UNIVERSITÀ DELLA CALABRIA
Dipartimento di ELETTRONICA, INFORMATICA E SISTEMISTICA

Models and Policies for Service Industries via Revenue Management

PhD thesis in Operational Research - MAT/09
cycle XXIV

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November 2011

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Preface

This Ph.D. thesis was written at the Department of Electronics, Informatics and Sistemistics (DEIS), at University of Calabria from November 2008 to November 2011 under the supervision of Professor Francesca Guerriero.

My special thank goes to my supervisor Francesca Guerriero for the much needed direction, guidance and support in the progression and realization of the research projects.

I would like to thank Giovanna Miglionico for her valuable collaboration and big contribution to the development of the research activities.

Also, I would like to thank my colleagues for making my work place so pleasant during the last three years.

Thanks.

Summary in Italian

L' attività di ricerca svolta durante il corso di dottorato, sotto la guida della Prof.ssa Francesca Guerriero e in collaborazione con l'Ing.Giovanna Miglionico, ha condotto alla stesura del presente lavoro di tesi dal titolo "Models and Policies for Service Industries via Revenue Management", sintesi dei risultati della ricerca concentrata sulla definizione, la progettazione, l'implementazione e la validazione di modelli e politiche di revenue management (RM) per aziende di servizi. In particolare, sono stati affrontanti i seguenti problemi:

1. gestione ottimale delle richieste di prenotazione di mezzi di trasporto per aziende di noleggio attraverso la definizione di:
   - politiche di RM e approcci di ottimizzazione robusta per l'autonoleggio;
   - politiche di RM per il noleggio di mezzi di trasporto con differente capacità di carico;
2. gestione ottimale delle richieste di prenotazione presso aziende operanti nel settore della ristorazione attraverso la definizione di politiche di RM per i ristoranti.

Per ogni problema sono stati definiti approcci innovativi di soluzione, sono state analizzate le proprietà teoriche dei metodi proposti e sono state validate le loro prestazioni in termini di efficienza ed efficacia mediante una articolata fase sperimentale.

Il presente lavoro di tesi è organizzato in 7 capitoli. Nel seguito viene riportata una breve descrizione del contenuto di ciascun capitolo.

Il Capitolo 1 presenta una introduzione al lavoro.

Nel Capitolo 2 viene presentato lo stato dell'arte relativo al RM e vengono analizzate e approfondite le differenti problematiche e i diversi campi di applicazione propri del RM ([34], [35],[75],[113], [157]), che svolge ormai un ruolo importante per molte aziende di servizio appartenenti a differenti settori [75], [73], [30]. Il RM può essere visto come un'efficace ed efficiente strumento per la massimizzazione del ricavo, obiettivo che viene raggiunto allocando in maniera ottimale la capacità limitata e deperibile tra i diversi prodotti/servizi, che vengono assegnati a segmenti eterogenei di clientela e generalmente venduti in anticipo rispetto al momento del consumo effettivo, tramite le tecniche di prenotazione, considerando i diversi livelli di incertezza della domanda. Le tecniche di RM inizialmente applicate al settore del trasporto aereo [123], sono state successivamente importate e sviluppate in altri contesti di interesse pratico, quali quello alberghiero, quello del noleggio di mezzi di trasporto e quello della ristorazione.

Nel Capitolo 3 viene introdotto il problema dell'autonoleggio e ci si sofferma sull' applicazione delle tecniche di RM al caso della gestione ottimale di una flotta di mezzi di trasporto (automobili) messa a disposizione da parte di un operatore logistico per servizi di noleggio. L'obiettivo principale è quello di definire approcci innovativi per gestire in maniera appropriata il servizio offerto in modo tale da massimizzare i ricavi. La decisione fondamentale alla base della gestione del rendimento consiste nel decidere se accettare o rifiutare una richiesta di prenotazione che arriva in un certo istante di tempo, e, in caso di rifiuto, gestire la disponibilità residua delle risorse in maniera opportuna. Le applicazioni del RM all'autonoleggio descritte in letteratura sono molto limitate, i contributi principali sono stati i lavori di Carroll e Grimes [27] e Geraghty e Johnson [49], che offrono una dettagliata descrizione dell'implementazione e dei contributi derivati dall'applicazione delle tecniche di RM nelle compagnie di noleggio internazionali Hertz e National, rispettivamente, ma non definiscono dei modelli nè dei metodi di validità generale per affrontare il problema descritto. In una prima fase del lavoro, sono stati definiti dei modelli di programmazione lineare per l'ottimizzazione di un classico servizio di noleggio di automobili e mezzi commerciali, suddivisi in gruppi a seconda del prezzo e delle caratteristiche. Ciascuna richiesta di noleggio, che può arrivare al sistema tramite una prenotazione (booking), può essere soddisfatta con il mezzo richiesto oppure con uno appartenente ad un gruppo superiore (upgrading). È stato formulato a tale scopo un modello di programmazione lineare intera che non è altro che l'estensione al campo del car rental del modello deterministico standard del RM [113]. Il modello deterministico permette di definire sulla base del rispetto dei vincoli sulla domanda e sulla disponibilità di mezzi, quali e quante richieste di noleggio accettare. Per gestire in modo adeguato l'incertezza della domanda, sono stati applicati al problema base prodotto in precedenza diversi criteri di robustezza [36], [9], [117], [116]. In particolare, sono stati considerati il criterio max-min (applicazione del criterio di robustezza assoluta in cui si considera la massimizzazione del caso peggiore), il criterio di deviazione robusta (min-max regret) e il criterio di p-robustezza. Inoltre, è stato formulato un modello che considera la deviazione media assoluta basata su criteri di variabilità [165], [97], che permette di individuare una sequenza di soluzioni progressivamente meno sensibili alle realizzazioni dei parametri del modello. È stata condotta un'ampia fase di sperimentazione e i risultati ottenuti mostrano una buona applicabilità dei modelli per gestire i problemi considerati.

Tali risultati sono stati riportati nel lavoro "Modelling and solving a car rental revenue optimisation problem" pubblicato sulla rivista internazionale Int. J. Mathematics in Operational Research [52].

Nel Capitolo 4 vengono descritti gli sviluppi per il modello presentato in [52] e gli approcci di RM utilizzati per definire politiche di gestione delle richieste e della disponibilità. In particolare, è stato sviluppato un modello di programmazione dinamica [130],

[151] e differenti politiche di RM (booking limits e bid prices [19], [143]) sono state definite e applicate al problema descritto. In particolare, con l'obiettivo di attenersi il più fedelmente possibile all'attività reale di un'agenzia di noleggio, è stato introdotto nei modelli di programmazione un aspetto tipico del car rental, quale il one-way rentals, cioè la possibilità di lasciare il mezzo noleggiato in un'agenzia o punto di rilascio diverso da quello di partenza, sviluppando di conseguenza i modelli dinamici e le dovute approssimazioni lineari che hanno portato alla definizione delle opportune politiche di gestione delle richieste e della capacità disponibile, quindi anche in questo caso politiche di controllo basate sul booking limit. Inoltre, anche per questo caso il modello contenente il one-way e le politiche associate sono state ampliate considerando la possibilità di trasferire i mezzi tra le diverse stazioni di noleggio, garantendo una certa mobilità della flotta e un più adeguato utilizzo delle risorse. I modelli di programmazione lineare intera definiti per il caso base di car rental sono stati inoltre modificati apportando un rilassamento dei vincoli di interezza dovuto alla proprietà della matrice dei vincoli di poter essere ridotta ad una matrice intervallo $0-1$, che si dimostra essere totalmente unimodulare. Gli esperimenti computazionali condotti su istanze che ricalcano l'attività di piccole-medie aziende di noleggio e di un'agenzia di noleggio locale opportunamente analizzata e i risultati ottenuti mostrano che l'applicazione pratica delle politiche decisionali portano ad un miglioramento del profitto e ad una più corretta utilizzazione dei mezzi disponibili, e sottolineano una gestione più efficace delle richieste rispetto ad approcci tradizionali quali la politica del first-come first-served o il caso della perfetta conoscenza della domanda. L'estensione e i miglioramenti del caso base appena descritti, hanno condotto alla stesura di un lavoro dal titolo "Revenue Models and Policies for the Car Rental Industry", attualmente in fase di revisione sulla rivista internazionale Mathematical Methods of Operations Research.

La rilevanza dei lavori appena descritti è sottolineata dall'assenza in letteratura di studi che considerano contemporaneamente gli aspetti tipici di un'attività di noleggio, quali l'upgrade, lo sharing, il one-way e il riposizionamento dei mezzi, nonchè l'adozione di politiche di RM basate sul booking limits e il bid prices, che richiedono di risolvere dinamicamente un modello di programmazione lineare a differenza di approcci di programmazione dinamica più noti in letteratura [151].

Nel Capitolo 5 vengono presentati i modelli e i metodi di RM definiti per il problema del noleggio di mezzi di trasporto. È stato analizzato il problema di un operatore logistico che possiede una flotta di mezzi, caratterizzati da una capacità di carico differente, con la quale deve soddisfare le richieste di trasporto di una certa quantità di merce da una origine ad una destinazione da parte dei propri clienti. In particolare, l'operatore deve decidere se accettare o rifiutare la richiesta di noleggio da parte di un potenziale cliente non sapendo se in futuro arriveranno richieste che potranno portare un ricavo maggiore. Il problema preso in esame è stato rappresentato mediante un modello di programmazione dinamica [113], [130]. Nel modello, inoltre, sono stati considerati i seguenti aspetti: ogni richiesta può essere soddisfatta con un mezzo con una capacità maggiore o uguale alla quantità da trasportare richiesta (upgrading); può essere effettuato lo sharing tra i mezzi, che consiste nell'utilizzo di uno stesso automezzo per soddisfare contemporaneamente la domanda di clienti diversi, che richiedono il trasporto verso una stessa destinazione e nello stesso istante di tempo. Inoltre, è stato formulato un modello in cui si tiene conto anche della possibilità e della convenienza per l'operatore di movimentare a "vuoto" mezzi per riportarli su un nodo dal quale potranno essere utilizzati per soddisfare altre richieste, che altrimenti rimarrebbero insoddisfatte per mancanza di risorse al nodo. Sono state sviluppate diverse tipologie di politiche proprie del RM, per supportare il decisore nell'accettazione o meno della richiesta e per la valutazione della corretta gestione dalle risorse disponibili. In particolare, sono state considerate le politiche di booking limits e bid prices per decidere se accettare o rifiutare una richiesta di trasporto in un certo periodo di tempo. Tali politiche sono basate su approssimazioni lineari della formulazione di programmazione dinamica. In particolare, il booking limits utilizza informazioni primali, mentre il bid prices duali. Le politiche sono state confrontate anche con la politica classica del first-come first-served. Inoltre, per il caso del noleggio di mezzi con una certa capacità di carico per il trasporto merci, sono stati formulati nuovi modelli che presentano nella stessa formulazione lo sharing e il riposizionamento dei mezzi non utilizzati. In entrambi i casi, sono state sviluppate le politiche di RM opportune. L'ampia gamma di esperimenti computazionali condotti su istanze generate in maniera random, dovute all'assenza in letteratura di lavori confrontabili, contiene risultati incoraggianti riguardo all'applicazione pratica delle politiche definite e ai miglioramenti che se ne potrebbero ricavare dal punto di vista della massimizzazione del ricavo, nonchè della più corretta gestione operativa delle richieste di prenotazione, considerazioni ottenute dal confronto tra le diverse politiche innovative proposte e quelle più tradizionali. Il lavoro presenta inoltre un'appendice in cui viene descritta un'analisi asintotica che conduce ad un'estensione del modello di programmazione lineare per il controllo della capacità su reti proposto da Talluri e van Ryzin in [144]. Inoltre, per il modello base viene riportata una formulazione di programmazione lineare con il rilassamento dei vincoli di interezza [119], [53].

Il lavoro dal titolo "Revenue Management Policies for the Truck Rental Industry" è stato pubblicato sulla rivista internazionale Transportation Research Part E [53].

Il lavoro di ricerca svolto ha permesso di sviluppare un'opportuna base di conoscenza dello stato dell'arte nell'ambito del RM con l'approfondimento delle metodologie, delle tecniche risolutive e delle politiche di gestione proprie di questo approccio, il cui obiettivo principale è la massimizzazione del ricavo. Inoltre, una più ampia conoscenza delle caratteristiche dei settori più adatti all'applicazione di tale strumento di gestione, ha condotto all'opportunità di individuare nuovi ambiti di sviluppo e applicabilità reali.

Nel Capitolo 6 vengono presentati gli studi condotti nell'ambito del settore della ristorazione preso in considerazione per le caratteristiche distintive che permettono un'efficace applicabilità dei principi del RM, nonchè per la bassa o recente attenzione mostrata in letteratura per alcuni aspetti che riguardano l'implementazione del RM in questo particolare settore. Il RM è stato ampiamente utilizzato nell'ambito delle linee aeree e degli hotels. Non sono moltissimi in letteratura i lavori che prendono in considerazione l'applicazione delle metodologie basate sul RM alla gestione dei ristoranti e sono tutti abbastanza recenti. I primi lavori sono i seguenti [73], [76] e [82], in cui si discute l'applicabilità del RM ai ristoranti e si introduce il RevPASH (the revenue per available seat-hour) come base per la corretta applicazione delle tecniche basate sulla massimizazione del profitto. La maggior parte dei lavori si concentra sulle politiche di gestione riguardanti l'arrivo delle richieste ([148], [20]) e le implicazioni della durata dei pasti non determinabile a priori ([72], [78]). Mentre, il pricing, strategia di RM fondamentale, è un aspetto considerato solo in pochi lavori [88]. Un' altro aspetto che riguarda i ristoranti, non ancora molto studiato dal punto di vista del RM, è il problema della configurazione ottimale dei tavoli (the table mix problem) [148], [149], [85], [86]. Lo studio condotto durante l'attività di ricerca ha portato alla definizione di modelli e di approcci risolutivi propri del RM applicati ai ristoranti. In particolare, il problema è stato affrontato da un punto di vista strategico e da un punto di vista operativo. Da un punto di vista strategico, il problema è quello di decidere la configurazione della sala ossia il numero di tavoli di diverse dimensioni che costituiranno il ristorante. In tale decisione intervengono numerose variabili quali la dimensione della sala, la domanda di tavoli di una data dimensione, la durata media del pasto di gruppi di varie dimensioni, la combinabilità o meno dei tavoli. In relazione al problema strategico sono state definite diverse formulazioni ottenute a partire da un modello base arricchito di volta in volta di nuovi aspetti. Dal punto di vista operativo, è stata considerata la necessità di prendere delle decisioni riguardanti l'assegnazione dei tavoli a gruppi di clienti nel modo più vantaggioso possibile. A questo scopo è stato formulato un modello di programmazione dinamica e successivamente una sua approssimazione lineare che permette di definire opportune politiche di accettazione/rifiuto di una prenotazione da parte di un gruppo di persone. In particolare, sono state definite politiche di booking limits e bid prices e un'estensione della politica booking limits che tiene in considerazione la possibilità di accettare una richiesta se il cliente accetta di attendere un certo tempo prima di occupare il tavolo. I risultati ottenuti sono piuttosto incoraggianti dal momento che mostrano un miglioramento in termini di rendimento rispetto a quelle che sono le tradizionali politiche di gestione delle richieste, come la first-come first-served.

Nel Capitolo 7 infine vengono riportate le conclusioni del lavoro di tesi.

Contents

Part I

Introduction

Chapter 1

Introduction

The work presented in this thesis collects the main results achieved in the three years of the PhD program in Operational Research. The research activities have been focused in developing and designing innovative revenue management (RM, for short) models and policies to handle the service sector, with focus in car rental and restaurant industries.

1.1 Motivation

RM represents a set of techniques of data collection, estimation, forecasting, optimization and controls that helps service and product companies to improve their profits through a better management of capacity and demand, by correctly identifying the customer groups that the company has to serve, establishing the right quantity of products and services and the optimal prices to be offered to these customers. RM has been practiced for many years in the airline and hospitality industries, but recently plays an important role in other industries. The RM literature suggests that many businesses are using RM approach and that many businesses would benefit from adopting RM systems.

In this thesis, we propose the application of RM in car rental and restaurant industries since, for these two service businesses, the literature provides a less number of RM applications and solution approaches compared to airlines or hotels. Furthermore, car rentals and restaurants have the framework and general characteristics to an effective and efficient implementation of a RM process.

One of the main objective of RM is managing the capacity allocation for various demand classes and one of the most important decision is to determine whether accept o reject a reservation request. In this thesis we develop mathematical models to allocate limited capacity, cars and tables in the context of rental and restaurant, respectively. We also provide innovative policies to manage booking requests, by using booking limits and bid prices controls.

The practice of RM was used first in a car rental setting in the early 1990s. The car rental industry displays similar characteristics to the airline industry, but there are not many papers in literature that provide RM models or policies to manage capacity allocation and the dynamics or the randomness of the booking process. Geraghty and Johnson [49] propose simple algorithms that implement models for demand forecasting, planned upgrades, pricing, and overbooking in order to integrate the collected information in a yield management system (YMS) useful for deciding the optimal combination and utilization of available cars. Furthermore, different papers provide RM pricing models applied to the car rental industry to set the price of renting a car. Another vein of the literature considers the fleet planning problem for a car rental.

Since scarce attention has been devoted to the operational problems arising in the car/truck rental industry, this thesis analyzes the operational steps related to receiving and meeting demand in the car/truck rental management process and proposes solution approaches based on RM methods and techniques.

In this thesis we provide new RM mathematical programming approaches and procedures for car and truck rental problems. We have two main areas of interest. The first is aimed at understanding and managing the car rental problem. The problem is how assign cars of several categories to different segments of customers, who make a rental request for a given type of car, for a given number of days at a given pickup time. The second area of interest is the problem of managing a fleet of trucks. We consider the problem of a logistic operator that offers a transportation service from a given set of origins to a given set of destinations. The transportation service consists in renting trucks of different capacities to different customers on a given time horizon. In both cases, the logistic operator has to decide whether to accept or reject a rental request.

First of all, we provide a detailed description of the typical car/truck rental activities. We show that the car/truck rental problem is characterized by several kinds of options. A car/truck may be rented by a booking made in advance of the day of rental. A rental booking specifies the car/truck group required, the start and end dates/times, the origin and destination stations of the rental. Companies usually offer an upgrade if car/truck required is not available. Optionally, the reservation may specify a one-way rental (in which the car is returned to a branch different from the pick-up branch) and may request a specific car model within the required group. Generally, the car/truck rental businesses have different rates depending on the type/group of car rented, and on the rental period.

For the first area, we propose two different approaches to car rental problem: robustness measures and the related scenario-based formulations; linear programming approximations and the related RM decision policies.

The robust approaches aim at balancing expected revenue with feasibility, performance or regret in given scenarios. To the best of our knowledge, this work represents the first attempt to apply robust optimization approaches to the car rental industry. We present a mathematical formulation of the considered car rental revenue optimization problem, which can be viewed as an extension of the deterministic linear programming model used in standard RM [144]. The proposed model allows to determine, for each category and for each time period, the number of vehicles to be rented and the number of upgrades, with the objective of maximizing the expected revenue, while satisfying demand and capacity constraints. Then, we focus our attention on proactive approaches to deal with uncertainty. In particular, the problem under study is represented by considering robust scenario-based formulations. Indeed, it is assumed that the uncertainty is represented by a set of possible realizations, called scenarios. Furthermore, different measures of robustness are considered. More specifically, the attention has been focused on the maxmin criterion (i.e., the application of a simple absolute robustness measure, maximizing the worst-case performance), the robust deviation criterion, the stochastic p-robustness criterion and a standard deviation based variability criterion. The results collected indicate that it is not possible to find a criterion that allows to obtained the best expected revenue value in all the test problems. In addition, on average the mean absolute deviation based variability criterion shows the best performance in terms of solution quality.

The second approach to car rental problem considers a dynamic programming formulation for the problem of assigning cars of several categories to different segments of customers, with the possibility of upgrading, i.e the rental firm can satisfy the demand for a given product with either the product requested or with a car of at most one category superior to that initially required. We also address the one-way rental scenario, which allows the possibility of the rental starting and ending at different locations, and we model a new real aspect of rental process, that is car transferring, i.e the ability to move the fleet from one station to another when the benefit of such operation is greater than the cost of moving the car. Since the proposed dynamic programming formulations are impractical due to the curse of dimensionality, linear programming approximations are used to derive RM decision policies to establish whether to accept or reject a rental request. Indeed, primal and dual acceptance policies are developed (i.e. booking limits, bid prices) and their effectiveness is assessed on the basis of an extensive computational phase. In order to evaluate their effectiveness, we compared these approaches with a typical first-come first-served policy and with the case of a perfect knowledge of the realized demand. We also assess the validity of the proposed policies in a real setting. Experiments reveal that the use of the proposed policies could help the logistic operator to control the capacity levels, to improve customer service and fleet utilization, by maximizing the revenue.

The second area of interest considers the problem of a logistic operator that rent trucks of different capacities for transportation service from a given set of origins to a given set of destinations on a given time horizon, with the possibility to apply the upgrading strategy, i. e to assign a truck of greater capacity to a certain customer. For this problem, a dynamic programming formulation and linear approximations are defined and primal and dual acceptance policies, that use partitioned booking limits and bid prices controls, are provided to accept or reject a request at a certain time. We also provide mathematical formulations and the related RM policies to consider specific aspects of truck rental process, such as the possibility of loading multiple demands on the same truck (i.e., truck sharing) and the repositioning of empty trucks from nodes, where they are not used, to nodes from which a new transportation request could be satisfied.

In a second part of the thesis, we consider the application of RM to restaurant industry. We investigate the restaurant RM problem from both a strategic and operational point of view. From a strategic perspective, we formulate different variants for the table mix problem by considering different aspects, such as the number of potential customers, the expected meals duration and the available space or restaurant dimensions. We also consider the possibility of combining tables, i.e satisfy the demand of a group of people not only by table with number of seats equal to the number of customers or greater but also by combining tables with seats less than customers. From an operational point of view, we assess the decision about how to assign tables or combination of tables to customers in a more profitable way. The problem is formulated as a dynamic program. Due to the curse of dimensionality, a linear programming approximation is proposed and RM policies, based on partitioned booking limits and bid prices controls, are presented. Moreover, we propose a new policy called later accommodation booking limit policy to consider the possibility of proposing to the customer, requiring for a table at a certain time of the meal horizon, a later starting time for the meal.

1.2 Goals

The scientific literature provides several solution approaches to solve the different RM problems. This management technique is widely applied in many service/product industries. Because of this, we analyze the RM applications in traditional and recent businesses industries from different perspectives in order to increase the knowledge on RM. In the light of RM success, we try to adapt RM approaches to the needs of car/truck rental and restaurant industries. Not a large body of literature deals with the RM applications in these two business fields. But we show how the RM methodologies can be implemented in the car rental and restaurant industries in a very innovative, effective and efficient manner.

The main goal of this thesis is to provide innovative RM models and policies to solve the capacity allocation problem, i.e how efficiently allocate a limited resource (cars or tables) among requests for service, by determining whether or not each service request received should be accepted or rejected, with the objective of increasing the organization's profit.

1.3 Contribution

In this thesis we propose innovative models and policies to address the following problems.

1. Optimal strategies to manage capacity and booking reservations for car/truck rental service through the definition of:

    - RM policies and robust optimization approach to deal with car rental problems;
    - RM policies to deal with truck (i.e means of transport with different load capacity) rental problems;

2. Optimal strategies to manage capacity and booking reservations for restaurant service through the definition of RM policies.

The proposed research problems are of both theoretical and practical importance. First of all, the detailed analysis and description of the car/truck rental process will contribute further to the effective application and growing understanding of RM techniques to manage capacity allocation and demand. This is the first time, to our knowledge, that RM techniques are applied to the problem of a car/truck rental on a network by considering at the same time several aspects of this business process. In fact, the possibility of considering upgrades, one way rentals, truck sharing, car transfer and repositioning of empty truck in the same mathematical model to optimally allocate limited capacity and the applicability of these models to develop acceptation policies to decide dynamically when to accept an incoming rental request, are our main contributions within the RM literature. Furthermore, dynamic programming models, that include upgrades and one-way rentals, are developed to represent the optimal management of a car/truck rental business, that are not presented within the RM literature up to now. In order to validate the performance of the developed policies and their applicability, an extensive computational study is carried out by considering a large set of randomly generated test problems, defined trying to be quite close to the reality of medium-sized car rental agencies. The computational results collected are very encouraging, showing that the proposed models can be used to address the problem under consideration and the proposed policies can be used to take profitable decisions in assigning resources. We also underline that, to the best of our knowledge, our robust optimization approach represents the first attempt to apply robust optimization to the car rental industry.

The main contribution of this thesis for RM restaurant field reside in presenting different models and solution approaches revenue based for two type of problems. The strategic problem, i.e to decide the best table configuration for a new restaurant. The operative problem, i.e to assign tables to customers in the more profitable way. For the first problem, we present different formulations at crescent level of details and the extensions at the tables combinability case. For the second problem, we give a dynamic formulation and a linear programming approximation of the novel parties mix problem to deal with the possibility of combining tables. In order to decide about accepting or denying booking requests, different control policies are developed.

1.4 Organization of the thesis

The rest of the thesis is organized as follows.

In chapter 2 we provide an overview of the literature, covering both RM, car rental and restaurant related papers. The chapter presents the origins, the key characteristics and the main business applications of RM and the necessity of extension of RM practices within the car rental and restaurant industries is emphasized.

Chapter 3 gives a description of the main characteristics of the car rental process.

In this chapter an innovative mathematical model, defined to represent the car rental RM problem, is presented. To incorporate demand uncertainty, several additional robustness measures and the related scenario-based formulations are provided.

Chapter 4 covers some strategic and tactical planning aspects of the car rental process considered in RM literature. First, the car rental problem, more detailed than the problem illustrated in the previous chapter, is described. Dynamic programming formulations of the problem are given for the basic rental problem (BRp, for short) and for an extension version, which considers the one-way rental strategy (OWRp, for short) and the related linear programming approximations are presented. Finally, in the chapter we provide a RM approach to these car rental problems with description of the proposed RM policies, based on the solution of the linear problems.

In Chapter 5 the trucks rental problem (TRP, for short) is discussed and its dynamic programming formulation is given. The linear programming formulation for the TRP, together with the description of some RM primal and dual acceptance policies, based on the solution of the linear problem are given. The theoretical issues of the proposed policies are also investigated. Furthermore, a new policy that considers the possibility to apply sharing is defined. New versions of the TRP, incorporating sharing and the repositioning of empty trucks, are also exploited and related partitioned booking limits and bid prices policies are defined.

In Chapter 6 the restaurant RM problem is investigated. Different formulations, at crescent level of details, of the table mix problem are given for a new restaurant that have to decide the best table configuration. Extensions at the tables combinability case are presented. We give a dynamic formulation of the new parties mix problem followed by a linear programming approximation of the problem. The chapter concludes with the description of the several proposed RM control policies.

Conclusions are given in Chapter 7. This chapter summarizes the results of research activities illustrated in the thesis and discusses the potential applications of the proposed models and policies. The Chapter concludes with direction for future research.

Chapter 2

Revenue Management: basic concepts and applications

2.1 Introduction

Every day all industries and firms face the challenge of maintaining or improving their revenues. Yet greater competitive pressures are making it more difficult to generate these additional revenues. This is especially true in industries where the inventory or sales opportunities are perishable, in that they cannot be used after a certain time: the potential revenue from an airline seat is lost if it is not filled by the time the flight leaves; a railway seat is lost if it is not filled by the time the train leaves; the revenue from a hotel room left empty for a night is lost; a rental car left idle during a day is a revenue loss.

To react to dynamic economic changes and increasing competition, firms usually implement actions intended to increase profits by reducing costs. In contrast, evidence indicates that companies that focus on revenue growth are more profitable than companies that concentrate on cost reduction. Traditional solutions such as cost-cutting will always play a role, but on their own they are unlikely to create a lasting competitive advantage because companies will be following similar strategies and may take turns to be market leader, often with lower prices and profits. As a result, more companies try to apply revenue management (RM), which can lead to a sustainable advantage, and this advantage is generally enhanced if competitors choose the same route. RM refers to the process of generating incremental revenues from existing inventory or capacity through a better administration of the sale of a perishable product or service and is the collection of strategies and tactics firms use to manage customer behavior and demand to maximize profits.

One definition of RM is defined as "an order acceptance and refusal process that employs differential pricing strategies and stops sales tactics to reallocate capacity, enhance delivery reliability and speed and realize revenue from change order responsiveness in order to maximize revenue from preexisting capacity" [59]. More simply, RM can be defined as "selling the right product to the right customer at the right time for the right price" [35]. Despite its widespread use, many researchers have asserted that there is no standard definition of RM in the literature (e.g [67], [157]), because the definition is evolving over the years depending on the focus of a specific RM paper.

RM tools and techniques developed in the airlines (see Section 2.2) but during the past decades spread to other industries, such as car rental, restaurant, banking, broadcasting, electric utilities, healthcare, hospitality, telecommunications, transportation, printing, etc. The general principles of RM are widely applicable. The application of RM principles depends on each company's competitive situation and specific activities, but the implementation process is similar for all firms. The applications of RM are discussed in the following Section 2.4.

RM has recently gained attention as one of the most economically significant and rapidly growing applications of operations research. RM deals with sales decisions, demand management decisions and supply decisions, so the term RM refers to a wide range of techniques, decisions, methods, processes and technologies. In reality "RM is a very old idea...What is new about RM is how these decisions are made...a technologically sophisticated, detailed, and intensely operational approach to making decisions driven by advances in science and technology managed by a human decision makers" [154].

In addition to researchers who examined RM in the context of specific industries (see Section 2.4), some provided overviews or general models and applications. The most comprehensive survey articles are those of Weatherford and Bodily, McGill and VanRyzin, van Ryzin and Talluri.

Weatherford and Bodily in [157] give an overview, examining over 40 articles, of the types of problems addressed and some of the models applied and, in order to categorize RM problems, propose a taxonomy of 14 distinguishing elements, such as the type of resources (discrete or continuous), the prices (predetermined, set optimally, set jointly with inventory decisions), the number of discount classes, decision rule types, etc. They examine some of the currently addressed problems and suggest realistic problems that have not been considered yet.

McGill and van Ryzin [113] give a research overview of transportation RM in its four key areas of forecasting, overbooking, seat inventory control and pricing. The survey reviews the forty-year history of RM, by analyzing over 190 references and includes a glossary of RM terminology.

van Ryzin and Talluri [154] provide an introduction to RM, based on excerpts from their book "The Theory and Practice of Revenue Management"[144]. They give an overview of the field, its origins, applications, models and methods used.

In addition, several books on RM have been published that address several issues about RM.

In "Revenue management: hard-core tactics for market domination" Cross [35], draws on case studies to present revenue generating strategies. Cross describes no-tech, low-tech, and high-tech methods that managers can use to increase revenue by meeting the challenge of matching supply with demand; to predict consumer behavior; to tap into new markets and to deliver products and services to customers effectively and efficiently.

The most important and best-known tome on RM is "The Theory and Practice of Revenue Management" of Talluri and van Ryzin [144]. It is the first comprehensive reference book to be published about RM. It covers theory, relevant research, industrial practices and details of implementation the entire field of RM. A central objective of the book is to unify the various forms of RM and to link them closely to each other and to the supporting fields of statistics and economics. The purpose of the book is to provide a comprehensive, accessible synthesis of the state-of-the-art in RM and of its related topics.

Another book that should be mentioned is "Revenue Management" of Danilo Zatta, who gives a comprehensive description of different aspects of RM and illustrates several case studies [167].

Finally, one of the most recent overviews on RM is the paper written by Chiang et al.[30]. The paper provides a comprehensive review of the recent development and progress of RM in different industries in recent years, especially after 1999 and examines 221 articles.

In the following, we focus on publications that are more related to our work, paying attention to point out the recent literature on different RM problems, methodologies and applications.

2.2 The origins of RM

RM, initially called yield management, originates from the airline industry in the $'70s$ [123]. The first application of RM came after the Airline Deregulation Act of 1978 in the USA as a need for airline companies to protect their business from aggressive new competitors in the post-deregulation era. Before 1978 the American airline routes, fares and frequency of flights were controlled by the Civil Aeronautics Board (CAB). The introduction of the Airline Deregulation Act established the elimination of all fare restrictions on domestic routes and the loss of CAB control and led to the introduction of new services to encourage new entrants into the airline business. These new airlines were able to price profitably much lower than the major carriers, by determining a significant reduction of price-sensitive or leisure travelers for the American Airline. In order to recapture these customers and to compete against the public charters, the strategy of American Airline was based on the control of surplus seats on each flight and of their fixed costs together with finding a way to increase revenue. To this aim, American Airline introduced different strategies like purchase restrictions, i.e additional discounts (Super Saver Fares) for travelers that purchased the seat a certain number of days before departure; capacity controlled fares, i.e number of discount seats that are limited on each flight, in fact, by offering the lower fare the airline could, in principle, risk losing the passengers paying the full fare. To this aim, American Airline segmented the market between leisure travelers, who were able to book early and who were more price sensitive, and business travelers, who were less price sensitive but needed seat availability at the last minute; used differentiated pricing to attack competitors, which allows every seat to be sold at a low fare. After few years, a Dynamic Inventory Allocation and Maintenance Optimizer (DINAMO) system was developed and implemented. This system constitutes the first large-scale RM system application. As a consequence of the positive results obtained in American Airline, which generated an additional incremental revenue of \$1.4 billion in three years [138], other major companies (United, Delta, Continental) implemented computerized RM systems. Furthermore, in the $'70s$, the deregulation of the air travel markets was taking place also in Europe and Asia and other airlines companies began to adopt RM. As an example, in 1972 British Overseas Airways Corporation (now British Airways) offered the so called "Earlybird" discount for bookings that arrive at least twenty-one days in advance. The effective control of discount seats required the development of solution methods for the airline RM problem and the application of specific seat inventory control rules. A first rule, proposed by Littlewood in [108], established that discount fare bookings should be accepted as long as their revenue value exceeded the expected revenue of future full fare bookings [113]. Even if a variety of solution methods have since been introduced, the Littlewood' rule constitutes the basis of modern RM stategies.

2.3 Key areas of RM

The four main research areas in RM are forecasting, overbooking, inventory control and pricing [113], [128]. These critical topics have been addressed independently in several works or some of these aspects were studied jointly.

Overbooking. The practice of overbooking involves selling more capacity than actually exists to counterbalance the effect of cancellations and no-shows. Overbooking balances the risks of spoilage and denied boarding. In fact, overbooking as an integral part of RM has received significant attention in the literature. This is the most studied RM problem, particularly in the airlines, and exhaustive list of references of overbooking are given in [157],[113],[30]. Rothstein in [133] provides the first dynamic programming formulation for the overbooking problem in airline and hotel industries. The paper of Karaesmen and van Ryzin is particularly interesting, who in [70] propose a generic method for coordinating overbooking and seat allocation decisions in general networks and provide a comprehensive overview of the main papers addressing appropriate overbooking and capacity control models and methods. The overbooking problem has been widely handled also in hotels [96], [95], [105]. In addition to surveys mentioned above, a recent publications about overbooking in the hotel industry is [65], in which mathematical models and techniques for calculating the optimal number of overbookings are proposed and a rich literature review is presented. Other recent papers are [55], which examines the practice of overbooking in hotels dealing with multiple tour-operators; [10] which develops two algorithms that integrate overbooking with the allocation decisions, by simulations modelling realistic hotel operating environments, and comparing the performance of five heuristics under 36 realistic hotel operating environments. Moreover, we can find good applications of overbooking in other service industries, for example [104] applies the thought of the real options analysis to construct a cruise line overbooking risk decision model.

Forecasting. One of the key principles of RM is the firm's ability to forecast demand. This issue is also strongly correlated to overbooking because correct overbooking calculations depend on forecasting of demand, cancellations and no-shows. Lee [100] divides RM forecasting methods into three types: historical booking models, advances booking models and combined models. For a review of forecasting in RM see [155] and [30]. In [113] models for demand distributions and customers' arrival processes are presented in chronological order, and aggregate and disaggregate forecasting in the airline industry are compared. Zaky [166] examines forecasting for airline RM; In [159], [158] a variety of forecasting methods for hotel RM systems using data from Choice Hotels and Marriott Hotels are presented and tested. A recent work about demand is [152], which reports some considerations on the limitations of RM current demand models. In [141] a novel model for airline demand forecasting by estimating the model from historical data is investigated.

Pricing. Most firm that practice RM rely on competitive pricing methods. The pricing problem is how to determine the price for each customers class and how to vary prices over time to maximize revenues. Research into pricing and related issues is extremely widespread and spans a variety of disciplines. The papers by McGill and van Ryzin, Elmghraby and Keskinocak, Bitran and Caldentey, and the book by Talluri and van Ryzin provide comprehensive overviews of the areas of dynamic pricing and RM. In [113], [30] a detailed collection of the main papers that investigate the pricing strategies in RM is given. Desiraju and Shugan [39] investigate pricing strategies based on RM, such as early discounting, overbooking and limiting early sales for capacity constrained services. Bitran and Caldentey [22] examines the research and results of different dynamic pricing models and policies and the different optimisation techniques for the deterministic solution of the pricing problem. Elmaghraby and Keskinocak in [44] provide an overview of dynamic pricing practices in the presence of inventory considerations. Anjos et al. in [5] present a family of continuous pricing functions for which the optimal pricing strategy can be explicitly characterized and easily implemented. A noteworthy work in these fields is the book of Phillips," Pricing and Revenue Optimization" [128], which can be considered the first comprehensive introduction to the concepts, theories, and applications of pricing and revenue optimization. Recent related papers are [33], where major milestones in the field of pricing and RM are identified and the adaptation of these concepts to diverse industries are described; in [112] multiproduct dynamic pricing models for a revenue maximizing monopolist firm are reviewed and a connection between these dynamic pricing models and the closely related model where prices are fixed is described. Moreover, it is possible for the seller to control dynamically how to allocate capacity to requests for the different products. Given the current importance of RM and dynamic pricing, a special issue of EJOR [102], is dedicated to them. This issue contains contributions in network airline passenger RM, air cargo overbooking, optimal pricing in airline and general service settings, dynamic pricing in competition, and dynamic pricing with strategic customers. See also Gallego and van Ryzin ([46], [47]) for discussions about how to address pricing and inventory decisions simultaneously, particularly in the airlines.

Capacity control. Another critical problem in RM is capacity control (or seat inventory control in the airline industry). The objective of capacity control methodologies is to define how to allocate capacity optimally to differentiated classes of demand. The aim of this thesis is to develop and apply capacity control policies based on RM in specific service settings. These policies aim to accept or reject requests for a given product, by optimally allocating the available capacity to customers classes.

Many articles on capacity control have been published by authors who provide various models and methods for making capacity decisions since 1972 when Littlewood proposed his rule for two fare classes [108]. Inventory control problems have usually been analyzed by considering two topics: single-resource and multiple resource or network capacity control. A review is provided by McGill and van Ryzin in [113] and for recent articles see [30].

Single-resource capacity control: refers to allocating capacity of a single resource to different demand classes, for example the sale of a single flight leg to different fare classes. Littlewood's rule for two fare classes, defined for airline problem, represents the earliest single-resource model. The question is how much demand for a class should be accepted so that the optimal mix of passengers is achieved and the highest revenue is obtained. By assuming demand in lower fare class precedes demand in higher fare classes, Littlewood suggests closing down a class when the certain revenue from selling another low fare seat is exceeded by the expected revenue of selling the same seat at the higher fare. This suggests that there is an optimal protection limit, i.e the total number of seats restricted to booking in one or more fare classes. Littlewood's model is limited to two classes. Belobaba in [14], [13] and [15] developed heuristic extensions of Littlewood's rule to multiple fare classes, assuming the probability distribution of demand in each fare class is known, and developed a model based on this rule called Expected marginal seat revenue (EMSR) which is an n-class model, and then correlated heuristics (EMSR-a, EMSR-b). Subsequently, a number of authors have discussed the single-resource capacity control, see [113] and [30] for a review. Netessine and Shumsky [120] focused on how allocate perishable inventory among a variety of customers segments and applied RM tools in airlines, hotels and car rentals, by considering the basic EMSR model. More recently, Talluri and van Ryzin [145] analyzed the multiple-fare class problem based on consumer choice. While this approach is more sophisticated and realistic than many, it requires information not only on the arrival process but also on the customers' choice behavior. Lan et al. [99] considered the classical multifare, single-resource (leg) problem in RM when the only information available is lower/upper bounds on demand. They provided also a consistent literature review. See also, [154] for a complete and deep recent examination of the single-resource RM problems and relative models and methods of solution.

To address the single-resource capacity case there are different RM approaches or types of controls (see [154] for relevant references). Among them, the most important are described in what follows. *Booking limits* that limit the amount of capacity that can be sold to any particular class at a given point in time. Booking limits are either partitioned or nested: a partitioned booking limit divides the available capacity into separate blocks (or buckets), one for each class, which can be sold only to the designated class. With a nested booking limit, the capacity available to different classes overlaps in a hierarchical manner, with higher ranked classes having access to all the capacity reserved for lower ranked classes (and perhaps more); *Protection levels* specify an amount of capacity to reserve (protect) for a particular class or set of classes. Again, protection levels can be nested or partitioned; *Bid Prices* are revenue-based, rather than class-based, controls. A bid-price control sets a threshold price (which may depend on variables such as the remaining capacity or time). Bid prices must be updated after each sale and possibly also with time.

The tutorial of van Ryzin and Talluri [154] contains a comprehensive review of models used to approach single-resource problem. In [154] the main static models, Littlewood' two-class model, n-class models and the RM heuristics used to calculate booking limits and protection levels are examined and the main assumptions that characterized the static models are explained. Among these, there are assumptions about demand for different classes that arrives in nonoverlapping intervals and in a low-to-high revenue order, about demands that are independent random variables, in the sense that do not depend on the availability of other classes; about the group bookings that can be partially accepted, etc.

Furthermore, the n-class models can be formulated as a dynamic program. The formulation and structural properties of dynamic models are described in [154], [28], [146]. Dynamic models are formulated to make optimal decisions over time under uncertain conditions and to control the state variables. In the RM field, dynamic models help to decide whether to accept or reject a booking reservation and to control the capacity. Dynamic programming considers an arbitrary order of arrival, the relaxation of the assumption of low-to-high revenue order, the limitations on the distributions of demand to use, and other assumptions that are similar to static case.

Multiple resource or Network capacity control: considers the capacity allocation when customers require a bundle of different resources simultaneously, for example a room occupied for multi-night stays in a hotel, a car rented for a certain number of days, two connecting flights etc. There are significant revenue benefits from using network approaches, but they require an increasing complexity and volume of data, in fact optimization methods are more complex and require several approximations because an exact optimization became computationally intractable [161], [146]. Several methods to address this RM problem have been developed from the origin-destination control problem in the airline industry. For the earliest contributions see [113], which presents a review of the Origin-Destination problem (also called origin-destination fare control ODF) mathematical formulations from 1982 to 1999 and three approaches that have been taken to address network RM problem: segment control, that considers a partial solution to the O-D control problem because does not involves connections between flights but only the single flight segment or leg; virtual nesting, which represents a technique to grouped multiple ODF classes into virtual buckets, i.e a set of fare classes, on the basis of revenue characteristics; bid-price methods, which establish marginal values for incremental resources by using dual variables of the basic linear programming model in order to accept or reject a booking request. The survey of Chiang et al. [30] contains references about network RM in other industries, like hotels [97]. For a simple examination of the major categories of network controls, see [146]. The authors consider the *Virtual nesting controls*, developed in airlines [138], uses single-resource nested booking controls at each resource in the network, the classes used are based on a set of virtual classes and products are assigned to a virtual class through a clustering (or indexing) process [113]; Furthermore, they analyzes the *Network bid-price controls* are a simple extension of their single-resource versions. These controls use dual prices from a deterministic LP model to estimate marginal values for incremental unit of the resources capacity [137],[161]. A request for a product is accepted if the revenue exceeds the sum of the bid prices related to the resources that constitute the product, otherwise it is rejected [113], [144].

In [146] the authors also describe the basic model of the network allocation problem and the construction of the typical dynamic program to determine optimal decisions for accepting or rejecting a request. Then they look at two approximations methods for dynamic formulations that cannot be solved exactly, i.e a deterministic linear programming model (DLP) and a randomized linear programming model (RLP). In the DLP stochastic demand is substituted by its mean value so that it became computationally simple to solve, the RLP method consists of simulating a sequence of realizations of itinerary demand and solving deterministic linear programs to allocate capacity to itineraries for each realization. The dual prices from this sequence are then averaged to form a bid price approximation [143]. Several models and algorithms for solving the stochastic and dynamic network RM problem have been provided by Bertsimas and Popescu in [19], they proposed a new efficient control algorithm, based on a certainty equivalent approximation and compared it with the widely used bid-price control policy.

A significant limitation of the applicability of these methods is the assumption, made for analytical convenience, of independent demand for different classes of customers [154]. In response to this, in recent years interest has arisen to incorporate *customer choice* into these models, further increasing their complexity, see for example [144], [109], [153], [168]. Recently, new tools to manage the more realistic situation of dependent demand RM have been designed [160].

RM problems are typically modeled as dynamic stochastic optimization problems (DP), with future demand unknown but that can be described by a stochastic process or a probability distribution. The DP usually suffers from intractability due to the curse of dimensionality, i.e. the size of the problem increases exponentially with the horizon considered. There is another important field of stochastic optimization, called *robust optimization*. The idea is to restore mathematical tractability, using mathematical programming techniques, while making fewer assumptions about the demand [147], [126]. Different decision-making criteria can be used: the maximin revenue, minimax regret, competitive ratio, etc. An overview of robust optimization approaches will be provided in Chapter 3.

2.4 The applications of RM

There are numerous demonstrations of the effective impact of RM in various industries. Generally, companies using RM have reported increases in revenue ranging from 2 to 5 percent [57], [138]. All major international airlines have already implemented RM systems. American Airlines, one of the pioneers, increased its yearly revenue by 500 million as a result. Delta Airlines has gained 100 million/ year [26]. Many other airlines in Europe and East Asia have reported significant gains. In the mid-1990s, at the Marriott Hotels, the execution of RM have added between $150 million and $200 million to the annual revenue [35], [62]. Orkin in [121] suggests that hotels can benefit from adopting RM systems. In particular, he claims that Hyatt Regency's average rate for all reservations increased after the adoption of RM techniques. He also states that many Hilton hotels have set revenue records since instituting RM. Harrahs Cherokee Casino and Hotel, in the USA has used RM and improved yield by 3-7 percent and revenue by 15 percent in 2008. IBM initiated a broad, analytics-based project called On TARGET. It provides a set of analytical models designed to improve revenue. These systems had a revenue impact of approximately $500 million. National Car Rental was saved from bankruptcy by RM and has increased its revenue by $56 million/year [49]. Ford Motor Co. has been quietly enjoying a huge surge in profitability, from 1995 to 1999 revenue was up from about $3 billion to $7.5 billion, of that $4.5 billion about $3 billion came from a series of RM initiatives [101]. National Broadcasting Corporation has increased its annual revenues by $50 million/year in improved ad sales from 1996 to 2000. The French National Railway has increased its revenue by 110 million francs/year. German Railways are using RM. GMAC in 2002 launched an early implementation of web-based RM in the financial services industry. Texas Children Hospitals is using RM had competitive advantage [25]. RM has improved the contribution to profit for a major steel company (Tata Steel) in India by $73 million in 1986/87 and given a cumulative impact of hundreds of millions of dollars in later years. In addition analytics driven Operations Research and RM techniques have helped Intel, CSX Railway, USA, Canadian Pacific Railway, Netherlands Railways, to improve their bottom line significantly.

"RM concepts will be applied to almost everything that can be sold and will prove to be such a powerful competitive weapon that major firms will be living, and in many cases dying, according to revenue management algorithms. The firms with the best revenue management will prosper and grow; the remainder will struggle to survive by restricting themselves to local or niche markets" [1].

2.4.1 RM preconditions

There are a number of characteristics common to industries where RM is traditionally applicable and these features seem to be valid pre-conditions and prerequisites for the

[1] Professor Peter Bell, Richard C. Ivey School of Business and former President of International Federation of Operational Research Societies.

success of RM applications [73],[157]:

- Perishable inventory. RM techniques can be applied to services or products that perish or lose all of their value after a specific date and cannot be stored. For example, an airline seat is unavailable for sale after a flight has departed, seats for the theater, a sporting event, or a restaurant; space on any means of transportation, in lodging, or for apartment rental; fashion or high technology goods; broadcast advertising time periods; etc.

- Relatively fixed capacity. RM is suitable for service industries where capacity is fixed or rather relatively fixed because it is possible to add or subtract inventory but it is usually very expensive or impractical in the short run.

- Segmentable markets. Customer heterogeneity is critical for implementing RM. Demand for the service can be divided into market segments characterized by different sensitivity to prices. The common mechanism used to segment customers in RM situations is the time of purchase, i.e the less price-sensitive customer generally waits until the last minute to make reservations, whereas the more price sensitive people make their reservations early for a reduced price.

- High fixed costs, low variable costs. The cost of selling an additional unit of the existing capacity is low relative to the price of the service.

- Advance reservations. Booking requests can be evaluated and accepted or rejected in advance of the performance of the service using logic programmed into the computerized reservation system. Advance purchasing also requires dealing with the following problem: accept a request and sell the product or service at the current rate, or reject and wait for a new better request until a later date and possibly a higher purchase price?

- Time variable demand. Demand varies over time (seasonally, weekly, daily, and so on). There are definite peaks and valleys in demand, which can be predicted, but not with a high degree of certainty so decisions became difficult. RM can be used to smooth the demand variability, by increasing the price during periods of high demand and decreasing the price during periods of low demand.

Equally important are the two following conditions underlined in [154]:

- Data and information system infrastructure: RM require database and transactions systems to collect and store demand data and automate demand decisions.

- Management culture: it is important to implement a RM system in a realistic context, that a firm's management had a culture of innovation and problem solving and a sufficient familiarity with the concepts of RM.

Many sectors, including transport, travel, hospitality, communication etc., are concerned mainly with the sale of perishable goods and services. Firms in these industries can increase their profits significantly by applying the principles of RM. More recently compared to development in airlines, RM practices can be encountered in many other service industries, such as car rental and hotels, considered with airlines traditional RM applications, and restaurants, cruise lines, sporting events, to name a few, considered recent and non-traditional applications of RM. The book "Revenue management and pricing: Case studies and applications" [163] contains a wealth of new cases, best practices and a broad range of experiences covering many industries and service sectors. It also complements the book "Yield Management: Strategies for the Services Industries" [64], as it shows how RM really works in practice.

In what follows some of the most relevant applications of RM in different industries are presented [30] [33].

2.4.2 RM in traditional industries

Airlines, hotels and car rentals industries represent the traditional applications of RM [120]. In the following sub-sections, some directions on recent RM problems and researches are given, paying particular attention to car rental industries.

Airlines. The development of RM techniques began in the airlines in the 1960s, with published work by Littlewood [108] on a two-fare single-leg problem and Rothstein [133] on overbooking policies. Since then, many researchers have extended RM techniques and the types of problems considered in the airlines. Its first application was optimizing revenue associated with individual passengers on a single flight leg [14]. A discussion of the earliest applications within airlines is provided by [138], that describes in detail the history and implementation of RM at American Airline. The paper contains the milestones in the airline industry that have affected the new RM techniques: overbooking, discount allocation, clustering fares, estimations and modeling of demand. An important survey paper of the past airline literature and main results in RM is provided by [113] and a recent research study that should be required reading is the overview of Chiang et al. [30].

The seats on a flight are perishable products which can be offered to different customer segments for different prices (fare classes). The seat inventory control problem concerns the allocation of the finite seat inventory to the demand that occurs over time before the flight is scheduled to depart. The objective is to find the right combination of passengers on the flights such that revenues are maximized. The optimal allocation of the seat inventory can be managed with a booking control policy, which determines whether or not to accept a booking request when it arrives. Airlines deal with the importance of demand forecasting because booking control policies make use of demand forecasts to determine the optimal booking control strategy. In order to prevent a flight from taking off with vacant seats, airlines tend to overbook a flight to avoid cancellation and no show. The level of overbooking for each type of passenger has been the topic of research for many years. Airlines also adopt pricing because price differentiation is the core issue of

RM [123]. The identification of these key issues and the applications of the basic principles to solve the problems in other industries determined the spread and the ongoing innovative implementation of RM.

Hotels. The hotel industry has made effective use of RM principles and Marriott International was a pioneer in implementing a RM system for the hospitality industry [8], [57], [35]. Recently, Chiang et al. in [30] and Anderson and Xie in [4], provide a chronological list of papers dealing with hotel RM; Cross et al. in [32] give a summary of the current and future status of hotel RM. [81] provides the results of an online survey of nearly 500 RM professionals and a framework on how hotels can best position themselves for the future. A general overview of RM practice in the hotel industry is provided by [73] and [41]. Orkin in [121] outlines some of the ideas behind RM for hotels and provides examples of the types of calculations. Early work on overbooking of hotel reservations was performed by [134], and stochastic cancellations of customers in a single day period was considered by [105]. Bitran and Gilbert in [24] modeled hotel reservations incorporating uncertain arrivals, and [23] extended previous models to include multiple day stays. Weatherford [156] concentrates on a booking control policy (see also [50]). Baker and Collier in [10] compare the performances of five booking control policies under 36 hotel operating environments. Hotels have to decide whether or not to accept a specific reservation request and the best use of the limited number of rooms, considering the number of reservations and customers show up without reservations (walk-ins). In hotels there is the possibility of downgrading rooms, i.e to give a customers a suite for the price of a standard room when the latter is not available. Rooms in hotels are perishability resources and overbooking is usually applied in order to maximize the expected profit. Three main segments can be identified in hotels: tourists, corporate travelers and groups, characterized by different types of room required and different prices for room, different rejection costs associated with turning them down and different days in which the booking is made.

2.4.3 Car Rentals

The first applications of RM in the car rental industry started in the early 1990s. In fact, only Carroll and Grimes and Geraghty and Johnson provide accounts of the state-of-the-art in car rental RM. In particular, Carroll and Grimes give a detailed description of the implementation of RM at National Car Rental, whereas Geraghty and Johnson analyze the RM approach in Hertz. Carroll and Grimes in [27] describe the story of car rental business, from the first local operators activity in 1918 to 1990 when Hertz began using price and RM system to control the price and availability of cars. Airline deregulation increased the demand for rental cars and small business low-priced entered the market because the percentage of leisure segments increased, changing company market shares. Hertz, initially the major car rental firm in the US, to respond to demand changes, developed system to support decisions about rental fleet and geographic redistribution of vehicles. Hertz developed RM decision support systems to decide: fleet size and composition using historical rental information; how to move cars among locations within a pool, by considering that demand can vary at different locations at different period of time;

rental car products to meet the needs of commercial (or corporate) customers and leisure customers, by establishing rates by car type and advance reservations and by offering price discounts; capacity control to set maximum availability for particular rates and car types, using a capacity management system (CAPS); integration between supply and demand information, developing a new Yield Management System (YMS). Carroll and Grimes give an accurate description of YMS, underlining its potentiality and benefits for Hertz. They also provide the main aspects of car rental RM and the distinct differences from its application to airlines. The rental car perishable product is the car rental day since an unrented car is considered to be a lost revenue opportunity to the business. A characteristic of RM is the ability of the business to segment the market into price-sensitive or time-sensitive customers. Specifically, in the car rental business, customers are categorized into corporate and leisure customers. The following characteristics differentiate the corporate (or business, or commercial) customer from the leisure customer: the different types of cars, the commercial customer typically rents full or mid-size cars, whereas the leisure customer prefers economy cars; to reserve in advance in order to get a discounted rate, commercial customers rent on weekdays for shorter time periods, whereas leisure customers can book multiple reservation with no prepayment required and they do not have to pay any cancellation fees, rent over weekends and for longer periods of time and book several days in advance to obtain discounted rates; the rate paid, corporate customers pay a fixed rate contracted with companies, whereas leisure users pay themselves.

The main differences of car rental RM from its application to airlines are the degree in which price changes, i.e. car rental firms have discrete price classes for the different types of cars (economy versus luxury), but actively change the prices within these classes on a daily basis. A car rental must deal with a restricted fleet but the mobility of inventory is greater; a car rental is characterized by a more decentralized management of inventory and by a large set of customers options: variable duration of rent and the possibility that a car can be returned to a different point from the departure location [27].

Geraghty and Johnson in [49] initially explain the state of the car rental industry in the U.S. from 1980 to 1990, when the car rental companies were taken over by automobile manufacturers, because low margins of profit and the disappearance of tax credits. Then they describe the crisis and the rebirth of National in 1993 due to RM techniques. The implementation of RM analytic models and the development of a RM system (RMS) to manage capacity, pricing and reservation saved National Car Rental. National had a revenue improvement of $56 million in the first year of RMS application. The National RMS processed the forecasting and analytical models to generate recommendations concerning availability, rate and length of rent control in order to make, accept or reject decisions. The analytic models were supported by sophisticated demand forecasts patterns. The business process was characterized by three steps: capacity management, that included fleet planning (how much of the available fleet should be at each inventory location) and planned upgrades (how many high-valued vehicles to make available to lower booking classes); overbooking (more reservations are accepted than can be accommodated to compensate for cancellations and no-shows); pricing, that recommends increased or decreased rates based on on-rent demand; reservations inventory control to identify the length of rent categories on each arrival day that provide the greatest revenue, it processes each booking request and decides if accept or reject it on the basis of a constrained day indicator. Geraghty and Johnson also propose algorithms that implement models for demand forecasting, planned upgrades, pricing, and overbooking.

Nobody, apart from Carroll and Grimes and Geraghty and Johnson, gives a detailed overview of car rental RM. In the recent literature new approaches to RM in the car rental business about problems of pricing and dynamic pricing and fleet planning are proposed.

Anderson and Blair in [2] give an accurate account of RM approaches to pricing for goods and/or services, analyze the practice of dynamically pricing a perishable product across different market segments and outlines Performance Monitor, a phased approach to performance measurement designed and implemented at Dollar Thrifty Automotive Group, Inc.

Anderson, Davison and Rasmussen [3], falls into the pricing model category of RM. They develop a new approach to dynamic pricing, one in which price itself is a random variable. They derived a novel approach to modeling price as a stochastic differential equation and applied it to car rentals. This model is based on the concepts of real option theory ([40], [1]) and is related to the swing options used in the power industries. The model produces minimally acceptable prices and the number of cars available for rent at a given price, but does not include multiple rental periods and multiple car classes.

The PhD thesis of Kopliku [93] evaluate and find an optimal management strategy for pricing problem for a car rental business using the principles of RM, by the following assumptions: the car rental business price like the competition at times of low demand, it offers only one product, i.e. the same type of car that can be rented only for one day, and finally the model does not account for cancellations, overbooking. They provide an overview of the existing literature of RM in the airline and car rental industries and review the theory of dynamic programming and use Bellmans principle of optimality to obtain a partial differential equation (pde) which describes the value of the car rental business with respect to reservation time and price and solve it with a finite difference scheme and Monte Carlo simulations.

One of the primary functions of RM for a car rental company is fleet planning, i.e. to determine the optimal mixture and size of the vehicle fleet that should be available for rent at each location on a daily basis. These locations can be independent or part of a pool, that are a group of rental agencies that shares a fleet of vehicles [122]. Pachon et al. [122] decompose the tactical fleet planning problem into two disjoint subproblems: the fleet deployment subproblem and the transportation subproblem and develop a heuristic to reduce the gap between the optimal solution and the solution provided by the decomposition of the problem into two disjoint subproblems. They present three extensions of the fleet deployment model to include the costs for unsatisfied demand and fleet surplus, service level, and a general price demand function.

For a large-scale car rental company, fleet resource sharing problem can be solved by pool segmentation. Leasing sites in the same pool can share a fleet of cars in a certain period, dispatching of vehicles among leasing sites are mainly realized through the specialized truck or return from remote places by clients naturally [162]. Yang et al. in [162] provide a dynamic model for pool segmentation in the car rental industry, and a heuristic algorithm is given to solve practical problems.

Fink and Reiners in [45] consider logistics management in the car rental business. After giving an overview of car rental operations, they present a novel quantitative decision model to solve efficiently short-term car rental logistics problems by means of network flow optimization. Their decision model includes essential practical aspects such as multi-period planning, a country-wide network, specific transportation relations, fleeting and defleeting, and different car groups.

Other researches closely related to the literature on RM and car rental industries are papers that consider general rental businesses, i.e. rental companies that acquires and maintains an inventory of items which are used by the customers for a limited period of time.

Savin et al. in [135] formulate the rental capacity allocation problem as a problem in the control of queues for a fleet of identical vehicles accessed by two customer classes whose arrival processes are independent, and use dynamic programming to investigate properties of the optimal control policies.

Gans and Savin in [48] consider a rental firm with multiple classes of two types of customers, contract customers and walk-in customers. Rental requests and durations are stochastic. They provide a stochastic model and policies for the treatment of the interaction between different classes of customers.

Papier and Thonemann in [124] present a stochastic model of a rental system with two customer classes that can choose between premium and classic service. Under premium service, customers reserve cars in advance, and they receive a service guarantee in return. Under classic service, customers do not make a reservation and do not receive a service guarantee. Because both demand classes access a common pool of cars, the company must decide which demands to accept and which to reject, without knowing the rental duration, which is an exponentially distributed random variable. They propose an ADI policy applied to a rail cargo operations.

2.4.4 RM recent industries adoptions

More recently, applications of RM are appearing in other non-traditional industries including restaurant (see Section 2.4.5), function space, spa or fitness centre, golf courses, casinos, palapas, tour operators, rail transport, freight and cargo, cruise line, TV broadcasting, Internet services, hospitals, etc. [88], [30], [4], [163]. In what follows some examples and literature references of RM non-traditional applications are provided.

Hospital and health care. This deals with large fluctuations in demand depending on time of day and day of week. Hospital surgeries are often overflowing on weekday mornings but sit empty and underutilized on the weekend. Hospitals may experiment with optimizing their inventory of services and products based on different demand points. Additionally, RM techniques allow hospitals to mitigate claim underpayments and denials, thus preventing significant revenue leakage. Stanciu in [140], provides models and heuristics implemented it in the healthcare industry, more specifically in the operating room area and develops advance patient scheduling and capacity allocation policies. Born et al. in [25] describe the collaboration between Texas Childrens Hospital and the tool PROS RM Solution and their interaction with the Hospital Optimization System (HOS) that have given Texas Childrens a competitive edge in the health-care marketplace, by using RM forecasting and optimization tools. Lieberman in [106], suggests that the application of RM can be extended to healthcare as well, by describing the case of a high quality primary care health centre in California, the Mid-Peninsula Medical Center (MMC).

Cruise lines. Cruise RM has received little attention in the academic literature. The primary goal of RM in the cruise-line industry is to maximise the net revenue received from the sale of cabins at each sailing. Lieberman and Dieck in [107] present decision support tools and modelling approach for the case of the purchase of airfare for cruise passengers. An optimal air planning programme is proposed with emphasis on decision support for routing passengers on flights and negotiating contract fares with the airlines. Toh et al. in [150] describe how rooms are managed in the North American cruise industry (individual as well as group reservations), and then explain why the cruise lines have out-performed the hotels in average cabin/room occupancies. Biehn [21] argues that cruise ships are not floating hotels (contrary to what is stated in Talluri and van Ryzin [144]). Ji and Mazarella in [66] investigate the unique characteristics of cruise line inventory and discuss how RM practices can be adapted to cruise inventory, present an effective solution for cruise inventory application that incorporate a nested class allocation (NCA) model and a dynamic class allocation (DCA) model. Maddah et al. in [111] develop a discrete-time dynamic capacity control model for a cruise ship characterized by multiple constraints on cabin and lifeboat capacities. Ayvaza and Huh in [7] develop a dynamic programming model to solve the resource allocation problem of a health-care facility, and presented the characteristics of the optimal policy and a simple heuristic policy that performs well. Li in [103] proposes a static model which considers two-dimensional capacity constraints, demand uncertainty and different customer types including families and singles and applies four methods to solve it: constrained programming, robust optimization, deterministic programming, and bid-price control. Then proposes a dynamic capacity allocation model to gain an accepting or rejecting optimal policy. Li in [104] employs the real options approach to construct a cruise line overbooking risk decision model with multiple price classes, incremental cost and nonlinear goodwill loss.

Park industry. The nature of the theme park industry suggests potential for enhancing revenue by exercising a variety of RM techniques. Heo and Lee [60] propose applications of RM to the theme park industry, and conduct an empirical analysis of theme park customers perceived fairness of RM.

Nonprofit Operations. Metters and Vargas in [115] extend RM concepts to the nonprofit sector. A general heuristic is presented to assist decision makers in pricing decisions.

The technique is demonstrated at a nonprofit child care center that provides discounts to low-income families. Véricourt and Lobo [38] investigate how a nonprofit organization should dynamically allocate its assets over time between its revenue-generating activities and its mission, in order to maximize the organizations social impact. They model this problem as a multiperiod stochastic dynamic program.

Tourism industry-Tour operators. Harewood [58] proposes a bid price control method for coordinating a decentralized tourism supply chain. In [4] some of the basic complexities that originate with the acquisition of hotel rooms for a reseller of bundled vacations are considered. They present a series of optimization models used by Sunquest Vacations, Canadas number one travel provider. The room-risk management problem is formulated as a math program with the objective of minimizing wasted rooms.

Financial Services. Offer a wide range of products to a wide range of customers. Banks have applied segmented pricing tactics to loan holders, often utilizing heavy amounts of data and modeling to project interest rates based on how much a customer is willing to pay [68].

Golf courses. The characteristics that make a business suited to RM are also common to the golf industry. Tee times are perishable inventory. Courses have a limited capacity and are booked via reservations. Different times and courses have different pricing structures. There is a negligible cost to booking an additional tee time, and unit pricing can have a dramatic impact on overall revenue. The golf course business is similar enough to hotel and airline operations that golf courses should be able to apply RM principles [77]. Kimes and Schruben in [84] explain that golf courses have two strategic levers, round duration control and demand-based pricing. This paper uses simulation to study the most controllable factor of capacity: the tee time interval. A dynamic simulation model is developed, which can be used to quantify the trade-offs in determining an appropriate tee time interval. Rasekh and Li in [132] give an analysis of golf course tee-time reservation practice and present a unique linear model that can be used to assign the demand to the available tee-times, and thus, maximize their utilization and the total revenue.

Broadcasting and Media. Advertisers place orders for commercials. Typically, each order consists of multiple spots, and the airdates of the spots are not fixed by the advertiser. Therefore, the channel has to decide simultaneously which orders to accept or to reject and when spots from accepted orders should be scheduled. Kimms and Muller in [92] present a mathematical model and five heuristics, develop a rigorous method to generate a test bed and evaluate the performance of the heuristics on over $10,000$ instances of various sizes. The broadcasting company allocates limited advertising space, called airtime or media capacity, between two customer classes: upfront (market) clients and scatter (market) clients. Optimally managing and valuing limited advertising space is one of the key problems faced by media companies today. Araman and Popescu in [6] provide formal models and solutions for the problem of managing broadcast advertising capacity.

For a review of RM innovations in *retailers*, *energy sector* and *manufacturing* industries ([59], [61]) we refer to [154]. For *casinos*, *function space and spa*, *palapas*, *table-games* applications of RM and related literature references, we recommend [4].

2.4.5 Restaurants

Restaurants are considered non-traditional RM applications, because the number and depth of studies on RM in restaurants are not extensive and recent when compared to airlines, hotels and car rentals. In restaurant RM the aim is to obtain higher revenue by optimizing the allocation of the tables among the different type of customers. This involves managing the time for which seats are occupied and optimizing menu options and prices.

[73], [83], [76] and [82] are the first articles addressing the issue and the potential benefits of restaurant RM. The first two papers discuss the applicability of RM techniques to restaurant. Kimes in [76] develops guidelines for restaurant operators to maximize revenue, in particular five steps to develop and apply a restaurant RM systems are provided. Kimes et al. in [82] define revenue per available seat-hour (RevPASH) as the best indicator of the revenue generating performance of a restaurant, it indicates the rate at which revenue is generated and is calculated as the ratio between revenue (or profit) for a certain time period and the number of seat-hours available during that interval. They also underline the importance of price and meal duration, the two main strategic levers [71] in restaurants, and the influence of duration uncertainty on RevPASH and on restaurant booking policies. They also provide some strategies for restaurant operators to deal with high and low RevPASH periods, using a combination of point of sale (POS) data and time study data.

One of the most important aspect of restaurant RM is considered the mix of table sizes [4] first analyzed by Thompson. Thompson in [148] focuses on restaurants with only walk-in customers (i.e. no reservations are taken), and examines the case of restaurants with tables of different sizes dedicated to particular party sizes and restaurant with the possibility of combining tables to seat larger parties. Thompson compares the two configurations by measuring performance based on the RevPASH. The results suggest that, in general, if one does not have the best mix of tables in ones restaurant, it is better to have combinable tables. He develops a restaurant table simulation model, called TABLEMIX, that simulates the best restaurant configuration (position of each table and table that can be combined) knowing the number of tables and the number of seats. TABLEMIX can be used to evaluate a specific restaurant configuration or it can be used to search for the best restaurant configuration. According to the result of Thompson's simulation study, fixed tables are best suited for large restaurant (200 seats), whereas small restaurants (50 seats) can benefit from combinable tables. Thompson in [149] identifies which tables should be combinable and resulted in better profitability, developing a guidelines that can help restaurant managers and designers to configure restaurants. He focuses on Contribution Margin per Available Seat Hour (CMPASH) instead of RevPASH, because RevPASH, as a measure, is limited to revenue maximization, rather than profit maximization. The study found that the optimal table configuration for profit improvement is the configurations with longer sequences of smaller combinable tables. Kimes el al. in [90] study the reduction of customer dining time as a duration control tool. If dining times can be reduced during peak periods, the number of customers served and profitability will increase. If customers feel rushed, however, satisfaction will decrease and the restaurant may lose future business. This paper first examines the relationship between meal duration and consumer behaviour and then proposes a model for the measurement of time sensitivity, called " Time Sensitivity Measurement" or TSM to derive the expected dining time, the optimal and indifference duration points. The results show that the time could be decreased by approximately 20 per cent without a decrease in customer satisfaction. Bertsimas and Shioda in [20] develop optimization models using mathematical models and simulated real data to determine when to seat a party to achieve higher revenue, taking into account expected waiting time and perceived fairness. They develop integer programming, stochastic programming, and approximate dynamic programming models to provide efficiently implementable policies for restaurant RM and propose a stochastic gradient algorithm to address with reservations. The results show that models improve revenue, compared to FCFS policies usually used to seat parties, without increasing the average waiting time. Pricing, although an important aspect of any RM strategy, has received limited attention. Kimes and Wirtz in [87] investigate the reaction and perceptions of fairness of customers when restaurants adopt differential RM pricing policies. They compare surcharges with discounts for different period of day or week and for different table location and analyze the customers perception. The results indicate restaurants should be able to smooth demand by encouraging off-pick dining through promotions. Kimes and Wirtz in [88], test a variety of demand-based pricing policies and found the most approaches were generally considered to be fair. In particular, coupons, time of day and lunch/dinner pricing were considered as fair; weekday/weekend was perceived as neutral and table location pricing was considered unfair. They investigate the customers expectations in Asia, America and Europe. Kimes and Wirtz in [89] describe the case of Prego Italian restaurant and the development of a RM strategy to increase revenues without damage diner satisfaction. The case deals with the typical challenges in demand and supply in capacity management in the restaurant business and services management. Susskind et al. in [142] evaluate the ability of price to balance demand by offering discounts menus, unusual food and service. They study the reaction of restaurant guests to these tactics and show that more than 77 percent of clients would be willing to shift their dining time to off-peak hours in exchange for discounts on menu items and promotional offers. Kimes and Robson in [72] examines the dining table characteristics and the measurable effects on duration and average check, which were combined to show average spending per minute (SPM). The analysis performed on four weeks of POS data found that the SPM for parties at booths was slightly higher than average, while the SPM for diners at banquette tables was below the average. These findings are based solely on a single restaurant, but they suggest that there may be interesting relationships between a restaurant's environment and its customers' behavior. Kimes in [78] discusses how Chevys Freshmex Restaurant developed, implemented, and evaluated a RM program involving process analysis and duration control at one of its restaurants. She analyzes the restaurant's baseline performance, including seat occupancy, revenue per available seat hour (RevPASH), party size mix, and dining duration. After reviewing the

RM strategies for duration control, Kimes discusses how managers implemented those strategies. Kimes and Thompson in [85] focus on developing an optimal supply (table) mix, applying a simulation-based table enumeration system that allowed to determine the optimal table mix using data from a 230 seat Chevys Freshmex Restaurant. Compared to the restaurant's existing table mix, the optimal mix resulted in more effective capacity and increased RevPASH with a 35% increase in customer volume and a 5.1% increase in revenue, without increasing waiting times. Kimes and Thompson in [86] provide different approaches to capacity planning and RM; they discuss the full-service restaurant table mix problem that finds the optimal number of different size tables for a restaurant to maximize its revenue; they examine the effectiveness of eight heuristic techniques for the problem; finally, they observe that altering the table mix on a daily basis increased performance by over 1% compared to maintaining the optimal weekly table mix. Hwang in [63] investigates the impact of table assignment polices on waiting time performance and the effects of key demand features (party size distribution and arrival rate) on best policy selection. A restaurant simulation model with the spatial priority concern for table location and combination showed the best policy varied by party size distribution and arrival rate. Kimes in [79] and Kimes in [80], provide a framework for the adoption of technology in the implementation of restaurant RM strategies and an analysis of the innovative online reservations, the use of mobile reservations applications and third-party sites that will continue to grow as restaurant IT systems begin to become more integrated.

Part II

Car Rental RM

Chapter 3

Large Modeling and Solving a Car Rental Revenue Optimization Problem [1]

Abstract

We address the problem of a car rental agency that is confronted with how to decide to accept or reject a booking request to optimise the revenue. An innovative integer programming model is devised, which incorporates particularities of the car rental business, like multi-day rents and non-cascading upgrades. To capture the randomness of the unknown demand, robustness measures and the related scenario-based formulations are presented. An extensive computational study is carried out, by considering a set of randomly generated instances. The collected computational results show the relation between problem size and computation time and the effect of risk-aversion on revenue.

Keywords: Revenue Management, Car Rental, Integer Linear Programming, Robust Optimization.

3.1 Introduction

In this paper, we consider a Revenue Management (RM) problem, arising in the context of the car rental industry. According to a classic definition, revenue management (RM, for short) is a strategy, that involves the application of quantitative techniques, aimed at

[1] Accepted for pubblication in the journal *Int. J. Mathematics in Operational Research, Vol. 3, No. 2, 2011*, DOI: 10.1504/IJMOR.2011.038911

maximizing revenue, generated from a limited capacity of a product over a finite horizon, by selling each product to the right customer at the right time for the right price ([73]). However, as underlined by Karadjov and Farahmand in [69], the application of this methodology allows companies to obtain sustainable benefits only if they integrate RM in all their business activities.

RM has been originally developed in the airline industry: the earliest works date back to the beginning of 1970s, with the paper of Littlewood [108], on a two-fare single-leg problem, and the contribution of Rothstein [133] on overbooking policies. Since then, RM techniques have been extended not only to other types of problems in the airlines (the reader is referred to the recent papers [118], [42] and [29]), but also other numerous retail and service industries, which have similar charateristics, began adopting RM strategies. These include hotels, car rental, restaurants, cargo, cruise ship, internet services, apartment renting, only to cite few of them ([113]; [30]; [91]; [12]; [114]).

It is worth noting that it is not possible to devise an unique standard revenue optimization model, that can be used in all sectors of applications. Indeed, it is necessary to develop tailored RM models and methods, defined by taking into account the specific characteristics of the application under study.

As mentioned above, the car rental industry represents one of the application areas of RM. Indeed, it possesses the characteristics that, if managed opportunely, could determine the success of a process of RM (relatively fixed capacity, perishable inventory, advanced reservations, time variable and stochastic demand, segmentable market, appropriate cost and pricing structure ([73]). But despite that, the RM concepts in the car rental industry has not well developed ([30]) and the number of scientific works, considering the definition of RM techniques, tailored to the car rental business, are very limited.

The pioneer contributions are due to Carroll and Grimes [27] and Geraghty and Johnson [49], who provide a detailed description of RM techniques, tailored for the car rental process, and analyze the impact of adopting RM to real car rental companies. Carroll and Grimes in [27] consider the implementation of a RM system at Hertz car rentals, whereas Geraghty and Johnson in [49] define RM policy for the National car rentals. Even though the aforementioned papers address all aspects of the business managed through RM (i.e., demand forecast, upgrades, pricing and overbooking), the descriptions are very specific to the considered car rental companies.

Recently, the development of methods for the dynamic pricing, in the car rental business, has been addressed by Anderson, Davison and Rasmussen [3] and Kopliku [93]. In particular, the car rental operator is considered as “the holder of a swing-like option on car rental” and the price process is modelled as a stochastic differential equation.

On the basis of the previous considerations, in this work we propose an innovative integer programming model, to mathematically represent the revenue optimization problem in the context of the car rental industry. The developed model allows to determine the appropriate number of cars to be rented and incorporates the possibility of implementing an upgrading policy.

To handle demand uncertainty, some robustness criteria are applied to the formulated problem and the related scenario-based representations are described. These robust approaches aim at balancing expected revenue with feasibility, performance or regret in given scenarios. To the best of our knowledge, this work represents the first attempt to apply robust optimization approaches to the car rental industry.

The remainder of the paper is organized as follows. In the next section, we outline the main characteristics of the car rental process. Section 3 is devoted to a detailed description of the innovative mathematical model, defined to represent the car rental revenue management problem. In Section 4, to incorporate demand uncertainty, several additional scenario-based formulations are presented. The proposed models are evaluated, on the basis of an extensive computational experimentation: the related results are presented and discussed in Section 5. The paper closes with some concluding remarks in Section 6.

3.2 Problem description

The problem under consideration is that of a car rental agency, that wants to rent its products, i.e., cars and commercial vehicles, in order to maximize the total revenue. In what follows, the terms cars, commercial vehicles and vehicles are used interchangeably, to indicate the products of the rental agency.

The car rental capacity is essentially given by a limited number of vehicles, that is fixed in the sense that it cannot be subjected to excessive variations, that imply high costs. It is organized in groups/categories. Each group is characterized by a given number of vehicles, that is determined at the beginning of the booking horizon by the decision maker on the basis of the requirements of the rental agency. The vehicles, belonging to the same group, can be of different models, but they are charged at the same rates and at the same rental conditions. Thus, each group is identified above all by the rate. Consequently, a *lower group/category* is a category of smaller price but also of smaller performance and value, while the *advanced categories* have greater rates and higher performances.

A car may be rented by a booking made in advance or by a walk-in customer on the day of rental. A rental booking specifies the car group required, the start and end dates/times of the rental. Optionally, the reservation may specify a one-way rental (in which the car is returned to a branch different from the pick-up branch) and may request a specific car model within the required group. One-way rentals are not within the scope of this paper and we do not consider walk-in customers; thus we focus only on the customers that use a reservation system. For each rental request the management of the rental company decides, based on available rental capacity, whether to accept or to deny the request.

The revenue management process of a typical car rental agency, when a car rental reservation request arrives, can be viewed as characterized by the execution of the following main steps.

- If a car in the specified group is available for the specified rental period, the request is accepted and the rental reservation is recorded.
- The rental request is rejected if:
  - all equipment units are rented out;
  - a car in the requested group is available, but the company's management feels that the rental capacity should be reserved for potentially more profitable requests;
  - a car in the requested group is not available and the customer is not willing to rent a different group car.
- The demand for car rental vehicles has a nested structure: the demand for vehicles of a specific type may be satisfied by
  - upgrading customers with higher priced vehicles (i.e., when customers cannot be provided with the type of car they had required, companies usually offer them an upgrade by providing them with a higher priced car at the same rate), but not vice versa;
  - offering an available lower category car at the corresponding lower price.

It is important to note that if either the rental capacity is entirely utilized or more cars have been requested than are available, the company's management can decide to:

- not satisfy the excess demand and to deny the reservations, with possible negative consequences above all for the image and the customer's satisfaction;
- to ask other branches to transfer to him/her a certain number of cars, if available. In this situation, a known cost has to be paid. In this paper, we don't consider this aspect.

In addition, the residual capacity, that is the set of vehicles that are un-rented in a given period, remains unused until a new rental request arrives and this represents a cost for the car rental agency. In this paper, we investigate only the maximization of the revenue.

Generally, the car rental businesses have different rates depending on the type/group of car rented, and on the rental period. In what follows, for the sake of simplicity, we assume that the rental rate depends only on the type of car rented and the revenue obtained from renting a car is equal to the rate charged per day for the given car, times the number of day the car was rented out.

3.3 Mathematical formulation

In this section, we present a mathematical formulation of the considered car rental revenue optimization problem, which can be viewed as an extension of the deterministic linear programming model used in standard revenue management ([144]). Following, the main elements of the proposed model are introduced.

The objective of the car rental agency is to maximize the total revenue, while satisfying the unknown demand on a given booking horizon. We assume that the infinite booking horizon is divided into finite planning intervals. Each of them is characterized by T time periods, indexed by $t = 1, \ldots, T-1$. Generally, a time period corresponds to a day. In each time period t, $t = 1, \ldots, T-1$, the car rental agency has to decide on accepting requests for a car belonging to a given group k, $k = 1, \ldots, K$ from the day i to the day j, $i = t, \ldots, T-1$, $j = i+1, \ldots, T$. A minimum rental period of 24 hours is required for all rentals.

To give a formal representation of the proposed mathematical models, in what follows the major notations for parameters and variables used in this paper are given.

Parameters

K available groups/categories of vehicles;

p^k daily rental rate for a car of category k, $k = 1, \ldots, K$;

Q^k total capacity of group k, $k = 1, \ldots, K$, (i.e., number of available cars belonging to the $k-th$ group);

$t = 1, \ldots, T-1$ time periods of the booking horizon;

$i = t, \ldots, T-1$ starting time of the rent period; $j = 2, \ldots, T$ ending time of the rent period; $1 \leq i < j \leq T$, $j = i+1, \ldots, T$; $(j-i)$ length of rent;

D^k_{ij} expected number of booking requests for cars belonging to the group k,$k = 1, \ldots, K$, from day i to day j;

Decision Variables

x^{lk}_{ij} number of cars belonging to the group k, $k = 2, \ldots, K$, that are rented from day i to day j to satisfy the rental request of cars belonging to the group l, $l = k-1$ or $l = k$, in the same period. The variable x^{11}_{ij} is associated to the first category, for which a lower group is not defined.

In the proposed models, by following an approach similar to the one presented in [97], [98], [16], the check-ins and the check-outs are viewed as the flows in and out of the nodes in a network.

Since the number of available cars of the group k is limited by the group capacity Q^k, the following capacity constraints are defined:

$$\sum_{i=1}^{t}\sum_{j=t+1}^{T}(x_{ij}^{k-1,k}+x_{ij}^{kk})\leq Q^{k},\;\; t=1,\ldots,T-1,k=2,\ldots,K; \tag{3.1}$$

$$\sum_{i=1}^{t}\sum_{j=t+1}^{T}x_{ij}^{11}\leq Q^{1},\;\; t=1,\ldots,T-1. \tag{3.2}$$

Constraints (3.1) and (3.2) establish that, for each group and for each time period, the number of vehicles used to satisfy the car rental requests cannot exceed the maximum available capacity. The aggregate demand constraints can be represented as follows:

$$x_{ij}^{ll}+x_{ij}^{l,l+1}\leq D_{ij}^{l},\;\; i=1,\ldots,T-1,j=i+1,\ldots,T,l=1,\ldots K-1; \tag{3.3}$$

$$x_{ij}^{KK}\leq D_{ij}^{K};\quad i=1,\ldots,T-1,j=i+1,\ldots,T. \tag{3.4}$$

In particular, conditions (3.3) impose that, for each rent period $(j-i)$ and for each group k, the total number of vehicles rented to satisfy the demand of cars belonging to the k-*th* group (i.e., cars belonging to the groups k and $k+1$) must be smaller than or equal to the expected number of requests of vehicles of the group k. Indeed, since the upgrading is allowed, the $k-th$ group requests are satisfied with both the available vehicles of category k and with the vehicles belonging to the superior group of k, $l=k+1$.

On the other hand, conditions (3.4) establish that, for each rent length and for the $K-th$ group, for which a superior group does not exist and thus the upgrading is not possible, the number of vehicles used to satisfy the rental requests must be smaller than or equal to the total demand for this group.

The main goal is to maximize the revenue gained from all possible rentals, satisfying the demand and capacity constraints introduced above. The objective function represents the revenue, obtained from the rentals during the considered booking horizon. In particular, it can be represented mathematically as follows:

$$\textit{Maximize}\left[\sum_{i=1}^{T-1}\sum_{j=i+1}^{T}\sum_{k=2}^{K}(j-i)[p^{k-1}x_{ij}^{k-1,k}+p^{k}x_{ij}^{kk}]+\sum_{i=1}^{T-1}\sum_{j=i+1}^{T}(j-i)p^{1}x_{ij}^{11}\right]$$

The proposed model allows to determine, for each category and for each time period, the number of vehicles to be rented and the number of upgrades, with the objective of maximizing the expected revenue, while satisfying demand and capacity constraints.

On the basis of the previous considerations, the complete mathematical representation of the problem under study is given in what follows.

$$\textit{Maximize} \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{k=2}^{K} (j-i)[p^{k-1} x_{ij}^{k-1,k} + p^k x_{ij}^{kk}] + \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} (j-i) p^1 x_{ij}^{11} \right]$$

$$x_{ij}^{ll} + x_{ij}^{l,l+1} \leq D_{ij}^{l}, \;\; i = 1, \dots, T-1, j = i+1, \dots, T, l = 1, \dots, K-1 \tag{3.5}$$

$$x_{ij}^{KK} \leq D_{ij}^{K}, \;\; i = 1, \dots, T-1, j = i+1, \dots, T \tag{3.6}$$

$$\sum_{i=1}^{t} \sum_{j=t+1}^{T} (x_{ij}^{k-1,k} + x_{ij}^{k,k}) \leq Q^k, \;\; t = 1, \dots, T-1, k = 2, \dots, K \tag{3.7}$$

$$\sum_{i=1}^{t} \sum_{j=t+1}^{T} x_{ij}^{11} \leq Q^1, \;\; t = 1, \dots, T-1 \tag{3.8}$$

$$x_{ij}^{lk}, x_{ij}^{11} \geq 0, \; \textit{integer} \qquad k = 2, \dots, K, l = k-1 \;\; \textit{or} \;\; l = k, \tag{3.9a}$$

$$i = 1, \dots, T-1, j = i+1, \dots, T. \tag{3.9b}$$

3.4 Robust scenario-based formulations

The car rental optimization model, presented in the previous section, is an integer programming model. In addition, the parameters D_{ij}^{l}, $i = 1, \dots, T-1$, $j = i+1, \dots, T-1$, $l = 1, \dots, K$ in constraints (3.3) and (3.4) are usually unknown at the beginning of the booking horizon. The revenues may also not be fixed, since the car rental agency would like to set different pricing, which in turn results in different demands.

In general, two different strategies can be considered to deal with uncertainty: the reactive and the proactive approaches ([117]). In the former, the unknown parameters are replaced by appropriate estimates, generally the expected values. The main drawback of this approach is that the obtained solutions could be infeasible. Thus, a post-optimality analysis is required to identify some corrective actions. The latter tries to reduce the consequences of uncertainty, by determining solutions that are less sensitive to the model data than traditional mathematical programming models. Indeed, probabilistic information about the problem data are taken into account.

A conventional way to address problem with probabilistic information is represented by stochastic programming (see for instance [117], [116] and [36]), which employs a scenario decomposition method to solve large-scale probabilistic optimization problems. It is worth observing that, stochastic programming concerns only minimizing expected costs or maximizing expected profits, where the expectation is taken over the assumed probability distribution, which may not reflect the decision makers true utility function, it ignores decision maker's preferences toward risk and fails to reduce the variability of the solution and constraints.

In order to overcome the aforementioned drawbacks, in [117] and [116], an improved method, i.e., robust programming, has been introduced. This approach allows to represent the decision makers' favored risk aversion ([11]) and to generate a set of solutions, that are progressively less sensitive to realizations of the model data from a scenario set ([165]).

On the basis of the previous considerations, in this paper, we focus our attention on proactive approaches to deal with uncertainty. In particular, the problem under study is represented by considering robust scenario-based formulations. Indeed, it is assumed that the uncertainty is represented by a set of possible realizations, called *scenarios*. Each scenario occurs with a certain probability and provides one possible course of future events and can be defined as a possible realization, which describes the behavior of the aleatory components of the problem.

Robust optimization is defined as an approach to find a solution whose objective value is close to that of the optimal solution for each scenario ([116]). Indeed, a solution is robust if it achieves the best worst-case deviation from optimality. This definition is used in *regret models* of robust optimization ([94]), where the regret of a solution in a given scenario is the difference between the revenue of the optimal solution for that scenario and the revenue of the solution in that scenario.

In particular, Kouvelis and Yu [94] define two regret measures for robustness: the *robust deviation* and the *relative robustness* criteria. The former is defined as a criterion that selects the solution that achieves the smallest deviation from the best possible performance for each scenario. The latter is defined as a criterion that selects the solution that has the smallest percentage from the best possible behaviour for each scenario. There is also a definition of *absolute robustness* presented by Kouvelis and Yu [94], it evaluates the objective function value in each scenario without reference to the best possible decision that could have been made in that scenario. Absolute robustness defines a solution that maximizes (minimizes) the minimum (maximum) total profits (costs).

In [54] and [139] a different robustness measure is considered, that is the *stochastic p-robustness*. More specifically, the aim is to find a solution that is within $p\%$ of the optimal solution for any realizable scenario. It is worth observing that this last approach can lead to infeasibility for small value of p. An alternative definition of robustness is given in [9], where it is underlined that the main goal of robust optimization is to find a near-optimal solution that is not overly sensitive to any specific realization of the uncertainty. This definition is generally used in the so-called *variability models*, which include variability measures (i.e., variance, standard deviation, etc.) in the objective function ([117]).

Recently, Perakis and Roels [125] investigate the classical problem of allocating network capacity in RM. The authors propose robust formulations and control policies for this problem, considering the maxmin and the minmax regret criteria. Their numerical results indicate that the robust policies generally appear to be comparable and sometimes even better than traditional heuristics for network RM.

In this paper, different measures of robustness are considered. More specifically, the attention has been focused on the maxmin criterion (i.e., the application of a simple absolute robustness measure, maximizing the worst-case performance), the robust deviation criterion, the stochastic p-robustness criterion and a standard deviation based variability criterion.

With the aim to introduce the considered robust formulations, it is useful to introduce the following notations and definitions. In all the robust formulations, S represents the number of possible scenarios, which differ for the demand values and the daily rental rates. To each scenario $s = 1, \dots, S$ is associated the occurrence probability $P(s)$, such that $\sum_{s=1}^{S} P(s) = 1$. Π_s denotes a deterministic car rental revenue optimization problem, that depends on the considered scenario s. Indeed, for each scenario $s = 1, \dots, S$, there is a different problem Π_s. These problems have an identical structure, but they are characterized by different data, in particular demands and daily rental rates. For each scenario $s = 1, \dots, S$, z_s^* represents the optimal objective value for Π_s, whereas let X be a feasible solution for Π_s, $z_s(X)$ denotes the objective value of the problem Π_s under the solution X.

The models parameters and the decision variables, with their meanings, are reported in what follows.

Parameters

S number of possible scenarios $s = 1, \dots, S$, each of them is characterized by different demand values and the daily rental rates.

$P(s)$ probability of occurrence of scenario s, such as $\sum_{s=1}^{S} P(s) = 1$

K available groups/categories of vehicles;

$p^{k,s}$ daily rental price for a vehicle of group k, under scenario s

$D_{ij}^{k,s}$ number of requests (bookings) for vehicles of group k,$k = 1, \dots, K$, from i to j, under the scenario s

The decision variables are described below.

Decision variables

$x_{ij}^{l,k}$ vehicles of group k, $k = 2, \ldots, K$, rented from i to j to satisfy the request for group l ($l = k-1$ or $l = k$).

$x_{ij}^{1,1}$ vehicles of group 1, for which a lower group is not defined, rented from i to j.

3.4.1 The absolute robustness criterion

As mentioned above, the common measure of absolute robustness is the maxmin (or min-max in minimizing problem) criterion. In this case the problems yields very conservative solutions based on the anticipation that the worst-case will happen ([94]). This criterion will select a solution for which the minimum revenue, taken across all possible realizations, is as greater as possible. Indeed, the aim is to find a solution that maximizes the minimum revenue.

In this model, the optimal value will be the greatest revenue among the lowest revenue obtained in the various scenarios. The related formulation is reported in what follows.

$$Maximixe \quad \gamma$$

such that

$$\gamma \leq \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{k=2}^{K} [(j-i)(p^{k-1,s} x_{ij}^{k-1,k} + p^{k,s} x_{ij}^{kk})] + \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} (j-i) p^{1,s} x_{ij}^{11} \right]$$

(3.10a) $$x_{ij}^{ll} + x_{ij}^{l,l+1} \leq D_{ij}^{l,s} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$$

(3.10b) $$l = 1, \ldots, K-1, s = 1, \ldots, S$$

(3.11) $$x_{ij}^{KK} \leq D_{ij}^{K,s}, \; i = 1, \ldots, T-1, j = i+1, \ldots, T, s = 1, \ldots, S$$

(3.12) $$\sum_{i=1}^{t} \sum_{j=t+1}^{T} (x_{ij}^{k-1,k} + x_{ij}^{k,k}) \leq Q^k, \; t = 1, \ldots, T-1, k = 2, \ldots, K$$

(3.13) $$\sum_{i=1}^{t} \sum_{j=t+1}^{T} x_{ij}^{11} \leq Q^1, \; t = 1, \ldots, T-1$$

(3.14a) $x_{ij}^{lk}, x_{ij}^{11} \geq 0,\ integer \qquad k = 2, \ldots, K, l = k-1\ \ or\ \ l = k,$

(3.14b) $\qquad i = 1, \ldots, T-1, j = i+1, \ldots, T.$

The robust deviation criterion

The robust deviation measure was chosen in this work, because it incorporates more information in the solution than absolute robustness and thus it is believed to provide a better answer about the objective of the model. The main measure of the robustness deviation is the minmax regret (*MinMax* for short).

For a given optimization problem with uncertain parameters, the minmax regret problem is to find a solution that minimizes the maximum regret across all scenarios. The basic minmax models presented in [94] are conservative models that try to minimize the effect of the worst-case scenario by minimizing the largest observed deviation from the optimal for all scenarios. The first step is to compute the regret associated with each combination of decision and input data scenario. The regret of a scenario is measured as the closeness between the optimal objective function value for that scenario and the objective function value of the chosen solution for that scenario. The minimax criterion is then applied to the regret values and the decision with the least maximum regret is chosen.

The minmax regret formulation of the proposed car rental revenue management problem assumes the following form, in which the parameters and the variable have the same meaning introduced in the previous subsection.

$$Minimize \quad \beta$$

such that

(3.15) $$\beta \geq z_s^* - z_s(X); \;\; s = 1, \ldots, S$$

and constraints (3.10)-(3.14) should be also satisfied.

Stochastic p-robustness criterion

In this section, we apply to the proposed car rental revenue optimization problem the robustness measure, introduced by Snyder and Daskin [139] (i.e., the stocastic p-robustness criterion, *SpRC*, for short) that combines the advantages of stochastic and robust optimization approaches by maximizing the expected revenue, while bounding the relative regret in each scenario. The definition of stochastic p-robustness considers relative regret, but it could easily be modified to consider absolute regret or cost/profit ([139]).

The main aim is to find the maximum expected revenue solution, subject to the constraint that the chosen solution is p-robust, that is the corresponding relative regret should not exceed p in every scenario, for a given $p \geq 0$. To introduce the stochastic p-robustness measure, it is useful to introduce the following definition.

Definition 3.4.1. *Let $p \geq 0$ be a constant. Let X be a feasible solution of the problem Π_s for all s, $s = 1, \ldots, S$, and let $z_s(X)$ be the objective value of problem Π_s under solution X. X is said to be* p-robust *if for all s, $s = 1, \ldots, S$, the following condition holds:*

$\frac{z_s^* - z_s(X)}{z_s^*} \leq p$;

or equivalently

$z_s(X) \geq (1 - p) * z_s^*$.

From definition 3.4.1, it is evident that, in order to apply the stochastic p-robustness criterion, it is necessary to solve the problem for each scenario to obtain the value of z_s^*, that, together with the other parameters already introduced, represent the input for the mathematical formulation, reported in what follows.

$$\textit{Maximize} \sum_{s=1}^{S} P(s) \left\{ \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{k=2}^{K} [(j-i)(p^{k-1,s} x_{ij}^{k-1,k} + p^{k,s} x_{ij}^{kk})] + \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} (j-i) p^{1,s} x_{ij}^{11} \right] \right\}$$

such that

$$z_s(X) \geq (1 - p) * z_s^*, s = 1, \ldots, S \tag{3.16}$$

and constraints (3.10)-(3.14) should be also verified.

It is evident from the formulation given above that, the objective function (to be maximized) represents the sum of the revenues, calculated for all scenarios and that verify the robustness condition, multiplied for the relative probability of occurrence. Consequently, the final value represents the maximum expected revenue, for the setting values and the considered scenarios.

The value of the parameter p is established according to a sensibility analysis. However, it is reasonable to associate to p a value in the interval $[0, 1]$. Indeed, negative values are not admitted, because they would change the meaning of the constraints, and values greater than *1*, for which the constraint always would be verified.

A mean absolute deviation based variability criterion - MADbVC

The following robust optimization model has been formulated because it overcomes the limits of stochastic programming and includes a measure of variability rather than regret.

The idea of integrating goal programming formulations with a scenario-based description of the problem data in the context of robust optimization was firstly introduced by Mulvey, Vanderbei and Zenios [117]. The main aim is to generate a sequence of solutions that are progressively less sensitive to realizations of the model parameters. They introduce the concept of “solution robustness” as the case when the optimal overall solution is near optimal for every possible demand scenarios. They define “model robustness” as the case when the optimal overall solution is almost feasible for all scenarios. They add norms, such as variance or utility functions, to the objective function to encourage solution robustness. They also add a feasibility penalty function to the objective function to encourage model robustness.

Yu and Li (2000) reformulate the robust optimization model proposed in [117] into a linear program that requires only half as many variables. The main drawback of the proposed formulation is that it can be applied only to linear models. The aforementioned robust optimization concepts have been considered in [97] and [110], to address the hotel revenue optimization problem, under an uncertain environments and in [98] to handle the service firms revenue optimization problem. In particular, measurement of robustness and mean absolute deviation terms are used to transform the proposed mathematical formulations into robust optimization models.

By applying this approach to the problem under consideration, we obtain the robust formulation reported in what follows. In particular, we introduce an utility function that embodies a trade-off between mean value and variability in this mean value and a feasibility penalty function, used to penalize violations of the demand constraints under some of the scenarios.

$$Maximize \quad \sum_{s=1}^{S} P(s) * \xi^s -$$

$$\lambda * \sum_{s=1}^{S} P(s) * \left| \xi^s - \sum_{s=1}^{S} P(s) * \xi^s \right| - \sum_{s=1}^{S} P(s) * \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{l=1}^{K-1} w_{i,j}^{l} \left| D_{i,j}^{l,s} - (x_{i,j}^{l,l} + x_{i,j}^{l,l+1}) \right| -$$

$$\sum_{s=1}^{S} P(s) * \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} w_{i,j}^{K} \left| D_{i,j}^{K,s} - x_{i,j}^{K,K} \right|$$

(3.17a) $$x_{ij}^{ll} + x_{ij}^{l,l+1} \leq \min\{D_{ij}^{l,s}\} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$$

(3.17b) $$l = 1, \ldots, K-1, s = 1, \ldots, S$$

(3.18a) $$x_{ij}^{KK} \leq \min\{D_{ij}^{K,s}\} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$$
(3.18b) $$s = 1, \ldots, S$$

(3.19) $$\sum_{i=1}^{t} \sum_{j=t+1}^{T} (x_{ij}^{k-1,k} + x_{ij}^{kk}) \leq Q^k \; t = 1, \ldots, T-1, k = 2, \ldots, K$$

(3.20) $$\sum_{i=1}^{t} \sum_{j=t+1}^{T} x_{ij}^{11} \leq Q^1 \; t = 1, \ldots, T-1$$

(3.21a) $$x_{ij}^{lk}, x_{ij}^{11} \geq 0, \; integer \qquad k = 2, \ldots, K, l = k-1 \; or \; l = k,$$
(3.21b) $$i = 1, \ldots, T-1, j = i+1, \ldots, T$$

where

$$\xi^s = \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{k=2}^{K} [(j-i)(p^{k-1,s} x_{ij}^{k-1,k} + p^{k,s} x_{ij}^{kk})] + \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} (j-i) p^{1,s} x_{ij}^{11} \right]$$

and λ and $w_{i,j}^{s,k}$, $k = 1, \ldots, K$, are non-negative weighting parameters.

The first term in the objective function represents the expected revenue, while the second term is the mean absolute deviation of the revenue. The parameter λ can be regarded as a risk trade-off factor, between expected revenue and deviation, for the decision-maker. The absolute deviation in the third term is a model robustness measurement while the parameters $w_{i,j}^{s,k}$, $k = 1, \ldots, K$, are the penalty weights for the constraints violations.

By using the mean absolute values as penalties, the model can generate solutions which are robust in all scenarios. Since the mean absolute value increases complexity and the number of artificial variables, it is transformed in a linear term by a linearization method.

In particular, the method proposed in [165] to solve a goal programming (GP, for short) problem is considered. Such an approach is theoretically justified by the results reported in the following theorem, which proof can be found in [165].

Theorem 3.4.2. *A GP problem:*

$$Minimize \quad Z = |f(X) - g|$$

subject to

$$X \in F;$$

where F is a feasible set, can be linearized using the following form:

$$Minimize \quad ZZ = f(X) - g + 2\delta$$

subject to

$$g - f(X) - \delta \leq 0$$
$$\delta \geq 0$$

To apply Theorem 3.4.2 to our problem, we set $f(X) = \xi^s$ and $g = (\sum_{s=1}^{S} P(s)*\xi^s)$ in the objective function; $f(X) = \sum_{k=l}^{l+1} x_{ij}^{lk}$ or x_{ij}^{KK} and $g = D_{i,j}^{l,s}$ in the constraints. In addition, we introduce a set of non-negative variables z^s and $y_{i,j}^{s,k}$ for all scenarios where s, $s = 1, \ldots, S$. Our robust optimization model assumes the following form.

Maximize

$$\sum_{s=1}^{S} P(s)*\xi^s - \lambda * \sum_{s=1}^{S} P(s)* \left[\xi^s - (\sum_{s=1}^{S} P(s)*\xi^s) + 2z^s \right] -$$

$$\sum_{s=1}^{S} P(s) * \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{l=1}^{K-1} w_{i,j}^{l} \left[D_{i,j}^{l,s} - (x_{i,j}^{ll} + x_{i,j}^{l,l+1}) + 2y_{i,j}^{l,s} \right] -$$

$$\sum_{s=1}^{S} P(s) * \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} w_{i,j}^{K} \left[D_{i,j}^{K,s} - x_{i,j}^{K,K} + 2y_{i,j}^{K,s} \right]$$

(3.22a) $\quad (x_{i,j}^{ll} + x_{i,j}^{l,l+1}) \leq \min\{D_{ij}^{l,s}\} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$

(3.22b) $\quad l = 1, \ldots, K-1, s = 1, \ldots, S$

(3.23a) $\quad x_{ij}^{KK} \leq \min\{D_{ij}^{K,s}\} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$

(3.23b) $\quad s = 1, \ldots, S$

(3.24a) $\quad (x_{i,j}^{ll} + x_{i,j}^{l,l+1}) - y_{i,j}^{l,s} \leq D_{i,j}^{k,s} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$

(3.24b) $\quad l = 1, \ldots, K-1, s = 1, \ldots, S$

(3.25a) $\quad x_{ij}^{KK} - y_{i,j}^{K,s} \leq D_{ij}^{K,s} \qquad i = 1, \ldots, T-1, j = i+1, \ldots, T,$

(3.25b) $\quad s = 1, \ldots, S$

(3.26) $\quad \xi^s - (\sum_{s=1}^{S} P(s)*\xi^s) + z^s \geq 0 \qquad s = 1, \ldots, S$

(3.27) $$\sum_{i=1}^{t}\sum_{j=t+1}^{T}(x_{ij}^{k-1,k}+x_{ij}^{k,k})\leq Q^k \qquad t=1,\dots,T-1, k=2,\dots,K$$

(3.28) $$\sum_{i=1}^{t}\sum_{j=t+1}^{T}x_{ij}^{11}\leq Q^1 \qquad t=1,\dots,T-1$$

(3.29a) $$x_{ij}^{lk}, x_{ij}^{11}\geq 0,\ integer \qquad k=2,\dots,K, l=k-1 \text{ or } l=k,$$
(3.29b) $$i=1,\dots,T-1, j=i+1,\dots,T$$

(3.30a) $$z^s,\ y_{i,j}^{k,s}\geq 0 \qquad s=1\dots,S, k=1,\dots,K,$$
(3.30b) $$i=1,\dots,T-1, j=i+1,\dots,T$$

In order to satisfy the constraints (3.26), the variable z^s tends to take a value equal to the difference between ξ^s and $(\sum_{s=1}^{S}P(s)*\xi^s)$, when this difference assumes a negative value. It is zero when this difference is either positive or zero. In order to avoid a null penalty term corresponding to the first case, in the objective function the variable z^s is multiplied by two.

Thus, the following situations can occur:

- if $\xi^s \geq (\sum_{s=1}^{S}P(s)*\xi^s)$ then $z^s=0$ and the penalty term, related to the deviation from the expected revenue, depends on the difference between these two quantities, the value of λ and also on the probability value associated to each scenario;

- if $\xi^s < (\sum_{s=1}^{S}P(s)*\xi^s)$ then $z^s\neq 0$ and the penalty is given by the value of z^s , λ and $P(s)$.

The third and fourth terms of the objective function represent a penalty term related to the constraints violation.

In this case, the difference between the required number of car rental requests $(D_{i,j}^{l,s})$ (or $D_{i,j}^{K,K}$, for the last group) and the quantity of satisfied demand $(x_{ij}^{ll}+x_{ij}^{l,l+1})$(or x_{ij}^{KK}, for the last group) in each scenario are taken into account.

Consequently, the following three mutually excludently situations can occur:

1. if the difference $(D_{i,j}^{l,s}-(x_{ij}^{ll}+x_{ij}^{l,l+1}))$ (or $(D_{i,j}^{K,K}-x_{ij}^{KK})$, for the last group) is positive, that is the demand is greater than the number of car rental requests satisfied in each scenario, the variable $y_{i,j}^{k,s}$, $k = 1, \ldots, K$ are equal to zero and the penalty term in the objective function depends on this difference and on the parameters $w_{i,j}^{k}$, $k = 1, \ldots, K$;

2. if the difference ($D_{i,j}^{l,s} - (x_{ij}^{ll} + x_{ij}^{l,l+1}$)) (or $(D_{i,j}^{K,K} - x_{ij}^{KK})$, for the last group) is equal to zero, that is the car rental requests are satisfy exactly, the variable $y_{i,j}^{k,s}$ has a value equal to zero and the penalty term in the objective function will be zero, even if the parameters $w_{i,j}^{k}$ are not null;

3. if the difference ($D_{i,j}^{l,s} - (x_{ij}^{ll} + x_{ij}^{l,l+1}$)) (or $(D_{i,j}^{K,K} - x_{ij}^{KK})$, for the last group) is negative, that is the number of effective car rentals are greater than the number of requests in some scenario, the variable $y_{i,j}^{k,s}$ assumes a value equal to this difference in order to satisfy the constraints and to have the penalty term that depends only on $y_{i,j}^{k,s}$ and $w_{i,j}^{k}$. This case can be verified, because the tie between the effective rentals and demand is established by constraints (3.24) and (3.25); consequently, in some scenarios the number of rentals could exceed the requests, these cases involve a decrease in yield, but a penalty is associated to these situations. Obviously, the greater is the deviation, the greater will be the value associated to the variable and therefore the penalties associated to the violated constraints.

3.5 Computational experiments

In this section, we present the computational experiments carried out to assess the practical performance of the proposed models for the car rental industry.

The models have been implemented in the AIMMS mathematical modeling language (www.aimms.com) and solved by using the ILOG CPLEX solver (www.ilog.com). The AIMMS implementations were run on a PC Pentium IV with 3.2 GHz and 2 GB of RAM, under the Windows XP operating system.

To the best of our knowledge, the problem under consideration has not been taken into account in the revenue management literature previously. Thus, benchmark instances are not available. For this reason, the computational experiments have been carried on test examples, defined trying to be quite close to the reality of medium-sized car rental agencies. A static evaluation of the models is performed.

Three different classes of instances, defined by considering an increasing number of groups K, have been considered. In particular, K has been chosen equal to 8, 10 and 15. The total number of available cars Q has been set equal to 180 and a booking horizon of 7 time periods (i.e., a week) has been considered. The rental rates were randomly generated into the interval [50; 100] and increasing with the category of cars. The characteristics of the test problems are reported in Table 1.

Problem Class	K	T
1	8	7
2	10	7
3	15	7

Table 3.1: Characteristics of the test problems.

The number of requests has been randomly generated according to a normal distribution, with a given expected demand and a given coefficient of variation. The expected demand and the coefficient of variation have been generated randomly from the interval $[7, 20]$ and $[0, 1]$, respectively. Also, the demand requests within a time interval in a single scenario are generated independently one of each other. The process has been simulated 50 times.

3.5.1 Deterministic car rental revenue optimization model

In the first stage of the investigation, computational experiments have been carried out by considering a single scenario. Indeed, the car rental optimization problem under certain demand and daily rental rate has been considered.

The main aims of this experimentation are to give an idea of the models evaluating in terms of computational effort the proposed programming formulations; For the sake of brevity, in what follows we report the computational results obtained on a specific instance, that represents the case of a medium-large car rental agency, characterized by a total number of available cars equal to 180. The number of groups has been chosen equal to 4, 8, 15 and 30, whereas the value of the time periods has been set equal to 5, 15, 30 and 60. A similar behavior has been observed for the other considered instances.

A graphical representation of the computational effort required to solve the proposed model, by varying the number of groups, while keeping fixed the length of the booking horizon is reported in Fig. 3.1, whereas in Fig. 3.2, the case in which the number of group is fixed and the number of time periods varies is considered.

The results depicted in Fig.3.1 clearly underline that when a constant number of time periods are considered, if the number of groups is increased, the execution time increases.

A similar behavior has been observed when the capacity is kept fixed, while the number of time periods in the booking horizon is increased (see Fig.3.2). The collected results also underline that the influence of the number of time periods, that constitute the booking horizon, on the execution time is greater than the influence of the number of groups.

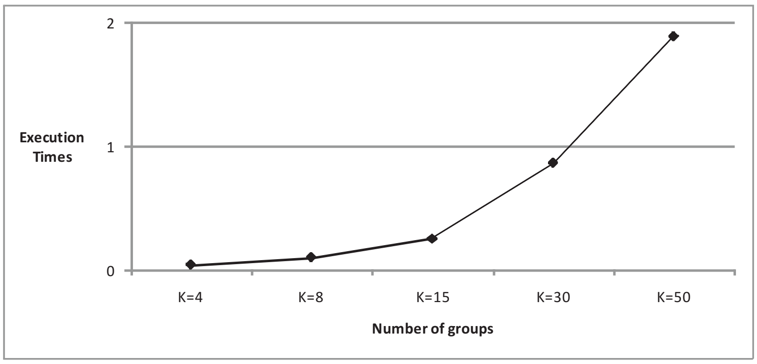

Figure 3.1: Execution times required to solve the proposed model, by varying the value of K, while keeping fixed the length of the booking horizon $T = 7$.

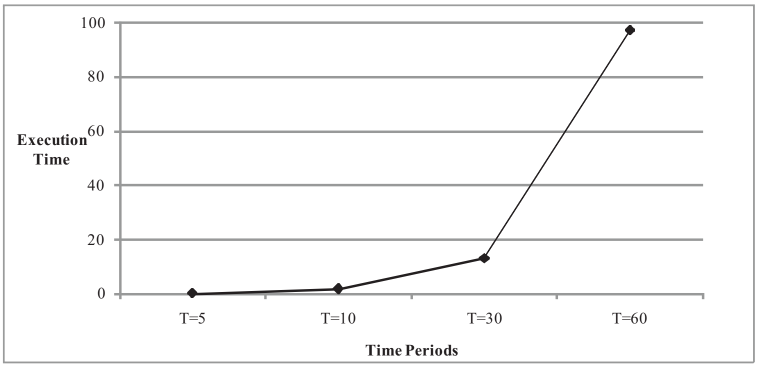

Figure 3.2: Execution times required to solve the proposed model, by varying the value of T, while keeping fixed the number of groups $K = 15$.

3.5.2 Robust car rental revenue optimization models

The computational results have been collected on the basis of an increasing number of scenarios. In particular, S has been set equal to 10, 20, and 30. We have considered equiprobable scenarios: to each scenario s, $s = 1, \ldots, S$ is associated the same probability of occurrence given by $P(s) = \frac{1}{S}$.

In the first stage of investigation, experiments have been carried out to evaluate the influence of the parameters' models on the solution quality. Under this respect, it is important to point out that the stochastic p-robustness model depends on the value of the parameter p, whereas the solution of the mean absolute deviation model depends on the value of λ and w_{ij}^k.

A sensibility analysis on the value of the parameter p has been conducted. Since this parameter naturally lies in the range $[0, 1]$, experiments have been carried out by considering p equal to 0.25, 0.50, 0.75. The best results, obtained by letting $p = 0.75$, are reported in Table 2, in which for each instances and for each value of S, the expected revenue values are highlighted.

S	K	Revenue
10	8	72652.05
	10	55929.33
	15	66684.44
20	8	70871.40
	10	76570.60
	15	49884.74
30	8	78552.35
	10	88519.15
	15	86738.10

Table 3.2: Revenue values obtained by applying the stochastic p-robustness criterion, by setting $p = 0.75$.

As far as the results obtained by applying the mean absolute deviation based variability model is concerned, it is important to note that the values of λ give a representation of the different degrees of the decision maker's risk aversion, whereas the penalty weights w_{ij}^k allow to control the feasibility robustness.

The relationship between the parameter λ and the expected revenue, when all weights w_{ij}^k are set equal to 1 is depicted in Fig.3.3, for $K = 10$ and $S = 10$. A similar behavior has been observed for the other instances.

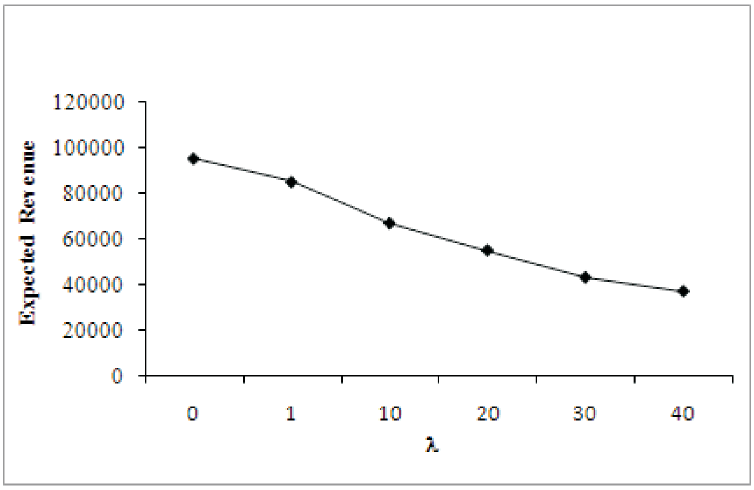

Figure 3.3: Expected revenue for different values of λ, $w_{ij}^k = 1$.

Fig.3.3 clearly underlines that the expected revenue decreases as the λ parameter increases. In addition, when λ is very large, the model gives a zero optimal solution and results in either zero or negative (given the presence of the fixed costs) expected revenue. Indeed, if the decision maker is very adverse to the risk and thus a high value of λ is chosen, the model suggests him/her to not run a business.

As far as the influence of the parameters w_{ij}^k is concerned, it is worth observing that

Problem Class	S	MinMax	MaxMin	MADbVC	SpRC
1	10	2.92	2.88	3.67	4.15
	20	3.54	3.25	5.20	5.85
	30	4.42	4.35	8.70	7.82
2	10	3.29	2.94	4.89	5.64
	20	4.18	4.47	8.14	7.92
	30	7.64	7.20	10.18	9.26
3	10	6.43	6.41	7.40	11.34
	20	9.41	9.44	12.89	12.17
	30	10.48	10.58	20.57	16.65

Table 3.3: Execution times required to solve the proposed robust optimization models.

the decision maker can choose these weights in such a way that some specific requests, characterized, for example, by a predetermined length of rent or related to a specific car group, are more likely to be satisfied than the others. In this case, it is sufficient to either increase the corresponding weight or decrease the weights associated to other requests.

In order to evaluate the performance of the proposed models, in terms of computational effort, we report in Table 3 the average execution time for each class of test problems and for each number of scenarios S. The related results have been collected by letting $\lambda = 1$ and $w_{ij}^k = 0.2$ in the $MADbVC$ model and $p = 0.75$ in the *SpRC* model.

The experimental results clearly underline that the computational cost depends on the size of the problem to be solved, on the number of scenarios and on the specific robust criterion adopted. In particular, if the number of groups is kept constant, the higher the number of scenarios, the higher the computational overhead. On the other hand, if we fix the number of scenarios and we increase the number of groups, the execution time is increased. In addition, Table 3 underlines that solving the mean absolute deviation based variability model is the most time consuming task, requiring in the worst case 20.57 minutes. The computational costs required to solve the other models are comparable. However, it is important to point out that, since the car rental agency has to solve the problem once a week, the execution time required to solve the problem can be considered acceptable.

In order to evaluate the performance of the proposed models in terms of solution quality, in the following table (i.e., Table 4), we report the expected revenue obtained by applying the considered robust criteria to each class of instances. The results reported in Table 4 indicate that it is not possible to find a criterion that allows to obtained the best expected revenue value in all the test problems. In addition, on average the $MADbVC$ shows the best performance in terms of solution quality.

Problem Class	S	MinMax	MaxMin	MADbVC	SpRC
1	10	90258.47	90259.54	92630.63	72652.05
	20	58836.60	58836.60	60428.00	55929.33
	30	77014.76	77014.93	80224.57	66684.44
2	10	69319.47	69319.47	70871.40	70871.40
	20	105938.10	100847.04	102664.33	76570.60
	30	56810.01	56810.01	58163.13	49884.74
3	10	97507.63	97507.63	98848.00	78552.35
	20	102208.80	102208.80	103572.00	88519.15
	30	92397.72	92397.72	93753.67	86738.10
Average		83365.73	82800.19	84572.86	71822.46

Table 3.4: Expected revenues obtained by applying the robust criteria to the considered test problems.

3.6 Concluding remarks and ongoing work

In the present paper, we have addressed a car rental management problem. More specifically, an integer linear programming model has been defined to mathematically represent the problem of a car agency that has to decide to accept or reject a booking request. The proposed model allows to determine the number of cars to be rented and incorporates the possibility of implementing an upgrade policy. In order to take into account the uncertainty in the demand, different measures of robustness are applied and the related scenario-based formulations are presented.

To assess the performance of the proposed models, in terms of efficiency and solution quality, an extensive computational phase has been carried out, by considering a large set of randomly generated problems. The computational results collected are very encouraging, showing that the proposed models can be used to address the problem under consideration.

It is important to point out that an alternative approach to handle the problem studied in this paper, relies on the definition of a so-called control policy, that is used to apply the results of the optimization model throughout the booking horizon. The definition and the computational evaluation of a primal control policy, based on partitioned booking limits, is the subject of current investigations.

Chapter 4

Revenue Models and Policies for the Car Rental Industry [1]

Abstract

In this paper, we consider the application of revenue management techniques in the context of the car rental industry. In particular, the paper presents a dynamic programming formulation for the problem of assigning cars of several categories to different segments of customers, with rental requests arising dynamically and randomly with time. Customers make a rental request for a given type of car, for a given number of days at a given pickup time. The rental firm can satisfy the demand for a given product with either the product requested or with a car of at most one category superior to that initially required, in this case an "upgrade" can take place. The one-way rental scenario, which allows the possibility of the rental starting and ending at different locations, is also addressed. In the framework considered, the logistic operator has to decide whether to accept or reject a rental request. Since the proposed dynamic programming formulations are impractical due to the curse of dimensionality, linear programming approximations are used to derive revenue management decision policies for the operator. Indeed, primal and dual acceptance policies are developed (i.e. booking limits, bid prices) and their effectiveness is assessed on the basis of an extensive computational phase.

Keywords: Car Rental, Dynamic Programming, Revenue Management Policies.

[1]Submitted for pubblication in the journal *Mathematical Methods of Operations Research* October 2011

4.1 Introduction

The main contribution of this paper is the description of innovative solution methodologies, which can support car hire operators in effectively and efficiently allocating their limited, perishable and reusable resources to satisfy the rental requests of different market segments, with the objective of maximizing total revenues.

In the literature, some strategic and tactical planning aspects of the car rental process (i.e., pool segmentation and allocation, multi-period planning, optimal mixture and size of the vehicle fleet) have been discussed and analysed in-depth. For example, we cite the work of Yang et al. [162], where a dynamic model and a heuristic algorithm to manage a leasing network, in which different rental sites share a fleet of cars in a certain period of time, are proposed in order to determine the most profitable way (revenue maximization) to dispatch the cars among all sites. An approach to modeling the short-term car rental logistics problem as a specific instance of the minimum cost network flow problem has been presented in [45]. Haensel et al., in [56], give a stochastic programming approach for a car rental network with the fleet capacity per station not fixed and the possibility of deciding the optimal fleet distribution on the network, day by day, at given transfer costs. In [122], models for tactical fleet planning, the deployment and the transportation of vehicles are provided. The models determine the number of vehicles to allocate to each location, in order to maximize the expected total pool revenue and to move between each pair of locations, in order to minimize the total transhipment.

Furthermore, in the literature several studies about the application of revenue management (RM) techniques to rental business can be found. We cite, for example, the papers [135], [136], which consider generic rental company problems, the work of Papier et al. [124], which analyzes the case of a rail cargo company and the contribution of Powell et al. [130], related to freight transportation.

It is important to observe that the car rental industry has some great features that make it different from other RM target industries.
One of the most important difference is the possibility of controlling the capacity ([27], [56]). Indeed, a car rental company can transfer the cars between different rental stations and modify the total fleet, in order to response to market needs, by buying or selling cars. The ability of fine-tuning capacity to customer needs is not present in the majority of RM settings. This is surely true for the hotel industry. Indeed, it is not possible to move rooms between different locations or build new rooms, to adjust capacity in the immediate term. It is important to observe that, when a single car station is considered, the car rental revenue optimization problem presents some similarities with the revenue maximization in the hotel. In particular, in the hotel context, the capacity is the room nights, whereas in the car rental industry it is given by the available days of cars; the length of stay in the hotel setting is equivalent to the length of the rent in the car rental industry. Despite the aforementioned common characteristics, some differences arise also in the single car station setting. Indeed, hotels and car rental firms have discrete price classes, but in the hotels the prices are fairly constant across the classes, whereas in the car rental industries

the prices within these classes change actively on a daily basis. In addition, in a car rental agency the number of different types of car is on average greater than the possible number of different rooms in a hotel.
Since scarce attention has been devoted to the operational problems arising in the car rental industry, this paper analyzes the operational steps related to receiving and meeting demand in the car rental management process and solution approaches are proposed based on RM methods and techniques.

RM originated in the airline industry [15], [113]. In particular, Littlewood in [108] began approaches for the management of inventories of perishable services and goods, introducing the result (Littlewood's rule) that a request for a seat should be fulfilled only if its revenue exceeds the expected future value of the seat. The beginning of intensive development of RM techniques dates from the deregulation of the American Airlines industry in the late 1970's [138]. For a detailed discussions about the origins and the basic concepts of RM and for an overview of the extensive RM literature, the reader is referred to [14], [22], [34], [73],[75], [113], [127], [144] and more recently [30].

From its origins in the airline industry, RM techniques have been extended to various areas of service and manufacturing businesses. Indeed, RM is suitable for many service/goods companies (i.e. restaurants, casinos, cargo, cruise lines, Internet services, broadcasting and media, gas trasmission etc.), which show similar characteristics (perishability, relatively fixed capacity, ability to segment markets, product sold in advance, variable demand [74]), allowing an appropriate and effective application of the RM key strategies: overbooking, pricing, capacity control and forecasting [30], [102], [157].

The car rental industry has been included in traditional RM applications, but only in the early 1990's the car rental industry began to pay attention to the development of RM techniques. Carroll and Grimes in [27] and Geraghty and Johnson in [49] provide accounts of the state-of-the-art in car rental RM. In particular, in [49] the application of a RM program to save National Car Rental and analytical models to manage capacity, pricing and reservation are described, whereas the development of a sophisticated decision support system to determine rental fleet size and strategies for geographically redistributing vehicles as demand varies is considered in [27].

Innovative RM approaches to model dynamic price strategies are defined and applied to car rentals in [3] and [93]. Anderson and Blair in [2] give an accurate account of the *Performance Monitor* approach implemented at the Dollar Thrifty Automotive Group to measure the impact of RM, considering the lost revenue opportunities of historic decisions and pricing activities. In [52], novel RM models, considering particularities of the car rental business, are defined and robust optimization approaches are provided.

The purpose of this paper is to devise optimal RM policies for a car rental operator to accept or reject a customer's class-specific rental booking request. The traditional booking rental process is enriched considering and modeling new real car rental aspects, such as one-way rental and car transferring. In addition, dynamic programming models are defined to represent the car rental process, by taking into account future possible booking decisions in evaluating a current decision ([130], [151], [164]).

This paper makes the following research contributions: first of all, dynamic programming models are developed to represent the optimal management of a car rental business; in addition, RM policies are developed to decide dynamically when to accept an incoming rental request; finally, in a computational study the performance of the developed policies are assessed and their applicability is illustrated by considering a real case study.

The rest of the paper is organized as follows: in the next section the problem under consideration is described; in section 3 the dynamic programming formulations of the problem are given for the basic rental problem (BRp, for short) and for an extension version, which considers the one-way rental strategy (OWRp, for short); while in section 4, the related linear programming approximations are presented; Section 5 contains the description of the proposed RM policies, based on the solution of the linear problems; Section 6 presents computational experiments and concluding remarks are found in Section 7.

4.2 Problem Description

This section presents a description of a typical car rental process, the relevant operations and introduces the core decision problems.

A typical selling strategy is aimed at selling services/products to customers arriving successively within a given booking period. As a customer arrives and requests a specific rate, the operator has to decide whether to accept or not the customer's request. When making this decision, companies do not know what types of booking requests will come in the future. If most of the customers' booking requests are accepted independently of rate class, the operator may lose a lot of customers who are willing to pay higher rates. On the other hand, if companies reject most of the lower rate booking requests, they run the risk of remaining with many unused resources. This is a typical decisional process that can be managed in accordance with the rules of RM. In particular, we consider the arrival of rental requests and the relative decision process of a car rental operator.

A car rental company usually operates with a certain number of car groups, where each group contains different cars of comparable quality (e.g., concerning size and equipment). Each group represents a homogeneous unit with a base rental rate per day. If a customer makes a reservation for a certain group in advance and no corresponding car is available at the time of check-out, an upgrade to a superior car group can be made. The car assigned to a customer becomes unavailable for the length of the rent. When the rental period ends, the unit becomes available again. A car rental company has often to manage the problems of no-shows, i.e. people who book inventory and then do not show up to use it or pay for it, and cancellations of booking. To compensate for no-shows and cancellations, firms have developed different overbooking policies ([27], [49]) that are concerned with increasing the total volume of sales by selling reservations above capacity. However, this aspect of the rental process are not within the scope of this paper.

In this work, we assume that the revenue due to a rental is the base rate per day multiplied by the rental days. In the case of an upgrade, the revenue is based on the rate of the car group originally reserved. In general, the rate may depend on different factors such as the season, day of the week, or special contracts with certain groups or companies [45]. This work does not consider these aspects.

A rental starts at some location where there is a car rental agency (usually an airport, railway station) and ends either at the same or at a different point where the car is returned. Here the case in which the car returns to the same check-out is separated from the case of one-way rentals, for which the initial and final station can be different. In particular, to illustrate these different situations, in the first case only, the starting and the ending time of the rental period is considered. In the case of one-way rental the set of origins and destinations nodes is specified, i.e the check-out and check-in stations of each request.

At each period of booking horizon, the numbers of arriving customers may be uncertain, and managers must develop effective policies for controlling the rental capacity and optimizing total revenue, taking into account that complete information about the future demand is not available.

4.3 Dynamic programming formulations

The use of dynamic programming in RM (see [18], [19], [20], [43], [135]) helps to decide whether to accept or reject an incoming booking reservation with more realism than other methods, at each point in time, taking the decision that would imply higher future expected revenues. In order to compute and optimize the expected revenue, obtained applying a control policy, we provide dynamic programming formulations to represent the problems related to the optimal management of a car rental process.

It is assumed that time is discrete and the booking horizon is divided into $\bar{T}$ decision periods $\bar{t} = 1, \ldots, \bar{T}$ such that, at most, one request arrives per period. It is important to point out that in our work the Bellman equation is evaluated in a forward manner. In each time period $\bar{t}$ of the booking horizon, the logistic operator has to decide on accepting the rental request of a car in a certain rental class, from day i to day j of the rental period [and from origin o to destination d in the one-way rental case] with the goal of maximizing the total expected revenue. The rental horizon is divided into T decision periods and $i = 1, \ldots, T-1$ represents the starting time of the rent period, $j = 2, \ldots, T$ is the ending time of the rent period, with $1 \leq i < j \leq T$, $j = i+1, \ldots, T$ and $(j-i)$ indicates the length of rent. We assume that the booking horizon and the rental horizon do not overlap. Demand for each product is time-dependent and modeled by a random variable.

The number of booking classes is denoted as $\bar{K}$ and represents the available groups/categories of vehicles. For each group, Q^k represents the total capacity of group k (i.e., number of available cars belonging to the $k-th$ group), $k = 1, \ldots, \bar{K}$.

A daily rental rate p^k, $k = 1, \ldots, \bar{K}$, is associated with each category of cars. We assume that the higher the category, the higher the rental rate, that is $p^1 < p^2 < \ldots < p^{\bar{K}}$. Each accepted booking request results in revenue $R^k_{ij} = (j-i)p^k$.

In what follows, the term "product" indicates a car of category k to be rented in a certain time interval $(j-i)$, $i = 1, \ldots, T-1$, $j = 2 \ldots, T$.

Customers can be viewed as partitioned into $\bar{K}$ different classes. A customer is of class k, $k = 1, \ldots, \bar{K}$, if he/she requires to rent a car of category k. A request for class k can be satisfied with cars of superior neighboring class, in this case we have an upgrade.

Let $\left[A^1|A^2|\ldots|A^{\bar{K}}\right]$, $A \in R^{\bar{K}\times(2\bar{K}-1)}$, denote a binary matrix, partitioned into $\bar{K}$ sub-matrices. Each sub-matrix ($A^k \in R^{\bar{K}\times 2}, k = 1\ldots, \bar{K}-1$ and $A^{\bar{K}} \in R^{\bar{K}\times 1}$) contains the set of possible products to satisfy the demand for a class k customer. In particular, sub-matrix A^1 contains the products that can be used to satisfy the first group rental request, that is, the first and the second car class, whereas the last sub-matrix $A^{\bar{K}}$ is the product constituted of the car of the highest category, a request for this group can be satisfy only with a car of the same category; thus $A^{\bar{K}}$ contains only one column.

We indicate each column of matrix A as A_{v_k}, $v_k = v_{min(k)}, \ldots, v_{max(k)}$, where $v_{min(k)} = (2k-1)$ for $k = 1, \ldots, \bar{K}$ and $v_{max(k)} = (2\bar{K}-1) - \sum_{s=k}^{k+1}(\bar{K}-s)$ for $k = 1, \ldots, \bar{K}-1$ and $v_{max(k)} = v_{min(k)}$ for $k = \bar{K}$.

Each element $a^k_{v_k}$, $k = 1, \ldots, \bar{K}$, $v_k = v_{min(k)}, \ldots, v_{max(k)}$, of matrix A is equal to one if car k is used in the product v_k, and 0 otherwise. It is worth noting that a product indicates the type of car that the logistic operator can use to satisfy the demand for a certain class k; in fact, owing to the upgrade, a class k request can be satisfied with cars of category k or next-higher class (i.e. $k+1$).

The use of this matrix to define the dynamic programming formulations for the basic and the one-way rental problems is highlighted in the following subsections.

4.3.1 A Dynamic Programming Formulation for the BRp

When a rental request for class k with starting day i and ending day j arrives at time $\bar{t}$, the car rental operator has to decide how to manage the available resources in order to obtain the maximum possible performance (i.e., maximize the revenue).

The state of the system is described by a matrix $Q = [Q_1|Q_2|\ldots|Q_T]$, where each column $Q_t = (q^1_t, \ldots, q^{\bar{K}}_t)^\top, \forall t = 1, \ldots, T$ represents the number of cars of type k, $k = 1, \ldots, \bar{K}$ available at time t.

It is assumed that at each booking period $\bar{t}$, at most one rental request arrives. We denote with $\lambda^{\bar{t}k}_{ij}$, the probability that at time $\bar{t}$ one rental request for class k with pickup day i and return day j, is made. It holds that $\sum_{\bar{t}=1}^{\bar{T}}\sum_{k=1}^{\bar{K}}\sum_{i=1}^{T}\sum_{j=i+1}^{T}\lambda^{\bar{t}k}_{ij} + \lambda^{\bar{t}}_0 = 1$, where $\lambda^{\bar{t}}_0$ is the probability that no request arrives at booking period $\bar{t}$.

Let us introduce boolean variables $u_{ij}^{\bar{t}v_k}$, with $u_{ij}^{\bar{t}v_k} = 1$ if the k rental request at time $\bar{t}$, from day i to day j, is accepted and satisfied by using product v_k and $u_{ij}^{\bar{t}v_k} = 0$ otherwise.

The problem can be formulated as a dynamic programming by letting $V_{\bar{t}}(Q)$ the maximum expected revenue obtainable from periods $\bar{t}, \bar{t}+1, \dots, \bar{T}$, given that at booking time $\bar{t}$ the capacity of the system is Q.

The Bellman equation for $V_{\bar{t}}(Q)$ is reported in what follows:

(4.1)
$$V_{\bar{t}}(Q) = \sum_{k=1}^{\bar{K}} \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \lambda_{ij}^{\bar{t}k} \max_{\substack{u_{ij}^{\bar{t}v_k} \in \{0,1\} \\ v_k \in \{v_{min(k)}, \dots v_{max(k)}\}}} \left[(j-i)p^k u_{ij}^{\bar{t}v_k} + V_{\bar{t}+1}(\tilde{Q}) \right] + \lambda_0^{\bar{t}} V_{\bar{t}+1}(Q)$$

with the following boundary conditions:

$$V_{\bar{t}}(0) = 0, \ \forall \bar{t};$$

$$V_{\bar{t}}(Q) = -\infty, \ if \ q_t^k < 0, \text{for some} t, k, \forall \bar{t};$$

$$V_{\bar{T}+1}(Q) = 0, \ if \ q_t^k \geq 0, \forall t, k.$$

$$V_{\bar{T}+1}(Q) = -\infty, \ if \ q_t^k < 0, \text{for some} t, k.$$

The update of Q at time $\bar{t}$ is related to the following events.

- When a certain request from time i is accepted, the capacity at time $\tilde{i}, \forall \tilde{i} = i, \dots, T$ is represented by the term $\tilde{Q}_{\tilde{i}} = (Q_{\tilde{i}} - A_{v_k} u_{ij}^{\bar{t}v_k})$. Indeed, the car used to satisfy the request will not be available anymore from time i until the end of the rental horizon.
- When a certain request with ending day j is accepted, the capacity at time $\tilde{j}, \forall \tilde{j} = j, \dots, T$ is represented by the term $\tilde{Q}_{\tilde{j}} = (Q_{\tilde{j}} + A_{v_k} u_{ij}^{\bar{t}v_k})$. Indeed, the car used to satisfy the request from day i to day j will be available from time j until the end of the rental horizon.
- When a request from i to j is accepted, the term $\tilde{Q}_l = Q_l, \forall l \neq [i, j]$ updates capacity on the rest of the system.

In other words, given a request, the decision maker either accepts the request, receiving the associated revenue and reducing the inventory level, or denies the request, moving to the next period with the same inventory.

4.3.2 A Dynamic Programming Formulation for the OWRp

While the previous subsection was focused on a simplified version of the rental process, in which it is assumed that the origin and destination of all rentals are the same, thus the rented car must be returned to the pick up agency, in this subsection, we analyze a typical aspect of the car rental process that arises when a customer needs to pick up the car at one location and leave it at a different destination point. In what follows, a dynamic model to represent this case (i.e. car rental with one-way possibility) is described.

Let N be the total number of stations and let O be a given set of origins and D a given set of destinations, we assumed that $O \equiv D$, i.e. each station can be both origin and destination for a rental.

We indicate with the term "product" a car of group k, $k = 1, \dots, \bar{K}$ to be rented in a certain time interval $(j-i)$, $i = 1, \dots, T-1$ and $j = 2, \dots, T$, from origin o to destination d. The state of the system is described by a matrix $Q = [Q_{11}|\dots|Q_{1|N|}|\dots|Q_{T1}|\dots|Q_{T|N|}]$, where each column $Q_{tn} = (q^1_{tn}, \dots, q^{\bar{K}}_{tn})^\top$, $\forall t = 1, \dots, T$, $n = 1, \dots, N$, represents the number of cars of type k, $k = 1, \dots, \bar{K}$ available at station n, that can be used to satisfy a car rental request starting at day t.

Similarly to the notations used in the basic car rental formulation, we define $\lambda^{\bar{t}kod}_{ij}$, the probability that at time $\bar{t}$ one rental request for class k from day i to day j, from station o to station d, is made; $\lambda^{\bar{t}}_0$ is the probability that no request arrives at time $\bar{t}$; boolean variables $u^{\bar{t}v_k}_{ij,od}$, with $u^{\bar{t}v_k}_{ij,od} = 1$ if the k rental request at time $\bar{t}$, from day i to day j and from origin o to destination d, is accepted and satisfied by using product v_k and $u^{\bar{t}v_k}_{ij,od} = 0$ otherwise. It is assumed that at each booking period $\bar{t}$, at most one request for rent can arrive, that is $\sum_{\bar{t}=1}^{\bar{T}} \sum_{k=1}^{\bar{K}} \sum_{i=1}^{T} \sum_{j=i+1}^{T} \sum_{o=1}^{N} \sum_{d=1}^{N} \lambda^{\bar{t}kod}_{ij} + \lambda^{\bar{t}}_0 = 1$

The Bellman equation for $V_{\bar{t}}(Q)$ is reported in what follows:

(4.2)
$$V_{\bar{t}}(Q) = \sum_{k=1}^{\bar{K}} \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{o=1}^{N} \sum_{d=1}^{N} \lambda^{\bar{t}kod}_{ij} \max_{\substack{u^{\bar{t}v_k}_{ij,od} \in \{0,1\} \\ v_k \in \{v_{min(k)}, \dots v_{max(k)}\}}} \left[(j-i)p^k u^{\bar{t}v_k}_{ij,od} + V_{\bar{t}+1}(\tilde{Q})\right]$$
$$+\lambda^{\bar{t}}_0 V_{\bar{t}+1}(Q)$$

The boundary conditions of the Bellman equation are the following:

$$V_{\bar{t}}(0) = 0, \ \forall \bar{t};$$
$$V_{\bar{t}}(Q) = -\infty, \ if \ q^k_{tn} < 0, \text{for some} t, n, k, \forall \bar{t};$$
$$V_{\bar{T}+1}(Q) = 0, \ if \ q^k_{tn} \geq 0, \forall t, n, k.$$
$$V_{\bar{T}+1}(Q) = -\infty, \ if \ q^k_{tn} < 0, \text{for some} t, n, k.$$

The update of Q at time $\bar{t}$ is related to the events reported below:

- $\tilde{Q}_{\tilde{i}o} = (Q_{\tilde{i}o} - A_{v_k} u^{\bar{t}v_k}_{ij,od}), \forall \tilde{i} = i, \ldots, T$: this term represents capacity at station o at time $\tilde{i}$ when a certain request from station o at pickup time i is accepted.
- $\tilde{Q}_{\tilde{j}d} = (Q_{\tilde{j}d} + A_{v_k} u^{\bar{t}v_k}_{ij,od}), \forall \tilde{j} = j, \ldots, T$: this term updates capacity at station d at time $\tilde{j}$ when a certain request to station d at return time j is accepted.
- $\tilde{Q}_{lm} = Q_{lm}, \forall l \neq [i, j], \forall m \neq o, d$: this term represents capacity on the rest of the system when a request from station o to station d, from time i to time j, is accepted.

4.4 Linear Programming Formulations

The proposed dynamic programming models are characterized by increasing computation difficulties according to the dimension of the problem. For this reason, owing to the large size of the solutions space, the dynamic models are unlikely to be solved optimally. This section presents linear programming approximations of the problems under consideration, which are used to define appropriate RM policies [144]. The proposed models are static but are solved in a "dynamic way" by appropriately updating the demand and the capacity information at the beginning of each time period. The aim is to support the logistic operator in taking decisions by adopting RM policies.

4.4.1 A Deterministic Linear Programming Formulation for the BRp

To illustrate the linear formulation we report the major notations for parameters and variables used. Let:

D^k_{ij} be the expected number of booking requests for cars belonging to the group k, $k = 1, \ldots, \bar{K}$, from day i to day j;

$x^{v_k}_{ij}$ number of products of type $v_k = v_{min(k)}, \ldots, v_{max(k)}$ to be used to satisfy the rental request for a k class customer, $k = 1, \ldots, \bar{K}$, from day i to day j.

We assume that there are no requests before day 1 and all the requests have to be satisfied on or before day T. It is also assumed that any type of car has to be booked for at least one day. On the basis of the previous considerations, the complete mathematical representation of the problem under study is given in what follows.

$$R^{BRp}(Q) = Max \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{k=1}^{\bar{K}} \sum_{v_k=v_{min(k)}}^{v_{max(k)}} (j-i) p^k x^{v_k}_{ij} \right] \tag{4.3}$$

$$\sum_{v_k=v_{min(k)}}^{v_{max(k)}} x^{v_k}_{ij} \leq D^k_{ij} \quad \forall i = 1, \ldots, T-1, j = i+1, \ldots, T, k = 1, \ldots, \bar{K} \tag{4.4}$$

(4.5)

$$\sum_{j=i+1}^{T}\sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{ij}^{v_k} + \sum_{t=1}^{i-1}\sum_{j=t+1}^{T}\sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{tj}^{v_k} - \sum_{j=1}^{i-1}\sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{ji}^{v_k} \leq Q_i^k \quad \forall k = 1,\ldots,\bar{K}, i = 1,\ldots,T-1$$

(4.6) $$x_{ij}^{v_k} \geq 0,\ integer \quad \forall v_k = 1,\ldots,2\bar{K}-1, i = 1,\ldots,T-1, j = i+1,\ldots,T$$

The objective function (5.2) represents the total revenue obtainable when the capacity on the network is Q. Conditions (5.3) impose that, a demand of class k can be satisfied with product v_k. Constraints (5.4) establish that, for each group and for each time period, the number of vehicles used to satisfy the car rental requests cannot exceed the maximum available capacity. In particular, constraints (5.4) are composed of three terms: the first indicates the total number of cars rented from day i to day j; the second indicates cars that start before i and end the rent after i; the third represents cars rented at time less than i and returned at time i.

The proposed model allows the number of vehicles to be rented and the number of upgrades to be determined, with the objective of maximizing the expected revenue, while satisfying demand and capacity constraints.

In the objective function (5.2), the total revenue is determined by introducing a rental rate per day p^k, $k = 1,\ldots,\bar{K}$ for each class k, that does not depend on the length of the rental period. However, in general, the higher the rental period, the lower the day rental price. The proposed model can be easily modified to handle this specific situation. In particular, it is sufficient to replace the objective function (5.2) with the following:

(4.7) $$Max\left[\sum_{i=1}^{T-1}\sum_{j=i+1}^{T}\sum_{k=1}^{\bar{K}}\sum_{v_k=v_{min(k)}}^{v_{max(k)}} (j-i)p^k(1-\pi_{j-i})x_{ij}^{v_k}\right]$$

where π_{j-i} is a non-negative scalar such that $\pi_1 = 0$ and $\pi_1 < \pi_2 < \ldots < \pi_T$.

In general, car rental companies can follow different upgrade policies; indeed, the next size car should be offered to the renter only when it is really necessary or upgrades should be encouraged. In the model presented above, the car rental company's attitude to making upgrades is not explicity represented. However, as will be shown in the computational experiments section, upgrades are driven by the fare structure, that is, the relative price difference between two consecutive car categories.
However, the optimization model (5.2)-(5.5) can be easily extended to represent the car rental companies' willingness to make upgrades ([52].

In the programming formulation for the BRp, reported above, we have imposed the satisfaction of integer constraints for the decision variables. However, given the specific structure of the constraint matrix (CM, for short) associated with BRp, its linear relaxation yields an integral solution, thus these constraints can be relaxed.

In particular, as reported in Appendix A, it is possible to show that CM can be reduced to an interval matrix, that is, a 0 - 1 matrix where the ones appear consecutively, that is totally unimodular [119].

4.4.2 A Linear Programming Formulation for the OWRp

To present the mathematical formulation of the problem, in what follows we report only the parameters and variables used to represent the one-way rentals problem. We do not report the notations that have already been introduced for the BRp.

Q_{io}^k total capacity of group k, $k = 1, \ldots, \bar{K}$, available at origin station $o = 1, \ldots, N$ at time $i = 1 \ldots, T$ (i.e., number of available cars belonging to the $k - th$ group);

$D_{ij}^{k,od}$ expected number of booking requests for cars belonging to the group k, $k = 1, \ldots, \bar{K}$, from day i to day j and from origin o to destination d.

$x_{ij}^{v_k,od}$ number of products of type $v_k = v_{min(k)}, \ldots, v_{max(k)}$ to be used to satisfy the rental request of k, $k = 1, \ldots, \bar{K}$, from day i to day j and from origin o to destination d.

$$R^{OWRp}(Q) = Max \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{o=1}^{N} \sum_{d=1}^{N} \sum_{k=1}^{\bar{K}} \sum_{v_k=v_{min(k)}}^{v_{max(k)}} (j-i) p^k x_{ij}^{v_k,od} \right] \tag{4.8}$$

$$\sum_{v_k=v_{min(k)}}^{v_{max(k)}} x_{ij}^{v_k,od} \leq D_{ij}^{k,od} \quad \forall k = 1, \ldots, \bar{K}, i = 1, \ldots, T-1, j = i+1, \ldots, T, o, d = 1, \ldots, N \tag{4.9}$$

$$\sum_{j=i+1}^{T} \sum_{d=1}^{N} \sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{ij}^{v_k,od} + \sum_{t=1}^{i-1} \sum_{j=t+1}^{T} \sum_{d=1}^{N} \sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{tj}^{v_k,od} - \sum_{j=1}^{i-1} \sum_{d=1}^{N} \sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{ji}^{v_k,do} \leq Q_{io}^k \tag{4.10a}$$

$$\forall k = 1, \ldots, \bar{K}, i = 1, \ldots, T-1, o = 1, \ldots, N \tag{4.10b}$$

$$x_{ij}^{v_k,od} \geq 0, integer \quad \forall v_k = 1, \ldots, 2\bar{K}-1, i = 1, \ldots, T-1, j = i+1, \ldots, T, o, d = 1, \ldots, N \tag{4.11}$$

Conditions (4.9) impose that a class k rental request from day i to day j and from station o to station d can be satisfied with product of type v_k. Constraints (4.10) control the availability of cars of category k at station o at time i.

An important operational problem that needs to be addressed in the car rental industry is related to the ability to move the fleet from one station to the other. Decisions for transferring cars among locations are crucial for different reasons: in the car rental with one-way possibility, rentals with different pickup and return stations can create chaos in the fleet distribution; at each station, the demand varies during the week between weekdays and weekends: there is greater demand for car rentals near the airports on weekdays and in city downtowns on weekends. The decision to transfer car from one station to another must take into account the trade-off between the incremental revenue and the transhipment cars costs.

The mathematical model introduced above can be modified to take into account the transfer of a car of category k from station o to station d on day i, when the benefit of such operation is greater than the cost of moving the car. To present the mathematical formulation, we need to introduce the following quantities:

- $z_i^{k,od}$ number of cars of category k moved from station o to station d on day i;
- γ_{od} the cost of moving a car from station o to station d;
- τ_{od} the average travel time from station o to station d.

It is assumed that the cost of moving a car from o to d does not depend on the car category, but it is related only to the distance travelled; the average travel time is an integer multiple of days. The new objective function takes the following form:

(4.12)
$$R^{OWR_p^{TR}}(Q) = Max \left[\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} \sum_{o=1}^{N} \sum_{d=1}^{N} \sum_{k=1}^{\bar{K}} \sum_{v_k=v_{min(k)}}^{v_{max(k)}} (j-i) p^k x_{ij}^{v_k,od} - \sum_{i=1}^{T} \sum_{o=1}^{N} \sum_{d=1}^{N} \sum_{k=1}^{\bar{K}} \gamma_{od} z_i^{k,od} \right]$$

Constraints (4.9), (4.11) are still valid whereas constraints (4.10) need to be restated as follows:

(4.13a)
$$\sum_{j=i+1}^{T} \sum_{d=1}^{N} \sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{ij}^{v_k,od} + \sum_{t=1}^{i-1} \sum_{j=t+1}^{T} \sum_{d=1}^{N} \sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{tj}^{v_k,od} - \sum_{j=1}^{i-1} \sum_{d=1}^{N} \sum_{v_k=1}^{2\bar{K}-1} a_v^k x_{ji}^{v_k,do}$$

(4.13b)
$$+ \sum_{d=1 d\neq o}^{N} z_i^{k,od} + \sum_{t=1}^{i-1} \sum_{d=1 d\neq o}^{N} z_t^{k,od} - \sum_{d=1 d\neq o}^{N} \sum_{t=1}^{i-\tau_{do}} z_t^{k,do} \leq Q_{io}^k$$

(4.13c)
$$\forall k = 1, \ldots, \bar{K}, i = 1, \ldots, T-1, o = 1, \ldots, N$$

4.5 Revenue-based primal and dual acceptance policies

The solution to the models presented in the previous sections can be used to implement two different forms of booking control that are used in practice, one based on booking limits and the other based on bid prices [18], [144]: both defined to accept or reject a request at a certain time. These policies are presented in the following.

At a certain point of the booking horizon, decisions about accepting or rejecting a rental request are to be made.

At each time $\bar{t}$, $R^{BRp}(Q)$ $[R^{OWRp}(Q)]$ is solved and its solution used to make a decision. In particular, a request arrived at a booking time $\hat{t}$ less than or equal to $\bar{t}$ is chosen and one of the following proposed policies is used to make the accept/deny decision.

It is important to observe that, in defining these policies, we relax the assumption that at most one rental request occurs in each booking period (see Section 3). The rationale is that, in real settings, it is not practical to solve the proposed models each time a booking request arrives.

From a primal viewpoint, the strategy to be adopted is a booking limits policy ($\mathcal{BLP}$, for short), which assumes the following form.

FOR $\bar{t} = 1, \ldots, \bar{T}$ DO

SOLVE $R^{BRp}(Q)$ $[R^{OWRp}(Q)]$. Let $x_{ij}^{\star v}$ $[x_{ij}^{\star v,od}$]denote its optimal solution.

FOR each rental request for class k from day i to day j [from origin o to destination d, for a request in OWRp] arrived at booking time $\hat{t}_{ij}^k \leq \bar{t}$ $[\hat{t}_{ij,od}^k \leq \bar{t}]$ DO

IF $x_{ij}^{\star v} > 0$ $[x_{ij}^{\star v,od} > 0$] for some $v= v_{min(k)}, \ldots, v_{max(k)}$ (where $v_{min(k)}= (2k-1)$ and $v_{max(k)} = 2k$ for $k = 1, \ldots, \bar{K}-1$ and $v_{min(k)}=v_{max(k)}$ for $k= \bar{K}$) THEN

ACCEPT the request with upgrade if $v > v_{min(k)}$;

SET $x_{ij}^v = x_{ij}^v - 1$ $[x_{ij}^{v,od} = x_{ij}^{v,od} - 1]$;

SET $\tilde{k} = v - v_{min(k)} + k$ i.e. determine the type of car used in product v;

UPDATE appropriately the capacity:
$Q_{\tilde{i}}^{\tilde{k}} = Q_{\tilde{i}}^{\tilde{k}} - 1, \forall\, \tilde{i} = i, \ldots, T-1$ and $Q_{\tilde{j}}^{\tilde{k}} = Q_{\tilde{j}}^{\tilde{k}} + 1, \forall\, \tilde{j} = j, \ldots, T$ and $j > i$; if $Q_{\hat{i}}^{\tilde{k}} > Q_{i}^{\tilde{k}}$ then $Q_{\hat{i}}^{\tilde{k}} = Q_{\hat{i}}^{\tilde{k}} - 1,\ \forall\, \hat{i} = 1, \ldots, i-1$.
$[Q_{\tilde{i},o}^{\tilde{k}} = Q_{\tilde{i},o}^{\tilde{k}} - 1, \forall\, \tilde{i} = i, \ldots, T-1$, and $Q_{\tilde{j},d}^{\tilde{k}} = Q_{\tilde{j},d}^{\tilde{k}} + 1, \forall\, \tilde{j} = j, \ldots, T$ and $j > i$; if $Q_{\hat{i},o}^{\tilde{k}} > Q_{i,o}^{\tilde{k}}$ then $Q_{\hat{i},o}^{\tilde{k}} = Q_{\hat{i},o}^{\tilde{k}} - 1,\ \forall\, \hat{i} = 1, \ldots, i-1]$.

CALCULATE the revenue obtained from accepting the request;

ELSE

REJECT the request.

END IF

END FOR

END FOR

We move to the next booking time period when there are no more requests, arrived before $\bar{t}$, that need to be evaluated.

It is important to point out that we have also defined a booking limits policy based on the solution of $R^{OWR_p^{TR}}(Q)$ and referred to as $\mathcal{BLP}^{\mathcal{TR}}$. In this case, $R^{OWR_p^{TR}}(Q)$ is solved instead of $R^{OWRp}(Q)$ and the capacity is updated by taking into account the cars that are transferred among stations.

An alternative booking policy is based on bid prices (i.e. the minimum amount of money to accept in exchange for a unit of capacity) [18], [137], [144].

It is not possible to define dual acceptance policies for both OWR_p^{TR} and OWR_p because they are formulated as integer programming problems. While, for the BRp, it is necessary to solve the problem $R^{BRp}(Q)$ and to deal with the dual variables associated with the capacity constraints. We will indicate with $\mathcal{BPP}$ the bid price policy associated with the linear programming formulation.

Letting $\pi_i^k, i = 1, \ldots, T-1, k = 1, \ldots, \bar{K}$, be the optimal values of the dual variables associated with constraints (5.4), we can use π_i^k as an estimate of the bid-prices of a unit of capacity in group k, which can be used to decide whether to accept or reject a request. The decision rule is that if the revenue from a rental request exceeds the sum of the bid prices of the request k from pickup day i to return day j that arrives at booking time $\bar{t}$, then we accept the car rental request subject to the capacity availability. The related decision strategy can be described as follows:

FOR $\bar{t} = 1, \ldots, \bar{T}$ DO

SOLVE the $R^{BRp}(Q)$ to obtain the dual variables $\pi_i^k, i = 1, \ldots, T-1, k = 1, \ldots, \bar{K}$.

FOR each rental request for class k from day i to day j arrived at booking time $\hat{t}_{ij}^k \leq \bar{t}$ DO

IF there is a $\tilde{k} \geq k$ and $\tilde{k} \leq k+1$ such that $(j-i)p^k \geq \sum_{\tau=i}^{j-1} \pi_\tau^{\tilde{k}}$ AND $Q_{\tilde{i}}^{\tilde{k}} > 0 \; \forall \; \tilde{i} = i, \ldots, T-1$ THEN

ACCEPT the request with upgrade if $\tilde{k} > k$;

UPDATE appropriately the capacity;

CALCULATE the revenue obtained from accepting the request;

ELSE

REJECT the request.

END IF

END FOR

END FOR

It is worth noting that the procedure used to update the capacity is the same as that described in the previous section for the booking limits policy.

In addition to the $\mathcal{BPP}$ outlined above, an alternative policy based on bid price has been defined. In this strategy, referred to as initial bid price policy ($\mathcal{IBPP}$, for short) an incoming rental request for class k from day i to day j is accepted if there is a $\tilde{k} \geq k$ and $\tilde{k} \leq k+1$ such that $(j-i)p^k \geq (j-i)\pi_i^{\tilde{k}}$.

4.6 Computational Experiments

We carried out computational experiments aimed at studying the performance of the developed models for the car rental problems and assessing the behaviour of the proposed RM policies.

The tests were carried out on a PC Pentium IV with 3.2 GHz and 2 GB of RAM, under the Windows XP operating system. The AIMMS 3.7 mathematical modeling language (www.aimms.com), with ILOG solver (www.ilog.com), was used to implement the proposal models and policies, whereas Cplex 10.1 was used to solve the implemented models.

The computational experiments were carried out on randomly generated test examples, defined trying to be quite close to the reality of medium-sized car rental agencies. Indeed, the problem under consideration has not been taken into account in the RM literature so far, thus, benchmark instances are not available.

In addition, to assess the validity of the proposed policies, a real case study has also been considered.

The experiments, on randomly generated test problems, were performed dynamically. In each simulation run, the rental requests are generated by applying the following procedure. First, the number of rental requests is randomly generated according to a normal distribution, with a given expected demand and a given coefficient of variation.

The expected demand was generated randomly in such a way as to have three different scenarios characterized by a low, medium and high load factor (defined as the the ratio of average expected demand to available capacity), whereas the coefficient of variation was generated randomly from the interval $[0, 1]$. Then, for each request, booking arrival times are randomly generated according to a uniform distribution.

For each test problem, the booking process was simulated 100 times. In each of 100 simulation runs, all the requests for each test example, generated by applying the strategy described above, are processed based on the policies presented in Section 5. In particular, at each booking period $\bar{t}$, a rental request with the booking arrival time less than or equal to the considered booking instant, is chosen and the accept/deny decision is made based on one of the proposed policies. The capacity is then updated and another booking request is processed. We move to the next booking time period when there are no more requests, arrived before $\bar{t}$, that need to be evaluated.

The daily price for the rental of one unit belonging to each category of cars is chosen in the interval $[80, 200]$, the higher the category the greater the price.

For each problem class all the proposed policies were implemented by considering a booking horizon of 2, 3, 5 and 7 time periods. Booking starts from the first day and demand forecasts for all products are updated every day of the booking horizon. In particular, on each day of the booking horizon we resolved the problems with updated capacities and future demand.

4.6.1 Numerical results for the BRp

The following three policies were implemented in the case of BRp:

- $\mathcal{BLP}$: booking limits policy;
- $\mathcal{BPP}$: bid price policy;
- $\mathcal{IBPP}$: initial bid price policy.

The test problems, used in the computational study, are characterized by an increasing number of groups $K = 5, 10, 15$. A rental horizon T of 7 periods (i.e., a week) was considered. In addition, the number of cars Q^k available for each group was randomly generated in the interval $[1, 10]$.

The characteristics of the test problems are reported in Table 4.1, where the number of groups and the load factor values are given.

Test Problem	Groups	Load Factor
T_1^L	5	0.235
T_1^M	5	0.784
T_1^H	5	1.520
T_2^L	10	0.381
T_2^M	10	0.702
T_2^H	10	1.227
T_3^L	15	0.461
T_3^M	15	0.709
T_3^H	15	1.264

Table 4.1: Characteristics of test problems

A first phase of the experimental tests was carried out with the aim of evaluating the influence of the length of the booking horizon on the revenue that is possible to achieve. In this respect it is worth observing that given the assumption that at most one rental request occurs in each booking period, the magnitude of a booking period should be small enough that the probability of having two bookings in the same period is negligible. However, this hypothesis is necessary to define the Bellman equation, but from a practical point of view solving the problem at the occurrence of each request is not a viable strategy, especially in real settings.

Fig.4.1 gives , for the test problem T_1^L, the trend of the average revenue value, obtained by applying the proposed policies, as a function of the length of the booking horizon. A similar trend has also been observed for the other test problems. From this figure, it is evident that a booking horizon of 7 periods gives the best compromise results between the computational overhead, that increases when the length of booking horizon is increased, and the revenues achieved for both the policies. For this reason, in what follows we focus on the results obtained by letting $\bar{T} = 7$.

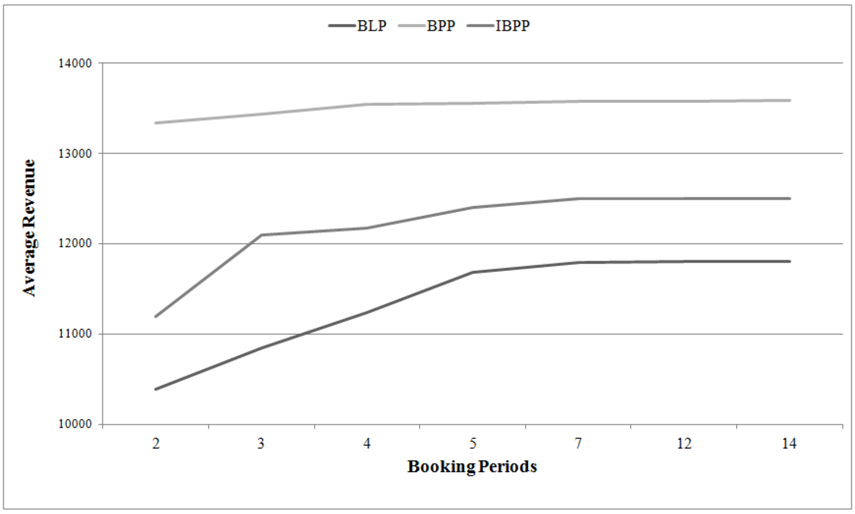

Figure 4.1: Trend of the average revenue values, obtained with the $\mathcal{BLP}$, $\mathcal{BPP}$ and $\mathcal{IBPP}$, as a function of the length of booking horizon for the test problem T_1^L.

The proposed optimization models do not take into account explicitely the car rental companies' attitude to make upgrades. In order to assess the influence of the fare structure on the number of upgrades, in Fig. 4.2 the trend of the average number of upgrades as function of the relative price difference (RPD, for short) between two consecutive car groups is depicted.
This figure clearly underlines that for $\mathcal{BLP}$ the higher the relative price difference between a car group and its superior neighboring class, the lower the average number of upgrades, whereas $\mathcal{BPP}$ and $\mathcal{IBPP}$ are less sensitive to this factor and the trend is fairly constant. Similar perfomance has also been obtained on the other test problems. These computational results highlight that the $\mathcal{BLP}$ allows the RPD to be properly taken into account. As far as the infuence of the load factor on the number of upgrades is concerned, Table 4.2 underlines that $\mathcal{BLP}$ performs upgrades only if they are really necessary. In particolar, for $\mathcal{BLP}$ the higher the load factor the lower the number of upgrades, whereas this trend

is inverted for $\mathcal{BPP}$ and $\mathcal{IBPP}$.

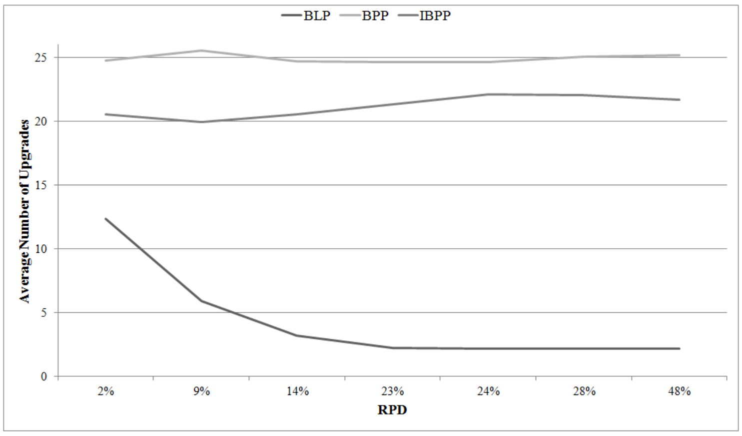

Figure 4.2: Trend of the average number of upgrades as function of the fare structure, for the test problem T_2^L.

Test Problem	$\mathcal{BLP}$	$\mathcal{BPP}$	$\mathcal{IBPP}$
T_1^L	4.88	14.33	10.19
T_1^M	5.86	17.63	15.49
T_1^H	0.12	15.07	14.56
T_2^L	13.45	24.62	20.73
T_2^M	6.54	21.31	20.33
T_2^H	3.34	23.65	21.68
T_3^L	16.18	43.46	35.30
T_3^M	9.06	40.40	35.95
T_3^H	3.28	47.84	41.92

Table 4.2: Average number of upgrades

In order to evaluate the quality of the policies, we compared $R^{\mathcal{BLP}}$, $R^{\mathcal{BPP}}$ and $R^{\mathcal{IBPP}}$, with the revenues obtained in case of a perfect knowledge of the realized demand $\mathcal{PKRD}$, and with that achieved when a first-come first-served policy, $\mathcal{FCFSP}$, is applied, i.e. all incoming requests are accepted as long as they can fulfil, by considering also the possibility of renting a car in at most one category superior to that initially required .

Table 4.3 gives the computational results obtained by the proposed policies. In particular, for each test problem, we report the average revenue value, obtained with the proposed policies (i.e., $R^{\mathcal{BLP}}$, $R^{\mathcal{BPP}}$ and $R^{\mathcal{IBPP}}$), and its 95% confidence interval.

Table 4.3 clearly underlines that $\mathcal{BLP}$ behaves the best, whereas $\mathcal{BPP}$ and $\mathcal{IBPP}$ show comparable performance. In particular, the average revenue value obtained with $\mathcal{BLP}$ is 32797.95, whereas the average revenues achieved when the bid price policies are considered, are equal to 29310.70 and 29313.56 for $\mathcal{BPP}$ and $\mathcal{IBPP}$, respectively.

Test Problem	$R^{\mathcal{BLP}}$	$R^{\mathcal{BPP}}$	$R^{\mathcal{IBPP}}$
T_1^L	12796.67	13577.12	12498.96
	12323.45 12869.89	13253.47 13900.77	12245.50 12752.42
T_1^M	16420.18	12497.97	12728.88
	16119.78 16720.58	12402.95 12592.99	12635.24 12822.52
T_1^H	16870.52	11968.14	13056.27
	16773.75 16967.29	11833.14 12103.14	12987.37 13125.17
T_2^L	32072.47	32420.49	31556.11
	31157.15 32987.79	32234.49 32606.49	31244.84 31867.38
T_2^M	35854.26	31933.87	33069.35
	34998 36710.5	31809.6 32058.11	32812.7 33326
T_2^H	37643.45	29690.42	31046.97
	36939.39 38347.51	29489.13 29891.71	30791.72 31302.22
T_3^L	46972.24	44848.47	44106.12
	45769.7 48174.7	44598.4 45098.56	43776.2 44436
T_3^M	47032.12	43837.08	42441.39
	45520.34 48543.90	43687.65 43986.51	42205.38 42677.40
T_3^H	50519.62	43022.73	43318.03
	49132.33 51906.91	42904.2 43141.26	43167.03 43469.03

Table 4.3: Average revenue value and its 95% confidence interval

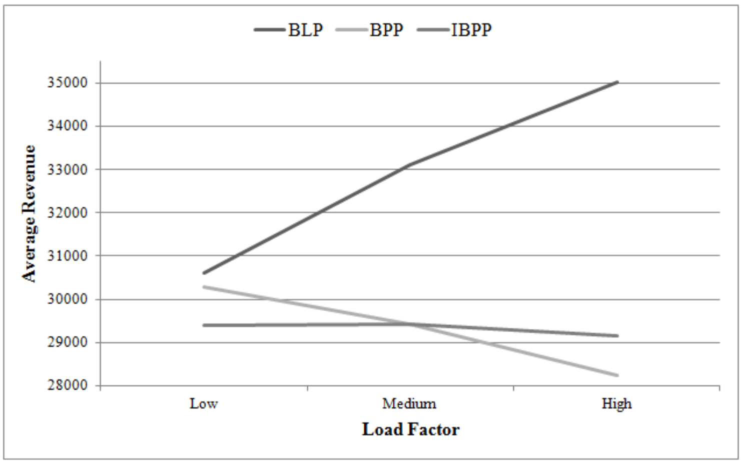

Figure 4.3: Trend of the average revenue as function of the load factor

The gain of $\mathcal{BLP}$ over $\mathcal{BPP}$ and $\mathcal{IBPP}$ generally increases with the load factor, as shown in Fig. 4.3, where the average revenue values as fuction of the load factor is given. The results reported in this figure underline that the higher the load factor the higher the revenue achieved by $\mathcal{BLP}$ and the lower that obtained by applying $\mathcal{BPP}$, whereas the behaviour of $\mathcal{IBBP}$ is relatively insensitive to the ratio between demand and capacity. In particular, for high load factor, an average 19.38% and 16.77% improvement over $\mathcal{BPP}$ and $\mathcal{IBPP}$, respectively, is observed.

In order to compare $\mathcal{BLP}$, $\mathcal{BPP}$ and $\mathcal{IBPP}$ with $\mathcal{PKRD}$ and $\mathcal{FCFSP}$, we evaluated the average percentage error (APE) and the average percentage gain (APG) defined as follows:

$$APE = \frac{R^{\mathcal{PKRD}} - R^i}{R^{\mathcal{PKRD}}} \times 100, i = \mathcal{BLP}, \mathcal{BPP}, \mathcal{IBPP}; \tag{4.14}$$

$$APG = \frac{R^i - R^{\mathcal{FCFSP}}}{R^{\mathcal{FCFSP}}} \times 100, i = \mathcal{BLP}, \mathcal{BPP}, \mathcal{IBPP}; \tag{4.15}$$

	$\mathcal{APG}$		
Test Problem	$\mathcal{BLP}$	$\mathcal{BPP}$	$\mathcal{IBPP}$
T_1^L	6.06%	12.53%	3.59%
T_1^M	34.29%	2.21%	4.10%
T_1^H	41.08%	0.08%	9.18%
T_2^L	9.60%	10.79%	7.84%
T_2^M	17.03%	4.23%	7.94%
T_2^H	26.84%	0.04%	4.61%
T_3^L	12.81%	7.71%	5.93%
T_3^M	14.03%	6.28%	2.90%
T_3^H	20.91%	2.97%	3.68%
Average	20.29%	5.21%	5.53%

Table 4.4: Average percentage gain values obtained by applyig the proposed policies.

The related computational results, reported in Tables 4.4 and 4.5, underline that the proposed policies outperform $\mathcal{FCFSP}$: the average percentage gain is equal to 20.29% , 5.21% and 5.53% for $\mathcal{BLP}$, $\mathcal{BPP}$ and $\mathcal{IBPP}$, respectively.

	$\mathcal{APE}$		
Test Problem	$\mathcal{BLP}$	$\mathcal{BPP}$	$\mathcal{IBPP}$
T_1^L	24.68%	20.08%	26.43%
T_1^M	7.89%	29.89%	28.59%
T_1^H	5.41%	32.90%	26.80%
T_2^L	15.24%	14.32%	16.60%
T_2^M	8.83%	18.80%	15.91%
T_2^H	5.07%	25.13%	21.71%
T_3^L	12.13%	16.11%	17.49%
T_3^M	13.14%	19.04%	21.62%
T_3^H	8.29%	21.90%	21.36%
Average	11.19%	22.02%	21.84%

Table 4.5: Average percentage error values obtained by applyig the proposed policies.

As for as the comparison with $\mathcal{PKRD}$ is concerned, the results of Table 4.5 highlight a good behaviour of the proposed policies. Indeed, an average percentage error of 22.02% is obtained in the worst case.

In order to assess the applicability of the proposed policies in real settings, it is useful to consider the results depicted in Table 4.6, in which the computational effort required by $\mathcal{BLP}$, $\mathcal{BPP}$ and $\mathcal{IBPP}$ as a function of the problems size and the load factor is given. From Table 4.6, it is evident that the execution time is very limited and increases with the number of groups and the load factor. In particular, the minimum average execution times is less than 1 second and it is achieved with problem classes of 5 groups and low load factor, whereas the maximum average execution time is of about 16 seconds and it was obtained for problem classes with 15 groups and high load factor.

Test Problem	$\mathcal{BLP}$	$\mathcal{BPP}$	$\mathcal{IBPP}$
T_1^L	0.13	0.17	0.08
T_1^M	1.41	1.29	1.54
T_1^H	5.90	6.89	5.35
Average	2.48	2.78	2.32
T_2^L	1.21	1.26	1.24
T_2^M	3.98	4.48	4.39
T_2^H	11.85	11.46	12.43
Average	5.68	5.73	6.02
T_3^L	1.92	1.91	1.86
T_3^M	4.77	5.14	5.01
T_3^H	15.54	11.88	13.13
Average	7.41	6.31	6.67

Table 4.6: Average computational time values (in sec), obtained with the $\mathcal{BLP}$, $\mathcal{BPP}$ and $\mathcal{IBPP}$

4.6.2 Numerical results for the OWRp

This section reports and discusses the computational results obtained by testing the policies defined for the OWR_p and the OWR_p^{TR} . It is important to point out that the mathematical models defined to represent OWR_p and OWR_p^{TR} are integer linear programming model, whose solution requires a great computational effort. In particular, the execution time increases with the number of stations. The minimum average computation time registered was about 50 seconds for an instance with 5 groups and 3 stations, whereas the maximum execution time was about 15000 seconds for an instance with 7 groups and 5 stations.

Consequently, in the computational experiments, we considered test problems of small size, characterized by an increasing number of groups K=5,7 and a number of stations equal to 3 and 5. A rental horizon of 7 periods was considered. Also in this case, three different demand scenarios, characterized by a low, medium and high load factor value, have been considered. The capacity of cars Q_o^k, available for each type of group, at each origin station, has been randomly generated into the interval $[1, 5]$ and we set $\gamma_{od} = 0$ and $\tau_{od} = 1$ for each pair of stations. The other parameters were chosen exactly as for the rental basic case.

The main characteristics of the considered test problems are given in Table 4.7, where for each instance the number of car groups, the number of stations and the load factor value are reported.

Test Problem	Groups	Stations	Load Factor
T^L_{5x3}	5	3	0.496
T^M_{5x3}	5	3	0.759
T^H_{5x3}	5	3	1.387
T^L_{7x3}	7	3	0.388
T^M_{7x3}	7	3	0.699
T^H_{7x3}	7	3	1.187
T^L_{5x5}	5	5	0.382
T^M_{5x5}	5	5	0.654
T^H_{5x5}	5	5	1.182
T^L_{7x5}	7	5	0.354
T^M_{7x5}	7	5	0.654
T^H_{7x5}	7	5	1.182

Table 4.7: Characteristics of the test problems

Similarly to what we did for the BRp, $\mathcal{BLP}$ and $\mathcal{BLP}^{\mathcal{TR}}$ were compared with $\mathcal{PKRD}$ and $\mathcal{FCFSP}$. The related results are reported in Table 4.8, where for each class of test problems the average revenue values are given.

Test Problem	$\mathcal{BLP}$	$\mathcal{BLP}^{\mathcal{TR}}$
T^L_{5x3}	23058.9	89927.8
T^M_{5x3}	24100.8	133048.9
T^H_{5x3}	23409.6	216569.9
T^L_{7x3}	27100.6	108160.8
T^M_{7x3}	28645.1	170258.0
T^H_{7x3}	28181.2	279235.2
T^L_{5x5}	24491.0	186404.8
T^M_{5x5}	27366.2	352404.3
T^H_{5x5}	27063.1	390405.0
T^L_{7x5}	47835.2	413808.0
T^M_{7x5}	50218.1	660745.2
T^H_{7x5}	55295.9	815973.2

Table 4.8: Average revenue values obtained by solving R^{OWR_p} and $R^{OWR_p^{TR}}$ and applying $\mathcal{BLP}$ and $\mathcal{BLP}^{\mathcal{TR}}$.

The results of Table 4.8 show that the booking limits policy incorporating car transferring behaves the best. Indeed, an average revenue of 32230.50 is obtained with $\mathcal{BLP}$, whereas the average revenue determined by applying $\mathcal{BLP}^{\mathcal{TR}}$ is equal to 318078.40. In addition, the benefit of using $\mathcal{BLP}^{\mathcal{TR}}$ is more evident when the load factors are high.

In order to compare $\mathcal{BLP}$ and $\mathcal{BLP}^{\mathcal{TR}}$ with $\mathcal{PKRD}$ and $\mathcal{FCFSP}$, we also determined the average percentage error $\mathcal{APE}$ and the average percentage gain $\mathcal{APG}$, on the basis of conditions (4.15) and (4.14). It is important to point that to evaluate $\mathcal{APE}$ the revenue achieved by applying $\mathcal{BLP}^{\mathcal{TR}}$ [$\mathcal{BLP}$] was compared with that determined by solving the $R^{OW_p^{TR}}$ [R^{OWR_p}] model and considering a perfect knowledge of the realized demand. The related computational results are reported in Table 4.9. They underline that the revenue obtained by applying $\mathcal{BLP}$ and $\mathcal{BLP}^{\mathcal{TR}}$ is quite close to that achieved in the case of perfect information on the demand. In addition, the proposed policies outperform the $\mathcal{FCFS}$. In particular, the average percentage gain is equal to 10.14% and 926.58% for $\mathcal{BLP}$ and $\mathcal{BLP}^{\mathcal{TR}}$, respectively. It is worth observing that the large value of $\mathcal{APG}$ in Table 4.9 are due the fact that in calculating the average percentage gain the comparison is made between the revenue determined by $\mathcal{BLP}^{\mathcal{TR}}$ that lets the possibility of car transferring among stations and the revenue obtained from $\mathcal{FCFS}$ that does not allow fleet reallocation.

	$\mathcal{BLP}$		$\mathcal{BLP}^{\mathcal{TR}}$	
Test Problem	$\mathcal{APE}$	$\mathcal{APG}$	$\mathcal{APE}$	$\mathcal{APG}$
T^L_{5x3}	18.23%	3.44%	47.34%	303.41%
T^M_{5x3}	14.54%	7.20%	45.86%	491.83%
T^H_{5x3}	16.99%	6.18%	49.52%	882.34%
T^L_{7x3}	19.50%	4.35%	47.13%	316.46%
T^M_{7x3}	14.91%	13.59%	49.86%	575.13%
T^H_{7x3}	16.29%	12.50%	50.08%	1014.67%
T^L_{5x5}	24.74%	6.30%	49.62%	709.05%
T^M_{5x5}	15.91%	9.70%	33.19%	1312.63%
T^H_{5x5}	16.84%	12.08%	39.49%	1516.87%
T^L_{7x5}	28.00%	3.80%	40.56%	797.90%
T^M_{7x5}	24.41%	13.14%	42.90%	1388.67%
T^H_{7x5}	16.77%	29.43%	47.61%	1810.00%
Average	18.93%	10.14%	45.26%	926.58%

Table 4.9: Average percentage gain and error values obtained by applying the $\mathcal{BLP}$ and the $\mathcal{BLP}^{\mathcal{TR}}$.

4.6.3 A real case study

In this section, we assess the validity of the proposed policies in a real setting. To this aim, computational experiments have been carried out on a realistic scenario, by taking as use case the Maggiore, a leading car rental company in Italy. The real data, provided by the Maggiore agency of Cosenza (Italy), have been collected from December 2006 to January 2007.

Within the considered real context, the booking requests are processed by following a first-come first-served policy. This strategy is quite similar to the policy followed by medium size car rental agencies. Thus the conclusions drawn in this study can be considered of general validity.

The company operates with 12 car groups, whose characteristics are given in Table

4.10, where for each car class, an example of a car type in that class, the rental rate per day and the number of car available are given.

Groups	Car Type	Capacity	Daily Rate (in Euros)
Mini	Smart Passion 61cv	2	86
Economy	Volkswagen Polo 1.2 Comfortline	2	86
Economy Comfort	Fiat Grande Punto 1.3 Multijet Dynamic	9	86
Compact	Alfa 147 1.9 JTD M-jet Progression	9	104
Intermediate	Volkswagen Passat 1.9 TDI Trendline	8	119
Wagon	Ford Focus 1.6 TDCI Style Wagon	7	128
Comfort Wagon	Volvo V50	4	142
Fully Size	Mercedes C220 CDI Elegance Berlina	5	153
MonoVolume	Fiat Ulisse 2.0 JTD Dynamic	6	153
Premium	Alfa 166 2.4 JTD M-Jet Classic	3	160
Automatic	Mercedes E280 CDI Elegance	2	181
Minivans	Fiat Ducato Panorama 2.8 JTD	2	181

Table 4.10: Characteristics of the fleet for the real case study.

The real data collected refer to rental requests in which the rental starts and ends at the same station. Thus, the basic rental formulation and the corresponding policies can be applied.

A rental horizon of 70 days and a booking horizon of length 7 were considered. The customers' rental requests have been processed in the order of their arrival and we adopted the best performing policy, that the $\mathcal{BLP}$ to take a decision. In order to adequately represent the real situation under consideration, a modified version of model BR_p, in which the objective function (3) is replaced by (7), was considered. To represent the dependence of the rental rate per day on the length of the rental period, the values of the non-negative scalar $\pi_{(j-i)}$, introduced in (7), were calculated for each length of rent performed during the two months of agency's activity; the corresponding values are reported in Table 4.11.

The computational results collected are very encouraging and highlight the superiority of the $\mathcal{BLP}$ over the policy actually adopted by the car rental agency: indeed a revenue improvement of the 51% has been observed.

4.7 Conclusions

This paper considers the application of the revenue management methodologies to the context of the car rental industry. In particular, we present innovative mathematical models and solution approaches to handle the car rental company's problem of accepting or rejecting a car rental request. Two different situations were considered: 1) the origin and the destination of the rental request are the same; 2) the one-way rentals case, in which a rental car is returned to a location different from its origin. For both the scenarios, our objective is to maximize the revenue of the car rental agency by satisfying the demand and the capacity constraints. Dynamic programming models were proposed to represent math-

Length of rent	$\pi_{(j-i)}$
1	0
2	0.910
3	1.180
4	1.670
5	1.824
6	1.910
7	2.180
8	3.014
9	3.413
10	3.899
12	4.873
13	5.360
14	5.851
15	6.167
17	7.119
20	8.546
24	10.183
28	11.263
30	11.263

Table 4.11: Values of $\pi_{(j-i)}$ used in the computazional experiments.

ematically the problems under study. In order to deal with "the curse of dimensionality", linear programming approximations were defined. We have also developed appropriate primal and dual acceptance policies that use booking limits and bid price controls. In order to evaluate their effectiveness, we compared these approaches with a typical first-come first-served policy and with the case of a perfect knowledge of the realized demand. We presented numerical results collected on small-medium size test instances and on a real case. Experiments reveal that the use of the proposed policies could help the logistic operator to control the capacity levels, to improve customer service and fleet utilization, by maximizing the revenue.

Appendix A

In what follows we show that the constraint matrix CM can be reduced to an interval matrix, which is a 0 - 1 matrix where the ones appear consecutively [119].

The typical structure of the CM in the case of a car rental agency with *3* groups of vehicles in a rental horizon of *4* periods is reported in Figure 4.4. In particular, the rows of CM are associated with the model constraints (capacity and demand constraints, for each category of cars and for each period), whereas the columns are associated with the decision variables. CM is a square matrix of order $\bar{K}\left[\left(\frac{T!}{2!(T-2)!}\right)+T\right] \times (2\bar{K}-1)\frac{T!}{2!(T-2)!}$. In what follows, we indicate the quantity $\frac{T!}{2!(T-2)!}$ as ω.

3 groups 4 renting periods				Decision variables x(v,i,j)																													
				1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
				(1,1,2)	(1,1,3)	(1,1,4)	(1,2,3)	(1,2,4)	(1,3,4)	(2,1,2)	(2,1,3)	(2,1,4)	(2,2,3)	(2,2,4)	(2,3,4)	(3,1,2)	(3,1,3)	(3,1,4)	(3,2,3)	(3,2,4)	(3,3,4)	(4,1,2)	(4,1,3)	(4,1,4)	(4,2,3)	(4,2,4)	(4,3,4)	(5,1,2)	(5,1,3)	(5,1,4)	(5,2,3)	(5,2,4)	(5,3,4)
Capacity constraints (k,i)	C^1	(1,1)	1 |1|1|1|																												
		(1,2)	2		1	1	1	1																									
		(1,3)	3	1		1		1	1																								
		(1,4)	4	1	1		1																										
	C^2	(2,1)	1							1	1	1				1	1	1															
		(2,2)	2								1	1	1	1			1	1	1	1													
		(2,3)	3							1		1		1	1	1		1		1	1												
		(2,4)	4							1	1		1			1	1		1														
	C^3	(3,1)	1																			1	1	1				1	1	1			
		(3,2)	2																				1	1	1	1			1	1	1	1	
		(3,3)	3																			1		1		1	1	1		1		1	1
		(3,4)	4																			1	1		1			1	1		1		
Demand constraint (k,i,j)	D^1	(1,1,2)	1	1						1																							
		(1,1,3)	2		1						1																						
		(1,1,4)	3			1						1																					
		(1,2,3)	4				1						1																				
		(1,2,4)	5					1						1																			
		(1,3,4)	6						1						1																		
	D^2	(2,1,2)	1													1						1											
		(2,1,3)	2														1						1										
		(2,1,4)	3															1						1									
		(2,2,3)	4																1						1								
		(2,2,4)	5																	1						1							
		(2,3,4)	6																		1						1						
	D^3	(3,1,2)	1																									1					
		(3,1,3)	2																										1				
		(3,1,4)	3																											1			
		(3,2,3)	4																												1		
		(3,2,4)	5																													1	
		(3,3,4)	6																														1

Figure 4.4: Graphical representation of CM in case of 3 groups and 4 periods.

In the general case of $\bar{K}$ vehicle groups and T rental periods, CM assumes the form reported in Figure 4.5, where three different types of submatrices are highlighthed: matrices of type B, whose structure is reported in Figure 4.6, identity matrices I and null matrices 0.

More specifically, for the first group there is only a matrix of type B (related to the capacity constraints) and two identity matrices I (corresponding to the demand constraints), for the last group we have two matrices of type B and an identity matrix I, while for the other groups there are two matrices of type B and two identity matrices I (see C^k and D^k, $k = 1, \ldots\ldots, \bar{K}$, in Figure 4.5).

In order to define the general procedure to reduce CM to an 0 - 1 matrix, it is important to consider the specific structure of B (Figure 4.6). Some examples for different value of the rental periods are reported in Figure 4.7.

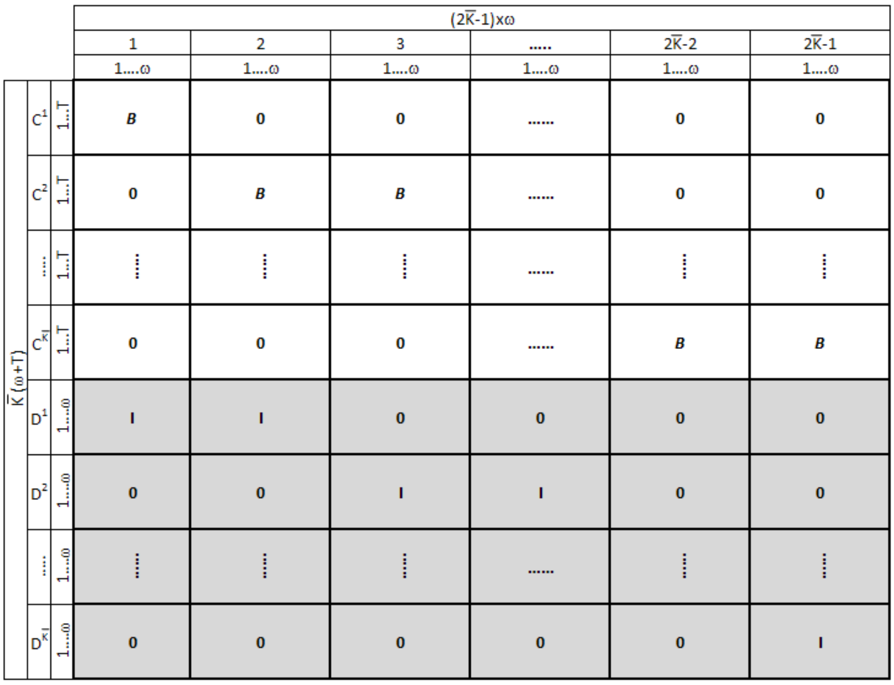

Figure 4.5: Graphical Representation of the original CM.

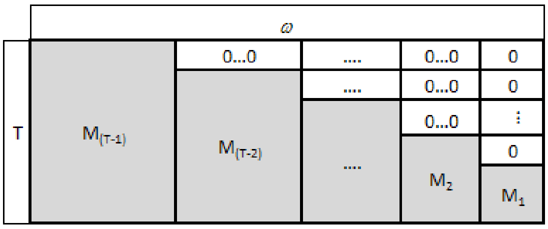

Figure 4.6: Graphical representation of a generic submatrix B.

As shown in Figure 4.8, B is a matrix of order $T\text{x}\omega$, characterized by $(T-1)$ submatrices of order $(n+1)\times n$, $n = T-1, T-2, \ldots, 1$, denoted as $M_r, r = T-1, \ldots, 1$ (see Figure 4.9).

The first row of $M_r, r = T-1, \ldots, 2$ is a unit vector of dimension r and the last n rows form a particular unit matrix U_r, $r = T-1, \ldots, 2$, of order $n \times n$, with the diagonal elments equal to 0 (see Figures 4.8 and 4.9). The submatrix M_1 reduces to the vector $[1, 0]^T$.

In what follows, we show that CM can be transformed into an interval matrix, by performing linear operations on its rows.

Denote the rows of the block C^k, $k = 1, \ldots, \bar{K}$ as $c_1^k, c_2^k, \ldots, c_T^k$ and the rows of each

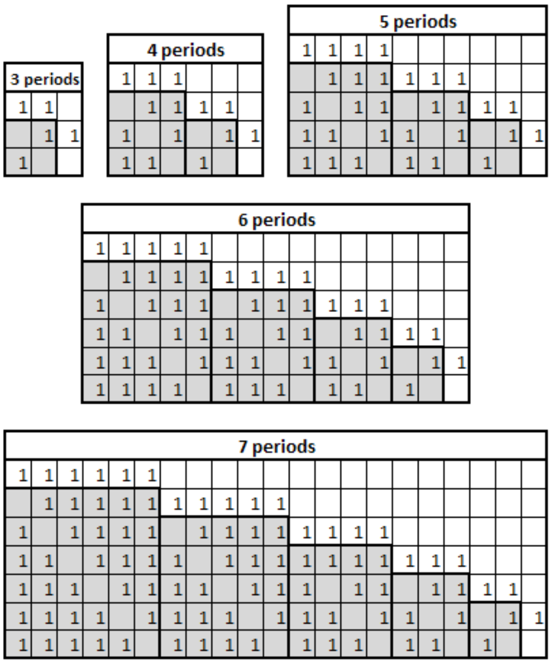

Figure 4.7: Examples of graphical representation of the block B for different T values.

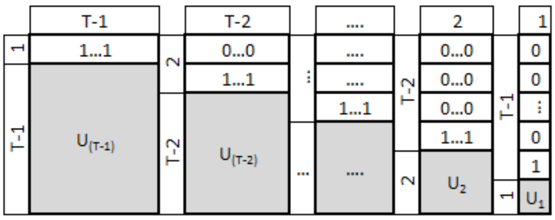

Figure 4.8: The dimensions of B.

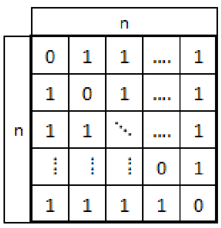

Figure 4.9: A graphical representation of matrix M_r.

block D^k, $k = 1, \dots, \bar{K}$, as $d_1^k, d_2^k, \dots, d_\omega^k$.

For each block $C^k, k = 1, \dots, \bar{K}$ execute the following operations.

- Add to each row c_l^k, $l = 1, \ldots, T$:
  - the row $c_l^{\tilde{k}}$ of the other blocks $C^{\tilde{k}}$, $\tilde{k} \neq k$ (see for example Figure 4.10);
  - the rows d_β^k, $\beta = 1, \ldots, \omega$, of each block D^k, $k = 1, \ldots, \bar{K}$, such as $\beta = \delta$, where δ, $\delta = 1, \ldots, \omega$, indicates the column of B, such that the element $b_{l,\delta}$ of B is equal to zero (see for example Figures 4.11, 4.12, 4.13, 4.14, 4.15).

C1	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0
	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1
	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0

Figure 4.10: Result of sum between the rows of C^1 and the rows of blocks C^2 and C^3.

		1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
C1	1	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0
	2	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
	3	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1
	4	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0
D^1	1	1						1																							
	2		1						1																						
	3			1						1																					
	4				1						1																				
	5					1						1																			
	6						1						1																		
D^2	1													1						1											
	2														1						1										
	3															1						1									
	4																1						1								
	5																	1						1							
	6																		1						1						
D^3	1																									1					
	2																										1				
	3																											1			
	4																												1		
	5																													1	
	6																														1

Figure 4.11: The first row of C^1 must be added to rows 4,5,6 of blocks D^1, D^2 and D^3.

For each block $D^k, k = 1, \ldots, \bar{K}$ execute the following operations.

- Add to row d_β^k, $\beta = 1, \ldots, \omega$:
  - the rows $d_h^k, h = \beta + 1, \ldots, \omega$ of the current block D^k;
  - the rows d_β^k, $\beta = 1, \ldots, \omega$ of the other blocks $D^{k+1}, \ldots, D^{\bar{K}}$ (see for example Figure 4.16). Obviously, this point is not applicable to rows of $D^{\bar{K}}$ (Figure 4.17).

The final matrix is an interval matrix and then it is totally unimodular (Figure 4.18).

		1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
C1	1	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0
	2	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
	3	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1
	4	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0

D1	1	1						1																							
	2		1						1																						
	3			1						1																					
	4				1						1																				
	5					1						1																			
	6						1						1																		
D2	1													1						1											
	2														1						1										
	3															1						1									
	4																1						1								
	5																	1						1							
	6																		1						1						
D3	1																									1					
	2																										1				
	3																											1			
	4																												1		
	5																													1	
	6																														1

Figure 4.12: The second row of C^1 must be added to rows 1, 6 of blocks D^1, D^2 and D^3.

		1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
C1	1	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0
	2	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
	3	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1
	4	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0

D1	1	1						1																							
	2		1						1																						
	3			1						1																					
	4				1						1																				
	5					1						1																			
	6						1						1																		
D2	1													1						1											
	2														1						1										
	3															1						1									
	4																1						1								
	5																	1						1							
	6																		1						1						
D3	1																									1					
	2																										1				
	3																											1			
	4																												1		
	5																													1	
	6																														1

Figure 4.13: The third row of C^1 must be added to rows 2, 4 of blocks D^1, D^2 and D^3.

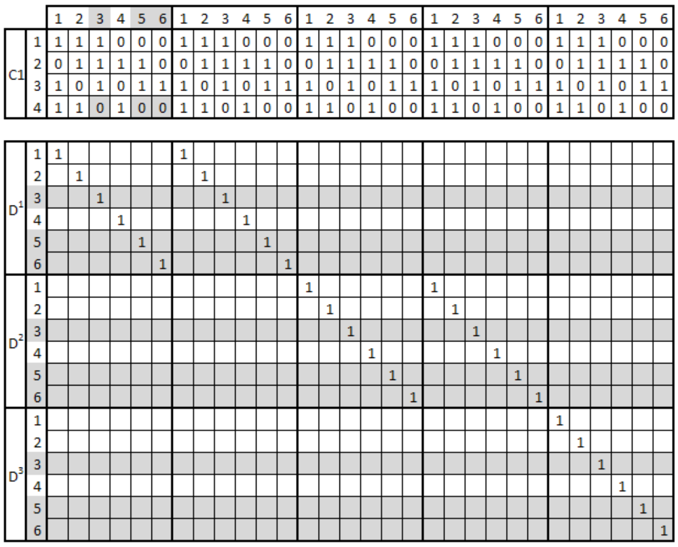

Figure 4.14: The fourth row of C^1 must be added to rows 3, 5, 6 of blocks D^1, D^2 and D^3.

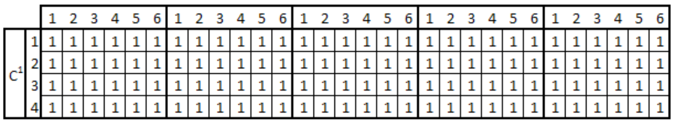

Figure 4.15: A graphical representation of C^1 after operations.

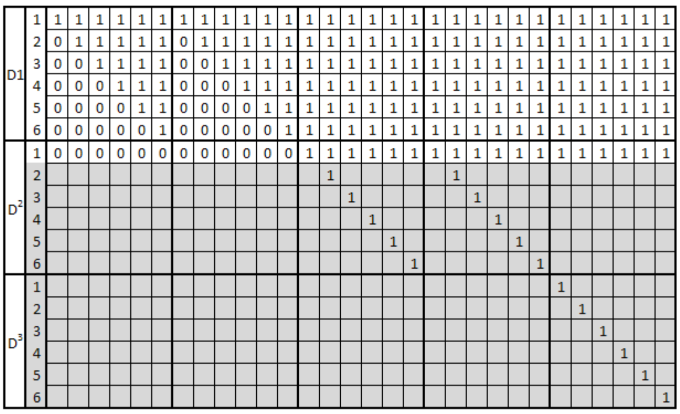

Figure 4.16: The second row of D^2 must be added to rows 3, 4, 5, 6 of blocks D^2 and rows 1,2,3,4,5,6 of block D^3.

D1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	0	0	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	5	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	6	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D^2	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
D^3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 4.17: The third row of D^3 must be added to rows 4, 5, 6 of block D^3.

C^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C^3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D^2	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
D^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 4.18: The resulting interval matrix for example in Figure 4.4

Chapter 5

Revenue Management Policies for the Truck Rental Industry [1]

Abstract

In this paper, we consider the problem of managing a fleet of trucks with different capacity to serve the requests of different customers that arise randomly over time. The problem is formulated via dynamic programming. Linear programming approximations of the problem are presented and their solutions are exploited to develop partitioned booking limits and bid prices policies. The numerical experiments show that the proposed policies can be profitably used in supporting the decision maker.

Keywords: Resource Allocation Problems, Dynamic Programming, Revenue Management, Booking Limits, Bid Prices

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5.1 Introduction

Revenue management methods are very effective to help companies in finding optimal policies to allocate their products in a given planning horizon. Indeed, revenue management capacity control is used by companies for maximizing revenue, by optimally allocating constrained and perishable capacity on differentiated products/services, that are targeted to heterogeneous customer segments and generally sold through advance booking in the face of uncertain levels of demand for service. One of the fundamental capacity control

[1] Accepted for pubblication in the journal *Transportation Research Part E: Logistics and Transportation Review, August 2011*, DOI: 10.1016/j.tre.2011.07.006

decision is either accept or reject an arriving booking request for a specific service and, in the latter case, preserve the availability for probably more valuable demand in subsequent periods.

Today, revenue management plays an important role for service firms in many different industries. While airlines have the longest history of development in revenue management, the techniques also apply to other industries with similar business characteristics, such as hotels, restaurants and car rental, freight transportation and passenger railways, telecommunications and financial services, internet service provision, electric utilities, broadcasting and even manufacturing companies ([30]).

McGill and van Ryzin in [113] give a comprehensive overview of the history of revenue management in transportation, where it has had the greatest impact.

In this paper, we apply revenue management methods and policies to a truck rental problem. We define, on the basis of the arrival process of the requests, the policy for either accept or reject a booking request of rent once it arrives. We address the question of how to coordinate the decisions on fleet management and to treat the randomness in the demand arrivals explicitly by decomposing the problem into time periods and assessing the impact of the current decisions on the future, through the managing of available capacity.

The problem addressed here is a dynamic resource allocation problem, that involves the assignment of a set of reusable resources (vehicles) to tasks (customer demands) that occur over time. The assignment of a resource to a task produces a reward, removes the task from the system, and modifies the state (typically, a geographical location) of the resource ([131]; [130]; [151]; [129], Chapter 12). We were confronted with this problem within the context of managing a fleet of trucks rented by a logistic operator to serve customers who request the freight transportation between different nodes in a network. It is a fleet management problem where a vehicle is assigned to a request from one location to another at a given time. The fleet is composed of different type of trucks. At each decision epoch, a certain number of customers arrive in, each requesting a transportation of a certain quantity of goods from a certain origin to a destination. Each customer demand can be satisfy with a truck with a capacity greater or equal to the request.

We give a dynamic formulation of the problem at hand. Dynamic models arise in a great variety of transportation applications as a result of the need to capture the evolution of activities over time. These models allow to find an answer to the following crucial question: *"Which truck should assign to a customer given the unknown but, probably, more profitable demand that will arrive in the system in the future?"* Due to "the curse of dimensionality", the dynamic programming model cannot be solved optimally. For this reason, in order to provide the decision maker with a tool useful in taking decisions, we develop a linear programming formulation of the problem and apply revenue management techniques to take the best decision.

The present work shares some similarities with that of Topaloglu and Powell ([151]). However, the following main differences can be found. First of all, in [151], it is assumed that the customers can ask for different types of vehicles, on the basis of their preferences.

In our work, instead, the logistic operator, by evaluating appropriately the convenience, can decide to assign a truck of greater capacity to a certain customer. Indeed, an "upgrade" can take place. In addition, to address the problem under consideration, we do not follow the approximate dynamic programming approach used by Topaloglu and Powell in [151]. In taking decisions we in fact adopt revenue management policies, based on booking limits and bid prices, that require to solve, dynamically, a linear programming model. A policy that allows the logistic operator to use the same truck to satisfy multiple demands is also devised. This possibility is not exploited in [151].

The rest of the paper is organized as follow. In section 5.2, the "Trucks Rental Problem" (TRP, for short) is introduced and its dynamic programming formulation is given. Section 5.3 contains the linear programming formulation for the TRP, together with the description of some revenue management policies, based on the solution of the linear problem. The theoretical issues of the proposed policies are also investigated. In the same section, a new policy that considers the "sharing", i.e. the possibility for the logistic operator of using a certain truck for serving multiple demand, is defined. New versions of the TRP, incorporating sharing and the repositioning of empty trucks, are also exploited in section 5.4. Numerical experiments are presented in section 5.5. Some concluding remarks are stated in section 5.6. The paper ends with an appendix containing some theoretical properties of the policies presented in section 5.3.

5.2 A dynamic programming formulation for the TRP

We consider the problem of a logistic operator that offers a transportation service from a given set of origins to a given set of destinations. The transportation service consists in renting trucks of different capacities to different customers on a given time horizon. Each customer is associated with a certain level of demand. At each time of the booking horizon, the transportation operator has to decide how to manage the overall capacity in the most profitable way, taking into account that complete information about the future demand are not available.

Let $O = \{1, \dots, o\}$ be a given set of origins and let $E = \{1, \dots, e\}$ denote a given set of destinations. It is assumed that $O \equiv E$, i.e. each node can be serve as origin and destination of the transportation request. The logistic operator transports goods from an origin i, $i \in O$ to a destination j, $j \in E$ by r types of trucks. A truck of type $k \in K = \{1, \dots, r\}$ is associated with a given value of capacity $a(k), k = 1, \dots r$.

Customers can be viewed as partitioned in r different classes. A customer is of class k if he/she requires the transportation of a quantity q_k of goods, such that $a(k-1) < q_k \leq a(k)$, $k = 1, \dots, r$ and $a(0) = 0$. The demand of a class k customer can be satisfied with trucks with capacity $a(k)$ or greater, i.e. an "upgrade" can take place. We also assume that customers requests cannot be partitioned among different trucks.

In each time period $t = 1, \dots, T$ of the booking horizon, the logistic operator has to decide on accepting the request of transferring a given quantity of goods from $i \in O$ to

$j \in E$ with departure time $\bar{t}, \bar{t} = 1, \dots \bar{T}$, with the goal of maximizing the total revenue. In the sequel we will refer to $1, \dots, \bar{T}$ as the "operation horizon" i.e. the horizon where the transportation service takes place. We assume that the booking horizon and the operation horizon do not overlap.

Let $A = [A^1|A^2|\dots|A^r]$, $A \in \mathcal{R}^{r\times u}$, $u = r + (r-1) + \dots + [r-(r-1)] = r^2 - \sum_{l=1}^{r-1} l = max_{prod}$ denote a binary matrix, partitioned in r sub-matrices. Each sub-matrix $A^k \in \mathcal{R}^{r\times[(r-k)+1]}$, $k = 1, \dots, r$ contains the set of possible products to satisfy the demand of a class k customer. In particular, the first column of sub-matrix A^k is the product constituted by the truck of minimum capacity $a(k)$ useful to satisfy the demand of class k, whereas the last column is the product constituted by the truck of maximum capacity $a(r)$ that can be used to satisfy the class k customer.

We indicate each column of matrix A as A_p $p = 1, \dots, max_{prod}$. Each element a_{kp}, $p = 1, \dots, max_{prod}$ of matrix A is equal to one if truck k is used in product p and 0 otherwise. It is worth noting that a product indicates the type of truck that the logistic operator can use to satisfy the demand of a certain class k; in fact, due to the upgrade, a class k request can be satisfied with trucks of capacity k or greater. In particular, given a class k request, the products $p = v_{kmin}, \dots, v_{kmax}$, with $v_{kmin} = (k-1)r - \sum_{s=1}^{k-1}[(k-1)-s]+1$ and $v_{kmax} = kr - \sum_{s=1}^{k-1}[k-s]$, can be used to meet the customer demand.

The state of the system is described by a matrix

$X = [X^1{}_1|X^1{}_2|\dots|X^1{}_{|E|}|\dots|X^{\bar{T}}{}_1|X^{\bar{T}}{}_2|\dots|X^{\bar{T}}{}_{|E|}]$,

each column $X_i^{\bar{t}} = (x_{i1}^{\bar{t}}, \dots, x_{ir}^{\bar{t}})^\top, \forall i \in O, \quad \bar{t} = 1, \dots \bar{T}$ representing the capacity of node i at time $\bar{t}$. In particular $x_{ik}^{\bar{t}}$ is the number of trucks of type k, $k = 1, \dots, r$ available at node $i \in O$ at time $\bar{t} = 1, \dots, \bar{T}$.

Time is discrete, there are T booking periods indexed by t, which runs forward; consequently, $t = 1$ is the first possible booking time.

In each time-period t, at most one request of transportation can arrive. Let $\lambda_{ij\bar{t}k}^t$ denote the probability that at time t one transportation request of class k from $i \in O$ to $j \in E$, with departure at time $\bar{t} = 1, \dots, \bar{T}$, is made. It holds that $\sum_{i\in E}\sum_{j\in E, j\neq i}\sum_{\bar{t}=1}^{\bar{T}}\sum_{k=1}^{r}\lambda_{ij\bar{t}k}^t + \lambda_0^t = 1$, where $\lambda_0^t = 1$, represents the probability that no request arrives at time t.

We, further, assume that the travel times are random and we indicate with μ_{ij}^τ the probability that the average travel time from node i, $i \in O$ to node j, $j \in E$ will be $\tau, \tau = 1, \dots, \bar{\tau}$ time units.

Let us introduce boolean variables $u_{ij\bar{t}p}^t$, with $u_{ij\bar{t}p}^t = 1$ if the transportation request from node i to node j with departure time at $\bar{t}$, is accepted, at time t, by using product p and $u_{ij\bar{t}p}^t = 0$ otherwise.

Let R^k_{ij} be the revenue obtained by satisfying a request from i to j with a truck of type k, $k = 1, \dots, r$.

The problem can be formulated as a dynamic program by letting $V_t(X)$ be the maximum expected revenue obtainable from periods $t, t+1, \dots, T$ given that, at time t, the capacity of the system is X.

The Bellman equation for $V_t(X)$ is reported in what follows:

$$V_t(X) = \sum_{k=1}^{r} \sum_{i \in E} \sum_{j \in E j \neq i} \lambda^t_{ij\bar{t}k} \max_{\substack{u^t_{ij\bar{t}p} \in \{0,1\} \\ p \in \{v_{kmin}, \dots v_{kmax}\}}} [R^k_{ij} u^t_{ij\bar{t}p} + V_{t+1}(\tilde{X})] + \lambda^t_{ij0k} V_{t+1}(X) \tag{5.1}$$

where

- $\tilde{X}^{\tilde{t}}_i = (X^{\tilde{t}}_i - A_p u^t_{ij\bar{t}p}), \forall \tilde{t} = \bar{t}, \dots, \bar{T}$. This term updates capacity on node i when a certain request from node i is accepted.
- $\tilde{X}^{\bar{\tau}}_j = (X^{\bar{\tau}}_j + A_p u^t_{ij\bar{t}p}), \forall \bar{\tau} = (\bar{t} + \sum_{\tau=1}^{\bar{\tau}} \tau \mu^\tau_{ij}), \dots, \bar{T}$. This term updates capacity on node j when a certain request to node j is accepted.
- $\tilde{X}^{\bar{t}}_l = X^{\bar{t}}_l, \forall \bar{t},\ l \neq i, j$. This term updates capacity on the rest of the system when a request from i to j is accepted.

It is worth noting that the update of X, at time t happens when a customer of class k requires the transportation service from i to j with departure time $\bar{t}$. In this case we need to change the state of the system by considering that, if the customer request is accepted by using product p, we need to update the capacity on node i considering that the truck used in product p will be not anymore in node i from time $\bar{t}$ (departure time) until the end of the operation horizon (unless the truck will return on node i in the future). Moreover we need to adjust the capacity on node j by considering that the truck moved from i to j will be on node j starting from a time equal to the departure time from node i plus the average travel time $\sum_{\tau=1}^{\bar{\tau}} \tau \mu^\tau_{ij}$ and until the end of the operation horizon. On the rest of the system the capacity will not be varied until the end of the operation horizon.

The boundary conditions of the Bellman equation are the following:

$$V_t(0) = 0, \forall t;$$

$$V_t(X) = -\infty \text{ if } x^{\bar{t}}_{ir} < 0 \text{ for some } i, \bar{t}, r; \forall t$$

$$V_{T+1}(X) = 0, \text{ if } x^{\bar{t}}_{ir} \geq 0\ \forall\ i, \bar{t}, r;$$

$$V_{T+1}(X) = -\infty \text{ if } x^{\bar{t}}_{ir} < 0 \text{ for some } i, \bar{t}, r.$$

The proposed dynamic programming (DP) model is unlikely to be solved optimally due to the curse of dimensionality. For this reason, in the next section, we propose a linear programming approximation of the DP which is an extension of well-known approximations for the DP of traditional network capacity management. In particular, we are interested in approximations by deterministic linear programming (DLP) ([37] and [161]). Solving the Bellman equation, by approximating the function $V_t(X)$, falls in the general class of approximate dynamic programming (ADP) methods ([17]), in which an approximate value to the exact value function is used in the Bellman equation. The main difference among various ADP methods comes from the specific approximating mathematical programming problem that is built and solved to calculate the value function. It is evident that the type of approximation used influences the complexity of the function evaluation. Our DLP approximation is a simpler alternative to other approximations (like those presented in [151] for the dynamic resource allocation problem) and offer the possibility to construct revenue management policies, based on easy to solve deterministic optimization problems, that perform well in comparison to optimal policies.

5.3 A linear programming formulation for the TRP

Starting from the dynamic programming problem, in the linear programming approximation, we replace stochastic quantities by their mean values and assume that capacity and demand are continuous.

Let be:

- d the random cumulative future demand at time t, and $\bar{d}$ its mean. In particular $d_{ij}^{\bar{t}k}$ is the aggregate number of transportation requests from i to j with departure time $\bar{t}$ belonging to class k.
- $y_{ij}^{\bar{t}p}$ the number of products of type $p = 1, \ldots, max_{prod}$ to be used to satisfy the transportation request from $i,\ i \in O$ to $j,\ j \in E$ with departure time $\bar{t}$.
- R_{ij}^k the revenue obtained by satisfying a transportation request from node i to node j with a truck of type k.
- $x_{ik}^{\bar{t}}$ the number of trucks of capacity $a(k)$ available on node i at time $\bar{t}$.
- $\bar{\tau}_{ij}$ the average travel time from node $i \in O$, to node $j \in E,\ \ i \neq j$.

In the sequel the assumption that each node is both an origin and a destination node, with $|O| = |E| = N$, will always hold.

The total revenue achievable by the logistic operator at time t, when the network capacity is x, can be calculated by solving the the following optimization problem:

$$R^{TRP}(x,t) = \max \sum_{\bar{t}=1}^{\bar{T}} \sum_{i \in O} \sum_{j \in E j \neq i} \sum_{k=1}^{r} \sum_{p=v_{kmin}}^{v_{kmax}} R_{ij}^{k} y_{ij}^{\bar{t}p} \tag{5.2}$$

$$\sum_{p=v_{kmin}}^{v_{kmax}} y_{ij}^{\bar{t}p} \leq \bar{d}_{ij}^{\bar{t}k} \quad \forall k, i \neq j,\ \bar{t} \tag{5.3}$$

$$\sum_{\tilde{t}=1}^{\bar{t}-1} \sum_{j \in E j \neq i} \sum_{p=1}^{max_{prod}} a_{kp} y_{ij}^{\tilde{t}p} \leq x_{ik}^{\bar{t}} + \sum_{j \in E j \neq i} \sum_{p=1}^{max_{prod}} a_{kp} \sum_{\tilde{t}=1}^{\bar{t}-\bar{\tau}_{ij}} y_{ji}^{\tilde{t}p} \quad \forall k, i,\ \bar{t} \tag{5.4}$$

$$y_{ij}^{\bar{t}p} \geq 0 \quad \forall\ \bar{t}, p, i, j \tag{5.5}$$

Constraints (5.3) state that the demand of class k can be satisfied with a truck of capacity $a(k)$ or greater. Constraints (5.4) control the availability of a truck of capacity $a(k)$ on node i at time t.

In the programming formulation for the TRP reported above, we have not imposed the satisfaction of integer constraints on the decision variables. Indeed, given the specific structure of the constraint matrix (CM, for short) associated with TRP, its linear relaxation yields an integral solution, thus these constraints can be relaxed. In particular, it is possible to demonstrate that CM can be reduced to an interval matrix (i.e., a $0-1$ matrix where the ones appear consecutively), that is totally unimodular ([119]). For the proof the reader is referred to the technical report version of this paper ([51]).

In a revenue management setting, based on the solution of a linear programming formulation of the problem, we define booking limits or bid price controls to accept or reject a request at a certain time.

In fact, it is well known ([144]) that by solving the DLP model we can use either the primal variables to construct a partitioned booking-limit control directly or the dual variables to define a bid-price control. In the partitioned booking-limit control, a fixed amount of capacity of each resource is allocated to every product offered. The demand for each product has access only to its allocated capacity and no other product may use this capacity. In contrast, a bid-price control policy sets a threshold price or bid price for each resource in the network ([143]). Roughly this bid price is an estimate of the marginal cost of consuming the next incremental unit of the resources capacity.

When a booking request for a product arrives, the revenue of the request is compared to the sum of the bid prices of all the resources required by the product. If the revenue exceeds the sum of the bid prices, the request is accepted provided that all the resources associated with the requested product are still available; if not, the request is rejected.

5.3.1 Revenue-based primal and dual acceptance policies

In the context of the DLP, optimal solutions $y_{ij}^{\star\bar{t}p}$ give partitioned booking limits while bid prices are formed from optimal dual variables $\pi_{ki}^{\bar{t}}, \bar{t} = 1, \ldots, \bar{T}$, $k = 1, \ldots, r$, $i \in O$ of constraints (5.4). The partitioned booking-limit policy and the bid-price policy based on the DLP can be formally defined as follows.

At a certain point of the booking horizon, decisions about either accepting or denying transportation requests are to be made.

The model is driven by the following event: a request of class $\bar{k}$ from origin node i to destination node j with departure time $\bar{t}$ arrives at time t to the transportation operator.

When this event happens, $R^{TRP}(x,t)$ is solved and its solution used to make a decision.

From a primal viewpoint, the strategy to be adopted is a partitioned booking limits policy ($\mathcal{BLP}$, for short), that assumes the following form.

$\mathcal{BLP}$ Scheme

Solve $R^{TRP}(x,t)$. Let $y_{ij}^{\star\bar{t}p}$ denote its optimal solution.

IF $y_{ij}^{\star\bar{t}p} > 0$ for some $p = v_{\bar{k}min}, \ldots, v_{\bar{k}max}$ THEN

ACCEPT the request with upgrade if $p > v_{\bar{k}min}$;

SET $y_{ij}^{\bar{t}p} = y_{ij}^{\bar{t}p} - 1$;

SET $\tilde{k} = p - v_{\bar{k}min} + \bar{k}$ i.e. determine the truck used in product p;

UPDATE appropriately the capacity:

$x_{i\tilde{k}}^{\tilde{t}} = x_{i\tilde{k}}^{\tilde{t}} - 1, \forall \tilde{t} = \bar{t}, \ldots \bar{T}$ and $x_{j\tilde{k}}^{\tilde{t}} = x_{j\tilde{k}}^{\tilde{t}} + 1 \quad \forall \tilde{t} = \bar{t} + \bar{\tau}_{ij}, \ldots, \bar{T}$; if $x_{i\tilde{k}}^{\hat{t}} > x_{i\tilde{k}}^{\bar{t}}$ then $x_{i\tilde{k}}^{\hat{t}} = x_{i\tilde{k}}^{\hat{t}} - 1 \; \forall \hat{t} = 1, \ldots \bar{t} - 1$.

CALCULATE the revenue obtained from accepting the request;

ELSE

DENY the request.

END IF

From a dual viewpoint, it is necessary to solve the problem $R^{TRP}(x,t)$ and to deal with the dual variables associated to the capacity constraints. We will indicate with $\mathcal{BPP}$ the bid price policy associated with the linear programming formulation.

Let us indicate with $\pi_{ki}^{\bar{t}}, \bar{t} = 1, \ldots, \bar{T}$, $k = 1, \ldots, r$, $i \in O$ the dual variables associated with constraints (5.4).

A possible strategy to either accept or deny a request of class $\bar{k}$, from origin node i to destination node j with departure time $\bar{t}$ that arrives at time t to the transportation operator, can be described as follows:

$\mathcal{BPP}$ **Scheme**

Solve the $R^{TRP}(x,t)$ to obtain the dual variables $\pi^{\bar{t}}_{ki}, \bar{t} = 1, \ldots, \bar{T}$, $k = 1, \ldots, r$, $i \in O$.

IF there exists a $\tilde{k} \geq \bar{k}$ such that $R^{\bar{k}}_{ij} \geq \pi^{\bar{t}}_{\tilde{k}i} - \pi^{\bar{t}+\tau_{ij}}_{\tilde{k}j}$ AND $x^{\tilde{t}}_{i\tilde{k}} > 0 \ \forall \tilde{t} = \bar{t}, \ldots, \bar{T}$ THEN

ACCEPT the request with upgrade if $\tilde{k} > \bar{k}$;
UPDATE appropriately the capacity;
CALCULATE the revenue obtained from accepting the request;

ELSE

DENY the request.

END IF

It is worth noting that the procedure used to update the capacity is the same as that in the previous case.

5.3.2 A partitioned booking limits policy with sharing

If more than one request, from the same origin i toward the same destination j with the same departure time $\bar{t}$, arrives to the transportation operator, multiple demands can be loaded on the same truck. Indeed, a policy, incorporating the truck sharing ($\mathcal{BLPS}$, for short) can be defined.

The model is always driven by the same event: a request of class $\bar{k}$ from origin node i to destination node j with departure time $\bar{t}$ arrives at time t to the transportation operator.

Let $Q^{\bar{t}}_{ij} = \{q^{1\bar{t}}_{ij}, \ldots, q^{\xi\bar{t}}_{ij}\}$ denote the set of the goods quantities that should be transported from the same origin i to the same destination j at the same departure time $\bar{t}$, belonging to the class $\bar{k}$ request.

The main operations executed by $\mathcal{BLPS}$ can be represented as follows.

$\mathcal{BLPS}$ **Scheme**

SOLVE $R^{TRP}(x,t)$. Let $y^{\star\bar{t}p}_{ij}$ denote its optimal solution.

IF $y^{\star\bar{t}p}_{ij} > 0$ for some $p = v_{\bar{k}min}, \ldots, v_{\bar{k}max}$ THEN

ACCEPT the class $\bar{k}$ request with upgrade if $p > v_{\bar{k}min}$;
SET $\tilde{k} = p - v_{\bar{k}min} + \bar{k}$ i.e. determine the truck $\tilde{k}$ used in product p;
SET $res_{cap} = a(\tilde{k}) - a(\bar{k})$;
REPEAT
SELECT a request $q^{\bar{\xi}\bar{t}}_{ij}$ from $Q^{\bar{t}}_{ij}$, such that $q^{\bar{\xi}\bar{t}}_{ij} \leq res_{cap}$;

DELETE $q_{ij}^{\bar{\xi}\bar{t}}$ from $Q_{ij}^{\bar{t}}$;
ACCEPT the new request;
SET $res_{cap} = res_{cap} - q_{ij}^{\bar{\xi}\bar{t}}$;
UNTIL $\{Q_{ij}^{\bar{t}} = \oslash$ or $res_{cap} <= 0\}$
SET $y_{ij}^{\bar{t}p} = y_{ij}^{\bar{t}p} - 1$;
UPDATE appropriately the capacity;
CALCULATE the revenue obtained from accepting all the selected requests;
ELSE
DENY the requests.
END IF

Following the same idea it is also possible to define a bid price policy incorporating sharing ($\mathcal{BPPS}$). We will consider this policy while making numerical experiments in section 5.5.

5.4 The TRP with Sharing (TRPS) and Repositioning (TRPR)

In section 5.3.2 we considered the possibility for the logistic operator of using the same truck for satisfying requests coming from customers that share origins, destinations and departure time. In this section we present a new formulation of the TRP that incorporate sharing in its decisions by making explicitly the total revenue dependent on the quantities transported between each couple of nodes.

Assume that the aggregate number of transportation requests $d_{ij}^{\bar{t}k}$ from i to j with departure time $\bar{t}$ belonging to class k, are available. Remembering that a customer is of class k if he/she requires the transportation of a quantity q_k of goods, such that $a(k-1) < q_k \leq a(k)$, $k = 1, \ldots, r$ and $a(0) = 0$, it is easy to associate to each request the quantity to be transported.

Let be:

- $M = \{1, \ldots, m\}$ the set of the trucks owned by the logistic operator;
- $b(m)$ the capacity of each truck in M;
- $c_{ij}^{\bar{t}} = \sum_{k=1}^{r} d_{ij}^{\bar{t}k}$ the aggregate number of transportation request from i to j with departure time $\bar{t}$ from period t to the end of the planning horizon;
- $Q_{ij}^{\bar{t}\delta}$ the quantity associated to the $\delta - th$ request from i to j with departure time $\bar{t}$;
- R_{ij} the revenue obtained by transferring a single unit of good from node i to node j;

- $x_{ij}^{\bar{t}\delta m} = 1$ if truck m is used to satisfy the $\delta - th$ request from i to j with departure time $\bar{t}$ and 0 otherwise;
- $y_{ij}^{\bar{t}m} = 1$ if $x_{ij}^{\bar{t}\delta m} = 1$ AND 0 otherwise. The introduction of variables $y_{ij}^{\bar{t}m}$ is necessary to control the availability of truck m on node i at time $\bar{t}$;
- $Q_i^m = 1$ if truck m is on node i at the beginning of the booking horizon and AND 0 otherwise;
- $\bar{\tau}_{ij}$ the average travel time from node $i \in O$, to node $j \in E, \;\; i \neq j$.

The revenue obtainable by the logistic operator at time t, when the network capacity is Q, can be calculated by solving the following problem:

$$R^{TRPS}(Q,t) = \max \sum_{\bar{t}=1}^{\bar{T}} \sum_{i \in O} \sum_{j \in E j \neq i} \sum_{\delta=1}^{c_{ij}^{\bar{t}}} \sum_{m=1}^{M} R_{ij} x_{ij}^{\bar{t}\delta m} Q_{ij}^{\bar{t}\delta} \tag{5.6}$$

$$x_{ij}^{\bar{t}\delta m} \leq Q_{ij}^{\bar{t}\delta} \quad \forall \; i \neq j, \; \bar{t}, \; m, \; \delta = 1, \ldots, c^{\bar{t}_{ij}} \tag{5.7}$$

$$\sum_{\delta=1}^{c_{ij}^{\bar{t}}} x_{ij}^{\bar{t}\delta m} Q_{ij}^{\bar{t}\delta} \leq b(m) \quad \forall \; m, \; i \neq j \; \bar{t} \tag{5.8}$$

$$\sum_{m=1}^{M} x_{ij}^{\bar{t}\delta m} \leq 1 \quad \forall \; i \neq j \; \bar{t}, \; \delta = 1, \ldots, c_{ij}^{\bar{t}} \tag{5.9}$$

$$y_{ij}^{\bar{t}m} \leq \sum_{\delta=1}^{c^{\bar{t}_{ij}}} x_{ij}^{\bar{t}\delta m} \quad \forall \; m, \; i \neq j \; \bar{t} \tag{5.10}$$

$$\sum_{\delta=1}^{c^{\bar{t}_{ij}}} x_{ij}^{\bar{t}\delta m} \leq G y_{ij}^{\bar{t}m} \quad \forall m \;, i \neq j \; \bar{t} \tag{5.11}$$

$$\sum_{j \in E, j \neq i} y_{ij}^{\bar{t}m} - \sum_{j \in E, j \neq i} \sum_{\tau=1}^{\bar{t}-\bar{\tau}_{ij}} y_{ji}^{\tau m} + \sum_{j \in E, j \neq i} \sum_{\tau=1}^{\bar{t}-1} y_{ij}^{\tau m} \leq Q_i^m \quad \forall \; m, \; i \neq j \; \bar{t} \tag{5.12}$$

$$y_{ij}^{\bar{t}m}, x_{ij}^{\bar{t}\delta m} \text{binary} \quad \forall\, \bar{t},\ m,\ \delta,\ i,\ j \tag{5.13}$$

The objective function (5.6) is the revenue obtainable at time t when the residual capacity on the network is Q. Notice that the total revenue increase while increasing the total quantity transferred on the network. Constraints (5.7) let the movement of a truck be possible only if the $\delta - th$ request refers to a quantity not equal to zero. Constraints (5.8) state that the total quantity transferred by truck m cannot exceed its capacity. Constraints (5.9)state that the $\delta - th$ request cannot be partitioned on different trucks. Constraints (5.10) and (5.11) (where $G >> 0$ is a large parameter) are introduced to impose the logical link between variables $x_{ij}^{\bar{t}\delta m}$ and $y_{ij}^{\bar{t}m}$. Constraints (5.12) controls the presence of each truck on each node of the network.

A further development of the model takes into account the repositioning of an empty truck on a node $\bar{i}$, when the benefit of such operation is greater than the cost of moving the truck empty.

To this aim, we need to introduce the following quantities:

- $z_{ij}^{\bar{t}m} = 1$ if truck m is moved empty from i to j at time $\bar{t}$ AND 0 otherwise;
- γ_{ij} the cost of moving an empty truck from i to j.

The new objective function takes the following form:

$$R^{TRPR}(Q,t) = \max \left[\sum_{\bar{t}=1}^{\bar{T}} \sum_{i \in O} \sum_{j \in E j \neq i} \sum_{\delta=1}^{c_{ij}^{\bar{t}}} \sum_{m=1}^{M} R_{ij} x_{ij}^{\bar{t}\delta m} Q_{ij}^{\bar{t}\delta} - \sum_{\bar{t}=1}^{\bar{T}} \sum_{i \in O} \sum_{j \in E j \neq i} \sum_{m=1}^{M} \gamma_{ij} z_{ij}^{\bar{t}m} \right] \tag{5.14}$$

Constraints (5.7), (5.9), (5.8),(5.10) and (5.11) are still valid, whereas constraints (5.12) need to be restated as follows:

$$\begin{aligned} &\textstyle\sum_{j \in E, j \neq i} y_{ij}^{\bar{t}m} - \sum_{j \in E, j \neq i} \sum_{\tau=1}^{\bar{t}-\bar{\tau}_{ij}} y_{ji}^{\tau m} + \sum_{j \in E, j \neq i} \sum_{\tau=1}^{\bar{t}-1} y_{ij}^{\tau m} + \\ &\textstyle\sum_{j \in E, j \neq i} z_{ij}^{\bar{t}m} - \sum_{j \in E, j \neq i} \sum_{\tau=1}^{\bar{t}-\bar{\tau}_{ij}} z_{ji}^{\tau m} + \sum_{j \in E, j \neq i} \sum_{\tau=1}^{\bar{t}-1} z_{ij}^{\tau m} \leq Q_i^m \quad \forall\, m,\ i \neq j\ \bar{t} \end{aligned} \tag{5.15}$$

A possible partitioned booking limits policy $\mathcal{BLPSM}$ based on $R^{TRPS}(Q,t)$ is reported below.

The model is again driven by the same event: a request of class $\bar{k}$ from origin node i to destination node j with departure time $\bar{t}$ arrives at time t to the transportation operator.

$\mathcal{BLPSM}$ **Scheme**

SOLVE $R^{TRPS}(Q,t)$. Let $x_{ij}^{*\bar{t}\delta m}$ denote its optimal solution.

IF $x_{ij}^{*\bar{t}\bar{\delta}\tilde{m}} = 1$ for some $\bar{\delta} \in c_{ij}^{\bar{t}}$ and for some $\tilde{m} \in M$ THEN

ACCEPT the class $\bar{k}$ request with upgrade if $b(\tilde{m}) > a(\bar{k})$;

SET $res_{cap} = a(\tilde{m})$-$a(\bar{k})$;

REPEAT

SELECT a request $\tilde{\delta}$ from $c_{ij}^{\bar{t}}$, such that $Q_{ij}^{\bar{t}\tilde{\delta}} \leq res_{cap}$;

DELETE $\tilde{\delta}$ from $c_{ij}^{\bar{t}}$;

ACCEPT the new request;

SET $res_{cap} = res_{cap} - Q_{ij}^{\bar{t}\tilde{\delta}}$;

UNTIL $\{c_{ij}^{\bar{t}} = \oslash$ or $res_{cap} <= 0\}$

SET set $x_{ij}^{*\bar{t}\bar{\delta}\tilde{m}} = 0$;

UPDATE appropriately capacity:$y_{ij}^{\bar{t}\tilde{m}} = 0 \; \forall \bar{t} = 1, \ldots, \bar{T}$ AND $y_{ij}^{\bar{t}\tilde{m}} = 1 \; \forall \tilde{t} = \bar{t} + \bar{\tau}_{ij}, \ldots, \bar{T}$;

CALCULATE the revenue obtained from accepting all the selected requests;

ELSE

DENY the requests.

END IF

Analogously, taking into account the variables z in updating capacity, it is possible to define a partitioned booking limits policy $\mathcal{BLPR}$ based on $R^{TRPR}(Q,t)$.

5.5 Computational experiments

In this section, we present the numerical results obtained by testing the policies described in sections 5.3.1, 5.3.2 and 5.4. All the numerical experiments have been carried out in AIMMS 3.7, with Cplex 10.1 as solver, on a Pentium Intel Core 2 T200 2.0 GHz PC.

To the best of our knowledge, the problem under consideration has not previously taken into account in the revenue management literature. Thus, benchmark instances are not available and the computational experiments have been carried out on a set of test problems randomly generated.

The test problems we considered, reported in Table 5.1, are characterized by an increasing number of origin and destination nodes, a different number of trucks type and different lengths of the *operation horizon* $\bar{T}$. Moreover, for each node $i \in O$ and for each type of truck k, the number of trucks x_{ik} available at node i is randomly chosen from the ranges $[a,\ b]$, given in Table 5.1.

In all the instances, we assume that each node can be both origin and destination for the transportation request, and that the booking horizon consists of two periods i.e. $T = 2$.

Test Problem	$\lvert O\rvert = \lvert E\rvert = N$	**Type of Trucks**	$\bar{T}$	**[a, b]**
T1	5	3	5	[1,5]
T2	5	5	5	[1,5]
T3	5	3	10	[1,5]
T4	5	5	10	[1,5]
T5	10	3	5	[1,3]
T6	10	5	5	[1,3]
T7	10	3	10	[1,3]
T8	10	5	10	[1,3]
T9	15	3	5	[1,2]
T10	15	5	5	[1,2]
T11	15	3	10	[1,2]
T12	15	5	10	[1,2]

Table 5.1: Characteristics of the test problems

For each truck k and for each origin-destination pair (i, j), $i \in O$ and $j \in E$, the value of the revenue R_{ij}^k was randomly generated into the interval [50, 330] and considering the revenue increasing with both the trucks capacity $a(k)$ and the distance between the nodes i and j. The average travel time from node i to node j was randomly generated into the interval $[1,\ \bar{T}]$.

For each test problem, the booking process was simulated 1000 times. In each simulation run, the transportation requests are randomly generated by applying a two phases procedure. In the first phase, for each origin-destination pair, and each departure time, the number of transportation requests for each class is randomly generated according to a normal distribution, with a given expected demand and a given coefficient of variation, chosen randomly from the interval [1, 10] and [0, 1], respectively. In the second phase, for each request, booking arrival times are randomly generated according to an uniform distribution.

The requests generated by the procedure outlined above are then processed.

In particular, at each time instant t in the booking horizon, a transportation request, for which the booking arrival time is less than or equal to the considered booking instant, is chosen and the accept/deny decision is made based on one of the proposed policies. The trucks' availability is then updated and another booking request is processed. We move to the next booking time period when there are no more requests, arrived before t, that need to be evaluated. It is worth observing that the value of the revenue is affected by the order in which the booking requests are processed. In our experiments, we solve the models, used to define the policies, a number of times equal to the length of the booking horizon.

In each of the 1000 simulation runs, all the requests for each test problem, are processed considering the policies $\mathcal{BLP}$, $\mathcal{BPP}$, $\mathcal{BLPS}$, $\mathcal{BPPS}$, $\mathcal{BLPSM}$ and $\mathcal{BLPR}$ and the 95% confidence intervals of average revenue are introduced.

Policies $\mathcal{BLPSM}$ and $\mathcal{BLPR}$ are based on the solution of the binary models $R^{TRPS}(Q,t)$ and $R^{TRPR}(Q,t)$ and even the MIP CPLEX solver was unable to find the optimal solution in a reasonable amount of time. Therefore a time-limit of 2-h CPU (referred as TO in Figure 5.2) was imposed and the obtained feasible solution used to implement the related policies. Moreover, the solver was able to find a feasible solution, within the imposed time limit, only for the 5 nodes instances T_1, T_2, T_3 and T_4.

Tables 5.2 contains the value of the average revenue values and its 95% confidence interval obtained applying each policies to our test problems. The last row ATP is the average revenue of each policy, on the 12 test problems.

The results of Table 5.2 show that, the partitioned booking limits policy incorporating the truck sharing outperforms the bid price counterpart, whereas the average revenue determined by applying $\mathcal{BPP}$ is on average 1.13 times greater than the one determined by $\mathcal{BLP}$. Moreover the policies incorporating sharing provide better solutions. Indeed, an average revenue of 42423.44 is obtained with $\mathcal{BLP}$, whereas the average revenue determined by applying $\mathcal{BPP}$ is equal to 48194.51. In addition an average revenue of 84213.37 and 67210.28 is determined by applying $\mathcal{BLPS}$ and $\mathcal{BPPS}$, respectively.

With reference to $\mathcal{BLPSM}$ and $\mathcal{BLPR}$, results suggest that for the smallest instances $T1 - T4$, higher revenue values can be obtained when truck sharing and truck repositioning are considered. Indeed, an average revenue of 47658.39 is obtained by applying $\mathcal{BLPR}$, whereas the average revenue is equal to 47504.30 when $\mathcal{BLPSM}$ is considered.

However, by considering the revenue values achieved with these two policies and those obtained by applying $\mathcal{BLPS}$, in which the possibility of satisfying different compatible requests by the same truck is considered only in the policy, it is evident that the performance are comparable. Consequently, in real applications, it is reasonable adopt the $\mathcal{BLPS}$, avoiding the excessive computational cost for solving $R^{TRPS}(Q,t)$ and $R^{TRPR}(Q,t)$,

In order to assess the quality of the partitioned booking limits and bid price strategies, we also compared the revenues $R_{\mathcal{BLP}}$, $R_{\mathcal{BLPS}}$, $R_{\mathcal{BPP}}$ and $R_{\mathcal{BPPS}}$ determined by applying respectively policies $\mathcal{BLP}$, $\mathcal{BLPS}$, $\mathcal{BPP}$ and $\mathcal{BPPS}$, with the revenue obtained by solving the R^{TRP} and considering a perfect knowledge of the realized demand ($R^{\mathcal{HSP}}$) and with that achieved when a first-come first-served policy is applied ($R^{\mathcal{FCFSP}}$).

In particular, we considered the average percentage error $\mathcal{APE}$ and the average percentage gain $\mathcal{APG}$ with respect to the revenue $R^{\mathcal{HSP}}$ and the revenue $R^{\mathcal{FCFSP}}$ defined as follows:

$$APE = \frac{R^{\mathcal{HSP}} - R_i}{R^{\mathcal{HSP}}} \times 100, \;\; i = \mathcal{BLP}, \mathcal{BPP}, \mathcal{BLPS}, \mathcal{BPPS};$$

$$APG = \frac{R_i - R^{\mathcal{FCFSP}}}{R^{\mathcal{FCFSP}}} \times 100, \;\; i = \mathcal{BLP}, \mathcal{BPP}, \mathcal{BLPS}, \mathcal{BPPS}.$$

The related results are reported in Tables 5.3 and 5.4. Table 5.3 clearly underlines that the revenue obtained by applying the partitioned bid price policy is quite close to the revenue achieved in the case of perfect information on the demand. Indeed, the average percentage error is equal to 31.60% and 38.36% for $\mathcal{BPP}$ and $\mathcal{BLP}$, respectively. In addition all the proposed policies outperform the $\mathcal{FCFSP}$. In particular, the average percentage gain is equal to 80.20% and 61.58% for $\mathcal{BPP}$ and $\mathcal{BLP}$, respectively. It can be noted that better performance are achieved with $\mathcal{BLPS}$ and $\mathcal{BPPS}$, as shown in Table 5.4. In particular, for almost all the test problem the revenue obtained from $\mathcal{BLPS}$ and $\mathcal{BPPS}$ is greater than the one with perfect information on demand. It is worth nothing that the negative values of APE in Table 5.4 are due to the fact that in calculating $\mathcal{APE}$ the comparison is made between the revenue that is obtained by applying the policy described in Section 5.3.2, that lets the possibility of load multiple demands on the same trucks, and the revenue obtained from the solution of the R^{TRP} model (that does not incorporate sharing) considering a perfect knowledge of the realized demand.

5.6 Conclusions

In this paper, we considered the optimal managing of a fleet of trucks rented by a logistic operator, to serve customers. The logistic operator has to decide whether to accept or reject a request and which type of truck should be used to address it. For this purpose, a dynamic programming formulation and linear approximations of the problem under consideration have been defined. Based on the proposed linear programming models, borrowing revenue management techniques primal and dual acceptance policies have been defined, that use partitioned booking limits and bid prices controls. The possibility of loading multiple demands on the same truck (i.e., "truck sharing") has been also exploited. The repositioning of empty trucks from nodes, where they are not used, to nodes from which a new transportation request could be satisfied, has been also considered. The contributions of the paper to the literature are several. Indeed this is the first time, to our knowledge, that revenue management techniques are applied to the problem of operating trucks on a network with the possibility of "upgrades" and consolidation. Moreover, the faced problem implies the possibility of reusing resources during the booking period. Finally, the paper exploits an alternative way to solve the dynamic Resource Allocation Problem by linear models and to use their solutions to define revenue based policies to take profitable decisions in assigning resources.

Table 5.2: Average revenue of the proposed policies and its 95% confidence interval

Test Problem	$\mathcal{BLP}$	$\mathcal{BLPS}$	$\mathcal{BPP}$	$\mathcal{BPPS}$	$\mathcal{BLPSM}$	$\mathcal{BLPR}$
T1	14334.75 [13938.71 , 14730.79]	20001.60 [17831.54, 22171.66]	13532.75 [13196.88, 13868.61]	22476.00 [21236.67, 23715.33]	18024.45 [17216.82, 18832.08]	17947.88 [17355.89, 18539.86]
T2	31534,80 [30695.78, 32373.81]	45896.20 [42577.42, 49214.98]	31649.30 [30973.47, 32325.13]	44540.16 [43432.45, 45647.86]	47418.81 [45524.32, 49313.30]	46718.88 [45535.14, 47902.61]
T3	24437.65 [23788.32, 25086.98]	43782.70 [40330.54, 47234.86]	26241.65 [25376.99, 27106.31]	35903.60 [34934.02, 36873.18]	39246.05 [37451.54, 41040.56]	39763.98 [38333.35, 41194.60]
T4	59912.80 [58806.62, 61018.98]	80472.00 [71001.00, 89942.99]	71937.15 [69681.18, 74193.12]	86347.00 [83639.14, 89054.86]	85327.9 [80381.42, 90274.38]	86202.80 [82223.18, 90182.42]
T5	18816.80 [18258.42, 19375.18]	28383.00 [26716.62, 30049.38]	21656.35 [20752.54, 22560.16]	31861.09 [30727.55, 32994.62]	TO TO	TO TO
T6	47271.50 [45842.09, 48700.90]	78260.50 [68259.64, 88261.36]	49957.45 [48055.72, 51859.18]	71596.25 [68870.80, 74321.70]	TO TO	TO TO
T7	36358.20 [34737.62, 37978.78]	95474.00 [91218.47, 99729.53]	45476.65 [43824.26, 47129.03]	62052.22 [59797.57, 64306.87]	TO TO	TO TO
T8	104621.75 [103915.68, 105327.82]	176747.20 [175554.37, 177940.02]	122458.60 [118511.24, 126405.96]	135077.15 [130723.04, 139431.26]	TO TO	TO TO
T9	19466.50 [18776.03, 20156.97]	44896.83 [43304.35, 46489.32]	21866.35 [21189.17, 22543.53]	40544.50 [39288.88, 41800.12]	TO TO	TO TO
T10	46412.75 [44355.97, 48469.53]	117488.63 [112282.13, 122695.13]	51711.90 [49771.82, 53651.98]	87167.27 [83897.01, 90437.53]	TO TO	TO TO
T11	37820.42 [36185.04, 39455.81]	102115.14 [97699.60, 106530.67]	41260.95 [39574.78, 42947.12]	68080.56 [65281.73, 70879.39]	TO TO	TO TO
T12	68093.30 [63158.41, 73028.19]	177042.58 [164211.88, 189873.28]	80585.00 [73891.15, 87278.84]	120877.50 [110836.73, 130918.27]	TO TO	TO TO
ATP	**42423.44**	**84213.37**	**48194.51**	**67210.28**	**-**	**-**

Test Problem	$\mathcal{BLP}$		$\mathcal{BPP}$	
	$\mathcal{APE}$	$\mathcal{APG}$	$\mathcal{APE}$	$\mathcal{APG}$
T1	28.23%	26.33%	32.25%	19.26%
T2	30.99%	74.10%	30.74%	74.73%
T3	36.64%	83.70%	31.97%	97.26%
T4	36.64%	66.43%	23.92%	99.83%
T5	39.42%	49.22%	30.28%	71.74%
T6	32.95%	77.53%	29.14%	87.62%
T7	41.84%	46.15%	27.26%	82.81%
T8	22.45%	133.84%	9.23%	173.70%
T9	40.91%	40.24%	33.62%	57.53%
T10	44.32%	57.61%	37.97%	75.60%
T11	48.93%	46.02%	43.70%	59.31%
T12	57.03%	37.77%	49.14%	63.04%
ATP	**38.36%**	**61.58%**	**31.60%**	**80.20%**

Table 5.3: Average percentage error and average percentage gain for $\mathcal{BLP}$ and $\mathcal{BPP}$

Test Problem	$\mathcal{BLPS}$		$\mathcal{BPPS}$	
	$\mathcal{APE}$	$\mathcal{APG}$	$\mathcal{APE}$	$\mathcal{APG}$
T1	-0.14%	76.27%	-12.52%	98.07%
T2	-0.44%	153.38%	2.53%	145.90%
T3	-13.51%	229.12%	6.92%	169.89%
T4	14.90%	123.54%	8.69%	139.86%
T5	8.63%	125.08%	-2.57%	152.66%
T6	-11.01%	193.91%	-1.55%	168.88%
T7	-52.72%	283.79%	0.74%	149.44%
T8	-31.01%	295.04%	-0.12%	201.91%
T9	-36.29%	223.44%	-23.08%	192.08%
T10	-40.94%	298.97%	-4.57%	196.00%
T11	-39.32%	294.27%	7.11%	162.86%
T12	-11.73%	258.20%	23.71%	144.56%
ATP	**-17.80%**	**212.92%**	**0.44%**	**160.18%**

Table 5.4: Average percentage error and average percentage gain for $\mathcal{BLPS}$ and $\mathcal{BPPS}$

Appendix

The aim of this appendix is to rewrite the linear programming problem presented in section 5.3 in a form that can be viewed as an extension of the well known deterministic linear programming model arising in the network capacity control ([144]) and for which the asymptotic analysis of network problems still holds. Indeed, we do not report any proof on the asymptotic analysis because the theoretical findings reported in [31] and [143] can be easily extended to our case.

The asymptotic optimality property basically states that the expected revenue generated from the partitioned booking limit policy, based on the optimal primal solution of the linear programming formulation, is asymptotically convergent to the optimal expected revenue $V_t(X)$, when both the available capacity and the demand are scaled up proportionally. Moreover, it can be shown that the bid-price policy is also asymptotically optimal, when capacities and demand are large provided, that the right bid prices are used.

It is worth noting that because we assumed that the sets of origins and destinations are the same, when we refer to a pair of nodes (i, j), we always assume that $i \neq j$. We

Δ	O	O	O	O	...
O	Δ	O	O	O	...
O	O	Δ	O	O	...
...	...	...	...	...	...
O	O	O	O	Δ	...
...	...	...	...	...	Δ

Figure 5.1: Graphical representation of the D Matrix

first concentrate our attention on constraints (5.3). Let D be the block matrix, with $[\bar{T} \times r \times \times N \times (N-1)]$ rows and $[N \times (N-1) \times max_{prod} \times \bar{T}]$ columns, reported in Figure 5.1. Each block of the Δ type is also a block matrix with $[\bar{T} \times r]$ rows and $[max_{prod} \times \bar{T}]$ columns, whose structure is given in Figure 5.2, whereas each block of the O type has $[\bar{T} \times r]$ rows and $[max_{prod} \times \bar{T}]$ with all zero entries.

The Δ type matrices are also block matrices with r rows and max_{prod} columns. Each row $k, k = 1, \ldots, r$ of the Δ matrices contains a block of ones with dimension v_{kmax} and the remaining elements equal to zero. Figure 5.3 reports the structure of matrix D, for the case $N = 3,\ r = 2,\ \bar{T} = 2$.

$11\ldots1$	$00\ldots0$	$00\ldots0$	$00\ldots0$
$00\ldots0$	$11\ldots1$	$00\ldots0$	$00\ldots0$
...	...	...	...
$00\ldots0$	$00\ldots0$	$00\ldots0$	1

Figure 5.2: Graphical representation of the structure of the Δ blocks

Also constraints (5.4) can be rewritten in compact form in the following way. Let C denote the block matrix with $[r \times N \times \bar{T}]$ rows and $[\bar{T} \times max_{prod} \times N \times (N-1)]$ columns,

110000	000000	000000	000000	000000	000000
001000	000000	000000	000000	000000	000000
000110	000000	000000	000000	000000	000000
000001	000000	000000	000000	000000	000000
000000	110000	000000	000000	000000	000000
000000	001000	000000	000000	000000	000000
000000	000110	000000	000000	000000	000000
000000	000001	000000	000000	000000	000000
000000	000000	110000	000000	000000	000000
000000	000000	001000	000000	000000	000000
000000	000000	000110	000000	000000	000000
000000	000000	000001	000000	000000	000000
000000	000000	000000	110000	000000	000000
000000	000000	000000	001000	000000	000000
000000	000000	000000	000110	000000	000000
000000	000000	000000	000001	000000	000000
000000	000000	000000	000000	110000	000000
000000	000000	000000	000000	001000	000000
000000	000000	000000	000000	000110	000000
000000	000000	000000	000000	000001	000000
000000	000000	000000	000000	000000	110000
000000	000000	000000	000000	000000	001000
000000	000000	000000	000000	000000	000110
000000	000000	000000	000000	000000	000001

Figure 5.3: Representation of the D Matrix when $N = 3,\ r = 2,\ \bar{T} = 2$

reported below:

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

where $C_i, i = 1, \ldots, N$ is a block matrix with $[r \times \bar{T}]$ rows and $[\bar{T} \times max_{prod} \times N \times (N-1)]$ columns, that takes the form given in Figure 5.4.

(1, 2)	...	(1, i)	...	(i-1, i)	...	(i, 1)	...	(i, N)	(i+1, 1)	...	(i+1, N)	...	(N, i)	...	(N, N-1)
[O	O	$-\tilde{A}$	O	$-\tilde{A}$	O	$\tilde{A}$	O	$\tilde{A}$	O	O	O	O	$-\tilde{A}$	O	O]

Figure 5.4: Graphical representation of the structure of the C_i block

The blocks of O type are matrices with all the entries equal to zero with $[\bar{T} \times r]$ rows and $[\bar{T} \times max_{prod}]$ columns while matrix $\tilde{A}$ is a block matrix with $[\bar{T} \times r]$ rows and $[\bar{T} \times max_{prod}]$ columns and has the form reported in Figure 5.5. Matrix A is the binary matrix described in section 5.2.

A	O	O	O	O	...
O	A	O	O	O	...
O	O	A	O	O	...
...	...	...	...	...	...
O	O	O	O	A	...
...	...	...	...	...	A

Figure 5.5: Graphical representation of the $\tilde{A}$ Matrix

In Figure 5.6, the structure of matrix C is highlighted in the case $N = 3,\ r = 2,\ \bar{T} = 2$.

100000	100000	000000	000000	000000	000000
011000	011000	000000	000000	000000	000000
000100	000100	-100000	000000	-100000	000000
000011	000011	0-1-1000	000000	0-1-1000	000000
000000	000000	100000	100000	000000	000000
000000	0001000	011000	011000	000000	000000
-100000	000000	000100	001000	000000	-100000
0-1-1000	000000	000011	000011	000000	0-1-1000
000000	000000	000000	000000	100000	100000
000000	000000	000000	000000	011000	011000
000000	-100000	000000	-100000	000100	000100
000000	0-1-1000	000000	0-1-1000	000011	000011

Figure 5.6: Representation of the C Matrix when $N = 3,\ r = 2,\ \bar{T} = 2$

The vectors of decision variables $y \in \mathcal{R}^{N(N-1)\bar{T}max_{prod}}$ and of the revenues $r \in \mathcal{R}^{N(N-1)\bar{T}max_{prod}}$ take the following form:

$$y = (y_{12}^{11} \dots y_{12}^{\bar{T}max_{prod}} \dots y_{1N}^{\bar{T}max_{prod}} y_{21}^{11} \dots y_{2N}^{\bar{T}max_{prod}} \dots y_{N1}^{\bar{T}max_{prod}} \dots y_{N(N-1)}^{\bar{T}max_{prod}})^{\top};$$

$$r = (r_{12}^{1} \dots r_{12}^{1} \dots r_{12}^{r} \dots r_{12}^{r} r_{12}^{1} \dots r_{12}^{1} \dots r_{12}^{r} \dots r_{12}^{r} \dots r_{N1}^{1} \dots r_{N1}^{1} \dots r_{N(N-1)}^{r} \dots r_{N(N-1)}^{r})^{\top}.$$

Revenues vector r contains, for each pair of origin and destination nodes (i, j), the same set of elements. In particular, for each (i, j), r contains, v_1 elements of the r^1 types representing the value of the revenue associated with a class 1 request, v_2 elements of the r^2 types representing the value of the revenue associated with a class 2 request and so on. The $v_1 + \ldots + v_r$ elements, for each pair of nodes are replicated $\bar{T}$ times.

The vectors of demand $d \in R^{N(N-1)\times\bar{T}\times r}$ and capacity $x \in \bar{T} \times N \times r$ at time t can be represented as follows:

$$d = (d_{12}^{11} \dots d_{12}^{1r} d_{12}^{21} \dots d_{12}^{\bar{T}r} \dots d_{1N}^{\bar{T}r} \dots d_{N(N-1)}^{\bar{T}r})^{\top}$$

$$x = (x_{12}^{1} \dots x_{1r}^{1} x_{21}^{1} \dots x_{2r}^{1} \dots x_{Nr}^{1} \dots x_{11}^{\bar{T}} \dots x_{Nr}^{\bar{T}})^{\top}$$

On the basis of the definitions and notations introduced, the model $R^{TRP}(x, t)$ can be rewritten in a compact form as follows:

$$\begin{aligned} R^{LP}(x,t) = max\ r^{\top} y & \\ Dy &\le d \\ Cy &\le x \\ y &\ge 0 \end{aligned}$$

It is easy to verify that the mathematical model reported above is an extension of the deterministic linear programming model for the network capacity control given in [144], Chapter 3, p. 93.

Part III

Restaurant RM

Chapter 6

Strategic and Operational Decisions in Restaurant Revenue Management

Abstract

Strategic decisions in restaurant are mainly related to choose the best table configuration while respecting space and capacity constraints. We propose several formulations for the “Tables Mix Problem” taking into account different features of the real problem. From an operational point of view, decisions are to be made in order to assign tables to customers in the more profitable way. Indeed the “Parties Mix Problem” consists in deciding on accepting or denying a booking request coming from a group of customers. A dynamic formulation of the “Parties Mix Problem” is presented together with a linear programming approximation which solution can be used in decision making. In fact, to decide on accepting or denying a customer booking request, different control policies, based on booking limits and bid prices, are defined. Computational results show that the proposed policies lead higher revenues than the traditional strategies used to support decision makers.

Keywords: restaurant, dynamic programming, revenue management policies

6.1 Introduction

Restaurant revenue management is a quite recent discipline. The first work in restaurant revenue management is [83] in which the authors discuss the applicability of revenue management techniques to restaurant and [76] in which the RevPASH (the revenue per available seat-hour) is defined as a good indicator of revenue performance and as base to establish strategies oriented to improve the revenue. In the same paper the author presents a five step approach to develop a revenue management strategy at the restaurants. This approach is then applied to a real case in [78].

In [148] questions related to the combinability of the tables are discussed. In particular, if it is better to have combinable or dedicated tables and how the better mix of tables change by considering dedicated or configurable table design. In [85] the best table mix is evaluated by using simulation on data coming from a real case. In [86] eight different heuristics are compared to find the best number of different size tables for a restaurant to maximize its revenue.

From an optimization point of view the most relevant paper on restaurant revenue management is [20] in which integer programming, stochastic programming and approximate dynamic programming methods are used to control the arrival of walk-in customers in order to maximize restaurant revenue. A reservation-booking model together with a stochastic gradient algorithm are then presented to determine the optimal booking level.

In this paper we investigate the restaurant revenue management problem from both strategic and operational point of view. In particular, from a strategic point of view we formulate the table mix problem for a new restaurant that have to decide the best table configuration. In choosing the best tables configuration we consider both the aspects concerning the expected demand and the available space. In section 6.2 we present different formulations of the problem at crescent level of details and the extensions at the tables combinability case. In section 6.3 we give a dynamic formulation of the parties mix problem followed by a linear programming approximation of the problem, due to the curse of dimensionality. We conclude the section with the description of the several revenue management control policies proposed. Finally, in 6.4, we present computational results. The last section summarizes our conclusions.

6.2 The Tables Mix Problem (TMP)

Strategic decisions in restaurant are typically connected with the opening of a new restaurant or with the renovation of an exiting one. The Tables Mix Problem is to find the combination of tables, with different sizes, that will constitute the restaurant. Decisions about the "best" table mix are influenced by several factors like the number of potential customers and the expected meal duration, the dimension and the layout of the restaurant, and the possibility of combining tables of different dimensions.

We start by considering the simplest formulation of the problem (**TMP1**). Let be:

- y_{ip} the number of parties of size p, $p \in P = \{1, 2, 3, 4...p_{max}\}$ that can fit in a table sized i, $i \in I = \{2, 4, 6, 8..\}$;
- R_p the revenue obtained from a party of size $p \in P$;

- s_i the occupancy (in square meters) of a table sized $i \in I$;
- TD the total dimension (in square meters) of the restaurant;
- AD_p the average number of a party sized $p \in P$.

The objective function, representing the revenue of the actual combination of tables, is:

$$\max \sum_{p \in P} R_p \sum_{i \geq p} y_{ip} - \eta \sum_{i \geq p} (i-p) y_{ip} \tag{6.1}$$

The term $\eta \sum_{i \in I} (i-p) y_{ip}$, for opportune values of η, discourages the assignment of parties sized p to tables greater than p.

The constraints are given by:

$$\sum_{i \in I} \sum_{\substack{p \in P \\ p \leq i}} y_{ip} s_i \leq TD; \tag{6.2}$$

$$\sum_{i \geq p} y_{ip} \leq AD_p \ \forall p \in P; \tag{6.3}$$

$$y_{ip} \geq 0, \ integer \ \forall i \in I, \ \forall p \in P. \tag{6.4}$$

Constraint (6.2) state that the occupancy of all the tables can not exceed the total area of the restaurant. Each table is considered to be rectangular. The dimension of each rectangle (representing a table of a given capacity) is increased to take also into account the space required among adjacent tables. Constraints (6.3) impose that the demand of a party sized p can be satisfied with tables sized p or greater.

Restaurant rooms have different shapes so it is important to appropriately consider the restaurant layout while deciding the best table mix. In the **TMP2** below, we impose space constraints on portions of the room restaurant so that it is possible to better fit tables into the restaurant area. To this aim let be:

- x_{is} the number of tables sized $i \in I$ assigned to space portion $s = 1, \ldots, S$;
- $z_{is} = 1$ if for some i, $x_{is} > 0$; $z_{is} = 0$ otherwise;
- l_i the length of a table sized $i \in I$;
- h_i the height of a table sized $i \in I$;
- L_s the length of the $s - th$ portion of space, $s = 1, \ldots, S$;

- H_s the height of the $s-th$ portion of space, $s = 1, \dots, S$.

The objective function, representing the revenue of the actual combination of tables, is:

$$\max \sum_{i \in I} \sum_{p \in P} R_p y_{ip} \tag{6.5}$$

$$\sum_{i \in I} l_i x_{is} \leq L_s \ \forall s = 1, \dots, S; \tag{6.6}$$

$$h_i z_{is} \leq H_s \ \forall i \in I, \ \forall s = 1, \dots, S; \tag{6.7}$$

$$\sum_{i \geq p} y_{ip} \leq AD_p \ \forall p \in P; \tag{6.8}$$

$$\sum_{p \leq i} y_{ip} = \sum_{s=1}^{S} x_{is} \ \forall i \in I; \tag{6.9}$$

$$x_{is} \leq M z_{is} \ \forall i \in I, \ \forall s = 1, \dots, S; \tag{6.10}$$

$$z_{is} \leq x_{is} \ \forall i \in I, \ \forall s = 1, \dots, S; \tag{6.11}$$

$$y_{ip} \geq 0, \ integer \ \forall i \in I, \ \forall p \in P; \tag{6.12}$$

$$x_{is} \geq 0, \ integer \ \forall i \in I, \ \forall s = 1, \dots, S; \tag{6.13}$$

$$z_{is} \ binary \ \forall i \in I, \ \forall s = 1, \dots, S. \tag{6.14}$$

A length and a height are associated to tables and to portions respecting the total area of the restaurant. Constraints (6.6) and (6.7), impose that the tables, used to satisfy the demand, are placed in an appropriate portion of the restaurant area.

Another important issue in formulating the TMP is the average duration of meals. The number of tables of a certain dimension depends on the turnover of the group that fit in that dimension, in fact could be possible to gain a greater revenue from small tables having a faster turnover than from larger tables, usually having a greater "unit value", but that are busy longer. Considering the average duration of meals, tables become reusable resources in a given planning horizon.

To state the problem **TMP3** let be:

- y_{ipt} the number of tables sized $i \in I$ combinable with parties sized $p \in P$ that start dining at time $t = 1, \ldots, T$;
- x_{it} the number of tables sized $i \in I$ at time t ;
- R_p the revenue obtained from a party of size $p \in P$;
- TD the total dimension (in square meters) of the restaurant;
- D_p the average meal duration of a party sized $p \in P$;
- AD_{pt} the average number of parties sized $p \in P$ at time t.

The total revenue of the actual combination of tables is:

$$\max \sum_{t=1}^{T} \sum_{i \in I} \sum_{\substack{p \in P \\ p \leq i}} R_p y_{ipt} \tag{6.15}$$

Constraints are given by:

$$\sum_{i \in I} x_{it} s_i \leq TD \quad \forall t = 1, \ldots, T; \tag{6.16}$$

$$\sum_{i \geq p} y_{ipt} \leq AD_{pt} \quad \forall p \in P, \ \forall t = 1, \ldots, T; \tag{6.17}$$

$$\sum_{p \leq i} y_{ipt} \leq x_{it} + \sum_{\substack{p \in P \\ p \leq i}} \sum_{\tau = 1}^{t - D_p} y_{ip\tau} \quad \forall i \in I, \ \forall t = 1, \ldots, T; \tag{6.18}$$

$$x_{it} \geq 0, \ integer \ \forall i \in I; \tag{6.19}$$

$$y_{ipt} \geq 0, \ integer \quad \forall i \in I, \ \forall p \in P, \ \forall t = 1, \ldots T. \tag{6.20}$$

Constraints (6.18) assure that the number of tables sized i, assignable to groups of dimension greater or equal to i, at time t, don't exceed the whole availability given by the number of tables sized i at t plus the number of tables sized i that, considering the average meal duration of the eating groups, will become free at t.

Also TMP3 can be extended by better considering the shape of the restaurant room. The new formulation **TMP4** can be stated as follows.

The objective function (6.15) and constraints, (6.7), (6.14), (6.17) e (6.20) are still valid, whereas constraints (6.6), (6.10), (6.11),(6.13) and (6.18) need to be restated as follows:

$$\sum_{i \in I} l_i x_{ist} \leq L_s \ \forall s = 1, \ldots, S \ \forall t = 1, \ldots T; \tag{6.21}$$

$$x_{ist} \leq M z_{is} \ \forall i \in I, \ \forall s = 1, \ldots, S \ \forall t = 1, \ldots T; \tag{6.22}$$

$$z_{is} \leq x_{ist} \ \forall i \in I, \ \forall s = 1, \ldots, S \ \forall t = 1, \ldots T; \tag{6.23}$$

$$x_{ist} \geq 0, \ integer \ \forall i \in I \ \forall s = 1, \ldots, S \ \forall t = 1, \ldots T; \tag{6.24}$$

$$\sum_{p \leq i} y_{ipt} \leq \sum_{s=1}^{S} x_{ist} + \sum_{\substack{p \in P \\ p \leq i}} \sum_{\tau = 1}^{t - D_p} y_{ip\tau} \quad \forall i \in I, \ \forall t = 1, \ldots, T. \tag{6.25}$$

We now consider the possibility of combining tables while deciding the number of tables that will constitute the restaurant. In particular, we admit the possibility of satisfying the demand of a group sized p not only by table sized p or greater but also by combining tables smaller than p. We associate to each group sized p a set of "products" that can be used to accommodate the group. A product is a table or a combination of tables.

We need to introduce the following quantities:

- y_{ptj} is the number of products j, $j = n_{p-1} + 1, \ldots, n_p$ with $n_0 = 0$, used to satisfy a group sized $p \in P$ at time $t = 1, \ldots, T$.
- a_{ij}^{p} is the number of tables sized $i \in I$ used in product j to accommodate a group sized $p \in P$.

Extensions of problems TMP1, TMP2, TMP3 and TMP4 to the combinability case are reported in what follows.

Problem **TMP1C** is:

$$\max \sum_{p=1}^{p_{max}} \sum_{j=n_(p-1)+1}^{n_p} R_p y_{pj} \tag{6.26}$$

$$\sum_{i \in I} x_i s_i \leq TD; \tag{6.27}$$

$$\sum_{j=n(p-1)+1}^{n(p)} y_{pj} \leq AD_p \ \forall p \in P; \tag{6.28}$$

$$\sum_{p=1}^{p_{max}} \sum_{j=n(p-1)+1}^{n(p)} a_{ij}^{p} y_{pj} \leq x_i \ \forall i \in I; \tag{6.29}$$

$$y_{pj} \geq 0, \ integer \ \forall i \in I, \ \forall p \in P; \tag{6.30}$$

$$x_i \geq 0, \ integer \ \forall i \in I. \tag{6.31}$$

Problem **TMP2C**, which is the extension of TMP2 to the case of combinability, can be formulated by considering the same objective function of TMP1C (6.26). Constraints (6.6), (6.7), (6.10), (6.11), (6.13), (6.14), (6.28) (6.30) are still valid, whereas constraints (6.9) need to be redefined as follows:

$$\sum_{p=1}^{p_{max}} \sum_{j=n_(p-1)+1}^{n_p} a_{ij}^{p} y_{pj} = \sum_{s=1}^{S} x_{is} \ \forall i \in I; \tag{6.32}$$

The extensions of TPM3 and TPM4 to the case of combinability are presented in what follows.

Problem **TMP3C** is:

$$\max \sum_{t=1}^{T} \sum_{p=1}^{p_{max}} \sum_{j=n_(p-1)+1}^{n_p} R_p y_{ptj} \tag{6.33}$$

$$\sum_{i\in I} x_{it}s_i \leq TD \ \forall t = 1,\dots,T; \tag{6.34}$$

$$\sum_{j=n(p-1)+1}^{n(p)} y_{ptj} \leq AD_{pt} \ \forall p \in P, \ \forall t = 1,\dots,T; \tag{6.35}$$

$$\sum_{p=1}^{p_{max}} \sum_{j=n(p-1)+1}^{n(p)} a^p_{ij} y_{ptj} \leq x_{it} + \sum_{p=1}^{p_{max}} \sum_{j=n(p-1)+1}^{n(p)} \sum_{\tau=1}^{t-D_p} a^p_{ij} y_{p\tau j}, \ \forall i \in I \ \forall t = 1,\dots,T; \tag{6.36}$$

$$y_{ptj} \geq 0, \ integer \ \forall i \in I, \ \forall p \in P, \ \forall t = 1,\dots T; \tag{6.37}$$

$$x_i \geq 0, \ integer \ \forall i \in I \tag{6.38}$$

For problem **TMP4C** the objective function (6.33) and constraints (6.6), (6.7), (6.10),(6.11), (6.13), (6.14), (6.35), (6.37) are still valid, whereas constraints (6.36) can to be stated as follows:

(6.39)
$$\sum_{p=1}^{p_{max}} \sum_{j=n_(p-1)+1}^{n_p} a^p_{ij} y_{ptj} \leq \sum_{s=1}^{S} x_{ist} + \sum_{p=1}^{p_{max}} \sum_{j=n_(p-1)+1}^{n_p} \sum_{\tau=1}^{t-D_p} a^p_{ij} y_{ip\tau}, \ \forall i \in I \ \forall t = 1,\dots,T.$$

6.3 The Parties Mix Problem (PMP)

Once the table mix problem is solved, i.e. the number of the tables of dimension $i, i \in I$ is decided, it is necessary to decide about how to assign tables to customers in the more profitable way. Every day restaurant manager have to deal with the problem of accepting or refusing booking requests coming from groups of customers.

In the sequel we assume that the restaurant have the possibility of combining tables of different dimensions. We present a formulation of what we called "The Parties Mix Problem".

Let $G = g_1, g_2, \dots, g_K$ the set of all possible groups dimension. We assume that the demand of a party sized g_k, $k = 1, \dots, K$ can be satisfied both with a table of dimension greater or equal to g_k or with a combination of tables, each of them, with size less than g_k. In our setting we will use the term "product" to indicate both a single table or a combination of two or more tables that can be used to satisfy the demand of a party sized g_k.

Therefore a product is a combination of the different sized tables (resources) that can be sold with a certain price that depends on the dimension of each party.

Let $A = [A^1|A^2|\dots|A^k|\dots|A^K]$ the product matrix. Sub matrix $A^k \in \mathcal{R}^{m\times n_k}, k = 1,\dots,K$ refers to a group sized g_k. In particular an element a^k_{ij} of sub-matrix A^k indicate the number tables of size i, $i = 1,\dots,m$ used in product j, $j = n_{(k-1)} + 1,\dots,n_k$, to satisfy the demand of a group sized g_k. A column of matrix A, ${A_j}^k$ $k = 1,\dots,K$ $j = n_{(k-1)} + 1,\dots,n_k, n_0 = 0$ indicates the $j - th$ product that can be used to satisfy the demand of a group sized g_k .

It is worth noting that a^p_{ij} introduced in Section 6.2, when models TMP1C, TMP2C, TMP3C, TMP4C are presented, have the same meaning of a^k_{ij}. Here k is substituted with p just for notation convenience.

Just for an example consider requests coming from group sized $2, 4, 6, 8$ and tables of dimension $4, 6, 8$. Matrix A is shown in Figure 6.1. The first product is a table of dimension 1 that can be used to satisfy a group sized 1. The same group could be satisfied with the product constituted by a table sized 4, or by table sized 6, or by a table sized 8. The third group, sized 6, could be satisfied with a product constituted by 2 tables sized 4, or by 1 table sized 6 or by 1 table sized 8; the fourth 4 sized 8 could be satisfied with the product constituted by 2 tables sized 4, or by 2 tables sized 6, or by 1 table sized 8 or by 1 table sized 4 and 1 table sized 6.

Matrix A	Groups												
	1			2			3			4			
Tables	1	2	3	4	5	6	7	8	9	10	11	12	13
1	**1**			**1**			**2**			**2**		**1**	
2		**1**			**1**			**1**			**2**	**1**	
3			**1**			**1**			**1**				**1**

Figure 6.1: A graphical representation of matrix A

The initial capacity of the network, i.e the number of tables sized i available at the beginning of the booking horizon is $c = (c_1,\dots,c_m)^\top$. The state of the network is described by a vector $x = (x_1,\dots,x_m)^\top$ of tables capacities. In each time period $t = 1,\dots,T$ of the booking horizon the restaurant has to decide on accepting the request of a group sized g_k that ask for a table at time $\bar{t}, \bar{t} = 1,\dots\bar{T}$, with the goal of maximizing the total revenue. In the sequel we will refer to $1,\dots,\bar{T}$ as the "meal horizon", i.e the horizon where customers consume their lunch/dinner. The state of the system is described by a matrix $X = [X_1|X_2|\dots|X_{\bar{T}}]$, each column $X_{\bar{t}} = (x^{\bar{t}}_1,\dots,x^{\bar{t}}_m)^\top, \forall i \in I,\ \ \bar{t} = 1,\dots\bar{T}$ representing the capacity, i.e the number of tables sized i available at time $\bar{t}$.

Time is discrete, there are T booking periods indexed by t, which runs forward; consequently, $t = 1$ is the first possible booking time.

In each time-period t, at most one request of eating can arrive. Let $\lambda^{\bar{t}}_{tk}$ denote the probability that at time t one eating request from a group sized g_k is made. It holds that $\lambda_{t0} + \sum_{\bar{t}=1}^{T}\sum_{k=1}^{K} = 1$, where $\lambda_{t0} = 1$ represents the probability that no request arrives at time t.

We, further, assume that the meal durations are random and indicate with $q_{\tau k}$ the probability that the meal duration of a group sized g_k will be $\tau, \tau = 1, \ldots, \bar{\tau}$ time units.

Let us introduce boolean variables $u_{tj}^{\bar{t}k}$, with $u_{tj}^{\bar{t}k} = 1$ if and only if a request, with lunch (dinner) time at $\bar{t}$, coming from a group sized g_k is satisfied at time t with product j.

Let R^k be the revenue obtained by satisfying a request of a group sized g_k.

The problem can be formulated as a dynamic program by letting $V_t(X)$ be the maximum expected revenue obtainable from periods $t, t+1, \ldots, T$ given that, at time t, the capacity of the system is X.

The Bellman equation for $V_t(X)$ is reported in what follows:

$$V_t(X) = \sum_{k=1}^{K} \lambda_{tk} \max_{\substack{u_{tj}^k \in \{0,1\} \\ j \in \{n_{(k-1)}+1, \ldots n_k\}}} [r^k u_{tj}^k + V_{t+1}(\tilde{X})] + \lambda_{t0} V_{t+1}(X) \tag{6.40}$$

with boundary conditions

$$V_t(0) = 0, \forall t;$$

$$V_t(X) = -\infty \text{ if } x_j^{\bar{t}} < 0 \text{ for some } j, \bar{t}; \forall t$$

$$V_{T+1}(X) = 0, \text{ if } x_j^{\bar{t}} \geq 0 \ \forall\ j, \bar{t};$$

$$V_{T+1}(X) = -\infty \text{ if } x_j^{\bar{t}} < 0 \text{ for some } j, \bar{t}.$$

where

- $\tilde{X}_{\tilde{t}} = (X_{\tilde{t}} - A_j^k u_{\bar{t}j}^k), \forall \tilde{t} = \bar{t}, \ldots, \bar{T}$. This term updates capacity when a certain request of products j from a group sized g_k is accepted.
- $\tilde{X}_{\tilde{l}} = X_{\tilde{l}} + A_j^k u_{\bar{t}j}^k, \forall \tilde{l} = (\bar{t} + \sum_{\tau=1}^{\bar{\tau}} \tau q_{\tau k}), \ldots, \bar{T}$. This term updates capacity when the release of products j from a group sized g_k happens.
- $\tilde{X}_w = X_w, \forall w \neq \tilde{t}, \tilde{l}$. This term updates capacity on the rest of the system when a request of products j from a group sized g_k is accepted.

It is worth noting that the update of X, at time t happens when a group sized g_k requires a meal with starting time $\bar{t}$. In this case we need to change the state of the system by considering that, if the group request is accepted by using product j, we need to update the capacity considering that the tables used in product j will be not anymore available from time $\bar{t}$ (start dining) until the end of the meal horizon (unless the product will return available in the future). Moreover, we need to adjust the capacity by considering that the tables used in product j will be available again starting from a time equal to the start dining time plus the average meal duration $\sum_{\tau=1}^{\bar{\tau}} \tau q_{\tau k}$ and until the end of the operation horizon. On the rest of the system the capacity will not be varied until the end of the operation horizon.

The proposed dynamic programming (DP) model is unlikely to be solved optimally due to the curse of dimensionality. For this reason, in the next section, we propose a linear programming approximation of the DP, which is an extension of well-known approximations for the DP of traditional network capacity management. In particular, we are interested in approximations by deterministic linear programming (DLP) [37], [161]. Solving the Bellman equation, by approximating the function $V_t(X)$, falls in the general class of approximate dynamic programming (ADP) methods [17], in which an approximate value to the exact value function is used in the Bellman equation. The main difference among various ADP methods comes from the specific approximating mathematical programming problem that is built and solved to calculate the value function. It is evident that the type of approximation used influences the complexity of the function evaluation. Our DLP approximation is a simpler alternative to other approximations (like those presented in [151] for the dynamic resource allocation problem) and offers the possibility to construct revenue management policies, based on easy to solve deterministic optimization problems, that perform well in comparison to optimal policies.

6.3.1 A Linear Programming Formulation for the Party Mix Problem (LPMP)

In the LPMP, we replace stochastic demand quantities by their mean values and assume that capacity and demand are continuous.

Let:

- d be the random cumulative future demand at time t, and $\bar{d}$ its mean. In particular $d_k^{\bar{t}}$ is the aggregate number of requests from a group sized g_k requiring to have a meal starting at $\bar{t}$.
- $y_j^{k\bar{t}}$ be the number of products of type $j = 1, \ldots, n_K$ used to satisfy the request of a group sized g_k with starting meal time $\bar{t}$.
- r^k be the revenue obtained by satisfying a request of a group sized g_k.
- $x_i^{\bar{t}}$ be the number of tables of dimension c_i, $i = 1, \ldots, m$ available at time $\bar{t}$.
- Dur_k be the average meal duration of a group sized g_k.

The total revenue obtainable at time t when the capacity of the restaurant is x can be obtained by solving the following optimization problem:

$$R^{LPMP}(x,t) = \max \sum_{\bar{t}=t}^{T} \sum_{k=1}^{K} \sum_{j=n_{(k-1)}+1}^{n_k} r^k y_j^{k\bar{t}} \tag{6.41}$$

$$\sum_{k=1}^{K}\sum_{j=n_{(k-1)}+1}^{n_k}\sum_{\tilde{t}=1}^{\bar{t}} a_{ij}^k y_j^{k\tilde{t}} \leq x_i^{\bar{t}} + \sum_{k=1}^{K}\sum_{j=n_{(k-1)}+1}^{n_k}\sum_{\tilde{t}=1}^{\bar{t}-Dur_k} a_{ij}^k y_j^{k\tilde{t}} \quad \forall i, \bar{t} = t, \dots, T \tag{6.42}$$

$$\sum_{j=n_{(k-1)}+1}^{n_k} y_j^{k\bar{t}} \leq d_k^{\bar{t}} \quad \forall k, \bar{t} = 1, \dots, T \tag{6.43}$$

$$y_j^{k\bar{t}} \geq 0 \;\; and \;\; integer \;\; \forall j, k, \bar{t} = t, \dots, T \tag{6.44}$$

The objective function (6.41) represents the total revenue obtainable at time t when the residual capacity of the restaurant is x. Constraints (6.43) state that the demand of a group sized g_k can be satisfied both with tables of dimension g_k or greater and with a combination of tables each of them with dimension less than g_k. Constraints (6.42) control the availability of a table of dimension s_i at time $\bar{t}$.

It is well known [144] that by solving the LPMP model we can use either the primal variables to construct a partitioned booking limit control directly or the dual variables to define a bid price control. In the partitioned booking limit control, a fixed amount of capacity of each resource is allocated to every product that is offered. The demand for each product has access only to its allocated capacity and no other product may use this capacity. In contrast, a bid price control policy sets a threshold price or bid price for each resource in the network. Roughly speaking this bid-price is an estimate of the marginal cost of consuming the next incremental unit of the resource's capacity. When a booking request for a product arrives, the revenue of the request is compared to the sum of the bid prices of all the resources required by the product. If the revenue exceeds the sum of the bid prices, the request is accepted provided that all the resources associated with the requested product are still available; if not, the request is rejected.

In the context of the PMP, optimal solutions y_j^{*kt} give partitioned booking limits while bid prices are formed from optimal dual variables of constraint (6.42). The revenue based policies are presented in the next section.

6.3.2 Revenue-based Primal and Dual Acceptation Policies

At a certain point of the planning horizon decisions about accepting or denying a meal request are to be made.

The model is driven by the arrival to the restaurant, at time t, of a booking request from a group sized g_k asking to have a meal at time $\bar{t}$.

When this event happens model $R^{LPMP}(x,t)$ is solved and its solution is used to take a decision.

From a primal viewpoint, the strategy to be adopted is a partitioned booking limits policy, $\mathcal{BLPMP}$, and assumes the following form.

$\mathcal{BLPMP}$ Scheme

Solve $R^{LPMP}(x,t)$. Let $y_{\bar{j}}^{*k\bar{t}}$ denote its optimal solution.

IF $y_{\bar{j}}^{*k\bar{t}} > 0$ for some $\bar{j} = n_{(k-1)} + 1, \ldots, n_k$ and there is enough capacity ($x_i^{\tilde{t}} > a_{i\bar{j}}^k$ $\forall i : a_{i\bar{j}}^k \neq 0$ and $\forall \tilde{t} = \bar{t}, \ldots, T$) THEN

ACCEPT the request;

SET $y_{\bar{j}}^{*k\bar{t}} = y_{\bar{j}}^{*k\bar{t}} - 1$;

UPDATE appropriately capacity:

$x_i^{\tilde{t}} = x_i^{\tilde{t}} - a_{i\bar{j}}^k$, $\forall i : a_{i\bar{j}}^k \neq 0$ and $\forall \tilde{t} = \bar{t}, \ldots T$;

CALCULATE the revenue obtained from accepting the request;

ELSE

DENY the request.

END IF

Now we present a new policy called Later Accommodation for Party Mix Problem ($\mathcal{L}APMP$). The model is driven by the arrival at time t of a request from a group sized g_k asking to have a meal at time $\bar{t}$. In the LABLP the restaurant has the possibility of suggesting later starting time t^{new} for the meal.

$\mathcal{LAPMP}$ Scheme

Solve $R^{LPMP}(x,t)$. Let $y_{\bar{j}}^{*k\bar{t}}$ denote its optimal solution.

IF $y_{\bar{j}}^{*k\bar{t}} > 0$ for some $\bar{j} = n_{(k-1)} + 1, \ldots, n_k$ and there is enough capacity THEN

ACCEPT the request;

SET $y_{\bar{j}}^{*k\bar{t}} = y_{\bar{j}}^{*k\bar{t}} - 1$;

UPDATE appropriately capacity i.e.

$x_i^{\tilde{t}} = x_i^{\tilde{t}} - a_{i\bar{j}}^k$, $\forall i : a_{i\bar{j}}^k \neq 0$ and $\forall \tilde{t} = \bar{t}, \ldots T$;

CALCULATE the revenue obtained from accepting the request;

ELSE

IF $y_{\tilde{j}}^{*kt^{new}} > 0$ for some $\tilde{j} = n_{(k-1)} + 1, \dots, n_k$ and $t^{new} > \bar{t}$ and $t^{new} \leq T$ THEN;

IF the group sized g_k accept to wait for $(t^{new} - \bar{t})$ and there is enough capacity THEN;

ACCEPT the request with later accommodation; SET $y_{\tilde{j}}^{*kt^{new}} = y_{\tilde{j}}^{*kt^{new}} - 1$;

UPDATE appropriately capacity i.e.

$x_i^{\hat{t}} = x_i^{\hat{t}} - a_{i\tilde{j}}^k, \forall i : a_{i\tilde{j}}^k \neq 0$ and $\forall \hat{t} = t^{new}, \dots T$;

CALCULATE the revenue obtained from accepting the request;

ELSE

DENY the request.

END IF

From a dual viewpoint, we solve the linear relaxation of the problem $R^{LPMP}(x, t)$ and use the dual variables associated to constraints (6.42). We will indicate with $\mathcal{BPPMP}$ the Bid Price Policy associated with the linear relaxation of the PMP problem.

Let us indicate with $\pi^{\bar{t}} = \pi_1^{\bar{t}}, \dots, \pi_m^{\bar{t}}$ $\bar{t} = t, \dots, T$ the vectors of dual variables associated with constraints (6.42).

A possible strategy to accept or deny a request from a group sized g_k, asking to have a meal at time $\bar{t}$, that arrives at time t to the restaurant is the following:

$\mathcal{BPPMP}$ Scheme.

Solve the linear relaxation of $R^{LPMP}(x, t)$ to obtain the dual variables $\pi^{\bar{t}}, \bar{t} = t, \dots, T$, $i = 1, \dots, m$.

IF $r^k \geq A_{\bar{j}}^k \pi^{\bar{t}}$ for some $\bar{j} = n_{(k-1)} + 1, \dots, n_k$ AND $x_i^{\tilde{t}} > 0$ $\forall \tilde{t} = \bar{t}, \dots, T$ THEN

ACCEPT the request;

UPDATE appropriately capacity;

CALCULATE the revenue obtained from accepting the request;

ELSE

DENY the request.

END IF

It is worth noting that the update of the capacity is the same that in the previous case.

We also implemented an Average Bid Price Policy $\mathcal{BPPMP}$. In this case an the average bid price value on the time interval $[\bar{t}, T]$ is considered and the revenue r^k is compared with $\pi_{average} = \frac{A_j^k(\pi^{\bar{t}}+\pi^{\bar{t}+1}+...+\pi^{\bar{t}+Dur_k})}{\{min[(\bar{t}+Dur_k),T]-\bar{t}\}}$.

6.4 Numerical Results

In this section, we report the numerical results obtained by testing the policies described in Section 6.3.2.

All the numerical experiments have been carried out in AIMMS 3.7, with Cplex 10.1 as solver, on a Pentium Intel(R) Core(TM) i7 CPU Q720 1.60GHz 4GB of RAM PC, under Windows 7 operating system.

The test problems we considered, reported in Table 6.1, refer to restaurant with capacity of 50, 100, and 200 seats. The dimension of the rooms, in square meters, are calculated by considering that Italian rules recommend 1.2 square meters for each seats.

In Table 6.2 are reported the tables characteristics, in particular for each table of a given dimension its length and height. These follows Italian rules for restaurants room and are comprehensive of the required distances between tables.

In Table 6.2 are reported the group characteristics, in particular the number of people constituting each group, the value of the revenue associated with each group and the expected meal duration.

In 6.3, with reference to TMP2, TMP4, TMP2C and TMP4C formulations, portions of different dimensions, which can be used to deal with restaurant rooms of different shapes, are reported. More in details, we considered the restaurant as constituted by three portions (rectangles), whose dimensions (in cm) are reported in Table 6.3.

Restaurant	**Capacity (Seats)**	**Dimension (m^2)**
1	50	60
2	100	120
3	200	240

Table 6.1: Characteristics of the restaurants

The meal horizon is taken as $T = 9$ periods of time. Each period of time corresponds to a quarter, so that, lunches can start at 12:00 a.m. and terminate at 14:00 a.m.

With reference to AD_p parameter, i.e the average number of a party sized $p \in P$, we considered both the case in which the number of customers is dimensioned equal to the number of seats (Low demand setting) and the case in which the number of customers is increased by the 80% of the number of seats (Hight demand setting). Moreover, for

Tables	**Seats**	**Length**	**Height**
1	4	237.5	215
2	6	307.5	215
3	8	367.5	215
Groups	**Number of people**	**Revenue**	**Duration meal**
1	2	60	2
2	4	100	2.5
3	6	150	3
4	8	210	3.5
5	10	260	3.5

Table 6.2: Characteristics of the tables and parties

Restaurant	**Dimensions**	**Portion 1**	**Portion 2**	**Portion 3**
1	Length	1000	1000	790.75
	Height	215	215	215
2	Length	2000	2000	1581.5
	Height	215	215	215
3	Length	4000	4000	3163
	Height	215	215	215

Table 6.3: Portion Characteristics

each demand setting, we generated three different scenarios, see Table 6.4. The first scenario corresponds to the case in which the higher percentage of demand is from groups of small dimensions, the second scenario is relative to the case of balanced percentage of demand between all the groups and the third scenario refers to the case in which the higher percentage of demand comes from groups of eight and ten people. The AD_{pt} parameters, i.e the average number of parties sized $p \in P$ at time t, was calculated by imposing that the 65% of customers ask for a meal between the third and the seventh period of time, the 20% of customers ask for a meal between the eight and the tenth period of time and the left 15% at the first and the last periods of time.

Group	**Scenario 1**	**Scenario 2**	**Scenario 3**
2	50%	20%	10%
4	30%	20%	10%
6	10%	20%	10%
8	6%	20%	40%
10	4%	20%	30%

Table 6.4: Demand scenarios

As discussed above, several formulations of TMP are considered and Section 6.4.1 presents numerical results for such problems. In addition, the results related to PMP and to the application of the primal and dual acceptation policies are illustrated in Section 6.4.2.

6.4.1 Preliminary Results for TMP

In what follows, we indicate the test problems for TMP with TP_sij, where the index i, $i = 1, 2, 3$, indicates the type of restaurant (see Table 6.1), the index j, $j = 1, 2, 3$, indicates the demand scenario (see Table 6.4) and s refers to strategic level. For instance, the test problem $TPs23$ is related to the case of a restaurant of 100 seats and 120 square meters and the values of demand are generated by considering the second scenario.

For models TMP1, we have considered $\eta = 0$.

The results of the three base cases are shown in Table 6.5 in which the values of the average revenue are reported.

As expected the average revenue for TMP3 and TMP4 is always greater than the revenue of TMP1 and TMP2 and in fact in the first two cases tables are managed as reusable resources in a given planning horizon. We can also observe that the revenue values for TMP2 and TMP4 are lower or equal than the revenue of TMP1 and TMP3, respectively, and this can be justified by the fact that considering a specific layout for a restaurant restricts the number of tables that can be used.

We also studied the effect of combining tables and results are summarized in Table 6.6. The average revenue in the case of combinability is always greater than the revenue without the possibility of combine tables. The benefit of combinability is shown in Table 6.7, it can be noticed that the gain in revenue is higher when the duration of meals is considered. Therefore we can conclude that combining tables is an important way to improve revenue.

6.4.2 Preliminary Results for PMP

In this subsection, we report the results related to the control policies defined in Section 6.3.2 for the PMP. We simulated the performances of BL, LABL, BPP and ABPP policies and compared the results with the performances of a simple first-come first-served policy (FCFSP) and of a hindsight policy (HSP) with the perfect knowledge of the realized demand.

We considered two possibility in accepting reservations: the *overlap* and the *no-overlap* settings. While in the case of no-overlap the decision about accept or reject a meal reservation is made in advance, the case of overlap between the meal and the booking horizon can be considered as a way of satisfy the walk-in customers.

To test the control booking policies, we considered an instance characterized by 3 tables sized $4, 6, 8$ seats, respectively, and 4 parties of $2, 4, 6, 8$ people. We considered the three types of restaurant with the same dimensions of the previous case (see Table 6.1), and the same scenarios (see Table 6.4). We report only the results obtained with the scenarios of low demand and high demand, in order to illustrate the two extreme situations.

For the test problems, we used the following notation. The first index indicates

Table 6.5: Average revenue for TMP1, TMP2, TMP3, TMP4

Test Problem	Low demand				High demand			
	TMP1	**TMP2**	**TMP3**	**TMP4**	**TMP1**	**TMP2**	**TMP3**	**TMP4**
TPs11	910.00	910.00	1390.00	1390.00	910.00	910.00	1390.00	1390.00
TPs12	1050.00	1050.00	1310.00	1310.00	1190.00	1190.00	1570.00	1570.00
TPs13	790.00	790.00	790.00	790.00	1250.00	1250.00	1310.00	1310.00
TPs21	1970.00	1970.00	2810.00	2810.00	1970.00	1970.00	2810.00	2810.00
TPs22	2120.00	2120.00	2180.00	2180.00	2400.00	2400.00	2440.00	2440.00
TPs23	1850.00	1850.00	1910.00	1910.00	2570.00	2570.00	2690.00	2690.00
TPs31	3870.00	3870.00	5370.00	5370.00	3950.00	3950.00	5630.00	5630.00
TPs32	4300.00	4300.00	4300.00	4300.00	4800.00	4800.00	5340.00	5340.00
TPs33	3800.00	3800.00	3800.00	3800.00	5180.00	5180.00	4940.00	4940.00
ATP	**2295.56**	**2295.56**	**2651.11**	**2651.11**	**2691.11**	**2691.11**	**3124.44**	**3124.44**

the type of restaurant while the second refers to the demand scenario (i.e TP11 indicates the test problem related to a restaurant of 50 seats, when the first demand scenario is considered, while TP32 indicates the test problem related to a restaurant of 200 seats, when the second type of demand scenario is considered). Moreover, the test problems for the overlap case are indicated with TPO, while TPNO is related to the no-overlap case.

Table 6.6: Average revenue for TMP1C, TMP2C, TMP3C, TMP4C

Test Problem	Low demand				High demand			
	TMP1C	TMP2C	TMP3C	TMP4C	TMP1C	TMP2C	TMP3C	TMP4C
TPs11	1120.00	1060.00	2500.00	2500.00	1120.00	1060.00	2500.00	2500.00
TPs12	1280.00	1270.00	2060.00	2060.00	1390.00	1390.00	2580.00	2580.00
TPs13	1250.00	1250.00	1370.00	1370.00	1410.00	1410.00	2410.00	2410.00
TPs21	2300.00	2240.00	4760.00	4760.00	2380.00	2380.00	5020.00	5020.00
TPs22	2640.00	2640.00	3820.00	3820.00	2780.00	2780.00	4860.00	4860.00
TPs23	2890.00	2890.00	3430.00	3430.00	2970.00	2930.00	4730.00	4730.00
TPs31	4560.00	4560.00	9480.00	9480.00	4840.00	4840.00	10000.00	10000.00
TPs32	5440.00	5440.00	7800.00	7800.00	5660.00	5660.00	9360.00	9360.00
TPs33	5790.00	5790.00	6810.00	6810.00	5940.00	5940.00	14610.00	14610.00
ATP	**3030.00**	**3015.56**	**4670.00**	**4670.00**	**3165.56**	**3154.44**	**6230.00**	**6230.00**

For the no-overlap case, the policies are tested by considering a booking horizon of 3 and 7 periods.

We measured the performances of policies over 2000 simulated booking processes

Table 6.7: Average percentage gain applying combinability

Test Problem	Low demand				High demand			
	APG1	**APG2**	**APG3**	**APG4**	**APG1**	**APG2**	**APG3**	**APG4**
TPs11	18.75%	14.15%	44.40%	44.40 %	18.75%	14.15 %	44.40%	44.40%
TPs12	17.97%	17.32%	36.41%	36.41 %	14.39%	14.39 %	39.15%	39.15%
TPs13	36.80%	36.80%	42.34%	42.34%	11.35%	11.35%	45.64%	45.64%
TPs21	14.35%	12.05%	40.97%	40.97%	17.23%	17.23%	44.02%	44.02%
TPs22	19.70%	19.70%	42.93%	42.93%	13.67%	13.67 %	49.79%	49.79%
TPs23	35.99%	35.99%	44.31%	44.31%	13.47%	12.29 %	43.13%	43.13%
TPs31	15.13%	15.13%	43.35%	43.35%	18.39%	18.39 %	43.70 %	43.70%
TPs32	20.96%	20.96%	44.87%	44.87%	15.19%	15.19%	42.95 %	42.95%
TPs33	34.37%	34.37%	44.20%	44.20%	12.79%	12.79 %	66.19 %	66.19%
ATP	**23.78%**	**22.94%**	**42.64%**	**42.64%**	**15.03%**	**14.38%**	**46.55%**	**46.55%**

for each test problem.

In each simulation run, the dinner requests are randomly generated by applying a two phases procedure. In the first phase, for each group of a given dimension, and each meal time, the number of dinner requests is randomly generated according to a normal distribution, with a given expected demand and a given coefficient of variation, chosen randomly from the interval [1, 10] and [0, 1], respectively. In the second phase, for each request, booking arrival times are randomly generated according to an uniform distribution.

The requests generated by the procedure outlined above are then processed.

In particular, when no-overlap is considered, at each time instant $\bar{t}$ in the booking horizon, a request, for which the booking arrival time is less than or equal to the considered booking instant, is chosen and the accept or deny decision is made based on one of the proposed policies. The tables availability is then updated and another booking request is processed. We move to the next booking time period when there are no more requests, arrived before $\bar{t}$, that need to be evaluated. Moreover, the value of the revenue is influenced by the order in which the booking requests are processed. In our experiments, we solve the models, used to define the policies, a number of times equal to the length of the booking horizon.

In the overlap case, the requests generated by the procedure described are processed by considering the accept or deny decision at each time instant t in the meal horizon.

The results for the optimal BL, LABL, BPP and ABPP booking control policies, are presented in Tables 6.9 and 6.10 for the no-overlap and overlap cases, respectively. For each test problem the average revenue values are given. We also report the average revenues for FCFSP and HSP in Table 6.8. We determined the 95% confidence intervals of average revenues and the average percentage error (APE) and the average percentage gain (APG) defined as follows:

$$APE = \frac{R^{\mathcal{HSP}} - R_i}{R^{\mathcal{HSP}}} \times 100, \quad i = \mathcal{BLP}, \mathcal{BPP}, \mathcal{ABPP}, \mathcal{LABLP};$$

$$APG = \frac{R_i - R^{\mathcal{FCFSP}}}{R^{\mathcal{FCFSP}}} \times 100, \quad i = \mathcal{BLP}, \mathcal{BPP}, \mathcal{ABPP}, \mathcal{LABLP}.$$

The related results to APE and APG are reported in Tables 6.11 and 6.12, for no-overlap and overlap cases, respectively. The Tables 6.13 and 6.14 contain the value of the average revenue and its 95% confidence interval, for no-overlap and overlap cases, respectively.

All the policies have required few seconds and this why we don't present evaluations on computational times.

Test Problem	FCFS	HSP
TP11	161.16	225.84
TP12	885.00	1239.36
TP21	545.52	773.88
TP22	943.65	1332.12
TP31	1376.40	1965.84
TP32	3524.28	5066.40

Table 6.8: Average revenue values for R^{LPMP} by applying HSP and FCFSP

Test Problem	BLP	BPP	ABPP	LABLP
TPNO11-b3	197.04	166.44	186.60	225.24
TPNO11-b7	188.52	167.28	185.40	219.24
TPNO12-b3	988.56	901.20	1034.40	1221.36
TPNO12-b7	969.96	895.68	1055.16	1193.52
TPNO21-b3	624.12	545.76	614.16	761.40
TPNO21-b7	622.44	547.08	627.84	757.92
TPNO22-b3	1048.20	974.52	1121.40	1246.08
TPNO22-b7	1050.00	973.32	1141.32	1288.80
TPNO31-b3	1556.04	1388.64	1584.48	1855.92
TPNO31-b7	1611.12	1387.80	1622.40	1882.32
TPNO32-b3	4017.24	3575.04	4280.16	4862.16
TPNO32-b7	4175.28	3610.68	4357.15	4940.52

Table 6.9: Average revenue for R^{LPMP} by applying BLP, BPP, ABPP and LABLP with no-overlap

Test Problem	BLP	BPP	ABPP	LABLP
TPO11	197.28	182.76	200.52	228.48
TPO12	983.88	947.59	1077.05	1173.76
TPO21	649.20	614.40	664.08	777.96
TPO22	1073.88	1011.00	1174.32	1257.36
TPO31	1635.60	1470.12	1685.64	1886.64
TPO32	4162.32	3827.76	4551.60	4908.72

Table 6.10: Average revenue for R^{LPMP} by applying BLP, BPP, ABPP and LABLP in case of overlap

Table 6.11 shows that in the no-overlap case, LABLP performs better on average than the all other policies. In particular, the better values for each policy are when a booking horizon of 7 periods is considered. On average, the LABLP obtains an error of 3.11% and a gain of 37.23%. The BPP does not seem to perform very well, with an error of 28.01% and a gain of only 1.95% for a booking horizon of 7 periods. The ABPP, when the booking periods are 7, is the policy with the greatest gain after LABLP and followed by BLP.

The same considerations can be written for the performances of the policies in the case of overlap.

It must be remembered that the LABLP considers the possibility for each party to wait for meal and to be served after a certain period of time, therefore the results obtained by comparing LABLP with the other policies, were predictable.

But by comparing the two cases, in particular the results for no-overlap case with 7 booking periods and the results for overlap, we can affirm that with overlap all the policies perform best.

It is worth nothing that the negative values of APE in Table 6.12 are due to the fact that in calculating APE the comparison is made between the revenue that is obtained by applying the policy LABLP, that lets the possibility of proposing to the customer a later starting time for the meal, and the revenue obtained from the solution of the LPMP model,that does not incorporate this possibility, considering a perfect knowledge of the realized demand.

Tables 6.13 and 6.14 contains the value of the average revenue values and its 95% confidence interval obtained applying each policies to our test problems for no-overlap and overlap cases, respectively. The last row ATP is the average revenue of each policy, on the test problems and for the two different booking horizon considered. The results show that the LABLP and the ABPP outperform BLP, while the average revenue for BPP is the worse than the other policies in both the cases. Moreover the policies with overlap provide better solutions.

Test Problem	**BLP**		**BPP**		**ABPP**		**LABLP**	
	APE	**APG**	**APE**	**APG**	**APE**	**APG**	**APE**	**APG**
TPNO11-b3	12.75%	22.26%	26.30%	3.28%	17.38%	15.79%	0.27%	39.76%
TPNO11-b7	16.52%	16.98%	25.93%	3.80%	17.91%	15.04%	2.92%	36.04%
TPNO12-b3	20.24%	11.70%	27.29%	1.83%	16.54%	16.88%	1.45%	38.01%
TPNO12-b7	21.74%	9.60%	27.73%	1.21%	14.86%	19.23%	3.70%	34.86%
TPNO21-b3	19.35%	14.41%	29.48%	0.04%	20.64%	12.58%	1.61%	39.57%
TPNO21-b7	19.57%	14.10%	29.31%	0.29%	18.87%	15.09%	2.06%	38.94%
TPNO22-b3	21.31%	11.08%	26.84%	3.27%	15.82%	18.84%	6.46%	32.05%
TPNO22-b7	21.18%	11.27%	26.93%	3.14%	14.32%	20.95%	3.25%	36.58%
TPNO31-b3	20.85%	13.05%	29.36%	0.89%	19.40%	15.12%	5.59%	34.84%
TPNO31-b7	18.04%	17.05%	29.40%	0.83%	17.47%	17.87%	4.25%	36.76%
TPNO32-b3	20.71%	13.99%	29.44%	1.44%	15.52%	21.45%	4.03%	37.96%
TPNO32-b7	17.59%	18.47%	28.73%	2.45%	14.00%	23.63%	2.48%	40.19%
ATP b=3	**19.20%**	**14.42%**	**28.12%**	**1.79%**	**17.55%**	**16.78%**	**3.24%**	**37.03%**
ATP b=7	**19.11%**	**14.58%**	**28.01%**	**1.95%**	**16.24%**	**18.64%**	**3.11%**	**37.23%**

Table 6.11: Average percentage gain and error values for BLP, BPP, ABPP, LABLP, with 3 and 7 booking periods

6.5 Conclusions

In this paper we studied the restaurant revenue management problem from both strategic and operational point of view. We formulated the table mix problem by considering different aspects like the expected demand, the available space and the tables combinability. From the operational point of view, we considered several booking control policies and showed how to apply them in the case of overlap and no-overlap between decision periods. Next to the well known deterministic model, we also looked at a dynamic model unlikely to be solved due to the curse of dimensionality so to construct the control policy we defined

Test Problem	BLP		BPP		ABPP		LABLP	
	APE	**APG**	**APE**	**APG**	**APE**	**APG**	**APE**	**APG**
TPO11	12.65%	22.41%	19.08%	13.40%	11.21%	24.42%	-1.17%	41.77%
TPO12	20.61%	11.17%	23.54%	7.07%	13.10%	21.70%	5.29%	32.63%
TPO21	16.11%	19.01%	20.61%	12.63%	14.19%	21.73%	-0.53%	42.61%
TPO22	19.39%	13.80%	24.11%	7.14%	11.85%	24.44%	5.61%	33.24%
TPO31	16.80%	18.83%	25.22%	6.81%	14.25%	22.47%	4.03%	37.07%
TPO32	17.84%	18.10%	24.45%	8.61%	10.16%	29.15%	3.11%	39.28%
ATP	**17.23%**	**17.22%**	**22.83%**	**9.28%**	**12.46%**	**23.99%**	**2.73%**	**37.77%**

Table 6.12: Average percentage gain and error for the BLP, BPP, ABPP, LABLP with overlap

Test Problem	BLP	BPP	ABPP	LAP
TPNO11-b3	205.44 [201.87, 209.00]	165.6 [162.15, 169.05]	183.84 [180.11, 187.57]	200.88 [196.51, 205.25]
TPNO11-b7	202.8 [199.31, 206.29]	170.52 [167.03, 174.01]	192.36 [188.76, 195.96]	194.28 [189.78, 198.78]
TPNO12-b3	1024.56 [1012.11, 1037.01]	928.44 [917.17, 939.71]	1056.12 [1045.16, 1067.08]	1149.12 [1131.89, 1166.35]
TPNO12-b7	1008.84 [996.39, 1021.29]	919.92 [908.85, 930.99]	1054.08 [1042.70, 1065.46]	1124.04 [1108.21, 1139.87]
TPNO21-b3	646.32 [637.82, 654.82]	552.36 [551.25, 563.31]	622.2 [629.28, 643.44]	710.76 [708.51, 728.85]
TPNO21-b7	643.32 [635.07, 651.57]	557.28 [551.25, 563.31]	636.36 [629.28, 643.44]	718.68 [708.51, 728.85]
TPNO22-b3	1071.96 [1061.41, 1082.51]	974.88 [965.59, 984.17]	1127.16 [1115.99, 1138.33]	1195.44 [1181.20, 1209.68]
TPNO22-b7	1074.12 [1063.22, 1085.02]	977.88 [968.60. 987.16]	1148.88 [1138.99, 1158.77]	1216.2 [1199.16, 1233.24]
TPNO31-b3	1600.44 [1585.42, 1615.46]	1398 [1388.14, 1407.86]	1597.68 [1585.75, 1609.61]	1805.16 [1789.96, 1820.36]
TPNO31-b7	1635.12 [1620.06, 1650.18]	1392.84 [1383.01, 1402.67]	1627.92 [1617.08, 1638.76]	1850.04 [1836.78, 1863.30]
TPNO32-b3	4059.96 [4026.48, 4093.44]	3586.56 [3563.34, 3609.78]	4302.96 [4275.53, 4330.39]	4829.64 [4801.69, 4857.59]
TPNO32-b7	4202.64 [4169.55, 4235.73]	3612.24 [3588.82, 3635.66]	4364.88 [4336.82, 4392.94]	4915.68 [4892.05, 4939.31]
ATP b=3	**1434.78**	**1267.64**	**1481.66**	**1648.50**
ATP b=7	**1461.14**	**1271.78**	**1504.08**	**1669.82**

Table 6.13: Average revenue of the proposed policies and its 95% confidence interval with no-overlap

a linear programming formulation of the DP. The performances of the different booking control policies are evaluated in a simulated environment. The results show that all the booking control policies perform better than the simple FCFS policy or than the case of

Test Problem	BLP	BPP	ABPP	LABLP
TPO11	202.80 [199.65, 205.95]	184.68 [181.54, 187.82]	202.08 [198.73, 205.43]	203.28 [198.73, 207.83]
TPO12	1019.88 [1008.12, 1031.64]	952.30 [940.43, 964.17]	1099.32 [1088.27, 1110.37]	1105.80 [1090.99, 1120.61]
TPO21	674.76 [667.12, 682.40]	616.08 [609.60, 622.56]	664.08 [656.82, 671.34]	727.56 [717.39, 737.73]
TPO22	1101.60 [1091.17, 1112.03]	1024.92 [1015.90, 1033.94]	1190.28 [1181.39, 1199.17]	1209.24 [1196.11, 1222.37]
TPO31	1663.20 [1649.61, 1676.79]	1477.56 [1468.00, 1487.12]	1692.00 [1680.78, 1703.22]	1849.56 [1835.64, 1863.48]
TPO32	4206.60 [4175.45, 4237.75]	3845.04 [3821.46, 3868.62]	4571.52 [4547.92, 4595.12]	4867.56 [4841.86, 4893.26]
ATP	**1478.14**	**1350.10**	**1569.88**	**1660.50**

Table 6.14: Average revenue of the proposed policies and its 95% confidence interval with overlap

perfect knowledge of demand. Especially the BLLAP and the ABPP seem to give the greatest benefits.

Part IV

Conclusions

Chapter 7

Conclusions

This thesis addressed the Revenue Management (RM, for short) optimization approaches and its main fields of applications. This work summarized the main results achieved in the three years of the Ph.D. program on this research topic. In particular, innovative RM optimization models and policies have been defined for two specific service sectors: car/truck rentals and restaurants.

More in details, in this thesis we considered:

- RM basic concepts and applications;
- A robust optimization approach for the car rental problem;
- A RM based approach for the car rental problem;
- A RM based approach for the truck rental problem;
- A RM based approach for strategic and operational decisions in restaurants.

Ad-hoc RM models and policies have been designed, developed and implemented. The obtained theoretical results have been validated by an extensive and exhaustive experimental phase in order to assess the behaviour of the proposed solution approaches in terms of robustness and practical applications in real contexts.

7.1 Summing up

In the thesis we started by summarizing the existing literature of RM and its main applications and solution approaches for service industries. Considerable attention was devoted to RM applications in car rentals and restaurants.

Regarding the car rental problem, we provided a detailed description of the typical car rental activities and proposed the first attempt to apply robust optimization to the car rental industry. To handle demand uncertainty, we proposed several robustness measures and the related scenario-based formulations for the car rental revenue optimization problem. More specifically, the attention has been focused on the maxmin criterion (i.e., the application of a simple absolute robustness measure, maximizing the worst-case performance), the robust deviation criterion, the stochastic p-robustness criterion and a standard deviation based variability criterion. The results collected indicate that these scenario-based formulations and these robustness measures can be used to obtained important considerations about expected revenue values during a car rental process.

Innovative RM models and policies to address the car rental problem have been proposed. We proceeded to evaluate the typical car rental process by considering same specific aspects, such as upgrades, one-way rentals and car transferring. In particular, we presented innovative mathematical models and solution approaches to manage a car rental. Dynamic programming models and the related linear programming approximations have been proposed to represent mathematically the problems under study. Primal and dual acceptance policies, which use booking limits and bid price controls, have been developed to handle the car rental company's problem of accepting or rejecting a car rental request. In order to evaluate the effectiveness of the proposed policies we compared these approaches with a typical first-come first-served policy and with the case of a perfect knowledge of the realized demand. An extensive computational phase has been conducted in order to demonstrate the validity of the proposed approaches as decisions support tools to maximize the revenue of the car rental agency by satisfying the demand and the capacity constraints.

Regarding the truck rental problem, we considered the optimal managing of a fleet of trucks, characterized by different load capacity, rented from a given set of origins to a given set of destinations, by a logistic operator, to serve customers. Many components of the truck rental business that affect its revenues have been considered. In particular, the possibility of loading multiple demands on the same truck and the repositioning of empty trucks have been exploited. The problems have been formulated as dynamic programs, unlikely to be solved optimally due to the curse of dimensionality and approximated by deterministic linear programming. In a RM settings, based on the solution of the linear programming formulations of the problems, we defined booking limits and bid price controls to accept or reject a request at a certain time. All defined revenue based linear models and policies suggested efficient and effective management approach to take profitable decisions in assigning resources and handling booking requests.

Regarding restaurant RM, we have suggested strategic and operational approaches to manage profitably the table mix problem and the booking requests problem. From the strategic point of view, the table mix problem has been formulated by considering several factors like the number of potential customers, the expected meal duration, the available space, i.e layout of the restaurant, and the tables combinability. We assumed that the restaurant have the flexibility of combining tables of different dimensions and we presented the new parties mix problem formulation. From the operational point of view we considered a dynamic formulation of the problem under consideration and several booking control policies: partitioned booking limit, bid price and the innovative later accommodation booking limit.

7.2 Concluding remarks

The contribution of this thesis can be summarized as follows.

The objective of the thesis was to evaluate and find an innovative and optimal management strategy for the car/truck rentals and restaurants businesses using the principles of RM. We believe we have fulfilled the objective, because the collected results are very encouraging and our models and policies are different than anything else in the RM literature in car/truck rental and restaurant fields.

Well studied aspects of the car/truck rental have been addressed and innovative RM models and policies have been proposed. To the best of our knowledge, the proposed robust approach represents the first attempt to apply robust optimization to the car rental industry to manage resources and demand in order to maximize the revenue. Furthermore, the dynamic programs, which consider in the same formulation the several and different aspects of a typical car/truck rental or restaurant process, have not previously taken into account in the RM literature.

For the aforementioned businesses no attention has been paid from the scientific literature to the definition of RM decision policies, such as booking limits and bid prices, to address the constrained and perishable capacity allocation problem. Thus, we believe the proposed RM acceptance policies be novel and valuable approaches.

7.3 Direction for future research

The main goal of future work will be to consider some of the aspects of rental industry not yet explored and to provide the definition and the computational evaluation of the related RM solution approaches and new controls to better respond to customers demand and management necessities.

Some things to consider are pricing, fleet planning, overbooking, bonus or penalty for upgrading. Furthermore, the validity of the proposed models and policies should be tested on a more realistic examples and practical settings.

Despite the encouraging results and the demonstrated great potential to apply RM approaches in the restaurant industries, this business field need to be studied further.

The models and policies presented in this thesis could be extended for modeling and solving the problems of other service industries.

A preliminary work related to the developments of models for planning problem at a tv broadcasting company was presented at Annual Conference of the Italian Operational Research Society (AIRO 2010 Conference). The aim of the work was to develop optimization models to support television networks in accepting and scheduling advertisements in Italy. Future research will be mainly focused on the development of efficient RM approaches and the definition of appropriate policies to support the negotiation process between advertisers and television networks.

By defining flexible RM pricing policy techniques, we will also address the problem for hotels to change dynamically the price in order to increase the total number of products sold and the customers willing to pay, by using mechanisms such as promotion, markdown, markup, to name a few.

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