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PhD Thesis

Fixed Point Iterations for Nonlinear Mappings

Scientific Sector: MAT/05 - Mathematical Analysis

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Sommario

In questa tesi vengono affrontati, e talvolta risolti, alcuni problemi sulla convergenza di algoritmi per punti fissi. A tali problemi, verrà affiancato inoltre l'ulteriore problema di stabilire quando tali algoritmi convergono a punti fissi che risultano essere soluzioni di disuguaglianze variazionali. I contributi scientifici personali apportati alla teoria dei punti fissi, riguardano essenzialmente la ricerca di ottenere convergenza forte di uno o più metodi iterativi, laddove non è nota convergenza, o qualora è nota la sola convergenza debole.
La struttura dei capitoli è articolata come segue:
Nel primo capitolo vengono introdotti gli strumenti di base e i cosiddetti spazi ambiente in cui verranno mostrati i principali risultati. Inoltre verranno fornite tutte le proprietà sulle mappe nonlineari utili nelle dimostrazioni presenti nei capitoli successivi.
Nel secondo capitolo, è presente una breve e mirata introduzione a quelli che sono alcuni dei risultati fondamentali sui metodi iterativi di punto fisso più noti in lettaratura.
Nel terzo capitolo, vengono mostrate le principali applicazioni dei metodi di approssimazione di punto fisso.
Nel quarto e nel quinto capitolo, vengono mostrati nei dettagli alcuni risultati riguardo un metodo iterativo di tipo Mann e il metodo iterativo di Halpern. In questi ultimi capitoli sono presenti i contributi dati alla teoria dell'approssimazione di punti fissi.

Contents

Introduction

Several problems in different areas of mathematics, and more generally applied sciences, can be redrafted in a fixed point problem involving nonlinear maps. Let C be a nonempty subset of a Hilbert space H and let be $T : C \to C$ a mapping. A point $x \in C$ is said to be *fixed point* of T if $Tx = x$. For istance, let us consider the differential equation

$$\frac{du}{dt} + Au\left(t\right) = 0$$

describing the evolution of a system where $A : H \to H$ is a monotone operator from an Hilbert space H in itself. At the equilibrium state,

$$\frac{du}{dt} = 0,$$

and then a solution of the equation $Au = 0$ describes the equilibrium's state of a system. This is very desiderable in many applications, as in economy sciences or in physical sciences, to mention a few. As a consequence, when A is a monotone operator, a solving approach to the equation $Au = 0$ becomes of considerable research interest. In general, however, A is a nonlinear operator, therefore we can not always find a closed form for the solution of the equation. Then, a standard technique is to consider the operator T defined by $T := I - A$, where I is the identity operator on H. A such map T is said to be *pseudo-contraction* (or pseudo-contractive map). Therefore it is clear that every zero of A is a fixed point of T. Due to this reason, the study of fixed point theory for pseudo-contrattive mappings, and the study about iterative methods for find such points, has attracted the interest of many mathematicians and was created a huge area of research, especially in the last 50 years.

Iteration means to repeat a process integrally over and over again. To iterate a function, we begin with a initial value for the iteration. This is a real number x_0, say. In the case of explicit methods, applying the function to x_0 yields the new number, x_1, say. Usually the iteration proceeds using the result of the previous computation as the input for the next. A sequence of numbers $x_0, x_1, x_2, ...$ is then generated. A very important question, then, is whether this sequence converges or diverges, and particularly for the purpose of this work, whether it converges to a fixed point or not. Moreover, another important question is whether a such point is a solution of a variational inequality as we shall see. This work focuses on fixed point theorems for maps defined on appropriate subsets of Hilbert spaces and satisfying a variety of conditions. A lot of convergence theorems have been obtained, more or less important from a theoretical point of view, but very important for a practical point of view.

An important subclass of the class of pseudo-contractive mappings is that of *nonexpansive* mappings, widely considered in the context of iterative methods to find a fixed point. We recall that, if C is a nonempty, closed and convex subset of an Banach space E, a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Apart from being an intuitive generalization of contractive maps, nonexpansive mappings are important, as noted by Bruck in [1], mainly for two reasons:

- Nonexpansive mappings are closely related with monotone method developed since 1960 and these constitute one of the first class of nonlinear mappings for wich fixed point theorems were obtained using only the geometric properties of Banach spaces involved, rather than compactness properties.

- Nonexpansive mappings are present in applications such as transition operators in initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $T(t)$ are, in general, multivalued and monotone.

However, it is not always guaranteed the existence of fixed points for certain maps, and even in cases in which it is guaranteed, the demonstrations are often not constructive. Thus arises the need to build iterative methods that allow, thanks to the properties of the maps involved and Banach space involved, to converge (weakly or strongly) to a specified fixed point. Of course, the convergence that we try to achieve is, where possible, that strong, as it has important practical applications. Thus, there are different types of algorithms, depending primarily on the properties of the maps involved.

This thesis studies the problems that arise in the interface between the fixed point theory for some classes of nonlinear mappings and the theory of a class of variational inequalities. In particular, the scientific contribution of this thesis about the problems mentioned above, can be summarized as follows:

- in F.Cianciaruso, G.Marino, A.Rugiano, B.Scardamaglia, *On Strong convergence of Halpern's method using averaged type mappings*, ([48], Journal of Applied Mathematics 05/2014; 2014), inspired by Iemoto and Takahashi [30], we faced the problem to study the Halpern's method to approximate strongly fixed points of a nonexpansive mapping and of a nonspreading mapping by the Halpern's method. Details are present in Chapter 5;

- in F.Cianciaruso, G.Marino, A.Rugiano, B.Scardamaglia, *On strong convergence of viscosity type method using averaged type mappings*, ([63], Journal of Nonlinear and Convex Analysis, Vol.16, Num.8 (2015), 1619-1640), we introduce a viscosity type algorithm to strongly approximate solutions of variational inequalities in the setting of Hilbert spaces. Moreover, depending on the hypothesis on the coefficients involved on the scheme, these solutions are common fixed points of a nonexpansive mapping and of a L-hybrid mapping or fixed points of one of them. Details are present in Section 5.1

- in G.Marino, B.Scardamaglia, E.Karapinar, *Strong convergence theorem for strict pseudo-contractions in Hilbert space*, ([81], Submitted for pubblications to Fixed Point Theory and Applications), inspired by Hussain, Marino et al. [70], we faced the problem to approximate strongly fixed points of strict pseudocontractive mappings, using 'the simplest' modification of the Mann algorithm. We have modified it because the original algorithm guarantees only the weak convergence. In [70] is shown that the same algorithm converges strongly to a fixed point of a nonexpansive mapping under suitable hypotheses on the coefficients. Here we give different assumptions on the coefficients, as well as the techniques of the proof. Details are present in Chapter 4

- in A.Rugiano, B.Scardamaglia, S.Wang, *Hybrid iterative algorithms for a finite family of nonexpansive mappings and for a countable family of nonspreading mappings In Hilbert spaces*, ([86], To appear in Journal of Nonlinear and Convex Analysis, Vol.16), taking inspiration again by Iemoto and Takahashi's iterative scheme for a nonexpansive mapping and a nonspreading mapping [30], we modify the iterative methods, introducing two hybrid iterative algorithms, to obtain strong convergence indeed weak convergence. Furthermore, the result holds for a finite family of nonexpansive mappings and for a countable family of nonspreading mappings in the setting of Hilbert spaces. Details are present in Section 5.1.

Chapter 1

Theoretical tools

In this chapter we collect some facts and Lemmas that will be used in studying iterative methods for fixed points. Throughout this chapter E will be a real Banach space with norm $\|.\|$. The dual space of E will be denoted by E^*, which is a Banach space itself, and where we will denote by $\langle x, x^* \rangle = x^*(x)$ the pointwise value of $x^*(x)$, with $x^* \in E^*$, $x \in E$. In the sequel, our main setting will be a Hilbert space denoted by H. By $x_n \to x$ and $x_n \rightharpoonup x$, we denote the strong and the weak convergence of (x_n) to x, respectively. The extended real line will be denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

1.1 Elements of convex analysis

We briefly introduce some notions in convex analysis which can be found in many text on the general theory of convex analysis.

Definition 1.1.1. A function $f : E \to \overline{\mathbb{R}}$ is said to be

- *proper* if its effective domain, $D(f) = \{x \in E : f(x) < \infty\}$, is nonempty;
- *convex* if
$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$
for all $\lambda \in (0,1)$ and $x, y \in D(f)$;
- *lower semicontinuous* at $x_0 \in D(f)$ if
$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$
We say that f is lower semicontinuous on $D(f)$ if it is so at every $x_0 \in D(f)$;
- *Gateaux differentiable* at $x_0 \in D(f)$ if there exists an element $f'(x_0) \in E^*$ such that
$$\lim_{t \to 0} \frac{f(x_0 + ty) - f(x_0)}{t} = \left\langle y, f'(x_0) \right\rangle, \quad \forall y \in D(f);$$

- *Frechet differentiable* at $x_0 \in D(f)$ if it is Gateaux differentiable and

$$\lim_{t\to 0}\sup_{\|y\|=1}\left|\frac{f(x_0+ty)-f(x_0)}{t}-\left\langle y, f'(x_0)\right\rangle\right|=0;$$

- *subdifferentiable* at $x_0 \in D(f)$ if there exists a functional $x^* \in E^*$, called subgradient of f at x_0, such that

$$f(x) \geq f(x_0) + \langle x - x_0, x^* \rangle, \quad \forall x \in E.$$

1.2 Classes of Banach spaces

Definition 1.2.1. A Banach space E

- is *smooth* if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1.1}$$

  exists for each $x, y \in S_E = \{v \in E : \|v\| = 1\}$;

- is uniformly smooth if the limit (1.1) is uniformly attained for $x, y \in S_E$;

- is uniformly convex if the *modulus of convexity*, $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\epsilon) := \inf\left\{1-\left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \quad \|y\| \leq 1, \quad \|x-y\| \geq \epsilon\right\},$$

  satisfies $\delta(\epsilon) > 0$ for all $\epsilon > 0$.

Definition 1.2.2. A Banach space E is said to have the *Opial property* if, whenever (x_n) is a sequence in E converging weakly to a $x_0 \in E$ and $x \neq x_0$, it follows that

$$\liminf_{n\to\infty}\|x_n - x_0\| < \liminf_{n\to\infty}\|x_n - x\|.$$

Alternatively, using the contrapositive, this condition may be written as

$$\liminf_{n\to\infty}\|x_n - x\| \leq \liminf_{n\to\infty}\|x_n - x_0\| \quad \Rightarrow \quad x = x_0.$$

It is know that every Hilbert space has the Opial property.

1.3 The metric projection on Hilbert spaces

Let C be a nonempty closed convex subset of a Hilbert space H. The *metric projection* onto C is the mapping $P_C : H \to C$ which assigns to each $x \in H$ the unique point $P_C x$ in C with the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}. \tag{1.2}$$

The following proposition characterize the metric projections (see e.g. [27]).

Proposition 1.3.1. *Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \textit{for all} \quad y \in C. \tag{1.3}$$

Proof. Let us start proving the necessary condition. Since $z = P_C x$ we can write $\|x - z\| = d(x, C)$. Then, for any $y \in C$ and $\lambda \in (0, 1)$, we have

$$\|x - y\| \le \|x - (1 - \lambda) z - \lambda y\|$$

and hence

$$\begin{aligned}\|x - z\|^2 &\le \|x - (1 - \lambda) z - \lambda y\|^2 = \|x - z + \lambda (z - y)\|^2\\ &= \|x - z\|^2 + 2\lambda \langle x - z, z - y\rangle + \lambda^2 \|z - y\|^2.\end{aligned}$$

So, we have

$$2\langle x - z, z - y\rangle \ge -\lambda \|z - y\|^2,$$

from which

$$2\langle x - z, y - z\rangle \le \lambda \|z - y\|^2.$$

Then, it follows that

$$\langle x - z, y - z\rangle \le 0 \quad \text{as} \quad \lambda \to 0.$$

In order to prove the sufficient condition, let $y \in C$. Then we have $\langle x - z, z - y\rangle \ge 0$. So, we obtain

$$\langle x - z, z - x\rangle + \langle x - z, x - y\rangle \ge 0$$

and hence

$$\|x - z\|^2 \le \langle x - z, x - y\rangle \le \|x - z\| \cdot \|x - y\|.$$

This implies $\|x - z\| \le \|x - y\|$. Therefore we have

$$\|x - z\| = d(x, C).$$

□

Taking in account the above proposition, we can prove the following.

Theorem 1.3.2. *Let C be a nonempty closed convex subset of H and let P_C be the metric projection onto C. Then the following hold:*

1. $P_C^2 = P_C$;
2. $\|P_C x - P_C y\| \leq \|x - y\|$ *for every* $x, y \in H$ *(i.e.* P_C *is nonexpansive);*
3. $x_n \rightharpoonup x_0$ *and* $P_C x_n \to y_0$ *imply* $P_C x_0 = y_0$.

Proof. (1) If $x \in C$, then $P_C x = x$; hence

$$P_C^2 z = P_C z$$

for all $z \in H$, i.e. $P_C^2 = P_C$.

(2) For every $x, y \in H$, by Proposition 1.3.1 we have

$$\langle x - y - (P_C x - P_C y), P_C x - P_C y \rangle = \langle x - P_C x, P_C x - P_C y \rangle + \langle y - P_C y, P_C y - P_C x \rangle \geq 0$$

hence

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \leq \|x - y\| \cdot \|P_C x - P_C y\|.$$

This implies that $\|P_C x - P_C y\| \leq \|x - y\|$ holds.

(3) By Proposition 1.3.1 we have

$$\langle x_n - P_C x_n, P_C x_n - z \rangle \geq 0 \quad \text{for every} \quad z \in C.$$

Since $x_n \rightharpoonup x_0$ and $P_C x_n \to y_0$, then

$$\langle x_0 - y_0, y_0 - z \rangle \geq 0 \quad \text{for every} \quad z \in C.$$

Using Proposition 1.3.1 again, we obtain $P_C(x_0) = y_0$. □

1.4 Monotone operators

Definition 1.4.1. Let $A : E \to 2^{E^*}$ be a set-valued operator with domain $D(A)$ and range $R(A)$ in E^*. The operator A is said to be

- *monotone* if for each $x, y \in D(A)$ and any $u \in A(x), v \in A(y)$,

$$\langle u - v, x - y \rangle \geq 0; \tag{1.4}$$

- *strictly monotone* if for each $x, y \in D(A)$ and any $u \in A(x), v \in A(y)$, holds the strict inequality of (1.4);
- *strongly monotone* if there exists a constant $\eta > 0$ such that for each $x, y \in D(A)$ and any $u \in A(x), v \in A(y)$,

$$\langle u - v, x - y \rangle \geq \eta \|x - y\|^2 ; \tag{1.5}$$

- *inverse strongly monotone* if there exists a constant $\nu > 0$ such that for all $x, y \in D(A)$ and any $u \in A(x), v \in A(y)$,

$$\langle x - y, u - v \rangle \geq \mu \|u - v\|^2 . \tag{1.6}$$

Definition 1.4.2. Let $A : E \to 2^E$ be a set-valued operator with domain $D(A)$ and range $R(A)$ in E. The operator A is said to be

- *accretive* if for each $u, v \in E$, $x \in A(u), y \in A(v)$,

$$\langle x - y, J(u - v) \rangle \geq 0, \tag{1.7}$$

  where J is a duality mapping of E into E^*, i.e., J is a mapping of E into E^* such that for each $u \in X$, $\|Ju\| = \|u\|$ and $\langle Ju, u \rangle = \|u\|^2$. (In Hilbert space, since J is the identity single valued mapping, this becomes the more transparent condition

$$\langle A(u) - A(v), u - v \rangle \geq 0, \tag{1.8}$$

  for all u and v in the domain of A).
- *co-accretive* if for each $u, v \in E$, $x \in A(u), y \in A(v)$,

$$\langle u - v, J(x - y) \rangle \geq 0. \tag{1.9}$$

1.5 Some lemmas about convergence

Lemma 1.5.1. *[18] Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where,

- $(\alpha_n)_{n\in\mathbb{N}} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- $\limsup_{n\to\infty} \sigma_n \leq 0$;
- $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then,

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 1.5.2. *Let C be a nonempty closed and convex subspace of H, T and S nonlinear mappings from C into itself such that $I-T$ and $I-S$ are demiclosed. Let $(y_n) \subset C$ be a bounded sequence. Then:*

1. *If $\|y_n - Ty_n\| \to 0$, then*

$$\limsup \langle \bar{p} - u, y_n - \bar{p} \rangle \geq 0,$$

where $\bar{p} = P_{Fix(T)}u$ is the unique point in $Fix(T)$ that satisfies the variational inequality

$$\langle \bar{p} - u, x - \bar{p} \rangle \geq 0 \quad \text{for all} \quad x \in Fix(T). \tag{1.10}$$

2. *If $\|y_n - Sy_n\| \to 0$, then*

$$\limsup \langle \hat{p} - u, y_n - \hat{p} \rangle \geq 0,$$

where $\hat{p} = P_{Fix(S)}u$ is the unique point in $Fix(S)$ that satisfies the variational inequality

$$\langle \hat{p} - u, x - \hat{p} \rangle \geq 0 \quad \forall x \in Fix(S)$$

.

3. If $\|y_n - Sy_n\| \to 0$ *and* $\|y_n - Ty_n\| \to 0$*, then*

$$\limsup \langle p_0 - u, y_n - p_0 \rangle \geq 0,$$

where $p_0 = P_{Fix(T)\cap Fix(S)}u$ *is the unique point in* $Fix(T) \cap Fix(S)$ *that satisfies the variational inequality*

$$\langle p_0 - u, x - p_0 \rangle \geq 0 \quad \forall x \in Fix(T) \cap Fix(S)$$

.

Proof. (1) Let $\bar{p}$ satisfying (1.10). Let (y_{n_k}) be a subsequence of (y_n) for wich

$$\limsup_n \langle \bar{p} - u, y_n - \bar{p} \rangle = \lim_k \langle \bar{p} - u, y_{n_k} - \bar{p} \rangle .$$

Select a subsequence $(y_{n_{k_j}})$ of (y_{n_k}) such that $y_{n_{k_j}} \rightharpoonup v$ (this is possible by boundedness of (y_n)). By the hypothesis $\|y_n - Ty_n\| \to 0$ and by demiclosedness of T we have $v \in Fix(T)$, and

$$\limsup_n \langle \bar{p} - u, y_n - \bar{p} \rangle = \lim_j \left\langle \bar{p} - u, y_{n_{k_j}} - \bar{p} \right\rangle = \langle \bar{p} - u, v - \bar{p} \rangle ,$$

so the claim follows by (1.10).
(2) Is the same of (1) since S is also demiclosed.
(3) Select a subsequence (y_{n_k}) of (y_n) such that

$$\limsup_n \langle p_0 - u, y_n - p_0 \rangle = \lim_k \langle p_0 - u, y_{n_k} - p_0 \rangle ,$$

where p_0 satisfies (3). Now select a subsequence $(y_{n_{k_j}})$ of (y_{n_k}) such that $y_{n_{k_j}} \rightharpoonup w$. Then by demiclosedness of T and S, and by the hypothesis $\|y_n - Ty_n\| \to 0$ and the hypothesis $\|y_n - Sy_n\| \to 0$, we obtain that $w = Tw = Sw$, i.e. $w \in Fix(T) \cap Fix(S)$. So,

$$\limsup_n \langle p_0 - u, y_n - p_0 \rangle = \lim_j \left\langle p_0 - u, y_{n_{k_j}} - p_0 \right\rangle = \langle p_0 - u, w - p_0 \rangle .$$

□

From the Lemma above we have the following.

Corollary 1.5.3. *Let* C *a nonempty closed and convex subspace of* H*,* T *a mapping from* C *into itself such that* $I - T$ *is demiclosed at 0, let* $(y_n) \subset C$ *be a bounded sequence.*

If $\|y_n - Ty_n\| \to 0$*, then*

$$\limsup_n \langle -\bar{p}, y_n - \bar{p} \rangle \leq 0,$$

where $\bar{p} = P_{Fix(T)}(0)$ *is the unique point in* $Fix(T)$ *that satisfies the variational inequality*

$$\langle -\bar{p}, x - \bar{p} \rangle \leq 0 \qquad \forall x \in Fix(T). \tag{1.11}$$

Lemma 1.5.4. *[36] Let* (x_n) *and* (y_n) *be bounded sequences in a Banach space* E *and let* $(\gamma_n) \subset [0,1]$ *be a sequence with* $0 < \liminf_n \gamma_n \leq \limsup_n \gamma_n < 1$. *Assume that* $x_{n+1} = \gamma_n y_n + (1-\gamma_n) x_n$, *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\|x_n - y_n\| \to 0$.

The following is very useful in many results of this work.

Lemma 1.5.5. *[34] Let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers such that there exists a subsequence* $(\gamma_{n_j})_{j\in\mathbb{N}}$ *of* $(\gamma_n)_{n\in\mathbb{N}}$ *such that* $\gamma_{n_j} < \gamma_{n_j+1}$, *for all* $j \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $(m_k)_{k\in\mathbb{N}}$ *of* $\mathbb{N}$ *such that* $\lim_{k\to\infty} m_k = \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$\gamma_{m_k} \leq \gamma_{m_k+1} \quad \text{and} \quad \gamma_k \leq \gamma_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1, ..., k\}$ *such that the condition* $\gamma_n < \gamma_{n+1}$ *holds.*

1.6 Facts about nonlinear mappings

Definition 1.6.1. A nonlinear mapping $T : C \to C$ with $Fix(T) \neq \emptyset$ is said to be *quasi-nonexpansive* if it is nonexpansive on the set $Fix(T)$, i.e.,

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \quad p \in Fix(T).$$

This concept which we have labeled quasi-nonexpansiveness was essentially introduced (along with some related ideas) by Diaz and Metcalf [73]. It is straightforward that a nonexpansive mapping $T : C \to C$ with at least one fixed point in C is quasi-nonexpansive; but there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings which are not nonexpansive, for example the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = \frac{x}{2}\sin\frac{1}{x}$, $x \neq 0$, and $T(0) = 0$. We have

Theorem 1.6.2. *[72] If* C *is a closed convex subset of a strictly convex normed linear space, and* $T : C \to C$ *is quasi-nonexpansive, then* $Fix(T) = \{p : p \in C \quad \text{and} \quad Tp = p\}$ *is a nonempty closed convex set on which* T *is continuous.*

Proof. It follows immediately from the definition of quasi-nonexpansiveness that $Fix(T) \neq 0$ and that T is continuous at each $p \in Fix(T)$. Suppose $Fix(T)$ is not closed. Then there is a cluster point x of $Fix(T)$ which does not lies in $Fix(T)$. Since C is closed, $x \in C$; so $x \notin Fix(T)$ implies $Tx \neq x$. Let $r = \frac{1}{3}\|Tx - x\| > 0$. There exists $y \in Fix(T)$ such that $\|x - y\| < r$. Since T is quasi-nonexpansive we have

$$\|Tx - y\| \leq \|x - y\| < r,$$

and hence we get

$$3r = \|Tx - x\| \leq \|Tx - y\| + \|y - x\| < 2r.$$

This contradiction establishes that $Fix(T)$ is closed.
We now prove that $Fix(T)$ is convex. Suppose $a, b \in Fix(T)$, $a \neq b$, and $0 < t < 1$. Then

$$c = (1 - t)\, a + tb \in C$$

since C is convex. Since T is quasi-nonexpansive we have

$$\|Tc - a\| \leq \|c - a\| \qquad and \qquad \|Tc - b\| \leq \|c - b\|.$$

Noting that $c - a = t(b - a)$ and $Tc - b = (1 - t)(a - b)$, we have

$$\|b - a\| \leq \|b - Tc\| + \|Tc - a\| \leq \|c - b\| + \|c - a\| = \|b - a\|.$$

Hence, we get

$$\|(b - Tc) + (Tc - a)\| = \|b - Tc\| + \|Tc - a\|.$$

If $b - Tc = 0$, then $\|Tc - a\| = \|b - a\| \leq \|c - a\| = t\,\|b - a\|$, whence $1 \leq t$ which is not true. Similarly, $Tc - a = 0$ implies $1 \leq 1 - t$, whence $t \leq 0$ which is not true. Thus, since the space is strictly convex, there exists $r > 0$ such that $Tc - a = r(b - Tc)$; whence $Tc = (1 - s)a + sb$ where $s = \frac{r}{l+r}$. We have $Tc - a = s(b - a)$, and so

$$s\,\|b - a\| = \|rc - a\| \leq \|c - a\| = t\,\|b - a\|,$$

which gives $s \leq t$. Using $Tc - b = (l - s)(a - b)$, a similar argument gives $s \geq t$. Thus $s = t$, and so $Tc = (l - t)a + tb = c$, i.e. $c \in Fix(T)$.

□

Demiclosedness principle

A remarkable result in the theory of nonexpansive mappings is Browder's demiclosedness principle.

Definition 1.6.3. A mapping $T : C \to E$ is said to be *demiclosed* at y if the conditions that (x_n) converges weakly to x and that (Tx_n) converges strongly to y imply that $x \in C$ and $Tx = y$. Moreover, we say thet E satisfies the *demiclosedness principle* if for any closed convex subset C of E and any nonexpansive mapping $T : C \to E$, the mapping $I - T$ is demiclosed.

The demiclosedness principle plays a relevant role about convergence of iterative method for nonexpansive mapping and other classes of nonlinear mappings as well.

Definition 1.6.4. [80] Let C be a subset of a real normed linear space X. A mapping $f : C \to X$ is said to be *demicompact* at $x \in X$ if, for any bounded sequence $(x_n) \subset C$ such that $x_n - f(x_n) \to h$ as $n \to \infty$, there exist a subsequence (x_{n_j}) and an $x \in C$ such that $x_{n_j} \to x$ as $j \to \infty$ and $x - f(x) = h$.

Definition 1.6.5. Let H be a Hilbert space. A mapping $T : H \to H$ is said to be firmly type nonexpansive [32] if for all $x, y \in D(T)$, there exists $k \in (0, +\infty)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k \|(x - Tx) - (y - Ty)\|^2 .$$

Definition 1.6.6. Let C be a closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to be strongly nonexpansive if:

1. T is nonexpansive;
2. $x_n - y_n - (Tx_n - Ty_n) \to 0$, whenever $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in C such that $(x_n - y_n)_{n\in\mathbb{N}}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$.

Definition 1.6.7. [33] Let C be a nonempty closed subset of a Hilbert space H. Then a mapping $T : C \to C$ is said to be a nonspreading mapping if:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C.$$

The following Lemma is an useful characterization of a nonspreading mapping.

Lemma 1.6.8. *[30] Let C be a nonempty closed subset of a Hilbert space H. Then a mapping $T : C \to C$ is nonspreading if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{1.12}$$

Observe that if T is a nonspreading mapping from C into itself and $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive, i.e.

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \quad \forall p \in Fix(T).$$

Lemma 1.6.9. *[28][29] Let C be a nonempty closed convex subset of H and let $T : C \to C$ be nonexpansive. Then:*

1. *$I - T : C \to H$ is $\frac{1}{2}$-inverse strongly monotone, i.e.,*

$$\frac{1}{2}\|(I - T)x - (I - T)y\|^2 \leq \langle x - y, (I - T)x - (I - T)y\rangle,$$

for all $x, y \in C$;

2. moreover, if $Fix(T) \neq \emptyset$*,* $I - T$ *is demiclosed at* 0*, i.e. for every sequence* $(x_n)_{n\in\mathbb{N}}$ *weakly convergent to* p *such that* $x_n - Tx_n \to 0$ *as* $n \to \infty$*, it follows* $p \in Fix(T)$*.*

If C is a nonempty, closed and convex subset of H and T is a nonlinear mapping of C into itself, we can define the *averaged type mapping* as follows

$$A_T = (1-\delta)I + \delta T = I - \delta(I - T) \tag{1.13}$$

where $\delta \in (0,1)$. We notice that $Fix(T) = Fix(A_T)$ and that if T is a nonexpansive mapping also A_T is nonexpansive.

If S is a nonspreading mapping of C into itself and $Fix(S) \neq \emptyset$, we observe that A_S is quasi-nonexpansive and further the set of fixed points of A_S is closed and convex. The following Lemma shows the demiclosedness of $I - S$ at 0.

Lemma 1.6.10. *[30] Let* C *be a nonempty, closed and convex subset of* H*. Let* $S : C \to C$ *be a nonspreading mapping such that* $Fix(S) \neq \emptyset$*. Then* $I - S$ *is demiclosed at* 0*.*

In the sequel we use the following property of $I - S$.

Lemma 1.6.11. *[30] Let* C *be a nonempty, closed and convex subset of* H*. Let* $S : C \to C$ *be a nonspreading mapping. Then*

$$\|(I-S)x - (I-S)y\|^2 \leq \langle x - y, (I-S)x - (I-S)y\rangle + \frac{1}{2}\bigg(\|x - Sx\|^2 + \|y - Sy\|^2\bigg),$$

for all $x, y \in C$*.*

If $Fix(S)$ is nonempty, Osilike and Isiogugu [31] proved that the averaged type mapping A_S is quasi-firmly type nonexpansive mapping, i.e. is a firmly type nonexpansive mapping on fixed points of S. On the same line of the proof in [31], we prove the following:

Proposition 1.6.12. *Let* C *be a nonempty closed and convex subset of* H *and let* $S : C \to C$ *be a nonspreading mapping such that* $Fix(S)$ *is nonempty. Then the averaged type mapping* A_S

$$A_S = (1-\delta)I + \delta S, \tag{1.14}$$

is quasi-firmly type nonexpansive mapping with coefficient $k = (1-\delta) \in (0,1)$*.*

Proof. We obtain

$$
\begin{aligned}
\|A_S x - A_S y\|^2 &= \|(1-\delta)(x-y) + \delta (Sx - Sy)\|^2 \\
\text{(by Lemma 1.6.22)} &= (1-\delta)\|x-y\|^2 + \delta \|Sx - Sy\|^2 \\
&- \delta (1-\delta)\|(x - Sx) - (y - Sy)\|^2 \\
\text{(by (1.12))} &\leq (1-\delta)\|x-y\|^2 + \delta \left[\|x-y\|^2 + 2\langle x - Sx, y - Sy\rangle\right] \\
&- \delta (1-\delta)\|(x - Sx) - (y - Sy)\|^2 \\
&= \|x-y\|^2 + \frac{2}{\delta}\langle \delta (x - Sx), \delta (y - Sy)\rangle \\
&- \frac{1-\delta}{\delta}\|\delta (x - Sx) - \delta (y - Sy)\|^2 \\
\text{(by (1.14))} &= \|x-y\|^2 + \frac{2}{\delta}\langle x - A_S x, y - A_S y\rangle \\
&- \frac{1-\delta}{\delta}\|(x - A_S x) - (y - A_S y)\|^2 \\
&\leq \|x-y\|^2 + \frac{2}{\delta}\langle x - A_S x, y - A_S y\rangle \\
&- (1-\delta)\|(x - A_S x) - (y - A_S y)\|^2 .
\end{aligned}
$$

Hence, we have

$$\|A_S x - A_S y\|^2 \leq \|x-y\|^2 + \frac{2}{\delta}\langle x - A_S x, y - A_S y\rangle - (1-\delta)\|(x - A_S x) - (y - A_S y)\|^2. \quad (1.15)$$

In particular, choosing $y = p$, where $p \in Fix(S) = Fix(A_S)$ we obtain

$$\|A_S x - p\|^2 \leq \|x - p\|^2 - (1-\delta)\|x - A_S x\|^2 . \quad (1.16)$$

□

Definition 1.6.13. [51] Let $T : H \to H$ be a mapping and $L \geq 0$ a nonnegative number. T is said L-hybrid, signified as $T \in \mathcal{H}_L$, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in H, \quad (1.17)$$

or equivalently

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\left(1 - \frac{L}{2}\right)\langle x - Tx, y - Ty\rangle. \quad (1.18)$$

Notice that for particular choices of L we obtain several important classes of nonlinear mappings. In fact

- $\mathcal{H}_0$ is the class of the nonexpansive mappings;
- $\mathcal{H}_2$ is the class of the nonspreading mappings;
- $\mathcal{H}_1$ is the class of the hybrid mappings.

Moreover

- if $Fix(T) \neq \emptyset$, each L-hybrid mapping is quasi-nonexpansive mapping (see, [51]), i.e.
$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C \quad \text{and} \quad p \in Fix(T);$$
- the set of fixed points of a quasi-nonexpansive mapping is closed and convex (see, [38]);
- if $T \in \mathcal{H}_L$, then $T_\delta := (1-\delta)I + \delta T$ belongs to $\mathcal{H}_{\frac{L}{\delta}}$ for $\delta > 0$ (see, [50]).

Lemma 1.6.14. *[28][39] Let C be a nonempty closed convex subset of H and let $T : C \to C$ be nonexpansive. Then:*

1. *$I - T : C \to H$ is $\frac{1}{2}$-inverse strongly monotone, i.e.,*
$$\frac{1}{2}\|(I-T)x - (I-T)y\|^2 \leq \langle x - y, (I-T)x - (I-T)y\rangle,$$
for all $x, y \in C$;
in particular, if $y \in Fix(T) \neq \emptyset$, we get,
$$\frac{1}{2}\|x - Tx\|^2 \leq \langle x - y, x - Tx\rangle; \tag{1.19}$$
2. *moreover, if $Fix(T) \neq \emptyset$, $I - T$ is demiclosed at 0, i.e. for every sequence $(x_n)_{n\in\mathbb{N}}$ weakly convergent to p such that $x_n - Tx_n \to 0$ as $n \to \infty$, it follows $p \in Fix(T)$.*

Lemma 1.6.15. *[51] Let C be a nonempty, closed and convex subset of H. Let $S : C \to H$ be a L-hybrid mapping such that $Fix(S) \neq \emptyset$. Then $I - S$ is demiclosed at 0.*

Lemma 1.6.16. *[30] Let C be a nonempty, closed and convex subset of H. Let $S : C \to C$ be a L-hybrid mapping. Then*

$$\begin{aligned}\|(I-S)x - (I-S)y\|^2 &\leq \langle x - y, (I-S)x - (I-S)y\rangle + \frac{1}{2}\Bigg(\|x - Sx\|^2 + \|y - Sy\|^2 \\ &- 2\left(1 - \frac{L}{2}\right)\langle x - Sx, y - Sy\rangle\Bigg),\end{aligned}$$

for all $x, y \in C$.

Proof. Put $A = I - S$. For all $x, y \in C$ we have

$$\begin{aligned} \|Ax - Ay\|^2 &= \langle Ax - Ay, Ax - Ay\rangle \\ &= \langle (x-y) - (Sx - Sy), Ax - Ay\rangle \\ &= \langle x - y, Ax - Ay\rangle - \langle Sx - Sy, Ax - Ay\rangle. \end{aligned} \tag{1.20}$$

Using , we obtain

$$\begin{aligned} 2\langle Sx - Sy, Ax - Ay\rangle &= 2\langle Sx - Sy, (x-y) - (Sx - Sy)\rangle \\ &= 2\langle Sx - Sy, x - y\rangle - 2\|Sx - Sy\|^2 \\ \text{(by Lemma (1.6.22))} \quad &\geq \|x - Sy\|^2 + \|y - Sx\|^2 - \|x - Sx\|^2 - \|y - Sy\|^2 \\ \text{(by (1.18))} \quad &- \Big(\|x - Sy\|^2 + \|y - Sx\|^2 - 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle\Big) \\ &= -\|x - Sx\|^2 - \|Sy - y\|^2 + 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle \\ &= -\|Ax\|^2 - \|Ay\|^2 + 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle. \end{aligned} \tag{1.21}$$

So, from (1.20) and (1.21), we can conclude

$$\begin{aligned} \|(I-S)x - (I-S)y\|^2 \leq\ & \langle x - y, (I-S)x - (I-S)y\rangle + \frac{1}{2}\Big(\|x - Sx\|^2 + \|y - Sy\|^2 \\ -\ & 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle\Big). \end{aligned}$$

□

Remark. If $p \in Fix(S) \neq \emptyset$, we have,

$$\|(I-S)x\|^2 \leq \langle x - p, (I-S)x\rangle + \frac{1}{2}\|(I-S)x\|^2,$$

in particular,

$$\langle x - p, (I-S)x\rangle \geq \frac{1}{2}\|(I-S)x\|^2. \tag{1.22}$$

Proposition 1.6.17. *Let C be a nonempty closed and convex subset of H and let $S : C \to C$ be a L-hybrid mapping such that $Fix(S)$ is nonempty. Then the averaged type mapping S_δ*

$$S_\delta = (1 - \delta)I + \delta S, \tag{1.23}$$

is quasi-firmly type nonexpansive mapping with coefficient $k = (1 - \delta) \in (0, 1)$.

Proof. We obtain

$$
\begin{aligned}
\|S_\delta x - S_\delta y\|^2 &= \|(1-\delta)(x-y) + \delta(Sx - Sy)\|^2 \\
\text{(by Lemma 1.6.22)} &= (1-\delta)\|x-y\|^2 + \delta\|Sx - Sy\|^2 \\
&- \delta(1-\delta)\|(x-Sx)-(y-Sy)\|^2 \\
\text{(by (1.17))} &\le (1-\delta)\|x-y\|^2 + \delta\left[\|x-y\|^2 + L\langle x-Sx, y-Sy\rangle\right] \\
&- \delta(1-\delta)\|(x-Sx)-(y-Sy)\|^2 \\
&= \|x-y\|^2 + \frac{L}{\delta}\langle \delta(x-Sx), \delta(y-Sy)\rangle \\
&- \frac{1-\delta}{\delta}\|\delta(x-Sx) - \delta(y-Sy)\|^2 \\
\text{(by (1.23))} &= \|x-y\|^2 + \frac{L}{\delta}\langle x - S_\delta x, y - S_\delta y\rangle \\
&- \frac{1-\delta}{\delta}\|(x-S_\delta x) - (y - S_\delta y)\|^2 \\
&\le \|x-y\|^2 + \frac{L}{\delta}\langle x - S_\delta x, y - S_\delta y\rangle \\
&- (1-\delta)\|(x-S_\delta x) - (y - S_\delta y)\|^2.
\end{aligned}
$$

In particular, we have

$$
\|S_\delta x - S_\delta y\|^2 \le \|x-y\|^2 + \frac{L}{\delta}\langle x - S_\delta x, y - S_\delta y\rangle - (1-\delta)\|(x-S_\delta x)-(y-S_\delta y)\|^2. \quad (1.24)
$$

Observe that S_δ is $\frac{L}{\delta}$-hybrid. Moreover, choosing in (1.24) $y = p$, where $p \in Fix(S) = Fix(S_\delta)$ we obtain

$$
\|S_\delta x - p\|^2 \le \|x-p\|^2 - (1-\delta)\|x - S_\delta x\|^2. \quad (1.25)
$$

□

Definition 1.6.18. A nonlinear mapping $T : C \to C$ is said to be $k-$strict pseudocontractive (in the sense of Browder-Petryshyn) if there exist $k \in [0,1)$ such that

$$
\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \quad (1.26)
$$

Note that the class of strict pseudocontractions includes the class of nonexpansive mappings, which are mappings T on C such that

$$
\|Tx - Ty\| \le \|x-y\| \quad \forall x, y \in C.
$$

Lemma 1.6.19. *[59] Assume C is a closed convex subset of a Hilbert space H and $T : C \to C$ be a self-mapping of C. If T is a $k-$strict pseudocontraction, then:*

1. *T satisfies the Lipschitz condition:*

$$\|Tx - Ty\| \leq \frac{1+k}{1-k} \|x - y\| ;$$

2. *the mapping $I - T$ is demiclosed at* 0*. That is, if (x_n) is a sequence in C such that $x_n \rightharpoonup \hat{x}$ and $(I-T)x_n \to 0$, then $T\hat{x} = \hat{x}$;*

3. *the set $Fix(T) = \{x \in C : Tx = x\}$ is closed and convex, so that the projection $P_{Fix(T)}$ is well defined.*

Moreover, we have the following auxiliary result:

Lemma 1.6.20. *Let be $T : C \to C$ a $k-$strict pseudocontractive self-mapping of a closed and convex subset of a Hilbert space H, and suppose that $F(T) \neq \emptyset$; then*

$$(1-k)\|Tx - x\|^2 \leq 2\langle x - p, x - Tx\rangle \quad \forall p \in Fix(T), \quad \forall x \in C \tag{1.27}$$

Proof. Let $p \in Fix(T)$. Putting $y = p$ in the definition of T, we get

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2$$

so

$$\begin{aligned} &\langle Tx - p, Tx - p\rangle \leq \langle x - p, x - Tx\rangle + \langle x - p, Tx - p\rangle + k\|x - Tx\|^2 , \\ &\Rightarrow \langle Tx - p, Tx - x\rangle \leq \langle x - p, x - Tx\rangle + k\|x - Tx\|^2 , \\ &\Rightarrow \langle Tx - x, Tx - x\rangle + \langle x - p, Tx - x\rangle \leq \langle x - p, x - Tx\rangle + k\|x - Tx\|^2 , \end{aligned}$$

from which we get (1.27). □

Definition 1.6.21. Let E be a Banach space and C a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be *pseudocontractive*, if $I - T$ is a accretive operator, where I is the identity map.

We recall also some inequalities in a Hilbert space H.

Lemma 1.6.22. *Let H be a Hilbert space. The following known results hold:*

1. $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$,
   for all $x, y \in H$ and for all $t \in [0, 1]$.

2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,
   for all $x, y \in H$.

3. $2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$,
   for all $x, y, z, w \in H$.

Chapter 2

Convergence of iterations

We recall that a mapping $T : C \to C$ is said to be a contraction if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\| < k\|x - y\|, \quad \forall x, y \in C.$$

These maps have always a single fixed point, as is ensured by the following:

Theorem 2.0.23. *(The Banach contraction mapping principle)*
Let (M, d) be a complete metric space and $T : M \to M$ be a contraction. Then T has a unique fixed point, i.e. there exists a unique $\bar{x} \in M$ such that $T\bar{x} = \bar{x}$. Moreover, for arbitrary $x_0 \in M$, the sequence (x_n) defined iteratively by $x_{n+1} = Tx_n, n \geq 0$, converges to the unique fixed point of T.

Proof. Let $k_d(T) = k$, select $x_0 \in M$ and define the iterative sequence (x_n) by $x_{n+1} = Tx_n$ (equivalently, $x_n = T^n x_0$), $n = 0, 1, 2, \ldots$ Observe that for any indices $n, p \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+p}) &= d\left(T^n x_0, T^{n+p} x_0\right) = d\left(T^n x_0, T^n \circ T^p x_0\right) \leq k\left(T^n\right) d\left(x_0, T^p x_0\right) \\ &\leq k^n \left[d\left(x_0, Tx_0\right) + d\left(Tx_0, T^2 x_0\right) + \ldots + d\left(T^{p-1} x_0, T^p x_0\right)\right] \\ &\leq k^n \left(1 + k + \ldots + k^{p-1}\right) d\left(x_0, Tx_0\right) \\ &\leq k^n \left(\frac{1 - k^p}{1 - k}\right) d\left(x_0, Tx_0\right). \end{aligned} \tag{2.1}$$

This shows that (x_n) is a Cauchy sequence, and since M is complete there exists $x \in M$ such that $\lim_n x_n = x$. To see that x is the unique fixed point of T observe that

$$x = \lim_n x_n = \lim_n x_{n+1} = \lim_n Tx_n = Tx$$

and, moreover, $x = Tx$ and $y = Ty$ imply

$$d(x, y) = d(Tx, Ty) \leq kd(x, y),$$

yielding $d(x, y) = 0$.
Letting $p \to +\infty$ in (2.1) yields

$$d(x_n, x) = d(T^n x_0, x) \leq \frac{k^n}{1-k} d(x_0, Tx_0). \tag{2.2}$$

□

Remark 2.0.24. The *a priori* estimate

$$d(x_{n+p}, x_n) \leq \frac{k^n}{1-k} d(x_1, x_0) \tag{2.3}$$

shows that, when starting from an initial guess $x_0 \in M$, the approximation error of the n^{th} iterate is completely determined by the contraction constant k and the initial displacement $d(x_1, x_0)$.
Similarly, the *a posteriori* estimate

$$d(x_n, x) \leq \frac{k}{1-k} d(x_{n-1}, x_n) \tag{2.4}$$

shows that, in order to obtain the desired error approximation of the fixed point by means of Picard iteration, that is, to have $d(x_n, x) < \epsilon$, we need to stop the iterative process at the step N for which the displacement between two consecutive iteratives is at most $\frac{1-k}{k}\epsilon$.
So, the a posteriori estimation offers a direct stopping criterion for the iterative approximation of fixed points by Picard iterations, while the a priori estimation indirectly gives a stopping criterion. It is easy to see that a posteriori estimation is better then the a priori one, in the sense that from (2.4) we can obtain (2.3) in the sense of

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0). \tag{2.5}$$

Each of the estimations (2.3), (2.4), (2.5), shows that the rate of convergence of the Picard iteration is at least as quick as that of the geometric series

$$\sum_{n \in \mathbb{N}} k^n$$

Remark 2.0.25. An analysis of the proof reveals that the assumption $k(T) < 1$ is stronger than necessary. It suffices to assume $k(T^n) < 1$ for at least one fixed $n \in \mathbb{N}$. This implies that T^n is a contraction and, by the theorem above, has a unique fixed point x. But

$$Tx = T^{n+1}x = T^n \circ Tx,$$

so Tx is also a fixed point of T^n. Hence $x = Tx$, proving x is also a fixed point of T (and the unique one).

As can be deduced from the last observation, to ensure the existence of a fixed point for nonexpansive mappings we need additional assumptions. In this regard, we show in the following, the well known Browder-Gohde-Kirk theorem ([75],[76],[77]).

Theorem 2.0.26. *Let C be a nonempty closed convex and bounded subset of a uniformly convex Banach space E, and let T be a nonexpansive mapping from C into itself. Then T has a fixed point.*

A very nice proof of the theorem above is present in [78].

Unlike the case of Banach contraction principle, some examples show that the sequence of successive approximations

$$x_{n+1} = Tx_n, n \geq 0$$

(so called *Picard iterations*), for a nonexpansive mapping $T : C \to C$ (where C is a nonempty convex closed and bounded subset of a Banach space) for which we assume that have a fixed point, it may not converge to this fixed point. More precisely, we have the following example:

Example 2.0.27. *Let $B := x \in \mathbb{R}^2 : \|x\| \leq 1$ and let T be the counterclockwise rotation of $\frac{\pi}{4}$ around the origin of the coordinates. Then T is nonexpansive and have the origin as unique fixed point. In addition, the sequence (x_n) defined by $x_{n+1} = Tx_n$, where $x_0 = (1, 0) \in B, n \geq 0$, does not converge to zero.*

However Krasnoselskii, in [3], showed that in this example, we can obtain a convergent sequence of successive approximations if we substitute the nonexpansive mapping T with the auxiliary nonexpansive mapping $\frac{1}{2}(I + T)$, where I denotes the identical transformation of the plan, i.e., if the sequence of successive approximations is defined by $x_0 \in C$,

$$x_{n+1} = \frac{1}{2}\left(x_n + Tx_n\right), \qquad n = 0, 1, ... \tag{2.6}$$

instead of the usual Picard iteration (in which at second member appears only the quantity Tx_n). It can be shown as a simple exercise that the mappings T and $\frac{1}{2}(I + T)$ have the same set of fixed points, therefore the limit of the convergent sequence defined by (2.6) is necessarily a fixed point of T. More generally, if X is a normed linear space, and C is a convex subset of X, a generalization of equation (2.6) that 'works' for approximating fixed points (when they exist) of nonexpansive mappings $T : C \to C$, is the following method: $x_0 \in C$,

$$x_{n+1} = (1 - \lambda)\, x_n + \lambda T x_n, \quad n = 0, 1, ...; \lambda \in (0, 1) \tag{2.7}$$

(see, e.g., Schaefer [4]). In this way were proved the following.

Theorem 2.0.28. *Let C be a bounded closed and convex subset of a uniformly convex Banach space E, and let T be a nonexpansive and demicompact mapping from C into itself. For any given $x_0 \in C$ and any fixed number λ with $0 < \lambda < 1$, the Krasnoselski iteration given by*

$$x_{n+1} = (1-\lambda)\, x_n + \lambda T x_n, \quad n = 0, 1, ...$$

converges strongly to a fixed point of T.

With this type of iterations, except in special cases, it is difficult or in such cases not possible to give an a priori estimate of the speed of convergence of the iterates. In this regard, it is more convenient to compare the speed of convergence of different iterative methods. However Oblomskaya [5], working on reflexive Banach spaces, shows an example in which the convergence is slower than $n^{-\alpha}, \alpha < 1$.
Edelstein [6], improves the works of Schaefer and Krasnoselskii, weakening the geometric properties of the initial space, i.e., he prove that the uniform convexity of the Banach space can be replaced by strict convexity. We recall that a strictly convex Banach space is a Banach space that for $\|x\| = \|y\| = 1$ and $x \neq y$ then $\|x + y\| < 2$. Anyway, a kind of iteration even more general that we can consider is the following

$$x_0 \in C, \quad x_{n+1} = (1 - c_n)\, x_n + c_n T x_n, \tag{2.8}$$

where $(c_n) \subset (0, 1)$ is a real sequence that satisfies appropriate conditions (see for example [9],[10]).
This type of algorithm is also known as Mann-Dotson process [7],[8], and is one of the most analyzed in the literature.

The important and intuitive question of when the strict convexity can be removed from the assumptions remained unanswered for many years. In 1967, it was resolved in the affirmative by the following.

Theorem 2.0.29. *(Ishikawa,[9]) Let C be a subset of a Banach space X and let T be a nonexpansive mapping from C to X. Taking an initial value $x_0 \in C$, one define the sequence (x_n) as in (2.8), where the real sequence (c_n) satisfies:*

1. *$\sum_{n=0}^{+\infty} c_n$ diverges;*
2. *$0 \leq c_n \leq b < 1$ for all $n \geq 1$;*
3. *$x_n \in C$ for all $n \geq 0$.*

If (x_n) is a bounded sequence, then $x_n - Tx_n \to 0$ for $n \to +\infty$.

A consequence of Theorem 2.0.29 is that if C is convex and compact, the sequence (x_n) defined by (2.8) converges strongly to a fixed point of T. Another consequence of

Theorem 2.0.29 is that for a convex C, and T from C into a bounded subset of X, the iterates of the mapping

$$S_\lambda = \lambda I + (1-\lambda)\,T, \quad \lambda \in (0,1)$$

, are *asymptotically regular at* x, i.e.

$$\left\| S_\lambda^{n+1}x - S_\lambda^n x \right\| \to 0 \quad for \quad n \to +\infty$$

. The concept of asymptotic regularity was introduced by Browder and Petryshyn [11]. The importance of the asymptotic regularity about the existence of a fixed point for T can be reasoned by the following theorem.

Theorem 2.0.30. *Let M be a metric space and $T : M \to M$ a continuous mapping which is asymptotically regular at x_0. Then every cluster point of $(T^n x_0)$ is a fixed point of T.*

It follows that for continuous mappings T, the asymptotic regularity of S_λ at some x_0 implies $S_\lambda(p) = p$ for some cluster point p of $(S_\lambda^n x_0)_{n\in\mathbb{N}}$. Therefore, the asymptotic regularity of a sequence, it is useful not only to prove that there are fixed points but also to show that in certain cases, the sequence of iterates at a point converges to a fixed point. Now we consider the following example (for details see [55]).

Example 2.0.31. *There is a closed bounded and convex set C in the Hilbert space l_2, a nonexpansive self-map T of C and a point $x_0 \in C$ such that $\left(S_{\frac{1}{2}}^n x_0\right)$ does not converge in the norm topology.*

While example of Genel and Lindenstrauss shows that we cannot, in general, get strong convergence of the sequence defined by

$$x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T x_n, \quad n = 1, 2, ...$$

to a fixed point of T, Theorem 2.0.32 allows us to conclude that the sequence converges weakly to a fixed point of T if E have the Opial property.

Theorem 2.0.32. *[10] Let E be a space which have the Opial property and let T a nonexpansive mapping of a weakly compact convex subset C of E in itself. Then, for any $x \in C$, the sequence $(S_\lambda^n x)$ converges weakly to a fixed point of T.*

Remark 2.0.33. The fixed point to which the Mann iterative process converges depends, in general, on the initial approximation x_0, as well as on the sequence c_n that determine the Mann iteration. Moreover, the Mann iteration need not converge to the fixed point of T 'nearest' x_0, as shown by the following example.

Example 2.0.34. *Let X the space $\mathbb{R}^2$ equipped with the Euclidean norm, and with (r,θ) denoting the polar coordinates. Let*

$$C := \left\{(r,\theta) : \quad 0 \le r \le 1, \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right\}.$$

The mapping $T : C \to C$ defined by

$$T(r,\theta) = \left(r, \frac{\pi}{2}\right),$$

for each point (r,θ) in C, is nonexpansive as it is easily to see, and the set of its fixed points is the line segment

$$Fix(T) = \left\{\left(r, \frac{\pi}{2}\right) : \quad 0 \le r \le 1\right\}.$$

Take $x_0 = (r_0, \theta_0) = (1, \frac{\pi}{2})$ and $c_n \in [0,1]$ for $n \ge 1$, and construct the Mann sequence (x_n) by

$$x_{n+1} = (1 - c_n)\, x_n + c_n T x_n, \quad n \ge 0$$

which gives

$$x_{n+1} = (r_{n+1}, \theta_{n+1}) = \left(r_n, \theta_n + c_n\left(\frac{\pi}{2} - \theta_n\right)\right).$$

Hence $r_n = r_0 = 1$ for all $n \ge 0$ and

$$\theta_{n+1} = c_n \frac{\pi}{2} + (1 - c_n)\,\theta_n, \quad n \ge 0 \quad \text{and} \quad \theta_0 = \frac{\pi}{4}.$$

(1) *For $c_n = 1$ we get $\theta_n = \frac{\pi}{2}$, $n \ge 0$ and so*

$$x_n \to \left(1, \frac{\pi}{2}\right) \in Fix(T)$$

which is not the nearest fixed point of T to x_0, because the nearest one is the point $p = \left(\frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)$.

(2) *The same happens when $c_n = \frac{1}{2}$, when we find*

$$\theta_n = \frac{\pi}{2^{n+2}} + \frac{\pi}{2}\frac{2^{n-1}}{2^n}, \quad n \ge 0$$

and hence

$$x_n \to \left(1, \frac{\pi}{2}\right) \in Fix(T)$$

which is also not the nearest fixed point of T to x_0.

(3) *For $c_n = 0$, we get $\theta_n = \frac{\pi}{4}$ and hence*

$$\lim_{n\to\infty} x_n = \left(1, \frac{\pi}{4}\right) \notin C.$$

An important class of mapping generalizing the class of nonexpansive mappings is the class of *Lipschitz pseudocontractive* maps, which are mappings that are both Lipschitz and both pseudocontractive. It is not difficult to check that every nonexpansive mapping is a Lipschitz pseudocontraction. All attempts to use the Mann formula, which has been successfully employed for nonexpansive mappings, to approximate fixed point of a Lipschitz pseudocontractive mapping, proved abortive. In 1974, Ishikawa [83] proved the following theorem.

Theorem 2.0.35. *Let C be a nonempty compact convex subset of a real Hilbert space H and $T : K \to K$ be a Lipschitz pseudocontractive mapping. Let the sequence (x_n) be defined by $x_0 \in C$,*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T y_n \tag{2.9}$$

$$y_n = (1 - \beta_n)\, x_n + \beta_n T x_n, \quad n \geq 1 \tag{2.10}$$

where (α_n) and (β_n) are real sequences satisfying the following conditions:

1. $0 \leq \alpha_n \leq \beta_n < 1$ *for all* $n \geq 1$*;*
2. $\sum_{n \in \mathbb{N}} \alpha_n \beta_n = \infty$*;*
3. $\lim_n \beta_n = 0$.

Then (x_n) converges strongly to a fixed point of T.

Remark 2.0.36. It is clear that the recursion formulas (2.9) and (2.10) of the Ishikawa scheme are more cumbersome than the Mann formula. However, since it was not known whether or not the simpler Mann sequence would always converge to fixed points of Lipschitz pseudocontractive mappings, the sophisticated Ishikawa algorithm was applied for this class of mappings. The question of whether or not the simpler Mann sequence had actually failed for this class of mappings remained open for many years. This was resolved in 2001 by Chidume and Mutangadura [85] who produced an example of a Lipschitz pseudocontractive mapping defined on a compact convex subset of $\mathbb{R}^2$ with a unique fixed point for which no Mann sequence converges.

Remark 2.0.37. We first observe that if we set $\beta_n = 0$ for all n in the recursion formula (2.10) then condition (1) in Theorem 2.0.35 shows that $\alpha_n = 0$ for all n and so (2.9) and (2.10) reduce to $x_{n+1} = x_n$ for all n. Therefore (x_n) converges to x_0, the initial approximation which may not be a fixed point of T. Since the Ishikawa formulas were used successfully in approximating a fixed point of T in Theorem 2.0.35, several authors started studying a modification of it in which condition (1) i replaced by the condition (1)*: $0 \leq \alpha_n, \beta_n < 1$, and condition (2) is modified accordingly. In thi modification, α_n and β_n are independent and it is permissible to set $\beta_n = 0$ for all n. They still called such a modified formula an Ishikawa-type formula. But this is questionable. In fact, to see this, it suffices to set $\beta_n = 0$ for all n and see that the sequence obtained from the modified scheme will not converge to a fixed point of T in Theorem 2.0.35. In particular, if $\beta_n = 0$ for all n, the modified formula generally reduces to the Mann formula and then the example of Chidume and Mutangadura [85] shows that the modified formula will not converge to a fixed point of T in the setting of Theorem 2.0.35.

Remark 2.0.38. The order of convergence of the Picard sequence is that of a geometric progression, that of the Mann sequence is of the form $O(\frac{1}{n})$, while that of the Ishikawa sequence is of the form $O(\frac{1}{\sqrt{n}})$. Furthermore, whenever Picard sequence converges, it is preferred to the Mann sequence which itself is preferred to the Ishikawa formula whenever it converges, because the preferred recursion formula is simpler (consequently, it requires less computation and therefore reducing cost of computation).

Chapter 3

Applications

3.1 Variational inequality problems

In this paragraph we assume that the involving space H is a Hilbert space and $C \subset H$ is a closed convex set. Given a monotone operator $A : H \to H$, the variational inequality problem $VIP(A, C)$ consists of finding $p \in C$ such that

$$\langle Ap, p - x \rangle \leq 0, \quad \forall x \in C. \tag{3.1}$$

Variational inequalities were initially studied by Stampacchia [43] and there after the problem of existence and uniqueness of solutions of $VIP(A, C)$ has been widely investigated by many authors in different disciplines as partial differential equations, optimal control, optimization, mechanics and finance.

If $f : H \to \overline{\mathbb{R}}$ is a lower semicontinuous convex function, a necessary and sufficient condition for the constrained convex minimization problem

$$\min_{x \in C} f(x), \tag{3.2}$$

is the $VIP(A, C)$, where the operator A is the subdifferential of f. This means that solving the minimization problem (3.2) is equivalent to finding a solution of a variational inequality. Thus, in order to solve a broad rouge of convexly constrained nonlinear inverse problems in real Hilbert spaces, Yamada [13] presented an hybrid steepest descent method for approximating solutions to the variational inequality problem $VIP(g, Fix(T))$, for an operator g and the fixed point set of a nonexpansive mapping $T : C \to C$. In particular, he proved that, when g is strongly monotone and Lipschitz continuous, the sequence (x_n) defined by the algorithm

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n), \quad n \geq 0, \tag{3.3}$$

converges strongly to the fixed point q of T, which is the unique solution to the inequality

$$\langle g(q), x - q\rangle \geq 0 \quad \forall x \in Fix(T). \tag{3.4}$$

Let us consider now the following particular variational inequality problem. Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$, $\Psi : H \to H$ be a contraction and A be a Lipschitz self operator on H wich is strongly monotone. Then the $VIP(A - \gamma\Psi, Fix(T))$

$$\langle (A - \gamma\Psi) q, q - x\rangle \leq 0, \quad \forall x \in Fix(T), \tag{3.5}$$

where $\gamma > 0$ is the optimality condition for the minimization problem

$$\min_{x \in Fix(T)} f(x) - h(x),$$

where f is a differentiable function with subdifferential $\partial f = A$ and h is a potential function for $\gamma\Psi$ (i.e. $h^{'}(x) = \gamma\Psi(x)$ for $x \in H$). Marino and Xu [52] presented an iterative method to solve the $VIP(A - \gamma\Psi, Fix(T))$ for a linear bounded operator:

Theorem 3.1.1. *Let H be a Hilbert space, let A be a bounded linear operator on H, and let T be a nonexpansive mapping on H. Assume the set $Fix(T)$ of fixed points of T is nonempty, and assume that the operator A is strongly positive with constant $\overline{\gamma}$. Let f be a α-contraction on H and let γ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let (x_n) generate by the algorithm*

$$x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), \quad n \geq 0,$$

where α_n is a sequence in $(0, 1)$ that satisfying the following conditions:

- $\alpha_n \to 0$;
- $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
- *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ *or* $\frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then (x_n) converges strongly to a fixed point $\overline{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f) \overline{x}, \overline{x} - z\rangle \leq 0, \quad z \in Fix(T).$$

3.2 Multiple-set split feasibility problem

The multiple-sets split feasibility problem requires finding a point closest to a family of closed convex sets in one space such that its image under a linear transformation will be closest to another family of closed convex sets in the image space. It can be a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range. It generalizes the convex feasibility problem as well as the two-sets split feasibility problem. This problem referred as the multiple-sets split feasibility problem (MSSFP) is formulated as follows:

$$\text{Find an } x \in \bigcap_{i=1}^{N} X_i \quad \text{such that} \quad Ax \in \bigcap_{j=1}^{M} H_j,$$

which was first investigated by Censor et al. [20]. There are a great deal of literature on the MSSFP, see e.g., [20], [21], [22], [23].
Next, we focus on the multiple-set split feasibility problem (MSSFP) which is to find a point x^* such that

$$x^* \in C = \bigcap_{i=1}^{N} C_i \text{ and } Ax^* \in Q = \bigcap_{j=1}^{M} Q_j \tag{3.6}$$

where $N, M \geq 1$ are integers, the $C_i (i = 1, 2, \cdots, N)$ are closed convex subsets of H_1, the $Q_j (j = 1, 2, \cdots, M)$ are closed convex subsets of H_2 and and $A : H_1 \to H_2$ is a bounded linear operator. Assume the MSSFP is consistent, i.e., it is solvable, and S denotes its solution set. The case where $N = M = 1$, called split feasibility problem (SFP), was introduced by Censor and Elfving [24], modeling phase retrieval and other image restoration problems, and further studied by many researchers.

We use Γ to denote the solution set of the SFP. Let $\gamma > 0$ and assume that $x^* \in \Gamma$. Thus, $Ax^* \in Q_1$ which implies the equation $(I - P_{Q_1})Ax^* = 0$ which in turns implies the equation $\gamma A^*(I - P_{Q_1})Ax^* = 0$, hence the fixed point equation $(I - \gamma A^*(I - P_{Q_1})A)x^* = x^*$. Requiring that $x^* \in C_1$, we consider the fixed point equation:

$$P_{C_1}(I - \gamma A^*(I - P_{Q_1})A)x^* = x^*. \tag{3.7}$$

We will see that solutions of the fixed point equation (3.7) are exactly solutions of the SFP. The following proposition is due to Byrne [25] and Xu [26].

Proposition 3.2.1. *Given $x^* \in H_1$. Then x^* solves the SFP if and only if x^* solves the fixed point equation (3.7).*

This proposition reminders us that (MSSFP) (3.6) is equivalent to a common fixed point problem of finitely many nonexpansive mappings, as we show below.

Decompose (MSSFP) into N subproblems $(1 \leq i \leq N)$:

$$x_i^* \in C_i \quad \text{and} \quad Ax_i^* \in Q := \bigcap_{j=1}^{M} Q_j. \tag{3.8}$$

For each $1 \leq i \leq N$, we define a mapping T_i by

$$T_i x = P_{C_i}(I - \gamma_i \nabla f)x = P_{C_i}\left(I - \gamma_i \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A\right)x \tag{3.9}$$

where f is defined by

$$f(x) = \frac{1}{2}\sum_{j=1}^{M} \beta_j \|Ax - P_{Q_j}Ax\|^2$$

with $\beta_j > 0$ for all $1 \le j \le M$. Note that the gradient of ∇f is

$$\nabla f(x) = \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Ax$$

which is L-Lipschitz continuous with constant

$$L = \sum_{j=1}^{M} \beta_j \|A\|^2. \tag{3.10}$$

It is known that if $0 < \gamma_i \le 2/L$, T_i is nonexpansive. Therefore fixed point algorithms for nonexpansive mappings can be applied to (MSSFP) (3.6).

The Picard iteration for MSSFP:

$$\begin{aligned} x_{n+1} &= T_N \cdots T_1 x_n \\ &= P_{C_N}\left(I - \gamma\sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})A\right) \cdots P_{C_1}\left(I - \gamma\sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})A\right)x_n \end{aligned} \tag{3.11}$$

Theorem 3.2.2. *([22]) Assume that the MSSFP (3.6) is consistent. Let $\{x_n\}$ be the sequence generated by the Algorithm 3.11, where $0 < \gamma < 2/L$ with L given by (3.10). Then $\{x_n\}$ converges weakly to a solution of the MSSFP (3.6).*

Note that the above algorithm only have weak convergence. Here, we will show an algorithm with strong convergence

The Halpern iteration for MSSFP:

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1 - \alpha_n)T_n x_n \\ &= \alpha_n u + (1 - \alpha_n)P_{C[n+1]}\left(I - \gamma\sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})A\right)x_n, n \ge 0. \end{aligned} \tag{3.12}$$

Theorem 3.2.3. *Assume that the MSSFP (3.6) is consistent, $0 < \gamma < 2/L$ with L given by (3.10), and $\{\alpha_n\}$ satisfies the conditions (for instance, $\alpha_n = 1/n$ for all $n \ge 1$)*

$(C1)$: $\lim_{n\to\infty} \alpha_n = 0$,

$(C2)$: $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and*

$(C3)$: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ generated by the Algorithm 3.12 converges strongly to a solution of the MSSFP (3.6) that is closest to u from the solution set of the MSSFP (3.6).

Chapter 4

Results for Mann's type iterations

As we mentioned earlier, the problem of finding fixed points of nonexpansive mappings via Mann's algorithm [7]-[8] has been widely investigated in literature (see e.g.[60]). Mann's algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive formula

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1, \tag{4.1}$$

where $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1)$.
If $T : C \to C$ is a nonexpansive mapping with a fixed point in a closed and convex subset of a uniformly convex Banach space with a Frechet differentiable norm, and if the control sequence $(\alpha_n)_{n\in\mathbb{N}}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence (x_n) generated by Mann's algorithm converges weakly to a fixed point of T [60]. However, this convergence is in general not strong (there is a counterexample in [55]).

On the other hand, iterative algorithm for strict-pseudocontractions, are still less developed than those for nonexpansive mappings, despite to the fact that the pioneering work of Browder and Petryshyn [69] dating 1967. However, strict pseudo-contractions have many applications, do in solving inverse problems [68] and their ties with inverse strongly monotone operators [67].
Marino and Xu [59] proved that the Mann algorithm, has weak convergence also in the broader setting of strict pseudo-contractions mapping, containing the nonexpansive mappings:

Theorem 4.0.4. *(Marino and Xu [59], 2007, Mann method)*
Let C be a closed and convex subset of a Hilbert space H. Let $T : C \to C$ be a k-strict pseudo-contraction for some $0 \leq k < 1$. Assume that T admits a fixed point in C. Let (x_n) be the sequence generated by $x_0 \in C$ and the Mann's algorithm

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\, Tx_n.$$

Assume that the control sequence (α_n) *is chosen so that* $k < \alpha_n < 1$ *for all* n *and*

$$\sum_{n=0}^{+\infty} (\alpha_n - k)(1 - \alpha_n) = +\infty.$$

Then (x_n) *converges to a fixed point of* T.

It is not possible, in general, to obtain strong convergence, thanks to the celebrated counterexample of Genel and Lindenstrauss [55].
So, to obtain strong convergence, one can try to modify the Mann's algorithm and strengthen the hypotheses on the mapping.
We recall here some obtained results:

Theorem 4.0.5. *(Li et al. [66], 2013, Modified Halpern method)*
Let C *be a closed and convex subset of a real Hilbert space* H, $T : C \to C$ *be a* k*-strict pseudo-contraction such that* $Fix(T) \neq \emptyset$. *For an arbitrary initial value* $x_0 \in C$ *and fixed anchor* $u \in C$, *define iteratively a sequence* (x_n) *as follows:*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n,$$

where (α_n), (β_n), (γ_n) *are three real sequence in* $(0,1)$ *satisfying* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $0 < k < \frac{\beta_n}{\beta_n + \gamma_n}$. *Suppose that* (α_n) *satisfies the conditions:*

$$\lim_{n\to\infty} \alpha_n = 0 \quad , \qquad \sum_{n=1}^{+\infty} \alpha_n = +\infty.$$

Then (x_n) *converges strongly to* $x^* = P_{Fix(T)} u$, *where* $P_{Fix(T)}$ *is the metric projection from* H *onto* $Fix(T)$.

Theorem 4.0.6. *(Marino and Xu [59], 2007, CQ method)*
Let C *be a closed convex subset of a Hilbert space* H. *Let* $T : C \to C$ *be a* k*-strict pseudo-contraction for some* $0 \leq k < 1$ *and assume that* $Fix(T) \neq \emptyset$. *Let* (x_n) *be the sequence generated by the following (CQ) algorithm :*

$$\begin{cases} x_0 \in C \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)(k - \alpha_n) \|x_n - T x_n\|^2 \right\} \\ Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \} \\ x_{n+1} = P_{C_n \cap D_n} x_0. \end{cases}$$

Assume that the control sequence (α_n) *is chosen so that* $\alpha_n < 1$ *for all* n. *Then* (x_n) *converges strongly to* $P_{Fix(T)} x_0$.

Theorem 4.0.7. *(Shang [65], 2007, Viscosity method)*
Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strict pseudo-contraction with $Fix(T) \neq \emptyset$. Let $f : C \to C$ be a contraction. The initial value $x_0 \in C$ is chosen arbitrarily, and given sequences (α_n) and (β_n) satisfying the following conditions:

1. $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{+\infty} \alpha_n = +\infty$;
2. $0 < a < \beta_n < \gamma$ *for some* $a \in (0, \gamma]$ *and* $\gamma = \min\{1, 2k\}$;
3. $\sum_{n=1}^{+\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ *and* $\sum_{n=1}^{+\infty} |\beta_{n+1} - \beta_n| < +\infty$.

Let (x_n) be the composite process defined by

$$\begin{cases} y_n = (1 - \beta_n)\, x_n + \beta_n T x_n \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\, y_n. \end{cases}$$

Then (x_n) converges strongly to a fixed point $p \in Fix(T)$.

Theorem 4.0.8. *(Osilike and Udomene [64], 2001, Ishikawa type method)*
Let H be a Hilbert space. Let C be a nonempty closed convex subset of H, $T : C \to C$ a demicompact k-strict pseudo-contraction with
$Fix(T) \neq \emptyset$. Let (α_n) and (β_n) be real sequences in $[0, 1]$ satisfying the following conditions:

1. $0 < a < \alpha_n \leq b < (1 - k)(1 - \beta_n) \quad \forall n \geq 1$ *and for some costants* $a, b \in (0, 1)$;
2. $\sum_{n=1}^{+\infty} \beta_n < +\infty$.

Then the sequence (x_n) genereted from an arbitrary $x_1 \in K$ by the Ishikawa iteration method

$$\begin{cases} y_n = (1 - \beta_n)\, x_n + \beta_n T x_n \\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases}$$

converges strongly to a fixed point of T.

The recalled results are probably neither the most general, nor the more recent, but certainly they represent very well some of the different modifications of the original Mann's approximation method, made to get strong convergence.
We like to point out that the differences with the original method are remarkable. So it is quite surprising that recently, in [70] was obtained a strong convergence method for nonexpansive mappings that is 'almost' the Mann's method (the difference is given only by a smaller and smaller amount). In [70] was proved the convergence of this method only for nonexpansive mappings:

Theorem 4.0.9. *(Hussain, Marino et al. [70], 2015)*
Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping. Let $(\alpha_n), (\mu_n)$ be sequences in $(0, 1]$ such that

- $\lim_{n\to\infty} \alpha_n = 0$;
- $\sum_{n=1}^{+\infty} \alpha_n\mu_n = +\infty$;
- $|\mu_{n+1} - \mu_n| = o(\mu_n)$;
- $|\alpha_{n+1} - \alpha_n| = o(\alpha_n\mu_n)$.

Then the sequence (x_n) generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n - \alpha_n\mu_n x_n$$

strongly converges to a point $x^* \in Fix(T)$ with minimum norm

$$\|x^*\| = \min_{x\in Fix(T)} \|x\| .$$

We would like to emphasize that:

1. In general, the mapping T cannot be defined on a closed convex subset C of H, since x_{n+1} is not a convex combination of two elements in C. However, since we can write
$$x_{n+1} = \alpha_n (1 - \mu_n) x_n + (1 - \alpha_n) T x_n,$$
then x_{n+1} is meaning full if $T : C \to C$ is a self-mapping defined on a cone C, that is a particular convex set, closed with respect to linear combinations with positive coefficients.

2. The proof of Theorem 4.0.9 use easily the properties of nonexpanive mappings and cannot be adjusted to the strict pseudo-contractive mappings. Purpose of the present paper is to show that the result is true also for strict pseudo-contractions. The proof uses completely different techniques, as well as the assumptions on coefficients. For all we know, this is the algorithm most similar (and the most easy to implement) to the original iterative Mann's method, providing strong convergence.

3. Our techniques can also be used to clarify the proofs of main results in [48] and [63].

Now we can prove our theorem. We use the notation $\omega_l(x_n)$ to denote the set of weak limit points of (x_n).

Theorem 4.0.10. *Let H be a Hilbert space and let C be a nonempty closed cone of H. Let $T : C \to C$ be a $k-$strict pseudo-contractive mapping such that $Fix(T) \neq \emptyset$. Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are real sequences respectively in $(k, 1)$ and in $(0, 1)$ satisfying the conditions:*

1. $k < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
2. $\lim_{n\to\infty} \mu_n = 0$,
3. $\sum_{n=1}^{\infty} \mu_n = \infty$,

we define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$x_1 \in C, \qquad x_{n+1} = \alpha_n (1 - \mu_n) x_n + (1 - \alpha_n) T x_n \quad n \in \mathbb{N}. \tag{4.2}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\bar{x} \in F(T)$, that is the unique solution of the variational inequality

$$\langle -\bar{x}, y - \bar{x} \rangle \leq 0, \qquad \forall y \in F(T).$$

Proof. We begin to prove that $(x_n)_{n\in\mathbb{N}}$ is bounded.

First of all, observe that from the conditions $\mu_n \to 0$ and $k < \liminf \alpha_n \leq \limsup \alpha_n < 1$, it follows that there exist an integer $n_0 \in \mathbb{N}$ such that

$$\mu_n \leq 1 - \frac{k}{\alpha_n} \qquad \forall n \geq n_0,$$

i.e.

$$k - \alpha_n (1 - \mu_n) \leq 0. \tag{4.3}$$

Let be $p \in F(T)$ and put $r = \max \{\|x_{n_0} - p\|, \|p\|\}$. We have

$$\begin{aligned} x_{n+1} - p &= \alpha_n [(1 - \mu_n) x_n - p] + (1 - \alpha_n) [T x_n - p] \\ &= \alpha_n [(1 - \mu_n)(x_n - p) + \mu_n (-p)] + (1 - \alpha_n) [T x_n - p], \end{aligned}$$

Regarding lemma 1.6.22 (ii), we derive that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \alpha_n \|(1-\mu_n)(x_n-p)+\mu_n(-p)\|^2 + (1-\alpha_n)\|Tx_n-p\|^2 \\
&- \alpha_n(1-\alpha_n)\|(1-\mu_n)x_n - Tx_n\|^2 \\
&\le \alpha_n\left[(1-\mu_n)\|x_n-p\|^2 + \mu_n\|p\|^2 - \mu_n(1-\mu_n)\|x_n\|^2\right] \\
&+ (1-\alpha_n)\left[\|x_n-p\|^2 + k\|x_n-Tx_n\|^2\right] \\
&- \alpha_n(1-\alpha_n)\|(1-\mu_n)(x_n-Tx_n)+\mu_n(-Tx_n)\|^2 \\
&= \alpha_n\left[(1-\mu_n)\|x_n-p\|^2 + \mu_n\|p\|^2 - \mu_n(1-\mu_n)\|x_n\|^2\right] \\
&+ (1-\alpha_n)\left[\|x_n-p\|^2 + k\|x_n-Tx_n\|^2\right] \\
&- \alpha_n(1-\alpha_n)\left[(1-\mu_n)\|x_n-Tx_n\|^2 + \mu_n\|Tx_n\|^2 - \mu_n(1-\mu_n)\|x_n\|^2\right] \\
&\le \alpha_n(1-\mu_n)\|x_n-p\|^2 + \alpha_n\mu_n\|p\|^2 + (1-\alpha_n)\|x_n-p\|^2 \\
&+ (1-\alpha_n)k\|x_n-Tx_n\|^2 - \alpha_n(1-\alpha_n)(1-\mu_n)\|x_n-Tx_n\|^2 \\
&= (1-\alpha_n\mu_n)\|x_n-p\|^2 + \alpha_n\mu_n\|p\|^2 + (1-\alpha_n)\left[k-\alpha_n(1-\mu_n)\right]\|x_n-Tx_n\|^2 \\
\text{(from 4.3)} &\le (1-\alpha_n\mu_n)\|x_n-p\|^2 + \alpha_n\mu_n\|p\|^2 \\
&\le \max\left\{\|x_n-p\|^2, \|p\|^2\right\} \le \max\left\{\|x_{n_0}-p\|^2, \|p\|^2\right\} = r^2.
\end{aligned}$$

Thus, we conclude that the sequence (x_n) is bounded.

Now we shall prove that, for $p \in Fix(T)$:

$$\begin{aligned}
(1-\alpha_n)(\alpha_n-k)\|x_n-Tx_n\|^2 &\le \left(\|x_n-p\|^2 - \|x_{n+1}-p\|^2\right) \\
&- 2\alpha_n\mu_n\langle x_n, x_{n+1}-p\rangle. \qquad (4.4)
\end{aligned}$$

Regarding (4.2) , we easily observe that

$$\begin{aligned}
x_{n+1}-p &= \alpha_n(1-\mu_n)x_n + (1-\alpha_n)Tx_n - p \\
&= \left[1-(1-\alpha_n(1-\mu_n))\right]x_n + (1-\alpha_n)Tx_n - p \\
&= (x_n-p) - (1-\alpha_n)(x_n-Tx_n) - \alpha_n\mu_n x_n,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \text{(by 1.6.22)} \le \|(x_n-p)-(1-\alpha_n)(x_n-Tx_n)\|^2 - 2\alpha_n\mu_n\langle x_n, x_{n+1}-p\rangle \\
&= \|x_n-p\|^2 - 2(1-\alpha_n)\langle x_n-Tx_n, x_n-p\rangle \\
&+ (1-\alpha_n)^2\|x_n-Tx_n\|^2 - 2\alpha_n\mu_n\langle x_n, x_{n+1}-p\rangle \\
\text{(from 1.27)} &\le \|x_n-p\|^2 + (1-\alpha_n)(k-\alpha_n)\|x_n-Tx_n\|^2 - 2\alpha_n\mu_n\langle x_n, x_{n+1}-p\rangle,
\end{aligned}$$

so (4.4) is proved. Moreover, since $\alpha_n \in (k, 1)$:

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - 2\alpha_n \mu_n \langle x_n, x_{n+1} - p \rangle.$$

Now we prove the strong convergence of (x_n) concerning two cases:

CASE 1 suppose that $\|x_n - p\|$ is monotone non increasing. Then $\|x_n - p\|$ converges and hence

$$\lim_{n\to\infty} \|x_{n+1} - p\|^2 - \|x_n - p\|^2 = 0$$

From this and from the assumptions $\lim_n \mu_n = 0$, and
$k < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$, by (4.4) we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0;$$

from this and boundedness of (x_n), thanks to demiclosedness of $I - T$ we deduce $\omega_l(x_n) \subseteq Fix(T)$.

Now we put

$$z_n = \alpha_n x_n + (1 - \alpha_n) T x_n = (1 - (1 - \alpha_n)) x_n + (1 - \alpha_n) T x_n$$

from which we have

$$z_n - x_n = (1 - \alpha_n)(Tx_n - x_n). \tag{4.5}$$

Hence, we find that

$$\begin{aligned} x_{n+1} &= z_n - \alpha_n \mu_n x_n \\ &= (1 - \alpha_n \mu_n) z_n + \alpha_n \mu_n (z_n - x_n) \\ \text{(from (4.5))} &= (1 - \alpha_n \mu_n) z_n + \alpha_n \mu_n (1 - \alpha_n)(Tx_n - x_n). \end{aligned} \tag{4.6}$$

Let $\bar{x} = P_{Fix(T)}(0) \in Fix(T)$ the unique solution of the variational inequality

$$\langle -\bar{x}, y - \bar{x} \rangle \leq 0 \quad \forall y \in Fix(T). \tag{4.7}$$

From definition of z_n,

$$\begin{aligned} \|z_n - \bar{x}\|^2 &= \|x_n - \bar{x} - (1 - \alpha_n)(x_n - Tx_n)\|^2 \\ &= \|x_n - \bar{x}\|^2 - 2(1 - \alpha_n)\langle x_n - Tx_n, x_n - \bar{x} \rangle + (1 - \alpha_n)\|x_n - Tx_n\|^2 \\ \text{(from(1.27))} &\leq \|x_n - \bar{x}\|^2 - (1 - \alpha_n)[(1 - k) - (1 - \alpha_n)]\|x_n - Tx_n\|^2 \\ &\leq \|x_n - \bar{x}\|^2. \end{aligned} \tag{4.8}$$

So,

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= (\text{from } (4.6)) = \|(1-\alpha_n\mu_n)\, z_n + \alpha_n\mu_n\,(1-\alpha_n)\,(Tx_n - x_n) - \bar{x}\|^2 \\
&= \|(1-\alpha_n\mu_n)\,(z_n-\bar{x}) + \alpha_n\mu_n\,[(1-\alpha_n)\,(Tx_n-x_n)-\bar{x}]\|^2 \\
(\text{from lemma } 1.6.22) &\le (1-\alpha_n\mu_n)^2\,\|z_n-\bar{x}\|^2 + 2\alpha_n\mu_n\,\langle(1-\alpha_n)\,(Tx_n-x_n)\,, x_{n+1}-\bar{x}\rangle \\
&+ 2\alpha_n\mu_n\,\langle-\bar{x}, x_{n+1}-\bar{x}\rangle \\
(\text{from } (4.8)) &\le (1-\alpha_n\mu_n)\,\|x_n-\bar{x}\|^2 \\
&+ 2\alpha_n\mu_n\,((1-\alpha_n)\,\langle Tx_n-x_n, x_{n+1}-\bar{x}\rangle + \langle-\bar{x}, x_{n+1}-\bar{x}\rangle) \qquad (4.9)
\end{aligned}
$$

Now, since (x_n) is bounded and $\omega_l(x_n) \subseteq Fix(T)$, there exists an appropriate subsequence $x_{n_k} \rightharpoonup p_0 \in Fix(T)$ such that

$$
\limsup_n \langle-\bar{x}, x_{n+1}-\bar{x}\rangle = \lim_k \langle-\bar{x}, x_{n_k}-\bar{x}\rangle = \langle-\bar{x}, p_0-\bar{x}\rangle \le 0. \qquad (4.10)
$$

From this, it follows that all the hypothesis of Lemma 1.5.1 are satisfied and finally by (4.9) we can conclude

$$
\lim_{n\to\infty}\|x_n-\bar{x}\| = 0.
$$

Let now $\bar{x} \in Fix(T)$ defined by the variational inequality (4.7).

CASE 2 If $\|x_n-\bar{x}\|$ does not be monotone non-increasing, there exist a subsequence (x_{n_k}) such that $\|x_{n_k}-\bar{x}\| < \|x_{n_k+1}-\bar{x}\|\ \forall k \in \mathbb{N}$. So by Lemma 5.33 $\exists\tau(n)\uparrow+\infty$ such that

1. $\left\|x_{\tau(n)}-\bar{x}\right\| < \left\|x_{\tau(n)+1}-\bar{x}\right\|$
2. $\left\|x_{n)}-\bar{x}\right\| < \left\|x_{\tau(n)+1}-\bar{x}\right\|$

Now, we have

$$
\begin{aligned}
0 &\le \liminf_n \left(\left\|x_{\tau(n)+1}-\bar{x}\right\| - \left\|x_{\tau(n)}-\bar{x}\right\|\right) \\
&\le \limsup_n \left(\left\|x_{\tau(n)+1}-\bar{x}\right\| - \left\|x_{\tau(n)}-\bar{x}\right\|\right) \\
&\le \limsup_n \left(\|x_{n+1}-\bar{x}\| - \left\|x_{n)}-\bar{x}\right\|\right) \\
&\le \limsup_n (\|x_n-\bar{x}\| + \sqrt{\mu_n}M - \|x_n-\bar{x}\|) = 0.
\end{aligned}
$$

Thus, we derive that

$$
\left\|x_{\tau(n)+1}-\bar{x}\right\|^2 - \left\|x_{\tau(n)}-\bar{x}\right\|^2 \longrightarrow 0,
$$

from which

$$
\left\|x_{\tau(n)} - Tx_{\tau(n)}\right\| \longrightarrow 0. \qquad (4.11)
$$

Now, from (4.9), we get that

$$
\begin{aligned}
\left\|x_{\tau(n)+1}-\bar{x}\right\|^2 \leq & \left(1-\alpha_{\tau(n)}\mu_{\tau(n)}\right)\left\|x_{\tau(n)}-\bar{x}\right\|^2 \\
& + 2\alpha_{\tau(n)}\mu_{\tau(n)}\left(1-\alpha_{\tau(n)}\right)\left\langle Tx_{\tau(n)}-x_{\tau(n)}, x_{\tau(n)+1}-\bar{x}\right\rangle \\
& + 2\alpha_{\tau(n)}\mu_{\tau(n)}\left\langle -\bar{x}, x_{\tau(n)+1}-\bar{x}\right\rangle \\
& = \left\|x_{\tau(n)}-\bar{x}\right\|^2 + 2\alpha_{\tau(n)}\mu_{\tau(n)}\left(1-\alpha_{\tau(n)}\right)\left\langle Tx_{\tau(n)}-x_{\tau(n)}, x_{\tau(n)+1}-\bar{x}\right\rangle \\
& + 2\alpha_{\tau(n)}\mu_{\tau(n)}\left\langle -\bar{x}, x_{\tau(n)+1}-\bar{x}\right\rangle \\
& - 2\alpha_{\tau(n)}\mu_{\tau(n)}\left(\frac{\left\|x_{\tau(n)}-\bar{x}\right\|^2}{2}\right). \qquad (4.12)
\end{aligned}
$$

Putting in (4.12)

$$
\begin{aligned}
A_{\tau(n)} = & \left(1-\alpha_{\tau(n)}\right)\left\langle Tx_{\tau(n)}-x_{\tau(n)}, x_{\tau(n)+1}-\bar{x}\right\rangle \\
& + \left\langle -\bar{x}, x_{\tau(n)+1}-\bar{x}\right\rangle - \frac{\left\|x_{\tau(n)}-\bar{x}\right\|^2}{2},
\end{aligned}
$$

we have

$$
\left\|x_{\tau(n)+1}-\bar{x}\right\|^2 \leq \left\|x_{\tau(n)}-\bar{x}\right\|^2 + 2\alpha_{\tau(n)}\mu_{\tau(n)}A_{\tau(n)} \qquad (4.13)
$$

Notice that we can not use Lemma 1.5.1 as in the case 1 (or in [48], [63]) since we could not guarantee that $\sum_{n=1}^{+\infty}\mu_{\tau(n)} = +\infty$. So, we change reasoning. Assume by contradiction that $\left\|x_{\tau(n)}-\bar{x}\right\|$ does not converge to 0. Then there exist (n_j) and a $\epsilon > 0$ such that

$$
\left\|x_{\tau(n_j)}-\bar{x}\right\| \geq 2\epsilon. \qquad (4.14)
$$

By (4.10) and (4.11) we know that there exist $n_0, n_1 \in \mathbb{N}$ such that

$$
\left(1-\alpha_{\tau(n)}\right)\left\langle Tx_{\tau(n)}-x_{\tau(n)}, x_{\tau(n)+1}-\bar{x}\right\rangle < \frac{\epsilon}{3} \quad \forall n \geq n_0 \qquad (4.15)
$$

and

$$
\left\langle -\bar{x}, x_{\tau(n)+1}-\bar{x}\right\rangle < \frac{\epsilon}{3} \quad \forall n \geq n_1. \qquad (4.16)
$$

Hence, if we take $n_{j_o} \geq \max\{n_0, n_1\}$ one obtain by definition of $A_{\tau(n)}$,

$$
A_{\tau(n)} < \frac{\epsilon}{3} + \frac{\epsilon}{3} - \epsilon = -\frac{\epsilon}{3} < 0, \quad \forall n \geq n_{j_0}.
$$

So, by (4.13) we have $\left\|x_{\tau(n)+1}-\bar{x}\right\|^2 \leq \left\|x_{\tau(n)}-\bar{x}\right\|^2$ that contradicts $\left\|x_{\tau(n)}-\bar{x}\right\| < \left\|x_{\tau(n)+1}-\bar{x}\right\| \quad \forall n$. This implies that

$$
\left\|x_{\tau(n)}-\bar{x}\right\| \longrightarrow 0,
$$

and so, using $\|x_n - \bar{x}\| < \left\|x_{\tau(n)+1} - \bar{x}\right\|$, we finally obtain

$$\|x_n - \bar{x}\| \longrightarrow 0.$$

□

Example 4.0.11. *The mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = -2x$ is $\frac{1}{3}-$strict pseudo-contractive. Taking $\alpha_n = \frac{1}{2}$, $\mu_n = \frac{1}{n}$, our algorithm becomes*

$$x_{n+1} = -\frac{1}{2}\frac{n+1}{n}x_n$$

that goes to $0 = Fix(T)$ swinging around it.

OPEN QUESTIONS

1. Holds the result in Banach spaces?
2. Holds the result for families of strict pseudo-contractive mappings?
3. Holds the result for Lipschitzian pseudo-contractive mappings?

Chapter 5

Results for Halpern's type iterations

Let E be a real Banach space, C a closed convex subset of E and $T : K \to K$ a nonexpansive mapping. For fixed $t \in (0, 1)$ and arbitrary $u \in C$, let $z_t \in K$ denote the unique fixed point of T_t defined by $T_t x := tu + (1 - t)Tx$, $x \in C$. Assume $Fix(T) \neq \emptyset$. Browder [10] proved that if $E = H$, a Hilbert space, then $\lim_{t\to 0} z_t$ exists and is a fixed point of T. Reich [11] extended this result to uniformly smooth Banach spaces. Kirk [12] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range. For a sequence (α_n) in $[0, 1]$ and an arbitrary $u \in C$, let the sequence (x_n) in C be iteratively defined by $x_0 \in C$,

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0. \tag{5.1}$$

Concerning this process, Reich posed the following question.
Question. Let E be a Banach space. Is there a sequence (α_n) such that whenever a weakly compact convex subset C of E has the fixed point property for nonexpansive mappings, then the sequence (x_n) defined by (5.1) converges to a fixed point of T for arbitrary fixed $u \in C$ and all nonexpansive $T : C \to C$?
Halpern [14] was the first to study the convergence of the algorithm (5.1) in the framework of Hilbert spaces. He proved the following theorem.

Theorem 5.0.12. *Let C be a bounded, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. For any initialization $x_1 \in C$ and anchor $u \in C$, define a sequence (x_n) in C by*

$$x_{n+1} = n^{-\theta} u + (1 - n^{-\theta}) T x_n, \quad \forall n \geq 1,$$

where $\theta \in (0, 1)$. Then (x_n) converges strongly to the element of $Fix(T)$ nearest to u.

An iteration method with recursion formula of the form (5.1) is now referred to as a Halpern-type iteration method. Lions [15] improved the theorem above, still in Hilbert spaces, by proving strong convergence of (x_n) to a fixed point of T if the real sequence (α_n)satisfies the following conditions:

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
3. $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0$.

Reich gave an affirmative answer to the above question in the case when E is uniformly smooth and $\alpha_n = n^{-a}$, with $0 < a < 1$. It was observed that both Halpern's and Lions conditions on the real sequence (α_n) excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [16] who proved, still in Hilbert spaces, the strong convergence of (x_n) if (α_n) satisfies the following conditions:

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
3. $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Reich [17] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly continuous duality maps (e.g. $l_p, 1 < p < \infty$), where the sequence (α_n) is required to satisfy the Wittmann's conditions and to be decreasing.

Xu [18],[19] showed that the result of Halpern holds in uniformly smooth Banach spaces if condition (3) of Lions is replaced with condition $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

Remark. Halpern showed that the conditions

$$(1) \quad \lim_{n\to\infty} \alpha_n = 0 \quad , \quad (2) \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$

are necessary for the convergence of the sequence (x_n) defined by (5.1). It is not know if generally they are sufficient. Some authors have established that if in the recursion formula (5.1), Tx_n is replaced with

$$T_n x_n := \left(\frac{1}{n}\right) \sum_{k=0}^{n-1} T^k x_n,$$

then Halpen's conditions (1) and (2) are sufficient.

Iemoto and Takahashi [30] approximated common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space using Moudafi's iterative scheme [45]. They obtained the following Theorem that states the weak convergence of their iterative method:

Theorem 5.0.13. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Assume that $Fix(S) \cap Fix(T) \neq \emptyset$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:*

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n S x_n + (1-\beta_n)T x_n], \end{cases}$$

for all $n \in \mathbb{N}$, where $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} \subset [0,1]$. Then, the following hold:

(i) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $p \in Fix(S)$;

(ii) If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $p \in Fix(T)$;

(iii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $p \in Fix(S) \cap Fix(T)$.

Here, we show an iterative method of Halpern's type to approximate strongly fixed points of a nonexpansive mapping T and a nonspreading mapping S. A crucial tool to prove the strong convergence of our iterative scheme is the use of averaged type mappings A_T and A_S which have a regularizing role.

Theorem 5.0.14. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a nonspreading mapping such that $Fix(S) \cap Fix(T) \neq \emptyset$. Let A_T and A_S be the averaged type mappings, i.e.*

$$A_T = (1-\delta)I + \delta T,\ A_S = (1-\delta)I + \delta S,\ \delta \in (0,1).$$

Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ is a real sequence in $(0,1)$ satisfying the conditions:

1. $\lim_{n\to\infty} \alpha_n = 0$,
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$, we define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n A_T x_n + (1-\beta_n)A_S x_n], \quad n \in \mathbb{N}. \end{cases}$$

Then, the following hold:

(i) If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$*, then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $p \in Fix(T)$*;*

(ii) If $\sum_{n=1}^{\infty}\beta_n < \infty$*, then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $p \in Fix(S)$*;*

(iii) If $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$*, then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $p \in Fix(T)\cap Fix(S)$*.*

Proof. We begin to prove that $(x_n)_{n\in\mathbb{N}}$ is bounded.
Put

$$U_n = \beta_n A_T + (1-\beta_n)A_S. \tag{5.2}$$

Notice that U_n is quasi-nonexpansive, for all $n \in \mathbb{N}$.
For $q \in Fix(T)\cap Fix(S)$, we have

$$\begin{aligned}\|x_{n+1}-q\| &= \|\alpha_n(u-q)+(1-\alpha_n)(U_nx_n-q)\|\\ &\le \alpha_n\|u-q\|+(1-\alpha_n)\|U_nx_n-q\|\\ &\le \alpha_n\|u-q\|+(1-\alpha_n)\|x_n-q\| \end{aligned}\tag{5.3}$$

Since

$$\|x_1-q\| \le \max\{\|u-q\|,\|x_1-q\|\},$$

and by induction we assume that

$$\|x_n-q\| \le \max\{\|u-q\|,\|x_1-q\|\},$$

then

$$\begin{aligned}\|x_{n+1}-q\| &\le \alpha_n\|u-q\|+(1-\alpha_n)\max\{\|u-q\|,\|x_1-q\|\}\\ &\le \alpha_n\max\{\|u-q\|,\|x_1-q\|\}+(1-\alpha_n)\max\{\|u-q\|,\|x_1-q\|\}\\ &= \max\{\|u-q\|,\|x_1-q\|\}.\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is bounded. Consequently, $(A_Tx_n)_{n\in\mathbb{N}}$, $(A_Sx_n)_{n\in\mathbb{N}}$ and $(U_nx_n)_{n\in\mathbb{N}}$ are bounded as well.
Proof of (i) We introduce an auxiliary sequence

$$z_{n+1} = \alpha_n u + (1-\alpha_n)A_Tx_n,\ n \in \mathbb{N}$$

and we study its properties and the relationship with the sequence $(x_n)_{n\in\mathbb{N}}$.
We shall divide the proof into several steps.
Step 1. $\lim_{n\to\infty}\|x_n - z_n\| = 0$.
Proof of Step 1. Observe that

$$\lim_{n\to\infty}\|z_{n+1}-A_Tx_n\| = \lim_{n\to\infty}\alpha_n\|u-A_Tx_n\| = 0. \tag{5.4}$$

Then we get

$$\begin{aligned}\|z_{n+1}-x_{n+1}\| &= \|\alpha_n u+(1-\alpha_n)A_Tx_n-\alpha_n u-(1-\alpha_n)Ux_n\| \\ &= (1-\alpha_n)\|A_Tx_n-Ux_n\| \\ &= (1-\alpha_n)\|A_Tx_n-\beta_nA_Tx_n-(1-\beta_n)A_Sx_n\| \\ &= (1-\alpha_n)(1-\beta_n)\|A_Tx_n-A_Sx_n\|. \end{aligned} \tag{5.5}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, we have

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{5.6}$$

So, also $(z_n)_{n\in\mathbb{N}}$ is bounded.
Step 2. $\lim_{n\to\infty}\|z_n-A_Tz_n\|=0$.
Proof of Step 2. We begin to prove that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.
Let $p\in Fix(T)=Fix(A_T)$. We have

$$\begin{aligned}\|z_{n+1}-p\|^2 &= \|\alpha_n u+(1-\alpha_n)(1-\delta)x_n+(1-\alpha_n)\delta Tx_n-p\|^2 \\ &= \|[(1-\alpha_n)\delta(Tx_n-x_n)+x_n-p]+\alpha_n(u-x_n)\|^2 \\ (\text{ by Lemma 1.6.22}) \quad &\leq \|(1-\alpha_n)\delta(Tx_n-x_n)+x_n-p\|^2 \\ &+ 2\alpha_n\langle u-x_n,z_{n+1}-p\rangle \\ &\leq (1-\alpha_n)^2\delta^2\|Tx_n-x_n\|^2+\|x_n-p\|^2 \\ &- 2(1-\alpha_n)\delta\langle x_n-p,x_n-Tx_n\rangle \\ &+ 2\alpha_n\|u-x_n\|\|z_{n+1}-p\| \\ &= (1-\alpha_n)^2\delta^2\|x_n-Tx_n\|^2+\|x_n-p\|^2 \\ ((I-T)p=0) \quad &- 2(1-\alpha_n)\delta\langle x_n-p,(I-T)x_n-(I-T)p\rangle \\ &+ 2\alpha_n\|u-x_n\|\|z_{n+1}-p\| \\ (\text{ by Lemma 1.6.9}) \quad &\leq \|x_n-p\|^2+(1-\alpha_n)^2\delta^2\|x_n-Tx_n\|^2 \\ &- (1-\alpha_n)\delta\|(I-T)x_n-(I-T)p\|^2 \\ &+ 2\alpha_n\|u-x_n\|\|z_{n+1}-p\| \\ &= \|x_n-p\|^2-(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Tx_n\|^2 \\ &+ 2\alpha_n\|u-x_n\|\|z_{n+1}-p\| \end{aligned}$$

and hence

$$\begin{aligned}&(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Tx_n\|^2-2\alpha_n\|u-x_n\|\|z_{n+1}-p\| \\ \leq\ &\|x_n-p\|^2-\|z_{n+1}-p\|^2.\end{aligned}$$

Set

$$L_n = (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 - 2\alpha_n\|u - x_n\|\|z_{n+1} - p\|;$$

let us consider the following two cases.
a) If $L_n \le 0$, for all $n \ge n_0$ large enough, then

$$(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \le 2\alpha_n\|u - x_n\|\|z_{n+1} - p\|.$$

So, since $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty}(1-\alpha_n)\delta[1-\delta(1-\alpha_n)] = \delta(1-\delta)$,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

b) Assume now that that there exists a subsequence $(L_{n_k})_{k\in\mathbb{N}}$ of $(L_n)_{n\in\mathbb{N}}$ taking all its positive terms; so $L_{n_k} > 0$ for every $k \in \mathbb{N}$, then

$$0 < L_{n_k} \le \|x_{n_k} - p\|^2 - \|z_{n_k+1} - p\|^2. \tag{5.7}$$

Summing (5.7) from $k = 1$ to N, we obtain

$$\begin{aligned}
\sum_{k=1}^{N} L_{n_k} &\le \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1}\big(\|x_{n_k+1} - p\|^2 - \|z_{n_k+1} - p\|^2\big) - \|z_{n_N+1} - p\|^2 \\
&\le \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1}\big(\|x_{n_k+1} - p\| + \|z_{n_k+1} - p\|\big)\|x_{n_k+1} - z_{n_k+1}\| \\
&\le \|x_{n_1} - p\|^2 \\
\text{(by (5.5))} \quad &+ \sum_{k=1}^{N-1}(1-\alpha_{n_k})(1-\beta_{n_k})\big(\|x_{n_k+1} - p\| + \|z_{n_k+1} - p\|\big)\|A_T x_{n_k} - A_S x_{n_k}\| \\
&\le \|x_{n_1} - p\|^2 + K\sum_{k=1}^{N-1}(1-\beta_{n_k}),
\end{aligned}$$

where $K = \sup_{k\in\mathbb{N}}\big\{(\|x_{n_k+1} - p\| + \|z_{n_k+1} - p\|)\|A_T x_{n_k} - A_S x_{n_k}\|\big\}$.
Since $\sum_{k=1}^{\infty}(1-\beta_{n_k}) < \infty$,

$$\sum_{k=1}^{\infty}\big((1-\alpha_{n_k})\delta[1-\delta(1-\alpha_{n_k})]\|x_{n_k} - Tx_{n_k}\|^2 - 2\alpha_{n_k}\|u - x_{n_k}\|\|z_{n_k+1} - p\|\big) < \infty.$$

Thus,

$$\lim_{k\to\infty}\big((1-\alpha_{n_k})\delta[1-\delta(1-\alpha_{n_k})]\|x_{n_k} - Tx_{n_k}\|^2 - 2\alpha_{n_k}\|u - x_{n_k}\|\|z_{n_k+1} - p\|\big) = 0,$$

and since $\lim_{k\to\infty} \alpha_{n_k} = 0$ and $\lim_{k\to\infty}(1-\alpha_{n_k})\delta[1-\delta(1-\alpha_{n_k})] = \delta(1-\delta)$, also in this case we get

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

Since the remanent terms of the sequence $\|x_n - Tx_n\|$ are not positive, from the case a) we can conclude that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Consequently,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - A_T x_n\| &= \lim_{n\to\infty} \|x_n - (1-\delta)x_n - \delta T x_n\| \\ &= \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \end{aligned} \tag{5.8}$$

Furthermore, from (5.6) and (5.8) and

$$\begin{aligned} \|z_n - A_T z_n\| &\le \|z_n - x_n\| + \|x_n - A_T x_n\| + \|A_T x_n - A_T z_n\| \\ &\le \|z_n - x_n\| + \|x_n - A_T x_n\| + \|x_n - z_n\| \end{aligned}$$

we get

$$\lim_{n\to\infty} \|z_n - A_T z_n\| = 0. \tag{5.9}$$

Now, define the real sequence

$$t_n = \sqrt{\|z_n - A_T z_n\|}, \quad n \in \mathbb{N}. \tag{5.10}$$

Let $z_{t_n} \in C$ be the unique fixed point of the contraction V_{t_n} defined su C by

$$V_{t_n} x = t_n u + (1-t_n)A_T x. \tag{5.11}$$

From Browder's Theorem, $\lim_{n\to\infty} z_{t_n} = p_0 \in Fix(A_T)$; now we prove that:
Step 3. $\limsup_{n\to\infty} \langle u - p_0, z_n - p_0\rangle \le 0$.
Proof of Step 3. From (5.11), we have

$$z_{t_n} - z_n = t_n(u - z_n) + (1-t_n)(A_T z_{t_n} - z_n).$$

We compute

$$\begin{aligned}
\|z_{t_n} - z_n\|^2 &= \|t_n(u - z_n) + (1-t_n)(A_T z_{t_n} - z_n)\|^2 \\
\text{(by Lemma 1.6.22)} \quad &\leq (1-t_n)^2\|A_T z_{t_n} - z_n\|^2 + 2t_n\langle u - z_n, z_{t_n} - z_n\rangle \\
&\leq (1-t_n)^2\big(\|A_T z_{t_n} - A_T z_n\| + \|A_T z_n - z_n\|\big)^2 \\
&+ 2t_n\langle u - z_n, z_{t_n} - z_n\rangle \\
&= (1-t_n)^2\big[\|A_T z_{t_n} - A_T z_n\|^2 + \|A_T z_n - z_n\|^2 \\
&+ 2\|A_T z_n - z_n\|\|A_T z_{t_n} - A_T z_n\|\big] \\
&+ 2t_n\langle u - z_{t_n}, z_{t_n} - z_n\rangle + 2t_n\langle z_{t_n} - z_n, z_{t_n} - z_n\rangle \\
(A_T \text{ nonexpansive}) \quad &\leq (1-t_n)^2\big[\|z_{t_n} - z_n\|^2 + \|A_T z_n - z_n\|^2 \\
&+ 2\|A_T z_n - z_n\|\|z_{t_n} - z_n\|\big] \\
&+ 2t_n\|z_{t_n} - z_n\|^2 + 2t_n\langle u - z_{t_n}, z_{t_n} - z_n\rangle \\
&= (1+t_n^2)\|z_{t_n} - z_n\|^2 \\
&+ \|A_T z_n - z_n\|\big(\|A_T z_n - z_n\| + 2\|z_{t_n} - z_n\|\big) \\
&+ 2t_n\langle u - z_{t_n}, z_{t_n} - z_n\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle u - z_{t_n}, z_n - z_{t_n}\rangle &\leq \frac{t_n}{2}\|z_{t_n} - z_n\|^2 \\
&+ \frac{\|A_T z_n - z_n\|}{2t_n}\big(\|A_T z_n - z_n\| + 2\|z_{t_n} - z_n\|\big).
\end{aligned}$$

From (5.10) and by the boundedness of $(z_{t_n})_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ and $(A_T z_n)_{n\in\mathbb{N}}$ we have

$$\limsup_{n\to\infty}\langle u - z_{t_n}, z_n - z_{t_n}\rangle \leq 0. \tag{5.12}$$

Furthermore,

$$\begin{aligned}
\langle u - z_{t_n}, z_n - z_{t_n}\rangle &= \langle u - p_0, z_n - z_{t_n}\rangle + \langle p_0 - z_{t_n}, z_n - z_{t_n}\rangle \\
&= \langle u - p_0, z_n - p_0\rangle + \langle u - p_0, p_0 - z_{t_n}\rangle + \langle p_0 - z_{t_n}, z_n - z_{t_n}\rangle
\end{aligned} \tag{5.13}$$

Since $\lim_{n\to\infty} z_{t_n} = p_0 \in Fix(A_T)$, we get

$$\lim_{n\to\infty}\langle p_0 - z_{t_n}, z_n - z_{t_n}\rangle = \lim_{n\to\infty}\langle u - p_0, p_0 - z_{t_n}\rangle = 0. \tag{5.14}$$

We conclude from (5.12), (5.13) and (5.14)

$$\limsup_{n\to\infty}\langle u - p_0, z_n - p_0\rangle \leq 0. \tag{5.15}$$

Step 4. $(z_n)_{n\in\mathbb{N}}$ converges strongly to $p_0 \in Fix(T)$.
Proof of Step 4. We compute

$$\begin{aligned}
\|z_{n+1}-p_0\|^2 &= \|\alpha_n(u-p_0)+(1-\alpha_n)(A_Tx_n-p_0)\|^2 \\
\text{(by Lemma 1.6.22)} &\leq (1-\alpha_n)^2\|A_Tx_n-p_0\|^2+2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle \\
(A_T \text{ nonexpansive}) &\leq (1-\alpha_n)\|x_n-p_0\|^2+2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle \\
&\leq (1-\alpha_n)\big(\|x_n-z_n\|+\|z_n-p_0\|\big)^2 \\
&+ 2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle \\
&\leq (1-\alpha_n)\|x_n-z_n\|^2+(1-\alpha_n)\|z_n-p_0\|^2 \\
&+ 2(1-\alpha_n)\|x_n-z_n\|\|z_n-p_0\| \\
&+ 2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle \\
&\leq (1-\alpha_n)\|z_n-p_0\|^2 \\
\text{(by (5.5))} &+ (1-\alpha_n)(1-\alpha_{n-1})^2(1-\beta_{n-1})^2\|A_Tx_{n-1}-A_Sx_{n-1}\|^2 \\
&+ 2(1-\alpha_n)(1-\alpha_{n-1})(1-\beta_{n-1})\|A_Tx_{n-1}-A_Sx_{n-1}\|\|z_n-p_0\| \\
&+ 2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle \\
&\leq (1-\alpha_n)\|z_n-p_0\|^2+(1-\beta_{n-1})\|A_Tx_{n-1}-A_Sx_{n-1}\|^2 \\
&+ 2(1-\beta_{n-1})\|A_Tx_{n-1}-A_Sx_{n-1}\|\|z_n-p_0\| \\
&+ 2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle \\
&\leq (1-\alpha_n)\|z_n-p_0\|^2+M(1-\beta_{n-1}) \\
&+ 2\alpha_n\langle u-p_0, z_{n+1}-p_0\rangle,
\end{aligned}$$

where $M := \sup_{n\in\mathbb{N}}\big\{\|A_Tx_{n-1}-A_Sx_{n-1}\|^2+2\|A_Tx_{n-1}-A_Sx_{n-1}\|\|z_n-p_0\|\big\}$.
Since by hypothesis $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, from (5.15) we can apply Lemma 1.5.1 and conclude that

$$\lim_{n\to\infty}\|z_{n+1}-p_0\|=0.$$

By $\lim_{n\to\infty}\|x_n-z_n\|=0$, we have

$$\lim_{n\to\infty}\|x_n-p_0\|=0.$$

Hence, $(x_n)_{n\in\mathbb{N}}$ converges strongly to $p_0 \in Fix(T)$. □

Proof of (ii)
Again we introduce an other auxiliary sequence

$$s_{n+1}=\alpha_n u+(1-\alpha_n)A_Sx_n, \tag{5.16}$$

and we study its properties and the relationship with the sequence $(x_n)_{n\in\mathbb{N}}$.
Recall that $A_S = (1-\delta)I + \delta S$, with $\delta \in (0,1)$.
We shall divide the proof into several steps.

Proof. **Step 1.** $\lim_{n\to\infty} \|x_n - s_n\| = 0$.
Proof of Step 1. We observe that

$$\lim_{n\to\infty} \|s_{n+1} - A_S x_n\| = \lim_{n\to\infty} \alpha_n \|u - A_S x_n\| = 0. \tag{5.17}$$

We compute

$$\begin{aligned}\|x_{n+1} - s_{n+1}\| &= \|\alpha_n u + (1-\alpha_n)Ux_n - \alpha_n u - (1-\alpha_n)A_S x_n\| \\ &= (1-\alpha_n)\|Ux_n - A_S x_n\| \\ &= (1-\alpha_n)\|\beta_n A_T x_n + (1-\beta_n)A_S x_n - A_S x_n\| \\ &= (1-\alpha_n)\beta_n \|A_T x_n - A_S x_n\|. \end{aligned} \tag{5.18}$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$,

$$\lim_{n\to\infty} \|x_n - s_n\| = 0. \tag{5.19}$$

This shows that also $(s_n)_{n\in\mathbb{N}}$ is bounded.

Step 2. $\lim_{n\to\infty} \|x_n - A_S x_n\| = 0$.
Proof of Step 2. We begin to prove that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Let $p \in Fix(S) = Fix(A_S)$. We compute

$$
\begin{array}{rcl}
\|s_{n+1}-p\|^2 & = & \|\alpha_n u+(1-\alpha_n)(1-\delta)x_n+(1-\alpha_n)\delta Sx_n-p\|^2 \\
& = & \|[(1-\alpha_n)\delta(Sx_n-x_n)+x_n-p]+\alpha_n(u-x_n)\|^2 \\
(\text{ by Lemma 1.6.22}) & \leq & \|(1-\alpha_n)\delta(Sx_n-x_n)+x_n-p\|^2 \\
& + & 2\alpha_n\langle u-x_n, s_{n+1}-p\rangle \\
& \leq & (1-\alpha_n)^2\delta^2\|Sx_n-x_n\|^2+\|x_n-p\|^2 \\
& - & 2(1-\alpha_n)\delta\langle x_n-p, x_n-Sx_n\rangle \\
& + & 2\alpha_n\|u-x_n\|\|s_{n+1}-p\| \\
& = & (1-\alpha_n)^2\delta^2\|x_n-Sx_n\|^2+\|x_n-p\|^2 \\
((I-S)p=0) & - & 2(1-\alpha_n)\delta\langle x_n-p,(I-S)x_n-(I-S)p\rangle \\
& + & 2\alpha_n\|u-x_n\|\|s_{n+1}-p\| \\
(\text{ by Lemma 1.6.11}) & \leq & \|x_n-p\|^2+(1-\alpha_n)^2\delta^2\|x_n-Sx_n\|^2 \\
& - & 2(1-\alpha_n)\delta\Bigg[\|(I-S)x_n-(I-S)p\|^2 \\
& - & \frac{1}{2}\Bigg(\|x_n-Sx_n\|^2+\|p-Sp\|^2\Bigg)\Bigg] \\
& + & 2\alpha_n\|u-x_n\|\|s_{n+1}-p\| \\
& = & \|x_n-p\|^2+(1-\alpha_n)^2\delta^2\|x_n-Sx_n\|^2 \\
& - & (1-\alpha_n)\delta\|x_n-Sx_n\|^2+2\alpha_n\|u-x_n\|\|s_{n+1}-p\| \\
& = & \|x_n-p\|^2-(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Sx_n\|^2 \\
& + & 2\alpha_n\|u-x_n\|\|s_{n+1}-p\|
\end{array}
$$

and hence

$$
\begin{array}{rl}
& (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Sx_n\|^2-2\alpha_n\|u-x_n\|\|s_{n+1}-p\| \\
\leq & \|x_n-p\|^2-\|s_{n+1}-p\|^2.
\end{array}
$$

Set

$$L_n=(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Sx_n\|^2-2\alpha_n\|u-x_n\|\|s_{n+1}-p\|;$$

let us consider the following two cases.

a) If $L_n \leq 0$, for all $n \geq n_0$ large enough, then

$$(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Sx_n\|^2 \leq 2\alpha_n\|u-x_n\|\|s_{n+1}-p\|.$$

So, since $\lim\limits_{n\to\infty}\alpha_n=0$ and $\lim\limits_{n\to\infty}(1-\alpha_n)\delta[1-\delta(1-\alpha_n)]=\delta(1-\delta)$,

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

b) Assume now that there exists a subsequence $(L_{n_k})_{k\in\mathbb{N}}$ of $(L_n)_{n\in\mathbb{N}}$ taking all its positive terms; so $L_{n_k} > 0$ for every $k \in \mathbb{N}$, then

$$0 < L_{n_k} \leq \|x_{n_k} - p\|^2 - \|s_{n_k+1} - p\|^2. \tag{5.20}$$

Summing (5.20) from $k = 1$ to N, we obtain

$$\begin{aligned}
\sum_{k=1}^{N} L_{n_k} &\leq \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1} \big(\|x_{n_k+1} - p\|^2 - \|s_{n_k+1} - p\|^2\big) - \|s_{n_N+1} - p\|^2 \\
&\leq \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1} \big(\|x_{n_k+1} - p\| + \|s_{n_k+1} - p\|\big)\|x_{n_k+1} - s_{n_k+1}\| \\
&\leq \|x_{n_1} - p\|^2 \\
\text{(by(5.18))} \quad &+ \sum_{k=1}^{N-1}(1-\alpha_{n_k})\beta_{n_k}\big(\|x_{n_k+1} - p\| + \|s_{n_k+1} - p\|\big)\|A_T x_{n_k} - A_S x_{n_k}\| \\
&\leq \|x_{n_1} - p\|^2 + K\sum_{k=1}^{N-1} \beta_{n_k},
\end{aligned}$$

where $K = \sup_{k\in\mathbb{N}} \big\{(\|x_{n_k+1} - p\| + \|s_{n_k+1} - p\|)\|A_T x_{n_k} - A_S x_{n_k}\|\big\}$.
Since $\sum_{k=1}^{\infty} \beta_{n_k} < \infty$,

$$\sum_{k=1}^{\infty} \big((1-\alpha_{n_k})\delta[1-\delta(1-\alpha_{n_k})]\|x_{n_k} - Sx_{n_k}\|^2 - 2\alpha_{n_k}\|u - x_{n_k}\|\|s_{n_k+1} - p\|\big) < \infty.$$

Thus,

$$\lim_{k\to\infty} \big((1-\alpha_{n_k})\delta[1-\delta(1-\alpha_{n_k})]\|x_{n_k} - Sx_{n_k}\|^2 - 2\alpha_{n_k}\|u - x_{n_k}\|\|s_{n_k+1} - p\|\big) = 0,$$

and since $\lim_{k\to\infty} \alpha_{n_k} = 0$ and $\lim_{k\to\infty}(1-\alpha_{n_k})\delta[1-\delta(1-\alpha_{n_k})] = \delta(1-\delta)$, also in this case we get

$$\lim_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0.$$

As in i), we can conclude that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Consequently,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - A_S x_n\| &= \lim_{n\to\infty} \|x_n - (1-\delta)x_n - \delta S x_n\| \\
&= \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0.
\end{aligned} \tag{5.21}$$

Moreover, from (5.19) and (5.21),

$$\lim_{n\to\infty} \|s_n - A_S s_n\| = \lim_{n\to\infty} \delta \|s_n - S s_n\| = 0.$$

Step 3. $\limsup_{n\to\infty} \left\langle u - P_{Fix(S)}u, s_n - P_{Fix(S)}u \right\rangle \le 0.$

Proof of Step 4. We may assume without loss of generality that there exists a subsequence $(s_{n_j})_{j\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ such that $s_{n_j} \rightharpoonup v$ and

$$\begin{aligned} \limsup_{n\to\infty} \left\langle u - P_{Fix(S)}u, s_n - P_{Fix(S)}u \right\rangle &= \lim_{j\to\infty} \left\langle u - P_{Fix(S)}u, s_{n_j} - P_{Fix(S)}u \right\rangle \\ &= \left\langle u - P_{Fix(S)}u, v - P_{Fix(S)}u \right\rangle \end{aligned}$$

Since $\lim_{n\to\infty} \|s_n - S s_n\| = 0$ and from $I - S$ is demiclosed at 0, $v \in Fix(S) = Fix(A_S)$. Then by (1.3), we have

$$\limsup_{n\to\infty} \left\langle u - P_{Fix(S)}u, s_n - P_{Fix(S)}u \right\rangle = \left\langle u - P_{Fix(S)}u, v - P_{Fix(S)}u \right\rangle \le 0. \tag{5.22}$$

Step 4. $(s_n)_{n\in\mathbb{N}}$ converges strongly to $P_{Fix(S)}u$.

Proof of Step 5. We compute

$$\begin{aligned} \|s_{n+1} - P_{Fix(S)}u\|^2 &= \|\alpha_n(u - P_{Fix(S)}u) + (1-\alpha_n)(A_S x_n - P_{Fix(S)}u)\|^2 \\ \text{(by Lemma 1.6.22)} \quad &\le (1-\alpha_n)^2 \|A_S x_n - P_{Fix(S)}u\|^2 \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ (A_S \text{ nonexpansive}) \quad &\le (1-\alpha_n)\|x_n - P_{Fix(S)}u\|^2 \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\le (1-\alpha_n)\big(\|x_n - s_n\| + \|s_n - P_{Fix(S)}u\|\big)^2 \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\le (1-\alpha_n)\|x_n - s_n\|^2 + (1-\alpha_n)\|s_n - P_{Fix(S)}u\|^2 \\ &+ 2(1-\alpha_n)\|x_n - s_n\|\|s_n - P_{Fix(S)}u\| \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\le (1-\alpha_n)\|s_n - P_{Fix(S)}u\|^2 \\ \text{by (5.18)} \quad &+ (1-\alpha_n)(1-\alpha_{n-1})^2 \beta_{n-1}^2 \|A_T x_{n-1} - A_S x_{n-1}\|^2 \\ &+ 2(1-\alpha_n)(1-\alpha_{n-1})\beta_{n-1}\|A_T x_{n-1} - A_S x_{n-1}\|\|s_n - P_{Fix(S)}u\| \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\le (1-\alpha_n)\|s_n - P_{Fix(S)}u\|^2 + \beta_{n-1}\|A_T x_{n-1} - A_S x_{n-1}\|^2 \\ &+ 2\beta_{n-1}\|A_T x_{n-1} - A_S x_{n-1}\|\|s_n - P_{Fix(S)}u\| \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\le (1-\alpha_n)\|s_n - P_{Fix(S)}u\|^2 + M\beta_{n-1} \\ &+ 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle, \end{aligned}$$

where $M := \sup_{n\in\mathbb{N}} \{\|A_T x_{n-1} - A_S x_{n-1}\|^2 + 2\|A_T x_{n-1} - A_S x_{n-1}\| \|s_n - P_{Fix(S)}u\|\}$.
Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$ and from (5.22) we can apply Lemma 1.5.1 and we conclude that

$$\lim_{n\to\infty} \|s_{n+1} - P_{Fix(S)}u\| = 0.$$

So, $(s_n)_{n\in\mathbb{N}}$ converges strongly to $P_{Fix(S)}u \in Fix(S)$. Since $\lim_{n\to\infty} \|x_n - s_n\| = 0$, we have

$$\lim_{n\to\infty} \|x_n - P_{Fix(S)}u\| = 0,$$

i.e. $(x_n)_{n\in\mathbb{N}}$ converges strongly to $P_{Fix(S)}u \in Fix(S)$. □

Proof of (iii)

Proof. Let $q \in Fix(S) \cap Fix(T)$.
Since in this last case, the techniques used in i) and ii) fail, we turn our attention on the monotony of the sequence $(\|x_n - q\|)_{n\in\mathbb{N}}$. We consider the following two cases.

Case 1. $\|x_{n+1} - q\| \leq \|x_n - q\|$, for every $n \geq n_0$ large enough.

Case 2. There exists a subsequence $(\|x_{n_j} - q\|)_{j\in\mathbb{N}}$ of $(\|x_n - q\|)_{n\in\mathbb{N}}$ such that

$$\|x_{n_j} - q\| < \|x_{n_j+1} - q\| \text{ for all } j \in \mathbb{N}.$$

Case 1. $\lim_{n\to\infty} \|x_n - q\|$ exists finite and hence

$$\lim_{n\to\infty} (\|x_{n+1} - q\| - \|x_n - q\|) = 0. \tag{5.23}$$

We shall divide the proof into several steps.
Step 1. $\lim_{n\to\infty} \|x_n - A_S x_n\| = 0$.
Proof of Step 1. Consider

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\beta_n \big(A_T x_n + (1 - \beta_n) A_S x_n\big). \tag{5.24}$$

We compute

$$\begin{aligned}
\|\beta_n(A_Tx_n - q) + (1-\beta_n)(A_Sx_n - q)\|^2 &= \beta_n\|A_Tx_n - q\|^2 \\
&+ (1-\beta_n)\|A_Sx_n - q\|^2 \\
&- \beta_n(1-\beta_n)\|A_Tx_n - A_Sx_n\|^2 \\
(A_T \text{ nonexpansive and by (1.25)}) &\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\|x_n - q\|^2 \\
&- (1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2 \\
&- \beta_n(1-\beta_n)\|A_Tx_n - A_Sx_n\|^2 \\
&= \|x_n - q\|^2 - (1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2 \\
&- \beta_n(1-\beta_n)\|A_Tx_n - A_Sx_n\|^2
\end{aligned}$$

We recall that $U_n = \beta_nA_T + (1-\beta_n)A_S$.
So, we get

$$\|U_nx_n - q\|^2 \leq \|x_n - q\|^2 - (1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2 - \beta_n(1-\beta_n)\|A_Tx_n - A_Sx_n\|^2. \tag{5.25}$$

We have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|U_nx_n - q + \alpha_n(u - U_nx_n)\|^2 \\
&\leq \|U_nx_n - q\|^2 + \alpha_n(\alpha_n\|u - U_nx_n\|^2 + 2\|U_nx_n - q\|\|u - U_nx_n\|) \\
(\text{by (5.25)}) &\leq \|x_n - q\|^2 - (1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2 \\
&- \beta_n(1-\beta_n)\|A_Tx_n - A_Sx_n\|^2 + \alpha_nM,
\end{aligned} \tag{5.26}$$

where $M := \sup_{n\in\mathbb{N}}\{\alpha_n\|u - U_nx_n\|^2 + 2\|U_nx_n - q\|\|u - U_nx_n\|\}$. From (5.26), we derive

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - (1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2 + \alpha_nM,$$

hence

$$(1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_nM. \tag{5.27}$$

From (5.23) and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}((1-\beta_n)(1-\delta)\|x_n - A_Sx_n\|^2) = 0.$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty}\|x_n - A_Sx_n\| = \lim_{n\to\infty}\delta\|x_n - Sx_n\| = 0. \tag{5.28}$$

Step 2. $\lim_{n\to\infty} \|A_T x_n - A_S x_n\| = 0.$

Proof of Step 2. Moreover, from (5.26), we also can derive

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \beta_n(1-\beta_n)\|A_T x_n - A_S x_n\|^2 + \alpha_n M,$$

hence

$$\beta_n(1-\beta_n)\|A_T x_n - A_S x_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n M.$$

As above, we can conclude that

$$\lim_{n\to\infty} \|A_T x_n - A_S x_n\| = \lim_{n\to\infty} \delta\|Tx_n - Sx_n\| = 0. \tag{5.29}$$

From (5.28) and (5.29), it follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{5.30}$$

Let $F = Fix(T) \cap Fix(S)$.

Step 3. $\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \leq 0.$

Proof of Step 3. We may assume without loss of generality that there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_j} \rightharpoonup v$ and

$$\begin{aligned} \limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle &= \lim_{j\to\infty} \left\langle u - P_F u, x_{n_j} - P_F u\right\rangle \\ &= \langle u - P_F u, v - P_F u\rangle . \end{aligned} \tag{5.31}$$

By (5.30) and (5.28) and by the demiclosedness of $I-T$ at 0 and of $I-S$ at 0, $v \in F = Fix(T) \cap Fix(S)$. Then we can conclude that

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle = \langle u - P_F u, v - P_F u\rangle \leq 0.$$

Step 4. $(x_n)_{n\in\mathbb{N}}$ converges strongly to $P_F u$.

Proof of Step 4. We compute

$$\begin{aligned} \|x_{n+1} - P_F u\|^2 &= \|\alpha_n(u - P_F u) + (1-\alpha_n)(U_n x_n - P_F u)\|^2 \\ \text{(by Lemma 1.6.22)} \quad &\leq (1-\alpha_n)^2\|U_n x_n - P_F u\|^2 \\ &+ 2\alpha_n\langle u - P_F u, x_{n+1} - P_F u\rangle \\ (U_n \text{ quasi-nonexpansive}) \quad &\leq (1-\alpha_n)\|x_n - P_F u\|^2 \\ &+ 2\alpha_n\langle u - P_F u, x_{n+1} - P_F u\rangle. \end{aligned} \tag{5.32}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we can apply Lemma 1.5.1 and conclude that

$$\lim_{n\to\infty} \|x_{n+1} - P_F u\| = 0.$$

Finally, $(x_n)_{n\in\mathbb{N}}$ converges strongly to $P_F u$.

Case 2. Let $q = P_F u$. Then there exists a subsequence $(\|x_{n_j} - P_F u\|)_{j\in\mathbb{N}}$ of $(\|x_n - P_F u\|)_{n\in\mathbb{N}}$ such that

$$\|x_{n_j} - P_F u\| < \|x_{n_j+1} - P_F u\| \text{ for all } j \in \mathbb{N}.$$

By Lemma 5.33, there exists a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} m_k = +\infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\|x_{m_k} - P_F u\| \le \|x_{m_k+1} - P_F u\| \quad \text{and} \quad \|x_k - P_F u\| \le \|x_{m_k+1} - P_F u\|. \tag{5.33}$$

Consequently,

$$\begin{aligned}
0 &\le \lim_{k\to\infty} \big(\|x_{m_k+1} - P_F u\| - \|x_{m_k} - P_F u\|\big) \\
&\le \limsup_{n\to\infty} \big(\|x_{n+1} - P_F u\| - \|x_n - P_F u\|\big) \\
\text{(by (5.42))} &\le \limsup_{n\to\infty} \big(\alpha_n\|u - P_F u\| + (1-\alpha_n)\|x_n - P_F u\| - \|x_n - P_F u\|\big) \\
(\alpha_n \to 0) &= \limsup_{n\to\infty} \alpha_n\big(\|u - P_F u\| - \|x_n - P_F u\|\big) = 0.
\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \big(\|x_{m_k+1} - P_F u\| - \|x_{m_k} - P_F u\|\big) = 0. \tag{5.34}$$

As in the **Case 1.**, we can prove that

$$\lim_{k\to\infty} \|x_{m_k} - Sx_{m_k}\| = \lim_{k\to\infty} \|x_{m_k} - Tx_{m_k}\| = 0$$

and by the demiclosedness of $I - T$ at 0 and of $I - S$ at 0, we obtain that

$$\limsup_{k\to\infty} \langle u - P_F u, x_{m_k} - P_F u\rangle \le 0. \tag{5.35}$$

We replace in (5.64) n with m_k, then

$$\|x_{m_k+1} - P_F u\|^2 \le (1-\alpha_{m_k})\|x_{m_k} - P_F u\|^2 + 2\alpha_{m_k}\langle u - P_F u, x_{m_k+1} - P_F u\rangle.$$

In particular, we get

$$\begin{aligned}
\alpha_{m_k}\|x_{m_k} - P_F u\|^2 &\le \|x_{m_k} - P_F u\|^2 - \|x_{m_k+1} - P_F u\|^2 \\
&\quad + 2\alpha_{m_k}\langle u - P_F u, x_{m_k+1} - P_F u\rangle \\
\text{(by (5.33))} &\le 2\alpha_{m_k}\langle u - P_F u, x_{m_k+1} - P_F u\rangle.
\end{aligned} \tag{5.36}$$

Then, from (5.35), we obtain

$$\limsup_{k\to\infty} \|x_{m_k} - P_F u\|^2 \le 2\limsup_{k\to\infty} \langle u - P_F u, x_{m_k+1} - P_F u\rangle \le 0.$$

Thus, from (5.33) and (5.34), we conclude that

$$\limsup_{k\to\infty} \|x_k - P_F u\|^2 \le \limsup_{k\to\infty} \|x_{m_k+1} - P_F u\|^2 = 0,$$

i.e., $(x_n)_{n\in\mathbb{N}}$ converges strongly to $P_F u$.

□

5.1 Viscosity type method

If the constant vector u in the Halpern-type recursion formula is replaced by $f(x_n)$, where $f : K \to K$ is a contraction, an iteration method involving the resulting formula is called the *viscosity method*. We make first the following remarks concerning this method.

Remark 5.1.1. The recursion formula with $f(x_n)$ involves more computation at each stage of the iteration than that with u and does not result in any improvement in the speed or rate of convergence of the scheme. Consequently, from the pratical point of view, it is undesirable.

Remark 5.1.2. When a Theorem has been proved using a Halpern-type iterative formula with a constant vector, say u, the proof of the same Theorem with u replaced by $f(x_n)$, the so-called viscosity method, generally does not involve any new ideas or method. Such a proof is generally an unnecessary repetition of the proof when the vector u is used.

Remark 5.1.3. The so-called viscosity method may be useful in other different iteration processes, which involve for example, family of operators or combinations of different operators as we shall see. But for the approximations of fixed points of a nonexpansive operator, there seems to be no justification for studying it.

Given a nonexpansive self-mapping T on a closed convex subset C, a real number $t \in (0, 1]$ and a contraction ψ on C, define the mapping $T_t : C \to C$ by

$$T_t x := t\psi(x) + (1-t)\,Tx, \quad x \in C.$$

It's easy to see that T_t is a contraction; hence T_t has a unique fixed point which is denoted by x_t. That is, x_t is the unique solution to the fixed point equation

$$x_t = t\psi(x_t) + (1-t)\,Tx_t, \quad t \in (0,1]. \tag{5.37}$$

The explicit iterative discretization of (5.37) is

$$x_{n+1} = \alpha_n\psi(x_n) + (1-\alpha_n)\,Tx_n, \quad n \ge 0, \tag{5.38}$$

where $(\alpha_n) \in [0, 1]$. Note that these two iterative processes (5.37) and (5.38) have Halpern iteration as special case by taking $\psi(x) = u \in C$ for any $x \in C$. The convergence of the implicit (5.37) and explicit (5.38) algorithms has been the subjects of many papers because under suitable conditions these iterations converge strongly to the unique solution $q \in Fix(T)$ of the variational inequality

$$\langle (I-\psi)\, q, x-q\rangle \geq 0 \quad \forall x \in Fix(T).$$

This fact allows us to apply this method to convex optimization, linear programming and monotone inclusions.

G. Marino and L. Muglia in [46] considered a viscosity type iterative method:

$$z_0 \in H,\ z_{n+1} = \alpha_n(I-\mu_n D)z_n + (1-\alpha_n)W_n z_n$$

where W_n is an appropriate family of mappings and they proved the strong convergence to the unique solution of the variational inequality (**??**) on the set of common fixed points of a family of mappings.

Inspired by [46] and [48], in we introduce the following viscosity type algorithm

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n(I-\mu_n D)x_n + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n], \quad n \in \mathbb{N}, \end{cases}$$

where $T_\delta = (1-\delta)I + \delta T,\ S_\delta = (1-\delta)I + \delta S$, $\delta \in (0,1)$ are the averaged type mappings with T is a nonexpansive mapping and S is a L-hybrid mapping and D is a β-strongly monotone and a ρ-Lipschitzian operator.

The introduced iterative scheme is very interesting because some well known iterative methods can be obtained by it.

For example, if $\mu_n = \mu$ and $D = \frac{I-u}{\mu}$, where u is the constant contraction, we obtain the algorithm proposed in [48].

Instead, if we consider $\mu_n = \mu$ and $D = \frac{I-f}{\mu}$, where f is a contraction, we have a viscosity method

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n]. \tag{5.39}$$

For $\beta_n = 1$ this scheme (5.39) was proposed by A. Moudafi in [45]. He proved the strong convergence of algorithm (5.39) to the unique solution $x^* \in C$ of the variational inequality

$$\langle (I-f)x^*, x^*-x\rangle \leq 0, \quad \forall x \in C,$$

where f is a contraction, in Hilbert spaces.

In [42] Xu extended Moudafi's results in a uniformly smooth Banach space.

Moreover, for $D = (I-\gamma f)$, where f is a contraction with coefficient α and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, we have

$$x_{n+1} = \alpha_n y_n + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n]$$

where $y_n = \mu_n \gamma f(x_n) + (1-\mu_n)x_n$.

If $A : H \to H$ is a strongly positive operator, i.e. there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

and f is a contraction, we can take $D = A - \gamma f$ that is a strongly monotone operator (see, [52]).
In [47], [52], [35], the authors consider iterative methods approximating a fixed point of nonexpansive mappings that is also the unique solution of the variational inequality problem

$$\langle (A - \gamma f)x^*, x^* - x \rangle \leq 0, \quad \forall x \in C. \tag{5.40}$$

(5.40) is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf, i.e., $h'(x) = \gamma f(x)$ for $x \in H$.
Depending on the choice of controll coefficients α_n and β_n, we prove the strong convergence of our iterative method to the unique solution of a variational inequality (**??**) on the set of common fixed points of T and S or on the fixed points of one of them. As in [48], the regularization with the averaged type mappings plays a crucial role for the strong convergence of our iterative method.

In all calculation below, $(\beta_n)_{n \in \mathbb{N}} \subset [0, 1]$ denotes a real sequence and $U_n : C \to C$ denotes the convex combination of T_δ and S_δ, i.e.

$$U_n = \beta_n T_\delta x_n + (1 - \beta_n) S_\delta x_n, \quad n \in \mathbb{N}.$$

Further we assume that

- $Fix(S) \cap Fix(T) \neq \emptyset$;
- $(\alpha_n)_{n \in \mathbb{N}} \subset (0, 1)$ a real sequence such that $\lim_{n \to \infty} \alpha_n = 0$;
- $O(1)$ is any bounded real sequence;
- $D : H \to H$ is a β-strongly monotone and ρ-Lipschitzian operator;
- $(\mu_n)_{n \in \mathbb{N}}$ is a sequence in $(0, \mu)$ such that $\mu < \frac{2\beta}{\rho^2}$.

We start with the two following Lemmas.

Lemma 5.1.4. *Let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined by*

$$x_{n+1} = \alpha_n (1 - \mu_n D) x_n + (1 - \alpha_n) U_n x_n,$$

where

$$U_n = \beta_n T_\delta + (1 - \beta_n) S_\delta.$$

Then

1. *U_n is quasi-nonexpansive for all $n \in \mathbb{N}$.*

2. *$(x_n)_{n\in\mathbb{N}}, (Sx_n)_{n\in\mathbb{N}}, (Tx_n)_{n\in\mathbb{N}}, (S_\delta x_n)_{n\in\mathbb{N}}, (T_\delta x_n)_{n\in\mathbb{N}}, (U_n x_n)_{n\in\mathbb{N}}$ are bounded sequences.*

Proof. (1)
To simplify the notation, we set

$$B_n := I - \mu_n D. \tag{5.41}$$

We observe that U_n is quasi-nonexpansive, for all $n \in \mathbb{N}$, since T_δ is a nonexpansive mapping and S_δ is a quasi-nonexpansive mapping.
(2)
First we prove that $(x_n)_{n\in\mathbb{N}}$ is bounded.
Moreover, we recall that B_n is a contraction.
For $q \in Fix(T) \cap Fix(S)$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(B_n x_n - q) + (1-\alpha_n)(U_n x_n - q)\| \\
&\leq \alpha_n\|B_n x_n - q\| + (1-\alpha_n)\|U_n x_n - q\| \\
(U_n \text{ quasi-nonexpansive}) \quad &\leq \alpha_n\|B_n x_n - q\| + (1-\alpha_n)\|x_n - q\| \\
&\leq \alpha_n\|B_n x_n - B_n q\| + \alpha_n\|B_n q - q\| + (1-\alpha_n)\|x_n - q\| \\
(B_n \text{ contraction}) \quad &\leq \alpha_n(1-\mu_n\tau)\|x_n - q\| + \alpha_n\|B_n q - q\| + (1-\alpha_n)\|x_n - q\| \\
&= (1-\alpha_n\mu_n\tau)\|x_n - q\| + \alpha_n\|B_n q - q\| \\
(\text{by } (5.41)) \quad &= (1-\alpha_n\mu_n\tau)\|x_n - q\| + \alpha_n\mu_n\tau\frac{\|Dq\|}{\tau}
\end{aligned} \tag{5.42}$$

Since

$$\|x_1 - q\| \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\},$$

and by induction we assume that

$$\|x_n - q\| \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\},$$

then

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} + \alpha_n\mu_n\tau\frac{\|Dq\|}{\tau} \\
&\leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} + \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} \\
&= 2\max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\}.
\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is bounded. Consequently, $(T_\delta x_n)_{n\in\mathbb{N}}$, $(S_\delta x_n)_{n\in\mathbb{N}}$, $(U_n x_n)_{n\in\mathbb{N}}$ and $(B_n x_n)_{n\in\mathbb{N}}$ are bounded as well.

□

Now, we prove our main Result.

Theorem 5.1.5. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a L-hybrid mapping such that $Fix(S) \cap Fix(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings, i.e.*

$$T_\delta = (1-\delta)I + \delta T,\ \ S_\delta = (1-\delta)I + \delta S,\ \delta \in (0,1).$$

Let $D : H \to H$ be a β-strongly monotone and ρ-Lipschitzian operator. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0,\mu)$, $\mu < \frac{2\beta}{\rho^2}$, and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$, satisfying the conditions:

1. $\lim_{n\to\infty} \alpha_n = 0$,
2. $\lim_{n\to\infty} \dfrac{\alpha_n}{\mu_n} = 0$,
3. $\sum_{n=1}^{\infty} \alpha_n \mu_n = \infty$.

If $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$, we define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n(I - \mu_n D)x_n + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n], \quad n \in \mathbb{N}. \end{cases}$$

Then, the following hold:

- *(i) If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p} \in Fix(T)$ which is the unique solution in $Fix(T)$ of the variational inequality $\langle D\overline{p}, \overline{p} - x\rangle \le 0$, for all $x \in Fix(T)$.*
- *(ii) If $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\widehat{p} \in Fix(S)$ which is the unique solution in $Fix(S)$ of the variational inequality $\langle D\widehat{p}, \widehat{p} - x\rangle \le 0$, for all $x \in Fix(S)$.*
- *(iii) If $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $p_0 \in Fix(T) \cap Fix(S)$ which is the unique solution in $Fix(T) \cap Fix(S)$ of the variational inequality $\langle Dp_0, p_0 - x\rangle \le 0$, for all $x \in Fix(T) \cap Fix(S)$.*

Proof. Proof. of (i)
We rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n)T_\delta x_n + (1-\beta_n)E_n, \tag{5.43}$$

where $E_n = (1-\alpha_n)(S_\delta x_n - T_\delta x_n)$ is bounded, i.e. $\|E_n\| \le O(1)$.
We begin to prove that $\lim_{n\to\infty} \|x_n - T_\delta x_n\| = 0$.
Let $\overline{p}$ the unique solution in $Fix(T) = Fix(T_\delta)$ of the variational inequality

$$\langle D\overline{p}, \overline{p} - x\rangle \le 0, \quad \forall x \in Fix(T).$$

We have

$$\begin{aligned}
\|x_{n+1} - \overline{p}\|^2 &= \|\alpha_n B_n x_n + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta T x_n + (1-\beta_n)E_n - \overline{p}\|^2 \\
&= \|[(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}] + [\alpha_n(B_n x_n - x_n) + (1-\beta_n)E_n]\|^2 \\
\text{(by Lemma 1.6.22)} \quad &\le \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
&+ 2\langle \alpha_n(B_n x_n - x_n) + (1-\beta_n)E_n, x_{n+1} - \overline{p}\rangle \\
&= \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
&+ 2\alpha_n\langle B_n x_n - x_n, x_{n+1} - \overline{p}\rangle + 2(1-\beta_n)\langle E_n, x_{n+1} - \overline{p}\rangle \\
&\le (1-\alpha_n)^2\delta^2\|Tx_n - x_n\|^2 + \|x_n - \overline{p}\|^2 \\
&- 2(1-\alpha_n)\delta\langle x_n - \overline{p}, x_n - Tx_n\rangle \\
&+ 2\alpha_n\|B_n x_n - x_n\|\|x_{n+1} - \overline{p}\| + 2(1-\beta_n)\|E_n\|\|x_{n+1} - \overline{p}\| \\
\text{(by (1.19))} \quad &\le \|x_n - \overline{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Tx_n\|^2 \\
&- (1-\alpha_n)\delta\|x_n - Tx_n\|^2 + \alpha_n O(1) + (1-\beta_n)O(1) \\
&= \|x_n - \overline{p}\|^2 - (1-\alpha_n)\delta[1 - \delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \\
&+ \alpha_n O(1) + (1-\beta_n)O(1)
\end{aligned}$$

and hence

$$\begin{aligned}
0 \le (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 &\le \|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 + \\
&+ \alpha_n O(1) + (1-\beta_n)O(1).
\end{aligned} \tag{5.44}$$

We turn our attention on the monotony of the sequence $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$.
We consider the following two cases.

Case A. $\|x_{n+1} - \overline{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \overline{p}\| < \|x_{n_k+1} - \overline{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n-\overline{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty}\|x_n-\overline{p}\|^2$ exists. From (5.44), $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n-Tx_n\|^2\Big) \\ &\leq \limsup_{n\to\infty}\Big(\|x_n-\overline{p}\|^2-\|x_{n+1}-\overline{p}\|^2 \\ &+ \alpha_nO(1)+(1-\beta_n)O(1)\Big)=0, \end{aligned}$$

so, we can conclude that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0,$$

and

$$\lim_{n\to\infty}\|x_n-T_\delta x_n\|=\lim_{n\to\infty}\delta\|x_n-Tx_n\|=0. \tag{5.45}$$

Since $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ and from $I-T$ is demiclosed at 0, we can use Lemma 1.5.2 (i), so we get

$$\limsup_{n\to\infty}\langle D\overline{p},\overline{p}-x_n\rangle\leq 0. \tag{5.46}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p}$.
We compute

$$\begin{aligned} \|x_{n+1}-\overline{p}\|^2 &\leq \|\alpha_n(B_nx_n-\overline{p})+(1-\alpha_n)(T_\delta x_n-\overline{p})+(1-\beta_n)E_n\|^2 \\ \text{(by Lemma 1.6.22)} &\leq \|(1-\alpha_n)(T_\delta x_n-\overline{p})\|^2 \\ &+ 2\langle(1-\beta_n)E_n+\alpha_n(B_nx_n-\overline{p}),x_{n+1}-\overline{p}\rangle \\ &\leq (1-\alpha_n)^2\|T_\delta x_n-\overline{p}\|^2+2\alpha_n\langle B_nx_n-\overline{p},x_{n+1}-\overline{p}\rangle \\ &+ (1-\beta_n)O(1) \\ (T_\delta\text{ nonexpansive}) &\leq (1-\alpha_n)^2\|x_n-\overline{p}\|^2+2\alpha_n\langle B_nx_n-B_n\overline{p},x_{n+1}-\overline{p}\rangle \\ &+ 2\alpha_n\langle B_n\overline{p}-\overline{p},x_{n+1}-\overline{p}\rangle+(1-\beta_n)O(1) \\ (B_n\text{ contraction}) &\leq (1-\alpha_n)^2\|x_n-\overline{p}\|^2 \\ &+ 2\alpha_n(1-\mu_n\tau)\|x_n-\overline{p}\|\|x_{n+1}-\overline{p}\| \\ &+ 2\alpha_n\langle B_n\overline{p}-\overline{p},x_{n+1}-\overline{p}\rangle+(1-\beta_n)O(1) \\ (B_n:=(1-\mu_nD)) &\leq (1-\alpha_n)^2\|x_n-\overline{p}\|^2 \\ &+ \alpha_n(1-\mu_n\tau)\Big[\|x_n-\overline{p}\|^2+\|x_{n+1}-\overline{p}\|^2\Big] \\ &- 2\alpha_n\mu_n\langle D\overline{p},x_{n+1}-\overline{p}\rangle+(1-\beta_n)O(1), \end{aligned}$$

Then it follows that

$$\begin{aligned}
\|x_{n+1}-\overline{p}\|^2 &\leq \frac{1-(1+\mu_n\tau)\alpha_n+\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}\|x_n-\overline{p}\|^2 \\
&- \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle D\overline{p}, x_{n+1}-\overline{p}\rangle \\
&+ \frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1) \\
&\leq \frac{1-(1+\mu_n\tau)\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\|x_n-\overline{p}\|^2 \\
&- \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle D\overline{p}, x_{n+1}-\overline{p}\rangle \\
&+ \frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1) + \frac{\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}O(1) \\
&\leq \left(1-\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\right)\|x_n-\overline{p}\|^2 \\
&+ \frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\left[-\frac{1}{\tau}\langle D\overline{p}, x_{n+1}-\overline{p}\rangle + \frac{\alpha_n}{2\mu_n\tau}O(1)\right] \\
&+ \frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1)
\end{aligned}$$

Notice that by

$$\lim_{n\to\infty}\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}=0,$$

it follows that

$$0<\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}<1, \quad definitively.$$

Moreover, using $\sum_{n=1}^{\infty}\alpha_n\mu_n=\infty$, $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, (5.46) and $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n}=0$, we can apply Lemma 1.5.1 and conclude that

$$\lim_{n\to\infty}\|x_{n+1}-\overline{p}\|=0.$$

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k}-\overline{p}\|<\|x_{n_k+1}-\overline{p}\| \text{ for all } k\in\mathbb{N}.$$

Then by Maingé Lemma 5.33 there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(a) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;

(b) $\lim_{n\to\infty} \tau(n) = \infty$;

(c) $\|x_{\tau(n)} - \overline{p}\| < \|x_{\tau(n)+1} - \overline{p}\|$;

(d) $\|x_n - \overline{p}\| < \|x_{\tau(n)+1} - \overline{p}\|$.

Consequently,

$$\begin{array}{rcl} 0 & \leq & \liminf_{n\to\infty} \Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\| \Big) \\ & \leq & \limsup_{n\to\infty} \Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\| \Big) \\ & \leq & \limsup_{n\to\infty} \Big(\|x_{n+1} - \overline{p}\| - \|x_n - \overline{p}\| \Big) \\ \text{(by (5.43))} & = & \limsup_{n\to\infty} \Big(\|\alpha_n(B_n x_n - T_\delta x_n) + T_\delta x_n - \overline{p} + (1-\beta_n)E_n\| - \|x_n - \overline{p}\| \Big) \\ (T_\delta \text{ nonexpansive}) & \leq & \limsup_{n\to\infty} \Big(\alpha_n O(1) + \|x_n - \overline{p}\| + (1-\beta_n)O(1) - \|x_n - \overline{p}\| \Big) = 0, \end{array}$$

so

$$\lim_{n\to\infty} \Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\| \Big) = 0. \tag{5.47}$$

By (5.44), we have

$$\begin{array}{rcl} 0 & \leq & (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\ & \leq & \|x_{\tau(n)} - \overline{p}\|^2 - \|x_{\tau(n)+1} - \overline{p}\|^2 + \alpha_n O(1) + (1-\beta_{\tau(n)})O(1), \end{array}$$

from (5.47), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$ we get

$$\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0. \tag{5.48}$$

By (5.48), as in the Case A, we have

$$\limsup_{n\to\infty} \langle D\overline{p}, \overline{p} - x_{\tau(n)} \rangle \leq 0$$

and

$$\lim_{n\to\infty} \|x_{\tau(n)} - \overline{p}\| = 0;$$

then, in the light of property (d) of Maingé Lemma 5.33 and (5.47) we conclude that

$$\lim_{n\to\infty} \|x_n - \overline{p}\| = 0.$$

□

Proof. of (ii)
Now, we rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n) S_\delta x_n + \beta_n E_n, \tag{5.49}$$

where $E_n = (1-\alpha_n)(T_\delta x_n - S_\delta x_n)$ is bounded, i.e. $\|E_n\| \leq O(1)$.
We begin to prove that $\lim_{n\to\infty} \|x_n - S_\delta x_n\| = 0$.

Let $\widehat{p}$ the unique solution in $Fix(S) = Fix(S_\delta)$ of the variational inequality $\langle D\widehat{p}, \widehat{p} - x\rangle \leq 0$, for all $x \in Fix(S)$. We have

$$\begin{aligned}
\|x_{n+1} - \widehat{p}\|^2 &= \|\alpha_n B_n x_n + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta S x_n + \beta_n E_n - \widehat{p}\|^2 \\
&= \|[(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}] + [\alpha_n(B_n x_n - x_n) + \beta_n E_n]\|^2 \\
\text{(by Lemma 1.6.22)} \quad &\leq \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
&+ 2\langle \alpha_n(B_n x_n - x_n) + \beta_n E_n, x_{n+1} - \widehat{p}\rangle \\
&\leq \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
&+ 2\alpha_n\langle B_n x_n - x_n, x_{n+1} - \widehat{p}\rangle + 2\beta_n\langle E_n, x_n - \widehat{p}\rangle \\
&\leq (1-\alpha_n)^2\delta^2\|Sx_n - x_n\|^2 + \|x_n - \widehat{p}\|^2 \\
&- 2(1-\alpha_n)\delta\langle x_n - \widehat{p}, x_n - Sx_n\rangle \\
&+ \alpha_n O(1) + \beta_n O(1) \\
\text{(by 1.22)} \quad &\leq \|x_n - \widehat{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Sx_n\|^2 \\
&- (1-\alpha_n)\delta\|x_n - Sx_n\|^2 + \alpha_n O(1) + \beta_n O(1) \\
&= \|x_n - \widehat{p}\|^2 - (1-\alpha_n)\delta[1 - \delta(1-\alpha_n)]\|x_n - Sx_n\|^2 \\
&+ \alpha_n O(1) + \beta_n O(1)
\end{aligned}$$

and hence

$$\begin{aligned}
0 \leq (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 &\leq \|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 + \\
&+ \alpha_n O(1) + \beta_n O(1).
\end{aligned} \tag{5.50}$$

Again, we turn our attention on the monotony of the sequence $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$. We consider the following two cases.

Case A. $\|x_{n+1} - \widehat{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - \widehat{p}\|^2$ exists. From (5.44), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, we have

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 \Big) \\ &\le \limsup_{n\to\infty} \Big(\|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 \\ &+ \alpha_n O(1) + \beta_n O(1) \Big) = 0, \end{aligned}$$

hence

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{5.51}$$

Since $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and from $I - S$ is demiclosed at 0, we can use Lemma 1.5.2 (i) and we have

$$\limsup_{n\to\infty} \langle D\widehat{p}, \widehat{p} - x_n\rangle \le 0. \tag{5.52}$$

Finally, we can prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\widehat{p}$ as in the proof (i).

Using $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, (5.52) and $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0$, we can apply Lemma 1.5.1 and conclude that

$$\lim_{n\to\infty} \|x_{n+1} - \widehat{p}\| = 0.$$

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.$$

Then by Maingé Lemma there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(a) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;

(b) $\lim_{n\to\infty} \tau(n) = \infty$;

(c) $\|x_{\tau(n)} - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$;

(d) $\|x_n - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$.

Consequently,

$$\begin{array}{rcl} 0 & \leq & \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big) \\ & \leq & \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big) \\ & \leq & \limsup_{n\to\infty}\Big(\|x_{n+1}-\widehat{p}\|-\|x_n-\widehat{p}\|\Big) \\ \text{(by (5.49))} & = & \limsup_{n\to\infty}\Big(\|\alpha_n(B_nx_n-S_\delta x_n)+S_\delta x_n-\widehat{p}+\beta_nE_n\|-\|x_n-\widehat{p}\|\Big) \\ (S_\delta \text{ quasi-nonexpansive}) & \leq & \limsup_{n\to\infty}\Big(\alpha_nO(1)+\|x_n-\widehat{p}\|+\beta_nO(1)-\|x_n-\widehat{p}\|\Big)=0, \end{array}$$

so

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big)=0. \tag{5.53}$$

By (5.50), we obtain

$$\begin{array}{rcl} 0 & \leq & (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)}-Sx_{\tau(n)}\|^2 \\ & \leq & \|x_{\tau(n)}-\widehat{p}\|^2-\|x_{\tau(n)+1}-\widehat{p}\|^2+\alpha_nO(1)+\beta_{\tau(n)}O(1), \end{array}$$

from (5.53), $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\beta_n<\infty$ we get

$$\lim_{n\to\infty}\|x_{\tau(n)}-Sx_{\tau(n)}\|=0. \tag{5.54}$$

By(5.54), as in the Case A, we get

$$\limsup_{n\to\infty}\langle D\widehat{p},\widehat{p}-x_{\tau(n)}\rangle\leq 0,$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)}-\widehat{p}\|=0,$$

then, from property (d) of Maingé Lemma and (5.53) it follows that

$$\lim_{n\to\infty}\|x_n-\widehat{p}\|=0.$$

□

Proof. *of (iii)* We recall that the sequence $(x_{n+1})_{n\in\mathbb{N}}$ is defined as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n)U_n x_n, \tag{5.55}$$

where $U_n = \beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n$.
We first show that $\lim_{n\to\infty} \|x_n - T_\delta x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - S_\delta x_n\| = 0$.

Let $p_0 \in Fix(T)\cap Fix(S)$ is the unique solution of the variational inequality $\langle Dp_0, p_0 - x\rangle \leq 0$, for all $x \in Fix(T)\cap Fix(S)$. We compute

$$\begin{aligned}
\|U_n x_n - p_0\|^2 &= \|\beta_n(T_\delta x_n - p_0) + (1-\beta_n)(S_\delta x_n - p_0)\|^2 \\
(\text{ by Lemma 1.6.22}) &= \beta_n\|T_\delta x_n - p_0\|^2 + (1-\beta_n)\|S_\delta x_n - p_0\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\
(T_\delta \text{ nonexpansive and by (1.25)}) &\leq \beta_n\|x_n - p_0\|^2 + (1-\beta_n)\|x_n - p_0\|^2 \\
&- (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\
&= \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2.
\end{aligned}$$

So, we get

$$\|U_n x_n - p_0\|^2 \leq \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 - \beta_n(1-\beta_n)\|T_\delta x_n - S - \delta x_n\|^2. \tag{5.56}$$

We have

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &= \|U_n x_n - p_0 + \alpha_n(B_n x_n - U_n x_n)\|^2 \\
&\leq \|U_n x_n - p_0\|^2 + \alpha_n(\alpha_n\|B_n x_n - U_n x_n\|^2 + 2\|U_n x_n - p_0\|\|B_n x_n - U_n x_n\|) \\
&\leq \|U_n x_n - p_0\|^2 + \alpha_n O(1) \\
(\text{by (5.56)}) &\leq \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 + \alpha_n O(1),
\end{aligned} \tag{5.57}$$

From (5.57), we derive

$$(1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \tag{5.58}$$

and

$$\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \tag{5.59}$$

Now, also we consider two cases.

Case A. $\|x_{n+1} - p_0\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - p_0\|^2$ exists. From (5.58), $\lim_{n\to\infty} \alpha_n = 0$ and since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ we conclude

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{5.60}$$

Furthermore, from (5.59) we have

$$\lim_{n\to\infty} \|S_\delta x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|Sx_n - Tx_n\| = 0; \tag{5.61}$$

since

$$\|x_n - Tx_n\| \le \|x_n - Sx_n\| + \|Sx_n - Tx_n\|,$$

by (5.60) and (5.61) we obtain

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{5.62}$$

By (5.62) and (5.60) and by the demiclosedness of $I - T$ at 0 and of $I - S$ at 0, we can conclude using Lemma 1.5.2 *(ii)*

$$\limsup_{n\to\infty} \langle Dp_0, p_0 - x_n\rangle \le 0. \tag{5.63}$$

Finally, $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

We compute

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &= \|\alpha_n(B_n x_n - p_0) + (1-\alpha_n)(U_n x_n - p_0)\|^2 \\
\text{(by Lemma 1.6.22)} &\le (1-\alpha_n)^2\|U_n x_n - p_0\|^2 \\
&+ 2\alpha_n\langle B_n x_n - p_0, x_{n+1} - p_0\rangle \\
&= (1-\alpha_n)^2\|U_n x_n - p_0\|^2 \\
&+ 2\alpha_n\langle B_n x_n - B_n p_0, x_{n+1} - p_0\rangle + 2\alpha_n\langle B_n p_0 - p_0, x_{n+1} - p_0\rangle \\
(U_n \text{ quasi-nonexpansive}) &\le (1-\alpha_n)^2\|x_n - p_0\|^2 \\
(B_n \text{ contraction}) &+ 2\alpha_n(1-\mu_n\tau)\|x_n - p_0\|\|x_{n+1} - p_0\| \\
&+ 2\alpha_n\langle B_n p_0 - p_0, x_{n+1} - p_0\rangle \\
(B_n := (1-\mu_n D)) &\le (1-\alpha_n)^2\|x_n - p_0\|^2 + \alpha_n(1-\mu_n\tau)(\|x_n - p_0\|^2 + \|x_{n+1} - p_0\|^2) \\
&- 2\alpha_n\mu_n\langle Dp_0, x_{n+1} - p_0\rangle
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &\leq \frac{1-(1+\mu_n\tau)\alpha_n + \alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}\|x_n - p_0\|^2 \\
&- \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle Dp_0, x_{n+1} - p_0\rangle \\
&\leq \frac{1-(1+\mu_n\tau)\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\|x_n - p_0\|^2 \\
&+ \frac{\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}O(1) - \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle Dp_0, x_{n+1} - p_0\rangle \\
&\leq \left(1 - \frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\right)\|x_n - p_0\|^2 \\
&+ \frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\left[-\frac{1}{\tau}\langle Dp_0, x_{n+1} - p_0\rangle + \frac{\alpha_n}{2\mu_n\tau}O(1)\right]. \quad (5.64)
\end{aligned}$$

Using $\sum_{n=1}^{\infty}\alpha_n\mu_n = \infty$, (5.63) and $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n} = 0$, we can apply Lemma 1.5.1 and conclude that

$$\lim_{n\to\infty}\|x_{n+1} - p_0\| = 0.$$

So, $(x_n)_{n\in\mathbb{N}}$ converges strongly to a fixed point of $Fix(T) \cap Fix(S)$.

Case B. $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ does not be definitively nonincreasing. This means that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

Then by Maingé Lemma 5.33 there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies some properties defined previous.
Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}\left(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\right) \\
&\leq \limsup_{n\to\infty}\left(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\right) \\
&\leq \limsup_{n\to\infty}\left(\|x_{n+1} - p_0\| - \|x_n - p_0\|\right) \\
\text{(by (5.55))} \quad &= \limsup_{n\to\infty}\left(\|\alpha_n(B_n x_n - p_0) + (1-\alpha_n)(U_n x_n - p_0)\| - \|x_n - p_0\|\right) \\
(U_n \text{ quasi-nonexpansive}) \quad &\leq \limsup_{n\to\infty}\left(\alpha_n O(1) + \|x_n - p_0\| - \|x_n - p_0\|\right) = 0,
\end{aligned}$$

hence

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1}-p_0\|-\|x_{\tau(n)}-p_0\|\Big)=0. \tag{5.65}$$

By (5.58) we get

$$(1-\beta_{\tau(n)})(1-\delta)\|x_{\tau(n)}-S_\delta x_{\tau(n)}\|^2\leq\|x_{\tau(n)}-p_0\|^2-\|x_{n+1}-p_0\|^2+\alpha_{\tau(n)}O(1), \tag{5.66}$$

and by(5.59) we have

$$\beta_{\tau(n)}(1-\beta_{\tau(n)})\|T_\delta x_{\tau(n)}-S_\delta x_{\tau(n)}\|^2\leq\|x_{\tau(n)}-p_0\|^2-\|x_{n+1}-p_0\|^2+\alpha_{\tau(n)}O(1), \tag{5.67}$$

As in the Case A., we get

(a) $\lim_{n\to\infty}\|x_{\tau(n)}-Sx_{\tau(n)}\|=0,$

(b) $\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0.$

By (a) and (b), as in the Case A, we have

$$\limsup_{n\to\infty}\langle Dp_0,p_0-x_{\tau(n)}\rangle\leq 0. \tag{5.68}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.
As in the Case A., using $\sum_{n=1}^{\infty}\alpha_n\mu_n=\infty$, $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n}=0$, and (5.68) we can apply Xu's Lemma 1.5.1 and we yield that

$$\lim_{n\to\infty}\|x_{\tau(n)}-\widehat{p}\|=0,$$

then, from property (d) of Maingé Lemma and (5.65) we can derive

$$\lim_{n\to\infty}\|x_n-p_0\|=0.$$

□

□

Example 5.1.6. *The sequences*

$$\alpha_n=\frac{1}{n^{\frac{2}{3}}},\quad \mu_n=\frac{\beta}{\rho^2n^{\frac{1}{3}}},\quad \forall n\in\mathbb{N},$$

satisfy the conditions:

1. $(\alpha_n)_{n\in\mathbb{N}}$ *is a sequence in* $(0, 1)$,

2. $(\mu_n)_{n\in\mathbb{N}}$ *is a sequence in* $(0, \mu)$, $\mu < \frac{2\beta}{\rho^2}$,

3. $\lim_{n\to\infty} \alpha_n = 0$,

4. $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0$,

5. $\sum_{n=1}^{\infty} \alpha_n \mu_n = \infty$.

Remark 5.1.7. We remark that (i) and (ii) of Theorem 5.1.5 actually hold for a wide class of nonlinear mappings. In fact, in (i) we can substitute a L-hybrid mapping S with a quasi-nonexpansive mapping because we use only the boundedness of $(S_\delta x_n)_{n\in\mathbb{N}}$.
For the same reason, in (ii) we can replace a nonexpansive mapping T with a quasi-nonexpansive mapping.

Now, consider again the Iemoto-Takahashi theorem and we proceed in another direction.
The concept of W_n-mappings was introduced in [57], [58]. It is one of the main tools in studying convergence of iterative methods to approach a common fixed point of nonlinear mappings, see [54], [47], [56], [62]. Atsushiba and Takahashi, [54], defined a mapping W_n as follows:

Definition 5.1.8. Let C be a nonempty convex subset of a Hilbert space and let $\lambda_{n,1}, ..., \lambda_{n,N} \in]0, 1], n \in \mathbb{N}$. Let $(T_i)_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. We define the mappings $U_{n,1}, ..., U_{n,N}$, for each n, by

$$\begin{aligned} U_{n,1} &:= \lambda_{n,1} T_1 + (1 - \lambda_{n,1}) I, \\ U_{n,2} &:= \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) I, \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ U_{n,N-1} &:= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\ W_n &:= U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I. \end{aligned} \tag{5.69}$$

Such a mapping W_n is called the W_n-mapping generated by $T_1, ..., T_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$.

The next properties of W_n and W are crucial in the proof of Theorems (5.1.15) and (5.1.16).

Lemma 5.1.9. *([54])Let C be a nonempty closed convex set of strictly convex Banach space. Let $T_1, ..., T_N$ be nonexpansive mappings of C into itself and let $\lambda_1, ..., \lambda_N$ be a real numbers such that $0 \leq \lambda \leq 1$ for every $i = 1, ..., N$. Let W be the W-mapping, i.e., the mapping of C into itself generated, following the previous scheme (5.69), by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then, $U_1, ..., U_{N-1}$ e W are also nonexpansive.*

Lemma 5.1.10. *([54])Let C be a nonempty closed convex set of strictly convex Banach space. Let $T_1, ..., T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be a real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let W be the $W-$mapping of C into itself generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(W) = \bigcap\limits_{i=1}^{N} Fix(T_i)$.*

Lemma 5.1.11. *([47])Let C be a nonempty closed convex set of strictly convex Banach space. Let $T_1, ..., T_N$ be nonexpansive mappings of C into itself and $(\lambda_{n,i})_{i=1}^{N}$ be a sequences in $[0, 1]$ such that $\lambda_{n,i} \to \lambda_i (i = 1, ..., N)$. Moreover for every $n \in N$, let W and W_n, generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$ and $T_1, ..., T_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, respectively. Then for every $x \in C$, it follows that*

$$\lim_{n\to\infty} \|W_n x - W x\| = 0.$$

Let E be a Banach space and K be a nonempty closed convex subset of E. Let $(T_n)_{n\in\mathbb{N}} : K \to K$ be a family of mappings. Then $(T_n)_{n\in\mathbb{N}}$ is said to satisfy the $AKTT$-condition, see [53], if for each bounded subset B of K, one has

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : \quad z \in B\} < \infty. \tag{5.70}$$

The following is an important result on a family of mappings $(T_n)_{n\in\mathbb{N}}$ satisfying the $AKTT$-condition.

Lemma 5.1.12. *Let K be a nonempty and closed subset of Banach space E, and let $(T_n)_{n\in\mathbb{N}}$ be a family of mappings of K into itself which satisfies the $AKTT$-condition (5.70). Then, for each $x \in K$, $(T_n x)_{n\in\mathbb{N}}$ converges strongly to a point in K. Moreover, let the mapping $T : K \to K$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in K,$$

for each bounded subset B of K,

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : \quad z \in B\} = 0.$$

If a family of mappings $(T_n)_{n\in\mathbb{N}}$ satisfy the $AKTT$-condition, then it is unnecessary that $Fix(T)=\bigcap_{i=1}^{\infty}Fix(T_i)$.

Example 5.1.13. *Let $E=\mathbb{R}$ and $K=[0,2]$. Let $(T)_{n\in\mathbb{N}}:K\to K$ a family of mapping defined by*

$$T_1x=0,\quad T_nx=\frac{1}{n}(1+x),\quad n\geq 2.$$

We compute

$$\begin{aligned}\sum_{n=1}^{\infty}\sup\Big\{|T_{n+1}x-T_nx|:\quad x\in K\Big\} &= \sup\Big\{|T_2x-T_1x|:\quad x\in K\Big\}\\ &+ \sum_{n=2}^{\infty}\sup\Big\{|T_{n+1}x-T_nx|:\quad x\in K\Big\}\\ &= \sup\Big\{\frac{1}{2}(1+x):\quad x\in K\Big\}\\ &+ \sum_{n=2}^{\infty}\sup\Big\{\Big|\frac{-1}{n(n+1)}(1+x)\Big|:\quad x\in K\Big\}\\ &= \frac{3}{2}+\sum_{n=2}^{\infty}\frac{3}{n(n+1)}<\infty\end{aligned}$$

Then $(T_n)_{n\in\mathbb{N}}$ satisfy the $AKTT$-condition (5.70)*.*
For each $x\in K$, $\lim_{n\to\infty}T_nx=0$.
If we define $T:K\to K$ as follows: $Tx=\lim_{n\to\infty}T_nx$, we get $Tx=0$, for all $x\in K$.
We conclude $Fix(T)\neq\bigcap_{n=1}^{\infty}Fix(T_n)$.

Definition 5.1.14. $((T_n)_{n\in\mathbb{N}},T)$ satisfy the $AKTT$-condition (5.70) if $(T_n)_{n\in\mathbb{N}}$ satisfy the $AKTT$-condition with $Fix(T)=\bigcap_{i=1}^{\infty}Fix(T_i)$.

Therefore, our results improve and generalize the Iemoto and Takahashi's Theorem in two directions:

- We get strong convergence;
- Strong convergence holds for a finite family of nonexpansive mappings and for a countable family of nonspreading mappings.

Theorem 5.1.15. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(S_j)_{j=1}^{N}:C\to C$ be a finite family of nonexpansive mappings and* *let* $(T_i)_{i=1}^{\infty} : C \to C$ *be a countable family of nonspreading mappings such that* $F = [\bigcap_{j=1}^{N} Fix(S_j)] \cap [\bigcap_{i=1}^{\infty} Fix(T_i)] \neq \emptyset$. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad \textit{chosen} \quad \textit{arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_i x_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$, W_n *is generated by* $S_1, ..., S_N$ *and* $\lambda_{n,1}, ..., \lambda_{n,N}$, *as in the scheme (5.69). Assume that* β_n *is strictly decreasing and* $\beta_0 = 1$. *Then, the following hold:*

i) If $\liminf_{n\to\infty} \alpha_n > 0$ *and* $\lim_{n\to\infty} \beta_n = 0$, *then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$;

ii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n > 0$, *then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $q \in F$.

Proof Since C_n is closed and convex, D_n is closed and convex. To show that $F \subset D_n$ for all $n \in \mathbb{N}$, we prove that $F \subset C_n$ for all $n \in \mathbb{N}$. Indeed, noting that $\beta_n + \sum_{i=1}^{n}(\beta_{i-1} -$

$\beta_i) = 1$, by the nonexpansivity of W_n and T_i, for $i = 1, ..., N$, we have, for each $p \in F$,

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n\big[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n\big] \\
&- \alpha_n\big(\beta_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\big)p + \alpha_n p - p\| \\
&= \big\|(1-\alpha_n)(x_n - p) + \alpha_n\big[\beta_n(W_n x_n - p) + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - p)\big]\big\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\big\|\beta_n(W_n x_n - p) + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - p)\big\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\big[\beta_n\|W_n x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - p\|\big] \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\big[\beta_n\|x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|x_n - p\|\big] \\
&= \|x_n - p\|. \qquad (5.71)
\end{aligned}$$

This implies that $p \in C_n, \quad \forall n \in \mathbb{N}$. Hence, C_n is nonempty for all $n \in \mathbb{N}$.
From $x_{n+1} = P_{D_n}x$, we have

$$\|x_{n+1} - x\| \leq \|v - x\|, \quad \forall v \in D_n, \quad \forall n \in \mathbb{N}.$$

Since $P_F x \in F \subset D_n$,

$$\|x_{n+1} - x\| \leq \|P_F x - x\|, \quad \forall n \in \mathbb{N}. \qquad (5.72)$$

So, $(x_n)_{n\in\mathbb{N}}$ is bounded and from (5.71), also, $(y_n)_{n\in\mathbb{N}}$ is bounded.
Since $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$,

$$x_{n+2} = P_{D_{n+1}}x \in D_{n+1} \subset D_n, \quad \forall n \in \mathbb{N}.$$

This implies that

$$\|x_{n+1} - x\| \leq \|x_{n+2} - x\|, \quad \forall n \in \mathbb{N}. \qquad (5.73)$$

It follows from (5.72) and (5.73) that the limit of $(x_n - x)_{n\in\mathbb{N}}$ exists.
From $x_{m+1} = P_{D_m}x \in D_m \subset D_n$ for all $m \geq n$ and by Lemma (1.3), we deduce

$$\langle x_{n+1} - x, x_{m+1} - x_{n+1}\rangle \geq 0, \quad \forall m \geq n. \qquad (5.74)$$

From (5.74) il follows that

$$\begin{aligned}
& \|x_{m+1} - x_{n+1}\|^2 \\
= \;& \|x_{m+1} - x - (x_{n+1} - x)\|^2 \\
= \;& \|x_{m+1} - x\|^2 + \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x\rangle \\
= \;& \|x_{m+1} - x\|^2 + \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x_{n+1} + x_{n+1} - x\rangle \quad (5.75) \\
= \;& \|x_{m+1} - x\|^2 - \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x_{n+1}\rangle \\
\leq \;& \|x_{m+1} - x\|^2 - \|x_{n+1} - x\|^2.
\end{aligned} \tag{5.76}$$

So, since the limit of $\|x_n - x\|$ exists, we have

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0.$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $q \in C$ such that

$$x_n \to q, \quad as \quad n \to \infty. \tag{5.77}$$

Putting $m = n + 1$ in (5.75),

$$\lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = 0,$$

i.e.,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{5.78}$$

So, from the fact that $x_{n+1} = P_{D_n}x \in D_n \subset C_n$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0, \quad as \quad n \to \infty,$$

and we derive

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad as \quad n \to \infty. \tag{5.79}$$

It follows, from (5.77) and (5.79), that

$$\lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|y_n - x_n + x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = \|q - p\|, \quad \forall p \in F. \tag{5.80}$$

We prove (i). We compute

$$\begin{aligned}
y_n \;&=\; (1 - \alpha_n)x_n + \alpha_n\big[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n\big] \pm \alpha_n(1 - \beta_n)x_n \\
&=\; (1 - \alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n)] + \alpha_n(1 - \beta_n)x_n \\
&=\; (1 - \alpha_n\beta_n)x_n + \alpha_n\beta_n W_n x_n + \alpha_n \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n),
\end{aligned}$$

we deduce

$$\alpha_n \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)(T_i x_n - x_n) = y_n - (1-\alpha_n\beta_n)x_n - \alpha_n\beta_n W_n x_n$$
$$= (1-\alpha_n\beta_n)(y_n - x_n) + \alpha_n\beta_n(y_n - W_n x_n). \quad (5.81)$$

For all $p \in \bigcap_{i=1}^{N} Fix(T_i)$, by the quasi-nonexpansivity of T_i, for $i = 1, ..., N$,

$$\begin{aligned} \|x_n - p\|^2 &\geq \|T_i x_n - T_i p\|^2 = \|T_i x_n - p\|^2 = \|T_i x_n - x_n + (x_n - p)\|^2 \\ &= \|T_i x_n - x_n\|^2 + \|x_n - p\|^2 + 2\langle T_i x_n - x_n, x_n - p\rangle, \end{aligned}$$

and so, we have

$$\|T_i x_n - x_n\|^2 \leq 2\langle x_n - T_i x_n, x_n - p\rangle, \quad \forall i \in \mathbb{N}. \quad (5.82)$$

Since $(\beta_n)_{n\in\mathbb{N}}$ is strictly decreasing and from (5.82) and (5.81), we obtain

$$\begin{aligned} \alpha_n(\beta_{i-1}-\beta_i)\|T_i x_n - x_n\|^2 &\leq \alpha_n \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_i x_n - x_n\|^2 \\ &\leq 2\alpha_n \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\langle x_n - T_i x_n, x_n - p\rangle \\ &= 2\langle (1-\alpha_n\beta_n)(y_n - x_n) + \alpha_n\beta_n(y_n - W_n x_n), x_n - p\rangle, \end{aligned}$$

so,

$$\begin{aligned} \|T_i x_n - x_n\|^2 &\leq \frac{2}{\alpha_n(\beta_{i-1}-\beta_i)}[(1-\alpha_n\beta_n)\langle y_n - x_n, x_n - T_i p\rangle \quad (5.83) \\ &+ \alpha_n\beta_n\langle y_n - W_n x_n, x_n - p\rangle], \quad \forall i \in \mathbb{N}. \end{aligned}$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\lim_{n\to\infty} \beta_n = 0$, it follows, from (5.79) and (5.83),

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0, \quad \forall i \in \mathbb{N}. \quad (5.84)$$

From (5.77) and (5.84) and since $(T_i)_{i=1}^{\infty}$ are nonspreading,

$$\|T_i q - T_i x_n\|^2 \leq \|x_n - q\|^2 + 2\langle q - T_i q, x_n - T_i x_n\rangle, \quad \forall i \in \mathbb{N},$$

it implies that

$$\lim_{n\to\infty} \|T_i q - T_i x_n\| = 0, \quad \forall i \in \mathbb{N}. \quad (5.85)$$

Hence, we conclude

$$0 \leq \|q - T_i q\| \leq \|q - x_n\| + \|x_n - T_i x_n\| + \|T_i x_n - T_i q\|, \quad \forall i \in \mathbb{N}, \quad (5.86)$$

i.e. $\|q - T_i q\| = 0$. Hence,

$$q \in \bigcap_{i=1}^{\infty} Fix(T_i).$$

Now, we prove (ii). We need show that

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0.$$

For every p, using again $\sum_{i=1}^{n}(\beta_{i-1} - \beta_i) + \beta_n = 1$, we compute

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\alpha_n\beta_n)x_n + \alpha_n\beta_n W_n x_n + \alpha_n \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)(T_i x_n - x_n) \\
&- (\sum_{i=1}^{n}(\beta_{i-1}-\beta_i) + \beta_n)p\|^2 \\
&= \|(\beta_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i) - \alpha_n\beta_n)x_n + \beta_n(\alpha_n W_n x_n - p) \\
&+ \alpha_n \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)(T_i x_n - x_n) - \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)p\|^2 \\
&= \Big\|\beta_n[(1-\alpha_n)x_n + \alpha_n W_n x_n - p] + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)[(1-\alpha_n)x_n + \alpha_n T_i x_n - p]\Big\|^2 \\
&\leq \beta_n\big\|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2 + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|[(1-\alpha_n)x_n + \alpha_n T_i x_n \\
&- (1-\alpha_n+\alpha_n)p]\big\|^2 \\
&\leq \beta_n\|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2 + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)[(1-\alpha_n)\|x_n - p\|^2 \\
&+ \alpha_n\|T_i x_n - p\|^2] \\
&\leq \beta_n\|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2 + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|x_n - p\|^2 \\
&\leq \beta_n\|(1-\alpha_n)x_n + \alpha_n W_n x_n - (1-\alpha_n+\alpha_n)p\|^2 + (1-\beta_n)\|x_n - p\|^2 \qquad (5.87) \\
&\leq \beta_n(1-\alpha_n)\|x_n - p\|^2 + \alpha_n\beta_n\|W_n x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \qquad (5.88) \\
&\leq \|x_n - p\|^2,
\end{aligned}$$

Since, by (5.87), we have

$$\|y_n - p\|^2 - \|x_n - p\|^2 \leq \beta_n\|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2 - \beta_n\|x_n - p\|^2,$$

then, from (5.88), we obtain

$$\begin{aligned} 0 &\leq \|x_n - p\|^2 - \beta_n\|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2 - (1-\beta_n)\|x_n - p\|^2 \\ &= \beta_n\big[\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2\big] \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \to 0. \end{aligned} \tag{5.89}$$

Since $\liminf\limits_{n\to\infty} \beta_n > 0$, we get, from (5.89),

$$\lim_{n\to\infty} \big(\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2\big) = 0. \tag{5.90}$$

By (5.90) and (5.83)

$$\|(1-\alpha_n)x_n+\alpha_n W_n x_n - p\|^2 = (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|W_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - W_n x_n\|^2,$$

we conclude

$$\begin{aligned} \alpha_n(1-\alpha_n)\|x_n - W_n x_n\|^2 &= \big(\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2\big) \\ &- \alpha_n\|x_n - p\|^2 + \alpha_n\|W_n x_n - p\|^2 \\ &\leq \big(\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2\big) \\ &- \alpha_n\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 \\ &= \|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n W_n x_n - p\|^2 \to 0. \end{aligned}$$

Since $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0. \tag{5.91}$$

Since $\|y_n - x_n\| \to 0$, as $n \to \infty$, and for (5.91), we have

$$\lim_{n\to\infty} \|y_n - W_n x_n\| = 0.$$

Since $x_n \to q \in \bigcap\limits_{i=1}^{\infty} Fix(T_i)$, as $n \to \infty$, and for (5.91), we derive

$$\|q - W_n q\| \leq \|q - x_n\| + \|x_n - W_n x_n\| + \|W_n x_n - W_n q\| \leq 2\|q - x_n\| + \|x_n - W_n x_n\|,$$

which implies that

$$q \in Fix(W_n),$$

and we conclude that $q \in F$.

Theorem 5.1.16. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $(S_j)_{j=1}^N : C \to C$ be finite family of nonexpansive mappings and let $(T_i)_{i=1}^\infty : C \to C$ be a countable family of nonexpansive mappings such that $F = [\bigcap_{j=1}^N Fix(S_j)] \cap [\bigcap_{i=1}^\infty Fix(T_i)] \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + (1-\beta_n)T_n x_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^n C_j, \\ x_{n+1} = P_{D_n} x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$, W_n is generated by $S_1, ..., S_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, as in the scheme (5.69). Assume that either $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the AKTT-condition. Then the following hold:

i) If $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in \bigcap_{i=1}^\infty Fix(T_i)$;

ii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in F$.

Proof By a similar process as in the proof of Theorem (5.1.15) we can say that $(x_n)_{n\in\mathbb{N}}$ is bounded, converges strongly to some $q \in C$ and

$$\|x_n - y_n\| \to 0, \quad as \quad n \to \infty.$$

We first prove (i). We compute

$$y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + (1-\beta_n)T_n x_n] \pm \alpha_n\beta_n x_n,$$

$$y_n - x_n = \alpha_n(1-\beta_n)(T_n x_n) + \alpha_n\beta_n(W_n x_n - x_n),$$

$$T_n x_n - x_n = \frac{1}{\alpha_n(1-\beta_n)}(y_n - x_n) - \frac{\beta_n}{1-\beta_n}(W_n x_n - x_n)$$

so

$$\|T_n x_n - x_n\| \leq \frac{1}{\alpha_n(1-\beta_n)}\|y_n - x_n\| + \frac{\beta_n}{1-\beta_n}\|W_n x_n - x_n\|.$$

Since $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, we obtain

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{5.92}$$

Furthermore, by Lemma (5.1.12) and (5.92) we have

$$\begin{aligned}\|x_n - Tx_n\| \leq & \ \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ \leq & \ \|x_n - T_n x_n\| + \sup\{\|T_n z - Tz\| : z \in (x_n)_{n\in\mathbb{N}}\},\end{aligned}$$

so

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{5.93}$$

Since each T_n is a nonspreading mapping, Lemma (**??**) shows that T is a nonspreading mapping, so

$$\|Tq - Tx_n\|^2 \leq \|x_n - q\|^2 + 2\langle q - Tq, x_n - Tx_n\rangle, \quad \forall n \in \mathbb{N},$$

$$\lim_{n\to\infty} \|Tq - Tx_n\| = 0. \tag{5.94}$$

Now, from (5.92) and (5.94) we get

$$\|q - Tq\| \leq \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\|, \tag{5.95}$$

this shows that $q \in Fix(T)$. Since $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the $AKTT$-condition, follows that $q \in \bigcap_{i=1}^{\infty} Fix(T_i) = Fix(T)$. This completes (i).
Now we show (ii). Using a similar process as in the proof of Theorem (5.1.15) and by (5.93),(5.94),(5.95), we can say that

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0, \ \lim_{n\to\infty} \|x_n - T_n x_n\| = 0, \ \lim_{n\to\infty} \|Tx_n - Tq\| = 0, \ \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Finally, from

$$\|q - W_n q\| \leq \|q - x_n\| + \|x_n - W_n x_n\| + \|W_n x_n - W_n q\| \leq 2\|x_n - q\| + \|x_n - W_n x_n\|,$$

we get

$$\lim_{n\to\infty} \|q - W_n\| = 0$$

i.e. $q \in Fix(W_n)$. Moreover,

$$\|q - Tq\| \leq \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\|,$$

where the last terms go to zero as n go to infinity, so $q \in Fix(T)$. Whereas $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the $AKTT$-condition, $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$. Then, we conclude that $q \in F$.
This complete (ii).

Putting $T_i = T$ for all $i \in \mathbb{N}$ in Theorem (5.1.15) and Theorem (5.1.16), we obtain the following.

Corollary 5.1.17. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(S_j)_{j=1}^N : C \to C$ be a finite family of nonexpansive mappings and let $T : C \to C$ be a nonspreading mapping such that $F = [\bigcap_{j=1}^{N} Fix(S_j)] \cap Fix(T) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + (1-\beta_n)Tx_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$, W_n is generated by $S_1, ..., S_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, as in the scheme (5.69). Then, the following hold:

i) If $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in Fix(T)$;

ii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in F$.

Putting $S_j = I$, for $j = 1, ..., N$, in Theorem (5.1.15) and Theorem (5.1.16), we have the following.

Corollary 5.1.18. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(T_i)_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n(1-\beta_n))x_n + \alpha_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)T_i x_n, \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$. Assume that β_n is strictly decreasing and $\beta_0 = 1$. If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

Corollary 5.1.19. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(T_i)_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n(1-\beta_n))x_n + \alpha_n(1-\beta_n)T_n x_n, \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n} x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$. Assume that $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the AKTT-condition. If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

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