\int_{\Omega_1} j_\xi(x,u,\nabla u)\nabla T_k(u) + \int_{\Omega_1} j_s(x,u,\nabla u)T_k(u) =$$
$$\int_{\Omega_1} j_\xi(x,u,\nabla u)\nabla u + \int_{\Omega_1} j_s(x,u,\nabla u)u,$$

and

$$\int_{\Omega_2} j_\xi(x,u,\nabla u)\nabla T_k(u) + \int_{\Omega_2} j_s(x,u,\nabla u)T_k(u) = \int_{\Omega_2} j_s(x,u,\nabla u)\,k\,\frac{u}{|u|}.$$

Then, for k large enough, $|\Omega_1|$ converges to $|\Omega|$ and $|\Omega_2|$ converges to 0.

Now we will prove that $|dJ|(u) \geq \sigma$. Fixed $\varepsilon > 0$ we first want to show that there exists $\delta_1 > 0$ such that

$$\|T_k(w)\|_{1,2} \leq (1+\varepsilon)\|T_k(u)\|_{1,2}, \tag{2.37}$$

$$\int_\Omega j_\xi(x, w, \nabla w)\nabla T_k(w) + \int_\Omega j_s(x, w, \nabla w)T_k(w) > \sigma\|T_k(u)\|_{1,2} \tag{2.38}$$

for every $w \in H_0^1(\Omega)$ with $\|w - u\|_{1,2} < \delta_1$. Indeed, if we take $\{w_n\} \subset H_0^1(\Omega)$ such that w_n converges to u in $H_0^1(\Omega)$, then $T_k(w_n)$ converges to $T_k(u)$ in $H_0^1(\Omega)$. Hence, let us fix $\varepsilon > 0$ there exists $n \geq 1$ such that

$$\|T_k(w_n) - T_k(u)\|_{1,2} < \varepsilon\|T_k(u)\|_{1,2}.$$

Then

$$\|T_k(w_n)\|_{1,2} - \|T_k(u)\|_{1,2} < \varepsilon\|T_k(u)\|_{1,2}$$

and the thesis follows. Moreover, observe that, if $|w_n| \geq R$, by (2.6) we have

$$j_s(x, w_n(x), \nabla w_n(x))w_n(x) \geq 0,$$

and if $|w_n| \leq R$ from (2.5) it follows that

$$|j_s(x, w_n(x), \nabla w_n(x))w_n(x)| \leq R\,\beta(R)|\nabla w_n|^2$$

Hence, we can conclude that

$$j_s(x, w_n(x), \nabla w_n(x))w_n(x) \geq -R\,\beta(R)|\nabla w_n|^2.$$

Since w_n converges to u in $H_0^1(\Omega)$, by (2.27) and applying Fatou's Lemma we get

$$\liminf_{n\to\infty}\left\{\int_\Omega j_\xi(x, w_n, \nabla w_n)\nabla T_k(w_n) + \int_\Omega j_s(x, w_n, \nabla w_n)T_k(w_n)\right\}$$

$$\geq \int_\Omega j_\xi(x, u, \nabla u)\nabla T_k(u) + \int_\Omega j_s(x, u, \nabla u)T_k(u) > \sigma\|T_k(u)\|_{1,2}.$$

Thus, inequality (2.38) holds. Let us now consider the continuous map

$$\mathscr{H} : B(u, \delta_1) \times [0, \delta_1] \to H_0^1(\Omega)$$

defined as

$$\mathscr{H}(w, t) = w - \frac{t}{\|T_k(u)\|_{1,2}(1+\varepsilon)} T_k(w). \tag{2.39}$$

From (2.37) and (2.38) we deduce that there exists $\delta < \delta_1$ such that

$$d(\mathscr{H}(w,t), w) \leq t \tag{2.40}$$

and

$$J(\mathscr{H}(w,t)) - J(w) \leq -\frac{\sigma}{1+\varepsilon} t \tag{2.41}$$

for every $t \in [0, \delta]$ and $w \in H_0^1(\Omega)$ with $\|w - u\|_{1,2} < \delta$ and $J(w) < J(u) + \delta$. The inequality (2.40) is trivial. As far as concerns (2.41) notice that

$$\lim_{t \to 0} \frac{J(\mathscr{H}(w,t)) - J(w)}{t} = J'(w) \left(-\frac{T_k(w)}{\|T_k(u)\|_{1,2}(1+\varepsilon)} \right)$$

and

$$J'(w) \left(-\frac{T_k(w)}{\|T_k(u)\|_{1,2}(1+\varepsilon)} \right) =$$

$$-\frac{1}{\|T_k(u)\|_{1,2}(1+\varepsilon)} \left[\int_\Omega j_\xi(x, w, \nabla w) \nabla T_k(w) + j_s(x, w, \nabla w_n) T_k(w) \right] \leq -\frac{\sigma}{1+\varepsilon}.$$

Hence, the arbitrariness of ε yields $|dJ|(u) \geq \sigma$. Therefore, for every $k \geq 1$ we get

$$\int_\Omega j_\xi(x, u, \nabla u) \nabla T_k(u) + \int_\Omega j_s(x, u, \nabla u) T_k(u) \leq |dJ|(u) \|T_k(u)\|_{1,2}.$$

Taking the limit as k tends to $+\infty$ and using the Monotone Convergence Theorem we obtain inequality (2.36). □

Notice that in general a generalized solution u (see Definition 2.1.2) is not a distributional solution. This, because a test function $v \in W_u$ may not belongs to C_0^∞. Thus, it is natural to study the conditions under which it is possible to enlarge the class of admissible test functions. This kind of argument was introduced in [8]. More precisely, let us suppose to have a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla z + \int_\Omega j_s(x,u,\nabla u)z = \langle w, z\rangle \quad \forall z \in V_u \tag{2.42}$$

where V_u is defined in (2.28) and $w \in H^{-1}(\Omega)$. A natural question is whether or not we can take as test function $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$. The next result gives an answer to this question.

Theorem 2.3.7. *Assume that conditions (2.2),(2.3),(2.5) are satisfied. Let $w \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ that satisfies (2.42). Moreover, suppose that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ and that there exist $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $\eta \in L^1(\Omega)$ such that*

$$j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \geq \eta. \tag{2.43}$$

Then

$$j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v \in L^1(\Omega)$$

and

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w, v\rangle.$$

Proof. Since $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, then $H\left(\frac{u}{k}\right)v \in V_u$, where H is defined in (2.29). By (2.42) we have

$$\int_\Omega j_\xi(x,u,\nabla u)H'\left(\frac{u}{k}\right)\cdot\nabla u\cdot\frac{v}{k} + \int_\Omega j_\xi(x,u,\nabla u)H\left(\frac{u}{k}\right)\nabla v$$

$$+\int_\Omega j_s(x,u,\nabla u)H\left(\frac{u}{k}\right)v = \left\langle w, H\left(\frac{u}{k}\right)v\right\rangle \tag{2.44}$$

for every $k \geq 1$. We observe that

$$\int_\Omega \left|j_\xi(x,u,\nabla u)H'\left(\frac{u}{k}\right)\cdot\nabla u\cdot\frac{v}{k}\right| \leq \frac{2}{k}\|v\|_\infty \int_\Omega j_\xi(x,u,\nabla u)\nabla u.$$

We notice that by (2.27) we have $j_\xi(x,u,\nabla u)\nabla u \geq 0$, hence in the last term there is not the absolute value. Since $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ we can apply the Lebesgue Convergence Theorem and to get:

$$\lim_{k\to\infty}\int_\Omega j_\xi(x,u,\nabla u)H'\left(\frac{u}{k}\right)\cdot\nabla u\cdot\frac{v}{k} = 0$$

and

$$\lim_{k\to\infty}\left\langle w, H\left(\frac{u}{k}\right)v\right\rangle = \langle w,v\rangle.$$

Now we consider the other term in (2.44), that is

$$\int_\Omega j_\xi(x,u,\nabla u)H\left(\frac{u}{k}\right)\nabla v + \int_\Omega j_s(x,u,\nabla u)H\left(\frac{u}{k}\right)v.$$

By (2.43) e taking into account that

$$-\eta^- = \begin{cases} 0, & \eta > 0 \\ \eta, & \eta < 0 \end{cases},$$

we have

$$[j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v]H\left(\frac{u}{k}\right) \geq H\left(\frac{u}{k}\right)\eta \geq -\eta^- \in L^1(\Omega).$$

Thus, applying Fatou's Lemma we obtain

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v \leq \langle w,v\rangle$$

The previous inequality and (2.43) imply that

$$j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v \in L^1(\Omega). \tag{2.45}$$

Now, notice that

$$\left|[j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v]H\left(\frac{u}{k}\right)\right|$$
$$\leq |j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v|.$$

From (2.45) we deduce that we can use Lebesgue Dominated Convergence Theorem to pass to the limit in (2.44) and to obtain the conclusion. □

In the next result, we find the conditions under which we can use $v \in H^1_0(\Omega)$ in (2.42). Moreover. we prove, under suitable hypotheses, that if u satisfies (2.42) then u is a generalized solution (see Definition 2.1.2) of the corresponding problem. In the next theorem we use the following Lemma proved in [8].

Lemma 2.3.8. *Let $T \in H^{-1} \cap L^1_{loc}(\Omega)$ and let $u \in H^1_0(\Omega)$ be such that for a.e. in $\Omega : T(x) \cdot u(x) \geq \mu(x)$ for some $\mu \in L^1(\Omega)$. Then $T \cdot u \in L^1(\Omega)$ and we have*

$$\langle T, u\rangle = \int_\Omega T(x) \cdot u(x)\, dx$$

Theorem 2.3.9. *Assume conditions (2.2),(2.3),(2.5),(2.6). Let $w \in H^{-1}(\Omega)$ and let $u \in H^1_0(\Omega)$ be such that (2.42) is satisfied. Moreover, suppose that $j_\xi(x, u, \nabla u) \cdot \nabla u \in L^1(\Omega)$, and that there exist $v \in H^1_0(\Omega)$ and $\eta \in L^1(\Omega)$ such that*

$$j_s(x, u, \nabla u)v \geq \eta, \qquad j_\xi(x, u, \nabla u)\nabla v \geq \eta. \tag{2.46}$$

Then $j_s(x, u, \nabla u)v \in L^1(\Omega), j_\xi(x, u, \nabla u)\nabla v \in L^1(\Omega)$ and

$$\int_\Omega j_\xi(x, u, \nabla u)\nabla v + \int_\Omega j_s(x, u, \nabla u)v = \langle w, v\rangle. \tag{2.47}$$

In particular, it results $j_s(x, u, \nabla u)u, j_s(x, u, \nabla u) \in L^1(\Omega)$ and

$$\int_\Omega j_\xi(x, u, \nabla u)\nabla u + \int_\Omega j_s(x, u, \nabla u)u = \langle w, u\rangle.$$

Moreover, u is a generalized solution of the problem

$$\begin{cases} -div(j_\xi(x, u, \nabla u)) + j_s(x, u, \nabla u) = w & in \quad \Omega \\ \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{2.48}$$

Proof. Let $k \geq 1$ be fixed. For every $v \in H^1_0(\Omega)$, by definition of $T_k(s)$ it follows that $T_k(v) \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $-v^- \leq T_k(v) \leq v^+$. Then, by (2.46), we get

$$j_s(x,u,\nabla u)T_k(v) \geq -\eta^- \in L^1(\Omega). \tag{2.49}$$

Moreover,

$$j_\xi(x,u,\nabla u)\nabla T_k(v) \geq -[j_\xi(x,u,\nabla u)\nabla T_k(v)]^- \geq -\eta^- \in L^1(\Omega). \tag{2.50}$$

Then, applying Theorem 2.3.7 we obtain

$$\int_\Omega j_s(x,u,\nabla u)T_k(v) + \int_\Omega j_\xi(x,u,\nabla u)\nabla T_k(v) = \langle w, T_k(v)\rangle \tag{2.51}$$

for every $k \geq 1$. By using again (2.49) and (2.50) and by arguing as in Theorem 2.3.7 we get

$$j_s(x,u,\nabla u)v \in L^1(\Omega) \qquad \text{and} \qquad j_\xi(x,u,\nabla u)\nabla v \in L^1(\Omega).$$

Thus, we can use Lebesgue Dominated Convergence Theorem to pass to the limit in (2.51) and to get (2.47). In particular by (2.5),(2.6),(2.27) we get

$$j_s(x,u,\nabla u)u + j_\xi(x,u,\nabla u)\nabla u$$

$$\geq -R\,\beta(R)|\nabla u|^2 + \alpha_0|\nabla u|^2 \in L^1(\Omega). \tag{2.52}$$

Then by Lemma 2.3.8 we can choose $v = u$. Finally, since

$$j_s(x,u,\nabla u) = j_s(x,u,\nabla u)\cdot\chi_{\{|u|<1\}} + j_s(x,u,\nabla u)\cdot\chi_{\{|u|\geq 1\}}$$

and

$$|j_s(x,u,\nabla u)\cdot\chi_{\{|u|\geq 1\}}| \leq |j_s(x,u,\nabla u)u|,$$

by (2.5) it results also $j_s(x,u,\nabla u) \in L^1(\Omega)$. Now, we note that if $v \in W_u$ we can take $\eta = \min\{j_\xi(x,u,\nabla v)\cdot\nabla v, j_s(x,u,\nabla u)v\}$, so that (2.47) is satisfied. Thus, u is a generalized solution of Problem (2.48) □

2.4 A compactness result for J

In this section, we will prove the following compactness result for J. We will follow an argument similar to the one used in [10], and in [28].

Theorem 2.4.1. *Assume that conditions (2.2), (2.3),(2.5),(2.6) holds. Let* $\{u_n\} \subset H_0^1(\Omega)$ *be a bounded sequence with* $j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n \in L^1(\Omega)$ *and let* $\{w_n\} \subset H^{-1}(\Omega)$ *be such that*

$$\forall v \in V_{u_n} : \int_\Omega j_s(x, u_n, \nabla u_n)v + j_\xi(x, u_n, \nabla u_n)\nabla v = \langle w_n, v\rangle. \tag{2.53}$$

If w_n *is strongly convergent in* $H^{-1}(\Omega)$*, then, up to a subsequence,* u_n *is strongly convergent in* $H_0^1(\Omega)$*.*

Proof. Let w be the limit of $\{w_n\}$ and let $L > 0$ be such that

$$\|u_n\|_{1,2} \leq L \quad \text{for every} \quad n \geq 1. \tag{2.54}$$

By (2.54) we deduce that there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in} \quad H_0^1(\Omega). \tag{2.55}$$

Step 1: First of all, let us prove that u is such that

$$\int_\Omega j_s(x, u, \nabla u)\psi + \int_\Omega j_\xi(x, u, \nabla u)\nabla\psi = \langle w, \psi\rangle \quad \forall \psi \in V_u. \tag{2.56}$$

From Rellich Compact Embedding Theorem, up to a subsequence, we have

$$\begin{cases} u_n \to u, & \text{in } L^q(\Omega) \quad \forall q \in [1, 2n/(n-2)); \\ u_n(x) \to u(x), & \text{for a.e. } x \in \Omega. \end{cases} \tag{2.57}$$

We now want to prove that, up to a subsequence, there holds

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega. \tag{2.58}$$

Let $h \geq 1$. For every $v \in C_c^\infty(\Omega)$ we have that $H\left(\frac{u_n}{h}\right) v \in V_{u_n}$, where H is again the function defined in (2.29). Then

$$\int_\Omega H\left(\tfrac{u_n}{h}\right) j_\xi(x, u_n, \nabla u_n)\nabla v$$
$$= -\int_\Omega \left[H\left(\frac{u_n}{h}\right) j_s(x, u_n, \nabla u_n) + H'\left(\frac{u_n}{h}\right) j_\xi(x, u_n, \nabla u_n) \cdot \frac{\nabla u_n}{h}\right] v$$
$$+ \left\langle w_n, H\left(\frac{u_n}{h}\right) v\right\rangle.$$

It is well know that the elements of $H^{-1}(\Omega)$ may be characterized as the derivatives of functions of $L^2(\Omega)$ in the distributional sense. Then we have that $w_n = -div(F_n)$, with (F_n) strongly convergent in $L^2(\Omega, \mathbb{R}^n)$. We have

$$\left\langle w_n, H\left(\frac{u_n}{h}\right) v\right\rangle = \int_\Omega -div(F_n) H\left(\frac{u_n}{h}\right) v =$$

$$\int_\Omega F_n \cdot \left[H'\left(\frac{u_n}{h}\right) \frac{\nabla u_n}{h} v + H\left(\frac{u_n}{h}\right) \cdot \nabla v\right].$$

Hence we can write

$$\int_\Omega H\left(\tfrac{u_n}{h}\right) j_\xi(x, u_n, \nabla u_n)\nabla v$$
$$= \int_\Omega \left[H'\left(\frac{u_n}{h}\right) (F_n - j_\xi(x, u_n, \nabla u_n))) \cdot \frac{\nabla u_n}{h} - H\left(\frac{u_n}{h}\right) j_s(x, u_n, \nabla u_n)\right] v$$
$$+ \int_\Omega H\left(\frac{u_n}{h}\right) F_n \cdot \nabla v.$$

Since the square bracket is bounded in $L^1(\Omega)$ and $\left(H\left(\frac{u_n}{h}\right), F_n\right)$ is strongly convergent in $L^2(\Omega, \mathbb{R}^n)$ we can apply Theorem 5 of [14],with

$$b_n(x, \xi) = H\left(\frac{u_n(x)}{h}\right) j_\xi(x, u_n(x), \xi) \quad \text{and} \quad E = E_h = \{x \in \Omega : |u(x)| \leq h\}$$

and we deduce (2.58) by the arbitrariness of $h \geq 1$. Notice that, by Theorem 2.3.9, and by (2.53) for every $n \in \mathbb{N}$ we have

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n = \langle w_n, u_n\rangle - \int_\Omega j_s(x, u_n, \nabla u_n) u_n.$$

Then, in view of (2.6), one has

$$\sup_{n\geq 1}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n<\infty. \tag{2.59}$$

Let now $k\geq 1,\varphi\in C_c^\infty(\Omega),\varphi\geq 0$ and consider

$$v=\varphi e^{-M_k(u_n+R)^+}H\left(\frac{u_n}{k}\right),$$

where

$$M_k=\frac{\beta(2k)}{\alpha_0}. \tag{2.60}$$

Note that $v\in V_{u_n}$ and

$$\nabla v=\nabla\varphi e^{-M_k(u_n+R)^+}H\left(\frac{u_n}{k}\right)-M_k\varphi e^{-M_k(u_n+R)^+}\nabla(u_n+R)^+H\left(\frac{u_n}{k}\right)$$
$$+\varphi e^{-M_k(u_n+R)^+}H'\left(\frac{u_n}{k}\right)\frac{\nabla u_n}{k}.$$

Taking v as test function in (2.53), we obtain

$$\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot e^{-M_k(u_n+R)^+}H\left(\frac{u_n}{k}\right)\nabla\varphi$$
$$+\int_\Omega[j_s(x,u_n,\nabla u_n)-M_kj_\xi(x,u_n,\nabla u_n)\cdot\nabla(u_n+R)^+]\varphi e^{-M_k(u_n+R)^+}H\left(\frac{u_n}{k}\right)$$
$$=-\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\varphi e^{-M_k(u_n+R)^+}H'\left(\frac{u_n}{k}\right)\frac{\nabla u_n}{k}$$
$$+\left\langle w_n,\varphi e^{-M_k(u_n+R)^+}H\left(\frac{u_n}{k}\right)\right\rangle. \tag{2.61}$$

Since

$$\left|\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\varphi e^{-M_k(u_n+R)^+}H'\left(\frac{u_n}{k}\right)\frac{\nabla u_n}{k}\right|\leq\frac{2}{k}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\|\varphi\|_\infty\cdot\nabla u_n,$$

taking into account (2.59), there exists a positive constant C such that

$$\left| \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \varphi e^{-M_k(u_n+R)^+} H'\left(\frac{u_n}{k}\right) \frac{\nabla u_n}{k} \right| \leq \frac{C}{k}.$$

Then we can write (2.61) as follows

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \nabla \varphi$$

$$+ \int_\Omega [j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla(u_n + R)^+] \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right)$$

$$\geq -\frac{C}{k} + \left\langle w_n, \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \right\rangle. \tag{2.62}$$

Observe that

$$[j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla(u_n + R)^+] \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \leq 0.$$

Indeed, the assertion follows from (2.6), for a.e. x such that $u_n(x) \leq -R$ while, for almost every x in $\{x : -R \leq u_n(x) \leq 2k\}$ from (2.5),(2.27),and (2.60) we get

$$j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla(u_n + R)^+$$

$$\leq (\beta(2k) - \alpha_0 M_k)|\nabla u_n|^2 = 0.$$

Moreover, from (2.26),(2.54),(2.57) and (2.58) it follows

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \nabla \varphi \to \int_\Omega j_\xi(x, u, \nabla u) e^{-M_k(u+R)^+} H\left(\frac{u}{k}\right) \nabla \varphi,$$

$$\left\langle w_n, \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \right\rangle \to \left\langle w, \varphi e^{-M_k(u+R)^+} H\left(\frac{u}{k}\right) \right\rangle$$

as n tends to ∞. We take the superior limit in (2.62) and we apply Fatou's lemma to obtain

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot e^{-M_k(u+R)^+} H\left(\frac{u}{k}\right) \nabla \varphi + \int_\Omega j_s(x, u, \nabla u) \varphi e^{-M_k(u+R)^+} H\left(\frac{u}{k}\right)$$

$$-M_k \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla(u+R)^+ e^{-M_k(u+R)^+} H\left(\frac{u}{k}\right)$$

$$\geq -\frac{C}{k} + \left\langle w, \varphi e^{-M_k(u+R)^+} H\left(\frac{u}{k}\right)\right\rangle \tag{2.63}$$

for every $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. Then, by density the previous inequality holds for every $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ with $\varphi \geq 0$. We now choose in (2.63) the admissible test function

$$\varphi = e^{M_k(u+R)^+}\psi, \quad \psi \in V_u, \quad \psi \geq 0.$$

It results

$$\int_\Omega j_\xi(x,u,\nabla u) H\left(\frac{u}{k}\right)\nabla\psi + \int_\Omega j_s(x,u,\nabla u) H\left(\frac{u}{k}\right)\psi$$

$$\geq -\frac{C}{k} + \left\langle w, H\left(\frac{u}{k}\right)\psi\right\rangle. \tag{2.64}$$

Notice that:

$$\left| j_\xi(x,u,\nabla u)\cdot H\left(\frac{u}{k}\right)\nabla\psi\right| \leq |j_\xi(x,u,\nabla u)||\nabla\psi|,$$

$$\left| j_s(x,u,\nabla u) H\left(\frac{u}{k}\right)\psi\right| \leq |j_s(x,u,\nabla u)\psi|.$$

Since $\psi \in V_u$, and from (2.5) and (2.26) we deduce that we can apply Lebesgue Dominated Convergence Theorem and, passing to the limit in (2.64) as $k \to \infty$, we obtain

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\psi + \int_\Omega j_s(x,u,\nabla u)\psi \geq \langle w,\psi\rangle \quad \forall\psi \in V_u, \quad \psi \geq 0.$$

In order to show the opposite inequality, we can take $v = \varphi e^{-M_k(u_n-R)^-} H\left(\frac{u_n}{k}\right)$ as test function in (2.53) and we can repeat the same argument as before. Thus, (2.56) follows.

Step 2: In this step we will prove that u_n converges to u strongly in $H_0^1(\Omega)$. From (2.27),(2.59) and Fatou's Lemma we have

$$0 \leq \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u \leq \liminf_n \int_\Omega j_\xi(x,u_n,\nabla u_n)\nabla u_n < \infty,$$

so that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$. Therefore by Theorem 2.3.9 we deduce

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u = \langle w,u\rangle. \tag{2.65}$$

In order to prove that u_n converges to u strongly in $H_0^1(\Omega)$ we follow the argument of [[28], Theorem 3.2] and we consider the function $\zeta:\mathbb{R}\to\mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} Ms & \text{if } 0 < s < R, \qquad M = \dfrac{\beta(R)}{\alpha_0}, \\ MR & \text{if } s \geq R \\ -Ms & \text{if } -R < s < 0 \\ MR & \text{if } s \leq -R \end{cases} \tag{2.66}$$

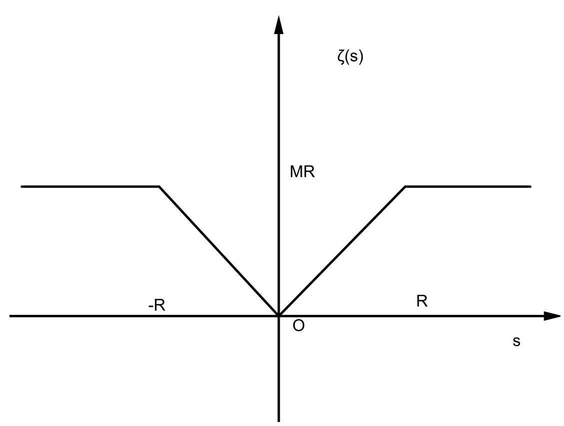

Figure 2.2: The function $\zeta(s)$

We have that $v_n = u_n e^{\zeta(u_n)}$ belongs to $H_0^1(\Omega)$, and conditions (2.5), (2.6), (2.26) imply that hypotheses of Theorem 2.3.9 are satisfied. Then, we can use v_n as test function in (2.53). It results

$$\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} =$$

$$\langle w_n, v_n\rangle - \int_\Omega [j_s(x,u_n,\nabla u_n) + j_\xi(x,u_n,\nabla u_n)\nabla u_n \zeta'(u_n)]v_n. \tag{2.67}$$

Now we observe that from (2.54) v_n is a bounded sequence and hence it converges to $ue^{\zeta(u)}$ weakly in $H^1_0(\Omega)$ and almost everywhere in Ω. Moreover, conditions (2.5), (2.6) and (2.66) allow us to apply Fatou's Lemma and to get that

$$\limsup_n \int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)}$$

$$\leq \langle w, ue^{\zeta(u)}\rangle - \int_\Omega [j_s(x,u,\nabla u) + j_\xi(x,u,\nabla u)\nabla u\zeta'(u)]ue^{\zeta(u)}. \tag{2.68}$$

On the other hand (2.65) and (2.66) imply that

$$\begin{cases} j_\xi(x,u,\nabla u)\nabla[ue^{\zeta(u)}] + j_s(x,u,\nabla u)ue^{\zeta(u)} \in L^1(\Omega) \\ \\ j_\xi(x,u,\nabla u)\nabla[ue^{\zeta(u)}] \in L^1(\Omega). \end{cases} \tag{2.69}$$

Therefore from Theorem 2.3.9 there holds

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla[ue^{\zeta(u)}] + \int_\Omega j_s(x,u,\nabla u)ue^{\zeta(u)} = \langle w, ue^{\zeta(u)}\rangle. \tag{2.70}$$

Thus, (2.68) and (2.70) imply that

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla u e^{\zeta(u)} \leq \liminf_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\nabla u_n e^{\zeta(u_n)}$$

$$\leq \limsup_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\nabla u_n e^{\zeta(u_n)} \leq \int_\Omega j_\xi(x,u,\nabla u)\nabla u e^{\zeta(u)},$$

namely

$$\lim_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\nabla u_n e^{\zeta(u_n)} = \int_\Omega j_\xi(x,u,\nabla u)\nabla u e^{\zeta(u)}.$$

By (2.27) we have $\alpha_0|\nabla u_n|^2 \le j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)}$. Then, using Fatou,s Lemma, we conclude that

$$\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^2 \le \int_\Omega |\nabla u|^2,$$

and the assertion is proved. □

2.5 Proof of the main result

In this section we prove the main result of this chapter. In order to apply variational methods, let us introduce a natural adaptation of Palais-Smale condition, that is the Concrete Palais-Smale condition. Hence, we first give the definition of a Concrete Palais-Smale sequence, then we study the relation between a Palais-Smale sequence and a Concrete Palais-Smale sequence, and finally we prove that f satisfies the $(PS)_c$ for every $c \in \mathbb{R}$.

Let us consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(v) = -\int_\Omega G(x,v) - \langle \Lambda, v\rangle,$$

where $\Lambda \in H^{-1}(\Omega), G(x,s) = \int_0^s g(x,t)$ and g satisfies assumption (2.7). Then (2.3) implies that the functional $f : H_0^1(\Omega) \to \mathbb{R}\cup\{+\infty\}$ defined by $f(v) = J(v) + I(v)$ is lower semicontinuous. In order to apply the abstract theory, it is crucial the following:

Theorem 2.5.1. *Assume conditions (2.2),(2.3),(2.6) and (2.7). Then, for every $(u,\eta) \in epi(f)$ with $f(u) < \eta$, it results*

$$|d\mathscr{G}_f|(u,\eta) = 1.$$

Moreover, if $j(x,-s,-\xi) = j(x,s,\xi), g(x,-s) = -g(x,s)$ and $\Lambda = 0,$ for every $\eta > f(0)$ one has $|d_{\mathbb{Z}_2}\mathscr{G}_f|(0,\eta) = 1$.

Proof. Since $\int_\Omega G(x,u)$ is of class C^1, Theorem 2.2.1 and Proposition 1.4.1 imply the result. □

Moreover, since $\int_\Omega G(x,u)$ is a C^1 functional, as a consequence of Proposition 2.3.5 one has the following

Proposition 2.5.2. *Assume that conditions (2.3), (2.5),(2.7), (2.26) hold, and consider $u \in dom(f)$ with $|df|(u) < \infty$. Then there exists $w \in H^{-1}(\Omega)$ such that $\|w\|_{1,2} \leq |df|(u)$ and*

$$\forall v \in V_u : \int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v - \int_\Omega g(x,u)v - \langle \Lambda, v\rangle = \langle w, v\rangle.$$

Proof. Given $u \in \mathrm{dom}(f)$ with $|df|(u) < \infty$, let

$$\hat{J}(v) = J(v) - \int_\Omega g(x,u)v - \langle \Lambda, v\rangle,$$

$$\hat{I}(v) = I(v) + \int_\Omega g(x,u)v + \langle \Lambda, v\rangle.$$

Then, since $\hat{I}$ is of class C^1 with $\hat{I}' = 0$, by (c) of Proposition 1.4.1 we get $|df|(u) = |d\hat{J}|(u)$. By Proposition 2.3.5 there exists $w \in H^{-1}(\Omega)$ with $\|w\|_{-1,2} \leq |df|(u)$ and

$$\forall v \in V_u : \int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v - \int_\Omega g(x,u)v - \langle \Lambda, v\rangle = \langle w, v\rangle$$

and the assertion is proved. □

We now give the definition of the Concrete Palais-Smale condition.

Definition 2.5.3. *Let $c \in \mathbb{R}$. We say that $\{u_n\} \subset H_0^1(\Omega)$ is a Concrete Palais-Smale sequence for f at level c ($(CPS)_c$-sequence for short) if there exists $w_n \in H^{-1}(\Omega)$ with $w_n \to 0$ such that $j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n \in L^1(\Omega)$ for every $n \geq 1$ and*

$$f(u_n) \to c, \tag{2.71}$$

$$\int_\Omega j_\xi(x,u_n,\nabla u_n)\nabla v + \int_\Omega j_s(x,u_n,\nabla u_n)v - \int_\Omega g(x,u_n)v - \langle \Lambda, v\rangle$$

$$= \langle w_n, v\rangle, \quad \forall v \in V_{u_n}. \tag{2.72}$$

We say that f satisfies the Concrete Palais-Smale condition at level c ($(CPS)_c$

for short) if every $(CPS)_c$-sequence for f admits a strongly convergent subsequence in $H_0^1(\Omega)$.

Proposition 2.5.4. *Assume that conditions (2.3),(2.5),(2.6),(2.26) hold. If $u \in dom(f)$ satisfies $|df|(u) = 0$, then u is a generalized solution of*

$$\begin{cases} -div(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) + \Lambda & in \quad \Omega \\ \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{2.73}$$

Proof. It is sufficient to combine Lemma 2.3.6, Proposition 2.5.2, and Theorem 2.3.9. □

The following result concerns the relation between the $(PS)_c$ condition and the $(CPS)_c$ condition.

Proposition 2.5.5. *Assume conditions (2.3),(2.5),(2.7),(2.26). Then if f satisfies the $(CPS)_c$ condition, it satisfies the $(PS)_c$ condition.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ that satisfies the conditions in Definition 1.2.3. Then from Lemma 2.3.6 and Proposition 2.5.2 we get that u_n satisfies the conditions in Definition 2.5.3. Hence, there exists a subsequence, which converges in $H_0^1(\Omega)$. □

We now want to prove that f satisfies the $(CPS)_c$ condition at every level c. In order to do this, let us consider a $(CPS)_c$-sequence $\{u_n\} \subset \mathrm{dom}(f)$.
As consequence of Theorem 2.4.1 we have

Proposition 2.5.6. *Assume that conditions (2.2)(2.3),(2.5),(2.6),(2.7) are satisfied. Let $\{u_n\}$ be a $(CPS)_c$-sequence for f, bounded in $H_0^1(\Omega)$. Then $\{u_n\}$ admits a strongly convergent subsequence in $H_0^1(\Omega)$.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ be a concrete Palais-Smale sequence for f at level c. Taking into account that by (2.7) the map $\{u \mapsto g(x,u)\}$ is compact from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, it suffices to apply Theorem 2.4.1 to see that $\{u_n\}$ is strongly compact in $H_0^1(\Omega)$. □

Proposition 2.5.7. *Assume conditions (2.2)(2.3),(2.5),(2.6),(2.7)(2.8) and (2.10). Then every $(CPS)_c$-sequence $\{u_n\}$ for f is bounded in $H_0^1(\Omega)$.*

Proof. Conditions (2.6) and (2.27) allow us to apply Theorem 2.3.9 to deduce that we may choose $v = u_n$ as test functions in (2.72). Taking into account conditions (2.7),(2.8),(2.10),(2.71), the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ follows by arguing as in [[28], Lemma 4.3]. □

Remark 2.5.8. Notice that we use condition (2.10) only in Proposition 2.5.7

We can now state the following

Theorem 2.5.9. *Assume that conditions(2.2)(2.3),(2.5),(2.6),(2.7),(2.8) and (2.10)hold. Then the functional f satisfies the $(PS)_c$ condition at every level $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ be a Concrete Palais-Smale sequence for f at level c. From Proposition 2.5.7 it follows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Proposition 2.5.6 f satisfies the Concrete Palais-Smale condition. Finally Proposition 2.5.5 implies that f satisfies the $(PS)_c$ condition. □

We are now able to prove the main Theorem:

Theorem 2.5.10. *Assume conditions (2.2)-(2.10). Moreover, let*

$$j(x,-s,-\xi) = j(x,s,\xi) \qquad \textit{and} \qquad g(x,-s) = -g(x,s) \tag{2.74}$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Then there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ of generalized solutions of problem (2.1) with $f(u_n) \to +\infty$.

Proof. We will prove this Theorem as a consequence of Theorem 1.4.4. First note that (2.3), and (2.7) imply that f is lower semicontinuous. Moreover, from (2.74) we deduce that f is an even functional, and from Theorem 2.5.1 we deduce that (1.7) and (d) of Theorem 1.4.4 are satisfied. Hypotheses (2.4) and (2.9) implies that condition (b) of Theorem 1.4.4 is verified. Let (λ_h, φ_h) be the sequence of solutions of $-\triangle u = \lambda u$ with homogeneous Dirichlet boundary conditions. Moreover, let us consider $V^+ = \overline{span}\{\varphi_h \in H_0^1(\Omega) : h \geq h_0\}$ and note that V^+ has finite codimension. In order to prove (a) of Theorem 1.4.4 it is enough to show that there exist $h_0, \gamma > 0$ such that for all $u \in V^+$ with $\|\nabla u\|_2 = 1$ there holds $f(u) \geq \gamma$. First, note that condition (2.7) implies that, for every $\varepsilon > 0$, we find $a_\varepsilon^{(1)} \in C_c^\infty(\Omega)$ and $a_\varepsilon^{(2)} \in L^{2n/n+2}(\Omega)$ with $\|a_\varepsilon^{(2)}\|_{\frac{2n}{n+2}} \leq \varepsilon$ and

$$|g(x,s)| \leq a_\varepsilon^{(1)}(x) + a_\varepsilon^{(2)}(x) + \varepsilon |s|^{\frac{n+2}{n-2}}.$$

Now, let $u \in V^+$ and notice that there exist two positive constants c_1, c_2 such that

$$\begin{aligned} f(u) &\geq \alpha_0 \|\nabla u\|_2^2 - \int_\Omega G(x,u) \\ &\geq \alpha_0 \|\nabla u\|_2^2 - \int_\Omega \left((a_\varepsilon^{(1)} + a_\varepsilon^{(2)})|u| + \frac{n-2}{2n} \varepsilon |u|^{\frac{2n}{n-2}} \right) \\ &\geq \alpha_0 \|\nabla u\|_2^2 - \|a_\varepsilon^{(1)}\|_2 \|u\|_2 - c_1 \|a_\varepsilon^{(2)}\|_{\frac{2n}{n+2}} \|\nabla u\|_2 - \varepsilon c_2 \|\nabla u\|_2^{\frac{2n}{n-2}} \\ &\geq \alpha_0 \|\nabla u\|_2^2 - \|a_\varepsilon^{(1)}\|_2 \|u\|_2 - c_1 \varepsilon \|\nabla u\|_2 - \varepsilon c_2 \|\nabla u\|_2^{\frac{2n}{n-2}}. \end{aligned}$$

Then if h_0 is sufficiently large, since $\lambda_h \to +\infty$, for all $u \in V^+, \|\nabla u\|_2 = 1$ implies $\|a_\varepsilon^{(1)}\|_2 \|u\|_2 \leq \alpha_0/2$. Thus, for $\varepsilon > 0$ small enough, $\|\nabla u\|_2 = 1$ implies $f(u) \geq \gamma$ for some $\gamma > 0$. Then also (a) of Theorem 1.4.4 is satisfied. Theorem 2.5.9 implies that f satisfies $(PS)_c$ condition at every level c, so that we get the existence of a sequence of critical points $\{u_h\} \subset H_0^1(\Omega)$ with $f(u_h) \to +\infty$. Finally Proposition 2.5.4 yields the assertion. □

Chapter 3

Existence of critical points for some noncoercive functionals

For a bounded domain Ω in $\mathbb{R}^n$, with $n > 2$, minimization problems in the Sobolev space $H_0^1(\Omega)$ for integral functional whose principal part depends on x, v and ∇v as

$$\frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2, \tag{3.1}$$

under the weak assumption $a(x,s) \geq \alpha_0 > 0\,(x \in \Omega, s \in \mathbb{R})$ (that is, of degenerate coerciveness), are now classic. Conversely, the study of critical points is quite more recent (see [3],[4],[16]). In this framework, the main difficulties are that the functional is not differentiable on the whole $H_0^1(\Omega)$, but only in $H_0^1(\Omega)\cap L^\infty(\Omega)$, even if $a(x,s)$ is smooth and that the associated differential operator

$$-\mathrm{div}(a(x,u)\nabla u) + \frac{1}{2}\frac{\partial a}{\partial s}(x,u)|\nabla u|^2,$$

involves a lower order term with quadratic growth in the gradient, which may not be in the dual space $H^{-1}(\Omega)$. Minimization results for integral functionals having principal part as (3.1) were proved in [6]. Specifically, the authors considered functionals whose model is

$$f(u) = \frac{1}{2}\int_{\Omega} \frac{|\nabla u|^2}{(b(x)+|u|)^{2\alpha}} - \int_{\Omega} hu, \quad u \in H_0^1(\Omega),$$

where $\alpha \in (0, 1/2), 0 < \beta_1 \leq b(x) \leq \beta_2$ and $h \in L^p(\Omega)$ for some $p \geq 1$. This functional, which is clearly well defined thanks to Sobolev embedding if $p \geq (2^*)'$, is however non coercive on $H_0^1(\Omega)$. We recall that a functional f, defined on reflexive Banach space, is coercive if $f(u)$ tends to ∞ as norm of u goes to ∞, where $u \in \mathrm{dom}(f)$. Instead in our case there exists a function h, and a sequence $\{u_n\}$ whose norm diverges in $H_0^1(\Omega)$ such that $f(u_n)$ tends to $-\infty$ (see Example 3.1.6).Thus, even if f is lower semicontinuous on $H_0^1(\Omega)$ as a consequence of a result of De Giorgi Theorem (see [15]), the lack of coerciveness implies that f may not attain its minimum on $H_0^1(\Omega)$ even in the case in which f is bounded from below (see Example 3.1.5). The structure of the functional has however enough properties in order to prove that if $h \in L^p(\Omega)$, with $p \geq [2^*(1-\alpha)]'$, then f (suitably extended) is coercive on $W_0^{1,q}(\Omega)$ for some $q < 2$ depending on α. Thus, f attains its minimum on this larger space. Moreover if f is regular enough, then any minimum is bounded, so that (as a consequence of the structure of the functional) it belongs to $H_0^1(\Omega)$. If we "decrease" appropriately the summability of h, the minima are no longer bounded, but they still belong to the "energy space" $H_0^1(\Omega)$. Finally, there is a range of summability for f such that the minima are neither bounded, nor in $H_0^1(\Omega)$. The idea of to try to minima in a large space then $H_0^1(\Omega)$ it follows also in [5]. In this paper the authors considered critical points problems for some integral functionals with principal part having degenerate coerciveness, whose model is

$$f(u) = \frac{1}{2}\int_{\Omega} \frac{|\nabla u|^2}{(b(x)+|u|)^{2\alpha}} - \frac{1}{p}\int_{\Omega} |v|^p, \quad u \in H_0^1(\Omega),$$

with $1 < p < 2^*(1-\alpha)$. The derivative of f is given by

$$\langle f'(u), v \rangle = \int_\Omega \frac{\nabla u \nabla v}{(b(x) + |u|)^{2\alpha}} - \alpha \int_\Omega \frac{|\nabla u|^2 uv}{(b(x) + |u|)^{1+2\alpha} |u|} - \int_\Omega |u|^{p-2} uv,$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. This explains the concept adapted of critical point of f, that is a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying $\langle f'(u), v \rangle = 0$ for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In this way, we can see the critical points of f as a solutions of the boundary value problem

$$\begin{cases} -\text{div}\left(\frac{\nabla u}{(b(x) + |u|)^{2\alpha}}\right) - \alpha \frac{|\nabla u|^2}{(b(x) + |u|)^{1+2\alpha}} \frac{u}{|u|} = |u|^{p-2} u \quad \text{in} \quad \Omega \\ \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases}$$

Remark that, even if the function $\frac{|\xi|^2}{(b(x) + |s|)^{1+2\alpha}} \cdot \frac{s}{|s|}$ is not continuous for $s = 0$, the term

$$\frac{|\nabla u|^2}{(b(x) + |u|)^{1+2\alpha}} \cdot \frac{u}{|u|}$$

appearing in the Euler equation of f is well defined and measurable since where $u = 0$ we have $\nabla u = 0$ almost everywhere (Stampacchia Theorem). The behaviour of the functional f may be different depending on the assumptions made on p. If p is " small enough", it shows, using the approach as [6], that f as a global minimum on $H_0^1(\Omega)$. On the other hand, if $p > 2(1 - \alpha)$ the functional is indefinite and global minimization is no longer possible. In the particular case $2 < p < 2^*(1 - \alpha)$, it possible to apply a version of the Ambrosetti-Rabinowitz Mountain Pass Theorem (see [1], [26]) given in [4] for functionals which are not differentiable along every directions . In such way it shows the existence of a non trivial critical point in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

3.1 A classical results

Let Ω be a bounded, open subset of $\mathbb{R}^n, n > 2$. Let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$\frac{\beta_0}{(1+|s|)^{2\alpha}} \leq a(x,s) \leq \beta_1, \tag{3.2}$$

for almost every x in Ω and for every s in $\mathbb{R}$, where α, β_0 and β_1 are positive constants with $0 < \alpha < \frac{1}{2}$. Let $\tau : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $\tau(0) = 0$, and

$$\beta_2|\xi|^2 \leq \tau(\xi) \leq \beta_3(1+|\xi|^2), \tag{3.3}$$

for every ξ in $\mathbb{R}^n$, where β_2 and β_3 are two positive constants.
Examples of functions a and τ are the follows:

$$a(x,s) = \frac{\beta_0}{(b(x)+|s|)^{2\alpha}}, \qquad \tau(\xi) = \beta_2|\xi|^2,$$

where b is a measurable function on Ω such that

$$0 < \beta_4 \leq b(x) \leq \beta_5 \qquad \text{for almost every} \quad x \quad \text{in} \quad \Omega,$$

with β_4 and β_5 two positive constants. Let h be a function in $L^p(\Omega)$, with $p \geq [2^*(1-\alpha)]'$. We define, for $u \in H_0^1(\Omega)$, the functional

$$f(u) = \int_\Omega a(x,u) \cdot \tau(\nabla u)\, dx - \int_\Omega hu\, dx.$$

By the assumptions on a, τ and h, the functional f turns out to be defined on the whole $H_0^1(\Omega)$. We extend the definition of f to a larger space, namely $W_0^{1,q}(\Omega)$, with $q = \frac{2n(1-\alpha)}{n-2\alpha} < 2$, in the following way

$$\hat{f}(u) = \begin{cases} f(u), & \text{if } f(u) \text{ is finite;} \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.4}$$

where $u \in W_0^{1,q}(\Omega)$.

Throughout this section, c denotes a non negative constant that depends on the data of the problem, and whose value may vary from line to line.

Theorem 3.1.1. *Let* $q = \frac{2n(1-\alpha)}{n-2\alpha}$. *Suppose that* h *belongs to* $L^p(\Omega)$, *with* $p \geq [2^*(1-\alpha)]'$. *Then* $\hat{f}$ *is coercive and weakly lower semicontinuous on* $W_0^{1,q}(\Omega)$. *As consequence there exists a minimum* u *of* $\hat{f}$ *on* $W_0^{1,q}(\Omega)$.

Proof. We begin with the coerciveness of $\hat{f}$, that is, we want to prove that for every M in $\mathbb{R}$ the set

$$E_M = \{v \in W_0^{1,q}(\Omega) : \hat{f}(v) \leq M\}$$

is bounded. Since for every u in $W_0^{1,q}(\Omega)$ we have

$$\int_\Omega hu < +\infty,$$

due to the assumption on p and to the fact that $q^* = 2^*(1-\alpha)$, we have that if u belongs to E_M, then

$$\int_\Omega a(x,u)\tau(\nabla u) < +\infty.$$

For these u, we have by (3.2) and (3.3), and Hölder inequality,

$$\int_\Omega |\nabla u|^q = \int_\Omega \frac{|\nabla u|^q}{(1+|u|)^{\alpha q}}(1+|u|)^{\alpha q} \leq \left(\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}}\right)^{\frac{q}{2}} \left(\int_\Omega (1+|u|)^{\frac{2\alpha q}{2-q}}\right)^{1-\frac{q}{2}}$$

$$\leq c\left(\int_\Omega a(x,u)\cdot\tau(\nabla u)\right)^{\frac{q}{2}} \cdot \left(1 + \int_\Omega |u|^{\frac{2\alpha q}{2-q}}\right)^{1-\frac{q}{2}}.$$

Since q is such that $\frac{2\alpha q}{2-q} = q^*$, the preceding inequality becomes

$$\int_\Omega |\nabla u|^q \leq c\left(\int_\Omega a(x,u)\cdot\tau(\nabla u)\right)^{\frac{q}{2}} \cdot \left(1 + \int_\Omega |u|^{q^*}\right)^{1-\frac{q}{2}},$$

which implies, by Sobolev embedding,

$$\int_\Omega |\nabla u|^q \leq c\left(\int_\Omega a(x,u)\cdot\tau(\nabla u)\right)^{\frac{q}{2}} \cdot \left(1 + \left(\int_\Omega |\nabla u|^q\right)^{\frac{q^*}{q}}\right)^{1-\frac{q}{2}}.$$

If the norm of u in $W_0^{1,q}(\Omega)$ is greater than one, this implies

$$\int_\Omega |\nabla u|^q \leq c\left(\int_\Omega a(x,u)\cdot\tau(\nabla u)\right)^{\frac{q}{2}} \cdot \left(\int_\Omega |\nabla u|^q\right)^{\left(1-\frac{q}{2}\right)\frac{q^*}{q}},$$

so that, by definition of q,

$$\int_\Omega a(x,u)\cdot\tau(\nabla u) \geq c\left(\int_\Omega |\nabla u|^q\right)^{\frac{2}{q}\left[1-\frac{q^*(2-q)}{2q}\right]} = c\|u\|_{1,q}^{2(1-\alpha)}.$$

Since $p \geq (q^*)'$, one has, again by Sobolev embedding,

$$\int_\Omega hu \leq \|h\|_{(q^*)'}\|u\|_{q^*} \leq c\|h\|_p\|u\|_{1,q}.$$

Hence,

$$\hat{f}(u) \geq c\|u\|_{1,q}^{2(1-\alpha)} - c\|h\|_p\|u\|_{1,q}$$

for every u in E_M of norm greater than 1. Since $\alpha < \frac{1}{2}$ implies $2(1-\alpha) > 1$, then $\hat{f}(u) > M$ if

$$\int_\Omega a(x,u)\cdot\tau(\nabla u) < +\infty,$$

and the norm of u in $W_0^{1,q}(\Omega)$ is large enough. Thus, there exists $R = R(M)$ such that E_M is contained in the ball of $W_0^{1,q}(\Omega)$ of radius R; hence E_M is bounded. Now we turn to the weak lower semicontinuity of $\hat{f}$ on $W_0^{1,q}(\Omega)$. Since $q^* = 2^*(1-\alpha)$, the assumption on p and the Sobolev embedding imply that the application

$$u \mapsto \int_\Omega hu,$$

is weakly continuous on $W_0^{1,q}(\Omega)$. On the other hand, the term

$$\int_\Omega a(x,u)\cdot\tau(\nabla u),$$

is weakly lower semicontinuous on $W_0^{1,q}(\Omega)$, since the assumptions on a and τ allow to apply the De Giorgi lower semicontinuity Theorem for integral functionals (see [15]). By standard results (see for example [13]), we thus have that there exists the minimum of $\hat{f}$ on $W_0^{1,q}(\Omega)$, that is, there exists u in $W_0^{1,q}(\Omega)$ such that

$$\hat{f}(u) = min\{\hat{f}(v) : v \in W_0^{1,q}(\Omega).\} \tag{3.5}$$

□

Once we have proved the existence of a minimum u, we can give some regularity results, depending on the summability of h.

Theorem 3.1.2. *Suppose that h belongs to $L^p(\Omega)$, with $p > \frac{n}{2}$. Then any minimum u of $\hat{f}$ on $W_0^{1,q}(\Omega)$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$; thus f attains its minimum on $H_0^1(\Omega)$.*

Theorem 3.1.3. *Suppose that u is a minimum of $\hat{f}$ on $W_0^{1,q}(\Omega)$, and that h belongs to $L^p(\Omega)$, with $[2^*(1-\alpha)]' \leq p < \frac{n}{2}$. Then one has:*

(a) *If $\left(\frac{2^*}{1+2\alpha}\right)' \leq p < \frac{n}{2}$, then u belongs to $H_0^1(\Omega)$ and to $L^s(\Omega)$, with*

$$s = \frac{np(1-2\alpha)}{n-2p}.$$

Thus, f attains its minimum on $H_0^1(\Omega)$.

(b) *If $[2^*(1-\alpha)]' \leq p < (\frac{2^*}{1+2\alpha})'$, then u belongs to $W_0^{1,\rho}(\Omega)$, with*

$$\rho = \frac{np(1-2\alpha)}{n-p(1+2\alpha)}.$$

Example 3.1.4. Uniqueness of minima for the model case.
Let us consider the model functional

$$f(u) = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}} - \int_\Omega hu^+, \qquad u \in H_0^1(\Omega),$$

with $0 < \alpha < \frac{1}{2}$, and h a nonnegative function in $L^p(\Omega), p \geq [2^*(1-\alpha)]'$. Let $\hat{f}$ be the extension of f to $W_0^{1,q}(\Omega)$, with $q = \frac{2n(1-\alpha)}{n-2\alpha}$, given by (3.4). Then, as we have shown in the previous section, there exists

$$m = \min\{\hat{f}(u) : u \in W_0^{1,q}(\Omega)\}.$$

If we define

$$F = \{u \in W_0^{1,q}(\Omega) : \hat{f}(v) < +\infty\},$$

it is then clear, by definition of $\hat{f}$, that

$$m = \min\{\hat{f}(u) : u \in F\} = \min\{f(u) : u \in F\}.$$

Furthermore, observe that since $\int_\Omega hu$ is always finite on $W_0^{1,q}(\Omega)$ by the assumptions on q and on the summability of h, then

$$F = \left\{ u \in W_0^{1,q}(\Omega) : \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}} < +\infty \right\}.$$

Let now

$$\psi(s) = \frac{(1+|s|)^{1-\alpha} - 1}{1-\alpha} sgn(s), \qquad \varphi(s) = [(1-\alpha)|s| + 1]^{\frac{1}{1-\alpha}} - 1 sgn(s),$$

so that $\psi(\varphi(s)) = s$. If u belongs to F, then

$$|\nabla\psi(u)|^2 = \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}} \in L^1(\Omega),$$

and if w belongs to $H_0^1(\Omega)$, then

$$\frac{|\nabla\varphi(w)|^2}{(1+|\varphi(w)|)^{2\alpha}} = |\nabla w|^2 \in L^1(\Omega),$$

so that the application $N : F \to H_0^1(\Omega)$ defined by $u \mapsto \psi(u)$ is both well defined and bijective. Now we change variables and consider the new functional

$$L(u) = \hat{f}(\varphi(u)) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega h \cdot \varphi(u^+).$$

Since $\varphi(s)$ grows as $|s|^{\frac{1}{1-\alpha}}$, and since $\frac{1}{1-\alpha} < 2$ due to the assumption $\alpha < \frac{1}{2}$, then L turns out to be weakly lower semicontinuous and coercive on $H_0^1(\Omega)$. Hence, there exists the minimum of L on $H_0^1(\Omega)$. Since N is bijective, we obviously have

$$\min\{L(u) : u \in H_0^1(\Omega)\} = m.$$

Any function w that realizes the minimum of L is also a solution of the Euler equation for L, that is, of the problem

$$\begin{cases} -\triangle w = h(x)\varphi'(w^+), & \text{in} \quad \Omega \\ w = 0, & \text{on} \quad \Omega \end{cases} \tag{3.6}$$

where

$$\varphi'(s^+) = [(1-\alpha)s^+ + 1]^{\frac{\alpha}{1-\alpha}}.$$

Since h is nonnegative, as is $\varphi'(s^+)$, then w is a nonnegative function. Since $\varphi'(s^+)$ is concave, it is well known (see for example [2], Lemma 3.3) that w is the unique positive solution of (3.6). Hence, w is the unique minimum of L on $H_0^1(\Omega)$. This implies that $u = \varphi(w)$ is the unique minimum of $\hat{f}$ on $W_0^{1,q}(\Omega)$. Thus, at least in the model example, we have proved a uniqueness result for the minimum point of $\hat{f}$. In [7] it has proved that if w is a solution of (3.6), and if h belongs to $L^p(\Omega)$, with $p > \frac{n}{2}$, then w belongs to $L^\infty(\Omega)$. If we consider now $u = \varphi(w)$, the minimum of $\hat{f}$, we easily obtain by definition of φ that also u belongs to $L^\infty(\Omega)$. Thus, since u is the minimum of $\hat{f}$, we have

$$\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}} \le \int_\Omega hu^+ \le c\|u\|_\infty,$$

since h belongs at least to $L^1(\Omega)$. The latter inequality then implies

$$\int_\Omega |\nabla u|^2 \le c(1+\|u\|_\infty)^{2\alpha+1},$$

so that u belongs to $H_0^1(\Omega)$. Hence, we have proved (by means of a change of variable, and in the model case) that the minimum u of $\hat{f}$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ if h belongs to $L^p(\Omega)$, with $p > \frac{n}{2}$. This explains the result of Theorem 3.1.2. Using other results of [7], and performing again a change of variable, it is possible to obtain for the minimum u of $\hat{f}$ the same results of Theorem 3.1.3.

Example 3.1.5. The infimum may not be achieved.
If h belongs to $L^p(\Omega)$, with $p \ge [2^*(1-\alpha)]'$, then the functional f is bounded from below on $H_0^1(\Omega)$. Indeed, since on $H_0^1(\Omega)$ we have that f coincides with $\hat{f}$, defined in (3.4), then

$$\inf\{f(v) : v \in H_0^1(\Omega)\} \ge \min\{\hat{f}(v) : v \in W_0^{1,q}(\Omega)\} > -\infty,$$

by (3.5). If, moreover h belongs to $L^p(\Omega)$, with $p \ge \bar{p} = \left(\frac{2^*}{1+2\alpha}\right)'$, then the results of Theorem 3.1.2, and Theorem 3.1.3 (a), state f attains its minimum

on $H_0^1(\Omega)$. Let now h belongs to $L^p(\Omega)$, with $p < \bar{p}$. The result of Theorem 3.1.3 (b), states that the minimum u of $\hat{f}$ does not belong to $H_0^1(\Omega)$. We are going to give an example in which we show that, in this case, the infimum of f on $H_0^1(\Omega)$ is not achieved.
Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, let $\rho = |x|$ and let

$$h(\rho) = \frac{c}{\rho^\beta}, \qquad \beta = \frac{n(1-2\alpha)+2(1+2\alpha)}{2},$$

with c a positive constant to be chosen later. It easy to see that h belongs to $L^p(\Omega)$ for every $p < \bar{p} = (\frac{2^*}{1+2\alpha})'$, but is not in $L^{\bar{p}}(\Omega)$. A straightforward calculation implies that it is possible to choose c such that the function

$$w(\rho) = \frac{1}{1-\alpha}\left(\frac{1}{\rho^\gamma} - 1\right), \qquad \text{with} \quad \gamma = \frac{(n-2)(1-\alpha)}{2},$$

is a solution of

$$-\triangle w = h(\rho)\varphi'(w) \quad \text{in} \quad \Omega,$$

with $\bar{g}$ as in Example 3.1.4. Since w is positive, then, as stated in Example 3.1.4, w is the unique solution of the above problem, so that $\bar{u} = \varphi(w)$ is the unique minimum point on $W_0^{1,q}(\Omega)$ of the functional $\hat{f}$, which is the extension, as in (3.4), of the functional

$$f(u) = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}} - \int_\Omega hu^+ \qquad u \in H_0^1(\Omega).$$

Performing the calculations, we get

$$\bar{u}(\rho) = \frac{1}{\rho^{\frac{n-2}{2}}} - 1.$$

Such a function does not belong to $H_0^1(\Omega)$. Let $n \in \mathbb{N}$, and consider the function

$$u_n = T_n(\bar{u}),$$

which belongs to $H_0^1(\Omega)$. We then have, by straightforward calculations,

$$\lim_{n\to+\infty} f(u_n) = \hat{f}(\bar{u}) = min\{\hat{f}(v) : v \in W_0^{1,q}(\Omega)\}.$$

Thus,

$$\inf\{f(u) : u \in H_0^1(\Omega)\} = min\{\hat{f}(v) : v \in W_0^{1,q}(\Omega)\}.$$

but the infimum is not achieved since the unique minimum point $\bar{u}$ of $\hat{f}$ does not belong to $H_0^1(\Omega)$

Example 3.1.6. $\hat{f}$ may be unbounded from below.
If h belongs to $L^p(\Omega)$, with $(2^*)' \leq p < [2^*(1-\alpha)]'$, then the functional $\hat{f}$ may be unbounded from below on $W_0^{1,q}(\Omega)$, with $q = \frac{2n(1-\alpha)}{n-2\alpha}$.
As before, let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, let $\rho = |x|$, and let

$$h(\rho) = \frac{C}{\rho^\beta}, \qquad \beta = \frac{n+2-2n\alpha}{2(1-\alpha)},$$

with C a positive constant to be chosen later. Then h does not belong to $L^p(\Omega), p = [2^*(1-\alpha)]'$. Moreover, let

$$u(\rho) = \frac{1}{\rho^\gamma} - 1, \qquad \text{with} \quad \gamma = \frac{n-2}{2(1-\alpha)}.$$

Let $m \in \mathbb{N}$, and let $u_m = T_m(u)$, which belongs to $H_0^1(\Omega)$. If $r_m \in (0,1)$ is such that $u(r_m) = m$, we then have

$$\frac{1}{2}\int_\Omega \frac{|\nabla u_m|^2}{(1+|u_m|)^{2\alpha}}dx = \frac{\gamma^2\omega_n}{2}\int_{r_m}^1 \frac{1}{\rho}d\rho = -\frac{\gamma^2\omega_n}{2}ln(r_m),$$

where ω_n is the $(n-1)$ dimensional measure of the unit sphere in $\mathbb{R}^n$. On the other hand,

$$\int_\Omega hu_m = C\omega_n \int_{r_m}^1 f\, u\, \rho^{n-1}d\rho + C\omega_n \cdot m \int_0^{r_m} h\rho^{n-1}d\rho,$$

and it is easily seen that we have

$$\int_\Omega h\, u_m = -C\,\omega_n ln(r_m) + \text{terms bounded with respect to} \quad m.$$

Thus, if $C > \frac{\gamma^2}{2}$, we have proved that there exists a positive constant $\bar{C}$ such that

$$\hat{f}(u_m) = \bar{C}ln(r_m) + \text{terms bounded with respect to} \quad m.$$

Since r_m converges to zero, $\hat{f}$ is not bounded from below on $W_0^{1,q}(\Omega)$. Observe that since the norm of u_m tends to infinity in $W_0^{1,q}(\Omega)$, hence in $H_0^1(\Omega)$, then f is not coercive on $H_0^1(\Omega)$.

3.2 The existence of a global minimum for the functional of Arcoya-Boccardo-Orsina

Let us state the precise assumptions on the functional f that we will study below. Let Ω be a bounded, open subset of $\mathbb{R}^n, n > 2$. Let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$\frac{c_1}{(1+|t|)^{2\alpha}} \leq a(x,t) \leq c_2 \tag{3.7}$$

for almost every x in Ω, for every t in $\mathbb{R}$, where c_1 and c_2 are positive constants, and $0 \leq \alpha < \frac{n}{2n-2}$ (note that $\frac{n}{2n-2} \in \left(\frac{1}{2}, 1\right)$, for every $n > 2$);
we also assume that the function $t \mapsto a(x,t)$ is differentiable on $\mathbb{R}$ for almost every x in Ω, and its derivative $a_t(x,t) \equiv \frac{\partial a}{\partial t}(x,t)$ is such that

$$-2\beta a(x,t) \leq a_t(x,t)(1+|t|)sgn(t) \leq 0 \tag{3.8}$$

for almost every $x \in \Omega$, for every $t \in \mathbb{R}$, with $|t| \geq R$, where R and β are positive constants such that $0 < \beta < c_1$.
As examples of functions a satisfying assumptions (3.7) and (3.8) we can consider either

$$a(x,t) = \frac{1}{(b(x)+|t|)^{2\alpha}},$$

with $0 < \beta_1 \leq b(x) \leq \beta_2$, or

$$a(x,t) = \frac{1}{(1+t^2)^{\alpha}}.$$

Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following assumption:

$$|G(x,t)| \leq \frac{a_1}{p}|t|^p + b_1, \tag{3.9}$$

for almost every x in Ω, for every t in $\mathbb{R}$, where a_1, b_1 are positive constants and $1 < p < 2^*$. We define, for $u \in H_0^1(\Omega)$, the functional

$$f(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \int_\Omega G(x,u). \tag{3.10}$$

Observe that by assumption (3.7) and (3.9), f is well defined on $H_0^1(\Omega)$. Furthermore, by the assumptions on a and G, f is also differentiable along directions in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and its derivative is given by

$$\langle f'(u), v\rangle = \int_\Omega a(x,u)\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega a_s(x,u)|\nabla u|^2\, v - \int_\Omega g(x,u)v, \quad (3.11)$$

for every u in $H_0^1(\Omega)$ and for every v in $H_0^1(\Omega) \cap L^\infty(\Omega)$, where $g(x,s) = \frac{\partial G}{\partial t}(x,t)$

In the sequel we use the following technical result

Lemma 3.2.1. *If* $\alpha \in \left(0, \frac{n}{2n-2}\right)$ *and*

$$q = \frac{2n(1-\alpha)}{n-2\alpha}, \quad (3.12)$$

then for every measurable set $A \subset \Omega$, *and for every* $v \in H_0^1(\Omega)$ *and* $u \in W_0^{1,q}(\Omega)$, *we have*

$$\int_A |\nabla v|^q \le \left(\int_A \frac{|\nabla v|^2}{(1+|u|)^{2\alpha}}\right)^{\frac{q}{2}} \left(\int_A (1+|u|)^{q^*}\right)^{1-\frac{q}{2}} \quad (3.13)$$

Proof. If $v \in H_0^1(\Omega)$ and $u \in W_0^{1,q}(\Omega)$, we have from the Hölder inequality that

$$\int_A |\nabla v|^q = \int_A \frac{|\nabla v|^q}{(1+|u|)^{\alpha q}}(1+|u|)^{\alpha q}$$
$$\le \left(\int_A \frac{|\nabla v|^2}{(1+|u|)^{2\alpha}}\right)^{\frac{q}{2}} \left(\int_A (1+|u|)^{\frac{2\alpha q}{2-q}}\right)^{1-\frac{q}{2}}$$

for every measurable set $A \subset \Omega$. We conclude the proof by observing that, by (3.12), $q^* = \frac{2\alpha q}{2-q}$. □

Now we prove the existence of a global minimum for the functional f defined as in (3.10).

Theorem 3.2.2. *Let assume (3.7) and (3.9) with* $1 < p < 2(1-\alpha)$. *Suppose furthermore that*

$$\lim_{t\to 0} \frac{G(x,t)}{t^2} = +\infty, \qquad \textit{uniformly with respect to } x \textit{ in } \Omega, \quad (3.14)$$

then f *has a global nontrivial minimum* u *in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Let q be as in the previous Lemma. Reasoning as in previous section, let us define the following functional on $W_0^{1,q}(\Omega)$:

$$\hat{f}(u) = \begin{cases} \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \int_\Omega G(x,u), & \text{if } \int_\Omega a(x,u)|\nabla u|^2 < +\infty; \\ \\ +\infty & \text{otherwise.} \end{cases}$$

We are going to prove that $\hat{f}$ is both coercive and weakly lower semicontinuous on $W_0^{1,q}(\Omega)$ so that the existence of a minimum will follows from standard results. The weak lower semicontinuity is a consequence of a Theorem by De Giorgi (see [15]) and it follows as in Theorem 3.1.1. As far as the coerciveness is concerned, it is enough to consider u in $W_0^{1,q}(\Omega)$ such that $\hat{f}(u)$ is finite. Reasoning as Lemma 3.2.1 we obtain, also using Sobolev embedding,

$$\int_\Omega |\nabla u|^q \le C_1\left(\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}}\right)^{\frac{q}{2}}\left(1+\left(\int_\Omega |\nabla u|^q\right)^{\frac{q^*}{q}}\right)^{1-\frac{q}{2}},$$

which implies that if $B = \|u\|_{1,q}$, we have from (3.7)

$$B^q \le C_2\left(\int_\Omega a(x,u)|\nabla u|^2\right)^{\frac{q}{2}}(1+B^{q^*})^{1-\frac{q}{2}}. \tag{3.15}$$

On the other hand, since $p < 2(1-\alpha) < 2^*(1-\alpha) = q^*$, one has

$$\int_\Omega |u|^p \le C_3\left(\int_\Omega |u|^{q^*}\right)^{\frac{p}{q^*}} \le C_4\left(\int_\Omega |\nabla u|^q\right)^{\frac{p}{q}},$$

that is

$$\int_\Omega |u|^p \le C_4 B^p.$$

Thus, by (3.9) and (3.15) we obtain

$$\hat{f}(u) \ge C_5\frac{B^2}{(1+B^{q^*})^{\frac{2}{q}-1}} - C_6B^p - C_7.$$

Using the definition of q, it is easy to check that

$$2 - q^*\left(\frac{2}{q}-1\right) = 2(1-\alpha) > p$$

so that

$$\lim_{B\to+\infty} \hat{f}(u) = +\infty,$$

that is, $\hat{f}$ is coercive on $W_0^{1,q}(\Omega)$. Let now u be a minimum of $\hat{f}$ on $W_0^{1,q}(\Omega)$. Let φ_1 be the first eigenfunction of the Laplacian in Ω, which we suppose to have chosen with norm equal to one in $H_0^1(\Omega)$. Then, since φ_1 belongs to $H_0^1(\Omega)$,

$$\hat{f}(t\varphi_1) = f(t\varphi_1) = \frac{t^2}{2}\int_\Omega a(x, t\varphi_1)|\nabla\varphi_1|^2 - \int_\Omega G(x, t\varphi_1).$$

Using assumption (3.14), it is easy to see that there exists $t > 0$ such that $\hat{f}(t\varphi_1) < 0$, and so $u \neq 0$. Furthermore, reasoning as [6] it can be proved that u belongs to $L^\infty(\Omega)$. This implies by (3.7) that u also belongs to $H_0^1(\Omega)$, thus concluding the proof of the theorem. □

Remark 3.2.3. We remark explicitly that inequality (3.15) holds under assumptions (3.7), for every $u \in W_0^{1,q}(\Omega)$ such that $\int_\Omega a(x,u)|\nabla u|^2 < +\infty$

3.3 Mountain Pass type critical point

In this section we study the existence of a nontrivial critical point.
Let us suppose that $g(x,t)$ satisfies the follows hypothesis

$$|g(x,t)| \le b|t|^{p-1} + d, \tag{3.16}$$

for almost every $x \in \Omega$ and for every $t \in \mathbb{R}$, where b, d are positive constants, and $1 < p < 2^*(1-\alpha)$. We note that this condition implies (3.9). Moreover let assume that there exist $\nu > 2, R > 0$ such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \ge R$ we have

$$0 < \nu G(x,t) \le tg(x,t). \tag{3.17}$$

In order to prove the existence of nontrivial critical point if $2 < p < 2^*(1-\alpha)$, one of the main difficulties is to check that a suitable compactness condition

of Palais-Smale type holds. Here the keystone in this approach will be the proof of the boundedness of the cluster points of the Palais-Smale sequences. An additional difficulty also arises: the degenerate coerciveness in $H_0^1(\Omega)$ of the principal part of the differential operator, which will lead us to extend the functional to a larger space, namely $W_0^{1,q}(\Omega)$ for some $q < 2$.
Hence the first result is the proof that sequences of Palais-Smale type for f are convergent. If $k > 0$ we consider the function $T_k(s)$ as defined in (2.16) and $G_k : \mathbb{R} \to \mathbb{R}$ defined as $G_k(s) = s - T_k(s)$.

Lemma 3.3.1. *Assume conditions (3.7),(3.8),(3.16) with* $2 < p < 2^*(1-\alpha)$ *and (3.17) hold. Let* q *as in Lemma 3.2.1. Then the functional* f *satisfies the following compactness condition:*
every sequence $\{u_n\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ *satisfying*

$$\lim_{n\to\infty} f(u_n) = c \tag{3.18}$$

and, for some sequence $\{\varepsilon_n\} \subset (0, \infty)$ *converging to zero,*

$$|\langle f'(u_n), v\rangle| \leq \varepsilon_n \frac{\|v\|_{1,2} + \|v\|_\infty}{\|u_n\|_{1,2} + \|u_n\|_\infty}, \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.19}$$

possesses a subsequence which is weakly convergent in $W_0^{1,q}(\Omega)$ *to some critical point* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of* f *with level* $f(u) = c$.

Proof. The proof is divided into 5 steps:

Step 1. The sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$.
Step 2. Up to a subsequence,$\{u_n\}$ weakly converges in $W_0^{1,q}(\Omega)$ to some $u \in L^\infty(\Omega)$.
Step 3. The function u belongs to $H_0^1(\Omega)$.
Step 4. For fixed $k \geq \|u\|_{L^\infty(\Omega)}$, the following convergences hold:

$$\int_\Omega a(x, u_n)|\nabla G_k(u_n)|^2 \to 0, \tag{3.20}$$

$$\int_\Omega a_t(x, u_n)v|\nabla G_k(u_n)|^2 \to 0, \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.21}$$

$$\|T_k(u_n) - u\|_{1,2} \to 0, \tag{3.22}$$

as n tends to infinity.

Step 5. The function u is a critical point of f with level $f(u) = c$.
For more details see [5], Lemma 3.1. We observe that in this case it is crucial to prove that the limit point is bounded in order to prove the compactness condition. □

In the sequel we present a modified version (see [4]) of the well-known Mountain Pass Theorem of Ambrosetti-Rabinowitz in order to study the existence of critical points for functionals not differentiable in all directions.

Theorem 3.3.2. *Let $(X, \|\cdot\|_X)$ be a Banach space and $Y \subset X$ a subspace, which is itself a Banach space endowed with a norm $\|\cdot\|_Y$ such that one has $\|y\|_X \leq \|y\|_Y$ for every $y \in Y$. Assume that $f : X \to \mathbb{R}$ is a functional on X such that $f|_{(Y,\|\cdot\|_Y)}$ is continuous and satisfies*

(a) *f has a directional derivative $\langle f'(u), v\rangle$ at each $u \in X$ through any direction $v \in Y$,*

(b) *For fixed $u \in X$, the function $\langle f'(u), v\rangle$ is linear in $v \in Y$, and, for fixed $v \in Y$, the function $\langle f'(u), v\rangle$ is continuous in $u \in X$.*

Let K be a compact metric space and $K_0 \subset K$ a closed subset. Moreover let $\gamma_0 : K_0 \to (Y, \|\cdot\|_Y)$ a continuous map. Consider the set

$$\gamma = \{\gamma : K \to (Y, \|\cdot\|_Y) : \gamma \quad \textit{is continuous and} \quad \gamma|_{K_0} = \gamma_0\}.$$

If

$$c = \inf_{\gamma\in\Gamma} \max_{t\in k} f(\gamma(t)) > c_1 = \max_{t\in k_0} f(\gamma_0(t)),$$

then, for every $\varepsilon > 0$ and $\gamma \in \Gamma$ such that

$$c \leq \max_{t\in k} f(\gamma(t)) \leq c + \frac{1}{2}\varepsilon,$$

there exist $\bar{\gamma}_\varepsilon \in \Gamma$ *and* $u_\varepsilon \in \bar{\gamma}_\varepsilon(K) \subset Y$ *satisfying*

$$c \le \max_{t\in K} f(\bar{\gamma}_\varepsilon(t)) \le \max_{t\in K} f(\gamma(t)) \le c + \frac{1}{2}\varepsilon,$$

$$\max_{t\in K} \|\bar{\gamma}_\varepsilon(t) - \gamma(t)\| \le \sqrt{\varepsilon},$$

$$c - \varepsilon \le f(u_\varepsilon) \le c + \frac{1}{2}\varepsilon, \quad |\langle f'(u_\varepsilon), v\rangle| \le \sqrt{\varepsilon}\|v\|_Y \quad \forall v \in Y.$$

Now we can to prove the main result of this section.

Theorem 3.3.3. *Let us assume that conditions (3.7), (3.8), (3.16) with* $2 < p < 2^*(1-\alpha)$, *(3.17) hold. Suppose furthermore that* G *satisfies the following assumptions:*

$$\lim_{t\to 0} \frac{G(x,t)}{t^2} = 0, \qquad \textit{uniformly with respect to} \quad x \quad \textit{in}\, \Omega, \tag{3.23}$$

$$\lim_{t\to +\infty} \frac{G(x,t)}{t^2} = +\infty, \qquad \textit{uniformly with respect to} \quad x \quad \textit{in}\, \Omega. \tag{3.24}$$

Then f *has at least a nontrivial critical point* u *belonging to* $H^1_0(\Omega) \cap L^\infty(\Omega)$.

Proof. We want to apply the previous Theorem. In order to do this, let u be in $H^1_0(\Omega)$, and set $B = \|u\|_{1,q}$ where q is as in Lemma 3.2.1. Then by (3.15) (see Remark 3.2.3) we have

$$\int_\Omega a(x,u)|\nabla u|^2 \ge \frac{C_1 B^2}{(1+B^{q^*})^{\frac{2}{q}-1}}.$$

On the other hand, using the growth condition (3.9) and (3.23), we observe that for every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $G(x,t) \le \varepsilon t^2 + K_\varepsilon t^p$ for every t in $\mathbb{R}$. Therefore, we have

$$f(u) \ge \frac{C_1 B^2}{(1+B^{q^*})^{\frac{2}{q}-1}} - \frac{\varepsilon}{\lambda_1} B^2 - C_2 B^p,$$

from which, by choosing ε sufficiently small and since $p > 2$ and $q^* = 2^*(1-\alpha) > 2$, it is easily deduced the existence of $B \in (0,1)$ and $\delta > 0$ such that

$$f(u) \ge \delta > 0 = f(0),$$

for every u in $H_0^1(\Omega)$ such that $\|u\|_{1,q} = B$. In addition, from (3.7) and (3.24) it follows

$$f(t\varphi_1) \le t^2 \left[c_2 \int_\Omega |\nabla \varphi_1|^2 - \int_\Omega \frac{G(x, t\varphi_1)}{t^2} \right] < 0,$$

provided that $t > 0$ is large enough. Hence we can choose $t_0 > \frac{B}{\|\varphi_1\|_{1,q}}$ such that

$$f(t_0\varphi_1) < 0.$$

Consider now the set

$$\Gamma = \{\gamma \in C^0([0,1], H_0^1(\Omega) \cap L^\infty(\Omega)) : \gamma(0) = 0,\ \gamma(1) = t_0\varphi_1\}.$$

Then, by the embedding of $H_0^1(\Omega)$ into $W_0^{1,q}(\Omega)$ and standard connectedness arguments, we get

$$c \equiv \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)) \ge \delta > 0 = \min\{f(0), f(t_0\varphi_1)\}.$$

Take now a sequence $\{\gamma_n\}$ of paths in Γ such that

$$c \le \max_{t\in[0,1]} f(\gamma_n(t)) \le c + \frac{1}{2n}, \qquad \forall n \in \mathbb{N}.$$

For fixed $n \in \mathbb{N}$, consider

$$M_n = \max_{t\in[0,1]} [\|\gamma_n(t)\|_{1,2} + \|\gamma_n(t)\|_\infty] \ge t_0(1 + \|\varphi_1\|_\infty),$$

and observe that $|||\cdot||| = (\|\cdot\|_{1,2} + \|\cdot\|_\infty)/M_n$ is a norm in $H_0^1(\Omega) \cap L^\infty(\Omega)$ which is equivalent to $\|\cdot\|_{1,2} + \|\cdot\|_\infty$. then, applying the previous Theorem, we deduce the existence of a path $\bar{\gamma}_n \in \Gamma$ and a function $u_n = \bar{\gamma}_n(t_n) \in \bar{\gamma}_n([0,1])$ satisfying

$$c \le \max_{t\in[0,1]} f(\bar{\gamma}_n(t)) \le \max_{t\in[0,1]} f(\gamma_n(t)) \le c + \frac{1}{2n},$$

$$\max_{t\in[0,1]} |||\bar{\gamma}_n(t) - \gamma_n(t)|||_n \le \sqrt{\frac{1}{n}},$$

$$c - \frac{1}{n} \le f(u_n) \le c + \frac{1}{2n}$$

$$|\langle f'(u_n), v\rangle| \le \sqrt{\frac{1}{n}} |||v|||_n, \qquad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

and for $n \in \mathbb{N}$ large enough,

$$\|u_n\|_{1,2} + \|u_n\|_\infty = \|\bar{\gamma}_n(t_n)\|_{1,2} + \|\bar{\gamma}_n(t_n)\|_\infty$$

$$\leq \|\bar{\gamma}_n(t_n) - \gamma_n(t_n)\|_{1,2} + \|\bar{\gamma}_n(t_n) - \gamma_n(t_n)\|_\infty$$

$$+\|\bar{\gamma}_n(t_n)\|_{1,2} + \|\gamma_n(t_n)\|_\infty \leq 2M_n.$$

Therefore,

$$|\langle f'(u_n), v\rangle| \leq \sqrt{\frac{2}{n}} \frac{\|v\|_{1,2} + \|v\|_\infty}{\|u_n\|_{1,2} + \|u_n\|_\infty}, \qquad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

From Lemma 3.3.1 it then follows the existence of a critical point u belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ of f with critical level $c > 0$. Clearly, $u \neq 0$ and the proof is finished. □

Chapter 4

Multiplicity of critical points for some integral functional noncoercive

In this Chapter, motivated by the results of Chapter 3 we will improve the result of Theorem 3.3.3 of Arcoya, Boccardo and Orsina and we will prove the existence of multiple critical points in the " energy space" $H_0^1(\Omega)$ for the following functional

$$f(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2\,dx - \int_\Omega \int_0^{u(x)} g(x,t)dt\,dx, \tag{4.1}$$

whose model is

$$f(u) = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(b(x)+|u|)^{2\alpha}} - \frac{1}{p}\int_\Omega |v|^p, \quad u \in H_0^1(\Omega),$$

with $2 < p < 2^*(1-\alpha)$. These critical points are the weak solutions for the boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) + \frac{1}{2}a_t(x,u)|\nabla u|^2 = g(x,u) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.2}$$

As we pointed out in the previous Chapter it is impossible to apply standard techniques of critical point theory. Our approach is different from that of [5], because with their techniques is not possible to obtain a result of multiplicity. The main tool is the nonsmooth critical point theory as developed in Chapter 1. The key argument in our proof is the Theorem 1.4.4. In order to use this Theorem we take advantage from the techniques developed in [25], and presented in Chapter 2, even if we have not their condition

$$\frac{\partial a}{\partial s}(x,u)u|\nabla u|^2 \geq 0$$

when $|s| \geq R$, where R is a positive constant. In fact in our case we have a sign condition opposed. Performing a suitable change variable, $u = \varphi(v)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism we overcome this obstacle. Then we will show that the functional

$$\tilde{f}(v) = f(\varphi(v))$$

can be studied by means of nonsmooth critical point theory for lower semi-continuous functionals. An other difference with the method proposed by [25] is that we have not the condition (2.10) that is fundamental in order to prove the boundness of the $(CPS)_c-$sequence. Fortunately the structure of the functional $\tilde{f}$ allows us to obtain this result with a brilliant technique. Then we prove that $\tilde{f}$ has infinitely critical point in $H_0^1(\Omega)$. Finally, by Theorem 1.2.5 which connect the weak slopes of $\tilde{f}$ and f we will prove that for every critical point of $\tilde{f}$ there exist a critical point of f.

4.1 The main result

Let Ω be a bounded open subset of $\mathbb{R}^n, n \geq 3$ and let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function, that satisfies the conditions (3.7), (3.8) of Chapter 3. Suppose also that there exists a positive constant c_3 such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ with $|s| \leq R$

$$|a_s(x,s)| \leq c_3. \tag{4.3}$$

and that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$

$$a(x,s) = a(x,-s)\,; \tag{4.4}$$

let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be another Carathéodory function that satisfies the assumption (3.16), with $2 < p < 2^*(1-\alpha)$, and (3.17) and such that that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ verifies the following condition

$$g(x,-s) = -g(x,s). \tag{4.5}$$

Remark 4.1.1. The assumption (3.16) implies that

$$b(u) = \int_\Omega G(x,u)$$

is of class C^1.

Let $j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be the function defined as

$$j(x,s,\xi) = \frac{1}{2} a(x,s)|\xi|^2.$$

Remark 4.1.2. The assumption (3.8) implies the following sign condition

$$j_s(x,s,\xi)s \leq 0 \quad \text{for every} \quad s \in \mathbb{R} \quad \text{with} \quad |s| \geq R.$$

Indeed, we have

$$j_s(x,s,\xi)s = \frac{1}{2} a_s(x,s)s|\xi|^2$$

and by hypothesis (3.8) it follows that

$$\begin{cases} \dfrac{2\beta a(x,s)}{1+s} \leq a_s(x,s) \leq 0 & \text{if} \quad s \geq R \\[2ex] 0 \leq a_s(x,s) \leq \dfrac{2\beta a(x,s)}{1-s} & \text{if} \quad s \leq -R \end{cases}$$

Now we give the definition of weak solution.

Definition 4.1.3. *We say that u is a weak solution of (4.2), if $u \in H_0^1(\Omega)$ and*

$$-div(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u)$$

in $\mathscr{D}'(\Omega)$, that is

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v = \int_\Omega g(x,u)v$$

for every $v \in C_0^\infty(\Omega)$.

The next result allows us to connect these "concrete" notion with the abstract critical point theory

Theorem 4.1.4. *Assume conditions (3.7), (3.8), (4.3) and (3.16). Then for every $u \in H_0^1(\Omega)$ we have*

$$|df|(u) \geq$$

$$\sup\left\{\int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v - \langle w, v\rangle : v \in C_c(\Omega), \|v\|_{1,2} \leq 1\right\}.$$

Proof. See,[9], Theorem 1.5. □

We immediately draw the obvious conclusion

Corollary 4.1.5. *Assume that conditions (3.7), (3.8) , (4.3) and (3.16) hold. If $u \in dom(f)$ satisfies $|df|(u) = 0$, then u is a weak solution*

$$\begin{cases} -div(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) & in \quad \Omega \\ \\ u = 0 & on \quad \partial\Omega \end{cases}$$

Let us state the main result of the chapter

Theorem 4.1.6. *Assume that conditions (3.7), (3.8), (4.3),(4.4),(4.5),(3.16) with $2 < p < 2^*(1-\alpha)$ and (3.17) hold. Then there exists a sequence $\{u_h\} \subset H_0^1(\Omega)$ of critical points of f such that $f(u_h)$ goes to $+\infty$.*

4.2 Change of variable

Let us consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that φ is a diffeomorphism $\varphi(0) = 0, \varphi(s)$ is odd function, $\varphi(s), \varphi''(s) \geq 0$ if $s \geq 0$ and the following properties are satisfied

$$\varphi'(s) \geq \gamma(1 + |\varphi(s)|)^{\alpha} \tag{4.6}$$

where γ is positive constant;

$$\varphi(s) \geq s \quad \text{if} \quad s \geq 0 \tag{4.7}$$

$$\varphi''(s) \geq \frac{\beta(\varphi'(s))^2}{1 + \varphi(s)}, \quad \text{if} \quad s \geq R \tag{4.8}$$

where R, β are defined in (3.8) ;

for all $r \in \mathbb{R}$ such that $|s| \leq r$

$$\varphi'(s) \leq \varphi'(r) \tag{4.9}$$

and

$$\begin{cases} |\varphi''(s)| \leq \alpha & \text{if} \quad 0 \leq \alpha < \frac{1}{2} \\ \\ |\varphi''(s)| \leq |\varphi''(r)| & \text{if} \quad \frac{1}{2} \leq \alpha < \frac{n}{2n-2} \end{cases} \tag{4.10}$$

There exist $\bar{R} > 1$ such that if $s \geq \bar{R}$

$$\varphi'(s) < \mu s^{\frac{\alpha}{1-\alpha}} \tag{4.11}$$

where μ is a positive constant.

Remark 4.2.1. By definition of φ it follows that

$$h(s) = s\varphi'(s) - \varphi(s)$$

is increasing function.

Remark 4.2.2. A simple model is the function

$$\varphi(s) = \{[(1-\alpha)|s|+1]^{\frac{1}{1-\alpha}} - 1\}sgn(s)$$

with $\gamma \leq 1, \beta \leq \alpha$, where γ and β are defined in (4.6), and (3.8) respectively. We observe that

$$\psi(s) = \frac{(1+|s|)^{1-\alpha} - 1}{1-\alpha} sgn(s)$$

is the inverse of φ. Moreover, since

$$\varphi'(s) = [(1-\alpha)|s|+1]^{\frac{\alpha}{1-\alpha}}$$

and

$$\varphi'(s) = \alpha[(1-\alpha)|s|+1]^{\frac{2\alpha-1}{1-\alpha}} \cdot sgn(s),$$

it follows that (4.6),(4.8),(4.9),(4.10) are obviously satisfied.

We prove now (4.7). Since the function

$$t(\sigma) = \left(1 + \frac{R}{\sigma}\right)^{\sigma}, \quad \text{with} \quad \sigma > 0$$

is increasing,we prove that

$$[(1-\alpha)s+1]^{\frac{1}{1-\alpha}} \geq 1 + s.$$

We put $\sigma = \frac{1}{1-\alpha}$. From $0 \leq \alpha < \frac{n}{2n-2}$ one has $1 \leq \sigma < 4$ and the assertion follows because $t(\sigma)$ is increasing. Now let $\bar{R} > 1$ then $\varphi'(s) \leq [(2-\alpha)s]^{\frac{\alpha}{1-\alpha}}$. Putting $\mu = (2-\alpha)^{\frac{\alpha}{1-\alpha}}$ it follows inequality (4.11).

Let us take $u = \varphi(v)$ in (4.1), then we have

$$\tilde{f}(v) = \frac{1}{2}\int_{\Omega} a(x, \varphi(v))(\varphi'(v))^2|\nabla v|^2 - \int_{\Omega}\int_0^{\varphi(v)} g(x,t)dt.$$

Setting $t = \varphi(s)$ in the last integral one has

$$\tilde{f}(v) = \frac{1}{2}\int_\Omega a(x,\varphi(v))(\varphi'(v))^2|\nabla v|^2 - \int_\Omega \int_0^v g(x,\varphi(s))\varphi'(s)ds.$$

Then we obtain that the functional f can be rewritten in the new variable as

$$\tilde{f}(v) = \frac{1}{2}\int_\Omega A(x,v)|\nabla v|^2 - \int_\Omega \tilde{G}(x,v)$$

where

$$A(x,s) = a(x,\varphi(s)) \cdot (\varphi'(s))^2 \quad \text{and} \quad \tilde{G}(x,s) = G(x,\varphi(s)) = \int_0^s \tilde{g}(x,t)dt,$$

with $\tilde{g}(x,t) = g(x,\varphi(s)) \cdot \varphi'(s)$.

Now let us consider the functional $\tilde{J} : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{J}(v) = \int_\Omega \tilde{j}(x,v,\nabla v). \tag{4.12}$$

where

$$\tilde{j}(x,s,\xi) = \frac{1}{2}A(x,s)|\xi|^2$$

Remark 4.2.3. For almost every $x \in \Omega$ and every s in $\mathbb{R}$ the function

$$\{\xi \to \tilde{j}(x,s,\xi)\}$$

is strictly convex.

Remark 4.2.4. It is readily seen that hypothesis (4.6) and the left inequality of (3.7) imply that for almost every $x \in \Omega$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$

$$\tilde{j}(x,s,\xi) \geq \alpha_0|\xi|^2,$$

where $\alpha_0 = \dfrac{\gamma^2 c_1}{2}$

Remark 4.2.5. The assumptions (3.8), (4.7) and (4.8) imply that almost everywhere in Ω and for every $\xi \in \mathbb{R}^n$

$$\tilde{j}_s(x,s,\xi)s \geq 0, \quad \text{for every} \quad s \in \mathbb{R} \quad \text{with} \quad |s| \geq R.$$

Indeed

$$\tilde{j}_s(x,s,\xi) = \frac{1}{2}A_s(x,s)|\xi|^2 =$$

$$\frac{1}{2}[a_t(x,\varphi(s)) \cdot (\varphi'(s))^3 + 2\varphi'(s) \cdot \varphi''(s) \cdot a(x,s)]|\xi|^2$$

and from (3.8) and (4.7) one has:

if $s \geq R$ then

$$\tilde{j}_s(x,s,\xi) \geq a(x,\varphi(s))\varphi'(s)\left[-\frac{\beta(\varphi'(s))^2}{1+\varphi(s)} + \varphi''(s)\right]|\xi|^2,$$

if $s \leq -R$ then

$$\tilde{j}_s(x,s,\xi) \leq a(x,\varphi(s))\varphi'(s)\left[\frac{\beta(\varphi'(s))^2}{1-\varphi(s)} + \varphi''(s)\right]|\xi|^2.$$

So that, taking into account that $a(x,\varphi(s))$ and $\varphi'(s)$are positive functions, the assertion follows by (4.8).

Proposition 4.2.6. *Assume that conditions (3.7),(3.8) hold. Then there exist $R' > 0$ such that*

$$s \leq -R' \Rightarrow \nu\tilde{j}(x,s,\xi) - \tilde{j}_s(x,s,\xi)s - \tilde{j}_\xi(x,s,\xi)\xi \geq \alpha_0(\nu-2)|\xi|^2$$

and

$$s \geq R' \Rightarrow \nu\tilde{j}(x,s,\xi) + \tilde{j}_s(x,s,\xi)s + \tilde{j}_\xi(x,s,\xi)\xi \geq \alpha_0(\nu+2)|\xi|^2$$

where ν is defined in (3.17).

Proof. We can suppose $R' \geq R$. In view of (4.7) and (3.8) one has

$$\nu\tilde{j}(x,s,\xi) - \tilde{j}_s(x,s,\xi)s - \tilde{j}_\xi(x,s,\xi)\xi =$$
$$\frac{1}{2}[(\nu-2)A(x,s) - A_s(x,s)s]|\xi|^2 \geq$$
$$\frac{1}{2}A(x,s)\left[(\nu-2) - \frac{2\beta}{1-\varphi(s)}\varphi'(s)s - 2\frac{\varphi''(s)}{\varphi'(s)}s\right]|\xi|^2.$$

and

$$\nu\tilde{j}(x,s,\xi) + \tilde{j}_s(x,s,\xi)s + \tilde{j}_\xi(x,s,\xi)\xi =$$
$$\frac{1}{2}[(\nu+2)A(x,s) + A_s(x,s)s]|\xi|^2 \geq$$
$$\frac{1}{2}A(x,s)\left[(\nu+2) - \frac{2\beta}{1+\varphi(s)}\varphi'(s)s + 2\frac{\varphi''(s)}{\varphi'(s)}s\right]|\xi|^2.$$

The assertions follows by (4.8) and Remark 4.2.4. □

Remark 4.2.7. From (3.17) one has

$$0 < \nu\tilde{G}(x,s) \leq \tilde{g}(x,s)\frac{\varphi(s)}{\varphi'(s)}.$$

We observe that if $s \geq R$ then $\tilde{g}(x,s) \geq 0$ and if $s \leq -R$ then $\tilde{g}(x,s) \leq 0$

Proposition 4.2.8. *Assume that condition (3.16) hold.*

$$a)\; |s| \geq R \Rightarrow \tilde{g}(x,s)s - \nu\tilde{G}(x,s) \geq 0$$

and there exists a positive constant $\bar{\mu} > 0$ such that

$$b)\; s \geq R \Rightarrow \tilde{g}(x,s)s + \nu\tilde{G}(x,s) \leq \bar{\mu}\cdot s^{2^*}$$

Proof. a) From Remark 4.2.7 we get

$$\tilde{g}(x,s)s - \nu\tilde{G}(x,s) \geq \tilde{g}(x,s)\left[s - \frac{\varphi(s)}{\varphi'(s)}\right],$$

and the result follows by Remark 4.2.1.
b) By Remark 4.2.7 it follows

$$\tilde{g}(x,s)s + \nu\tilde{G}(x,s) \leq g(x,\varphi(s))[\varphi'(s)s + \varphi(s)]$$

and by Remark 4.2.1 we obtain

$$\tilde{g}(x,s)s + \nu \tilde{G}(x,s) \leq 2g(x,\varphi(s)) \cdot \varphi'(s)s. \tag{4.13}$$

By (4.13) and applying again Remark 3.16 it immediately follows

$$\tilde{g}(x,s)s + \nu \tilde{G}(x,s) \leq 2[bs^p(\varphi'(s))^p + d\varphi'(s)s)].$$

We can suppose that $R > 1$ then $s^p > s$, and hence using (4.11) we obtain the assertion. □

4.3 Study of the functional $\tilde{J}$

In this section we will prove some important facts concerning the functional $\tilde{J}$ defined in (4.12). Since Remark 4.2.4 and by Fatou's Lemma we obtain that $\tilde{J}$ is lower semicontinuous functional. We now prove that $\tilde{J}$ satisfies condition (1.7).

Theorem 4.3.1. *Assume that conditions (3.7) and (3.8) hold. Then, for every $(u,\eta) \in epi(\tilde{J})$ with $\tilde{J}(u) < \eta$, there holds*

$$|d\mathscr{G}_{\tilde{J}}|(u,\eta) = 1.$$

Moreover, if (4.4) is satisfied then $\forall \eta > \tilde{J}(0)$ it results $|d_{\mathbb{Z}_2}\mathscr{G}_{\tilde{J}}|(0,\eta) = 1$.

Proof. From (3.7) and (4.9) we have

$$\tilde{j}(x,v,\nabla T_k(v)) \leq \frac{c_2}{2}(\varphi'(k))^2|\nabla v|^2,$$

where T_k is defined as in (2.16). Moreover from definition of φ and (4.4) it follows that $\tilde{j}(x,-s,-\xi) = \tilde{j}(x,s,\xi)$. Then arguing as in the proof of Theorem 2.2.1 the thesis follows. □

The next results allows us to connect the weak slope $|d\tilde{J}|(u)$ with the directional derivatives of the functional $\tilde{J}$. We note that, because of our assumptions, we use different estimates than those used in the corresponding Propositions of Chapter 2.

Remark 4.3.2. From Remark 4.2.4 it follows

$$\tilde{j}_\xi(x, s, \xi) \cdot \xi \geq 2\alpha_0 |\xi|^2 \tag{4.14}$$

Indeed, we have

$$\tilde{j}_\xi(x, s, \xi) \cdot \xi = 2\tilde{j}(x, s, \xi).$$

Now we consider the subspace V_u of $H_0^1(\Omega)$, defined in (2.28) We set

$$A = \max\left\{\frac{c_1 c_2}{(1+R)}, \frac{c_3}{2}\right\} \quad \text{and} \quad B = \max\{\varphi''(\|u\|_\infty + \|v\|_\infty), \alpha\}. \tag{4.15}$$

Proposition 4.3.3. *Assume that conditions (3.7),(3.8),(4.3) hold. Then there exists $\tilde{J}'(u)(v)$ for every $u \in dom(\tilde{J})$ and $v \in V_u$. Furthermore, we have*

$$\tilde{j}_s(x, u, \nabla u)v \in L^1(\Omega) \quad \textit{and} \quad \tilde{j}_\xi(x, u, \nabla u)\nabla v \in L^1(\Omega)$$

and

$$\tilde{J}'(u)(v) = \int_\Omega \tilde{j}_\xi(x, u, \nabla u)\nabla v + \int_\Omega \tilde{j}_s(x, u, \nabla u)v.$$

Proof. Taking into account Proposition 2.3.4 in Chapter 2, it suffice to prove that the functions

$$F(x,t) = \tilde{j}(x, u(x) + tv(x), \nabla u(x) + t\nabla v(x))$$

and

$$\frac{\partial F}{\partial t}(x,t) = \tilde{j}_s(x, u + tv, \nabla u + t\nabla v)v + \tilde{j}_\xi(x, u + tv, \nabla u + t\nabla v) \cdot \nabla v$$

belong to $L^1(\Omega)$. Since $v \in V_u$ using (3.7) and by (4.9), it follows that

$$F(x,t) \leq \frac{c_2}{2}(\varphi'(\|u\|_\infty + \|v\|_\infty))^2(|\nabla u| + |\nabla v|)^2$$

whence $F(x,t) \in L^1(\Omega)$. Moreover, it results

$$\frac{\partial F}{\partial t}(x,t) = \left[\frac{1}{2}a_t(x,\varphi(u+tv))(\varphi'(u+tv))^3 + a(x,\varphi(u+tv))\cdot\varphi'(u+tv)\cdot\right.$$

$$\left.\varphi''(u+tv)\right]\cdot(\nabla u+t\nabla v)^2 v + a(x,\varphi(u+tv))(\varphi'(u+tv))^2(\nabla u+t\nabla v)\nabla v.$$

From hypothesis (3.7),(3.8),(4.3),(4.9) and (4.10) we get that for every $x \in \Omega$ with $v(x) \neq 0$, it results

$$\left|\frac{\partial F}{\partial t}(x,t)\right| \leq [A(\varphi'(\|u\|_\infty+\|v\|_\infty))^3+Bc_2\varphi'(\|u\|_\infty+\|v\|_\infty)]\cdot\|v\|_\infty(|\nabla u|+|\nabla v|)^2$$

$$+c_2\varphi'(\|u\|_\infty+\|v\|_\infty)^2(|\nabla u|+|\nabla v|)|\nabla v|$$

Since the function in the right-hand side of the previous inequality belongs to $L^1(\Omega)$, then also $\frac{\partial F}{\partial t}(x,t) \in L^1(\Omega)$. □

Proposition 4.3.4. *Assume conditions (3.7),(3.8),(4.3). Then we have*

$$|d(\tilde{J}-w)|(u)$$

$$\geq \sup\left\{\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla v + \int_\Omega \tilde{j}_s(x,u,\nabla u)v - \langle w,v\rangle : v \in V_u, \|v\|_{1,2} \leq 1\right\}$$

for every $u \in dom(\tilde{J})$ and every $w \in H^{-1}(\Omega)$.

Proof. Following the argument of Proposition 2.3.5 it suffices to prove that the inequality is satisfied if $|d(\tilde{J}-w)|(u) < \infty$ and

$$\sup\left\{\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla v + \int_\Omega \tilde{j}_s(x,u,\nabla u)v - \langle w,v\rangle : v \in V_u, \|v\|_{1,2} \leq 1\right\} > 0.$$

Let $u \in \text{dom}(\tilde{J})$ and let $\eta \in \mathbb{R}^+$ be such that $\tilde{J}(u) < \eta$. Then we can consider $\bar{\sigma} > 0$ and $\bar{v} \in V_u$ such that $\|\bar{v}\|_{1,2} \leq 1$ and

$$\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla\bar{v} + \int_\Omega \tilde{j}_s(x,u,\nabla u)\bar{v} - \langle w,\bar{v}\rangle < -\bar{\sigma}. \tag{4.16}$$

We set $v_k = H\left(\frac{u}{k}\right)\bar{v}$, where $H(s)$ is defined as in (2.29). Using the conditions (3.7),(3.8),(4.3),(4.9) and (4.10), we get

$$|\tilde{j}_s(x,u,\nabla u)v_k| \leq [A(\varphi'(\|u\|_\infty))^3 + Bc_2\varphi'(\|u\|_\infty)]\|\bar{v}\|_\infty|\nabla u|^2,$$

$$|\tilde{j}_\xi(x,u,\nabla u)\cdot \nabla v_k| \le c_2(\varphi'(\|u\|_\infty))^2[|\nabla u||\nabla \bar{v}| + 2|\nabla u|^2\|\bar{v}\|_\infty].$$

Then, following the same approach of the Proposition 2.3.5, we obtain that fixed $\varepsilon > 0$, there exists $k_0 \ge 1$ such that

$$\left\| H\left(\frac{u}{k_0}\right)\bar{v}\right\|_{1,2} < 1 + \frac{\varepsilon}{2}, \tag{4.17}$$

and

$$\int_\Omega \tilde{j}_s(x,u,\nabla u) H\left(\frac{u}{k_0}\right)\bar{v}$$
$$+ \int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla\left(H\left(\frac{u}{k_0}\right)\bar{v}\right) - \left\langle w, H\left(\frac{u}{k_0}\right)\bar{v}\right\rangle < -\bar{\sigma}. \tag{4.18}$$

Now we take $u_n \in \tilde{J}^n$ such that u_n converges to u in $H^1_0(\Omega)$ and set

$$v_n = H\left(\frac{u_n}{k_0}\right)\bar{v}$$

Using again (3.7),(3.8),(4.3),(4.9) and (4.10), it follows

$$|\tilde{j}_s(x,u_n,\nabla u_n)v_n| \le [A(\varphi'(2k_0))^3 + Bc_2\varphi'(2k_0)]\cdot \|\bar{v}\|_\infty |\nabla u_n|^2$$

$$|\tilde{j}_\xi(x,u_n,\nabla u_n)\nabla v_n| \le c_2(\varphi'(2k_0))^2\left[|\nabla u_n||\nabla \bar{v}| + \frac{2}{k_0}|\nabla u_n|^2\|\bar{v}\|_\infty\right].$$

Arguing again as in Proposition 2.3.5 we have that for fixed $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$\left\| H\left(\frac{z}{k_0}\right)\bar{v}\right\|_{1,2} < 1 + \varepsilon, \tag{4.19}$$

and

$$\int_\Omega \tilde{j}_s(x,z,\nabla z) H\left(\frac{z}{k_0}\right)\bar{v}$$
$$+ \int_\Omega \tilde{j}_\xi(x,z,\nabla z)\nabla\left(H\left(\frac{z}{k_0}\right)\bar{v}\right) - \left\langle w, H\left(\frac{z}{k_0}\right)\bar{v}\right\rangle < -\bar{\sigma}. \tag{4.20}$$

for every $z \in B(u,\delta_1)\cap \tilde{J}^\eta$. Then the continuous function $\mathscr{H} : B(u,\delta)\cap \tilde{J}^\eta \times [0,\delta] \to H^1_0(\Omega)$ given by

$$\mathscr{H}(z,t) = z + \frac{t}{1+\varepsilon} H\left(\frac{z}{k_0}\right)\bar{v}.$$

satisfies all the hypotheses of Proposition 1.2.4. and hence we can conclude that $|d(\tilde{J} - w)|(u) \geq \frac{\bar{\sigma}}{1+\varepsilon}$, and the thesis follows from the arbitrariness of ε. □

Lemma 4.3.5. *Assume conditions (3.7),(3.8),(4.3) hold and let $u \in dom(\tilde{J})$. Then*

$$\int_\Omega \tilde{j}_\xi(x, u, \nabla u)\nabla u + \int_\Omega \tilde{j}_s(x, u, \nabla u)u \leq |d\tilde{J}|(u)\|u\|_{1,2}. \tag{4.21}$$

In particular, if $|d\tilde{J}|(u) < +\infty$ it results

$$\tilde{j}_\xi(x, u, \nabla u)\nabla u \in L^1(\Omega), \qquad \tilde{j}_s(x, u, \nabla u)u \in L^1(\Omega)$$

Proof. The proof follows arguing as in Proposition 2.3.6. It suffices to prove that, if $u \in \text{dom}(\tilde{J})$ with $|d\tilde{J}|(u) < +\infty$ and

$$\int_\Omega \tilde{j}_\xi(x, u, \nabla u)\nabla u + \int_\Omega \tilde{j}_s(x, u, \nabla u)u > 0,$$

then $|d\tilde{J}|(u) < +\infty$. Using the same device of Proposition 2.3.6 we have that, let us fixing $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\|T_k(w)\|_{1,2} \leq (1+\varepsilon)\|T_k(u)\|_{1,2}. \tag{4.22}$$

Let us take $w_n \in H^1_0(\Omega)$ such that $w_n \to u$ in $H^1_0(\Omega)$. Then, if $|w_n| \geq R$ from Remark 4.2.5 we have

$$\tilde{j}_s(x, w_n(x), \nabla w_n(x))w_n(x) \geq 0,$$

and if $|w_n| \leq R$ from (3.7),(3.8),(4.3),(4.9) and (4.10), it follows that

$$|\tilde{j}_s(x, w_n(x), \nabla w_n(x))w_n(x)| \leq [A(\varphi'(R))^3 + Bc_2\varphi'(R)] \cdot |\nabla w_n|^2 \cdot R$$

Hence, we can conclude that

$$\tilde{j}_s(x, w_n(x), \nabla w_n(x))w_n(x) \geq -R[A(\varphi'(R))^3 + Bc_2\varphi'(R)] \cdot |\nabla w_n|^2.$$

Since $w_n \to u$ in $H_0^1(\Omega)$, from (4.14) and by applying Fatou Lemma we get

$$\liminf_{n\to\infty}\left\{\int_\Omega \tilde{j}_\xi(x,w_n,\nabla w_n)\nabla T_k(w_n) + \int_\Omega \tilde{j}_s(x,w_n,\nabla w_n)T_k(w_n)\right\}$$

$$\geq \int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla T_k(u) + \int_\Omega \tilde{j}_s(x,u,\nabla u)T_k(u) > \sigma\|T_k(u)\|_{1,2}.$$

Thus,

$$\int_\Omega \tilde{j}_\xi(x,w,\nabla w)\nabla T_k(w) + \int_\Omega \tilde{j}_s(x,w,\nabla w)T_k(w) \geq \sigma\|T_k(u)\|_{1,2} \tag{4.23}$$

holds. Then, if we consider the continuous map defined as in (2.39) the thesis follows. □

Now we want to study the conditions under which it is possible to take as test function $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$. In order to do this , suppose we have a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla z + \int_\Omega \tilde{j}_s(x,u,\nabla u)z = \langle w,z\rangle \,\forall z\in V_u \tag{4.24}$$

where V_u is defined in (2.28) and $w \in H^{-1}(\Omega)$.

Theorem 4.3.6. *Assume the following conditions (3.7),(3.8),(4.3). Let $w \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ that satisfies (4.24). Moreover, suppose that $\tilde{j}_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ and that there exist $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $\eta \in L^1(\Omega)$ such that*

$$\tilde{j}_s(x,u,\nabla u)v + \tilde{j}_\xi(x,u,\nabla u)\cdot\nabla v \geq \eta. \tag{4.25}$$

Then

$$\tilde{j}_\xi(x,u,\nabla u)\nabla v + \tilde{j}_s(x,u,\nabla u)v \in L^1(\Omega)$$

and

$$\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla v + \int_\Omega \tilde{j}_s(x,u,\nabla u)v = \langle w,v\rangle.$$

Proof. The assertions follows arguing as in Theorem 2.3.7 □

In the next result, we find the conditions under which we can use $v \in H_0^1(\Omega)$ in (4.24).

Theorem 4.3.7. *Assume the following conditions (3.7),(3.8),(4.3). Let $w \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be such that (4.24) is satisfied. Moreover, suppose that $\tilde{j}_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, and there exist $v \in H_0^1(\Omega)$ and $\eta \in L^1(\Omega)$ such that*

$$\tilde{j}_s(x,u,\nabla u)v \geq \eta \qquad \text{and} \quad \tilde{j}_\xi(x,u,\nabla u)\nabla v \geq \eta. \tag{4.26}$$

Then $\tilde{j}_s(x,u,\nabla u)v \in L^1(\Omega), \tilde{j}_\xi(x,u,\nabla u)\nabla v \in L^1(\Omega)$ and

$$\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla v + \int_\Omega \tilde{j}_s(x,u,\nabla u)v = \langle w, v\rangle. \tag{4.27}$$

In particular, it results $\tilde{j}_s(x,u,\nabla u)u, \tilde{j}_s(x,u,\nabla u) \in L^1(\Omega)$ and

$$\int_\Omega \tilde{j}_\xi(x,u,\nabla u)\nabla u + \int_\Omega \tilde{j}_s(x,u,\nabla u)u = \langle w, u\rangle.$$

Proof. Arguing as in Theorem 2.3.9 and applying Theorem 4.3.6 we obtain (4.27). Notice that if $|u| \leq R$ from (3.8),(4.3) (4.9) and (4.10) we get

$$\tilde{j}_s(x,u,\nabla u)u \geq -[A(\varphi'(R))^3 + Bc_2\varphi'(R)]|\nabla u|^2 R. \tag{4.28}$$

Then taking into account Remark 4.3.2 and arguing again as in Theorem 2.3.9 we have $\tilde{j}_s(x,u,\nabla u)u, \tilde{j}_s(x,u,\nabla u) \in L^1(\Omega)$. □

Now we will prove a compactness property for $\tilde{J}$. In order to do this we will follow the argument in the Theorem 2.4.1. In our case we will have to modify the test functions used in this Theorem.

Theorem 4.3.8. *Assume conditions (3.7), (3.8), (4.3). Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded sequence with $j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n \in L^1(\Omega)$ and let $\{w_n\} \subset H^{-1}(\Omega)$ be such that*

$$\forall v \in V_{u_n} : \int_\Omega \tilde{j}_s(x,u_n,\nabla u_n)v + \tilde{j}_\xi(x,u_n,\nabla u_n)\nabla v = \langle w_n, v\rangle. \tag{4.29}$$

If w_n is strongly convergent in $H^{-1}(\Omega)$, then, up to a subsequence, u_n is strongly convergent in $H_0^1(\Omega)$.

Proof. Up to a subsequence,u_n is convergent to some u weakly in $H^{-1}(\Omega)$, strongly in $L^q(\Omega)$ for every $q \in [1, 2n/(n-2))$ and a.e. in Ω. Let $h \geq 1$. For every $v \in C_c^\infty(\Omega)$ we have that $H(\frac{u_n}{h})v \in V_{u_n}$, where H is again the function defined in (2.29). Then using $H(\frac{u_n}{h})v$ as test function we have

$$\int_\Omega H\left(\tfrac{u_n}{h}\right)\tilde{j}_\xi(x,u_n,\nabla u_n)\nabla v$$

$$= \int_\Omega \left[H'\left(\frac{u_n}{h}\right)(F_n - \tilde{j}_\xi(x,u_n,\nabla u_n))) \cdot \frac{\nabla u_n}{h} - H\left(\frac{u_n}{h}\right)\tilde{j}_s(x,u_n,\nabla u_n)\right] v$$

$$+ \int_\Omega H\left(\frac{u_n}{h}\right) F_n \cdot \nabla v,$$

where we used the fact the $w_n = -div(F_n)$, with (F_n) strongly convergent in $L^2(\Omega, \mathbb{R}^n)$. Now, by definition of $H(s)$, and from (3.8), (4.3)(4.9), (4.10) and Remark 4.3.2 it follows that

$$\left| H'\left(\frac{u_n}{h}\right)(F_n - \tilde{j}_\xi(x,u_n,\nabla u_n))) \cdot \frac{\nabla u_n}{h} - H\left(\frac{u_n}{h}\right)\tilde{j}_s(x,u_n,\nabla u_n)\right| \leq$$

$$2|\nabla u_n| \cdot |F_n| + 4\alpha_0|\nabla u_n|^2 + [A(\varphi'(2h))^3 + Bc_2\varphi'(2h)]|\nabla u_n|^2 \in L^1(\Omega).$$

Then, by Theorem 5 in [14], up to a further subsequence, $\nabla u_n \to \nabla u$ a.e. Let now $k \geq 1, \varphi \in C_c^\infty(\Omega), \varphi \geq 0$ and consider

$$v = \varphi e^{-M(u_n+R)^+} H\left(\frac{u_n}{k}\right),$$

where

$$M_k = \frac{A(\varphi'(2k))^3 + Bc_2\varphi'(2k)}{2\alpha_0}. \tag{4.30}$$

Observe that

$$[\tilde{j}_s(x,u_n,\nabla u_n) - M_k\tilde{j}_\xi(x,u_n,\nabla u_n) \cdot \nabla(u_n+R)^+]\varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \leq 0.$$

Indeed, the assertion follows from Remark 4.2.5, for almost every x such that $u_n(x) \leq -R$ while, for almost every x in $\{x : -R \leq u_n(x) \leq 2k\}$ from 4.14, (3.8), (4.3)(4.9),(4.10) and (4.30) we get

$$[\tilde{j}_s(x, u_n, \nabla u_n) - M_k \tilde{j}_\xi(x, u_n, \nabla u_n) \cdot \nabla(u_n + R)^+] \leq$$

$$[A(\varphi'(2k))^3 + Bc_2\varphi'(2k) - 2\alpha_0 M_k] \cdot |\nabla u_n|^2 = 0$$

Moreover, if $|u_n| \leq 2k$, by (3.7) and (4.9) we have

$$|\tilde{j}_\xi(x, u_n, \nabla u_n)| \leq c_2(\varphi'(2k))^2 \nabla u_n,$$

then using the same device of Theorem 2.4.1 we have that

$$\int_\Omega \tilde{j}_s(x, u, \nabla u)\psi + \int_\Omega \tilde{j}_\xi(x, u, \nabla u)\nabla\psi = \langle w, \psi \rangle \quad \forall \psi \in V_u. \tag{4.31}$$

Moreover considering the function $\zeta : \mathbb{R} \to \mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} Ms & \text{if } 0 < s < R, \qquad M = \dfrac{A(\varphi'(R))^3 + Bc_2\varphi'(R)}{2\alpha_0}, \\ MR & \text{if } s \geq R \\ -Ms & \text{if } -R < s < 0 \\ MR & \text{if } s \leq -R \end{cases} \tag{4.32}$$

We have that $v_n = u_n e^{\zeta(u_n)}$ belongs to $H_0^1(\Omega)$. Moreover by Remark 4.3.2 we have

$$\tilde{j}_\xi(x, u_n, \nabla u_n)\nabla v_n \geq 0 \in L^1(\Omega),$$

and by (3.8), (4.3) and Remark 4.2.5 we obtain

$$\tilde{j}_s(x, u_n, \nabla u_n)v_n \geq -R[A(\varphi'(R))^3 + Bc_2\varphi'(R)] \cdot |\nabla u_n|^2 e^{\zeta(R)} \in L^1(\Omega).$$

Then the hypotheses of Theorem 4.3.7 are satisfied and we can use v_n as test function in (4.29). Hence arguing again as in Theorem 2.4.1 we obtain that

$$\lim_{n\to\infty} \int_\Omega \tilde{j}_\xi(x, u_n, \nabla u_n)\nabla u_n e^{\zeta(u_n)} = \int_\Omega \tilde{j}_\xi(x, u, \nabla u)\nabla u e^{\zeta(u)}, \tag{4.33}$$

and by Remark 4.3.2 we have the conclusion. □

4.4 The Palais-Smale condition

In this section we prove that $\tilde{f}$ satisfies the $(PS)_c$ for every $c \in \mathbb{R}$. Let us consider the functional $\tilde{I} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\tilde{I}(v) = -\int_\Omega \tilde{G}(x, v) - \langle \Lambda, v \rangle,$$

where $\Lambda \in H^{-1}(\Omega)$. We observe that from Remark (4.1.1) and since $\varphi \in C^1$ it follows that $\int_\Omega \tilde{G}(x, u)$ is also of class C^1, because is it obtained as composition of C^1 functions. Then condition (3.7) implies that the functional $\tilde{f} : H_0^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by $\tilde{f}(v) = \tilde{J}(v) + \tilde{I}(v)$ is lower semicontinuous.

As consequence of Proposition 4.3.4 and following the argument in Proposition 2.5.2 one has the following

Proposition 4.4.1. *Assume conditions (3.7), (3.8), (4.3) and (3.16) and consider $u \in dom(\tilde{f})$ with $|d\tilde{f}|(u) < \infty$. Then there exists $w \in H^{-1}(\Omega)$ such that $\|w\|_{1,2} \leq |d\tilde{f}|(u)$ and*

$$\forall v \in V_u : \int_\Omega \tilde{j}_\xi(x, u, \nabla u)\nabla v + \int_\Omega \tilde{j}_s(x, u, \nabla u)v - \int_\Omega \tilde{g}(x, u)v - \langle \Lambda, v \rangle = \langle w, v \rangle.$$

We can now give the definition of the Concrete Palais-Smale condition for functional $\tilde{f}$.

Definition 4.4.2. *Let $c \in \mathbb{R}$. We say that $\{u_n\}$ is a Concrete Palais-Smale sequence for $\tilde{f}$ at level c ($(CPS)_c$-sequence for short) if there exists $w_n \in H^{-1}(\Omega)$ with $w_n \to 0$ such that $\tilde{j}_\xi(x, u_n, \nabla u_n)\nabla u_n$ for every $n \geq 1$ and*

$$\tilde{f}(u_n) \to c, \tag{4.34}$$

$$\int_\Omega \tilde{j}_\xi(x, u_n, \nabla u_n)\nabla v + \int_\Omega \tilde{j}_s(x, u_n, \nabla u_n)v - \int_\Omega \tilde{g}(x, u_n)v - \langle \Lambda, v \rangle$$

$$= \langle w_n, v \rangle, \quad \forall v \in V_{u_n}. \tag{4.35}$$

We say that $\tilde{f}$ satisfies the Concrete Palais-Smale condition at level c ($(CPS)_c$

for short) if every $(CPS)_c$-sequence for $\tilde{f}$ admits a strongly convergent subsequence in $H^1_0(\Omega)$.

The following result concerns the relation between the $(PS)_c$ condition and the $(CPS)_c$ condition.

Proposition 4.4.3. *Assume that conditions (3.7), (3.8), (4.3) and (3.16). Then if $\tilde{f}$ satisfies the $(CPS)_c$ condition, it satisfies the $(PS)_c$ condition.*

Proof. Combining Definition 1.2.3, Lemma 4.3.5 and Proposition 4.4.1 we get that u_n satisfies the conditions in Definition 4.4.2. Hence, there exists a subsequence, which converges in $H^1_0(\Omega)$. □

We now want to prove that $\tilde{f}$ satisfies the $(CPS)_c$ condition at every level c. In order to do this, let us consider a $(CPS)_c$-sequence $\{u_n\} \subset \mathrm{dom}(\tilde{f})$.
As consequence of Theorem 4.3.8 we have

Proposition 4.4.4. *Assume that conditions (3.7), (3.8), (4.3) (3.16) are satisfied. Let $\{u_n\}$ ba a $(CPS)_c$-sequence for $\tilde{f}$, bounded in $H^1_0(\Omega)$. Then $\{u_n\}$ admits a strongly convergent subsequence in $H^1_0(\Omega)$.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(\tilde{f})$ be a concrete Palais-Smale sequence for $\tilde{f}$ at level c. From (3.16), with $2 < p < 2^*(1-\alpha)$, and (4.11) it follows that

$$|\tilde{g}(x,s)| \leq d_1 |s|^{2^*-1} \qquad \text{for} \quad s \geq \bar{R}$$

Then the map $\{u \mapsto \tilde{g}(x,u)\}$ is compact from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$. Therefore it suffices to apply Theorem 4.3.8 to see the $\{u_n\}$ is strongly compact in $H^1_0(\Omega)$. □

Lemma 4.4.5. *Assume conditions (3.7), (3.8),(4.3),(3.16) and (3.17). Then every $(CPS)_c$-sequence u_n for $\tilde{f}$ is bounded in $H^1_0(\Omega)$.*

Proof. Since $\tilde{f}(u_n)$ is a convergent sequence in $\mathbb{R}$, there exists c_1 in $\mathbb{R}$ such that $|\tilde{f}(u_n)| \leq c_1$, thus:

$$\nu \int_\Omega \tilde{j}(x, u_n, \nabla u_n) - \nu \int_\Omega \tilde{G}(x, u_n) \leq c_2 \tag{4.36}$$

where ν is as in (3.17). Theorem 4.3.7 yields:

$$-\int_\Omega \tilde{j}_\xi(x,u_n,\nabla u_n)\nabla u_n - \int_\Omega \tilde{j}_s(x,u_n,\nabla u_n)u_n + \int_\Omega \tilde{g}(x,u_n)u_n + \langle \Lambda, u_n\rangle \le \varepsilon_n \|u_n\| \tag{4.37}$$

and

$$\int_\Omega \tilde{j}_\xi(x,u_n,\nabla u_n)\nabla u_n + \int_\Omega \tilde{j}_s(x,u_n,\nabla u_n)u_n - \int_\Omega \tilde{g}(x,u_n)u_n - \langle \Lambda, u_n\rangle \le \varepsilon_n \|u_n\|, \tag{4.38}$$

where $\lim_n \varepsilon_n = 0$. Adding (4.36) and (4.37) we have

$$\int_\Omega \nu\tilde{j}(x,u_n,\nabla u_n) - \tilde{j}_\xi(x,u_n,\nabla u_n)\nabla u_n - \tilde{j}_s(x,u_n,\nabla u_n)u_n -$$

$$\int_\Omega [\nu\tilde{G}(x,u_n) - \tilde{g}(x,u_n)u_n] \le c_2 + (\varepsilon_n + \|\Lambda\|_{-1,2})\|u_n\| \tag{4.39}$$

Adding (4.36) and (4.38) we have

$$\int_\Omega \nu\tilde{j}(x,u_n,\nabla u_n) + \tilde{j}_\xi(x,u_n,\nabla u_n)\nabla u_n + \tilde{j}_s(x,u_n,\nabla u_n)u_n -$$

$$\int_\Omega [\nu\tilde{G}(x,u_n) + \tilde{g}(x,u_n)u_n] \le c_2 + (\varepsilon_n + \|\Lambda\|_{-1,2})\|u_n\|. \tag{4.40}$$

If $u_n \le -R$, by Proposition 4.2.6 and Proposition 4.2.8 we get

$$\alpha_0(\nu-2)\|u_n\|^2_{1,2} \le c_2 + (\varepsilon_n + \|\Lambda\|_{-1,2})\|u_n\|.$$

If $u_n \ge R$, again by Proposition 4.2.6 and Proposition 4.2.8 we get

$$\alpha_0(\nu+2)\|u_n\|^2_{1,2} \le +c_2 + (\varepsilon_n + \|\Lambda\|_{-1,2})\|u_n\| + \mu\cdot\|u_n\|^{2^*}_{2^*}.$$

If

$$|u_n| \le R,$$

by condition (3.16) we have that $\int_\Omega \tilde{G}(x,u_n) \le c_4$ and from (4.36) and Remark 4.2.4 we have

$$\alpha_0\|u\|^2_{1,2} \le c_5$$

Then the assertion follows. □

We can now state the following

Theorem 4.4.6. *Assume that conditions (3.7), (3.8), (4.3),(3.16) and (3.17) hold. Then the functional* $\tilde{f}$ *satisfies the* $(PS)_c$ *condition at every level* $c \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset \text{dom}(\tilde{f})$ be a Concrete Palais-Smale sequence for $\tilde{f}$ at level c. From Lemma 4.4.5 it follows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Proposition 4.4.4 $\tilde{f}$ satisfies the Concrete Palais-Smale condition. Finally Proposition 4.4.3 implies that $\tilde{f}$ satisfies the $(PS)_c$ condition. □

4.5 Proof of the main result

In this section we prove the main result. First of all we prove that $\tilde{f}$ satisfies the fundamental condition (1.7) in order to apply variational methods.

Theorem 4.5.1. *Assume conditions (3.7),(3.8),(3.16). Then, for every* $(u, \eta) \in epi(\tilde{f})$ *with* $\tilde{f}(u) < \eta$*, it results*

$$|d\mathscr{G}_{\tilde{f}}|(u, \eta) = 1.$$

Moreover, if (4.3) and (4.5) are satisfied and $\Lambda = 0,$ *for every* $\eta > \tilde{f}(0)$ *one has* $|d_{\mathbb{Z}_2}\mathscr{G}_{\tilde{f}}|(0, \eta) = 1.$

Proof. Since $\int_\Omega \tilde{G}(x, u)$ is of class C^1, Theorem 4.3.1 and Proposition 1.4.1 imply the result. □

We are now to prove a multiplicity of critical point for the functional $\tilde{f}$.

Proposition 4.5.2. *Assume conditions (3.7),(3.8),(4.3),(4.4),(3.16), (3.17) and (4.5). Then there exists a sequence* $u_h \subset H_0^1(\Omega)$ *of critical points of* $\tilde{f}$ *with* $\tilde{f}(u_h) \to +\infty$.

Proof. First, note that (3.7) and (3.16) imply that $\tilde{f}$ is lower semicontinuous. Moreover from ,(4.3) and (4.5) we deduce that $\tilde{f}$ is an even functional, and from Theorem 4.5.1 we deduce that (1.7) and (d) of Theorem 1.4.4

are satisfied. Moreover, by Theorem 4.4.6, the functional $\tilde{f}$ satisfied $(PS)_c$ for every $c \in \mathbb{R}$. It remains to prove the geometrical hypotheses of Theorem 1.4.4. Let (λ_h, φ_h) be the sequence of solutions of $-\triangle u = \lambda u$ with homogeneous Dirichlet boundary conditions. Moreover, let us consider the subspaces $W_h = span\{\varphi_1, \ldots, \varphi_h\}$ and $V_h = \overline{span}\{\varphi_{h+1}, \varphi_{h+2}, \ldots\}$. We note that $\{W_h\}$ is a strictly increasing sequence of finite-dimensional subspaces of $H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\{V_h\}$ is a a strictly decreasing sequence of closed subspaces of $H_0^1(\Omega)$ such that $H_0^1(\Omega) = W_h \oplus V_h$ and $\bigcap_h V_h = \{0\}$. In order to prove (a) of Theorem 1.4.4 let us consider $u_h \in V_h$ with $\|u_h\|_{1,2} = 1$. Then $\{u_h\}$ is bounded in $H_0^1(\Omega)$, hence, up to a subsequence, weakly convergent in $H_0^1(\Omega)$ and almost everywhere to some u. Since $u \in \bigcap_h V_h$, it must be $u = 0$. Hence, u_h weakly convergent $H_0^1(\Omega)$ and almost everywhere to 0. Since $\int_\Omega \tilde{G}(x,u)$ is of class C^1 one has

$$\int_\Omega \tilde{G}(x, u_h) \to \int_\Omega \tilde{G}(x, 0) = 0.$$

On the other hand from Remark 4.2.4 we have

$$\int_\Omega \tilde{j}(x, u_h, \nabla u_h) \geq \alpha_0 \|u_h\|_{1,2}^2 \geq \alpha_0$$

if $u_h \in V_h$. Thus, for every $\varepsilon > 0$ there exists h_0 such that if $h \geq h_0$ one has $\int_\Omega \tilde{G}(x, u_h) < \varepsilon$, for all $u_h \in V_h$ with $\|u_h\|_{1,2}^2 = 1$. Now, let $\varepsilon > 0$ such that $\alpha_0 - \varepsilon > 0$, and let $V = V_{h_0}$. Then we obtain for every $u \in V_{h_0}$ with $\|u\|_{1,2}^2 = 1$,

$$\tilde{f}(u) \geq \alpha_0 - \varepsilon > 0.$$

Finally, we observe that $H_0^1(\Omega) = W_{h_0-1} \oplus V$. From (4.2.7) integrating and taking into account (4.7) one has that there exist positive constant d_1 such that

$$\tilde{G}(x, s) \geq d_1 |s|^q.$$

Now we verify the second geometrical condition. Let $w \in W_h$. By definition of W_h and since $L^q(\Omega) \subset L^2(\Omega)$ for $q > 2$ we have that there exist a positive constant d_2 such that

$$\|w\|_{1,2}^2 \leq d_2 \, \lambda_h^2 \, \|w\|_q^2.$$

Since $W_h \subset L^\infty(\Omega)$ one has

$$\tilde{f}(u) \leq \frac{c_2}{2}(\varphi'(\|w\|_\infty))^2\|w\|_{1,2}^2 - d_2\|w\|_q^q \leq$$

$$C_w\|w\|_{1,2}^2 - d_3\|w\|_{1,2}^q$$

Since $q > 2$ there exists R_h, sufficiently large, such that:

$$f(u) \leq 0, \quad \text{for every} \quad u \in W_h \quad \text{with} \quad \|u\|_{1,2} > R_h.$$

□

Now we prove the main result

Theorem 4.5.3. *Assume conditions (3.7), (3.8), (4.3),(4.4),(4.5),(3.16), with $2 < p < 2^*(1-\alpha)$, and (3.17) hold. Then there exists a sequence $u_h \subset H_0^1(\Omega)$ of critical points of f such that $f(u_h)$ goes to $+\infty$.*

Proof. Because of Lemma 4.5.2 the functional $\tilde{f}$ has the infinitely critical points v_h in $H_0^1(\Omega)$ with $f(v_h)$ goes to $+\infty$. By Theorem 1.2.5 we have that $\{u_h\} = \{\varphi(v_h)\}$ is a sequence of critical point of f and since $f(u_h) = f(\varphi(v_h)) = \tilde{f}(v_h)$ we have $f(u_h)$ goes to $+\infty$. □

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