Università della Calabria

Dottorato di Ricerca in Matematica ed Informatica

XVIII Ciclo

Tesi di Dottorato (S.S.D. MAT/05 Analisi Matematica)

Lower Hausdorff norm and Hausdorff norm of retractions in Banach spaces of continuous functions

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A.A. 2006-2007

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Ph.D. Thesis

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November, 30 2006

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Introduction

Introduction

This thesis is concerned with the following two problems of nonlinear analysis:

Problem 1: The estimates of the *lower Hausdorff norm* and of the *Hausdorff norm* of a retraction from the closed unit ball of an infinite dimensional Banach space onto its boundary;

Problem 2: The evaluation of the *Wosko constant* $W(X)$ for an infinite dimensional Banach space X, where $W(X)$ is the infimum of all $k \geq 1$ for which there exists a retraction of the closed unit ball onto its boundary with Hausdorff norm less than or equal to k.

Let X be a Banach space and let

$$B(X) = \{x \in X : \|x\| \leq 1\} \quad \text{and} \quad S(X) = \{x \in X : \|x\| = 1\}.$$

A continuous map $R : B(X) \to S(X)$ is a retraction if $Rx = x$, for all $x \in S(X)$
If $R : B(X) \to S(X)$ is a retraction, then $-R$ is a continuous fixed point free self-mapping of the closed unit ball. Therefore, by the Brouwer's fixed point principle, if X is finite dimensional there is no retraction from $B(X)$ onto $S(X)$.
The Scottish Book [36] contains the following question raised around 1935 by Ulam: *"Is there a retraction of the closed unit ball of an infinite dimensional Hilbert space onto its boundary?"*
In 1943, Kakutani [32] gave a positive answer to this question. Later Dugundji [21](1951) and Klee [33](1955) gave a positive answer to Ulam's question in

the more general setting of infinite dimensional Banach spaces.

Assume that X is infinite dimensional and set $\inf \emptyset = \infty$. Given a retraction $R : B(X) \to S(X)$, we denote by

$$Lip(R) = \inf\{k \geq 1 : \|Rx - Ry\| \leq k\|x - y\| \text{ for all } x, y \in B(X)\}$$

the *Lipschitz norm of* R.
In [9] Benyamini and Sternfeld, following Nowak [37], proved that for any Banach space X there is a retraction $R : B(X) \to S(X)$ with $Lip(R) < \infty$.
It is of interest in the literature the problem of evaluating the following quantitative characteristic

$$L(X) = \inf\{k \geq 1 : \text{ there is a retraction } R : B(X) \to S(X) \text{ with } Lip(R) \leq k\},$$

called the *Lipschitz constant of the space* X.
A general result states that in any Banach space X, $3 \leq L(X) \leq 256 \times 10^9$ (see [29],[2]). In special Banach spaces more precise estimates have been obtained by means of constructions depending on each space. We refer to [29] for a collection of results on this problem and related ones.
Recall that the *Hausdorff measure of noncompactness* $\gamma_X(A)$ of a bounded subset A of a Banach space X is defined by

$$\gamma_X(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{−net in } X\},$$

where by a finite $\varepsilon-$net for A we mean, as usual, a finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$A \subset \bigcup_{i=1,\ldots,n} (x_i + B_\varepsilon(X))$$

with $B_\varepsilon(X) = \{x \in X : \|x\| \le \varepsilon\}$.
Given a retraction $R : B(X) \to S(X)$, we denote by

$$\begin{aligned} \underline{\gamma}_X(R) &= \sup\{k \ge 0 : \gamma_X(R(A)) \ge k\gamma_X(A) \text{ for } A \subseteq B(X)\} \\ \text{and} & \\ \gamma_X(R) &= \inf\{k \ge 0 : \gamma_X(R(A)) \le k\gamma_X(A) \text{ for } A \subseteq B(X)\}, \end{aligned}$$

the *lower Hausdorff norm* and the *Hausdorff norm* of R, respectively.
Actually $\underline{\gamma}_X(R)$ is a maximum, while $Lip(R)$ and $\gamma_X(R)$ are minima.
Then two interesting problems arise in nonlinear analysis (see for example [22],[41],[4],[15]) : the estimates of $\underline{\gamma}_X(R)$ and $\gamma_X(R)$ for a given retraction R and, in connection with the Hausdorff norm, the evaluation of the following quantitative characteristic

$$W(X) = \inf\{k \ge 1 : \text{ there is a retraction } R : B(X) \to S(X) \text{ with } \gamma_X(R) \le k\},$$

called the *Wosko constant of the space* X. The constant was introduced by Wosko in [42].
From

$$\gamma_X(R(B(X))) = \gamma_X(S(X)) = \gamma_X(\overline{\text{co}}(S(X))) = \gamma_X(B(X)),$$

where $\overline{\text{co}}(X)$ is the closed convex hull of $S(X)$, it follows that $W(X) \ge 1$ for every space X.
Observe that for a given retraction R, we have $\gamma_X(R) \le 2Lip(R)$ which becomes $\gamma_X(R) \le Lip(R)$, when X has the ball intersection property (see [38]), infact, in this case $\gamma_X(A) = \gamma_{B(X)}(A)$ for every $A \subset B(X)$, where

$$\gamma_{B(X)}(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{−net in } B(X)\}.$$

Therefore $W(X) \le 2L(X)$ for any space X. In particular $W(X) \le L(X)$, if the space X has the ball intersection property.
Concerning general results about the evaluation of $W(X)$, in [41] it was proved that $W(X) \le 6$ for any Banach space X, and $W(X) \le 4$ for sepa-

rable or reflexive Banach spaces. It has also been proved that $W(X) \leq 3$ whenever X contains an isometric copy of l^p with $p \leq (2 - \frac{\log 3}{\log 2})^{-1} \simeq 2.41\ldots$ Moreover it has been proved that $W(X) = 1$ in some spaces of measurable functions ([14]) and in Banach spaces whose norm is monotone with respect to some basis ([4]).
Regarding Banach spaces of real continuous functions, which in the sequel we assume to be equipped with the sup norm, we cite that in $\mathcal{C}([0,1])$ for any $k > 1$, there exists a retraction $R : B(\mathcal{C}([0,1])) \to S(\mathcal{C}([0,1]))$ with $\gamma_{\mathcal{C}([0,1])}(R) \leq k$ so that $W(\mathcal{C}([0,1])) = 1$ (see [42]). However we point out that the first evaluation of the Wośko constant of the space $\mathcal{C}([0,1])$ has been given by Furi and Martelli in 1974.
In [26], [25] they have proved that $W(\mathcal{C}([0,1]))) \leq 9$.
If E is a finite dimensional normed space and K a convex compact set in E with nonempty interior, the same result has been obtained in the Banach space $\mathcal{C}(K)$ of all real continuous functions on K (see [40]), and in the Banach space $\mathcal{BC}([0,\infty))$ of all real bounded continuous functions on the noncompact interval $[0,\infty)$ (see [16]). In particular, in [16] it is shown that for any $k > 1$ the same retraction R can be chosen with $\gamma_{\mathcal{BC}([0,\infty))}(R) \leq k$ and $\underline{\gamma}_{\mathcal{BC}([0,\infty))}(R) > 0$.
The aim of this thesis is to construct retractions with *positive lower Hausdorff norms* and *small Hausdorff norms* in Banach spaces of real continuous functions which domains are not necessarily bounded or finite dimensional. Moreover by means of some examples we give explicit formulas for the *lower Hausdorff norms* and the *Hausdorff norms* of such maps.
Let S and T, with $T \subset S$, be nonempty subsets of a topological space. In the following we will denote by $\mathcal{BC}(S)$ and $\mathcal{BCU}(S)$ the Banach spaces of real all functions defined on S which are, respectively, *bounded* and *continuous*, *bounded* and *uniformly continuous*. Moreover we denote by $\mathcal{BC}_T(S)$ the Banach space of all real bounded functions that are *continuous* on S and *uniformly continuous* on T.
Assume that K is a set in a normed space E containing the closed unit ball $B(E)$. The main result of Chapter 2 is the following: For any $u > 0$ there is a

retraction $R_u : B\left(\mathcal{BC}_{B(E)}(K)\right) \to S\left(\mathcal{BC}_{B(E)}(K)\right)$ such that

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}\ (K)}(R_u) = \begin{cases} \frac{1}{2}, u \leq 4 \\ \frac{2}{u}, u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{u+8}{u}$$

In particular we have that $W(\mathcal{BC}_{B(E)}(K)) = 1$.

As corollaries we obtain similar results in the case of the space $\mathcal{BCU}(E)$ and in the space $\mathcal{BC}(E)$ when E is a finite dimensional normed space. In particular, if $E = \mathbb{R}$, we obtain [16] and, if K is a convex compact set in a finite dimensional normed space E with nonempty interior, we obtain [40]. Moreover, by the invariance of the Hausdorff measure of noncompactness under isometries, we obtain the same result in the Banach space $\mathcal{BC}_{h(B(E))}(M)$ being M a metric space homeomorphic to K under a map h, which is bi-lipschitzian when restricted to $B(E)$.

Chapter 3 is devoted to the construction of retractions from the closed unit ball onto the unit sphere in the Banach space $\mathcal{C}(P)$ of all real continuous functions defined on the Hilbert cube P. As in the previous chapter, we obtain explicit formulas for the *lower Hausdorff norms* and the *Hausdorff norms* of such retractions.

Our main result is the following: For any $u > 0$ there is a retraction R_u : $B(C(P)) \to S(C(P))$ such that

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, \text{ if } u \leq 4 \\ \frac{4}{u}, \text{ if } u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}.$$

In particular, we have that $W(C(P)) = 1$.

Let K be a metrizable infinite dimensional compact convex set in a topological linear space and assume K is an absolute retract (e.g. a compact

convex subset of a normed space is an absolute retract). Then K is homeomorphic to the Hilbert cube ([20]), so that the Banach space $\mathcal{C}(K)$ of all real continuous functions defined on K and the space $\mathcal{C}(P)$ are isometric. Therefore, since the Hausdorff measure of noncompactness is invariant under isometries, the previous result holds also in the Banach space $\mathcal{C}(K)$.
We observe that a retraction R which has positive *lower Hausdorff norm* is a proper map, which means that the preimage $R^{-1}(M)$ of any compact set $M \subseteq X$ is compact. Thus, all the retractions we construct are proper maps. Finally we remark that the following questions are still open:

- Is $W(X) = 1$ for any infinite dimensional Banach space X?
- Is $W(X) = 1$ a minimum for any infinite dimensional Banach space X?

In [14], it is shown that the second question has a positive answer in a class of Orlicz spaces, which contains the classical Lebesgue spaces $L^p([0,1])$ $(p \geq 1)$.

Chapter 1

Preliminary topics

1.1 Introduction

This chapter contains some basic concepts and background used throughout this thesis, concerning measures of noncompactness, some metric and topological quantitative characteristics of nonlinear operators, Lipschitz constant and Wośko constant of an infinite dimensional Banach space, and isometries. References for this material are Banaś and Goebel [7], Akhmerov et al. [1], Acedo et al. [5] and Banaś [6] for the measure of noncompactness; Goebel and Kirk [29] and Appel et al. [3] as well as papers [8],[9], [10], [11], [12], [13], [24], [28], [27], [30], [34], [37], [4], [35], [39], [40], [15], [41], [42] for the quantitative characteristics of nonlinear operators and for the constants of an infinite dimensional Banach space ; and Fleming and Jamison [23] for the isometries.

1.2 Notations

Throughout this thesis we shall use the following notations. We will denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural and real numbers, respectively. Moreover X will denote an infinite dimensional Banach space with the norm $\|\cdot\|$ and the zero element 0. If $A \subset X$ is a non-empty set, we denote by $\overline{A}$, coA, $\overline{co}A$ and $diamA$ the closure, the convex hull, the closed convex hull and the diameter of A, respectively. Further, we wil use the notation

$$B_{r,x_0}(X) = \{x \in X : \|x - x_0\| \leq r\}$$

for the closed ball centered at x_0 and of radius $r > 0$ in X, and

$$S(X) = \{x \in X : \|x\| = 1\}$$

for the unit sphere in X. In the case $x_0 = 0$ we simply write $B_{r,0}(X) = B_r(X)$ and $B_{1,0}(X) = B(X)$.

Given a subset A of X, a finite $\varepsilon-net$ for A in X is a finite set $\{y_1, y_2, ..., y_m\}$ in X such that

$$A \subset \bigcup_{i=1}^{m} (y_k + B_\epsilon(X)).$$

Whenever E is a topological space, $\mathcal{BC}(E)$ (or $\mathcal{C}(E)$, if E is compact) will denote the Banach space of real bounded continuous functions on E, endowed with the $\sup$ norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty := \sup_{x \in E} |f(x)|.$$

Further, let T and S, with $T \subseteq S$, be nonempty sets of a topological space. In the following we will denote by $\mathcal{BC}(S)$ and $\mathcal{BCU}(S)$ the Banach spaces of all real functions defined on S which are, respectively, bounded and continuous, bounded and uniformly continuous. Moreover we denote by $\mathcal{BC}_T(S)$ the space of all real bounded functions that are continuous on S and uniformly continuous on T.

1.3 Measures of noncompactness and quantitative characteristics

Roughly speaking, a measure of noncompactness is a tool to understand how far is a set from being compact. The notion of measure of noncompactness plays a central role in this thesis, thus we start giving a formal definition.

Definition 1 *Let X be a Banach space. A non negative function ψ defined on the bounded subsets of X is called a measure of noncompactness on X, if it satisfies the following properties, for $A, B \subset X$:*

1. $\psi(A) = 0$ *iff A is precompact;*
2. $\psi(A) = \psi(\overline{co}(A))$*;*
3. $\psi(A \cup B) = \max\{\psi(A), \psi(B)\}$*;*
4. $\psi(A + B) \leq \psi(A) + \psi(B)$*;*
5. $\psi(\lambda A) = |\lambda|\psi(A)$*, for all $\lambda \in \mathbb{R}$.*

The most important examples are the *Kuratowski measure of noncompactness*

$$\begin{aligned}\alpha(B) \quad = \inf \quad &\{\varepsilon > 0 : B \text{ can be covered by finitely} \\ &\text{many sets with diameter} \leq \varepsilon\},\end{aligned}$$

the *Istrătescu measure of noncompactness*

$$\begin{aligned}\beta(B) \quad = \sup \quad &\{\varepsilon > 0 : B \text{ contains a sequence} \\ &\{x_n\} \text{ with } \|x_m - x_n\| \geq \varepsilon \text{ for } m \neq n\},\end{aligned}$$

the *Hausdorff measure of noncompactness*

$$\begin{aligned}\gamma(B) \quad = \inf \quad &\{\varepsilon > 0 : B \text{ has a finite} \\ &\varepsilon\text{− net in } X\}\end{aligned}$$

or equivalently,

$$\begin{aligned}\gamma(B) \quad = \inf \quad &\{\varepsilon > 0 : B \text{ can be covered by finitely} \\ &\text{many balls with radii} \leq \varepsilon\}.\end{aligned}$$

When we need to relate a measure of noncompactness to a specific space X, we shall write α_X,β_X and γ_X, respectively.
We note that the following inequality holds for γ

Remark 2 *Let $A \subset B(X)$, then*

$$\gamma_X(A) \leq 2\gamma_{B(X)}(A),$$

where

$$\begin{aligned} \gamma_{B(X)}(A) \quad &= \inf \quad \{\varepsilon > 0 : B \textit{ has a finite} \\ &\qquad\qquad \varepsilon- \textit{ net in } B(X)\} \end{aligned}$$

Proof. Let $A \subset B(X)$ and $a > \gamma_X(A)$. By definition of γ_X, there exists a finite $a-$net $\{\xi_1, \xi_2, \dots \xi_n\}$ for A in X such that $B_{a,\xi_k} \cap A \neq \emptyset$, for $k = 1, 2, \dots, n$.
We show that A has a finite $2a-$net $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ in $B(X)$.
For every ξ_k, fix $\zeta_k \in A \cap B_{a,\xi_k}(X)$.
Let $x \in A$ and choose ξ_j such that

$$\|x - \xi_j\| \leq a.$$

Then

$$\|x - \zeta_j\| \leq \|x - \xi_j\| + \|\xi_j - \zeta_j\| \leq 2a.$$

Hence $\gamma_X(A) \leq 2a$ which proves $\gamma_X(A) \leq 2\gamma_{B(X)}$.

■

The measures α,β and γ are related by the relations explained below.

Remark 3 *The above measures of noncompactness are mutually equivalent in the sense that*

$$\gamma \leq \beta \leq \alpha \leq 2\gamma. \tag{1.1}$$

In the sequel, the letter ψ will always denote one of the three measures of noncompactness α, β or γ.
Given a Banach space X and an operator $F : D(F) \subset X \to X$, the quantita-

tive characteristics

$$
\begin{aligned}
Lip(F) \quad &:= \inf \quad \{k \geq 0 : \|F(x) - F(y)\| \leq k\|x - y\| \\
&\qquad\qquad \text{for } x, y \in D(F)\}\,, &(1.2)\\
\underline{\psi}(F) \quad &:= \sup \quad \{k \geq 0 : \psi(F(B)) \geq k\psi(B) \\
&\qquad\qquad \text{for every bounded } B \subset D(F)\}\,, &(1.3)\\
\psi(F) \quad &:= \inf \quad \{k \geq 0 : \psi(F(B)) \leq k\psi(B) \\
&\qquad\qquad \text{for every bounded } B \subset D(F)\}\,, &(1.4)
\end{aligned}
$$

are called the *Lipschitz norm*, the *lower ψ-norm* and the *ψ-norm* of the operator F, respectively.

Proposition 4 *Let X be a Banach space and*

$$F : D(F) \subset F \to X$$

a continuous map. Then

(a) $Lip(F)$ is a minimum, whenever it is finite;

(b) $\psi(F)$ is a minimum , whenever it is finite;

(c) $\underline{\psi}(F)$ is a maximum.

Proof.

(a). Put $\lambda = Lip(F)$ and $H = \{k \geq 1 : \|Fx - Fy\| \leq k\|x - y\|$ for all $x, y \in D(F)\}$.

Suppose that $\lambda \notin H$. Then there are $x, y \in D(F)$ such that

$$\|Fx - Fy\| > \lambda\|x - y\|. \tag{1.5}$$

On the other hand, we have that

$$\|Fx - Fy\| \leq \left(\lambda + \frac{1}{n}\right) \|x - y\| \tag{1.6}$$

holds for every $x, y \in D(F)$ and $n = 1, 2, \ldots$ By 1.5 and 1.6, it follows that

$$\lambda\|x-y\| < \|Fx - Fy\| \leq \left(\lambda + \frac{1}{n}\right)\|x-y\|.$$

Letting $n \to \infty$, we obtain

$$\|Fx - Fy\| = \lambda\|x-y\|,$$

which is a contradiction.

(b). Put $\lambda = \psi(F)$ and $H = \{k \geq 1 : \psi(F(A)) \leq k\psi(A) \text{ for } A \subset D(F)\}$. Suppose that $\lambda \notin H$. Then there are $A \subset D(F)$ such that

$$\psi(F(A)) > \lambda\psi(A). \tag{1.7}$$

On the other hand, we have that

$$\psi(C) \leq \left(\lambda + \frac{1}{n}\right)\psi(C) \tag{1.8}$$

holds for every $C \subset D(F)$ and $n = 1, 2, \ldots$ By 1.7 and 1.8, it follows that

$$\lambda\psi(A) < \psi(F(A)) \leq \left(\lambda + \frac{1}{n}\right)\psi(A).$$

Letting $n \to \infty$, we obtain

$$\psi(F(A)) = \lambda\psi(A),$$

which is a contradiction.

(c). Put $\lambda = \underline{\psi}(F)$ and $H = \{k > 0 : \psi(F(A)) \geq k\psi(A) \text{ for } A \subset D(F)\}$. Suppose that $\lambda \notin H$. Then there are $A \subset D(F)$ such that

$$\psi(F(A)) < \lambda\psi(A). \tag{1.9}$$

On the other hand, we have that

$$\psi(C) \geq \left(\lambda - \frac{1}{n}\right) \psi(C) \tag{1.10}$$

holds for every $C \subset D(F)$ and $n = 1, 2, \ldots$ By 1.9 and 1.10, it follows that

$$\lambda\psi(A) > \psi(F(A)) \geq \left(\lambda - \frac{1}{n}\right) \psi(A).$$

Letting $n \to \infty$, we obtain

$$\psi(F(A)) = \lambda\psi(A),$$

which is a contradiction.

■

Remark 5 *The following inequalities are obtained from Remark 3,*

(a) $\frac{1}{2}\gamma(F) \leq \alpha(F) \leq 2\gamma(F)$,

(b) $\frac{1}{2}\underline{\gamma}(F) \leq \underline{\alpha}(F) \leq 2\underline{\gamma}(F)$,

(c) $\frac{1}{2}\gamma(F) \leq \beta(F) \leq 2\gamma(F)$,

(d) $\frac{1}{2}\underline{\gamma}(F) \leq \underline{\beta}(F) \leq 2\underline{\gamma}(F)$.

Let X be a Banach space and set $\inf \emptyset = \infty$, the two quantitative characteristics

$$L(X) := \inf \{k \geq 1 : \text{ there is a retraction } R : B(X) \to S(X) \text{ with } Lip(R) \leq k\} \tag{1.11}$$

and

$$W_\psi(X) := \inf \{k \geq 1 : \text{ there is a retraction } R : B(X) \to S(X) \text{ with } \psi(R) \leq k\}, \tag{1.12}$$

are called the *Lipschitz constant* and the $\psi-constant$ of X, respectively. In the sequel we shall write $W(X)$ for $W_\gamma(X)$ and refer to it as the *Wosko constant* of the space X.

The following definition has been given by Nussbaum in [38],

Definition 6 *A Banach space X has the ball intersection property (BIP) if for every $r > 0$ and every $\varepsilon > 0$ the intersection of the unit ball with any ball of radius r can be covered with finitely many balls of radius $r + \varepsilon$ centered at points contained in the unit ball.*

We note that for $1 \leq p < \infty$, the Banach space l^p has the BIP, while the Banach spaces $\mathcal{C}([a, b]), \mathcal{BC}([0, \infty))$ and $L^p(\mathbb{R})$, for $p \neq 2$, fails to have the property.

The following Proposition relates $L(X)$ and $W_\psi(X)$ for a given Banach space X.

Proposition 7 *Let X be a Banach space, then*

(a) $W(X) \leq 2L(X)$ *and in particular* $W(X) \leq L(X)$*, if* X *has the BIP;*

(b) $W_\alpha(X) \leq L(X)$;

(c) $W_\beta(X) \leq L(X)$.

Proof. Let $C \subset B(X)$ and $L(X) > c$.By definition of $L(X)$, there exists a retraction $R : B(X) \to S(X)$ with $Lip(R) = c$.

(a). We show that $\gamma(R) \leq 2c$.
Suppose that $a < \gamma(C)a$, then by Remark 2, C has a finite $2a-$net $\{\zeta_1, \zeta_2, \dots, \zeta_n\} \subset B(X)$.
Let $x \in B(X)$ and $k \in \{0, 1, \dots n\}$ such that

$$\|x - \zeta_k\| \leq 2a,$$

then

$$\|R(x) - R(\zeta_k)\| \leq c\|x - \zeta_k\| \leq 2ac.$$

Hence $\{R\zeta_1, R\zeta_2, \dots, R\zeta_n\}$ is a $2ac-$net for $R(A)$ in X, so that

$$\gamma_X(R(C)) \leq 2c\gamma_X(C)$$

and

$$\gamma_X(R) \leq 2c. \tag{1.13}$$

Since 1.13 holds for any $c > L(X)$, it follows

$$W(X) \leq 2L(X).$$

Assume that X has the BIP, then a finite $a-$net for C in X is also a finite $a-$net for C in $B(X)$, so that

$$\gamma_X(R) \leq c$$

and, by a similar proof,

$$W(X) \leq L(X)$$

holds.

(b). Now we show that $\alpha(A) \leq c$.
Let $a > \alpha(C)$, then, by definition of α, there exist sets $C_1, C_2, \dots, C_n \subset C$ with diam$C_k \leq a$ ($k \in \{1, 2, \dots, n\}$) such that

$$C = \bigcup_{k=1,\dots,n} C_k.$$

Then

$$R(C) = \bigcup_{k=1,2,\dots n} R(C_k)$$

and

$$\|R(x) - R(y)\| \leq c\|x - y\| \leq c\mathrm{diam}(C_k) \leq ca,$$

for all $x, y \in C_k$ and $k = 1, 2, \dots n$.
Therefore,

$$\mathrm{diam}(R(C_k)) \leq ca \ (k = 1, 2, \dots, n).$$

Hence

$$\alpha(R(C)) \leq c\alpha(A),$$

so that $\alpha(R) \leq c$. It follows that $W_\alpha(X) \leq L(X)$.

(c). If $\beta(R(C)) > a$, then there exists a sequence $\{R(x_n)\} \subset R(C)$, such that $\|R(x_n) - R(x_m)\| \geq a$ for all $n \neq m$.
From

$$a \leq \|R(x_n) - R(x_m)\| \leq c\|x_n - x_m\|,$$

we obtain,

$$\frac{a}{c} \leq \beta(C)$$

and then

$$\frac{\beta(R(C))}{c} \leq \beta(C)$$

so that $W_\beta(X) \leq L(X)$

■

From Remark 5, we also obtain the following inequalities:

1. $W_\alpha(X) \leq 2W_\gamma(X)$,

2. $W_\beta(X) \leq 2W_\gamma(X)$.

We note that not every retraction from the unit ball onto its boundary has positive $\underline{\psi} - norm$:

Example 8 *Let X be a Banach space, then there exists at least a retraction $R : B(X) \to S(X)$ such that*

$$\underline{\psi}(R) = 0.$$

Proof. Let $h : X \to X\backslash\{0\}$ be a deleting homeomorphism, such that $h(x) = x$ for every $x \in X \setminus B(X)$.
We define a continuous map

$$R : B(X) \to S(X)$$

by,

$$R(x) := \phi(h(x)), \tag{1.14}$$

where $\phi : B(X) \setminus \{0\} \to S(X)$ is the radial retraction

$$\phi(x) := \frac{x}{\|x\|}.$$

We observe that R is a retraction since

$$R(x) = \phi(h(x)) = \phi(x) = x$$

for every $x \in S(X)$.
To show that $\underline{\psi}(R) = 0$, fix $x \in S(X)$.
We have

$$R^{-1}(x) = k^{-1}(\phi^{-1})(x) = h^{-1}((0, x]). \tag{1.15}$$

We note that $h^{-1}((0, x])$ is closed because $\{x\}$ is closed and R continuous, though it is not compact.
To show this, let $z_n := h^{-1}(\frac{x}{n})$ $(n = 1, 2, \ldots)$.
Assume that there exists a convergent subsequence $\{z_{n_k}\}$ of $\{z_n\}$, since h is continuous, we should have

$$h(\lim_{k\to\infty} z_{n_k}) = \lim_{k\to\infty} \frac{x}{n_k} = 0.$$

But it is impossible, since $0 \notin h(X)$. This show that R is not proper, hence $\underline{\psi}(R) = 0$. ■
We observe that there are retractions with $Lip(R) = \infty$. In order to give an example, we need the following well known theorem ([19])

Theorem 9 *Let C be a non-empty bounded closed convex subset of a Banach space X, ψ a measure of noncompactness on X and $F : C \to C$ a continuous mapping such that $\psi(F) < 1$. Then F has at least one fixed point.*

Example 10 *Let $L_p([0, 1])$ $(1 \le p < \infty)$, be the classical Lebesgue space. Then there is a retraction*

$$R : B(L_p([0, 1])) \to S(L_p([0, 1]))$$

with $Lip(R) = \infty$.

Proof. Let $Q : B(L_p([0,1])) \to B(L_p([0,1]))$ and $P_p : B(L_p([0,1])) \to L_p([0,1])$ by setting, respectively,

$$Qf = \Big(\frac{2}{1+\|f\|_p}\Big)^{\frac{1}{p}} f_{\frac{2}{1+\|f\|_p}},$$

and

$$P_p f = \begin{cases} \Big(\frac{2}{1-\|f\|_p}(1-\|f\|_p^p)\Big)^{\frac{1}{p}} \chi_{\left(\frac{1+\|f\|_p}{2},1\right]} & \text{if} \quad f \in B(L_p([0,1])) \setminus S(L_p([0,1])) \\ \\ 0 & \text{if} \quad f \in S(L_p([0,1])). \end{cases}$$

In [14, Theorem 4.5] it has been proved that the mapping $R : B(L_p([0,1])) \to S(L_p([0,1]))$ defined by $Rf = Qf + P_p f$ satisfies $\gamma_{L_p}(R) = 1$.

Then following [42] set $T = -R$ take any $0 < \varepsilon < 1$ and consider the equation $x = (1-\varepsilon)Tx$. Since $\gamma_{L_p}((1-\varepsilon)T) = 1-\varepsilon$ by Theorem 9, the map $(1-\varepsilon)T$ has a fixed point. If $x = (1-\varepsilon)Tx$, then $\|x - Tx\|_p = \varepsilon$ and $T^2x = -Tx = Rx$. Suppose that R is lipschitzian with constant k, then so it is T. Hence we obtain $2 = \|T^2x - Tx\|_p \le k\|x - Tx\|_p = k\epsilon$. By the arbitrariness of ε we get a contradiction. Hence $Lip(R) = \infty$. ∎ Using a similar proof, the next remark can be obtained.

Remark 11 *Let X be a Banach space and ψ a measure of noncompactness on X. If*

$$R : B(X) \to S(X)$$

is a retraction such that $\psi(R) = 1$, then $Lip(R) = \infty$.

1.4 Isometries

The measures of noncompactness of a set in a Banach space is invariant under isometries.

In fact, assume that X and Y are two isometric Banach spaces and let $H : X \to Y$ be an isometry.

If ψ is the same measure of noncompactness on X and Y, then $\psi(A) =$

$\psi(H(A))$ for any subset A of X. Therefore all the quantitative caracteristics introduced in the previous section are invariant under isometries.
We recall that if two topological spaces, E_1 and E_2 are homeomorphic, then the Banach spaces $\mathcal{BC}(E_1)$ and $\mathcal{BC}(E_2)$ are isometric, though, we may find of particular interest to investigate about homeomorphisms.
Let l_2 be the real Hilbert space, with the usual norm $\|\cdot\|_2$ and canonical basis (e_n). Denote by

$$P = \{x = (x_n) \in l_2 : |x_n| \leq \frac{1}{n} \ \ (n = 1, 2, ...)\}$$

the Hilbert cube. Homeomorphisms between C and compact sets in infinite dimensional spaces have been widely studied.
A first result due to Klee ([33]) is the following,

Theorem 12 *Every infinite-dimensional compact convex subset of a normed space is homeomorphic to the Hilbert cube.*

For a generalization of the previous theorem, we recall the following,

Definition 13 *[[31]] A topological space E is an absolute retract (briefly, AR) if given any metric space M, a closed subspace $A \subset M$ and a continuous map*

$$f : A \to E,$$

there exists a continuous map

$$F : M \to E$$

such that $F(a) = f(a)$ for any $a \in A$.

Taking into account the next

Theorem 14 *[21] Let C be any convex subset of a locally convex linear topological space.*
Then for every metric space M and any closed $A \subset M$ each continuous

$$f : A \to C$$

has a continuous extension

$$F : M \to C.$$

In particular, if C is metrizable, then C is an AR.

It follows that the following result ([20]) is a proper generalization of Theorem 12.

Theorem 15 *An infinite dimensional compact metrizable convex subset K of a topological vector space is homeomorphic to the Hilbert cube, if it is a an AR.*

Let E be a topological space. Our work will be partially based on subspaces of $\mathcal{BC}(E)$, whose elements are uniformly continuous functions on some subsets of E.
In particular, we shall work on $\mathcal{BCU}(S)$ and $BC_T(S)$, where $T \subset S \subset E$.
The next Lemma will be useful in the sequel.

Lemma 16 *Let (M, d) a metric space. Assume that there are a set K, with $B(E) \subset K$, in a normed space E and an homeomorphism $h : K \to M$ such that both the restrictions $h_{|B(E)}$ and $h^{-1}_{|h(B(E))}$ are lipschitz.*
Then $H : \mathcal{BC}_{B(E)}(K) \to \mathcal{BC}_{h(B(E))}(M)$, defined by

$$H(f) := f \circ h \tag{1.16}$$

is an isometry.

Proof. It is well known that H maps isometrically $\mathcal{BC}(M)$ onto $\mathcal{BC}(K)$. For sake of completeness we shall give a proof.
Let $H^{-1}(g) = f \circ h^{-1}$. Then for every $f \in \mathcal{BC}(M)$ we have

$$H^{-1}(H(f)) = f \circ h^{-1} \circ h = f$$

and for every $g \in \mathcal{BC}(K)$, it follows

$$H(H^{-1}(f)) = f \circ h \circ h^{-1} = g.$$

Being linear, to prove that H and H^{-1} are continuous it is sufficient to show that they are norm preserving.

Let$f \in \mathcal{BC}(M)$. Then

$$\begin{aligned}\|H(f)\|_\infty &= \sup_{x\in K} |f \circ h(x)| \\ &= \sup_{y\in M} |f(h(h^{-1}(y))| = \|f\|_\infty.\end{aligned}$$

We shall show that, for any fixed $f \in \mathcal{BC}_{h(B(E))}(M)$ the map Hf is uniformly continuous on $B(E)$.
To prove the above, we let $\varepsilon > 0$. Since f is uniformly continuous, we may find $\delta_0 > 0$ such that if $w_1, w_2 \in h(B(E))$, with $d(w_1, w_2) < \delta_0$, then

$$|f(w_1) - f(w_2)| < \varepsilon. \tag{1.17}$$

Since h is bilipschitzian on $B(E)$, we may find two constants $l, L > 0$ such that for every $x_1, x_2 \in B(E)$, we have

$$l\|x_1 - x_2\| \leq d(h(x_1), h(x_2)) \leq L\|x_1 - x_2\|. \tag{1.18}$$

If we fix $\delta \leq \frac{\delta_0}{L}$, then for every $x_1, x_2 \in B(E)$, with $\|x_1 - x_2\| < \delta$, we have

$$d(h(x_1), h(x_2)) \leq L\|x_1 - x_2\| < L\delta \leq \delta_0,$$

with $h(x_1), h(x_2) \in h(B(E))$. Thus from 1.17, it follows

$$|H(f)(x_1) - H(f)(x_2)| = |f(h(x_1)) - f(h(x_2))| < \varepsilon.$$

Since from 1.18 it follows that

$$\frac{1}{L} d(w_1, w_2) \leq \|h^{-1}(w_1) - h^{-1}(w_2)\| \leq \frac{1}{l}\, d(w_1, w_2) \tag{1.19}$$

holds for every $w_1, w_2 \in B(E)$ we may prove that $H^{-1}(g)$ is also uniformly continuous on $h(B(E))$ whenever g is in $\mathcal{BC}_{B(E)}(K)$. To show this last, we can replace h and L by h^{-1} and $\frac{1}{l}$ in the proof above. ■

Chapter 2

Retractions in $\mathcal{BC}_{B(E)}(K)$

Since the work of Väth [41] it is known that, in any infinite dimensional Banach space X, for any $k > 6$ there exists a retraction $R : B(X) \to S(X)$ with *Hausdorff norm* $\gamma_X(R) \leq k$, so that the *Wośko constant* $W(X) = 6$. In the setting of infinite dimensional Banach spaces of real continuous functions, according to our knowledge, better estimates of the *Hausdorff norm* and of the *Wośko constant* are obtained only in some Banach space of real continuous functions which domains are *compact* or *finite dimensional*. Precisely:

- for any $u > 0$ there exists a retraction $R : B(C(K)) \to S(C(K))$ with $\gamma_{C(K)}(R) \leq \frac{u+4}{u}$, where K is a convex compact set in a finite dimensional normed space with nonempty interior (see [40],[42]);

- for any $u > 0$ there exists a retraction $R : B(BC([0, \infty])) \to S(BC([0, \infty)))$ with $\gamma_{BC([0,\infty))}(R) \leq \frac{u+4}{u}$ (see [16]). Therefore $W(C(K)) = 1$, in particular $W(C([0, 1])) = 1$, and
$W(BC([0, \infty))) = 1$.

Concerning the estimate of the *lower Hausdorff norm* in [16] it is proved that in the Banach space $BC([0, \infty))$ for any $u > 0$ the same retraction can be chosen with $\underline{\gamma}_{BC([0,\infty))}(R) \geq \frac{1}{u+2}$ and $\gamma_{BC([0,\infty))}(R) \leq \frac{u+4}{u}$. Consequently, as we have already observed, there is a retraction with positive lower Hausdorff and Hausdorff normarbitrarily close to 1.

This chapter is devoted to the study of "Problem 1" and "Problem 2" of the introduction, when we restrict our attention to the setting of infinite di-

mensional Banach spaces of real continuous functions which domains are *not necessarily bounded* or *finite dimensional*, in particular we generalize the results of [42] ,[40] and [16].
The reformulation, in our framework, of the above two problems is the following:

Problem 1: The estimates of the *lower Hausdorff norm* and of the *Hausdorff norm* of a retraction of the closed unit ball of the Banach space $\mathcal{BC}_{B(E)}(K)$ onto its boundary;

Problem 2: The evaluation of the *Wośko constant* $W(\mathcal{BC}_{B(E)}(K))$ of the space $\mathcal{BC}_{B(E)}(K)$.

In Section 2.2 we show that in the space $\mathcal{BC}_{B(E)}$ it is possible to construct a retraction from $B(\mathcal{BC}_{B(E)}(K))$ onto $S(\mathcal{BC}_{B(E)}(K))$ with positive lower Hausdorff norm andHausdorff norm arbitrarily close to 1. Moreover, by means of some "nice " examples contained in Section 2.3, we give the main result of this chapter: *precise formulas for the lower Hausdorff norms and for the Hausdorff norms of retractions we construct in the space* $\mathcal{BC}_{B(E)}(K)$. Infact we prove that for any $u > 0$ there is a retraction $R_u : B(\mathcal{BC}_{B(E)}(K)) \to S(\mathcal{BC}_{B(E)}(K))$ such that

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \begin{cases} \frac{1}{2}, & \text{if } u \leq 4 \\ \frac{2}{u}, & \text{if } u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{BC}_{B(E)}} = \frac{u+8}{u}.$$

Therefore the *Wosko constant* $W(\mathcal{BC}_{B(E)}(K)) = 1$.

Since the Hausdorff measure of noncompactness of a set is invariant under isometries, we obtain the same result in the Banach space $\mathcal{BC}_{h(B(E))}(M)$, where M is a metric space homeomorphic to K under a map h which is bilipschitz when restricted to $B(E)$. Indeed in this case the space $\mathcal{BC}_{h(B(E))}(M)$ is isometric to the space $\mathcal{BC}_{B(E)}(K)$ (see Lemma 16 of Chapter 1)

We observe that all our retractions are proper maps.

Finally we observe that the Example 30 of Section 2.3 shows that the construction of this chapter does not work in the case of the Banach space

$\mathcal{BC}(E)$, if E is an infinite dimensional normed space, or, as is easy to see, in the case of the Banach space $\mathcal{C}(K)$, when K is an infinite dimensional convex compact set in a normed space. The results of this chapter are contained in [18]

2.1 Preliminaries

Let $(E, \|\cdot\|)$ be a normed space and K a subset of E containing the closed unit ball $B(E)$. Now for each $a \in [0,1]$ we introduce the maps $\lambda_a, \lambda^a : E \to E$ by

$$\lambda_a(x) = \begin{cases} \frac{2}{1+a}\, x, & \text{if } \|x\| \leq \frac{1+a}{2} \\ \frac{x}{\|x\|}, & \text{if } \frac{1+a}{2} < \|x\| \leq 1 \\ x, & \text{if } \|x\| > 1 \end{cases}$$

and

$$\lambda^a(x) = \begin{cases} \frac{1+a}{2}\, x, & \text{if } \|x\| \leq 1 \\ x, & \text{if } \|x\| > 1. \end{cases}$$

Moreover for $f \in \mathcal{BC}_{B(E)}(K)$, we set

$$\begin{aligned} A_f &:= \left\{ f \circ (\lambda_a)_{|K} : a \in [0,1] \right\}, \\ A^f &:= \left\{ f \circ (\lambda^a)_{|K} : a \in [0,1] \right\}. \end{aligned}$$

Observe that $A_f \subseteq \mathcal{BC}_{B(E)}(K)$ and $A^f \subseteq \mathcal{L}_\infty(K)$, where $\mathcal{L}_\infty(K)$ is the space of all real essentially bounded functions defined on K.
The following lemmas are technical and the proofs are straightforward.

Lemma 17 *Let $a, b \in [0,1]$. Then for all $x, y \in E$*

(a) $\|\lambda_a(x) - \lambda_a(y)\| \leq \frac{4}{1+a} \|x - y\|$,

(b) $\|\lambda_a(x) - \lambda_b(x)\| \leq |a - b|$,

(c) $\left\|\lambda^a(x) - \lambda^b(x)\right\| \leq \frac{1}{2} |a - b|$.

Proof. (a) Let $x, y \in E$. Observe that

$$\left\| \frac{x}{\alpha} - \frac{y}{\beta} \right\| \leq \frac{2}{\beta} \|x - y\|, \tag{2.1}$$

for $\alpha, \beta \in (0, \infty)$ with $\|x\| \leq \alpha \leq \beta \leq \|y\|$.
Now if $\|x\|, \|y\| \leq \frac{1+a}{2}$ then $\|\lambda_a(x) - \lambda_a(y)\| = \frac{2}{1+a}\|x-y\|$. If $\|x\|, \|y\| > 1$ then $\|\lambda_a(x) - \lambda_a(y)\| = \|x-y\|$. Moreover, using (2.1) it is easy to check the following implications:

$$\begin{aligned}
\|x\| \leq \frac{1+a}{2},\ \frac{1+a}{2} \leq \|y\| \leq 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq \frac{4}{1+a}\|x-y\|, \\
\|x\| \leq \frac{1+a}{2},\ \|y\| > 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq 2\|x-y\|, \\
\frac{1+a}{2} \leq \|x\| \leq 1,\ \|y\| > 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq 2\|x-y\|, \\
\frac{1+a}{2} \leq \|x\| \leq 1,\ \frac{1+a}{2} \leq \|y\| \leq 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq \frac{4}{1+a}\|x-y\|.
\end{aligned}$$

(b) and (c) can be easily checked. ■

Lemma 18 *Let (a_m) be a sequence of elements of $[0,1]$ converging to a. Then $\|\lambda_{a_m} - \lambda_a\|_\infty \to 0$ and $\|\lambda^{a_m} - \lambda^a\|_\infty \to 0$.*

Proof. The lemma follows immediately by (b) and (c) of Lemma 17. ■

Lemma 19 *For all $f \in \mathcal{BC}_{B(E)}(K)$, the sets A_f and A^f are compact in $\mathcal{BC}_{B(E)}(K)$ and in $\mathcal{L}_\infty(K)$, respectively.*

Proof. Let (a_m) be a sequence of elements of $[0,1]$ converging to a. In order to prove the compactness of the set A_f we will show that

$$\left\| f \circ (\lambda_{a_m})_{|K} - f \circ (\lambda_a)_{|K} \right\|_\infty \to 0.$$

Let $\varepsilon > 0$. Since f is uniformly continuous on $B(E)$, there is $\delta > 0$ such that

$$|f(y) - f(z)| \leq \varepsilon, \tag{2.2}$$

for all $y, z \in B(E)$ with $\|y - z\| \leq \delta$. By Lemma 18 there is $\bar{m} \in \mathbb{N}$ such that

$$\|\lambda_{a_m}(x) - \lambda_a(x)\| \leq \delta, \tag{2.3}$$

for all $m \geq \overline{m}$ and $x \in B(E)$. By (2.2) and (2.3) we have

$$\left\| f \circ (\lambda_{a_m})_{|K} - f \circ (\lambda_a)_{|K} \right\|_\infty = \max_{x \in B(E)} |f(\lambda_{a_m}(x)) - f(\lambda_a(x))| \leq \varepsilon,$$

for all $m \geq \overline{m}$. Hence the thesis. The proof of the compactness of the set A^f in $\mathcal{L}_\infty(K)$ is similar. ■

2.2 Retractions in the space $\mathcal{BC}_{B(E)}(K)$.

Let K be a set in a normed space E such that $B(E) \subseteq K$. We begin by defining a map $Q : B(\mathcal{BC}_{B(E)}(K)) \to B(\mathcal{BC}_{B(E)}(K))$ as follows

$$Qf(x) := \begin{cases} f\left(\frac{2}{1+\|f\|_\infty} x\right), & \text{if } \|x\| \leq \frac{1+\|f\|_\infty}{2} \\ f\left(\frac{x}{\|x\|}\right), & \text{if } \frac{1+a}{2} < \|x\| \leq 1 \\ f(x), & \text{if } \|x\| > 1 \end{cases}$$

Then $\|Qf\|_\infty = \|f\|_\infty$ for all $f \in B(\mathcal{BC}_{B(E)}(K))$ and $Qf = f$ for all $f \in S(\mathcal{BC}_{B(E)}(K))$. Observe that the mapping Q can be written as

$$Qf = f \circ (\lambda_{\|f\|_\infty})_{|K}. \tag{2.4}$$

Proposition 20 *The map Q is continuous.*

Proof. Let (f_m) be a sequence of elements of $B(\mathcal{BC}_{B(E)}(K))$ such that $\|f_m - f\|_\infty \to 0$. Let $\varepsilon > 0$. Then there exists $m_1 \in \mathbb{N}$ such that $\|f_m - f\|_\infty \leq \frac{\varepsilon}{2}$ for all $m \geq m_1$. By the uniform continuity of f on $B(E)$ there is $\delta > 0$ such that if $\|x - y\| \leq \delta$ then

$$|f(x) - f(y)| \leq \frac{\varepsilon}{2},$$

for all $x, y \in B(E)$. Since $\|f_m\|_\infty \to \|f\|_\infty$, by Lemma 18 we have that $\left\|\lambda_{\|f_m\|_\infty} - \lambda_{\|f\|_\infty}\right\|_\infty \to 0$. Hence there is $m_2 \in \mathbb{N}$ such that

$$\left\|\lambda_{\|f_m\|_\infty} - \lambda_{\|f\|_\infty}\right\|_\infty \leq \delta,$$

for all $m \geq m_2$. Therefore

$$\left| f\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f\|_\infty}\right)_{|K}(x) \right| \leq \frac{\varepsilon}{2},$$

for all $x \in K$ and $m \geq m_2$. Then, for any $x \in K$ and $m \geq \max\{m_1, m_2\}$, we have

$$\begin{aligned}
& \left| f_m\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f\|_\infty}\right)(x) \right| \\
\leq & \left| f_m\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) \right| \\
+ & \left| f\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f\|_\infty}\right)_{|K}(x) \right| \\
\leq & \|f_m - f\|_\infty + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

So we obtain $\|Qf_m - Qf\|_\infty \to 0$. ■

By the following proposition we give lower and upper estimates of the Hausdorff measure of noncompactness of QA for a set A in $B(\mathcal{BC}_{B(E)}(K))$.

Proposition 21 *Let A be a subset of $B(\mathcal{BC}_{B(E)}(K))$. Then*

$$\frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A).$$

Proof. Let $A \subseteq B(\mathcal{BC}_{B(E)}(K))$. First we prove $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A)$. Fix $\alpha > \gamma_{\mathcal{BC}_{B(E)}(K)}(A)$ and let $\{f_1, ..., f_n\}$ be an α-net of A in $\mathcal{BC}_{B(E)}(K)$. By Proposition 19 the set $\cup_{i=1}^n A_{f_i}$ is compact in $\mathcal{BC}_{B(E)}(K)$. Hence given $\delta > 0$ we can choose a δ-net $\{g_1, ..., g_m\}$ of $\cup_{i=1}^n A_{f_i}$ in $\mathcal{BC}_{B(E)}(K)$.
We show that $\{g_1, ..., g_m\}$ is a $(\alpha + \delta)$-net of QA in $\mathcal{BC}_{B(E)}(K)$. To this end for $g \in A$ let $f \in A$ such that $Qf = g$ and fix $i \in \{1, ..., n\}$ such that $\|f - f_i\|_\infty \leq \alpha$. Since $f_i \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} \in A_{f_i}$ we can find $j \in \{1, ..., m\}$ such that $\left\| \left((f_i \circ \lambda_{\|f\|_\infty}\right)_{|K} - g_j \right\|_\infty \leq \delta$. Then

$$\begin{aligned}
& \|g_j - Qf\|_\infty \\
\leq & \left\| g_j - f_i \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} \right\|_\infty + \left\| f_i \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} - f \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} \right\|_\infty \\
\leq & \delta + \left\| (f_i - f) \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} \right\|_\infty \leq \delta + \varepsilon.
\end{aligned}$$

Therefore $\gamma_{\mathcal{BC}_{B(E)}(K)}(K)(QA) \leq \alpha + \delta$, so $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A)$. We now prove $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \geq \frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A)$. Fix $\beta > \gamma_{\mathcal{BC}_{B(E)}(K)}(QA)$ and let $\{h_1, ..., h_s\}$ be a β-net for QA in $\mathcal{BC}_{B(E)}(K)$. By Proposition 19 the set $\cup_{i=1}^{s} A^{h_i}$ is compact in $\mathcal{L}_\infty(K)$. Therefore given $\delta > 0$ we can choose a δ-net $\{p_1, ..., p_k\}$ for $\cup_{i=1}^{s} A^{h_i}$ in $\mathcal{L}_\infty(K)$. We now show that $\{p_1, ..., p_k\}$ is a $(\beta + \delta)$-net for A in $\mathcal{L}_\infty(K)$. Let $f \in A$. Fix $l \in \{1, ..., s\}$ such that $\|Qf_l - h_l\|_\infty \leq \beta$. Since $h_l \circ (\lambda^{\|f\|_\infty})_{|K} \in A^{h_l}$ we can find $m \in \{1, ..., k\}$ such that $\left\|h_l \circ (\lambda^{\|f\|_\infty})_{|K} - p_m\right\|_\infty \leq \delta$. Then

$$
\begin{aligned}
& \|f - p_m\|_\infty \\
\leq \; & \left\|f - h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right\|_\infty + \left\|h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K} - p_m\right\|_\infty \\
\leq \; & \left\|\left(f - h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right) \circ \left(\varphi_{\|f\|_\infty}\right)_{|K}\right\|_\infty + \delta \\
= \; & \sup_{\left\{x \in K : \|x\| \in \left[0, \frac{1+\|f\|_\infty}{2}\right] \cup (1,\infty)\right\}} \left|\left(\left(f - h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right) \circ \left(\lambda_{\|f\|_\infty}\right)_{|K}\right)(x)\right| + \delta \\
= \; & \sup_{\left\{x \in K : \|x\| \in \left[0, \frac{1+\|f\|_\infty}{2}\right] \cup (1,\infty)\right\}} \left|\left(f \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right)(x) - h_l(x)\right| + \delta \\
\leq \; & \|Qf - h_l\|_\infty + \delta \leq \beta + \delta.
\end{aligned}
$$

Therefore $\gamma_{\mathcal{L}_\infty(K)}(A) \leq \beta + \delta$, so we obtain $\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq 2(\beta + \delta)$ and consequently $\frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(QA)$. ■

Remark 22 Observe that for all $f \in \mathcal{BCU}(B(E))$ the set A^f is compact in $\mathcal{BCU}(B(E))$. Therefore

$$\gamma_{\mathcal{BCU}(B(E))}(QA) = \gamma_{\mathcal{BCU}(B(E))}(A),$$

for all $A \subseteq B(\mathcal{BCU}(B(E)))$.

Next, for a given $u \in (0, +\infty)$, we define a map $P_u : B(\mathcal{BC}_{B(E)}(K)) \to \mathcal{BC}_{B(E)}(K)$ by

$$
(P_u f)(x) = \begin{cases} 0, & \text{if } \|x\| \leq \frac{1+\|f\|_\infty}{2} \text{ or } \|x\| > 1, \\ u\left(\|x\| - \frac{1+\|f\|_\infty}{2}\right), & \text{if } \frac{1+\|f\|_\infty}{2} \leq \|x\| \leq \frac{3+\|f\|_\infty}{4}, \\ -u(\|x\| - 1), & \text{if } \frac{3+\|f\|_\infty}{4} \leq \|x\| \leq 1. \end{cases}
$$

Proposition 23 *The map P_u is compact, i.e., P_u is continuous and $P_uB(\mathcal{BC}_{B(E)}(K))$ is relatively compact.*

Proof. First we prove that the map P_u is continuous. Observe that if $f, g \in B(\mathcal{BC}_{B(E)}(K))$ with $\|f\|_\infty \le \|g\|_\infty$ and $\frac{1+\|g\|_\infty}{2} \le \frac{3+\|f\|_\infty}{4}$ we have

$$\|P_u f - P_u g\|_\infty = u\frac{\|g\|_\infty - \|f\|_\infty}{2}\,. \tag{2.5}$$

Let now (f_m) be a sequence in $B(\mathcal{BC}_{B(E)}(K))$ converging to f. Then $\|f_m\|_\infty \to \|f\|_\infty$. Moreover if $\|f\|_\infty = 1$ we have

$$\|P_u f_m - P_u f\|_\infty = \frac{u}{4}(1 - \|f_m\|). \tag{2.6}$$

On the other hand, if $\|f\|_\infty \neq 1$ there is $\overline{m} \in \mathbb{N}$ such that for all $m \ge \overline{m}$

$$\|f_m\|_\infty \le \|f\|_\infty \Longrightarrow \frac{1+\|f\|_\infty}{2} \le \frac{3+\|f_m\|_\infty}{4},$$

and

$$\|f\|_\infty \le \|f_m\|_\infty \Longrightarrow \frac{1+\|f_m\|_\infty}{2} \le \frac{3+\|f\|_\infty}{4}.$$

Thus by (2.5), if $\|f\|_\infty \neq 1$, it follows that

$$\|P_u f_m - P_u f\|_\infty = \frac{u}{2}\left|\|f_m\|_\infty - \|f\|_\infty\right|. \tag{2.7}$$

for all $m \ge \overline{m}$. By (2.6) and (2.7) it follows $\|P_u f_m - P_u f\|_\infty \to 0$. To complete the proof it remains to show that $P_uB(\mathcal{BC}_{B(E)}(K))$ is compact. Let (g_m) be a sequence in $P_uB(\mathcal{BC}_{B(E)}(K))$ and let (f_m) be a sequence in $B(\mathcal{BC}_{B(E)}(K))$ with $P_u f_m = g_m$. Without loss of generality we can assume that $\|f_m\|_\infty \to \|f\|_\infty$ where $f \in B(\mathcal{BC}_{B(E)}(K))$ and that (2.7) holds for all m. Then

$$\|g_m - P_u f\|_\infty = \|P_u f_m - P_u f\|_\infty = \frac{u}{2}\left|\|f_m\|_\infty - \|f\|_\infty\right| \to 0,$$

which completes the proof. ■

Now we are in the position to prove our main result.

Theorem 24 *Let K be a set in a normed space E such that $B(E) \subseteq K$. For any $u > 0$ there is a retraction $R_u : B\left(\mathcal{BC}_{B(E)}(K)\right) \to S\left(\mathcal{BC}_{B(E)}(K)\right)$ such that*

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \begin{cases} \frac{1}{2}, & \text{if } u \leq 4 \\ \frac{2}{u}, & \text{if } u > 4. \end{cases}$$

and

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{u+8}{u}. \tag{2.8}$$

In particular we have that $W(\mathcal{BC}_{B(E)}(K)) = 1$.

Proof. Let $u \in (0, +\infty)$. Define a map $T_u : B(\mathcal{BC}_{B(E)}(K)) \to \mathcal{BC}_{B(E)}(K)$ by $T_u f = Qf + P_u f$. Since P_u is compact, Proposition 21 implies

$$\frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(T_u A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A), \tag{2.9}$$

for any $A \subseteq B(\mathcal{BC}_{B(E)}(K))$. We have $T_u f = f$ for all $f \in S(\mathcal{BC}_{B(E)}(K))$ and for any $f \in B(\mathcal{BC}_{B(E)}(K))$ we find

$$\begin{aligned} & \|T_u f\|_\infty \\ = \; & \max\left\{ \sup_{\left\{x \in K, \|x\| \in \left[0, \frac{1+\|f\|_\infty}{2}\right] \cup (1,\infty)\right\}} |(T_u f)(x)|, \sup_{\left\{x \in K : \|x\| \in \left]\frac{1+\|f\|_\infty}{2}, 1\right]\right\}} |(T_u f)(x)| \right\} \\ \geq \; & \min_{f \in B(\mathcal{BC}_{B(E)}(K))} \max\left\{ \|f\|_\infty, \frac{u}{4}(1 - \|f\|_\infty) - \|f\|_\infty \right\} \geq \frac{u}{u+8}. \end{aligned}$$

We define a retraction $R_u : B(\mathcal{BC}_{B(E)}(K)) \to S(\mathcal{BC}_{B(E)}(K))$ by setting

$$R_u f = \frac{T_u f}{\|T_u f\|_\infty}.$$

Since $\|T_u f\|_\infty \geq \frac{u}{u+8}$ holds, the definition of R_u implies that

$$R_u A \subseteq \left[0, \frac{u+8}{u}\right] \cdot (Q + P_u)A.$$

Therefore using the monotonicity property of the Hausdorff measure of non-compactness and (2.9) we get

$$\begin{aligned}
& \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A) \\
\leq\ & \gamma_{\mathcal{BC}_{B(E)}(K)}\left(\left[0, \frac{u+8}{u}\right] \cdot (Q+P_u)A\right) = \frac{u+8}{u}\gamma_{\mathcal{BC}_{B(E)}(K)}\left((Q+P_u)\,A\right) \\
=\ & \frac{u+8}{u}\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \frac{u+8}{u}\gamma_{\mathcal{BC}_{B(E)}(K)}(A).
\end{aligned}$$

The latter inequality together with (2.13) of the Example 31 implies

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{u+8}{u}. \tag{2.10}$$

On the other hand,

$$\|T_u f\|_\infty \leq \max\left\{1, \frac{u}{4}\right\} = \begin{cases} 1, & \text{if } u \leq 4, \\ \frac{u}{4}, & \text{if } u > 4. \end{cases}$$

Fix $u > 4$. We have $(Q+P_u)A \subseteq \left[0, \frac{u}{4}\right] \cdot R_u A$. Using the monotonicity property of the Hausdorff measure of noncompactness and (2.9), we obtain

$$\begin{aligned}
\gamma_{\mathcal{BC}_{B(E)}(K)}(A) &\leq 2\gamma_{\mathcal{BC}_{B(E)}(K)}((Q+P_u)A) \\
&\leq 2\gamma_{\mathcal{BC}_{B(E)}(K)}\left(\left[0, \frac{u}{4}\right] \cdot R_u A\right) = \frac{u}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A).
\end{aligned}$$

Then for all $u > 4$ we have

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{2}{u}. \tag{2.11}$$

Now let $u \leq 4$. We have $(Q+P_u)A \subseteq [0,1] \cdot R_u A$. So $\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq 2$ $\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A)$ hence

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{1}{2}. \tag{2.12}$$

By Example 32 we obtain

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \begin{cases} \frac{1}{2}, & \text{if } u \leq 4 \\ \frac{2}{u}, & \text{if } u > 4. \end{cases}$$

Finally by (2.10), since $\lim_{u\to\infty}\frac{u+8}{u}=1$, we have $W(\mathcal{BC}_{B(E)}(K))=1$. ∎

The Example 30 of Section 2.3 shows that, given a set K in an infinite dimensional normed space E with $B(E)\subseteq K$, the map Q is not anymore continuous when considered from $B(\mathcal{BC}(K))$ into itself. Therefore our construction does not work in the case of the Banach space $\mathcal{BC}(K)$.

Since the Hausdorff measure of noncompactness of a set is invariant under isometries, the following corollary enlarges the class of spaces for which Theorem 24 holds.

Corollary 25 *Let M be a metric space. Assume that there are a set K in a normed space E containing $B(E)$ and an homeomorphism $h: K\to M$ such that both the restrictions $h_{/B(E)}$ and $h^{-1}_{/h(B(E))}$ are lipschitz, where h^{-1} is the inverse homeomorphism of h. Then for any $u>0$ there is a retraction $R_u: B\left(\mathcal{BC}_{h(B(E))}(M)\right)\to S\left(\mathcal{BC}_{h(B(E))}(M)\right)$ such that*

$$\underline{\gamma}_{\mathcal{BC}_{h(B(E))}(M)}(R_u)=\begin{cases}\frac{1}{2}, & \text{if } u\leq 4\\ \frac{2}{u}, & \text{if } u>4\end{cases}$$

and $\gamma_{\mathcal{BC}_{h(B(E))}(M)}(R_u)=\frac{u+8}{u}$. In particular, we have $W(BC_{h(B(E))}(M))=1$.

Proof. It is enough to observe that $i:\mathcal{BC}_{B(E)}(K)\to\mathcal{BC}_{h(B(E))}(M)$ defined by $i(f)=f\circ h$ is an isometry. ∎

Corollary 26 *Let $x_0\in E$, $r>0$ and K be a set in E such that $B_{x_0,r}(E):=\{x\in E:\|x-x_0\|\leq r\}\subseteq K$. For any $u>0$ there is a retraction $R_u: B(\mathcal{BC}_{B_{x_0,r}(E)}(K))\to S(\mathcal{BC}_{B_{\{x_0,r\}}(E)}(K))$ such that*

$$\underline{\gamma}_{\mathcal{BC}_{B_{x_0,r}(E)}(K)}(R_u)=\begin{cases}\frac{1}{2}, & \text{if } u\leq 4,\\ \frac{2}{u}, & \text{if } u>4,\end{cases}$$

and $\gamma_{\mathcal{BC}_{B_{x_0,r}(E)}(K)}(R_u)=\frac{u+8}{u}$.

Proof. It follows by Corollary 25 when we consider $h(x)=\frac{1}{r}(x-x_0)$ for all $x\in K$. ∎

Remark 27 As a particular case of Theorem 24, if E is a finite dimensional normed space, then for any $u > 0$ there is a retraction $R_u : B(\mathcal{BC}(E)) \to S(\mathcal{BC}(E))$ such that

$$\underline{\gamma}_{\mathcal{BC}(E)}(R_u) = \begin{cases} \frac{1}{2}, & \text{if } u \leq 4, \\ \frac{2}{u}, & \text{if } u > 4, \end{cases}$$

and $\gamma_{\mathcal{BC}(E)}(R_u) = \frac{u+8}{u}$.
This result with Corollary 26 yields $W(\mathcal{C}(K)) = 1$ ([40, Theorem 10]) when K is a convex compact set in E with nonempty interior, and also yields $W(\mathcal{BC}(\mathbb{R})) = 1$ ([16, Theorem 2.4]).

The following corollary covers the case of the space $\mathcal{BCU}(E))$. By repeating the proof of Theorem 24, taking into account Remark 22, and by slight modifications of Examples 31 and 32 we have a different evaluation of $\underline{\gamma}_{\mathcal{BCU}(E))}(R_u)$.

Corollary 28 *For any $u > 0$ there is a retraction $R_u : B\left(\mathcal{BCU}(E)\right) \to S(\mathcal{BCU}(E))$ such that*

$$\underline{\gamma}_{\mathcal{BCU}(E)}(R_u) = \begin{cases} 1, & \text{if } u \leq 4 \\ \frac{4}{u}, & \text{if } u > 4 \end{cases}$$

and $\gamma_{\mathcal{BCU}(E)}(R_u) = \frac{u+8}{u}$.

We point out that for all $u \in (0, +\infty)$ the retractions R_u we have considered are all proper maps.

At last, we observe that from Remark 5 of Chapter 1 the following holds:

Corollary 29 *For every $u > 0$, the retraction R_u defined in Theorem 24 satisfies,*

$$\underline{\psi}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \begin{cases} \frac{1}{4}, & \text{if } u \leq 4 \\ \frac{1}{u}, & \text{if } u > 4 \end{cases}$$

and

$$\psi_{\mathcal{BC}_{B(E)}(K)}(R_u) \leq \frac{2u + 16}{u},$$

for $\psi \in \{\alpha, \beta\}$. Thus

$$W_\alpha(\mathcal{BC}_{B(E)}(K)) \leq 2$$

and

$$W_\beta(\mathcal{BC}_{B(E)}(K)) \leq 2.$$

2.3 Examples

Example 30 Let (x_k) be a sequence of elements of $B(\mathcal{BC}(K))$ such that $\|x_k\| = \frac{1}{2}$ $(k = 1, 2, ...)$ and $\|x_i - x_j\| \geq \frac{1}{4}$ for all $i, j \in \mathbb{N}$ with $i \neq j$. Moreover let (a_k) be a monotone increasing sequence of elements of $(0, 1)$ such that $a_k \to 1$. Set $y_k = \frac{2}{1+a_k} x_k$ $(k = 1, 2, ...)$. We have that $\|y_k\| = \frac{1}{1+a_k} < 1$ and $\|x_k\| = \frac{1}{2} < \frac{1}{1+a_k}$ $(k = 1, 2, ...)$. Set $S = \{x_k : k = 1, 2, ...\} \cup \{y_k : k = 1, 2, ...\}$. Then S is a closed subset of K and the map $f : S \to \mathbb{R}$ defined by $f(x_k) = 1$ and $f(y_k) = 0$ $(k = 1, 2, ...)$ is continuous. By the Dugundji's theorem there is a continuous extension $\widetilde{f} : K \to [0, 1]$ of f. We have that $\|f\|_\infty = \left\|\widetilde{f}\right\|_\infty = 1$ and $\widetilde{f}$ is not uniformly continuous on K. In fact, fixed $\varepsilon \in (0, 1)$, since $\|x_k - y_k\| \to 0$, for all $\delta > 0$ there is $\overline{k}$ such that $\|x_{\overline{k}} - y_{\overline{k}}\| \leq \delta$ and $|f(x_{\overline{k}}) - f(y_{\overline{k}})| = 1 > \varepsilon$. Now we show that the map Q is not continuous. Put $\widetilde{f}_k = a_k \widetilde{f}$ $(k = 1, 2, ...)$. Then $\left\|\widetilde{f}_k - \widetilde{f}\right\|_\infty \to 0$ but $\left\|Q\widetilde{f}_k - Q\widetilde{f}\right\|_\infty = 1$ $(k = 1, 2, ...)$. In fact

$$\begin{aligned}
1 &\geq \left\|Q\widetilde{f}_k - Q\widetilde{f}\right\|_\infty \\
&\geq \left|Q\widetilde{f}_k(x_k) - Q\widetilde{f}(x_k)\right| = \left|\widetilde{f}_k\left(\widetilde{\lambda}_{\|f_k\|_\infty}(x_k)\right) - \widetilde{f}\left(\widetilde{\lambda}_{\|f\|_\infty}(x_k)\right)\right| \\
&= \left|\widetilde{f}_k\left(\widetilde{\lambda}_{a_k}(x_k)\right) - \widetilde{f}(x_k)\right| = \left|a_k\widetilde{f}\left(\frac{2}{1+a_k}x_k\right) - \widetilde{f}(x_k)\right| = 1.
\end{aligned}$$

In connection with Theorem 24 we have the following Examples 31 and 32.

Example 31 Let K be a set in a normed space E such that $B(E) \subseteq K$. Define the maps $f_n : K \to \mathbb{R}$ $(n = 3, 4, ...)$ by

$$f_n(x) = \begin{cases} \frac{u}{u+8}, & \text{if } \|x\| < \frac{1}{2} - \frac{1}{n}, \\ -n\frac{u}{u+8}\left(\|x\| - \frac{1}{2}\right), & \text{if } \frac{1}{2} - \frac{1}{n} \leq \|x\| < \frac{1}{2} + \frac{1}{n}, \\ -\frac{u}{u+8}, & \text{if } \|x\| \geq \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Then the following are the expression for Qf_n and, given $u > 0$, that for P_uf_n

respectively

$$Qf_n(x) = \begin{cases} \frac{u}{u+8}, & \text{if } \|x\| < \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} - \frac{1}{n}), \\ -2n\frac{\frac{u}{u+8}}{1+\frac{u}{u+8}}\left(\|x\| - \frac{1+\frac{u}{u+8}}{2}\frac{1}{2}\right), & \text{if } \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} - \frac{1}{n}) \leq \|x\| < \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} + \frac{1}{n}), \\ -\frac{u}{u+8}, & \text{if } \|x\| \geq \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} + \frac{1}{n}), \end{cases}$$

$$(P_u f_n)(x) = \begin{cases} 0, & \text{if } \|x\| \leq \frac{1+\frac{u}{u+8}}{2} \text{ or } \|x\| > 1, \\ u\left(\|x\| - \frac{1+\frac{u}{u+8}}{2}\right), & \text{if } \frac{1+\frac{u}{u+8}}{2} \leq \|x\| < \frac{3+\frac{u}{u+8}}{4}, \\ -u\left(\|x\| - 1\right), & \text{if } \frac{3+\frac{u}{u+8}}{4} \leq \|x\| \leq 1. \end{cases}$$

Hence we obtain

$$\|Qf_n + P_u f_n\|_\infty = \max\left\{\|f_n\|_\infty, \frac{u}{4}(1 - \|f_n\|) - \|f_n\|\right\} = \frac{u}{u+8}.$$

Setting $A = \{f_n : n \geq 3\}$, we have

$$\begin{aligned} R_u A &= \{R_u f_n : n \geq 3\} \\ &= \left\{\frac{Qf_n + P_u f_n}{\|Qf_n + P_u f_n\|_\infty} : n \geq 3\right\} = \frac{u+8}{u}(Q + P_u)A, \end{aligned}$$

and

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u(A)) = \frac{u+8}{u}\gamma_{\mathcal{BC}_{B(E)}(K)}(A). \tag{2.13}$$

Example 32 Let K be a set in a normed space E. Without loss of generality we may assume $B_2(E) \subseteq K$. Define the maps $f_{c,n} : K \to \mathbb{R}$ $(n = 3, 4, ...)$ by

$$f_{c,n}(x) = \begin{cases} -c, & \text{if } \|x\| < 1 - \frac{1}{n}, \\ nc(\|x\| - 1), & \text{if } 1 - \frac{1}{n} \leq \|x\| < 1 + \frac{1}{n}, \\ c, & \text{if } \|x\| \geq 1 + \frac{1}{n}. \end{cases}$$

We have that

$$Qf_{c,n}(x) = \begin{cases} -c, & \text{if } \|x\| < \frac{1+c}{2}(1-\frac{1}{n}), \\ 2n\frac{c}{1+c}(\|x\| - \frac{1+c}{2}), & \text{if } \frac{1+c}{2}(1-\frac{1}{n}) \leq \|x\| < \frac{1+c}{2}, \\ 0, & \text{if } \frac{1+c}{2} \leq \|x\| < 1, \\ nc(\|x\|-1), & \text{if } 1 \leq \|x\| < 1+\frac{1}{n}, \\ c, & \text{if } \|x\| \geq 1+\frac{1}{n}. \end{cases}$$

Set $A_c = \{f_{c,n} : n \geq 3\}$. Then $\gamma_{\mathcal{BC}_{B(E)}(K)}(A_c) = c$ and $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA_c) = \frac{c}{2}$. Let $u > 0$. We find

$$P_u f_n(x) = \begin{cases} 0, & \text{if } \|x\| < \frac{1+c}{2} \text{ or } \|x\| > 1, \\ u(\|x\| - \frac{1+c}{2}), & \text{if } \frac{1+c}{2} \leq \|x\| < \frac{3+c}{4}, \\ -u(\|x\|-1), & \text{if } \frac{3+c}{4} \leq \|x\| < 1. \end{cases}$$

Thus $\|P_u f_n\|_\infty = u\frac{1-c}{4}$. Moreover if $c \leq \frac{u}{u+4}$ then $\|Qf_n + P_u f_n\|_\infty = \max\left\{c, u\frac{1-c}{4}\right\} = u\frac{1-c}{4}$. Hence

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_c) = \frac{4}{u(1-c)}\gamma_{\mathcal{BC}_{B(E)}(K)}(QA_c) = \frac{2}{u(1-c)}\gamma_{\mathcal{BC}_{B(E)}(K)}(A_c).$$

Now by (2.11), $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{2}{u}$. Suppose $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{2}{u} + \sigma$ with $\sigma > 0$. Fix $\overline{c}$ such that $\frac{2}{u(1-\overline{c})} < \frac{2}{u} + \sigma$, then

$$\begin{aligned} \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}) &= \frac{2}{u(1-\overline{c})}\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \\ &< \left(\frac{2}{u} + \sigma\right)\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}), \end{aligned}$$

which is a contradiction. So that $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{2}{u}$. For each $u \leq 4$, by (2.12) we have $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{1}{2}$. Suppose $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{1}{2} + \sigma$ with $\sigma > 0$. Fix $\overline{c}$ such that $\frac{2}{u(1-\overline{c})} < \frac{1}{2} + \sigma$, then

$$\begin{aligned} \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}) &= \frac{2}{u(1-\overline{c})}\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \\ &< (\frac{1}{2} + \sigma)\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}), \end{aligned}$$

which is a contradiction so that $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{1}{2}$.

Chapter 3

Lower Hausdorff norm and Hausdorff norm of retractions in $\mathcal{C}(P)$

3.1 Introduction.

In this final chapter we consider the problems of evaluating the *lower Hausdorff norm* and the *Hausdorff norm* of a retraction of the closed unit ball of the Banach space $\mathcal{C}(P)$ onto its boundary, and that of the evaluation of the *Wośko constant* $W(\mathcal{C}(P))$. Section 3.3 is devoted to the construction of retractions from $B(\mathcal{C}(P))$ onto $S(\mathcal{C}(P))$. As in the previous chapter, by means of the examples contained in Section 3.4, we obtain explicit formulas for the *lower Hausdorff norms* and the *Hausdorff norms* of a such retractions. Our main result is the following: For any $u > 0$ there is a retraction $R_u : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ such that

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, & \text{if } u \leq 4 \\ \frac{4}{u}, & \text{if } u > 4 \end{cases}$$

and $\gamma_{\mathcal{C}(P)}=\frac{u+8}{u}$. Hence the *Wośko constant* $W(\mathcal{C}(P)) = 1$.

Let K be a topological space homeomorphic to the Hilbert cube P. Then the Banach spaces $\mathcal{C}(K)$and $\mathcal{C}(P)$ are isometric. Therefore, since the Hausdorff measure of noncompactness of a set is invariant under isometries, the previous result holds also in the space $\mathcal{C}(K)$. We observe that all our retractions are proper maps.

We conclude by observing that this chapter is in a certain sense complementary to the previous one. On the one hand we have observed that the construction we have made in Chapter 2 fails in the space $\mathcal{C}(K)$ when K is a convex compact infinite dimensional set in a normed space, but now the problem has been solved since such a K is homeomorphic to the Hilbert cube P. On the other hand Example 44 of Section 3.4 shows that the construction of this chapter does not work in the case of the Banach space $\mathcal{BC}(l_2)$ and in the case of the Banach spaces considered in the Chapter 2. Moreover it can be easily seen that this construction does not hold in the case of the Banach spaces considered in the previous chapter.

The results of this chapter are contained in [17]

3.2 Preliminaries.

Let l_2 be the real Hilbert space, with the usual norm $\|\cdot\|_2$ and canonical basis (e_n). Denote by

$$P = \{x = (x_n) \in l_2 : |x_n| \leq \frac{1}{n} \ \ (n = 1, 2, ...)\}$$

the Hilbert cube. We consider the Banach space $\mathcal{C}(P)$ of all real continuous function defined on P, endowed with the sup norm $\|\cdot\|_\infty$.

For $a \in [0, 1]$ define the maps φ_a and $\varphi^a : P \to P$ by

$$\varphi_a(t) := (\lambda_a(t_1), t_2, ..., t_n, ...),$$

$$\varphi^a(t) := (\lambda^a(t_1), t_2, ..., t_n, ...),$$

where

$$\lambda_a(t_1) := \begin{cases} -1, & \text{if } t_1 \in \left[-1, -\frac{1+a}{2}\right), \\ \frac{2}{1+a}t_1, & \text{if } t_1 \in \left[-\frac{1+a}{2}, \frac{1+a}{2}\right), \\ 1, & \text{if } t_1 \in \left[\frac{1+a}{2}, 1\right], \end{cases}$$

and

$$\lambda^a(t_1) := \frac{1+a}{2}t_1, \qquad t_1 \in [-1, 1]$$

Moreover for $f \in \mathcal{C}(P)$ set

$$\begin{aligned} A_f &= \{f \circ \varphi_a : a \in [0,1]\}, \\ A^f &= \{f \circ \varphi^a : a \in [0,1]\}. \end{aligned}$$

We begin with the following lemmas whose proofs are straightforward.

Lemma 33 *Let $a \in [0,1]$. Then*

(a) $|\lambda_a(x) - \lambda_a(y)| \leq 2\,|x-y|$,

(b) $|\lambda^a(x) - \lambda^a(y)| \leq \frac{1+a}{2}\,|x-y|$ *, for all* $x, y \in [-1,1]$,

(c) the maps φ_a and φ^a are continuous.

Proof. (a) Set $I_1 = \left[-1, -\frac{1+a}{2}\right)$, $I_2 = \left[-\frac{1+a}{2}, \frac{1+a}{2}\right]$ and $I_3 = \left(\frac{1+a}{2}, 1\right]$. If $x, y \in I_1$ or $x, y \in I_3$ then $|\lambda_a(x) - \lambda_a(y)| = 0$. If $x, y \in I_2$ then

$$|\lambda_a(x) - \lambda_a(y)| = \left|\frac{2}{1+a}x - \frac{2}{1+a}y\right| \leq \frac{2}{1+a}\,|x-y|.$$

If $x \in I_1$ and $y \in I_3$, since $|x-y| \geq 1+a$, we have that$|\lambda_a(x) - \lambda_a(y)| = 2 \leq 2\,|x-y|$. If $x \in I_2$ and $y \in I_3$ then

$$|\lambda_a(x) - \lambda_a(y)| = \left|\frac{2}{1+a}\left(x - \frac{1+a}{2}\right)\right| \leq 2\left|x - \frac{1+a}{2}\right| \leq 2\,|x-y|.$$

The case $x \in I_2$ and $y \in I_3$ is analogous.
(b) It is immediate.
(c) It follows by the definitions of the maps using (a) and (b), respectively.
■

Lemma 34 *Let (a_m) be a sequence in $[0,1]$ and $a_m \to a$. Then*

$$\|\varphi_{a_m} - \varphi_a\|_\infty \to 0 \text{ and } \|\varphi^{a_m} - \varphi^a\|_\infty \to 0.$$

Proof. Let $\varepsilon > 0$. Then there is $\overline{m}$ such that $|a_m - a| \leq \varepsilon$ for all $m \geq \overline{m}$. It is easy to verify that $|\lambda_{a_m}(t_1) - \lambda_a(t_1)| \leq |a_m - a|$ for any $t_1 \in [0,1]$, hence we obtain that

$$\|\varphi_{a_m}(t) - \varphi_a(t)\|_2^2 = |\lambda_{a_m}(t_1) - \lambda_a(t_1)|^2 \leq \varepsilon^2,$$

for all $m \geq \overline{m}$ and for all $t \in P$. Hence $\|\varphi_{a_m} - \varphi_a\|_\infty \to 0$. The proof of the continuity of the maps φ^a is similar, taking into account that $|\lambda^{a_m}(t_1) - \lambda^a(t_1)| \leq \frac{1}{2}|a_m - a|$ for any $t_1 \in [0,1]$. ■

Lemma 35 *Let $f \in \mathcal{C}(P)$. The sets A_f and A^f are compact in $\mathcal{C}(P)$.*

Proof. Let (a_m) be a sequence of elements of $[0,1]$ converging to a. In order to prove the compactness of the set A_f it will be sufficient to show that

$$\|f \circ \varphi_{a_m} - f \circ \varphi_a\|_\infty \to 0.$$

Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathcal{C}(P)$, there is $\delta > 0$ such that for all $s, t \in P$ such that $\|t - s\|_2 \leq \delta$ we have

$$|f(t) - f(s)| \leq \varepsilon. \tag{3.1}$$

By Lemma 33 (c) we can choose $\overline{m}$ such that

$$\|\varphi_{a_m}(t) - \varphi_a(t)\|_2 \leq \delta, \tag{3.2}$$

for any $m \geq \overline{m}$ and $t \in P$. By (3.1) and (3.2) it follows that

$$\|f \circ \varphi_{a_m} - f \circ \varphi_a\|_\infty = \max_{t \in P} |f(\varphi_{a_m}(t)) - f(\varphi_a(t))| \leq \varepsilon\ ,$$

for all $m \geq \overline{m}$. Hence $\|f \circ \varphi_{a_m} - f \circ \varphi_a\|_\infty \to 0$. The proof of the compactness of the set A^f in $\mathcal{C}(P)$ is similar. ■

3.3 Retractions in the space $\mathcal{C}(P)$.

In order to construct retractions in the space $\mathcal{C}(P)$ we now define a map $Q : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ as follows

$$Qf := f \circ \varphi_{\|f\|_\infty}. \tag{3.3}$$

It is easy to see that $\|Qf\|_\infty = \|f\|_\infty$ for all $f \in B(\mathcal{C}(P))$ and $Qf = f$ for all $f \in S(\mathcal{C}(P))$.

Proposition 36 *The map Q is continuous.*

Proof. Let (f_m) be a sequence in $B(\mathcal{C}(P))$ such that $\|f_m - f\|_\infty \to 0$. Let $\varepsilon > 0$ be given. Choose m_1 such that $\|f_m - f\|_\infty \leq \frac{\varepsilon}{2}$ for all $m \geq m_1$. Since f is uniformly continuous, we can find $\delta > 0$ such that $|f(s) - f(t)| \leq \frac{\varepsilon}{2}$ for all $s, t \in P$ with $\|t - s\|_2 \leq \delta$. Since $\|f_m\|_\infty \to \|f\|_\infty$, by Lemma 34 there is m_2 such that

$$\left\|\varphi_{\|f_m\|_\infty} - \varphi_{\|f\|_\infty}\right\|_\infty \leq \delta,$$

for any $m \geq m_2$. Hence we get

$$\left|f(\varphi_{\|f_m\|_\infty}(t)) - f(\varphi_{\|f\|_\infty}(t))\right| \leq \frac{\varepsilon}{2},$$

for all $t \in P$ and $m \geq m_2$. Therefore

$$\begin{aligned}
\left|f_m(\varphi_{\|f_m\|_\infty}(t)) - f(\varphi_{\|f\|_\infty}(t))\right| &\leq \left|f_m(\varphi_{\|f_m\|_\infty}(t)) - f(\varphi_{\|f\|_\infty}(t))\right| \\
+\left|f(\varphi_{\|f_m\|_\infty}(t)) - f(\varphi_{\|f\|_\infty}(t))\right| &\leq \|f_m - f\|_\infty + \frac{\varepsilon}{2} \leq \varepsilon,
\end{aligned}$$

for all $t \in P$ and $m \geq \max\{m_1, m_2\}$, which gives $\|Qf_m - Qf\|_\infty \to 0$. ■

We now are in a position to calculate the Hausdorff measure of noncompactness of QA for $A \subseteq B(\mathcal{C}(P))$.

Proposition 37 *Let A be a subset of $B(\mathcal{C}(P))$. Then $\gamma_{\mathcal{C}(P)}(QA) = \gamma_{\mathcal{C}(P)}(A)$.*

Proof. Let $A \subseteq B(\mathcal{C}(P))$. We start to prove the inequality $\gamma_{BC(P)}(QA) \leq \gamma_{BC(P)}(A)$. Let $\alpha > \gamma_{BC(P)}(A)$. There is an α-net $\{f_1, ..., f_n\}$ for A in $\mathcal{C}(P)$. By Proposition 35 the set $\cup_{i=1}^n A_{f_i}$ is compact in $\mathcal{C}(P)$. Hence for all $\delta > 0$ we can choose a δ-net $\{g_1, ..., g_m\}$ for $\cup_{k=1}^n A_{f_k}$in $\mathcal{C}(P)$. We now show that $\{g_1, ..., g_m\}$ is a $(\alpha + \delta)$-net for QA in $\mathcal{C}(P)$. Let $g \in A$ and let $f \in A$ such that $Qf = g$. Fix $i \in \{1, ..., n\}$ such that $\|f - f_i\|_\infty \leq \alpha$. Since $f_i \circ \varphi_{\|f\|_\infty} \in A_{f_i}$ we can find $j \in \{1, ..., m\}$ such that $\left\|f_i \circ \varphi_{\|f\|_\infty} - g_j\right\|_\infty \leq \delta$. Then

$$\begin{aligned}
\|g_j - Qf\|_\infty &\leq \left\|g_j - f_i \circ \varphi_{\|f\|_\infty}\right\|_\infty + \left\|f_i \circ \varphi_{\|f\|_\infty} - f \circ \varphi_{\|f\|_\infty}\right\|_\infty \\
&\leq \delta + \left\|(f_i - f) \circ \varphi_{\|f\|_\infty}\right\|_\infty \leq \delta + \varepsilon.
\end{aligned}$$

Therefore $\gamma_{\mathcal{C}(P)}(QA) \leq \alpha + \delta$ so that $\gamma_{\mathcal{C}(P)}(QA) \leq \gamma_{\mathcal{C}(P)}(A)$. Now we prove the opposite inequality $\gamma_{\mathcal{C}(P)}(QA) \geq \gamma_{\mathcal{C}(P)}(A)$. Let $\beta > \gamma_{\mathcal{C}(P)}(QA)$. There is a β-net $\{h_1, ..., h_s\}$ for QA in $BC(P)$. By Proposition 35 the set $\cup_{i=1}^{s} A^{h_i}$ is compact in $\mathcal{C}(P)$. Therefore for all $\delta > 0$ we can choose a δ-net $\{p_1, ..., p_k\}$ for $\cup_{i=1}^{s} A^{h_i}$ in $\mathcal{C}(P)$. We now show that $\{p_1, ..., p_k\}$ is a $(\beta + \delta)$-net for A in $\mathcal{C}(P)$. Let $f \in A$. Fix $l \in \{1, ..., s\}$ such that $\|Qf_l - h_l\|_\infty \leq \beta$. Since $h_l \circ \varphi^{\|f\|_\infty} \in A^{h_l}$ we can find $m \in \{1, ..., k\}$ such that $\left\|h_l \circ \varphi^{\|f\|_\infty} - p_m\right\|_\infty \leq \delta$. Then

$$\begin{aligned}
\|f - p_m\|_\infty &\leq \left\|f - h_l \circ \varphi^{\|f\|_\infty}\right\|_\infty + \left\|h_l \circ \varphi^{\|f\|_\infty} - p_m\right\|_\infty \\
&\leq \left\|(f - h_l \circ \varphi^{\|f\|_\infty}) \circ \varphi_{\|f\|_\infty}\right\|_\infty + \delta \\
&= \sup_{\left\{t \in P : t_1 \in \left[-\frac{1+\|f\|_\infty}{2}, \frac{1+\|f\|_\infty}{2}\right]\right\}} \left|((f - h_l \circ \varphi^{\|f\|_\infty}) \circ \varphi_{\|f\|_\infty})(t)\right| + \delta \\
&= \sup_{\left\{t \in P : t_1 \in \left[-\frac{1+\|f\|_\infty}{2}, \frac{1+\|f\|_\infty}{2}\right]\right\}} \left|(f \circ \varphi_{\|f\|_\infty})(t) - h_l(t)\right| + \delta \\
&\leq \|Qf - h_l\|_\infty + \delta \leq \beta + \delta.
\end{aligned}$$

Hence $\gamma_{\mathcal{C}(P)}(A) \leq \beta + \delta$ and $\gamma_{\mathcal{C}(P)}(A) \leq \gamma_{\mathcal{C}(P)}(QA)$. ■

Let $u \in (0, +\infty)$. Define the map $P_u : B(\mathcal{C}(P)) \to \mathcal{C}(P)$ by

$$(P_u f)(t) := \begin{cases}
u(t_1 + 1), & \text{if } t_1 \in \left[-1, -\frac{3+\|f\|_\infty}{4}\right), \\
-u(t_1 + \frac{1+\|f\|_\infty}{2}), & \text{if } t_1 \in \left[-\frac{3+\|f\|_\infty}{4}, -\frac{1+\|f\|_\infty}{2}\right), \\
0, & \text{if } t_1 \in \left[-\frac{1+\|f\|_\infty}{2}, \frac{1+\|f\|_\infty}{2}\right], \\
u(t_1 - \frac{1+\|f\|_\infty}{2}), & \text{if } t_1 \in \left(\frac{1+\|f\|_\infty}{2}, \frac{3+\|f\|_\infty}{4}\right], \\
-u(t_1 - 1), & \text{if } t_1 \in \left(\frac{3+\|f\|_\infty}{4}, 1\right].
\end{cases}$$

Proposition 38 *The map P_u is compact.*

Proof. The map P_u is continuous, in fact if (f_m) is a sequence in $B(\mathcal{C}(P))$ converging to f, then

$$\|P_u f_m - P_u f\|_\infty = \frac{u}{2} \left|\|f\|_m - \|f\|_\infty\right| \to 0,$$

since $\|f_m\|_\infty \to \|f\|_\infty$. To complete the proof we show that $P_u B(\mathcal{C}(P))$ is compact. Let (g_m) be a sequence in $P_u B(\mathcal{C}(P))$ and let $\{f_m\}$ be a sequence in

$B(\mathcal{C}(P))$ such that $Qf_m = g_m$. Being $0 \leq \|f_m\|_\infty \leq 1$ we can assume without loss of generality that $\|f_m\|_\infty \to a \in [0,1]$. Choose $f \in B(\mathcal{C}(P))$ such that $\|f\|_\infty = a$. Then

$$\|g_m - P_u f\|_\infty = \|P_u f_m - P_u f\|_\infty = u \left| \frac{\|f_m\|_\infty - \|f\|_\infty}{2} \right| \to 0,$$

and the proof is complete. ■

The next theorem is our main result.

Theorem 39 *For any $u > 0$ there is a retraction $R_u : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ with*

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, & \text{if } u \leq 4 \\ \frac{4}{u}, & \text{if } u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}.$$

In particular, we have $W(\mathcal{C}(P) = 1$.

Proof. Let $u \in (0, +\infty)$. Define the map $T_u : B(\mathcal{C}(P)) \to \mathcal{C}(P)$ by $T_u f = Qf + P_u f$. Since P_u is compact, Lemma 37 implies

$$\gamma_{\mathcal{C}(P)}(T_u A) = \gamma_{\mathcal{C}(P)}(A), \tag{3.4}$$

for any $A \subset B(\mathcal{C}(P))$.

Moreover $T_u f = f$ for all $f \in S(\mathcal{C}(P))$ and

$$\begin{aligned}
\|T_u f\|_\infty &= \max\Bigg\{ \sup_{\left\{t\in P: t\in\left[-\frac{1+\|f\|_\infty}{2}, \frac{1+\|f\|_\infty}{2}\right]\right\}} |(T_u f)(t)|, \sup_{\left\{t\in P: t\in\left[-1, -\frac{1+\|f\|_\infty}{2}\right)\right\}} |(T_u f)(t)|, \\
&\qquad \sup_{\left\{t\in P: \left]\frac{1+\|f\|_\infty}{2}, 1\right]\right\}} |(T_u f)(t)| \Bigg\} \\
&= \max\Bigg\{ \|f\|_\infty, \sup_{\left\{t\in P: t\in\left[-1, -\frac{1+\|f\|_\infty}{2}\right)\right\}} \left| f((-1, t_2, t_3, ...)) + \frac{u}{4}(1 - \|f\|_\infty) \right|, \\
&\qquad \sup_{\left\{t\in P: t\in\left]\frac{1+\|f\|_\infty}{2}, 1\right]\right\}} \left| f((1, t_2, t_3, ...)) + \frac{u}{4}(1 - \|f\|_\infty) \right| \Bigg\} \\
&\geq \max\left\{ \|f\|_\infty, \frac{u}{4}(1 - \|f\|_\infty) - \|f\|_\infty \right\} \geq \frac{u}{u+8},
\end{aligned}$$

for all $f \in B(\mathcal{C}(P))$. Consider the map $R_u : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ defined by

$$R_u f = \frac{T_u f}{\|T_u f\|_\infty}.$$

As $\|T_u f\|_\infty \geq \frac{u}{u+8}$, the definition of R_u implies

$$R_u A \subseteq \left[0, \frac{u+8}{u}\right] \cdot (Q + P_u)A.$$

Therefore

$$\begin{aligned}
\gamma_{\mathcal{C}(P)}(R_u A) &\leq \gamma_{\mathcal{C}(P)}(\left[0, \frac{u+8}{u}\right].((Q + P_u)A)) = \frac{u+8}{u}\gamma_{\mathcal{C}(P)}((Q + P_u)A) \\
&= \frac{u+8}{u}\gamma_{\mathcal{C}(P)}(QA) \leq \frac{u+8}{u}\gamma_{\mathcal{C}(P)}(A).
\end{aligned}$$

The latter inequality together with (3.7) of the Example 45 implies $\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}$. On the other hand,

$$\|T_u f\|_\infty \leq \max\left\{ \|f\|_\infty, \|f\|_\infty + \frac{u}{4}(1 - \|f\|_\infty) \right\} \leq \max\left\{1, \frac{u}{4}\right\}.$$

If $u > 4$ we have

$$(Q + P_u)A \subseteq \left[0, \frac{u}{4}\right] \cdot R_u A.$$

Then

$$\gamma_{\mathcal{C}(P)}(A) = \gamma_{\mathcal{C}(P)}((Q + P_u)A) \leq \gamma_{\mathcal{C}(P)}(\left[0, \frac{u}{4}\right] R_u A) = \frac{u}{4}\gamma_{\mathcal{C}(P)}(R_u A),$$

which gives

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) \geq \frac{4}{u} \tag{3.5}$$

By (3.9) of the Example 46 we have $\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \frac{4}{u}$. If $u \leq 4$ we have

$$(Q + P_u)A \subseteq [0, 1] \cdot R_u A.$$

Hence

$$\gamma_{\mathcal{C}(P)}(A) = \gamma_{\mathcal{C}(P)}((Q + P_u)A) \leq \gamma_{\mathcal{C}(P)}([0, 1] \cdot R_u A) = \gamma_{\mathcal{C}(P)}(R_u A),$$

which gives

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) \geq 1. \tag{3.6}$$

By (3.10) of the Example 46 we have $\underline{\gamma}(R_u) = 1$.
Finally, since $\lim_{u\to\infty} \frac{u+8}{u} = 1$ we obtain $W(\mathcal{C}(P) = 1$. ■

We point out that, in view of example 44, our construction does not work in the case of the Banach space $\mathcal{BC}(l_2)$.

Remark 40 We observe that a retraction which has positive $\underline{\gamma}_X$-norm is a proper map, i.e., the preimage $R^{-1}M$ of any compact set $M \subseteq X$ is compact. Thus, for all $u \in (0, +\infty)$, the retractions R_u defined above are proper maps.

Since the measure of noncompactness of a set is invariant under isometries and the Banach space $\mathcal{C}(K)$ of all real continuous functions defined on a Hausdorff space K homeomorphic to the Hilbert cube is isometric to $\mathcal{C}(P)$, we get the following corollary.

Corollary 41 *Let K be a Hausdorff space homeomorphic to the Hilbert cube P. For* *any $u > 0$ there is a retraction $R_u : B(\mathcal{C}(K)) \to S(\mathcal{C}(K))$ with*

$$\underline{\gamma}_{\mathcal{C}(K)}(R_u) = \begin{cases} 1, & \textit{if } u \leq 4 \\ \frac{4}{u}, & \textit{if } u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{C}(K)}(R_u) = \frac{u+8}{u}.$$

In particular, we have $W(\mathcal{C}(K) = 1$.

Remark 42 The previous Corollary applies in two particular important cases.
(i) Every infinite dimensional compact convex subset K of a normed space is homeomorphic to the Hilbert cube P (see [33]).
(ii) Let K be a metrizable infinite dimensional compact convex set in a topological linear space and assume K is an absolute retract, then K is homeomorphic to the Hilbert cube P (see [16]).

Taking into account Remark 5 of Chapter 1, it follows:

Corollary 43 *For every $u > 0$, the retraction R_u defined defined in Theorem 39 satisfies,*

$$\underline{\psi}_{\mathcal{C}(P)}(R_u) \geq \begin{cases} \frac{1}{2}, & \textit{if } u \leq 4 \\ \frac{2}{u}, & \textit{if } u > 4 \end{cases}$$

and

$$\psi_{\mathcal{C}(P)}(R_u) \leq \frac{2u+16}{u},$$

for $\psi \in \{\alpha, \beta\}$. Thus

$$W_\alpha(\mathcal{C}(P)) \leq 2$$

and

$$W_\beta(\mathcal{C}(P)) \leq 2.$$

3.4 Examples

Example 44 The map Q defined in (3.3) is not anymore continuous when considered from $B(\mathcal{BC}(l_2))$ into itself. Set $I_k = \left[\sum_{i=1}^{2k} \frac{1}{i}, \sum_{i=1}^{2k+1} \frac{1}{i}\right)$ and $J_k =$

$\left[\sum_{i=1}^{2k+1} \frac{1}{i},\ \sum_{i=1}^{2k+2} \frac{1}{i}\right)$, for each $k = 1, 2, ...$
Then let $f : l_2 \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0, & \text{if } \ \|t\|_2^2 \in \left[0,\ \frac{3}{2}\right), \\ (2k+1)(\|t\|_2^2 - \sum_{i=1}^{2k} \frac{1}{i}), & \text{if } \ \|t\|_2^2 \in I_k, \qquad (k = 1, 2, ...). \\ -(2k+2)(\|t\|_2^2 - \sum_{i=1}^{2k+2} \frac{1}{i}), & \text{if } \ \|t\|_2^2 \in J_k. \end{cases}$$

The map f is bounded, continuous and $\|f\|_\infty = 1$ but f is not uniformly continuous. In fact, let $0 < \epsilon < 1$. Given $\delta > 0$ find $k \in \mathbb{N}$ such that $\frac{1}{2k+1} \leq \delta$ and choose $t,\ s \in l_2$ such that $\|t\|_2^2 = \sum_{i=1}^{2k} \frac{1}{i}$ and $\|s\|_2^2 = \sum_{i=1}^{2k+1} \frac{1}{i}$. Then $f(t) = 0$ and $f(s) = 1$ so that $|f(t) - f(s)| = 1 > \varepsilon$.
Consider now the sequence (f_m), where $f_m : l_2 \to \mathbb{R}$ is defined by $f_m = (1 - \frac{1}{m})f \ \ (m = 1, 2, ...)$. Then $\|f_m\|_\infty = 1 - \frac{1}{m}$ and $\|f_m - f\|_\infty = \frac{1}{m} \to 0$. We now show that

$$\left\| f \circ \varphi_{1-\frac{1}{p}} - f \circ \varphi_{1-\frac{1}{q}} \right\|_\infty = 1,$$

for all $p, q \in \mathbb{N}$. Suppose $q < p$ so that $0 < \frac{1+(1-\frac{1}{q})}{2} < \frac{1+(1-\frac{1}{p})}{2}$ and $\frac{2}{1+(1-\frac{1}{q})} > \frac{2}{1+(1-\frac{1}{p})}$. Set

$$\delta_{q,p} = \left(\frac{2}{1+(1-\frac{1}{q})}\right)^2 - \left(\frac{2}{1+(1-\frac{1}{p})}\right)^2 > 0.$$

Fix $\overline{k}$ such that $(t_{1,\overline{k}})^2 = \frac{1}{\delta_{q,p}(2\overline{k}+1)}$ and $t_{1,\overline{k}} \in \left(0,\ \frac{1+(1-\frac{1}{q})}{2}\right)$. Set

$$t^{(\overline{k})} = (t_{1,\overline{k}},\ t(\sum_{i=1}^{2\overline{k}} \frac{1}{i} - (t_{1,\overline{k}})^2 (\frac{2}{1+(1-\frac{1}{p})})^2)^{\frac{1}{2}},\ 0,\ 0, ...).$$

Since

$$\left\| \varphi_{1-\frac{1}{p}}(t^{(\overline{k})}) \right\|_2^2 = (t_{1,\overline{k}})^2 (\frac{2}{1+(1-\frac{1}{p})})^2 + \sum_{i=1}^{2\overline{k}} \frac{1}{i} - (t_{1,\overline{k}})^2 (\frac{2}{1+(1-\frac{1}{p})})^2 = \sum_{i=1}^{2\overline{k}} \frac{1}{i},$$

we have

$$f(\varphi_{1-\frac{1}{p}}(t^{(\overline{k})})) = f(\sum_{i=1}^{2\overline{k}} \frac{1}{i}) = 0.$$

On the other hand

$$\begin{aligned}\left\|\varphi_{1-\frac{1}{q}}(t^{(\overline{k})})\right\|_2^2 &= (t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{q})})^2+\sum_{i=1}^{2\overline{k}}\frac{1}{i}-(t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{p})})^2\\ &= \sum_{i=1}^{2\overline{k}}\frac{1}{i}+\frac{1}{\delta_{q,p}(2\overline{k}+1)}((\frac{2}{1+(1-\frac{1}{q})})^2-(\frac{2}{1+(1-\frac{1}{p})})^2)=\sum_{i=1}^{2\overline{k}+1}\frac{1}{i},\end{aligned}$$

implies

$$f(\varphi_{1-\frac{1}{q}}(t^{(\overline{k})}))=f(\sum_{i=1}^{2\overline{k}+1}\frac{1}{i})=1.$$

Thus $\left|f(\varphi_{1-\frac{1}{p}}(t^{(\overline{k})}))-f(\varphi_{1-\frac{1}{q}}(t^{(\overline{k})}))\right|=1$ so that $\left\|f\circ\varphi_{1-\frac{1}{p}}-f\circ\varphi_{1-\frac{1}{q}}\right\|_\infty=1$. Suppose $Q:B(\mathcal{BC}(l_2))\to B(\mathcal{BC}(l_2))$ continuous. Then if $\|f_m-f\|_\infty\to 0$ we have that

$$\begin{aligned}\|Qf_m-Qf\|_\infty &= \left\|f_m\circ\varphi_{\|f_m\|_\infty}-f\circ\varphi_{\|f\|_\infty}\right\|_\infty\\ &= \left\|f_m\circ\varphi_{1-\frac{1}{m}}-f\right\|_\infty\to 0.\end{aligned}$$

Let $\varepsilon>0$. Fix $p,q\in\mathbb{N}$ such that

$$\|f-f_p\|_\infty+\|Qf_p-Qf_q\|_\infty+\|f_q-f\|_\infty<1.$$

Then

$$\begin{aligned}1 &= \left\|f\circ\varphi_{1-\frac{1}{p}}-f\circ\varphi_{1-\frac{1}{q}}\right\|_\infty\le\left\|f\circ\varphi_{1-\frac{1}{p}}-f_p\circ\varphi_{\|f_p\|_\infty}\right\|_\infty\\ &\quad+\left\|f_p\circ\varphi_{\|f_p\|_\infty}-f_q\circ\varphi_{\|f_q\|_\infty}\right\|_\infty+\left\|f_q\circ\varphi_{\|f_q\|_\infty}-f\circ\varphi_{1-\frac{1}{q}}\right\|_\infty\\ &= \left\|f\circ\varphi_{1-\frac{1}{p}}-f_p\circ\varphi_{1-\frac{1}{p}}\right\|_\infty+\|Qf_p-Qf_q\|_\infty\\ &\quad+\left\|f_q\circ\varphi_{1-\frac{1}{q}}-f\circ\varphi_{1-\frac{1}{q}}\right\|_\infty\\ &= \|f-f_p\|_\infty+\|Qf_p-Qf_q\|_\infty+\|f_q-f\|_\infty<1,\end{aligned}$$

which is a contradiction. Observe that the set $A^f=\{f\circ\varphi^a : a\in[0,1]\}$ is not compact.

In connection with Theorem 39 we have the following examples:

Example 45 Define the maps $f_n : P \to \mathbb{R}$ $(n = 3, 4, ...)$ by

$$f_n(t) = \begin{cases} \frac{u}{u+8}, & \text{if } |t_1| < \frac{1}{2} - \frac{1}{n}, \\ n\frac{u}{u+8}\left(|t_1| - \frac{1}{2}\right), & \text{if } \frac{1}{2} - \frac{1}{n} \le |t_1| < \frac{1}{2} + \frac{1}{n}, \\ -\frac{u}{u+8}, & \text{if } \frac{1}{2} + \frac{1}{n} \le |t_1| \le 1. \end{cases}$$

Then

$$Qf_n(x) \begin{cases} \frac{u}{u+8}, & \text{if } |t_1| < \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} - \frac{1}{n}), \\ -2n\frac{\frac{u}{u+8}}{1+\frac{u}{u+8}}\left(|t_1| - \frac{1+\frac{u}{u+8}}{2}\frac{1}{2}\right), & \text{if } \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} - \frac{1}{n}) \le |t_1| < \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} + \frac{1}{n}) \\ -\frac{u}{u+8}, & \text{if } \frac{1+\frac{u}{u+8}}{2}(\frac{1}{2} + \frac{1}{n}) \le |t_1| \le 1. \end{cases}$$

Moreover for $u > 0$ we find

$$(P_u f_n)(x) \begin{cases} 0, & \text{if } |t_1| \le \frac{1+\frac{u}{u+8}}{2}, \\ u\left(|t_1| - \frac{1+\frac{u}{u+8}}{2}\right), & \text{if } \frac{1+\frac{u}{u+8}}{2} \le |t_1| < \frac{3+\frac{u}{u+8}}{4}, \\ -u\left(|t_1| - 1\right), & \text{if } \frac{3+\frac{u}{u+8}}{4} \le |t_1| \le 1. \end{cases}$$

Therefore

$$\|Qf_n + P_u f_n\|_\infty = \frac{u}{u+8}.$$

Setting $A = \{f_n : n \ge 3\}$ we have

$$R_u A = \{R_u f_n : n \ge 3\} = \left\{ \frac{Qf_n + P_u f_n}{\|Qf_n + P_u f_n\|_\infty} : n \ge 3 \right\} = \frac{u+8}{u}(Q + P_u)A.$$

Hence

$$\gamma_{\mathcal{C}(P)}(R_u A) = \frac{u+8}{u}\gamma_{\mathcal{C}(P)}(A) \tag{3.7}$$

Example 46 Consider the maps $f_{c,n} : P \to \mathbb{R}$ $(n = 2, 3, ...)$ defined by

$$f_{c,n}(t) = \begin{cases} -c, & \text{if } |t_1| < 1 - \frac{1}{n}, \\ nc(|t_1| - 1), & \text{if } 1 - \frac{1}{n} \le |t_1| \le 1, \end{cases}$$

We have that

$$Qf_{c,n}(t) = \begin{cases} -c, & \text{if } |t_1| < \frac{1+c}{2}(1-\frac{1}{n}), \\ 2n\frac{c}{1+c}(|t_1| - \frac{1+c}{2}), & \text{if } \frac{1+c}{2}(1-\frac{1}{n}) \le |t_1| < \frac{1+c}{2}, \\ 0, & \text{if } \frac{1+c}{2} \le |t_1| \le 1. \end{cases}$$

Set $A_c = \{f_{c,n} : n \ge 2\}$. Observe that $\gamma_{\mathcal{C}(P)}(A_c) = \frac{c}{2}$ and $\gamma_{\mathcal{C}(P)}(QA_c) = \frac{c}{2}$. Let $u > 0$. Then

$$P_u f_n(t) = \begin{cases} 0, & \text{if } |t_1| < \frac{1+c}{2}, \\ u(|t_1| - \frac{1+c}{2}), & \text{if } \frac{1+c}{2} \le |t_1| < \frac{3+c}{4}, \\ -u(|t_1| - 1), & \text{if } \frac{3+c}{4} \le |t_1| \le 1, \end{cases}$$

and $\|P_u f_n\| = u\frac{1-c}{4}$. If $c \le \frac{u}{u+4}$ we have $\|Qf_n + P_u f_n\| = \max\left\{c, u\frac{1-c}{4}\right\} = u\frac{1-c}{4}$. Then

$$\gamma_{\mathcal{C}(P)}(R_u A_c) = \frac{4}{u(1-c)}\gamma_{\mathcal{C}(P)}(QA_c) = \frac{4}{u(1-c)}\gamma_{\mathcal{C}(P)}(A_c). \tag{3.8}$$

If $u > 4$, by (3.5) we have $\underline{\gamma}_{\mathcal{C}(P)}(R_u) \ge \frac{4}{u}$. Suppose $\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \frac{4}{u} + \sigma$ with $\sigma > 0$. Fix $\overline{c}$ such that $\frac{4}{u(1-\overline{c})} < \frac{4}{u} + \sigma$. Then using (3.8)

$$\begin{aligned} \gamma_{\mathcal{C}(P)}(R_u A_{\overline{c}}) &= \frac{4}{u(1-\overline{c})}\gamma_{\mathcal{C}(P)}(A_{\overline{c}}) \\ &< (\frac{4}{u} + \sigma)\gamma_{\mathcal{C}(P)}(A_{\overline{c}}) \le \gamma_{\mathcal{C}(P)}(R_u A_{\overline{c}}) \end{aligned}$$

which yields to a contradiction. So that

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \frac{4}{u}. \tag{3.9}$$

If $u \le 4$, by (3.6) we have

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) \ge 1.$$

Suppose $\underline{\gamma}_{\mathcal{C}_{(P)}(K)}(R_u) = 1 + \sigma$ with $\sigma > 0$. Fix $\bar{c}$ such that $\frac{2}{u(1-\bar{c})} < 1 + \sigma$, then

$$\begin{aligned} \gamma_{\mathcal{C}(P)(K)}(R_u A_{\bar{c}}) &= \frac{2}{u(1-\bar{c})} \gamma_{\mathcal{C}_{(P)}(K)}(A_{\bar{c}}) \\ &< (1+\sigma)\gamma_{\mathcal{C}(P)}(A_{\bar{c}}) \leq \gamma_{\mathcal{C}(P)}(R_u A_{\bar{c}}), \end{aligned}$$

which is a contradiction so that

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = 1. \tag{3.10}$$

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