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Topics in Metric Fixed Point Theory and Stability of Dynamical Systems

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Abstract

In this thesis we introduce iterative methods approximating fixed points for non-linear operators defined in infinite-dimensional spaces. The starting points are the Implicit and Explicit Midpoint Rules generating polygonal functions approximating a solution for an ordinary differential equation in finite-dimensional spaces.

Our study has the purpose of determining suitable conditions on

- the mapping,
- the underlying space,
- the coefficients defining the method,

in order to get strong convergence of the generated sequence to a common solution of a fixed point problem and a variational inequality.

The contributions to this topic appear in the papers:

G. Marino, R. Zaccone, *On strong convergence of some midpoint type methods for nonexpansive mappings*, J. Nonlinear Var. Anal., vol. 1 (2017), n. 2, 159-174;

G. Marino, B. Scardamaglia, R. Zaccone, *A general viscosity explicit midpoint rule for quasi-nonexpansive mappings*, J. Nonlinear and Convex Anal., vol. 18 (2017), n. 1, 137-148;

J. Garcia-Falset, G. Marino, R. Zaccone, *An explicit midpoint algorithm in Banach spaces*, to appear in J. Nonlinear and Convex Anal. (2017).

Not rarely a fixed point iteration scheme is used to find a stationary state for a dynamical system. However the fixed points may not be stable. In view of this, we study some conditions under which the asymptotic stability for the critical points of a certain dynamical system is ensured.

Our contribution to this topic appears in the paper:

R. P. Agarwal, G. Marino, H. K. Xu, R. Zaccone, *On the dynamics of love: a model including synergism*, J. Linear and Nonlinear Anal., vol. 2, n. 1 (2016), 1-16.

Contents

Introduction

The fixed point problem, at the basis of the Fixed Point Theory, can be formulated as:

Let X be a set, A and B two nonempty subsets of X with $A \cap B \neq \emptyset$, and $f : A \to B$ a map. When does a point $x \in A$ verify $f(x) = x$?

In Fixed Point Theory, two major areas can be distinguished:

(i) Topological Fixed Point Theory,

(ii) Metric Fixed Point Theory.

Historically, the borderline between these branches has been signed respectively by the discovery of the following theorems

(i) **Theorem 0.0.1 (Brouwer's Fixed Point Theorem, 1912)** *Let B be a closed ball in $\mathbb{R}^n$. Then any continuous function $f : B \to B$ has at least one fixed point.*

(ii) **Theorem 0.0.2 (Banach's Contraction Principle, 1922)** *Let (E, d) be a metric space and $f : E \to E$ be a contraction mapping. Then f has a unique fixed point in E and, for each $x_0 \in E$, the sequence of iterates $(f^n(x_0))_{n\in\mathbb{N}}$ converges to this fixed point.*

The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ takes the name of *Picard iterates* from the method of successive approximations for proving the existence and uniqueness of solutions of diffential equations used by Picard.

Recall that, given a metric space E with distance function (metric) d, a mapping $f : E \to E$ is said to be *lipschitzian* if there exists a constant $L \geq 0$ such that

$$d(f(x), f(y)) \leq Ld(x, y) \quad \forall x, y \in E.$$

The smallest L for which this condition holds is said Lipschitz constant for f. In particular, if $L < 1$ then f is called *contraction*.

A lipschitzian mapping with constant $L = 1$ is said to be nonexpansive. Explicitly, $T : E \to E$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \quad \forall x, y \in E.$$

From the definition, it follows that the class of nonexpansive mappings includes that of contractions and all isometries. On the other hand, under the hypotheses of Banach's Contraction Principle, it is not guaranted neither the existence nor the uniqueness of a fixed point for a nonexpansive mapping; besides the sequence of Picard iterates may fail to converge to the fixed point even if it exists. To this regard, the following examples can be mentioned:

Example 0.0.3 *[37, Example 2.1] Let $C[0,1]$ be the space of continuous real valued functions on $[0,1]$ with the standard supremum norm:*

$$\text{For } x \in C[0,1],\ \|x\|_\infty = \sup\{|x(t)|, t \in [0,1]\}$$

Set $C = \{x \in C[0,1] : 0 = x(0) \leq x(t) \leq x(1) = 1\}$. C is closed (also bounded and convex), and since $C[0,1]$ is complete in the metric induced by $\|\cdot\|_\infty$, so is C. The mapping $T : C \to C$ defined by $(Tx)(t) = tx(t)$, $x \in C$, $t \in [0,1]$, is both nonexpansive and fixed point free.

Example 0.0.4 *[19, Example 6.4] Let $B_1 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and let $T : B_1 \to B_1$ denote an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as unique fixed point, but the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} = Tx_n$ does not converge to the fixed point for any initial point $x_0 \in B_1 \setminus \{(0,0)\}$.*

In order to get significant results for the class of nonexpansive mappings, the condition to be considered on the space is that E must be a Banach space. Recognition of fixed point existence results for nonexpansive mappings has a significant line of research in the works of Browder [9], Gohde [36] and Kirk [49] published in 1965:

Theorem 0.0.5 (Browder-Gohde's theorem) *If C is a bounded, closed and convex subset of a uniformly convex Banach space X and if $T : C \to C$ is nonexpansive, then T has a fixed point.*

Theorem 0.0.6 (Kirk's theorem) *Let C be a weakly-compact, convex subset of a Banach space X. Assume that C has the normal structure property , then any nonexpansive mapping $T : C \to C$ has a fixed point.*

They were proved independently. Furthermore, with respect to the first result, the second one makes use of the normal structure property of a Banach space, introduced by Brodskij and Milman (1948) and satisfied by every uniformly convex Banach space.

Nonexpansive mappings, beyond to be a generalization of contractions, represent a class of interest for the following motivations (see Bruck's survey [16]):

> They constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties;
>
> They appear in applications as the transition operators for initial value problems of differential inclusions of the form
>
> $$0 \in \frac{du}{dt} + T(t)u,$$
>
> where the operators $T(t)$ are, in general, set-valued and are accretive or dissipative and minimally continuous.

Moreover, iterative methods for nonexpansive mappings can be applied to find a solution of a variational inequality and a minimizer of a convex function. For instance, let C a nonempty, closed and convex subset of a Hilbert space H and $A : C \to H$ a nonlinear operator. The variational inequality problem associated to A, and designated with $VIP(A, C)$, is stated as

$$\text{find } x^* \in C \text{ such that}$$

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{0.0.1}$$

Let λ a positive constant, $VIP(A, C)$ is equivalent to the following fixed point problem

$$x^* = P_C(x^* - \lambda Ax^*),$$

where P_C is the metric projection onto C, that is a nonexpansive mapping in this case.

If, in addition, $f : C \to \mathbb{R}$ is a convex and differentiable function, denoted the gradient operator of f with A, variational inequality (0.0.1) is the optimality condition for the minimization problem

$$\min_{x \in C} f(x).$$

These facts promoted the development of two basic research lines:

- Study of suitable assumptions regarding the structure of the underlying space X and/or restrictions on T to ensure the existence of at least a fixed point;
- Assumed that a fixed point exists, construction of iterative methods for approximating the fixed points of T.

In this thesis we focus our attention on the second survey. In this direction, one of the basic methods for approximating fixed points of nonexpansive mappings dates back to 1953 and is known as *Mann's method*, in light of Mann [55]. Let C be a nonempty, closed and convex subset of a Banach space X, Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n), \quad n \geq 0, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a real control sequence in $(0,1)$.

However Mann's method, without assuming additional assumptions on the mapping and/or the space, does not yield strong convergence, even in a Hilbert space setting, as shown in the celebrated counterexample by Genel and Linderstrauss [33]. Ever since, several modifications to the original Mann scheme have been proposed in order to get strong convergence; just to mention some of the most relevant, we recall the work of Ishikawa [44] , Halpern [38] and Moudafi [60]. In particular, Moudafi's method (including Halpern's one) is said to be a regularization method, since, with the introduction of a contraction function, it allows to select the fixed point of a nonexpansive mapping that is also the solution of a certain variational inequality.

In this thesis we continue to carry out the research along the same direction, following the investigation line opened by M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H-K Xu in [5]. They introduced a new fixed point iteration for nonlinear nonexpansive mappings in infinite-dimensional Hilbert spaces H following the formal definition of the Implicit Midpoint Rule (IMR), which generates polygonal functions approximating a solution of a differential equation in finite-dimensional spaces. The method proposed in [5] is implicit and is a Mann-type scheme, named Implicit Midpoint Rule for nonexpansive mappings:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{0.0.2}$$

where $T : H \to H$ is a nonexpansive mapping and $(t_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$.

For this procedure, the authors of [5] proved a weak convergence result in the setting of Hilbert spaces.

It is our purpose to provide a variation to (0.0.2) in order to get strong convergence.

In Chapter 1, following the same modification line adopted in 2015 by N. Hussain, G. Marino, L. Muglia, L. Alamri in their work [41], we introduce a modified IMR for nonexpansive mappings which differs from the previous ones available in literature. The framework is still that of a Hilbert space H. The proposed algorithm differs from scheme (0.0.2) for the introduction of a term $\alpha_n\mu_n(u-x_n)$ that can also be infinitesimal. The proposed scheme is

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}) + \alpha_n\mu_n(u-x_n), \quad n \geq 0, \end{cases} \tag{0.0.3}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$ and $T : H \to H$ is a nonexpansive mapping.

We show that, under suitable conditions on the parameters $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$, the sequence $(x_n)_{n\in\mathbb{N}}$, generated by (0.0.3), converges strongly to the fixed point of T nearest to u.

A different situation occurs if the starting point is the less known numerical Explicit Midpoint Rule (EMR) for constructing polygonal functions in finite-dimensional spaces. Through formal analogy with EMR, a recursive scheme for the fixed point problem $x = Tx$ is obtained and it is given by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1-t_n)x_n + t_n T(x_n) \\ x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n+\bar{x}_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{0.0.4}$$

with $(t_n)_{n\in\mathbb{N}} \in (0,1)$ and $T : H \to H$ is a nonexpansive mapping.

We designate it with Explicit Midpoint Rule for nonexpansive appings.

If the midpoint $\frac{x_n+\bar{x}_{n+1}}{2}$ in the evaluation of T in (0.0.4) is replaced with any convex combination between x_n and $\bar{x}_{n+1}$, then scheme (0.0.4) is named General Explicit Midpoint Rule for nonexpansive mappings.

We provide for the latter scheme the same formal modification as for the IMR for nonexpansive mappings, following [41]. The proposed method is given by

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) + \alpha_n\mu_n(u-x_n), \quad n \geq 0, \end{cases} \tag{0.0.5}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$.

Analougsly to the first case, the convergence result is also a localization result, since, under certain conditions on the coefficients, we show that the sequence generated by (0.0.5) strongly converges to the fixed point of T closest to u.

In Chapter 2, inspired by work [91] of H. K. Xu, M. A. Alghamdi, N. Shahzad and the paper [48] of Y. Ke and C. Ma, we propose an explicit iterative method, starting from the EMR for nonexpansive mappings. The method is obtained from this last, introducing a contraction f and replacing the midpoint between two terms in the evaluation point of the operator with any convex combination of the same terms. The resulting iterative scheme is said Generalized Viscosity Explicit Midpoint Rule (GVEMR) and is given by

$$\begin{cases} x_0 \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0. \end{cases} \tag{0.0.6}$$

The purpose is to approximate fixed points of quasi-nonexpansive mappings in Hilbert spaces. We recall that a mapping $T : C \to C$, with C a nonempty subset of a Banach space X, is said to be quasi-nonexpansive if T has at least a fixed point and verify

$$\|Tx - q\| \leq \|x - q\|, \quad \forall q \in Fix(T), \forall x \in C.$$

This class of mappings, besides for including the class of nonexpansive operators with at least a fixed point, is of interest for the researchers because they can be discontinuous (see, for example, the pioneering works [25], [27] and more recently the monograph [19]).

In a second stage, strong convergence results are proved for the class of quasi-nonexpansive mappings in the more general setting of p-uniformly convex Banach spaces, for $1 < p < \infty$, with new techniques with respect to those employed in a Hilbert spaces framework.

The first main result is applicable to l_p spaces; the second one, using the concept of ψ -expansive mappings (see [31] and references therein), is applicable to L_p spaces which fail to have a weakly continuous duality mapping.

The last part of this thesis concerns the stability of equilibrium solutions for a given dynamical system. Often the search for equilibrium solutions can be reduced to the solution of a fixed point problem for a certain operator. On the other hand, the stability of fixed points, without assuming additional assumptions, is not guaranteed. For instance, consider the sequence of nonexpansive mappings $(f_\alpha(x))_{\alpha\in(0,1)}$

defined by

$$f_\alpha(x) = \begin{cases} 0 & -1 \le x < 0, \\ \alpha x, & 0 \le x \le 1, \\ \frac{\alpha}{2}(x-1) + \alpha, & 1 < x \le 2. \end{cases}$$

It results that $F(f_\alpha) = \{z \in [-1, 2] : f_\alpha z = z\} = \{0\}$ for each α.

Moreover, for $\alpha \to 1$, the sequence $(f_\alpha(x))_{\alpha \in (0,1)}$ converges uniformly to the function

$$f(x) = \begin{cases} 0 & -1 \le x < 0, \\ x, & 0 \le x \le 1, \\ \frac{1}{2}(x+1), & 1 < x \le 2. \end{cases}$$

Nevertheless, the fixed points set of f is $F(f) = \{z \in [-1, 2] : fz = z\} = [0, 1]$.

In Chapter 3, we study the stability of critical points for a system of two first-order nonlinear differential equations describing the dynamics of an interpersonal relation between two individuals, according to a current trend introduced in 1988 by S. H. Strogatz. Under certain assumptions on the coefficients of the proposed model the solutions at equilibrium are asymptotically stable, while some other are ultimately unstable.

Chapter 1

Midpoint-type methods for Nonexpansive Operators

Preface.

Numerical methods are employed for approximating the solutions of an Initial Value Problem (IVP) of Ordinary Differential Equations (ODE's) as

$$\begin{cases} x'(t) = \Phi(x(t)), \\ x(t_0) = x_0. \end{cases}$$

Among the basic ones, the so called *Midpoint methods*, in their implicit and explicit formulation (IMR and EMR, respectively), show a considerable level of accuracy. In both cases the final aim is to generate a sequence of polygonal functions $(Y_N)_{N\in\mathbb{N}}$ converging to the exact solution of the differential problem under suitable hypotheses on the function Φ.

The stability analysis of such differential problems starts determining its critical points, that are the solutions of the equation $\Phi(x(t)) = 0$.

At the same time, if the function Φ is defined as $\Phi(x) = x - g(x)$, the problem of finding zeros for Φ can be reformulated as a fixed point problem for g, $g(x) = x$. In 2014, this consideration motivated M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H-K Xu to introduce a new iterative method for nonexpansive nonlinear operators following the formal definition of the IMR. It appeared for the first time in [5] and is called Implicit Midpoint Rule for nonexpansive mapping. For this procedure, the authors of [5] proved a weak convergence result in the setting of Hilbert spaces.

The purpose of our contribution, in [58], is to provide some modifications for the IMR and the EMR for nonexpansive mappings. In both cases, strong convergence results are proved under suitable assumptions on the control sequences, in the framework of Hilbert spaces. The same conditions allow us to select a particular fixed point of the operator.

1.1 Classical Midpoint methods for ODE's problems.

Consider the Initial Value Problem (IVP) of Ordinary Differential Equations (ODE's)

$$\begin{cases} x'(t) = \Phi(x(t)) \\ x(t_0) = x_0, \end{cases} \tag{1.1.1}$$

where $\Phi : \mathbb{R}^s \to \mathbb{R}^s$ is a continuous function.

A lot of equations like (1.1.1) cannot be solved in closed form. In this context, numerical integration turns out to be a fundamental tool in order to get informations about the solution trajectory.

This section is intended as a survey on some numerical methods among the basic ones: *Euler's method* and some of its improvements, called *Midpoint methods*. They belong to the more general class of discretization methods, since they reduce a differential problem to a finite difference scheme in order to approximate the solution of the former. In detail, assigned a range $[t_0, T]$ for the time variable t and fixed a positive integer N, they consist in

1. Determining the quantity $h = \frac{T-t_0}{N}$, also known as *step size*;
2. Constructing a mesh of t-values, $\{t_n\}_{n=0}^N$, called *time steps*, through the recursive formula
$$t_n = t_0 + nh, \quad n = 0, \cdots, N; \tag{1.1.2}$$
3. Computing a set of y-values, $\{y_n\}_{n=0}^N$, where y_n is the approximate value of $x(t_n)$, using a certain finite difference scheme, that is said to be

   Explicit, if defined as
   $$y_{n+1} = H(\Phi, y_n, y_{n-1}, \cdots, y_{n+1-m})$$
   where the value y_{n+1} is computed in terms of m previously determined quantities y_j, called *steps*, for $j = n+1-m, \cdots, n-1, n$;

   Implicit, if represented by
   $$G(\Phi, y_{n+1}, \cdots, y_{n+1-m}) = 0$$
   that cannot be put in the explicit form.

Let Y_N be the polygonal obtained joining via straight lines two consecutive nodes $(t_n, y_n), (t_{n+1}, y_{n+1})$ for $n = 0, \cdots, N-1$. Under suitable conditions on the function Φ defining the problem, the sequence $(Y_N)_{N\in\mathbb{N}}$ converges to the exact solution of problem (1.1.1).

The way to compute an approximate value y_n of the solution makes different a discretization method from another, in formal terms and accuracy.

Euler's method.

It is the easiest numerical approach for approximating the solutions of a differential problem. It essentially uses the derivative information at a certain time t to make a linear prediction about the value of the solution at the next time $t+h$.

In detail, it starts from the difference quotient approximation

$$x(t+h) - x(t) \approx hx'(t)$$

for h sufficiently small, and, since the direct estimate

$$x'(t) = \Phi(x(t))$$

holds, achieves the relation

$$x(t+h) \approx x(t) + h\Phi(x(t)). \tag{1.1.3}$$

Putting the indipendent variable t in (1.1.3), as in the discretization step (1.1.2), Euler's method generates a set of y-values through the scheme

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + h\Phi(y_n), \quad n = 0, \cdots N-1. \end{cases} \tag{1.1.4}$$

It is an explicit one-step method and its formulation seems to be quite simple, since only one evaluation of the function Φ is required for each step.

Nevertheless, it can be proved that the global truncation error is proportional to the step size h, within $O(h^2)$ (for a detailed analysis, we refer to book [18] of B. Carnahan and A. H. Luther). More accurate methods can be obtained modifying Euler's method, even to the strenght to decrease the step size h and to include more than one evaluation of Φ for each step, as described in the following procedures.

Explicit Midpoint Rule (EMR).

One way to derive the Explicit Midpoint Rule is using the properties of truncation error for scheme (1.1.4) (see book [64] of R. S. Palais, R. A. Palais).

Applying one step Euler's method with step size h and two steps with step size $\frac{h}{2}$ we reach the expressions

$$y_{n+1} = y_n + h\Phi(y_n) + Ch + O(h^2),$$
$$y_{n+1} = y_n + \frac{h}{2}\Phi(y_n) + \frac{h}{2}\Phi(y_n + \frac{h}{2}\Phi(y_n)) + C\frac{h}{2} + O(h^2).$$

Taking twice the latter minus the former:

$$y_{n+1} \approx y_n + h\Phi(y_n + \frac{h}{2}\Phi(y_n)) + O(h^2).$$

Neglecting the infinitesimal term of second order, we get the expression

$$y_{n+1} \approx y_n + h\Phi(y_n + \frac{h}{2}\Phi(y_n)). \tag{1.1.5}$$

Fixed the positive integer N, the finite difference scheme defined as

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + h\Phi(y_n), \\ y_{n+1} = y_{n+1} = y_n + h\Phi(\frac{y_n+\bar{y}_{n+1}}{2}), \quad n = 0, \cdots N-1 \end{cases} \tag{1.1.6}$$

is known as Explicit Midpoint Rule (EMR).

Analougsly to Euler's method, this procedure arises from an estimation for the difference quotient $\frac{x(t+h)-x(t)}{h}$, but differently from the first, this quantity is approximed with the slope of the solution at the midpoint of $[t, t+h]$, as follows

$$x(t+h) - x(t) \approx x'(t + \frac{h}{2}) \tag{1.1.7}$$

and hence

$$x(t+h) - x(t) \approx \Phi(x(t + \frac{h}{2})).$$

If we predict the value $x(t + \frac{h}{2})$ using Euler's method with step size $\frac{h}{2}$

$$x(t + \frac{h}{2}) = x(t) + \frac{h}{2}\Phi(x(t)),$$

relation (1.1.7) becomes

$$x(t+h) \approx x(t) + h\Phi(x(t) + \frac{h}{2}\Phi(x(t))).$$

Selecting discrete values $\{t_n\}_{n=0}^{N}$ for the variable t with time step h, as in (1.1.2), and rearraging the last estimation in terms of $y_n \approx x(t_n)$, we get finite difference scheme (1.1.6).

Implicit Midpoint Rule (IMR).

A different situation occurs if the value of the solution at $t + \frac{h}{2}$ for the evaluation point of Φ is estimated with the midpoint of the segment between $x(t)$ and $x(t+h)$. In this case we have

$$x(t+h) - x(t) \approx h\Phi(\frac{x(t) + x(t+h)}{2})$$

Providing the suitable substitutions in terms of t_n and y_n for $n = 0, \cdots N-1$, defined as above, we get the finite difference scheme

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + h\Phi(\frac{y_n + y_{n+1}}{2}), \quad n = 0, \cdots N-1 \end{cases} \tag{1.1.8}$$

known as Implicit Midpoint Rule (IMR).

The local error at each step of the midpoint method is of order $O(h^3)$, giving a global error of order $O(h^2)$. Thus, although more computationally intensive than Euler's method, the midpoint method's error generally decreases faster as $h \to 0$.

For both EMR and IMR, it is proved that if the the function Φ is Lipschitz continuous and sufficiently smooth, then the polygonal functions sequence $(Y_N)_{N\in\mathbb{N}}$, identified by the values $\{y_n\}_{n=0}^{N}$, generated by (1.1.6) or (1.1.8), converges to the exact solution of (1.1.1), as $N \to \infty$, uniformly on $t \in [t_0, T]$, for any fixed $T > 0$ (see [61]).

1.2 Iterative methods for nonexpansive operators.

Considerable research efforts, within the past 60 years, have been devoted to study methods of successive approximations for a nonexpansive mapping T with many kinds of additional conditions. We start the survey below with one of the most investigated methods.

In 1953, W. R. Mann [55], basing on recent well known works of Cesáro, Fejér and Toeplitz in the summation of divergent series, considered the problem of constructing a sequence in a convex and compact set C of a Banach space X, converging to a fixed point of a continuous transformation $T : C \to C$. To this aim, he introduced an infinite real matrix $A = [a_{n,j}]$ satisfying the following conditions:

(A_1) A is a lower matrix with nonnegative entries ($a_{n,j} \geq 0$ for all $n, j = 1, 2, 3, \ldots$, and $a_{n,j} = 0$ for all $j > n$), that is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ \dots & \dots & \dots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n,n} & \cdots & 0 \\ \dots & \dots & \dots & \dots & 0 \end{bmatrix};$$

(A_2) The sum of each row is 1, i.e., $\sum_{j=1}^{n} a_{n,j} = 1$ for all $n = 1, 2, 3, \dots$;

(A_3) $\lim_{n\to\infty} a_{n,j} = 0$ for all $j = 1, 2, 3, \dots$.

Definition 1.2.1 *[55, Mann iteration] Let $A = [a_{n,j}]$ be a infinite matrix satisfying (A_1), (A_2), (A_3), $T : C \to C$ a nonexpansive mapping and $x_1 \in C$. Mann iterative process, designated briefly by $M(x_1, A, T)$, is defined by*

$$v_n = \sum_{k=1}^{n} a_{n,k} x_k,$$

$$x_{n+1} = Tv_n, \ n \geq 1 \tag{1.2.1}$$

It can be regarded as a generalized iteration process because if A is the identity matrix, the sequence associated to $M(x_1, I, T)$ is just the ordinary iteration process $x_{n+1} = T^n x_1$, for $n \geq 0$.

For the sequence generated by (1.2.1), the following result holds:

Theorem 1.2.2 *[55, Theorem 1] If either of the sequences $(x_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converges, then the other also converges to the same point, and their common limit is a fixed point of T.*

An example of matrix satisfying hypotheses $(A_1), (A_2), (A_3)$ is the so called Cesáro matrix, given by

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \dots & \dots & \dots & \cdots & \cdots \\ \frac{1}{n} & \cdots & \frac{1}{n} & \cdots & 0 \\ \dots & \dots & \dots & \dots & 0 \end{bmatrix};$$

In this case the sequence (1.2.1), starting with $x_1 \in C$, becomes

$$\begin{cases} v_n = \frac{1}{n} \sum_{k=1}^{n} x_k, \\ x_{n+1} = T(v_n), \quad n \geq 1 \end{cases} \tag{1.2.2}$$

and it can be briefly denoted with $M(x_1, \Lambda, T)$.

From (1.2.2), follows that

$$v_{n+1} = \frac{1}{n+1}\sum_{k=1}^{n} x_k + \frac{1}{n+1}x_{n+1} = (1 - \frac{1}{n+1})v_n + \frac{1}{n+1}Tv_n. \tag{1.2.3}$$

and $(v_n)_{n\in\mathbb{N}}$ converges to the same fixed point of T as $(x_n)_{n\in\mathbb{N}}$.

For the sequence generated by (1.2.2), if X is the real axis and C a bounded and closed interval in X, Mann proved the following convergence result:

Theorem 1.2.1 *[55, Theorem 4] If T is a continuous function carrying the interval $[a, b]$ into itself and having a unique fixpoint, q, then $M(x_1, \Lambda, T)$ converges to q for all choices of x_1 in $[a, b]$.*

Franks and Marzec [29] showed that the restriction about the uniqueness of the fixed point is unnecessary, convergence is proved by considering only the continuity of T.

In 1970, Dotson improved Theorem 1.2.2:

Theorem 1.2.3 *[26, Theorem 1] Suppose X is a locally convex Hausdorff linear topological space, C is a closed, convex subset of X, $T : C \to C$ is continuous, $x_1 \in C$ and $A = [a_{n,j}]$ satisfies $(A_1), (A_2), (A_3)$. If either of the sequences $\{x_n\}_{n\in\mathbb{N}}$ or $\{v_n\}_{n\in\mathbb{N}}$ in the Mann iterative process $M(x_1, A, T)$ converges to a point q, then the other sequence also converges to q, and q is a fixed point of T.*

The following example shows that without the assumption of continuity for T, even if the sequence of Mann iterates converges strongly to an element z, it is not guaranteed that z belongs to $Fix(T)$.

Example 1.2.4 *[3, Example 6.3.6] Let $X = C = [0, 1]$ and $T : C \to C$ the mapping defined by*

$$Tx = \begin{cases} 0, & x = 0 \\ 1, & 0 < x < 1 \\ 0, & x = 1 \end{cases}$$

Then the maping T is discontinuous and has $Fix(T) = \{0\}$. Pick any $x_1 \in (0, 1)$, Mann *process $M(x_1, \Lambda, T)$ can be rewritten in the form*

$$\begin{cases} v_1 = x_1, \\ v_2 = \frac{x_1+1}{2}, \\ v_3 = \frac{x_1+2}{3} \\ \cdots \\ v_n = \frac{x_1+n-1}{n}, \\ \cdots \end{cases}$$

Therefore $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{x_1+n-1}{n} = 1 \notin Fix(T)$.

In 1970, Dotson generalized process (1.2.3) introducing the *normal Mann iteration*:

Definition 1.2.5 *[26] A Mann process $M(x_1, A, T)$ is said to be normal provided that $A = [a_{n,j}]$ satisfies $(A_1), (A_2), (A_3)$, and*

(A_4) $a_{n+1,j} = (1 - a_{n+1,n+1})a_{n,j}$ *for* $j = 1, \cdots, n$ *and* $n = 1, 2, 3, \cdots$;

(A_5) *Either* $a_{n,n} = 1$ *for all* n, *or* $a_{n,n} < 1$ *for all* $n > 1$.

Theorem 1.2.6 *[26, Theorem 2] The following are true:*

1. *In order that $M(x_1, A, T)$ be a normal process, it is necessary and sufficient that $A = [a_{n,j}]$ satisfy $(A_1), (A_2), (A_4), (A_5)$ and $(A_3)'$ $\sum_{n=1}^{\infty} a_{n,n}$ diverges;*
2. *The matrices $A = [a_{n,j}]$ (other than the infinite identity matrix) in all normal processes $M(x_1, A, T)$ are constructed as follows:*

   Choose a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $0 \leq \alpha_n < 1$ for all n and $\sum_{n=1}^{\infty} \alpha_n$ diverges, and define $A = [a_{n,j}]$ by

$$\begin{cases} a_{1,1} = 1, a_{1,j} = 0 \quad j > 1, \\ a_{n+1,n+1} = \alpha_n \quad n = 1, 2, 3, \cdots \\ a_{n+1,j} = a_{j,j} \prod_{i=j}^{n} (1 - \alpha_i) \quad j = 1, 2, \cdots, n, \\ a_{n+1,j} = 0 \quad j > n+1, n = 1, 2, \cdots \end{cases}$$

3. *The sequence $(v_n)_{n\in\mathbb{N}}$ in a normal Mann process $M(x_1, A, T)$ satisfies*

$$v_{n+1} = (1 - \alpha_n)v_n + \alpha_n T v_n \tag{1.2.4}$$

   for all $n = 1, 2, \cdots$, where $\alpha_n = a_{n+1,n+1}$.

In the sequel, we will only consider normal Mann processes that we simply call Mann iterations. Moreover, in light of Theorem 1.2.3, in order to apply Mann iterative process for nonexpansive mappings, one needs to prove the convergence of either $(x_n)_{n\in\mathbb{N}}$ or $(v_n)_{n\in\mathbb{N}}$. Thus, we will only study the convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ that will be denoted with $(x_n)_{n\in\mathbb{N}}$.

An important example of normal Mann iteration procedure can be obtained choosing the infinite matrix $A_\lambda = [a_{n,j}]$, for $\lambda \in (0,1)$, defined by

$$\begin{cases} a_{n,1} = \lambda^{n-1}, \\ a_{n,j} = \lambda^{n-j}(1-\lambda), \quad j = 2,3,\cdots,n \\ a_{n,j} = 0, \quad j > n, \quad n = 1,2,3,\cdots \end{cases}$$

for $n \geq 1$. Since the diagonal sequence of A_λ is given by $a_{n+1,n+1} = 1-\lambda$ for all $n = 1,2,3,\cdots$, by point (3) of Theorem 1.2.6, Mann process $M(x_1, A_\lambda, T)$ generates the sequence

$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n, \quad n \geq 1 \tag{1.2.5}$$

This sequence was already known, in 1957, as *general Krasnoselskij iteration*.

It included *Krasnoselskij iteration* introduced by Krasnoselskij in 1955:

$$x_{n+1} = \frac{x_n + Tx_n}{2}, \quad n \geq 1, \tag{1.2.6}$$

obtained from (1.2.5) replacing $\lambda = \frac{1}{2}$. Therefore (1.2.6) represents a particular case of Mann iteration and, in terms of this latter, can be denoted by $M(x_1, A_{\frac{1}{2}}, T)$. It can be noticed that, although, chronologically, it was introduced earlier than the geneal Krasnoselskij iteration, Mann iteration is formally a generalization of the latter.

Krasnoselskij showed the convergence of (1.2.6) in a uniformly convex Banach space with compact setting:

Theorem 1.2.7 *[51] Let C be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space X and T a nonexpansive mapping of C into a compact subset of C. Let $x_1 \in C$ be an arbitrary point in C. Then the sequence $(x_n)_{n\in\mathbb{N}}$ defined by* (1.2.6) *converges strongly to a fixed point of T in C.*

In 1966 Edelstein [28] achieved the same result relaxing the condition of uniform convexity to strictly convex Banach spaces.

We recall the following definitions for a normed space X:

Definition 1.2.8 *[19] A normed space X is called uniformly convex if for any $\epsilon \in (0,2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \epsilon$ then*

$$\|\frac{x+y}{2}\| \leq 1-\delta$$

Thus, a normed space is uniformly convex if for any two points $x \neq y$, in the unit sphere centred at the origin, the midpoint of the line segment between x and y is never on the sphere but is close to the sphere only if x and y are sufficiently close to each other.

Definition 1.2.9 *[19] A normed space X is called strictly convex if for all $x, y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$.*

Theorem 1.2.10 *[19, Theorem 1.6] Every uniformly convex space is strictly convex.*

In 1957, Schaefer proved an analogous result of Theorem 1.2.7 for the sequence given by (1.2.5):

Theorem 1.2.11 *[74] Let C be a closed, convex and bounded subset of a banach space X, $T : C \to C$ a nonexpansive mapping and $(x_n)_{n\in\mathbb{N}}$ defined by* (1.2.5) *with starting point x_1. Then*

1. *If X is strictly convex then $Fix(T)$ is convex;*

2. *If X is uniformly convex and $T(C)$ is compact, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T;*

3. *If $X = H$ is a real Hilbert space and T is weakly continuous on C, then $(x_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.*

In 1976, Ishikawa proved that Theorem 1.2.7 holds, even without assuming any convexity condition on the Banach space X, for the more general Mann iteration (1.2.4):

Theorem 1.2.12 *[43, Theorem 1] Let C a closed subset of a Banach space X, and let T be a nonexpansive mapping from C into a compact subset of X, If there exist $x_1 \in C$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ that satisfy*

1. *$0 \leq \alpha_n \leq b < 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,*

2. *$x_n \in C$ for all $n \geq 1$ with $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$,*

then T has a fixed point in C and $\{x_n\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Browder and Petryshyn, in [14], carried further the result obtained by Schaefer [74] and Krasnoselskij [51], studying the convergence of the Picard iterates and general Krasnoselskij iterations for nonexpansive operators satisfying the following property:

Definition 1.2.13 *[14] A mapping $T : C \subset X \to X$ is said to be asymptotically regular on C, if for each $x \in C$, $T^{n+1}x - T^n x \to 0$ strongly in X as $n \to +\infty$. $T : C \subset X \to X$ is asymptotically regular at $x_0 \in C$ if $T^{n+1}x_0 - T^n x_0 \to 0$ strongly in X as $n \to +\infty$.*

We recall that

Definition 1.2.2 *[13] Let X be a real Banach space and C a nonempty and closed subset of X. A mapping $T : C \to C$ is said to be demiclosed (at y), if for any $(x_n)_{n\in N}$, in C, the conditions $x_n \rightharpoonup x$ and $Tx_n \to y$ imply $Tx = y$.*

Theorem 1.2.14 *[14, Theorem 3] Let X be a Banach space, T a nonexpansive mapping of X into X such that T is asymptotically regular and $I - T$ is demiclosed. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence of Picard iterates for T starting at x_0. Suppose that T has at least one fixed point. Then the weak limit of any weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$ lies in $Fix(T)$. In particular, if X is reflexive and $Fix(T)$ consists of exactly one point z, $(x_n)_{n\in\mathbb{N}}$ converges weakly to z.*

Theorem 1.2.15 *[14, Theorem 6] Let X be a Banach space and T an asymptotically regular nonexpansive self-mapping of X. Suppose that T has a fixed point and that $I - T$ maps closed bounded subsets of X into closed subsets of X. Then for each $x \in X$, $(T^n x)_{n\in\mathbb{N}}$ converges strongly in X to a fixed point of T.*

In [62], Opial refined the results obtained by Browder and Petryshyn in [14], showing that if X is a uniformly convex Banach space with a weakly continuous duality map (and, in particular, a Hilbert space) and if T is a nonexpansive asymptotically regular map of a closed convex subset C of X into C with $Fix(T) \neq \emptyset$, then $(x_n)_{n\in\mathbb{N}}$ given by $x_{n+1} = T^n x_0$, with $x_0 \in C$ and $n \geq 0$, weakly converges to a fixed point of T.

If $(x_n)_{n\in\mathbb{N}}$ is given by (1.2.5), then it can be rewritten as the sequence of Picard iterates for the auxiliary operator $T_\lambda = \lambda I + (1-\lambda)T$, with $\lambda \in (0,1)$, that is $x_{n+1} = T_\lambda^n x_0$, with $x_0 \in C$. For each choice of $\lambda \in (0,1)$, T_λ is nonexpansive. In particular in [14, Theorem 5] is proved that T_λ is asymptotically regular and has the same fixed points set of T, that is $Fix(T_\lambda) = Fix(T)$. In light of the above Opial's result, the sequence $(x_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T. Therefore Opial's result is also an improvement of Theorem 1.2.11 obtained by Schaefer for the sequence (1.2.5).

For Mann iteration, Reich proved the following convergence result in a noncompact setting in 1979:

Theorem 1.2.16 *[68, Theorem 2] Let C be a closed convex subset of a uniformly convex Banach space X with a Frechet diferentiable norm, $T : C \to C$ a nonexpansive mapping* *with a fixed point and* $(\alpha_n)_{n\in\mathbb{N}}$ *a real sequence such that* $\alpha_n \in [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $x_1 \in C$ *and* $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$, *for* $n \geq 1$, *then* $(x_n)_{n\in\mathbb{N}}$ *converges weakly to a fixed point of* T.

This result has been generalized in 2001 [30, Theorem 6.4], under the same hypotheses of the parameters sequence, to a more general setting of Banach spaces.

In 1975, Genel and Linderstrauss showed that the compact condition about T or C cannot be eliminated, even in a Hilbert space, in order to get strong convergence of Mann iterative procedure. In particular, in [33] they constructed an example of a nonexpansive map T of the unit ball B_1 of l_2 into itself, for which the sequence $(x_n)_{n\in\mathbb{N}}$ defined by (1.2.6) does not converge to a fixed point of T although $Fix(T) = Fix(T_{\frac{1}{2}}) \neq \emptyset$ and $T_{\frac{1}{2}} = \frac{I+T}{2}$ is asymptotically regular.

Hence, to recap, we have seen that Mann iteration strongly converges to fixed points of nonexpansive mapping in finite-dimensional Banach spaces. For spaces of infinite dimension, continuity is not enough to guarantee convergence to a fixed point. Therefore, the successive purpose has been that of developping iterative procedures generating strongly convergent sequences in a Banach space.

First we recall a convergence result for nonexpansive mappings proved by Browder in 1967. In detail, Browder constructed a sequence of approximants converging strongly to a fixed point of T in the framework of Hilbert spaces:

Theorem 1.2.17 *[11, Theorem 1] Let* H *be a Hilbert space,* T *a nonexpansive mapping of* H *into* H. *Suppose that there exists a bounded closed convex subset* C *of* H *mapped by* T *into itself. Let* u *be an arbitrary point of* C, *and for each* t *with* $0 < t < 1$, *let*

$$T_t(x) = tu + (1-t)Tx.$$

Then T_t *is a strict contraction of* H *with constant* $1-t$, T_t *has an unique fixed point* z_t *in* C, *and* z_t *converges as* $t \to 0$ *strongly in* H *to a fixed point* z_0 *of* T *in* C. *The fixed point* z_0 *in* C *is uniquely specified as the fixed point of* T *in* C *closest to* u .

In 1980, Reich [72, Corollary 1] extended this result to uniformly smooth Banach spaces.

In 1967 Halpern gave another proof of Theorem 1.2.17 and initiated the study of a new (explicit) iterative procedure for approximating fixed points of a nonexpansive mapping:

Definition 1.2.18 (Halpern iteration) *Let* C *be closed and convex subset of a linear space* X *and* $T : C \to C$ *a nonexpansive mapping. Given* $(\alpha_n)_{n\in\mathbb{N}} \in (0,1)$ *and fixed* $x_0, u \in C$, *the sequence* $(x_n)_{n\in\mathbb{N}}$ *generated by Halpern's method is defined as*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.2.7}$$

In [38] Halpern established the following strong convergence result for the sequence generated by (1.2.7), for a suitable choice of the sequence $(\alpha_n)_{n\in\mathbb{N}}$:

Theorem 1.2.19 *[38, Theorem 3] Let C be a bounded, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. For any initialization $x_0 \in C$ and anchor $u \in C$, define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C by*

$$x_{n+1} = n^{-\theta}u + (1 - n^{-\theta})Tx_n, \quad n \geq 0$$

where $\theta \in (0, 1)$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the element of $Fix(T)$ nearest to u.

The focal point in algorithm (1.2.7) is that although it is not possible to have strong convergence for the sequence $(T^n x_1)_{n\in\mathbb{N}}$, a suitable convex combination of (Tx_0) with the point u leads to get strong convergence.

In addition, the author in [38] showed that conditions

(C_1) $\lim\limits_{n\to\infty} \alpha_n = 0$

(C_2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, or equivalently $\prod\limits_{n=0}^{\infty}(1 - \alpha_n) = 0$

are necessay for the strong convergence of algorithm (1.2.7) for a nonexpansive mapping $T : C \to C$.

In 1977, Lions in [52] improved Halpern's result 1.2.19, still in Hilbert spaces, by proving strong convergence of the sequence (1.2.7) to a fixed point of T under the assumption that the parameters $(\alpha_n)_{n\in\mathbb{N}}$ verify the conditions (C_1), (C_2) and the following one:

(C_3) $\lim\limits_{n\to\infty} \dfrac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0$

It can be noticed that, in both results, by Halpern [38] and Lions [52], the classical choice $\alpha_n = \frac{1}{n}$ is excluded. This limitation has been surmounted: indeed if C is a subset of a Banach space X, $T : C \to X$ is a nonexpansive mapping and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0, 1)$ satisfying (C_1), (C_2), then sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.2.7) strongly converges to a fixed point of T if

- (R. Wittmann, [84, Theorem 2]) $X = H$ is a Hilbert space and $(\alpha_n)_{n\in\mathbb{N}}$ satisfies in addition

  $(C_3)'$ $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$

  Remark 1.2.20 *Wittmann's result has been extended by Shioji and Takahashi in the framework of uniformly smooth Banach spaces (see [75]).*

- (H-K Xu, [88, Theorem 2.3]) X is a uniformly smooth Banach space, $T(C) \subset C$ e $(\alpha_n)_{n\in\mathbb{N}}$ satisfies also

$(C_3)''$ $\lim_{n\to\infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}} = 0$, or equivalently $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.$

In these cases the choice $(\frac{1}{n})_{n\in\mathbb{N}}$ is contemplated, since it satisfies both $(C_3)'$ and $(C_3)''$.

In the sequel, several mathematicians studied the problem of determining if the conditions C_1 and C_2 are also sufficient.

Until now, it has been claimed that C_1 and C_2 are sufficient if additional assumptions are assumed. In [84], Wittmann observed that if the sequence $(\alpha_n)_{n\in\mathbb{N}}$ satisfies C_1 and C_2 and is decreasing then condition $(C_3)'$ is ensured since

$$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| = \sum_{n=0}^{\infty} (\alpha_n - \alpha_{n+1}) = \alpha_0 - \lim_{N\to\infty} \alpha_N < \infty.$$

We mention also the work of C. E. Chidume and C. O. Chidume [20, Theorem 3.1] and Suzuki [81, Theorem 3], in which, independently, it is proved that the conditions C_1, C_2 are sufficient to get strong convergence to a fixed point of T for the iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n), \quad \forall n \geq 1$$

Recently, in the setting of Banach spaces, Song and Chai [76, Theorem 3.1], under conditions C_1 and C_2 but formulating stronger hypotheses on the mapping, achieved strong convergence for Halpern's iteration (1.2.7). In detail, they assumed that X is a real reflexive Banach space, with a uniformly Gateaux differentiable norm, having the fixed point property for nonexpansive self-mappings; and considered a subclass of nonexpansive mappings known as *firmly type nonexpansive mappings*:

Definition 1.2.21 *[76] Let T be a mapping with domain $D(T)$ and range $R(T)$ in Banach space X. T is called firmly type nonexpansive if for all $x, y \in D(T)$, there exists $k \in (0, +\infty)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k\|(x - Tx) - (y - Ty)\|^2$$

Another interesting issue, is to see the behavior of Mann iteration when the contractive condition on the mapping is weaker than the nonexpansivity.

To this regard, we recall, from Browder and Petryshyn's paper [15], that a mapping $T : C \subset H \to H$ is said to be

- *k-Strictly-pseudocontractive* is there exists a contant $k \in (0,1)$ such that for all $x, y \in C$
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$
- *Pseudocontractive* if
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$$
for all $x, y \in C$

Clearly, the second class contains the first one and the class of nonexpansive mappings. Hence, a nonexpansive mapping is pseudocontractive, but the converse does not hold. Indeed pseudocontractive mappings are not necessarirly continuous, and hence nonexpansive, as it can be seen in

Example 1.2.22 *[19, Example 8.1] The mapping $T : [0,1] \to \mathbb{R}$ defined as*
$$Tx = \begin{cases} x - \frac{1}{2}, & x \in [0, \frac{1}{2}) \\ x - 1, & x \in [\frac{1}{2}, 1] \end{cases}$$
is pseudocontractive, and not continuous.

In 1974, Ishikawa defined a new iterative method:

Definition 1.2.23 *[44, Ishikawa iteration] Given $x_0 \in C \subset H$, Ishikawa iteration generates a sequence through the scheme*
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n((1 - \beta_n)x_n + \beta_n Tx_n), \quad n \geq 0 \tag{1.2.8}$$
where $0 \leq \alpha_n \leq \beta_n \leq 1$.

For this procedure, Ishikawa established a strong convergence result for pseudocontractive mappings:

Theorem 1.2.24 *[44] If C is a compact convex subset of a Hilbert space H, $T : C \to C$ is a lipschitzian pseudocontractive map and x_0 is any point of C, then the sequence defined by* (1.2.8)*, where $0 \leq \alpha_n \leq \beta_n < 1$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, strongly converges to a fixed point of T.*

It should be emphasized the interest of may mathematicians for the class of pseudocontractive mappings. Indeed, in addition to generalize the nonexpansive mappings, the pseudocontractive ones are characterized by the fact that T is pseudocontractive if and only if $I - T$ is accretive [12, Proposition 1]. Therefore, the mapping theory for accretive operators is closely related to the fixed point theory of pseudocontractive mappings. It has been proved (see Deimling, [23]) that if A is an accretive operator, then the zeros of A correspond to the equilibrium points of some evolution systems.

Let (x, w) denote the pairing of the element x of X and the element w of the conjugate space X^* and for each $x \in X$. The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{w : w \in X^*, (w, x) = \|x\|^2, \|w\| = \|x\|\}$$

is called *normalized duality mapping*.

The accretive operators were introduced independently in 1967 by Browder and Kato:

Definition 1.2.25 (Accretive operator, [12]) *Let X be a Banach space. If A is a (possibly) nonlinear mapping with domain D(A) in X and with range R(A) in X, then A is said to be accretive if for each pair x and y in D(A),*

$$\langle Ax - Ay, w\rangle \geq 0, \tag{1.2.9}$$

for all $w \in J(x - y)$.

Definition 1.2.26 (Monotonic operator, [47]) *A mapping A, with domain $D(A)$ and range $R(A)$ in a Banach space X, is called monotonic if the inequality*

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\| \tag{1.2.10}$$

holds for every $x, y \in D(A)$ and $s > 0$.

Kato proved [47, Lemma 1.1] that condition (1.2.10) is equivalent to the following:

For each $x, y \in D(A)$ there is $w \in J(x - y)$ such that

$$\langle Ax - Ay, w\rangle \geq 0.$$

Hence, Definition 1.2.25 and Definition 1.2.26 are almost identical. Of course the two definitions coincide if J is single-valued.

In paper [40] of Kubicek and Hicks we find the following question

Does Mann iteration process always converge for continuous pseudocontractive mappings or Lipschitz pseudocontractive mappings ?

Chidume and Mutangadura in 2001, gave a negative answer:

Example 1.2.27 *[21] Let $B_1 \subset \mathbb{R}^2$ the unit ball centered in zero, $B_{\frac{1}{2}} \subset \mathbb{R}^2$ the ball centered in zero with radius $\frac{1}{2}$, $x^{\perp} = (b, -a)$ for each $(a, b) \in \mathbb{R}^2$, and $T : B_1 \to B_1$ defined by:*

$$Tx = \begin{cases} x + x^{\perp}, & x \in B_{\frac{1}{2}} \\ \frac{x}{\|x\|} - x + x^{\perp}, & x \in B_1 \setminus B_{\frac{1}{2}} \end{cases}$$

Then

- T is lipschitzian and pseudocontractive;
- The origin is the unique fixed point of T;
- No Mann sequence converges to the fixed point;
- No Mann sequence converges to any $x \neq 0$.

1.3 IMR for nonexpansive mappings.

In Section 1.1 we have recalled the finite difference scheme of the numerical IMR for approximating the solutions of a differential problem, that we reproduce below as

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n + h\Phi(\frac{y_n + y_{n+1}}{2}), \quad n = 0, \cdots N-1. \end{cases}$$

If the function Φ is written as $\Phi(x) = x - g(x)$, then it reduces to

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n + h\big[\frac{y_n + y_{n+1}}{2} - g(\frac{y_n + y_{n+1}}{2})\big], \quad n = 0, \cdots N-1, \end{cases} \tag{1.3.1}$$

and the critical points of IVP

$$\begin{cases} x'(t) = \Phi(x(t)) \\ x(t_0) = x_0, \end{cases}$$

are the fixed points of the function g, $x = g(x)$.

This consideration motivated M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H-K Xu, in [5], to transplant IMR scheme to the solution of the fixed point equation $x = Tx$, where T is a nonlinear operator in a Hilbert space.

Therefore, by (only formal) analogy, they introduced the IMR for the fixed point problem $x = Tx$, given by

$$\begin{cases} x_0 \in H \\ x_{n+1} = x_n - s_n\big[\frac{x_n + x_{n+1}}{2} - T(\frac{x_n + x_{n+1}}{2})\big], \quad n \geq 0, \end{cases} \tag{1.3.2}$$

where H is a Hilbert space and $(s_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$.

The authors of [5] observed that, putting $t_n = \frac{2s_n}{2+s_n}$, last scheme is equivalent to the Mann-type algorithm given by

$$\begin{cases} x_0 \in H \\ x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n+x_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{1.3.3}$$

with $(t_n)_{n\in\mathbb{N}} \in (0,1)$, known as *IMR for nonexpansive mappings*.

Focusing on (1.3.3), the following result holds:

Theorem 1.3.1 *[5, Theorem 2.6]*

Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by (1.3.3), with $(t_n)_{n\in\mathbb{N}} \in (0,1)$ satisfying the conditions

(H_1) $t_{n+1}^2 \leq at_n, \quad \forall n \geq 0$ *and some* $a > 0$,

(H_2) $\liminf_{n\to\infty} t_n > 0$.

The sequence $(x_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

Remark 1.3.1 *The conditions (H_1), (H_2) are not restrictive. As a matter of fact, it can be noticed that, for each $p > 0$, the sequence $t_n = 1 - \frac{1}{(n+2)^p}$, for $n \geq 0$, verifies (H_1) and (H_2).*

1.4 Our contributions.

Notations

Throgouth this section, will be denoted with

H, a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$;

C, a closed and convex subset of H;

T, a nonexpansive mapping, that is $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D(T)$;

$Fix(T)$, the fixed points set of T;

$\omega_w(x_n)$, the set of weak cluster points of $(x_n)_{n\in\mathbb{N}}$, that is $\omega_w(x_n) = \{z : \exists (x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup z, k \to \infty\}$;

$\to$, the strong convergence;

$\rightharpoonup$, the weak convergence.

1.4.1 Results.

In order to obtain strong convergence for the sequence generated by the IMR for nonexpansive mappings, we construct, in [58], a number of variations of (1.3.3). For this purpose, we follow the same modification line adopted by Hussain, Marino, Muglia and Alamri in [41], in which the authors introduced modified Mann iterations for nonexpansive mappings, generating strongly convergent sequences.

The framework is a real Hilbert space and the considered class of operators is that of nonexpansive mappings. The proposed algorithm differs from scheme (1.3.3) for the introduction of a term $\alpha_n\mu_n(u-x_n)$ that can also be infinitesimal. The addition of this term, that at first glance could appear non-significant, ensure the strong convergence of the method to a fixed point of the mapping.

In detail, let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be sequences in $(0,1]$, we define a modified implicit midpoint algorithm generating a sequence $(x_n)_{n\in\mathbb{N}}$ in the following way:

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}) + \alpha_n\mu_n(u-x_n), \quad n \geq 0, \end{cases} \tag{1.4.1}$$

where $T : H \to H$ is a nonexpansive mapping.

Remark 1.4.1 *Iterative method* (1.4.1) *is well defined. Indeed, if $T : H \to H$ is a nonexpansive mapping, y, z, w are given points in H and $\alpha \in (0,1)$, then the mapping $\tilde{T} : H \to H$ defined by*

$$\tilde{T}x = \alpha y + (1-\alpha)T(\frac{z+x}{2}) + w$$

is a contraction with constant $\frac{1-\alpha}{2}$. Therefore $\tilde{T}$ has a unique fixed point.

The following result is proved:

Theorem 1.4.2 *[58, Theorem 3.2] Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}} \in (0,1]$ satisfy the conditions*

(1) $\displaystyle\lim_{n\to\infty} \alpha_n = 0,$

(2) $\displaystyle\sum_{n=0}^{\infty} \alpha_n\mu_n = +\infty,$

(3) $\displaystyle\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n\mu_n} = 0,$

(4) $\displaystyle\lim_{n\to\infty} \frac{|\mu_n - \mu_{n-1}|}{\mu_n} = 0.$

Then the sequence $(x_n)_{n\in\mathbb{N}}$*, generated by* (1.4.1)*, strongly converges to the point* $q_u \in Fix(T)$ *nearest to* u*, that is* $\|u - q_u\| = \min_{x\in Fix(T)} \|u - x\|$*.*
Proof. *See page 33.* □

Remark 1.4.3 *If* $u = 0 \in H$*, under the same hypotheses of Theorem 1.4.2, we get that the sequence* $(x_n)_{n\in\mathbb{N}}$ *generated by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \alpha_n T(\frac{x_n + x_{n+1}}{2}) - \alpha_n \mu_n x_n \quad n \geq 0 \end{cases}$$

strongly converges to the point $q \in Fix(T)$ *nearest to* $0 \in H$*, that is, the fixed point of* T *with minimum norm* $\|q\| = \min_{x\in Fix(T)} \|x\|$*.*

A particular case is obtained from Theorem 1.4.2 when $\mu_n = 1$. The resulting algorithm is a Halpern-type iteration, for which the following convergence result holds:

Corollary 1.4.4 *[58, Corollary 3.4] Let* H *be a real Hilbert space and* $T : H \to H$ *a nonexpansive maping with* $Fix(T) \neq \emptyset$*. If the sequence* $(\alpha_n)_{n\in\mathbb{N}}, \subset (0,1]$ *satisfies the conditions*

(1) $\lim\limits_{n\to\infty} \alpha_n = 0,$

(2)′ $\sum\limits_{n=0}^{\infty} \alpha_n = +\infty,$

(3)′ $\lim\limits_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ *generated by*

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0 \end{cases} \tag{1.4.2}$$

strongly converges to the point $q_u \in Fix(T)$ *nearest to* u*, that is*
$\|u - q_u\| = \min_{x\in Fix(T)} \|u - x\|$

Remark 1.4.5 *Even in this case, it is considered the eventuality* $u = 0 \in H$*. Therefore, we get that, under the same assumptions of Corollary 1.4.4, the sequence* $(x_n)_{n\in\mathbb{N}}$ *generated by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n) T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0 \end{cases}$$

strongly converges to the point $q \in Fix(T)$ *nearest to* $0 \in H$*, that is the fixed point of* T *with minimum norm* $\|q\| = \min_{x\in Fix(T)} \|x\|$*.*

Going back to the idea born by Alghamdi et al. in [5], a different situation occurs if the starting point is the numerical Explicit Midpoint Rule (EMR), employed for numerically integrating an IVP of type (1.1.1). The finite difference scheme defining the EMR, as we have seen in Section 1.1, is given by:

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + h\Phi(y_n), \\ y_{n+1} = y_n + h\Phi(\frac{y_n+\bar{y}_{n+1}}{2}). \end{cases}$$

If the function Φ is written as $\Phi(x) = x - g(x)$, then the numerical EMR becomes

$$y_{n+1} = y_n + h\Big(\frac{y_n + \bar{y}_{n+1}}{2} - g(\frac{y_n + \bar{y}_{n+1}}{2})\Big) \tag{1.4.3}$$

and the critical points of IVP (1.1.1) are the fixed points of the function g, $x = g(x)$.

Therefore, as in the previous case, by formal analogy with (1.4.3), a recursive scheme for the Fixed Points Problem $x = Tx$, is obtained:

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1-t_n)\hat{x}_n + t_n T(\hat{x}_n), \quad n \geq 0 \\ \hat{x}_{n+1} = \hat{x}_n - t_n\big(\frac{\hat{x}_n+\bar{x}_{n+1}}{2} - T(\frac{\hat{x}_n+\bar{x}_{n+1}}{2})\big), \quad n \geq 0, \end{cases} \tag{1.4.4}$$

where $(t_n)_{n\in\mathbb{N}} \subset (0,1)$.

Replacing $\bar{x}_{n+1}$ in the last equation of (1.4.4), we obtain

$$\begin{aligned} \hat{x}_{n+1} &= \hat{x}_n - \frac{t_n}{2}\hat{x}_n - \frac{t_n}{2}((1-t_n)\hat{x}_n + t_n T(\hat{x}_n)) + t_n T(\frac{\hat{x}_n + \bar{x}_{n+1}}{2}) \\ &= (1-t_n)\hat{x}_n + t_n T(\frac{\hat{x}_n + \bar{x}_{n+1}}{2}) + t_n^2(\frac{\hat{x}_n - T(\hat{x}_n)}{2}) \end{aligned}$$

If we neglect the infinitesimal term of second order in t_n, we obtain the recursive scheme

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1-t_n)x_n + t_n T(x_n), \quad n \geq 0 \\ x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n+\bar{x}_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{1.4.5}$$

with $(t_n)_{n\in\mathbb{N}} \subset (0,1)$. We call it *Explicit Midpoint Rule* (EMR) for the search of fixed points. It should be stressed that scheme (1.4.5) constitutes again a Mann-type method.

With the purpose of obtaining strong convergence, as we have done for the IMR for nonexpansive mappings (1.3.3), we provide a variant of (1.4.5), on the same modification line adopted in [41], obtaining the following iterative scheme:

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) + \alpha_n \mu_n (u - x_n), \quad n \geq 0, \end{cases} \tag{1.4.6}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$. For this last algorithm we establish the following results:

Theorem 1.4.6 *[58, Theorem 4.2] Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Under the assumptions* $(1), (2), (3), (4)$ *of Theorem 1.4.2, if the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subset (0,1]$ satisfy also the hypotheses*

(5) $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{\alpha_n \mu_n} = 0$

(6) $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \mu_n} = 0$

(7) $\limsup_{n\to\infty} \beta_n(1-s_n) + s_n > 0,$

then $(x_n)_{n\in\mathbb{N}}$ generated by (1.4.6) *strongly converges to the point $x_u{}^* \in Fix(T)$ nearest to u, that is*

$$\|u - x_u{}^*\| = \min_{x\in Fix(T)} \|u - x\|$$

Proof. *See page 37.* ☐

Remark 1.4.7 *In case $u = 0$, under the same assumptions of Theorem 1.4.6, we obtain strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$, generated by*

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - \alpha_n \mu_n x_n, \quad n \geq 0, \end{cases}$$

to the point $x* \in Fix(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|x^*\| = \min_{x\in Fix(T)} \|x\|$.

As in the previous case, in the eventuality $\mu_n = 1$, for all $n \in \mathbb{N}$, we have the following convergence result for a Halpern-type method:

Corollary 1.4.8 *[58, Corollary 4.4] Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that conditions* $(1), (2)', (3)'$ *of Corollary 1.4.4 hold and that the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ satisfy also the hypotheses:*

$(5)'$ $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{\alpha_n} = 0$,

$(6)'$ $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$,

(7) $\limsup_{n\to\infty} \beta_n(1 - s_n) + s_n > 0$,

then the sequence $(x_n)_{n\in\mathbb{N}}$ *generated by*

$$\begin{cases} x_0, u \in H \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases} \tag{1.4.7}$$

strongly converges to the point $x_u{}^* \in Fix(T)$ *nearest to* u.

Remark 1.4.9 *For* $u = 0$*, under the same assumptions of* Corollary 1.4.8*, we obtain strong convergence of the sequence* $(x_n)_{n\in\mathbb{N}}$*, generated by*

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0, \\ x_{n+1} = (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases}$$

to the point $x* \in Fix(T)$ *nearest to* $0 \in H$*, that is the fixed point of* T *with minimum norm* $\|x^*\| = \min_{x\in Fix(T)} \|x\|$.

1.4.2 Tools.

For each $x \in H$ there exists a unique *nearest point* in C, denoted by $P_C(x)$, that is $\|x - P_C(x)\| \leq \|x - z\|$, $\forall z \in C$. The mapping $P_C : H \to C$ is called the *metric projection* of H onto C and it is characterized by the following property:

Lemma 1.4.10 *[37, Lemma 12.1]* $y = P_C(x)$ *if and only if for each* $z \in C$,

$$\langle z - y, y - x\rangle \geq 0.$$

For the structure of the fixed points set of a nonexpansive mapping we mention the following results:

Theorem 1.4.11 *[3, Theorem 5.2.27] Let* C *be a convex subset of a strictly convex Banach space* X *and* $T : C \to X$ *a nonexpansive mapping. Then* $Fix(T)$ *is either empty or convex.*

If the hypothesis of strict convexity of the Banach space is removed the result does not hold, as shown by the following:

Example 1.4.12 *[3, Example 5.2.30] Let $X = \mathbb{R}^2$ endowed with*

$$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\} \quad \forall (x_1, x_2) \in \mathbb{R}^2$$

and $T : X \to X$ defined by

$$T(x_1, x_2) = (|x_2|, x_2) \quad \forall (x_1, x_2) \in \mathbb{R}.$$

For each $x, y \in \mathbb{R}^2$, with $x \equiv (x_1, x_2)$ and $y \equiv (y_1, y_2)$, it results that

$$\begin{aligned}
\|T(x_1, x_2) - T(y_1, y_2)\| &= \|(|x_2|, x_2) - (|y_2|, y_2)\| \\
&= \|(|x_2| - |y_2|, x_2 - y_2)\| \\
&= \max\{||x_2| - |y_2||, |x_2 - y_2|\} \\
&= |x_2 - y_2| \\
&\leq \max\{|x_1 - y_1|, |x_2 - y_2|\} \\
&= \|(x_1 - y_1, x_2 - y_2)\| \\
&= \|(x_1, x_2) - (y_1, y_2)\|
\end{aligned}$$

Hence T is nonexpansive. Moreover $Fix(T)$ is nonempty. Indeed, the equality $(x_1, x_2) = T(x_1, x_2)$ is satisfied by $(x_1, x_2) \in \mathbb{R}^2$ such that $x_1 = |x_2|$. Hence $Fix(T) = \{(|x|, x) : x \in \mathbb{R}\}$. On the other hand, if we consider $(1, 1), (1, -1) \in Fix(T)$, any element in the segment between these points, of the type $(1, 2\lambda - 1)$ with $\lambda \in (0, 1)$, does not belong to $Fix(T)$. Hence $Fix(T)$ is not convex.

Remark 1.4.13 *By the continuity of T, $Fix(T)$ is always closed.*

Corollary 1.4.14 *[3, Corollary 5.2.29] Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then $Fix(T)$ is closed and convex.*

In the convergence analysis of the next section, we will apply a lemma for sequences of real numbers introduced by Xu in 2002:

Lemma 1.4.15 *[87, Lemma 2.5] Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \geq 0$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (\gamma_n)_{n\in\mathbb{N}}$ satisfy the conditions:

1. *$(\alpha_n)_{n\in\mathbb{N}} \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ or, equivalently, $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$;*

2. $\limsup_{n\to\infty} \beta_n \le 0$;

3. $\gamma_n \ge 0$, $\forall n \ge 0$ *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

We will often use Browder's *demiclosedness principle* for nonexpansive mappings:

Lemma 1.4.16 *(Demiclosedness Principle, [13, Theorem 3]) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X, and let $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed, that is*

$$(x_n)_{n\in\mathbb{N}} \subset C, \quad x_n \rightharpoonup x, \quad (I-T)x_n \to y \implies (I-T)x = y.$$

1.4.3 Proofs.

First of all, we introduce the following lemma, establishing the boundedness for the sequence generated by (1.4.1)

Lemma 1.4.17 *The sequence $(x_n)_{n\in\mathbb{N}}$ generated by* (1.4.1) *is bounded.*
Proof. *Let $q \in Fix(T)$, for arbitrary $n \in \mathbb{N}$, we compute:*

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(1-\mu_n)x_n + (1-\alpha_n)T\big(\frac{x_n + x_{n+1}}{2}\big) + \alpha_n\mu_n u - q\| \\
\text{($\pm\alpha_n(1-\mu_n)q$)} \quad &= \|\alpha_n(1-\mu_n)(x_n - q) + (1-\alpha_n)(T\big(\frac{x_n + x_{n+1}}{2}\big) - q\big) \\
&\quad +\alpha_n\mu_n(u-q)\| \\
\text{(T nonexpansive)} \quad &\le \alpha_n(1-\mu_n)\|x_n - q\| + (1-\alpha_n)\|\frac{x_n + x_{n+1}}{2} - q\| \\
&\quad +\alpha_n\mu_n\|u-q\| \\
&\le \alpha_n(1-\mu_n)\|x_n - q\| + \frac{1-\alpha_n}{2}\|x_n - q\| \\
&\quad +\frac{1-\alpha_n}{2}\|x_{n+1} - q\| + \alpha_n\mu_n\|u-q\|.
\end{aligned}$$

Thus, for $n \ge 0$, we have that

$$\big(\frac{1+\alpha_n}{2}\big)\|x_{n+1} - q\| \le \big(\frac{1+\alpha_n}{2} - \alpha_n\mu_n\big)\|x_n - q\| + \alpha_n\mu_n\|u-q\|,$$

and hence

$$\begin{aligned}
\|x_{n+1} - q\| &\le \big(1 - \frac{2\alpha_n\mu_n}{1+\alpha_n}\big)\|x_n - q\| + \frac{2\alpha_n\mu_n}{1+\alpha_n}\|u-q\| \\
&\le \max\{\|x_n - q\|, \|u-q\|\}.
\end{aligned}$$

Therefore, it results

$$\begin{aligned}\|x_{n+1}-q\| &\leq \max\{\|x_n-q\|,\|u-q\|\}\\ &\leq \max\{\|x_{n-1}-q\|,\|u-q\|\}.\end{aligned}$$

Hence, we have $\|x_{n+1}-q\| \leq \max\{\|x_0-q\|,\|u-q\|\}, \quad \forall n \geq 0$. *Then* $(x_n)_{n\in\mathbb{N}}$ *is bounded.* □

Proof.(Theorem 1.4.2) A possible choise of parameters satisfying the hypotheses of Theorem 1.4.2 is given by $\alpha_n = \mu_n = \frac{1}{\sqrt{n}}$.

From Lemma 1.4.17, it is known that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

We are going to prove that $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular, that is $(x_n)_{n\in\mathbb{N}}$ verify

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Indeed, it results that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n(1-\mu_n)x_n+(1-\alpha_n)T(\frac{x_n+x_{n+1}}{2})+\alpha_n\mu_n u\\
&\quad -\alpha_{n-1}(1-\mu_{n-1})x_{n-1}\\
&\quad -(1-\alpha_{n-1})T(\frac{x_{n-1}+x_n}{2})-\alpha_{n-1}\mu_{n-1}u\|\\
(\ \pm\alpha_n(1-\mu_n)x_{n-1}\ ) &= \|\alpha_n(1-\mu_n)(x_n-x_{n-1})+\alpha_n(\mu_{n-1}-\mu_n)x_{n-1}\\
(\ \pm\alpha_n(1-\mu_{n-1})x_{n-1}\ ) &\quad +(\alpha_n-\alpha_{n-1})((1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2}))\\
(\ \pm\alpha_n T(\tfrac{x_{n-1}+x_n}{2})) &\quad +(1-\alpha_n)(T(\frac{x_n+x_{n+1}}{2})-T(\frac{x_{n-1}+x_n}{2}))\\
(\pm\alpha_n\mu_{n-1}u) &\quad +\alpha_n(\mu_n-\mu_{n-1})u+\mu_{n-1}(\alpha_n-\alpha_{n-1})u\|\\
(\text{T nonexpansive}) &\leq \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|\\
&\quad +\alpha_n|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|)\\
&\quad +|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1}\\
&\quad -T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|)\\
&\quad +(1-\alpha_n)\|\frac{x_n+x_{n+1}}{2}-\frac{x_{n-1}+x_n}{2}\|\\
&\leq \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|\\
&\quad +\alpha_n|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|)\\
&\quad +|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1}\\
&\quad -T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|)\\
&\quad +\frac{(1-\alpha_n)}{2}\|x_n-x_{n-1}\|+\frac{(1-\alpha_n)}{2}\|x_{n+1}-x_n\|.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}(\frac{1+\alpha_n}{2})\|x_{n+1}-x_n\| &\leq (\frac{1+\alpha_n}{2}-\alpha_n\mu_n)\|x_n-x_{n-1}\| \\ &+\alpha_n|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|) \\ &+|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1} \\ &-T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|),\end{aligned}$$

from which it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\frac{2\alpha_n\mu_n}{1+\alpha_n})\|x_n-x_{n-1}\| \\ &+\frac{2\alpha_n}{1+\alpha_n}|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|) \\ &+\frac{2}{1+\alpha_n}|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1} \\ &-T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|)\end{aligned} \tag{1.4.8}$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exist two positive constants M and L such that

$$\|x_n\| \leq M,$$

$$\|(1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})\| \leq L,$$

for $n \geq 0$. Therefore, inequality (1.4.8) becomes

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\frac{2\alpha_n\mu_n}{1+\alpha_n})\|x_n-x_{n-1}\| \\ &+\frac{2\alpha_n}{1+\alpha_n}|\mu_n-\mu_{n-1}|(M+\|u\|) \\ &+\frac{2}{1+\alpha_n}|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|).\end{aligned}$$

From hypotheses $(1), (2)$ and since $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$, it follows that

- $\lim_{n\to\infty}\frac{\alpha_n\mu_n}{1+\alpha_n}=0,$
- $\sum_{n=0}^{\infty}\frac{\alpha_n\mu_n}{1+\alpha_n}=+\infty,$

These last, added to hypotheses (3) and (4), allows us (Lemma 1.4.15) to conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{1.4.9}$$

By definition of (1.4.1), it can be noticed that

$$\|x_{n+1} - T(\frac{x_n + x_{n+1}}{2})\| = \alpha_n \|(1-\mu_n)x_n + \mu_n u - T(\frac{x_n + x_{n+1}}{2})\|.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, we have that

$$\lim_{n\to\infty} \|x_{n+1} - T(\frac{x_n + x_{n+1}}{2})\| = 0. \tag{1.4.10}$$

On the other hand, the following estimation holds

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T(\frac{x_n + x_{n+1}}{2})\| + \|T(\frac{x_n + x_{n+1}}{2}) - Tx_{n+1}\| \\
&\le \|x_{n+1} - T(\frac{x_n + x_{n+1}}{2})\| + \frac{\|x_{n+1} - x_n\|}{2}.
\end{aligned}$$

Therefore, from (1.4.9) and (1.4.10), we get that

$$\lim_{n\to\infty} \|x_{n+1} - Tx_{n+1}\| = 0, \tag{1.4.11}$$

that is, $(x_n)_{n\in\mathbb{N}}$ is an approximating fixed point sequence for T. From the Demiclosedness Principle and (1.4.11), follows that that every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is a fixed point for T, i.e. $\omega_w(x_n) \subset Fix(T)$.

The next goal is to prove that

$$\lim_{n\to+\infty} \|x_n - q_u\| = 0,$$

where q_u is the point of $Fix(T)$ nearest to u. In this direction, we compute:

$$\begin{aligned}
\|x_{n+1} - q_u\|^2 &= \|\alpha_n(1-\mu_n)x_n + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}) + \alpha_n\mu_n u - q_u\|^2 \\
(\pm\alpha_n(1-\mu_n)q_u) &= \|\alpha_n(1-\mu_n)(x_n - q_u) + (1-\alpha_n)\big(T(\frac{x_n + x_{n+1}}{2}) - q_u\big) \\
&\quad + \alpha_n\mu_n(u - q_u)\|^2.
\end{aligned}$$

Since in a Hilbert space H, the inequality

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle \tag{1.4.12}$$

holds for all $x, y \in H$, we have that

$$\begin{aligned}
\|x_{n+1} - q_u\|^2 &\le \|\alpha_n(1-\mu_n)(x_n - q_u) + (1-\alpha_n)\big(T(\frac{x_n + x_{n+1}}{2}) - q_u\big)\|^2 \\
&\quad + 2\alpha_n\mu_n\langle u - q_u, x_{n+1} - q_u\rangle
\end{aligned}$$

Moreover, for each $x, y \in H$ and $\lambda \in [0,1]$, it results

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{1.4.13}$$

Hence we obtain

$$\begin{aligned}
\|x_{n+1} - q_u\|^2 &\leq \alpha_n\|(1-\mu_n)(x_n - q_u)\|^2 + (1-\alpha_n)\|T(\frac{x_n + x_{n+1}}{2}) - q_u\|^2 \\
&\quad +2\alpha_n\mu_n\langle u - q_u, x_{n+1} - q_u\rangle \\
\text{(T nonexpansive)} &\leq \alpha_n(1-\mu_n)^2\|x_n - q_u\|^2 + (1-\alpha_n)\|\frac{x_n + x_{n+1}}{2} - q_u\|^2 \\
&\quad +2\alpha_n\mu_n\langle u - q_u, x_{n+1} - q_u\rangle \\
(\mu_n \leq 1) &\leq \alpha_n(1-\mu_n)\|x_n - q_u\|^2 + (\frac{1-\alpha_n}{2})\|x_n - q_u\|^2 \\
&\quad +(\frac{1-\alpha_n}{2})\|x_{n+1} - q_u\|^2 + 2\alpha_n\mu_n\langle u - q_u, x_{n+1} - q_u\rangle \\
&\leq (\frac{1+\alpha_n}{2} - \alpha_n\mu_n)\|x_n - q_u\|^2 \\
&\quad +(\frac{1-\alpha_n}{2})\|x_{n+1} - q_u\|^2 + 2\alpha_n\mu_n\langle u - q_u, x_{n+1} - q_u\rangle,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\|x_{n+1} - q_u\|^2 &\leq (1 - \frac{2\alpha_n\mu_n}{1+\alpha_n})\|x_n - q_u\|^2 \\
&\quad \frac{4\alpha_n\mu_n}{1+\alpha_n}\langle u - q_u, x_{n+1} - q_u\rangle.
\end{aligned} \tag{1.4.14}$$

In order to apply Lemma 1.4.15 to the real sequence $(\|x_{n+1} - q_u\|^2)_{n\in\mathbb{N}}$, we claim that

$$\limsup_{n\to+\infty}\langle u - q_u, x_{n+1} - q_u\rangle \leq 0.$$

First of all, we designate with $(x_{n_k})_{k\in\mathbb{N}}$ the subsequence of $(x_n)_{n\in\mathbb{N}}$, such that

$$\limsup_{n\to+\infty}\langle u - q_u, x_{n+1} - q_u\rangle = \lim_{k\to+\infty}\langle u - q_u, x_{n_k+1} - q_u\rangle.$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists a subsequence, for semplicity of notation denoted with $(x_{n_k})_{k\in\mathbb{N}}$, such that $x_{n_k} \rightharpoonup \bar{x} \in H$, for $k \to \infty$. In particular, considering that $\omega_w(x_n) \subset Fix(T)$, $\bar{x} \in Fix(T)$.

Therefore, we have that

$$\begin{aligned}
\limsup_{n\to+\infty}\langle u - q_u, x_{n+1} - q_u\rangle &= \lim_{k\to+\infty}\langle u - q_u, x_{n_k+1} - q_u\rangle \\
&= \langle u - q_u, \bar{x} - q_u\rangle \\
\text{(Lemma 1.4.10)} &\leq 0.
\end{aligned}$$

Finally, from Lemma 1.4.15, we get that $\lim_{n\to\infty} \|x_{n+1} - q_u\| = 0$. $\square$

For the modified EMR (1.4.6), we first claim the boundedness of the generated sequence with the following:

Lemma 1.4.18 *The sequence $(x_n)_{n\in\mathbb{N}}$ generated by* (1.4.6) *is bounded.*
Proof. *Let $q \in Fix(T)$ and $n \in \mathbb{N}$, let us compute:*

$$
\begin{array}{rcl}
\|x_{n+1} - q\| & = & \|\alpha_n(1-\mu_n)x_n \\
& & +(1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) + \alpha_n\mu_n u - q\| \\
(\pm\alpha_n(1-\mu_n)q) & = & \|\alpha_n(1-\mu_n)(x_n - q) \\
& & +(1-\alpha_n)(T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q) \\
& & +\alpha_n\mu_n(u-q)\| \\
& \leq & \alpha_n(1-\mu_n)\|x_n - q\| \\
& & +(1-\alpha_n)\|T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q\| \\
& & +\alpha_n\mu_n\|u-q\| \\
\textit{(T nonexpansive)} & \leq & \alpha_n(1-\mu_n)\|x_n - q\| \\
& & +(1-\alpha_n)\|s_n x_n + (1-s_n)\bar{x}_{n+1} - q\| \\
& & +\alpha_n\mu_n\|u-q\| \\
& \leq & \alpha_n(1-\mu_n)\|x_n - q\| \\
& & +(1-\alpha_n)s_n\|x_n - q\| + (1-\alpha_n)(1-s_n)\|\bar{x}_{n+1} - q\| \\
& & +\alpha_n\mu_n\|u-q\| \\
& \leq & \alpha_n(1-\mu_n)\|x_n - q\| + (1-\alpha_n)s_n\|x_n - q\| \\
& & +(1-\alpha_n)(1-s_n)\beta_n\|x_n - q\| \\
& & +(1-\alpha_n)(1-s_n)(1-\beta_n)\|Tx_n - q\| \\
& \leq & (1-\alpha_n\mu_n)\|x_n - q\| + \alpha_n\mu_n\|u-q\| \\
& \leq & \max\{\|x_n - q\|, \|u-q\|\}.
\end{array}
$$

Consequently, it results that

$$\|x_{n+1} - q\| \leq \max\{\|x_0 - q\|, \|u - q\|\}.$$

Therefore the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. $\square$

Proof.(Theorem 1.4.6) An example of control sequences satisfying conditions $(1)-(7)$ is given by $\alpha_n = s_n = \mu_n = \frac{1}{\sqrt{n}}$ and $\beta_n = \dfrac{n}{n+1}$. The first goal is to prove that the sequence $(x_n)_{n\in\mathbb{N}}$, defined by (1.4.6), is asymptotically regular:

$$\begin{array}{rcl}
\|x_{n+1}-x_n\| & = & \|\alpha_n(1-\mu_n)x_n \\
& & +(1-\alpha_n)T(s_nx_n+(1-s_n)\bar{x}_{n+1}) \\
& & +\alpha_n\mu_nu-\alpha_{n-1}(1-\mu_{n-1})x_{n-1} \\
& & -(1-\alpha_{n-1})T(s_{n-1}x_{n-1} \\
& & +(1-s_{n-1})\bar{x}_n)-\alpha_{n-1}\mu_{n-1}u\| \\
\pm\alpha_n(1-\mu_n)x_{n-1} & = & \|\alpha_n(1-\mu_n)(x_n-x_{n-1})+\alpha_n(\mu_{n-1}-\mu_n)x_{n-1} \\
\pm\alpha_n(1-\mu_{n-1})x_{n-1} & & +(\alpha_n-\alpha_{n-1})((1-\mu_{n-1})x_{n-1} \\
& & -T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)) \\
\pm\alpha_nT(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n) & & +(1-\alpha_n)(T(s_nx_n+(1-s_n)\bar{x}_{n+1}) \\
& & -T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)) \\
\pm\alpha_n\mu_{n-1}u & & +\mu_{n-1}(\alpha_n-\alpha_{n-1})u+\alpha_n(\mu_n-\mu_{n-1})u\|
\end{array}$$

Hence we get that:

$$\begin{array}{rcl}
\|x_{n+1}-x_n\| & \leq & \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
& & +|\alpha_n-\alpha_{n-1}|\|(1-\mu_{n-1})x_{n-1} \\
& & -T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)\| \\
& & +(1-\alpha_n)\|s_nx_n+(1-s_n)\bar{x}_{n+1} \\
& & -s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)\|+\mu_{n-1}|\alpha_n-\alpha_{n-1}|\|u\| \\
& & +\alpha_n|\mu_n-\mu_{n-1}|\|u\| \\
& = & \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
& & +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|) \\
& & +(1-\alpha_n)\|s_nx_n+(1-s_n)\beta_nx_n \\
& & +(1-s_n)(1-\beta_n)Tx_n-s_{n-1}x_{n-1}-(1-s_{n-1})\beta_{n-1}x_{n-1} \\
& & -(1-s_{n-1})(1-\beta_{n-1})Tx_{n-1}\|
\end{array}$$

From which we obtain also:

$$\begin{array}{rcl}
\|x_{n+1}-x_n\| & \leq & \alpha_n(1-\mu_n)\|x_n-x_{n-1}\| \\
(\pm s_n x_{n-1}) & & +\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
(\pm(1-s_n)\beta_n x_{n-1}) & & +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|) \\
& & +(1-\alpha_n)\|s_n(x_n-x_{n-1}) \\
(\pm(1-s_n)(1-\beta_n)Tx_{n-1}) & & +(1-s_n)\beta_n(x_n-x_{n-1})+(1-s_n)(1-\beta_n)(Tx_n-Tx_{n-1}) \\
(\pm\beta_n s_{n-1}x_{n-1}) & & +(s_n-s_{n-1}+\beta_n-\beta_{n-1}+\beta_n(s_{n-1}-s_n) \\
(\pm\beta_n s_{n-1}Tx_{n-1}) & & +s_{n-1}(\beta_{n-1}-\beta_n))x_{n-1}+(s_{n-1}-s_n+\beta_{n-1}-\beta_n \\
& & +\beta_n(s_n-s_{n-1})+s_{n-1}(\beta_n-\beta_{n-1}))Tx_{n-1}\|.
\end{array}$$

Hence, we have

$$\begin{array}{rcl}
\|x_{n+1}-x_n\| & \leq & (\alpha_n(1-\mu_n)+1-\alpha_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
& & +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|)+((1-s_{n-1})|\beta_n-\beta_{n-1}| \\
& & +(1-\beta_n)|s_n-s_{n-1}|)M \\
& & +((1-s_{n-1})|\beta_n-\beta_{n-1}|+(1-\beta_n)|s_n-s_{n-1}|)N \\
& = & (1-\alpha_n\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
& & +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|)+((1-s_{n-1})|\beta_n-\beta_{n-1}| \\
& & +(1-\beta_n)|s_n-s_{n-1}|)(M+N),
\end{array}$$

where L and M are positive constants defined as previously, and $N>0$ is such that $\|Tx_n\|\leq N$ for all $n\geq 0$. From conditions $(1)-(6)$, thanks to Lemma 1.4.15, we finally get that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{1.4.15}$$

Moreover, it results that

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(s_n x_n + (1-s_n)\bar{x}_{n+1})\| \\
&\quad + \|T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|(1-\mu_n)x_n \\
&\quad + \mu_n u - T(s_n x_n + (1-s_n)\bar{x}_{n+1})\| \\
&\quad + \|s_n x_n + (1-s_n)\bar{x}_{n+1} - x_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|(1-\mu_n)x_n \\
&\quad + \mu_n u - T(s_n x_n + (1-s_n)\bar{x}_{n+1})\| \\
&\quad + (1-s_n)\|\bar{x}_{n+1} - x_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|(1-\mu_n)x_n \\
&\quad + \mu_n u - T(s_n x_n + (1-s_n)\bar{x}_{n+1})\| \\
&\quad + (1-s_n)(1-\beta_n)\|x_n - Tx_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
(\beta_n(1-s_n) + s_n)\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| \\
&\quad + \alpha_n \|(1-\mu_n)x_n + \mu_n u - T(s_n x_n + (1-s_n)\bar{x}_{n+1})\|.
\end{aligned}$$

Since condition (1) holds, and from (1.4.15), we deduce that

$$\limsup_{n\to\infty} (\beta_n(1-s_n) + s_n)\|x_n - Tx_n\| \leq 0.$$

By hypothesis (7), it follows that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Consequently, in light of the Demiclosedness Principle, we have that $\omega_w(x_n) \subset Fix(T)$.

Next purpose is to claim that $\lim_{n\to\infty} \|x_n - x_u{}^*\| = 0$, where $x_u{}^*$ is the unique point of $Fix(T)$ with the property

$$\|u - x_u{}^*\| = \min_{x \in Fix(T)} \|u - x\|.$$

Indeed, it holds that

$$\begin{aligned}
\|x_{n+1} - x_u{}^*\|^2 &= \|\alpha_n(1-\mu_n)x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) \\
&\quad +\alpha_n\mu_n u - x_u{}^*\|^2 \\
(\pm\alpha_n(1-\mu_n)x_u{}^*) &= \|\alpha_n(1-\mu_n)(x_n - x_u{}^*) \\
&\quad +(1-\alpha_n)(T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - x_u{}^*) \\
&\quad +\alpha_n\mu_n(u - x_u{}^*)\|^2 \\
&\leq \|\alpha_n(1-\mu_n)(x_n - x_u{}^*) \\
&\quad +(1-\alpha_n)(T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - x_u{}^*)\|^2 \\
&\quad +2\alpha_n\mu_n\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle \\
&\leq \alpha_n(1-\mu_n)^2\|x_n - x_u{}^*\|^2 \\
&\quad +(1-\alpha_n)\|T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - x_u{}^*\|^2 \\
&\quad +2\alpha_n\mu_n\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle \\
&\leq \alpha_n(1-\mu_n)\|x_n - x_u{}^*\|^2 \\
&\quad +(1-\alpha_n)\|s_n x_n + (1-s_n)\bar{x}_{n+1} - x_u{}^*\|^2 \\
&\quad +2\alpha_n\mu_n\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle \\
&\leq (1-\alpha_n\mu_n)\|x_n - x_u{}^*\|^2 \\
&\quad +2\alpha_n\mu_n\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle.
\end{aligned}$$

Argumenting as in the proof of Theorem 1.4.2, we are going to claim that

$$\limsup_{n\to\infty}\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle \leq 0.$$

First of all, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty}\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle = \lim_{k\to\infty}\langle u - x_u{}^*, x_{n_k} - x_u{}^*\rangle.$$

On the other hand, from Lemma 1.4.18, it is known that $(x_n)_{n\in\mathbb{N}}$ is bounded. From reflexivity of H it follows that there exists a subsequence, that we still denote with $(x_{n_k})_{k\in\mathbb{N}}$, for semplicity of notation, such that $x_{n_k} \rightharpoonup x \in H$, as $k \to \infty$. Since every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is a fixed point, it follows that $x \in Fix(T)$. Therefore, we have that

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - x_u{}^*, x_{n+1} - x_u{}^*\rangle &= \lim_{k\to\infty}\langle u - x_u{}^*, x_{n_k} - x_u{}^*\rangle \\
&= \langle u - x_u{}^*, x - x_u{}^*\rangle \\
\text{(Lemma 1.4.10)} &\leq 0.
\end{aligned}$$

From Xu's lemma 1.4.15, we finally conclude that $\lim_{n\to\infty} \|x_{n+1} - x_u{}^*\| = 0$. $\square$

Chapter 2

A Viscosity Midpoint Rule for Quasi-nonexpansive Mappings

Preface.

In this chapter we study an iterative algorithm for the class of quasi-nonexpansive mappings, basing on the finite difference scheme which defines the numerical EMR (scheme (1.1.6)); even though the proposed method generates a sequence instead of a polygonal function in $\mathbb{R}$, and the goal is not to numerically solve an initial value problem but to approximate fixed points of a nonlinear operator.

We stress some fundamental features of our contributions with respect to the previous works, for instance [5], [91], [48], dealing with midpoint rules for approximating fixed points:

- The proposed algorithm, named General Viscosity Explicit Midpoint Rule (GVEMR), is explicit rather than implicit;
- The considered class of mappings is wider than that of nonexpansive operators, since it includes these latter and, for instance, the so called nonspreading and L-hybrid mappings;
- We establish the convergence results in the framework of Hilbert spaces. At a later stage, we prove the same results in a more general Banach spaces setting, with new techniques with respect to those employed in Hilbert spaces;
- The results hold under very mild assumptions on the control sequences.

2.1 Iterative methods for Quasi-nonexpansive operators.

Quasi-nonexpansivity condition appeared for the first time in [82], on real functions, by Tricomi. In his work, Tricomi obtained a general result concerning iterations of a real function, a special case of which may be formulated as follows:

Theorem 2.1.1 *[82] Let g be a real-valued function on the (finite or infinite) interval $a < x < b$, whose values lie in the same interval; and such that g^k is continuous, for a certain positive integer k. Suppose*

- *there exists a number q, with $a < q < b$, such that $g(q) = q$;*
- *$|g(x) - q| < |x - q|$ for $a < x < b$, $x \neq q$.*

  Then, for every $a < x < b$,
  $$\lim_{m \to +\infty} g^m(x) = q,$$
  where g^m denotes the m-th iterate of the function g.

Definition 2.1.2 *[26] In a normed space X, a mapping $T : C \subset X \to C$ is said to be quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $\|Tx - q\| \leq \|x - q\|$ for all $q \in Fix(T)$ and $\forall x \in C$.*

Follows from the definition that a nonexpansive mapping, with at least a fixed point, is quasi-nonexpansive. A linear quasi-nonexpansive mapping on a subspace is nonexpansive on that subspace; but there exists continuous and discontinuous nonlinear quasi-nonexpansive mappings which are not nonexpansive. For instance we recall the following

Example 2.1.3 *[27]*
Let $X = \mathbb{R}$ and T defined as

$$Tx = \begin{cases} 0, & x = 0 \\ \frac{x}{2}\sin(\frac{1}{x}), & x \neq 0. \end{cases}$$

We notice that $Fix(T) = \{0\}$, since if exists a point $x \neq 0$ such that $Tx = x$ then it should be

$$x = \frac{x}{2}\sin(\frac{1}{x})$$

or

$$2 = \sin(\frac{1}{x})$$

that is impossible. Moreover T is quasi-nonexpansive since

$$|Tx - q| = |Tx| = |\frac{x}{2}||\sin(\frac{1}{x})| \leq |\frac{x}{2}| \leq |x| = |x - 0|$$

T is not nonexpansive, indeed for $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$ it results

$$|Tx - Ty| = \frac{2}{\pi}\sin(\frac{\pi}{2}) - \frac{2}{3\pi}\sin(\frac{3\pi}{2}) = \frac{2}{\pi}\frac{4}{3} = \frac{8\pi}{3},$$

instead

$$|x - y| = \frac{4\pi}{3}$$

Example 2.1.4 *Consider $X = \mathbb{R}$ and $T : X \to X$ defined by*

$$Tx = \begin{cases} 0, & x \leq 1 \\ \frac{1}{3}, & x > 1. \end{cases}$$

The fixed point equation $Tx = x$ is satisfied only by $q = 0$, hence $Fix(T) = \{0\}$. The quasi-nonexpansivity condition is satisfied in $x = 0$, and for $x \neq 0$:

if $x > 1$, $|Tx - 0| = |Tx| = \frac{1}{3} < |x|$

if $x \leq 1$, $|Tx - 0| = 0 < |x|$

Therefore T is quasi-nonexpansive. Anyway T is not continuous, and hence not nonexpansive.

Some of the iteration procedures studied for the class of nonexpansive mappings have successively been extended, under suitable conditions, to quasi-nonexpansive operators.

In 1967, Diaz and Metcalf [25], following the work of Tricomi in [82], started an investigation on the set of subsequential limit points for the sequence of Picard iterates of a given continuous operator T defined on a metric space E. This set is denoted with

$$\mathcal{L}(x) = \{y \in E : y = \lim_{i\to\infty} T^{n_i}(x)\}, \quad x \in E.$$

where $(T^{n_i}(x))_{i\in\mathbb{N}} \subset (T^n(x))_{n\in\mathbb{N}}$.

The authors of [25] first studied the structure of the set $\mathcal{L}(x)$, assuming hypotheses which do not guarantee the convergence of the entire sequence of iterates $(T^n(x))_{n\in\mathbb{N}}$. Anyway, the final aim is to determine conditions under which the sequence $(T^n(x))_{n\in\mathbb{N}}$ actually converges. The main result is expressed by

Theorem 2.1.5 *[25, Theorem 2] Let $T : E \to E$ be continuous. Suppose*

1. *$Fix(T)$ is nonempty and compact;*
2. *For each $x \in E$, with $x \notin Fix(T)$, it results that*

$$d(Tx, Fix(T)) < d(x, Fix(T))$$

Then, for $x \in E$, the set $\mathcal{L}(x)$ is a closed and connected subset of $Fix(T)$. Either $\mathcal{L}(x)$ is empty, or it contains exactly one point, or it contains uncountably many points. In case $\mathcal{L}(x)$ is just one point, then $\lim_{n\to\infty} T^n(x)$ exists and belong to $Fix(T)$. In case $\mathcal{L}(x)$ is uncountable, then it is contained in the boundary of $Fix(T)$.

A variation of this result could be obtained removing the hypothesis of compactness on $Fix(T)$ and replacing condition (2) with the uniform requirement of nonexpansivity of T with respect to $Fix(T)$:

Theorem 2.1.6 *[25, Theorem 3] Let $T : E \to E$ be continuous. Suppose*

1. *$Fix(T)$ is nonempty;*
2. *For each $x \in E$, with $x \notin Fix(T)$, and each $q \in Fix(T)$, one has $d(Tx, q) < d(x, q)$.*

Let $x \in E$. Then, either $(T^n(x))_{n\in\mathbb{N}}$ contains no convergent subsequence, or $\lim_{n\to\infty} T^n(x)$ exists and belongs to $Fix(T)$.

Remark 2.1.7 *It can be noticed the connection with a previous work of Edelstein [28, Theorem 1]. In this last, the author assumed that the mapping T satisfies*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in E$, with $x \neq y$, and that, for some $x \in E$, there exists a convergent subsequence of iterates $(T^{n_i}(x))_{i\in\mathbb{N}} \subset (T^n(x))_{n\in\mathbb{N}}$, that is $\mathcal{L}(x)$ is nonempty. He then concluded that $\lim_{n\to\infty} T^n(x)$ exists and is a fixed point of T. Hence Theorem 2.1.6 is a generalization of Edelstein's result.

In 1970, Dotson studied the behavior of normal Mann process for the class of quasi-nonexpansive mappings in different frameworks.

In the setting of strictly convex Banach spaces he proved a result that generalizes a previous one obtained by Edelstein [28] for the convergence of Krasnoselskij iterations for the class of nonexpansive mappings:

Theorem 2.1.8 *[26, Theorem 3] Suppose X is a strictly convex Banach space, C is a closed and convex subset of X, $T : C \to C$ is continuous and quasi-nonexpansive on C, and $T(C) \subset K \subset C$, where K is compact. Suppose $x_1 \in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\alpha_n = a_{n+1,n+1}$, for $n \geq 0$, and $(\alpha_n)_{n\in\mathbb{N}}$ clusters at some $t \in (0, 1)$. Then the sequences $(x_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in the process $M(x_1, A, T)$ converge strongly to a fixed point of T.*

If the framework is that of uniformly convex Banach spaces, it holds:

Theorem 2.1.9 *[26, Theorem 5] Suppose X is a uniformly convex Banach space , C is a closed, convex subset of X, $T : C \to C$ is quasi-nonexpansive on C, and $I - T$ is closed. Suppose $x_1 \in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\alpha_n = a_{n+1,n+1}$ for $n \geq 0$ and $(\alpha_n)_{n\in\mathbb{N}}$ is bounded away from 0 and 1. If $(v_n)_{n\in\mathbb{N}}$ clusters (strongly) to some $y \in C$, then $Ty = y$ and the sequences $(x_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge (strongly) to y.*

Theorem 2.1.10 *[26, Theorem 6] Suppose X is a uniformly convex Banach space, C is a closed, convex subset of X, $T : C \to C$ is quasi-nonexpansive on C. Suppose $x_1 \in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\alpha_n = a_{n+1,n+1}$ for $n \geq 0$ and $(\alpha_n)_{n\in\mathbb{N}}$ is bounded away from 0 and 1. Then the following are true:*

(a) There is a subsequence of $(v_n)_{n\in\mathbb{N}}$ which converges weakly to some $y \in C$ and if $I-T$ is demiclosed then each weak subsequential limit point of $(v_n)_{n\in\mathbb{N}}$ is a fixed point of T;

(b) If $I - T$ is demiclosed and T has only one fixed point $q \in C$, then the sequences $(x_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge weakly to q;

(c) If $I - T$ is weakly closed, then each weak cluster point of $(v_n)_{n\in\mathbb{N}}$ is a fixed point of T.

In Hilbert spaces, the following result holds:

Theorem 2.1.11 *[26, Theorem 8] Suppose H is a real Hilbert space, C is a closed and convex subset of H, $T : C \to C$ is quasi-nonexpansive on C, and $I - T$ is demiclosed. Supposed $x_1 \in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\alpha_n = a_{n+1,n+1}$ for $n \geq 0$ and $(\alpha_n)_{n\in\mathbb{N}}$ is bounded away from 0 and 1. Then the sequences $(x_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge weakly to a fixed point of T.*

Remark 2.1.12 *Since for a nonexpansive weakly continuous mapping T it holds that $I-T$ is demiclosed, this result generalizes Theorem 1.2.11 obtained by Schaefer for the class of nonexpansive mappings.*

In 1973, Petryshyn and Williamson studied the weak and strong convergence of the general Krasnoselskij iteration and Picard iterates for the class of quasi-nonexpansive mappings. The main results are characterizations of the strong convergence for these iterates:

Theorem 2.1.13 *[83, Theorem 1.1] Let C be a closed subset of a Banach space X and let T map C continuously into X such that*

1. $Fix(T) \neq \emptyset$;

2. *For each $x \in C$ and $q \in Fix(T)$,*

$$\|Tx - q\| \leq \|x - q\|;$$

3. *There exists a $x_0 \in C$ such that $x_n = T^n(x_0) \in C$ for all $n \geq 0$.*

Then $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T in C if and only if

$$\lim_{n\to\infty} d(x_n, Fix(T)) = 0.$$

Theorem 2.1.14 *[83, Theorem 1.1'] Let C be a closed and convex subset of a Banach space X and let T map C continuously into X such that*

1. *$Fix(T) \neq \emptyset$;*

2. *For each $x \in C$ and $q \in Fix(T)$,*

$$\|Tx - q\| \leq \|x - q\|;$$

3. *There exists a $x_0 \in C$ such that $x_n = T_\lambda^n(x_0) \in C$ for each $n \geq 0$ and some $\lambda \in (0, 1)$.*

Then $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T in C if and only if

$$\lim_{n\to\infty} d(T_\lambda^n(x_0), Fix(T)) = 0.$$

In 1997 Ghosh and Debnath, in [34, Theorem 3.1], proved that the same conditions employed by Petryshyn and Williamson ensure the convergence of Picard iterates for the mapping $T_{\lambda,\mu} = \lambda I + (1-\lambda)(\mu I + (1-\mu)T)$ with $\lambda, \mu \in (0, 1)$, namely the Ishikawa iterates, with T quasi-nonexpansive.

A consequence of Theorem 2.1.13 is the following result that is a proper generalization (see [83, Example 1.3]) of Theorem 2.1.6:

Theorem 2.1.15 *[83, Theorem 1.3] Let C be a closed subset of a Banach space X. Let T map C continuously into X such that*

1. *$Fix(T) \neq \emptyset$;*

2. *For each $x \in C$ and $q \in Fix(T)$,*

$$\|Tx - q\| \leq \|x - q\|,$$

3. *For every $x \in C \setminus Fix(T)$, there exists $q_x \in Fix(T)$ such that*

$$\|Tx - q_x\| < \|x - q_x\|,$$

4. *There exists a* $x_0 \in C$ *such that* $T^n(x_0) \in C$ *for each* $n \geq 0$*, and* $(x_n)_{n\in\mathbb{N}} = (T^n(x_0))_{n\in\mathbb{N}}$ *contains a subsequence* $(x_{n_j})_{j\in\mathbb{N}}$ *converging to some* $z \in C$*.*

Then $z \in Fix(T)$ *and* $(x_n)_{n\in\mathbb{N}}$ *converges to* z*.*

Remark 2.1.16 *Previous theorem can be formulated also for the sequence* $(x_n)_{n\in\mathbb{N}}$ *generated by* $(T_\lambda^n(x_0))_{n\in\mathbb{N}}$ *for a certain* $x_0 \in C$*.*

Petryshyn and Williamson used Theorem 2.1.13 and 2.1.15, both for the sequences $(T^n(x_0))_{n\in\mathbb{N}}$ and $(T_\lambda^n(x_0))_{n\in\mathbb{N}}$, to unify and to extend the results appeared in [51, 28, 74, 14] to the more general class of quasi-nonexpansive mappings (see [83, Lemma 2.1-2.2, Theorem 3.3]).

2.2 Viscosity methods for fixed points approximation.

Let H be a real Hilbert space, $C \subset H$ a closed and convex set. In Section 1.1, in order to introduce Halpern method, we have focused on the averaged type mapping $T_t : C \to C$, defined by

$$T_t(x) = tu + (1-t)Tx, \quad x \in C \tag{2.2.1}$$

where u is fixed in C, $t \in (0,1)$ and $T : C \to C$ is a nonexpansive mapping.

The development of viscosity approximation methods is based on replacing u in (2.2.1) with a certain contraction $f : C \to C$.

Let $f : C \to C$ be a contraction of parameter $\theta \in (0,1)$; the resulting averaged mapping, denoted with $T_t^f : C \to C$, is given by

$$T_t^f(x) = tf(x) + (1-t)Tx, \quad x \in C \tag{2.2.2}$$

for each $t \in (0,1)$. For semplicity of notation T_t^f will be designated by T_t.

The mapping T_t, for each $x, y \in C$, verifies

$$\begin{aligned} \|T_t x - T_t y\| &\leq t\|f(x) - f(y)\| + (1-t)\|Tx - Ty\| \\ &\leq ((1-t) + t\theta)\|x - y\| = (1 - (1-\theta)t)\|x - y\|. \end{aligned}$$

Hence T_t is a contraction with constant $1 - (1-\theta)t$. Then, there exists a unique element $x_t^f = x_t$ such that $x_t = T_t x_t$, that is

$$x_t = tf(x_t) + (1-t)Tx_t \tag{2.2.3}$$

The introduction of the contraction f, under suitable assumptions, allows to select the fixed point of T that satisfies a certain variational inequality, as it can be seen in work [60] of Moudafi, who first introduced viscosity methods.

Let C be a closed and convex subset of H, $T : C \to C$ a nonexpansive mapping and $f : C \to C$ a θ-contraction for a certain $\theta \in (0,1)$. First Moudafi introduced the implicit scheme

$$x_n = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} T x_n, \quad n \geq 1, \tag{2.2.4}$$

with $(\epsilon_n)_{n\in\mathbb{N}} \in (0,1)$.

For the sequence defined by (2.2.4), Moudafi claimed the following strong convergence result:

Theorem 2.2.1 *[60, Theorem 2.1] The sequence $(x_n)_{n\in\mathbb{N}}$ generated by* (2.2.4)*, with $(\epsilon_n)_{n\in\mathbb{N}} \in (0,1)$ such that $\lim_{n\to\infty} \epsilon_n = 0$, strongly converges to the unique solution $\tilde{x} \in Fix(T)$ of the variational inequality*

$$\langle \tilde{x} - f(\tilde{x}), \tilde{x} - x \rangle \leq 0 \quad \forall x \in Fix(T),$$

in other words, to the unique fixed-point of the operator $P_{Fix(T)} \circ f$.

Note that, as T is nonexpansive, $Fix(T)$ is a closed and convex subset of H (Corollary (1.4.14)), hence $P_{Fix(T)}$ is well-defined.

Moreover an explicit iterative method is given. Fixed arbitrarirly an initial point $x_0 \in C$, it generates a sequence $(x_n)_{n\in\mathbb{N}}$ by

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} T x_n, \quad n \geq 0, \tag{2.2.5}$$

where $(\epsilon_n)_{n\in\mathbb{N}} \in (0,1)$. We point out that, with respect to (2.2.5), Halpern's method can be obtained as a particular case.

Even in this case a strong convergence result holds:

Theorem 2.2.2 *[60, Theorem 2.2] Suppose*

(M_1) $\lim_{n\to\infty} \epsilon_n = 0$;

(M_2) $\sum_{n=1}^{\infty} \epsilon_n = +\infty$;

(M_3) $\lim_{n\to\infty} |\frac{1}{\epsilon_n} - \frac{1}{\epsilon_{n-1}}| = 0$.

Then for all $x_0 \in C$ the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.2.5) *converges strongly to the unique solution $\tilde{x} \in Fix(T)$ of the variational inequality*

$$\langle \tilde{x} - f(\tilde{x}), \tilde{x} - x \rangle \leq 0 \quad \forall x \in Fix(T),$$

in other words, to the unique fixed point of the operator $P_{Fix(T)} \circ f$.

Sequences of real numbers satisfying (M_1) and (M_2) are said *slowly vanishing*.

In 2004, Xu extended Moudafi's results [60] in the framework of Hilbert spaces. Moreover he proved strong convergence of the continuous scheme defined by (2.2.3) and the discrete iterative scheme (2.2.5) in a uniformly smooth Banach space setting.

We recall that

Definition 2.2.3 *[37] A Banach space X is called uniformly smooth if the limit*

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists uniformly in the set $\{(x,y)\in X : \|x\|=\|y\|=1\}$.

The first result that we mention is a continuous version of Theorem 2.2.1:

Theorem 2.2.4 *[89, Theorem 3.1] In a Hilbert space H, let x_t be given by* (2.2.3). *Then we have*

- $s-\lim\limits_{t\to 0} x_t=\tilde{x}$ *exists;*
- $\tilde{x}=P_{Fix(T)}f(\tilde{x})$ *, or equivalently, $\tilde{x}$ is the unique solution in $Fix(T)$ to the variational inequality*

  $$\langle (I-f)\tilde{x}, x-\tilde{x}\rangle \geq 0, \quad x\in Fix(T), \tag{2.2.6}$$

  where $P_{Fix(T)}$ is the metric projection from H onto $Fix(T)$.

The second is a strong convergence result for the discrete method

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Tx_n, \quad n\geq 0, \tag{2.2.7}$$

where $(\alpha_n)_{n\in\mathbb{N}}\in(0,1)$.

Theorem 2.2.5 *[89, Theorem 3.2] Let H be a Hilbert space, C a closed convex subset of H, $T: C\to C$ a nonexpansive mapping with $Fix(T)\neq\emptyset$, and $f: C\to C$ a contraction. Let $(x_n)_{n\in\mathbb{N}}$ be generated by* (2.2.7) *. Then under the hypotheses*

(H_1) $\lim\limits_{n\to\infty}\alpha_n=0$*;*

(H_2) $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$*;*

(H_3) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ *or* $\lim\limits_{n\to\infty}\dfrac{\alpha_{n+1}}{\alpha_n}=1$*;*

$\lim_{n\to\infty}x_n=\tilde{x}$*, where $\tilde{x}$ is the unique solution of the variational inequality* (2.2.6).

Remark 2.2.6 *Moudafi's conditions $(M_1) - (M_3)$ exclude the natural choice of $\epsilon_n = \frac{1}{n}$, while the conditions $(H_1) - (H_3)$, used by Xu, include the sequence $\alpha_n = \frac{1}{n}$.*

The following results are proved in the framework of a more general Banach space X. Before stating them we mention that

Definition 2.2.1 *[37] A nonempty subset K of $C \subset X$ is said to be a retract of C if there exists a continuous mapping $Q : C \to K$ with $K = Fix(Q)$. Any such mapping Q is a retraction of C onto K*

Theorem 2.2.7 *[89, Thoerem 4.1] Let X be a uniformly smooth Banach space, C a closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : C \to C$ a contraction. Then (x_t) defined by* (2.2.3) *converges strongly to a point in $Fix(T)$. If we define $Q : \Pi_C \to Fix(T)$ by*

$$Q(f) = \lim_{t \to 0} x_t, \quad f \in \Pi_C \tag{2.2.8}$$

where Π_C is the set of all contractions from C to C, then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - q)\rangle \leq 0, \quad f \in \Pi_C, q \in Fix(T).$$

In particular, if $f = u \in C$ is constant, then (2.2.8) *is reduced to the sunny nonexpansive retraction of Reich from C onto $Fix(T)$,*

$$\langle Q(u) - u, J(Q(u) - q)\rangle \leq 0, \quad u \in C, q \in Fix(T).$$

Theorem 2.2.8 *[89, Theorem 4.2] Let X be a uniformly smooth Banach space, C a closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f \in \Pi_C$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ defined by* (2.2.7)*, with $(\alpha_n)_{n\in\mathbb{N}} \in (0, 1)$ satisfying the conditions $(H_1), (H_2), (H_3)$, converges strongly to $Q(f)$, where $Q : \Pi_C \to Fix(T)$ is defined by $Q(f) = s - \lim_{t\to 0} x_t$.*

In 2006, Marino and Xu modified viscosity approximation method (2.2.7) introducing a strongly positive operator A, i.e. there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Let H be a real Hilbert space, C a nonempty, closed and convex subset of H and fixed $x_0 \in C$, the proposed algorithm generates a sequence via the scheme:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0. \tag{2.2.9}$$

where $T : C \to C$ is a nonexpansive mapping, $f \in \Pi_C$ and $(\alpha_n)_{n\in\mathbb{N}} \in [0, 1]$. For this method, Marino and Xu proved the following convergence result

Theorem 2.2.9 *[56, Theorem 3.4] Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, $f : H \to H$ a contraction with parameter $\alpha \in (0,1)$ and A a strongly positive, linear, bounded operator with constant $\bar{\gamma} > 0$. Let $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$.*

If $(\alpha_n)_{n\in\mathbb{N}} \in (0,1)$ satisfies conditions

1. $\lim_{n\to\infty} \alpha_n = 0$;

2. $\sum_{n=1}^{\infty} \alpha_n = \infty$;

3. $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

then the sequence generated by (2.2.9) *strongly converges to the unique solution $x^* \in Fix(T)$ of the variational inequality*

$$\langle (A - \gamma f)x^*, q - x^* \rangle \geq 0, \quad \forall q \in Fix(T),$$

that is the optimality condition for the minimization problem

$$\min_{q\in Fix(T)} \frac{1}{2}\langle Aq, q\rangle - h(q)$$

where h is the potential function for γf (that is $h'(x) = \gamma f(x)$, $\forall x \in H$) or, equivalently, $P_{Fix(T)}(I - A + \gamma f)x^* = x^*$.

Viscosity approximation method has been studied for the class of quasi-nonexpansive mappings by Maingé in 2010, who proved a strong convergence result for algorithm (2.2.7) in the setting of Hilbert spaces ([54, Theorem 3.1]). In [85], Saejung and Wongchan in [85] improved Maingé's result to a more general relaxation.

We recall that a mapping T is strongly quasi-nonexpansive (see [69]) if T is quasi-nonexpansive and $\|z_n - Tz_n\| \to 0$, as $n \to \infty$, whenever z_n is a bounded sequence in C such that $\lim_{n\to\infty}(\|z_n - q\| - \|Tz_n - q\|) = 0$ for some $q \in Fix(T)$.

Theorem 2.2.10 *[85, Theorem 2.3] Let C be a nonempty, closed and convex subset of a Hilbert space H, $T : C \to C$ a strongly quasi-nonexpasive mapping such taht $I - T$ is demiclosed at zero. Suppose that $f : C \to C$ is a contraction. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C given by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \tag{2.2.10}$$

with $x_0 \in C$ and $\alpha_n \in (0,1)$ such that

- $\lim_{n\to\infty} \alpha_n = 0$,

- $\sum_{n=1}^{\infty} \alpha_n = +\infty$

Then $(x_n)_{n\in\mathbb{N}}$ *converges strongly to an element* $x^* \in Fix(T)$ *and the following inequality holds:*

$$\langle (I-f)x^*, q-x^* \rangle \geq 0, \quad \forall q \in Fix(T).$$

2.3 Viscosity midpoint rules for nonexpansive mappings.

In 2015, with the purpose to get strong convergence, Xu, Alghamdi and Shahzad in [91], following Moudafi's line ([60]), modified the IMR for nonexpansive mappings introducing a viscosity term f in IMR scheme (1.3.3). The resulting algorithm is called Viscosity Implicit Midpoint Rule (VIMR) and is given by

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{2.3.1}$$

where C is a closed and convex subset of a Hilbert space H, $f : C \to C$ is a contraction and $(\alpha_n)_{n\in\mathbb{N}} \in (0,1)$.

It is proved that VIMR converges in norm to a fixed point of T which, in addition, also solves a certain variational inequality:

Theorem 2.3.1 *[90, Theorem 3.1] Let* H *be a real Hilbert space,* C *a closed and convex subset of* H*,* $T : C \to C$ *a nonexpansive mapping with* $Fix(T) \neq \emptyset$*, and* $f : C \to C$ *a contraction with parameter* $\theta \in [0,1)$*. Let* $(x_n)_{n\in\mathbb{N}}$ *be generated by* (2.3.1)*. Assume the conditions*

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
3. $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

Then $(x_n)_{n\in\mathbb{N}}$ *converges in norm to a fixed point* q *of* T*, which is also the unique solution of the variational inequality*

$$\langle (I-f)q, x-q \rangle \geq 0, \quad \forall x \in Fix(T). \tag{2.3.2}$$

In other words, q *is the unique fixed point of the contraction* $P_{Fix(T)}f$ *, that is* $P_{Fix(T)}f(q) = q$*.*

Later, Y. Ke and C. Ma ([48]) improved the VIMR replacing the midpoint of $[x_n, x_{n+1}]$, in the evaluation point of T, by any element of the same interval, obtaining the implicit scheme

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}), \quad n \geq 0, \end{cases} \tag{2.3.3}$$

where $f \in \Pi_C$ and $(\alpha_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subset (0,1)$, called Generalized Viscosity Implicit Midpoint Rule (GVIMR) for nonexpansive mappings in Hilbert spaces.

The authors of [48] proved that the sequence defined by GVIMR converges strongly to a fixed point of T, under certain assumptions imposed on the sequences of parameters, which also solve variational inequality (2.3.2):

Theorem 2.3.2 *[48, Theorem 3.1] Let H be a real Hilbert space, C a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $f : C \to C$ a contraction with parameter $\theta \in [0,1)$. Pick any $x_0 \in C$. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by* (2.3.3)*, where $(\alpha_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \in (0,1)$ satisfy the conditions*

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
3. $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
4. $0 < \epsilon \leq s_n \leq s_{n+1} < 1, \quad \forall n \geq 0$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a fixed point q of T, which is also the unique solution of the variational inequality (2.3.2)*. In other words, q is the unique fixed point of the contraction $P_{Fix(T)}f$, that is $P_{Fix(T)}f(q) = q$.*

2.4 Our contributions.

Notations.

Throgouth this section, will be denoted with

H, a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$;

X, a real Banach space with norm $\| \cdot \|$;

X^*, the dual space of X with duality pairing $\langle x, x^* \rangle = x^*(x)$ for each $x \in X$ and $x^* \in X^*$;

C, a closed and convex subset of H or X;

$T : C \to C$, a nonexpansive or a quasi-nonexpansive mapping;

$f : C \to C$, a θ-contraction for a certain $\theta \in [0, 1)$;

$S : C \to C$, a nonspreading mapping;

$Fix(T)$, the fixed points set of T;

$P_{Fix(T)}$, the metric projection of C onto $Fix(T)$;

J_ϕ, the duality mapping associated to the gauge function ϕ;

J, the normalized duality mapping;

$\omega_w(x_n)$, the set of weak cluster points of $(x_n)_{n\in\mathbb{N}}$, that is $\omega_w(x_n) = \{z : \exists (x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup z, k \to \infty\}$.

$\to$, the strong convergence;

$\rightharpoonup$, the weak convergence.

2.4.1 Results.

With the purpose of obtaining strong convergence for the EMR for nonexpansive mappings, we follow the same investigation line adopted by Ke and Ma in [48] for IMR. Hence we introduce in EMR scheme

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1 - t_n)x_n + t_n T(x_n), \quad n \geq 0, \\ x_{n+1} = (1 - t_n)x_n + t_n T(\frac{x_n + \bar{x}_{n+1}}{2}), \quad n \geq 0, \end{cases}$$

a viscosity term $f \in \Pi_C$ and, further, we replace the midpoint of $[x_n, \bar{x}_{n+1}]$ with a generic point of the same interval in the evaluation point of T.

The proposed iterative method is given by

$$\begin{cases} x_0 \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases} \tag{2.4.1}$$

that we call General Viscosity Explicit Midpoint Rule (GVEMR).

Let us consider the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\limsup_{n\to\infty} \beta_n(1-\beta_n)(1-s_n) > 0$.

A strong convergence result of the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.4.1) to a fixed point of a quasi-nonexpansive operator is proved in the framework of Hilbert spaces:

Theorem 2.4.1 *[57, Theorem 3.2] Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a quasi-nonexpansive mapping, with $I - T$ demiclosed at 0, and $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Let $(\alpha_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be three sequences in $(0,1)$, satisfying the conditions $(i), (ii), (iii)$. Then, the sequence $(x_n)_{n\in\mathbb{N}}$, defined in* (2.4.1)*, strongly converges to $\bar{q} \in Fix(T)$, which is the unique solution in $Fix(T)$ of the variational inequality (VI)*

$$\langle \bar{q} - f(\bar{q}), \bar{q} - x\rangle \leq 0, \quad \forall x \in Fix(T). \tag{2.4.2}$$

Proof. See page 65. □

Remark 2.4.2 *Conditions*

- *T is quasi-nonexpansive,*
- *$I - T$ is demiclosed at 0,*

appearing in the previous theorem, are not related. One could consider the following examples to confirm this:

Example 2.4.3 *[39, Example 2.3]*
Let $H = \mathbb{R}$, $C = [0, +\infty)$ and $T : C \to C$ a mapping defined by

$$Tx = \begin{cases} \frac{2x}{x^2+1} & x \in (1, +\infty) \\ 0 & x \in [0,1] \end{cases}$$

It results that

- *$Fix(T) = \{0\}$;*
- *T is discontinuous;*
- *T is quasi-nonexpansive, indeed if $x \in [0,1]$ then $|Tx - 0| = 0 \leq |x - 0|$; while, if $x \in (1, +\infty)$ then $|Tx - 0| = \frac{2x}{1+x^2} \leq 1 \leq |x - 0|$;*

- *Considering the sequence* $x_n = 1 + \frac{1}{n} \in (1, 2)$*, it results that* $x_n \to 1 \notin Fix(T)$ *and* $|x_n - Tx_n| \to 0$*, thus* $I - T$ *is not demiclosed at* 0*.*

Example 2.4.4 *[19, Example 8.2]*
Let $H = \mathbb{R}$*,* $C = [0, 1]$ *and* $T : C \to H$ *defined by* $Tx = 1 - x^{\frac{2}{3}}$*.*

- $Fix(T) = \{q\}$*, with* $q \in (0, 1)$*;*
- T *is a continuous pseudo-contraction, since* $I - T$ *is monotone; therefore* $I - T$ *is demiclosed at* 0 *(see [92, Demi-closedness Principle]);*
- T *is not quasi-nonexpansive since:*
  - *if* $x = 0$*, then* $|Tx - q| \leq |x - q|$ *implies* $1 - q \leq q$*, and hence* $q \geq \frac{1}{2}$
  - *if* $x = 1$*, then* $|Tx - q| \leq |x - q|$ *implies* $q \leq 1 - q$*, and hence* $q \leq \frac{1}{2}$

Thus it must be $q = \frac{1}{2}$*, that is a contradiction.*

Remark 2.4.5 *There exist mappings* $T : C \to C$*, with* $Fix(T)$ *nonempty, which are quasi-nonexpansive and such that* $I - T$ *is demiclosed at* 0*. Among these, in addition to nonexpansive mappings, let us mention*

- *Nonspreading mappings, introduced by Kohsaka and Takahashi in 2008:*

  Definition 2.4.6 *[50] Let* X *be a smooth, strictly convex and reflexive Banach space, let* J *be the duality mapping of* X *and let* C *a nonempty, closed and convex subset of* X*. Then, a mapping* $S : C \to C$ *is said to be nonspreading if*

  $$\Phi(Sx, Sy) + \Phi(Sy, Sx) \leq \Phi(Sx, y) + \Phi(Sy, x),$$

  for all $x, y \in C$*, where* $\Phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ *for all* $x, y \in X$*.*

  Particularly, if X *is a Hilbert space, it is known that* $\Phi(x, y) = \|x - y\|^2$ *for all* $x, y \in H$*. Then a nonspreading mapping* $S : C \to C$ *in a Hilbert space is defined as follows:*

  $$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,$$

  for all $x, y \in C$*.*

- *L-hybrid mappings, introduced by Aoyam et al. in 2010:*

  Definition 2.4.7 *[4] Let* $T : H \to H$ *be a mapping and* L *a nonnegative number. We will say that* T *is L-hybrid, signified as* $T \in \mathcal{H}_L$ *, if*

  $$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in H.$$

Convergence result 2.4.1 holds for these classes of mappings.

Remark 2.4.8 *In Theorem 2.4.1, no additional assumption has been formulated for the limit of the sequence $(s_n)_{n\in\mathbb{N}}$, hence it could be that $\lim_{n\to\infty} s_n = 0$ (see Example 2.4.18).*

Theorem 2.4.1 has been improved by Garcia-Falset, Marino and Zaccone in [32] in the more general setting of p-uniformly convex Banah spaces, in which inequalities analogous to (1.4.12) and (1.4.13), valid in Hilbert spaces, hold (see Xu's paper [86] for a detailed survey).

Theorem 2.4.9 *[32, Theorem 3.2] Let X be a p-uniformly convex Banach space, with $1 < p < +\infty$, having a weakly sequentially continuous duality mapping J_ϕ. Let C be a nonempty, closed and convex subset of X, $T : C \to C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0, and $f : C \to C$ a θ-contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ be generated by (2.4.1). Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (ii), (iii)$. If $Fix(T)$ is the sunny nonexpansive retract of C, with $Q : C \to Fix(T)$ sunny nonexpansive retraction, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $\bar{q} = Q(f(\bar{q}))$. Further $\bar{q}$ is the unique solution in $Fix(T)$ of the variational inequality

$$\langle f(\bar{q}) - \bar{q}, j_\phi(x - \bar{q})\rangle \leq 0, \quad \forall x \in Fix(T) \tag{2.4.3}$$

Proof. See page 70. □

Remark 2.4.1 *In order to have a strong convergence result in the setting of a p-uniformly Banach space X, we assume that X has weakly sequentially continuous duality mapping J_ϕ, for a certain gauge function ϕ. This hypothesis, compared to the weak continuity for the normalized duality map J, that is frequently assumed in trying to extend some results from the setting of Hilbert spaces to that of Banach spaces (see [45], [77] and other works), allows us to include, for instance, also the sequential spaces l_p. Indeed l_p spaces, for $p \neq 2$, fail to have weakly continuous map J, but they have generalized duality map J_p weakly sequentially continuous (for a detailed survey, see [90]).*

Concerning the hypotheses assumed for the control sequence in Theorem (2.4.9)*, as well as in Theorem 2.4.1, no additional assumption has been formulated for the limit of sequence $(s_n)_{n\in\mathbb{N}} \subset (0, 1)$, that, for instance, may converge to zero. (see Example 2.4.19).*

In light of Theorem (2.4.9), we include the following corollary:

Corollary 2.4.10 *[32, Corollary 3.4] Let X be a p-uniformly convex Banach space, for $1 < p < \infty$, having a weakly sequentially continuous duality mapping J_ϕ. Let C be a nonempty, closed and convex subset of X, $T : C \to C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0 and $Fix(T) = \{q\}$. Let $f : C \to C$ a θ-contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by (2.4.1).

If the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (ii), (iii)$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to q.

Remark 2.4.11 *The assert follows from Theorem* (2.4.9), *considered that $Fix(T) = \{q\}$ is a sunny nonexpansive retract of C, with sunny nonexpansive retraction the constant function $Q(x) = q$.*

If $X = H$ is a real Hilbert space, then X is 2-uniformly convex, with normalized duality mapping weakly sequentially continuous. Let C be a closed and convex subset of H. From [27, Theorem 1], it is known that if $T : C \to C$ is a quasi-nonexpansive then $Fix(T)$ is nonempty, closed and convex. Therefore the metric projection $P_{Fix(T)}$ is a sunny nonexpansive retraction from C onto $Fix(T)$. Since there is at most one sunny nonexpansive retraction (Lemma 2.4.15 in Section 2.4.2), we have that $Q \equiv P_{Fix(T)}$.

These considerations motivate the result that follows.

Corollary 2.4.12 *[32, Corollary 3.5] Let H be real Hilbert space. Let C be a nonempty, closed and convex subset of H, $T : C \to C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0, and $f : C \to C$ a θ-contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ be generated by 2.4.1. Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (ii)$ and (iii). Then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $q \in Fix(T)$. Further q is the unique solution in $Fix(T)$ of the variational inequality

$$\langle f(q) - q, x - q\rangle \leq 0, \quad \forall x \in Fix(T).$$

Therefore Theorem 2.4.1 can be considered as a particular case of Theorem 2.4.9.

For the eventuality in which the Banach space X fails to have weakly sequentially continuous duality map J_ϕ, as occours for L_p spaces, we establish a strong convergence result for the GVEMR, assuming the additional assumption that $I - T$ is ψ-expansive (for more details on this type of mapping, see [31] and references therein).

Theorem 2.4.13 *[32, Theorem 3.7] Let X be a p-uniformly convex Banach space, for $1 < p < \infty$, and $C \subset X$ a nonempty, closed and convex set. Let $T : C \to C$ a quasi-nonexpansive mapping such that $I - T$ is ψ-expansive, and $f : C \to C$ a θ-contraction for a certain $\theta \in (0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ the sequence generated by (2.4.1). If the parameters sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (iii)$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to to the unique fixed point of T.

Proof. *See page 76.* □

2.4.2 Tools.

Let $g : X \to (-\infty, \infty)$ be a proper functional and D a nonempty and convex subset of X.

Definition 2.4.2 *[19] The function g is said to be uniformly convex on D if there exists a function $\mu : [0, +\infty) \to [0, +\infty)$, with $\mu(t) = 0$ if and only if $t = 0$, such that*

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) - \lambda(1-\lambda)\mu(\|x - y\|)$$

for all $x, y \in D$ and $\lambda \in [0, 1]$.

Definition 2.4.3 *[22] The subdifferential of g is a map*

$$\partial g : X \to 2^{X^*}$$

defined by

$$\partial g(x) = \{x^* \in X^* : g(y) \geq g(x) + \langle y - x, x^* \rangle, \forall y \in X\}$$

The framework of the convergence results in Theorem 2.4.9 and Theorem 2.4.13 is constituted by p-uniformly convex Banach spaces. About the matter, we need to recall the following definitions:

Definition 2.4.4 *[37] The modulus of convexity of a Banach space X is the function δ_X : $(0, 2] \to (0, 1]$ defined by*

$$\delta_X(\epsilon) = \inf\left\{1 - \|\frac{x+y}{2}\|\} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}.$$

Definition 2.4.5 *[19]*

Let $p > 1$ be a real number. Then X is said to be p-uniformly convex if there is a constant $c > 0$ such that

$$\delta_X(\epsilon) \geq c\epsilon^p.$$

Example 2.4.6 *[19]*

If $X = L_p$ (or l_p), $1 < p < \infty$, then

1. $\delta_X(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$, *if* $1 < p < 2$,
2. $\delta_X(\epsilon) \geq \epsilon^p$, *if* $2 \leq p < \infty$.

In particular, for such class of Banach spaces, we mention that for all $x, y \in X$ and $\lambda \in [0, 1]$, the following inequality is verified ([86]):

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - \lambda(1-\lambda)c\|x - y\|^p \qquad (2.4.4)$$

for a certain positive constant c, for all $x, y \in X$ and $0 \leq \lambda \leq 1$.

The concept of duality mapping appeared for first time in the work of Beurling and Livingston ([8]).

Definition 2.4.7 *[19] A continuous and strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that*

$$\phi(0) = 0$$

$$\lim_{t \to \infty} \phi(t) = +\infty,$$

is called a gauge function (or weight function).

Definition 2.4.8 *[19] Given a gauge function ϕ, the mapping $J_\phi : X \to 2^{X^*}$ defined by*

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \|x^*\|; \phi(\|x\|) = \|x^*\|\}$$

is called the duality mapping with gauge function ϕ.

If the gauge ϕ is given by $\phi(t) = t^{p-1}$, $1 < p < +\infty$ for all $t \in [0, +\infty)$, then $J_\phi = J_p$ is known as pth generalized duality mapping; in particular, for $p = 2$ $J_2 = J$ is called normalized duality mapping.

When we deal with $j_\phi(x)$ we mean a (single-valued) selection of $J_\phi(x)$.

Lemma 2.4.9 *[22] Let ϕ a gauge function and $\Phi(t) = \int_0^t \phi(s)\, ds$, then Φ is a convex function.*

The duality mapping J_ϕ, associated to ϕ, can be also described in the following way:

Theorem 2.4.10 *[6] If J_ϕ is the duality mapping associated to a gauge ϕ, then*

$$J_\phi x = \partial \Phi(\|x\|) \quad \forall x \in X.$$

Thus a subdifferential inequality holds:

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x + y) \rangle, \quad j_\phi(x + y) \in J_\phi(x + y) \tag{2.4.5}$$

The following definition is due to Browder:

Definition 2.4.11 *[10]*

The duality mapping J_ϕ is said to be (sequentially) weak continuous if it is single-valued and maps weakly convergent sequences in X to $weak^*$ convergent sequences in X^*, that is, if $x_n \rightharpoonup x$ in X, then $J_\phi(x_n) \rightharpoonup^* J_\phi(x)$ in X^*.

Example 2.4.12 *[90] For each $1 < p < \infty$, the generalized duality map J_p of l_p is weakly continuous, instead that of L_p fails to be weakly continuous.*

Example 2.4.13 *[90] Let H be a real (infinite dimensional) Hilbert space. Then J_p is weakly continuous if and only if $p = 2$.*

For the fixed points set of a quasi-nonexpansive mapping the following result holds:

Theorem 2.4.14 *[27, Theorem 1]*
If C is a closed, convex subset of a strictly convex normed linear space, and $T : C \to C$ is quasi-nonexpansive, then $Fix(T) = \{z \in C : Tz = z\}$ is a nonempty, closed and convex set in which T is continuous.

It is known (see [22]) that a Banach space X is smooth if and only if each duality mapping J_ϕ is single-valued. In such spaces, a characterization for a sunny nonexpansive retraction is given by:

Lemma 2.4.15 *[67, Lemma 2.7]*
Let X be a smooth Banach space and let C a nonempty subset of X. Let $Q : X \to C$ a retraction and let J be the normalized duality map on X. The the following are equivalent:

1. *Q is sunny and nonexpansive,*
2. *$\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$ for all $x, y \in X$,*
3. *$\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in X$ and $y \in C$.*

Hence, there is at most one sunny nonexpansive retraction on C.

Remark 2.4.14 *Previous lemma holds even if the normalized duality map J is replaced with the duality map J_ϕ associated to a gauge function ϕ.*

Let us recall the definition of ψ-expansive mapping (see paper [31], of Garcia-Falset and Muniz-Perez, and references therein).

Definition 2.4.15 *[31] A mapping $A : D(A) \subset X \to X$ is said to be ψ-expansive if there exists a function $\psi : [0, +\infty) \to [0, +\infty)$ such that for every $x, y \in D(A)$, the inequality $\|Ax - Ay\| \geq \psi(\|x - y\|)$ holds, with ψ satisfying*

- *$\psi(0) = 0$;*
- *$\psi(r) > 0 \quad \forall r > 0$;*

- *Either ψ is continuous or it is nondecreasing.*

Next we recall a lemma, formulated and proved by Maingé in 2008, useful in the proof techniques of our convergence results.

Lemma 2.4.16 *[53, Lemma 3.1]*
Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_i})_{i\in\mathbb{N}}$ of $(\gamma_n)_{n\in\mathbb{N}}$ such that

$$\gamma_{n_i} < \gamma_{n_{i+1}}, \quad \textit{for all } i \in \mathbb{N}.$$

Also consider the sequence of integers $(\tau(n))_{n\geq n_0}$ defined by

$$\tau(n) := \max\{k \leq n : \quad \gamma_k < \gamma_{k+1}\}.$$

Then $(\tau(n))_{n\in\mathbb{N}}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$, the following two estimates hold

$$\gamma_{\tau(n)} < \gamma_{\tau(n)+1}, \forall\, n \geq n_0;$$

$$\gamma_n < \gamma_{\tau(n)+1}, \forall\, n \geq n_0.$$

2.4.3 Proofs.

First of all we prove that the sequence generated by (2.4.1) is bounded:

Lemma 2.4.17 *Let X be a Banach space, C a nonempty, closed and convex subset of X, $T : C \to C$ be a quasi-nonexpansive mapping, $f : C \to C$ a contraction with coefficient $\theta \in [0,1)$. Let $(\alpha_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be sequences in $(0,1)$. Then the sequence $(x_n)_{n\in\mathbb{N}}$, defined in (2.4.1), is bounded.*
Proof. *Let $q \in Fix(T)$, we have that*

$$\begin{array}{rcl}
\|x_{n+1} - q\| & = & \|\alpha_n(f(x_n) - q) + (1-\alpha_n)(T(s_n x_n + (1-s_n)\overline{x}_{n+1}) - q)\| \\
& \leq & \alpha_n\|f(x_n) - q\| + (1-\alpha_n)\|T(s_n x_n + (1-s_n)\overline{x}_{n+1}) - q\| \\
\textit{(T quasi-nonexp.)} & \leq & \alpha_n\|f(x_n) - q\| + (1-\alpha_n)\|s_n x_n + (1-s_n)\overline{x}_{n+1} - q\| \\
& = & \alpha_n\|f(x_n) - q\| + (1-\alpha_n)\|s_n(x_n - q) + (1-s_n)(\overline{x}_{n+1} - q)\| \\
& \leq & \alpha_n\|f(x_n) - q\| + (1-\alpha_n)s_n\|x_n - q\| \\
& + & (1-\alpha_n)(1-s_n)\|\overline{x}_{n+1} - q\|
\end{array}$$

From the definition of $\bar{x}_{n+1}$, we get

$$\begin{array}{rcl}\|x_{n+1}-q\| & \leq & \alpha_n\|f(x_n)-f(q)\|+\alpha_n\|f(q)-q\|+(1-\alpha_n)s_n\|x_n-q\| \\ & + & (1-\alpha_n)(1-s_n)\beta_n\|x_n-q\| \\ & + & (1-\alpha_n)(1-s_n)(1-\beta_n)\|Tx_n-q\| \\ (f \text{ contrac., } T \text{ quasi-nonexp.}) & \leq & (1-\alpha_n(1-\theta))\|x_n-q\|+\alpha_n(1-\theta)\dfrac{\|f(q)-q\|}{(1-\theta)},\end{array}$$

for all $n \geq 0$.

Hence, we obtain

$$\|x_{n+1}-q\| \leq \max\left\{\|x_n-q\|, \frac{\|f(q)-q\|}{(1-\theta)}\right\}.$$

Set $K_n = \max\{\|x_n-q\|, \frac{\|f(q)-q\|}{(1-\theta)}\}$, we have that

$$\|x_{n+1}-q\| \leq K_n.$$

Since $K_{n+1} \leq K_n$ for all $n \geq 0$, so that $K_n \leq K_0$, we finally obtain

$$\|x_{n+1}-q\| \leq \max\left\{\|x_0-q\|, \frac{\|f(q)-q\|}{(1-\theta)}\right\},$$

then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. □

Proof.(Theorem 2.4.1)

We denote by $O(1)$ any bounded real sequence.

Let q a fixed point of T, we compute:

$$\|x_{n+1}-q\|^2 = \|\alpha_n(f(x_n)-q)+(1-\alpha_n)(T(s_nx_n+(1-s_n)\overline{x}_{n+1})-q)\|^2.$$

From the well known inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

which holds for all $x, y \in H$, follows that

$$\begin{array}{rcl}\|x_{n+1}-q\|^2 & \leq & \|(1-\alpha_n)(T(s_nx_n+(1-s_n)\overline{x}_{n+1})-q)\|^2 \\ & + & 2\alpha_n\langle f(x_n)-q, x_{n+1}-q\rangle \\ & = & (1-\alpha_n)^2\|T(s_nx_n+(1-s_n)\overline{x}_{n+1})-q\|^2 \\ & + & 2\alpha_n\langle f(x_n)-f(q), x_{n+1}-q\rangle \\ & + & 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\ \text{(T quasi-nonexp, f contrac.)} & \leq & (1-\alpha_n)^2\|s_nx_n+(1-s_n)\overline{x}_{n+1}-q\|^2 \\ & + & 2\theta\alpha_n\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle.\end{array}$$

It is known that in a Hilbert space H the following identity

$$\|\lambda x+(1-\lambda)\,y\|^2=\lambda\,\|x\|^2+(1-\lambda)\,\|y\|^2-\lambda\,(1-\lambda)\,\|x-y\|^2\,,$$

holds for all $x, y \in H$ and $\lambda \in [0,1]$.

We deduce that

$$\begin{array}{rcl}
\|x_{n+1}-q\|^2 & \leq & (1-\alpha_n)^2 s_n\|x_n-q\|^2+(1-\alpha_n)^2(1-s_n)\|\overline{x}_{n+1}-q\|^2 \\
& - & (1-\alpha_n)^2 s_n(1-s_n)\|x_n-\overline{x}_{n+1}\|^2 \\
& + & \theta\alpha_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
& + & 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\
& = & (1-\alpha_n)^2 s_n\|x_n-q\|^2+(1-\alpha_n)^2(1-s_n)\beta_n\|x_n-q\|^2 \\
& + & (1-\alpha_n)^2(1-s_n)(1-\beta_n)\|Tx_n-q\|^2 \\
& - & (1-\alpha_n)^2(1-s_n)(1-\beta_n)\beta_n\|Tx_n-x_n\|^2 \\
& - & (1-\alpha_n)^2 s_n(1-s_n)(1-\beta_n)^2\|x_n-Tx_n\|^2 \\
& + & \theta\alpha_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
& + & 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\
& \leq & (1-\alpha_n)^2 s_n\|x_n-q\|^2 \\
& + & (1-\alpha_n)^2(1-s_n)\beta_n\|x_n-q\|^2 \\
& + & (1-\alpha_n)^2(1-s_n)(1-\beta_n)\|x_n-q\|^2 \\
& - & (1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)\|Tx_n-x_n\|^2 \\
& + & \theta\alpha_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
& + & 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\
& = & ((1-2\alpha_n)+\alpha_n\theta)\|x_n-q\|^2+\alpha_n^2\|x_n-q\|^2 \\
& - & (1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)\|Tx_n-x_n\|^2 \\
& + & \theta\alpha_n\|x_{n+1}-q\|^2+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle.
\end{array}$$

Summarizing, we get that

$$\begin{array}{rcl}
\|x_{n+1}-q\|^2 & \leq & ((1-2\alpha_n)+\alpha_n\theta)\|x_n-q\|^2+\alpha_n^2\|x_n-q\|^2 \\
& - & (1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)\|Tx_n-x_n\|^2 \\
& + & \theta\alpha_n\|x_{n+1}-q\|^2+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle
\end{array} \qquad (2.4.6)$$

Neglecting the non-positive term $-(1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)\|Tx_n-$

$x_n\|^2$ in the obtained relation, we get the following inequality:

$$\begin{aligned} \|x_{n+1}-q\|^2 &\leq \left(1-\frac{2\alpha_n(1-\theta)}{1-\alpha_n\theta}\right)\|x_n-q\|^2 \\ &+ \frac{\alpha_n^2}{1-\alpha_n\theta}O(1) \\ &+ \frac{2\alpha_n}{1-\alpha_n\theta}\langle f(q)-q, x_{n+1}-q\rangle. \end{aligned} \tag{2.4.7}$$

From relation (2.4.6) we also obtain

$$\begin{aligned} \frac{(1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)}{1-\alpha_n\theta}\|Tx_n-x_n\|^2 &\leq (\|x_n-q\|^2-\|x_{n+1}-q\|^2) \\ &+ \frac{\alpha_n^2}{1-\alpha_n\theta}O(1) \\ &+ \frac{2\alpha_n}{1-\alpha_n\theta}O(1). \end{aligned} \tag{2.4.8}$$

Thence we focus our attention on two cases that may arise.

Case I The sequence $\|x_{n+1}-q\|$ is definitively nonincreasing.

In this case, $\lim_{n\to\infty}\|x_{n+1}-q\|$ exists.

Since $\lim_{n\to\infty}\alpha_n=0$ and from relation (2.4.8), we get that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\frac{(1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)}{1-\alpha_n\theta}\|Tx_n-x_n\|^2 \\ &\leq \limsup_{n\to\infty}\left((\|x_n-q\|^2-\|x_{n+1}-q\|^2)+\frac{\alpha_n^2}{1-\alpha_n\theta}O(1)+\frac{2\alpha_n}{1-\alpha_n\theta}O(1)\right)=0. \end{aligned}$$

From this last, it follows

$$\limsup_{n\to\infty}\frac{(1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)}{1-\alpha_n\theta}\|Tx_n-x_n\|^2=0.$$

On the other hand, from hypothesis $\limsup_{n\to\infty}\beta_n(1-\beta_n)(1-s_n)>0$, we have that

$$\limsup_{n\to\infty}\frac{(1-\alpha_n)^2(1-s_n)(1-\beta_n)(\beta_n(1-s_n)+s_n)}{1-\alpha_n\theta}>0$$

Therefore, we get

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

In order to apply Xu's Lemma 1.4.15 to the sequence $a_n=\|x_n-\bar{q}\|$, where $\bar{q}$ is the unique solution of the variational inequality (2.4.2), we will claim claim that

$$\limsup_{n\to\infty}\langle\bar{q}-f(\bar{q}),\bar{q}-x_n\rangle\leq 0.$$

For this purpose, we select a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that

$$\limsup_{n\to\infty}\langle \bar{q}-f(\bar{q}),\bar{q}-x_n\rangle = \lim_{k\to\infty}\langle \bar{q}-f(\bar{q}),\bar{q}-x_{n_k}\rangle.$$

As $(x_{n_k})_{k\in\mathbb{N}}$ is a bounded sequence, there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ which converges weakly to a z. Since $\|x_n - Tx_n\| \to 0$, as $n\to\infty$, and $I-T$ is demiclosed at 0, we deduce that $z \in Fix(T)$. Therefore, by VI (2.4.2), we have that

$$\begin{aligned}\limsup_{n\to\infty}\langle \bar{q}-f(\bar{q}),\bar{q}-x_n\rangle &= \lim_{j\to\infty}\langle \bar{q}-f(\bar{q}),\bar{q}-x_{n_{k_j}}\rangle\\ &= \langle \bar{q}-f(\bar{q}),\bar{q}-z\rangle \le 0\end{aligned}$$

Applying Lemma 1.4.15, we deduce that $\lim_{n\to\infty}\|x_{n+1}-\bar{q}\| = 0$, that is the sequence $(x_n)_{n\in\mathbb{N}}$ strongly converges to a fixed point of T.

Case II There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\|x_{n_k}-\bar{q}\| < \|x_{n_k+1}-\bar{q}\| \text{ for all } k\in\mathbb{N}.$$

From Maingé's Lemma 2.4.16 it follows that there exists a nondecreasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying, for all $n\ge n_0$, the conditions:

$$\lim_{n\to\infty}\tau(n)=\infty \tag{2.4.9}$$

$$\|x_{\tau(n)}-\bar{q}\| < \|x_{\tau(n)+1}-\bar{q}\| \tag{2.4.10}$$

$$\|x_n-\bar{q}\| < \|x_{\tau(n)+1}-\bar{q}\| \tag{2.4.11}$$

We should consider a subsequence $(\tau(n_k))_{k\in\mathbb{N}} \subset (\tau(n))_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty}\Big(\|x_{\tau(n)+1}-\bar{q}\|^p - \|x_{\tau(n)}-\bar{q}\|^p\Big) = \lim_{k\to\infty}\Big(\|x_{\tau(n_k)+1}-\bar{q}\|^p - \|x_{\tau(n_k)}-\bar{q}\|^p\Big).$$

From condition (2.4.10) it results

$$\lim_{k\to\infty}\Big(\|x_{\tau(n_k)+1}-\bar{q}\|^p - \|x_{\tau(n_k)}-\bar{q}\|^p\Big) \ge 0. \tag{2.4.12}$$

On the other hand we can construct a strictly increasing sequence $(\tau(m_s))_{s\in\mathbb{N}}$ of $(\tau(n_k))_{k\in\mathbb{N}}$ defined as

$$m_1 = \min\{n_k\in\mathbb{N} : \tau(1) < \tau(n_k)\}$$

$$
\begin{aligned}
m_2 &= \min\{n_k \in \mathbb{N} : \tau(m_1) < \tau(n_k)\} \\
&\cdots \\
m_{s+1} &= \min\{n_k \in \mathbb{N} : \tau(m_s) < \tau(n_k)\}. \qquad (2.4.13)
\end{aligned}
$$

Bearing in mind this latter, from (2.4.12) and (2.4.7), we have that

$$
\begin{aligned}
0 &\leq \lim_{s\to\infty}\left(\|x_{\tau(m_s)+1} - \bar{q}\|^2 - \|x_{\tau(m_s)} - \bar{q}\|^2\right) \\
&\leq \limsup_{s\to\infty}\left(\|x_{\tau(m_s)+1} - \bar{q}\|^2 - \|x_{\tau(m_s)} - \bar{q}\|^2\right) \\
&\leq \limsup_{n\to\infty}\left(\|x_{n+1} - \bar{q}\|^2 - \|x_n - \bar{q}\|^2\right) \\
&\leq \limsup_{n\to\infty}\left(\frac{\alpha_n^2}{1-\alpha_n\theta}O(1) + \frac{2\alpha_n}{1-\alpha_n\theta}\langle f(\bar{q}) - \bar{q}, x_{n+1} - \bar{q}\rangle\right) \leq 0,
\end{aligned}
$$

hence it turns out that

$$
\lim_{n\to\infty}\left(\|x_{\tau(n)+1} - \bar{q}\|^2 - \|x_{\tau(n)} - \bar{q}\|^2\right) = 0. \qquad (2.4.14)
$$

If we replace n with $\tau(n)$ in (2.4.7), we obtain

$$
\begin{aligned}
\|x_{\tau(n)+1} - \bar{q}\|^2 &\leq \left(1 - \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}\right)\|x_{\tau(n)} - \bar{q}\|^2 \\
&+ \frac{\alpha_{\tau(n)}^2}{1-\alpha_{\tau(n)}\theta}O(1) \\
&+ \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}\langle f(\bar{q}) - \bar{q}, x_{\tau(n)+1} - \bar{q}\rangle \\
&\leq \left(1 - \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}\right)\|x_{\tau(n)+1} - \bar{q}\|^2 \\
&+ \frac{\alpha_{\tau(n)}^2}{1-\alpha_{\tau(n)}\theta}O(1) \\
&+ \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}\langle f(\bar{q}) - \bar{q}, x_{\tau(n)+1} - \bar{q}\rangle,
\end{aligned}
$$

and also

$$
\begin{aligned}
\frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}\|x_{\tau(n)+1} - \bar{q}\|^2 &\leq \frac{\alpha_{\tau(n)}^2}{1-\alpha_{\tau(n)}\theta}O(1) \\
&+ \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}\langle f(\bar{q}) - \bar{q}, x_{\tau(n)+1} - \bar{q}\rangle.
\end{aligned}
$$

Therefore, dividing both sides of the obtained inequality by $\alpha_{\tau(n)}$, we have

$$\begin{aligned} \frac{2}{1-\alpha_{\tau(n)}\theta}\|x_{\tau(n)+1}-\bar{q}\|^2 &\leq \frac{\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\theta}O(1) \\ &+ \frac{2}{1-\alpha_{\tau(n)}\theta}\langle f(\bar{q})-\bar{q}, x_{\tau(n)+1}-\bar{q}\rangle, \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}\langle f(\bar{q})-\bar{q}, x_{\tau(n)+1}-\bar{q}\rangle \leq 0$ (obtained following the same argument of Case I), it follows that $\lim_{n\to\infty}\|x_{\tau(n)+1}-\bar{q}\| = 0$.

This last, from inequality (2.4.11), ensures that $\lim_{n\to\infty}\|x_n-\bar{q}\| = 0$.

□

Proof.(Theorem 2.4.9) First of all, we notice that the problem $\bar{q} = Q(f(\bar{q}))$ is well posed since the mapping $Q\circ f$ is a contraction (Q is nonexpansive and f is a contraction). Thus, from Banach contraction principle, it has a unique fixed point.

From Lemma 2.4.15, characterizing sunny nonexpansive retractions in smooth spaces, we have that

$$\langle f(\bar{q})-\bar{q}, j_\phi(x-\bar{q})\rangle = \langle f(\bar{q})-Q(f(\bar{q})), j_\phi(x-Q(f(\bar{q})))\rangle \leq 0 \quad \forall x\in Fix(T),$$

hence variational inequality (2.4.3) holds.

Moreover, the solution $\bar{q}$ is unique. Indeed, if $\tilde{q}\in Fix(T)$ is another solution of (2.4.3), then

$$\langle f(\tilde{q})-\tilde{q}, j_\phi(x-\tilde{q})\rangle \leq 0$$

On the other hand, putting $x=\bar{q}$ in the last inequality, it reduces to

$$\langle f(\tilde{q})-\tilde{q}, j_\phi(\bar{q}-\tilde{q})\rangle \leq 0 \quad \forall x\in Fix(T). \tag{2.4.15}$$

Analouglsy, if $x=\tilde{q}$ in (2.4.3), we can write

$$\langle f(\bar{q})-\bar{q}, j_\phi(\tilde{q}-\bar{q})\rangle \leq 0. \tag{2.4.16}$$

Adding (2.4.15) and (2.4.16), we get

$$\begin{aligned} 0 \geq \langle f(\tilde{q})-\tilde{q}-f(\bar{q})+\bar{q}, j_\phi(\bar{q}-\tilde{q})\rangle &= \langle \bar{q}-\tilde{q}-(f(\bar{q})-f(\tilde{q})), j_\phi(\bar{q}-\tilde{q})\rangle \\ (f\ \theta\text{-contraction}) &\geq \|\bar{q}-\tilde{q}\|\phi(\|\bar{q}-\tilde{q}\|)-\theta\|\bar{q}-\tilde{q}\|\phi(\|\bar{q}-\tilde{q}\|) \\ &= (1-\theta)\|\bar{q}-\tilde{q}\|\phi(\|\bar{q}-\tilde{q}\|) \end{aligned}$$

Thus

$$(1-\theta)\|\bar{q}-\tilde{q}\|\phi(\|\bar{q}-\tilde{q}\|) \leq 0$$

from which we deduce $\bar{q} = \tilde{q}$.

From Lemma 2.4.17 we know the $(x_n)_{n\in\mathbb{N}}$ is bounded.

Let us designate with $O(1)$ any real bounded sequence.

For each $q \in Fix(T)$, we compute

$$\begin{aligned}
\|x_{n+1} - q\|^p &= \|\alpha_n(f(x_n) - q) + (1-\alpha_n)(T(s_n x_n + (1-s_n)\overline{x}_{n+1}) - q)\|^p \\
\textit{(convexity of } \|\cdot\|^p\textit{)} &\leq \alpha_n\|f(x_n) - q\|^p \\
&+ (1-\alpha_n)\|T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q\|^p \\
\textit{(}T\textit{ quasi-nonexpansive)} &\leq \alpha_n\|f(x_n) - q\|^p \\
&+ (1-\alpha_n)\|s_n x_n + (1-s_n)\bar{x}_{n+1} - q\|^p \\
&\leq \alpha_n\|f(x_n) - q\|^p + (1-\alpha_n)s_n\|x_n - q\|^p \\
&+ (1-\alpha_n)(1-s_n)\|\bar{x}_{n+1} - q\|^p \\
&\leq \alpha_n\|f(x_n) - q\|^p + (1-\alpha_n)s_n\|x_n - q\|^p \\
&+ (1-\alpha_n)(1-s_n)\|\beta_n(x_n - q) + (1-\beta_n)(Tx_n - q)\|^p.
\end{aligned}$$

For p-uniformly Banach spaces, we mention (see [86]) that for all $x, y \in X$ and $\lambda \in [0,1]$, the following inequality is verified:

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - \lambda(1-\lambda)c\|x - y\|^p \tag{2.4.17}$$

for a certain positive constant c, for all $x, y \in X$ and $0 \leq \lambda \leq 1$.

Therefore we get that

$$\begin{aligned}
\|x_{n+1} - q\|^p &\leq \alpha_n\|f(x_n) - q\|^p + (1-\alpha_n)s_n\|x_n - q\|^p \\
&+ (1-\alpha_n)(1-s_n)\beta_n\|x_n - q\|^p \\
&+ (1-\alpha_n)(1-s_n)(1-\beta_n)\|Tx_n - q\|^p \\
&- (1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n - Tx_n\|^p \\
&\leq \alpha_n\|f(x_n) - q\|^p + (1-\alpha_n)\|x_n - q\|^p \\
&- (1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n - Tx_n\|^p.
\end{aligned}$$

for a certain $c \in \mathbb{R}^+$.

From which we deduce that

$$\begin{aligned}
\|x_{n+1} - q\|^p &\leq \alpha_n\|f(x_n) - q\|^p + (1-\alpha_n)\|x_n - q\|^p \\
&- (1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n - Tx_n\|^p \qquad (2.4.18)
\end{aligned}$$

that is equivalent to state that

$$\begin{aligned}
(1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n - Tx_n\|^p &\leq (1-\alpha_n)\|x_n - q\|^p \\
&- \|x_{n+1} - q\|^p + \alpha_n O(1) \qquad (2.4.19)
\end{aligned}$$

Neverthless, neglecting the term $(1-\alpha_n)(1-s_n)(1-\beta_n)c\|x_n - Tx_n\|^p$, from (2.4.18), we also have

$$\|x_{n+1} - q\|^p \leq (1-\alpha_n)\|x_n - q\|^p + \alpha_n O(1) \tag{2.4.20}$$

Moreover, for each $q \in Fix(T)$, we have that

$$\begin{aligned}
\Phi(\|x_{n+1} - q\|) &= \Phi(\|\alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q\|) \\
&= \Phi(\|\alpha_n(f(x_n) - f(q)) \\
&+ (1-\alpha_n)(T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q) \\
&+ \alpha_n(f(q) - q)\|).
\end{aligned}$$

Since the subdifferential inequality

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x+y)\rangle, \quad j_\phi(x+y) \in J_\phi(x+y) \tag{2.4.21}$$

holds (see book [22] of Cioranescu), it results that

$$\begin{aligned}
\Phi(\|x_{n+1} - q\|) &\leq \Phi(\|\alpha_n(f(x_n) - f(q)) \\
&+ (1-\alpha_n)(T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q)\|) \\
&+ \alpha_n\langle f(q) - q, j_\phi(x_{n+1} - q)\rangle \\
(\Phi \textit{ increasing}) &\leq \Phi(\alpha_n\|f(x_n) - f(q)\| \\
&+ (1-\alpha_n)\|T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - q\|) \\
&+ \alpha_n\langle f(q) - q, j_\phi(x_{n+1} - q)\rangle \\
(T \textit{ quasi-nonexp., } f\ \theta\textit{-contrac.}) &\leq \Phi(\alpha_n\theta\|x_n - q\| \\
&+ (1-\alpha_n)\|s_n x_n + (1-s_n)\bar{x}_{n+1} - q\|) \\
&+ \alpha_n\langle f(q) - q, j_\phi(x_{n+1} - q)\rangle \\
&\leq \Phi(\alpha_n\theta\|x_n - q\| \\
&+ (1-\alpha_n)(s_n\|x_n - q\| + (1-s_n)\|\bar{x}_{n+1} - q\|) \\
&+ \alpha_n\langle f(q) - q, j_\phi(x_{n+1} - q)\rangle
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
\Phi(\|x_{n+1}-q\|) &\leq \Phi(\alpha_n\theta\|x_n-q\| \\
&+ (1-\alpha_n)s_n\|x_n-q\|+(1-\alpha_n)(1-s_n)\beta_n\|x_n-q\| \\
&+ (1-\alpha_n)(1-s_n)(1-\beta_n)\|Tx_n-q\|) \\
&+ \alpha_n\langle f(q)-q, j_\phi(x_{n+1}-q)\rangle \\
&= \Phi(\alpha_n\theta\|x_n-q\|+(1-\alpha_n)\|x_n-q\|) \\
&+ \alpha_n\langle f(q)-q, j_\phi(x_{n+1}-q)\rangle \\
&= \Phi((1-\alpha_n(1-\theta))\|x_n-q\|) \\
&+ \alpha_n\langle f(q)-q, j_\phi(x_{n+1}-q)\rangle \\
&\leq (1-\alpha_n(1-\theta))\Phi(\|x_n-q\|) \\
&+ \alpha_n\langle f(q)-q, j_\phi(x_{n+1}-q)\rangle.
\end{aligned}$$

Hence we get the inequality

$$\begin{aligned}
\Phi(\|x_{n+1}-q\|) &\leq (1-\alpha_n(1-\theta))\Phi(\|x_n-q\|) \\
&\quad +\alpha_n\langle f(q)-q, j_\phi(x_{n+1}-q)\rangle.
\end{aligned} \tag{2.4.22}$$

The following cases can be distinguished:

Case I Assume that the real sequence $(\|x_n-q\|)_{n\in\mathbb{N}}$ is definitively nonincreasing.

This fact, added to the boundedness of $(\|x_n-q\|)_{n\in\mathbb{N}}$, allows us to deduce that $\lim_{n\to+\infty}\|x_n-q\|$ exists.

Therefore, from relation (2.4.19) and hypothesis $\lim_{n\to\infty}\alpha_n=0$, it results:

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}(1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n-Tx_n\|^p \\
&\leq \limsup_{n\to\infty}(\|x_n-q\|^p-\|x_{n+1}-q\|^p+\alpha_n O(1))=0
\end{aligned}$$

Thus

$$\lim_{n\to\infty}(1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n-Tx_n\|^p=0$$

Since, by hypotheses

$$\lim_{n\to\infty}\alpha_n=0$$
$$\limsup_{n\to\infty}\beta_n(1-\beta_n)(1-s_n)>0,$$

it holds that

$$\limsup_{n\to\infty}(1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n>0,$$

we can conclude that

$$\lim_{n\to+\infty} \|x_n - Tx_n\| = 0, \tag{2.4.23}$$

that is, $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence for T.

The next goal is to prove that $\lim_{n\to+\infty} \|x_n - \bar{q}\| = 0$.

Let us put

$$a_{n+1} = \Phi(\|x_{n+1} - \bar{q}\|), \quad \forall n \geq 0.$$

Then relation (2.4.22) can be rewritten as

$$a_{n+1} \leq (1 - \alpha_n(1-\theta))a_n + \alpha_n \langle f(\bar{q}) - \bar{q}, j_\phi(x_{n+1} - \bar{q})\rangle.$$

In order to apply Lemma 1.4.15 to the real sequence $(a_n)_{n\in\mathbb{N}}$, we claim that

$$\limsup_{n\to+\infty} \langle f(\bar{q}) - \bar{q}, j_\phi(x_{n+1} - \bar{q})\rangle \leq 0.$$

Consider a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that

$$\limsup_{n\to+\infty} \langle f(\bar{q}) - \bar{q}, j_\phi(x_{n+1} - \bar{q})\rangle = \lim_{k\to+\infty} \langle f(\bar{q}) - \bar{q}, j_\phi(x_{n_k} - \bar{q})\rangle.$$

Since X is a reflexive space (Milman-Pettis Theorem, [59], [66]), we can consider a subsequence of $(x_n)_{n\in\mathbb{N}}$, that for semplicity of notation we will designate still with $(x_{n_k})_{k\in\mathbb{N}}$, weakly convergent to a point $\bar{x}$ of C.

From demiclosedness hypothesis of $I - T$ at 0 and (2.4.23), we deduce that $\bar{x} \in Fix(T)$.

Since J_ϕ is weakly continuous, for a certain gauge ϕ, and from variational inequality (2.4.3), we obtain that

$$\begin{aligned}\limsup_{n\to+\infty} \langle f(\bar{q}) - \bar{q}, j_\phi(x_{n+1} - \bar{q})\rangle &= \lim_{k\to+\infty} \langle f(\bar{q}) - \bar{q}, j_\phi(x_{n_k} - \bar{q})\rangle \\ &= \langle f(\bar{q}) - \bar{q}, j_\phi(\bar{x} - \bar{q})\rangle \leq 0.\end{aligned}$$

Therefore, we get that $\lim_{n\to\infty} \Phi(\|x_{n+1} - \bar{q}\|) = 0$ and hence

$$\lim_{n\to\infty} \|x_{n+1} - \bar{q}\| = 0.$$

Case II Assume that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\|x_{n_k} - \bar{q}\| < \|x_{n_{k+1}} - \bar{q}\| \quad \forall k \in \mathbb{N}.$$

From Lemma 2.4.16 it follows that there exists a nondecreasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying, for all $n \geq n_0$, the conditions (2.4.9), (2.4.10) and (2.4.11).

We can consider a subsequence $(\tau(n_k))_{k\in\mathbb{N}} \subset (\tau(n))_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty}\left(\|x_{\tau(n)+1}-\bar{q}\|^p - \|x_{\tau(n)}-\bar{q}\|^p\right) = \lim_{k\to\infty}\left(\|x_{\tau(n_k)+1}-\bar{q}\|^p - \|x_{\tau(n_k)}-\bar{q}\|^p\right).$$

From condition (2.4.10) it results

$$\lim_{k\to\infty}\left(\|x_{\tau(n_k)+1}-\bar{q}\|^p - \|x_{\tau(n_k)}-\bar{q}\|^p\right) \geq 0. \tag{2.4.24}$$

On the other hand we can construct a strictly increasing sequence $(\tau(m_s))_{s\in\mathbb{N}}$ of $(\tau(n_k))_{k\in\mathbb{N}}$ defined as in (2.4.13).

Therefore, from (2.4.24) and (2.4.20), we have that

$$\begin{aligned} 0 &\leq \lim_{s\to\infty}\left(\|x_{\tau(m_s)+1}-\bar{q}\|^p - \|x_{\tau(m_s)}-\bar{q}\|^p\right) \\ &\leq \limsup_{s\to\infty}\left(\|x_{\tau(m_s)+1}-\bar{q}\|^p - \|x_{\tau(m_s)}-\bar{q}\|^p\right) \\ &\leq \limsup_{n\to\infty}\left(\|x_{n+1}-\bar{q}\|^p - \|x_n-\bar{q}\|^p\right) \\ &\leq \limsup_{n\to\infty}\alpha_n O(1) = 0, \end{aligned}$$

hence it turns out that

$$\lim_{n\to\infty}\left(\|x_{\tau(n)+1}-\bar{q}\|^p - \|x_{\tau(n)}-\bar{q}\|^p\right) = 0. \tag{2.4.25}$$

Replacing n with $\tau(n)$ in (2.4.19) and (2.4.22), we get from the first that

$$\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

While, for the second:

$$\begin{aligned} \Phi(\|x_{\tau(n)+1}-\bar{q}\|) &\leq (1-\alpha_{\tau(n)}(1-\theta))\Phi(\|x_{\tau(n)}-\bar{q}\|) \\ &+ \alpha_{\tau(n)}\langle f(\bar{q})-\bar{q}, j_\phi(x_{\tau(n)+1}-\bar{q})\rangle \end{aligned}$$

Since, from Lemma 2.4.16

$$\|x_{\tau(n)}-\bar{q}\| < \|x_{\tau(n)+1}-\bar{q}\|$$

then it holds

$$\begin{aligned} \Phi(\|x_{\tau(n)+1}-\bar{q}\|) &\leq (1-\alpha_{\tau(n)}(1-\theta))\Phi(\|x_{\tau(n)+1}-\bar{q}\|) \\ &+ \alpha_{\tau(n)}\langle f(\bar{q})-\bar{q}, j_\phi(x_{\tau(n)+1}-\bar{q})\rangle. \end{aligned}$$

Hence we get that

$$\alpha_{\tau(n)}(1-\theta)\Phi(\|x_{\tau(n)+1}-\bar{q}\|) \quad \leq \quad \alpha_{\tau(n)}\langle f(\bar{q})-\bar{q}, j_\phi(x_{\tau(n)+1}-\bar{q})\rangle.$$

Dividing both sides for $\alpha_{\tau(n)}(1-\theta)$ it follows that

$$\Phi(\|x_{\tau(n)+1}-\bar{q}\|) \quad \leq \quad \frac{1}{(1-\theta)}\langle f(\bar{q})-\bar{q}, j_\phi(x_{\tau(n)+1}-\bar{q})\rangle.$$

Following the same argument of Case I, we have that

$$\limsup_{n\to\infty}\langle f(\bar{q})-\bar{q}, j_\phi(x_{\tau(n)+1}-\bar{q})\rangle \leq 0,$$

which implies that

$$\lim_{n\to+\infty}\Phi(\|x_{\tau(n)+1}-\bar{q}\|)=0,$$

thus

$$\lim_{n\to+\infty}\|x_{\tau(n)+1}-\bar{q}\|=0.$$

From

$$\|x_n-\bar{q}\|<\|x_{\tau(n)+1}-\bar{q}\|,$$

finally we get that

$$\lim_{n\to+\infty}\|x_n-\bar{q}\|=0.$$

□

Proof.(Theorem 2.4.13) First of all, from the definition of ψ-expansive mapping we have that, if $I-T$ is ψ-expansive, then T has a unique fixed point. Indeed, if q_1, $q_2 \in Fix(T)$ and $q_1 \neq q_2$, then

$$\psi(\|q_1-q_2\|) \leq \|(I-T)q_1-(I-T)q_2\|=0$$

from which it follows that

$$\|q_1-q_2\|=0$$

if and only if $q_1=q_2$, that is a contradiction.

Let q the unique fixed point of T.

Argumenting as in the preceeding theorems, we obtain inequality (2.4.19):

$$(1-\alpha_n)(1-s_n)(1-\beta_n)\beta_n c\|x_n-Tx_n\|^p \leq (1-\alpha_n)\|x_n-q\|^p-\|x_{n+1}-q\|^p+\alpha_n O(1)$$

with $c\in\mathbb{R}^+$.

As well as inequality (2.4.20):

$$\|x_{n+1}-q\|^p \leq (1-\alpha_n)\|x_n-q\|^p+\alpha_n O(1)$$

Also in this case, we distinguish two eventualities:

Case I Assume that the real sequence $(\|x_n - q\|)_{n\in\mathbb{N}}$ is definitively nonincreasing.

Following the same proof lines for Case I in Theorem 2.4.9, we get that

$$\lim_{n\to+\infty} \|x_n - Tx_n\| = 0. \tag{2.4.26}$$

Hence, $(x_n)_{n\in\mathbb{N}}$ generated by (2.4.1) is an approximate fixed point sequence for T. Moreover, since

$$\psi(\|x_n - q\|) \leq \|(I-T)x_n - (I-T)q\| = \|x_n - Tx_n\|,$$

then we deduce that

$$\lim_{n\to+\infty} \|x_n - q\| = 0.$$

Case II Assume that exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\|x_{n_k} - q\| < \|x_{n_{k+1}} - q\| \quad \forall k \in \mathbb{N}.$$

From Maingé's Lemma 2.4.16 it follows that there exists a nondecreasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying, for all $n \geq n_0$, the conditions:

a) $\lim_{n\to\infty} \tau(n) = \infty$,

b) $\|x_{\tau(n)} - q\| < \|x_{\tau(n)+1} - q\|$,

c) $\|x_n - q\| < \|x_{\tau(n)+1} - q\|$.

We can consider a subsequence $(\tau(n_k))_{k\in\mathbb{N}}$ of $(\tau(n))_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty}\left(\|x_{\tau(n)+1} - q\|^p - \|x_{\tau(n)} - q\|^p\right) = \lim_{k\to\infty}\left(\|x_{\tau(n_k)+1} - q\|^p - \|x_{\tau(n_k)} - q\|^p\right)$$

From condition (b) it results

$$\lim_{k\to\infty}\left(\|x_{\tau(n_k)+1} - q\|^p - \|x_{\tau(n_k)} - q\|^p\right) \geq 0$$

Considered a subsequence $(\tau(m_s))_{s\in\mathbb{N}}$ of $(\tau(n_k))_{k\in\mathbb{N}}$, defined as in (2.4.13), from relation (2.4.20), we get

$$\begin{aligned}
0 &\leq \lim_{s\to\infty}\left(\|x_{\tau(m_s)+1} - q\|^p - \|x_{\tau(m_s)} - q\|^p\right)\\
&\leq \limsup_{s\to\infty}\left(\|x_{\tau(m_s)+1} - q\|^p - \|x_{\tau(m_s)} - q\|^p\right)\\
&\leq \limsup_{n\to\infty}\left(\|x_{n+1} - q\|^p - \|x_n - q\|^p\right)\\
&\leq \limsup_{n\to\infty} \alpha_n O(1) = 0,
\end{aligned}$$

Consequently, from relation (2.4.19), in which n has been replaced by $\tau(n)$, we deduce that

$$\lim_{n\to+\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0$$

and from the fact that

$$\psi(\|x_{\tau(n)} - q\|) \leq \|(I-T)x_{\tau(n)} - (I-T)q\| = \|x_{\tau(n)} - Tx_{\tau(n)}\|$$

we obtain

$$\lim_{n\to+\infty}\|x_{\tau(n)} - q\| = 0 \tag{2.4.27}$$

Replacing n with $\tau(n)$ in expression (2.4.20), we get

$$\|x_{\tau(n)+1} - q\|^p \leq (1-\alpha_{\tau(n)})^p\|x_{\tau(n)} - q\|^p + \alpha_{\tau(n)}O(1)$$

From hypothesis (i) and (2.4.27), we have that $\lim_{n\to\infty}\|x_{\tau(n)+1} - q\|^p = 0$ and hence, by condition $c)$, it follows that

$$\lim_{n\to\infty}\|x_n - q\| = 0.$$

□

2.4.4 Examples.

Inspired to [42, Example 2] by Iemoto and Takahashi, we give an example for the convergence result stated in Theorem 2.4.1 for the class of nonspreading operators:

Example 2.4.18 *Let H be a Hilbert space. Assume that*

$$B_1 = \{x \in H \quad s.t. \quad \|x\| \leq 1\},$$
$$B_2 = \{x \in H \quad s.t. \quad \|x\| \leq 2\},$$
$$B_3 = \{x \in H \quad s.t. \quad \|x\| \leq 3\}.$$

The mapping defined as

$$Sx = \begin{cases} 0 \quad if \quad x \in B_2 \\ P_{B_1}(x) \quad if \quad x \in B_3 \setminus B_2, \end{cases}$$

with P_{B_1} the metric projection of H onto B_1, is a nonspreading operator with $Fix(S) = \{0\}$, hence it is quasi-nonexpansive. It is known that $I-S$ is demiclosed at 0.

Moreover S is discontinuous, hence it is not nonexpansive.

Let us choose $H = \mathbb{R}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n-1}{2n}$, $s_n = \frac{1}{n}$, $f(x) = \frac{x}{2}$, $x_0 = 3$.

The operator S can be written as

$$Sx = \begin{cases} 0 \quad if \quad x \in [-2,2] \\ P_{B_1}(x) \quad if \quad x \in [-3,3] \setminus [-2,2], \end{cases}$$

with $B_1 = [-1,1]$, $B_2 = [-2,2]$, $C = [-3,3]$.
While the sequence generated by (2.4.1) *is given by*

$$\bar{x}_{n+1} = \frac{n}{2(n+1)} x_n + \frac{n+2}{2(n+1)} S x_n, \quad n \geq 0$$

$$x_{n+1} = \frac{x_n}{2(n+1)} + \left(\frac{n}{n+1}\right) S \left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right), \quad n \geq 0.$$

Therefore the sequence generated by (2.4.1) *is given by*

$$x_0 = 3,$$

$$x_1 = \frac{3}{2},$$

$$x_2 = \frac{3}{8},$$

$$\cdots$$

$$x_{n+1} = \frac{3}{(n+1)!2^{n+1}}$$

that quickly converges to 0.

For Theorem (2.4.9) we include the following:

Example 2.4.19 *Consider the real Banach space $X = l_p$, for $1 < p < +\infty$ endowed with the norm $\|x\| = \|x\|_p = \left[\sum_{n=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}}$, for $x = (x_1, x_2, \cdots, x_n, \cdots)$.*
Set

$$B_1 = \{x \in l_p : \|x\| \leq 1\}$$

$$B_2 = \{x \in l_p : \|x\| \leq 2\}$$

$$B_3 = \{x \in l_p : \|x\| \leq 3\}$$

Let $T : B_3 \to B_3$ the mapping defined as

$$Tx = \begin{cases} 0_{l_p} \quad x \in B_2 \\ P_{B_1}(x) \quad x \in B_3 \setminus B_2 \end{cases}$$

where $P_{B_1}(x)$ is the metric projection of X onto B_1.
Let $f : B_3 \to B_3$ the map defined as $f(x) = \frac{1}{2}x$.
Let us put $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2(n+1)}$, $s_n = \frac{1}{n+1}$.
We point out the following considerations:

- *l_p, with $1 < p < \infty$, is a p-uniformly convex Banach space with weakly continuous duality mapping J_p,*
- *$Fix(T) = \{0\}$,*
- *T is quasi-nonexpansive,*
- *T is not nonexpansive since it is discontinuous,*
- *$I - T$ is demiclosed at 0.*

  Indeed if we consider $x_n \rightharpoonup x \in B_3 \setminus B_2$ then we have that

  $$\|x_n - Tx_n\| \geq |\|x_n\| - \|Tx_n\| \geq \|x_n\| - 1,$$

  so that $\liminf_{n\to+\infty} \|x_n - Tx_n\| \geq \|x\| - 1 \geq 1$.

  We approach to the same conlcusion if $x_n \rightharpoonup x \in B_2$, with $x \neq 0$, considering that $\liminf_{n\to+\infty} \|x_n - Tx_n\| \geq \|x\| > 0$.
- *f is a $\frac{1}{2}$-contraction,*
- *sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy conditions (i), (ii) and (iii) of Theorem 2.4.9.*

Fixed $x_0 = (3, 0, 0, \cdots)$, then algorithm (2.4.1) *is given by*

$$\begin{cases} \bar{x}_{n+1} = \frac{n}{2(n+1)}x_n + \frac{n+2}{2(n+1)}Tx_n, \quad n \geq 0 \\ x_{n+1} = \frac{x_n}{2(n+1)} + \frac{n}{n+1}T\left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right) \quad n \geq 0 \end{cases}$$

It generates the sequence:

$$\begin{aligned} x_0 &= (3, 0, 0, \cdots), \\ x_1 &= \frac{1}{2}(3, 0, 0, \cdots), \\ x_2 &= \frac{1}{8}(3, 0, 0, \cdots), \\ &\cdots \\ x_{n+1} &= \frac{1}{(n+1)!2^{n+1}}(3, 0, 0, \cdots), \end{aligned}$$

that converges to 0, as $n \to +\infty$.

For Theorem (2.4.13), we give the following example in a p-uniformly convex Banach space that fails to have a weakly continuous duality mapping:

Example 2.4.16 *Let $X = L_p([0,1])$, with $1 < p < +\infty$, endowed with the norm*

$$\|x\| = \|x\|_p = \Big[\int_{[0,1]} |x(s)|^p \, ds\Big]^{\frac{1}{p}}$$

and

$$B_1 = \{x \in L_p[0,1] : \|x\| \leq 1\}$$
$$B_2 = \{x \in L_p[0,1] : \|x\| \leq 2\}$$

Let $T : B_2 \to B_2$ the map defined as

$$Tx = \begin{cases} 0 & x \in B_1 \\ -x & x \in B_2 \setminus B_1 \end{cases} \tag{2.4.28}$$

and $f : B_2 \to B_2$ such that $f(x) = \frac{x}{2}$.

If we set $\alpha_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{n}{2(n+1)}$, $s_n = \frac{1}{n+1}$ then algorithm (2.4.1) becomes

$$\begin{cases} \bar{x}_{n+1} = \frac{n}{2(n+1)} x_n + \frac{n+2}{2(n+1)} T x_n, & n \geq 0 \\ x_{n+1} = \frac{x_n}{2(n+1)^2} + \left(\frac{n^2+2n}{(n+1)^2}\right) T \left(\frac{x_n}{n+1} + \frac{n}{n+1} \bar{x}_{n+1}\right) & n \geq 0 \end{cases}$$

We observe that

- *L_p is a p-uniformly convex Banach space that fails to have a weakly continuous duality mapping,*
- *$Fix(T) = \{0\}$,*
- *T is quasi-nonexpansive,*
- *T is discontinuous so it is not nonexpansive,*
- *$I - T$ is ψ-expansive. Indeed:*
  - *if $x, y \in B_1$, then $\|(I-T)x - (I-T)y\| = \|x - y\|$,*
  - *if $x, y \in B_2$, then $\|(I-T)x - (I-T)y\| = 2\|x - y\|$,*
  - *if $x \in B_1$ and $y \in B_2$, then*

$$\begin{aligned} \|(I-T)x - (I-T)y\| &= \|x - 2y\| \\ &\geq 2\|y\| - \|x\| \\ &\geq 1 \end{aligned}$$

Since

$$\|x - y\| \leq 3 \quad \forall x \in B_1, y \in B_2,$$

then $\|(I-T)x - (I-T)y\| \geq \frac{1}{3}\|x - y\|$.
Therefore $I - T$ is ψ-expansive with $\psi(t) = \frac{t}{3}$, for $t \in [0, +\infty)$,

- *f is a $\frac{1}{2}$-contraction,*
- *The sequences α_n, β_n and s_n satisfy conditions 1) and 2) of the preceeding theorem.*

  Fixed the constant function $x_0 = 2$ in $L_p([0,1])$, then the other terms of the sequence generated by (2.4.28) are given by

$$
\begin{aligned}
x_0 &= 2 \\
x_1 &= 1 \\
x_2 &= \frac{1}{8} \\
x_3 &= \frac{1}{144} \\
&\dots
\end{aligned}
$$

It can be noticed that, for $n \geq n_0 \in \mathbb{N}$, the term $T\left(\frac{x_n}{n} + \frac{n-1}{n}\bar{x}_{n+1}\right)$ vanishes since $\|\frac{x_n}{n} + \frac{n-1}{n}\bar{x}_{n+1}\| \leq 1$, then $x_{n+1} = \frac{x_n}{2n^2}$ and $\lim_{n\to+\infty} \|\frac{x_n}{2n^2}\| = 0$.

Chapter 3

A modelling approach for love dynamics

Preface.

In their paper of 2012, entitled "Dynamics of Love and Happiness: A Mathematical Analysis", D. Satsangi and Arun K. Sinha proposed a dynamical system describing a love affair between two individuals. Similar to previous Rinaldi's model, it included the phenomena of oblivion, return (both described by linear functions) and instinct (supposed constant), but introduced a nonlinear component for each member of the couple, with the meaning of synergism and emotional interaction.

The governing equations appear to be more realistic than the other models, on the same topic, known in literature; however, the stability analysis is wrong.

The purpose of our contribution, in [2], is to follow the same modelling approach proposed by Satasangi and Sinha in order to study the dynamics of the feelings between two people, introducing a variation in the original model, and to provide corrections for the the stability analysis, assuming certain hypotheses on the parameters of the system.

3.1 Linear Two-Dimensional Dynamical Systems.

A two-dimensional dynamical system is a system of the form

$$\begin{cases} \dot{x_1}(t) = g_1(t, x_1(t), x_2(t)), \quad t \in [t_0, T] \\ \dot{x_2}(t) = g_2(t, x_1(t), x_2(t)) \end{cases} \tag{3.1.1}$$

If the functions g_1 and g_2 do not depend on time t then the system is said to be *autonomous*. Thus, two-dimensional autonomous systems are of the form

$$\begin{cases} \dot{x_1}(t) = g_1(x_1(t), x_2(t)), \quad t \in [t_0, T] \\ \dot{x_2}(t) = g_2(x_1(t), x_2(t)) \end{cases} \tag{3.1.2}$$

In addition to system (3.1.2) there may also be given initial conditions of the form

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \tag{3.1.3}$$

where $(x_1^0, x_2^0) \in \mathbb{R}^2$.

Differential system (3.1.2) together with the initial condition (3.1.3) forms an initial value problem.

We primarily focus our attention on two-dimensional autonomous linear dynamical systems, that can be written as

$$\begin{cases} \dot{x_1}(t) = a_{1,1}x_1(t) + a_{1,2}x_2(t) \\ \dot{x_2}(t) = a_{2,1}x_1(t) + a_{2,2}x_2(t) \end{cases}, \tag{3.1.4}$$

where $a_{i,j}$ for $i, j = 1, 2$ are parameters.

It is *linear* in the sense that if $\overline{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ and $\overline{z} = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ are solutions of (3.1.4) then so is any linear combination

$$c_1 \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + c_2 \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

with $c_1, c_2 \in \mathbb{R}$.

System (3.1.4) can be written more compactly in matrix form as

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) \\ \bar{x}(t_0) = \bar{x}_0 \end{cases}$$

where

- $\bar{x}(t) = (x_1(t), x_2(t))$ is the *state vector*,
- $\dot{\bar{x}}(t) = \frac{d\bar{x}(t)}{dt}$ is its derivative,
- $A = [a_{i,j}]_{i,j=1,2}$ is the *dynamic matrix*,
- $\bar{x}(t_0) = \bar{x}_0 \in \mathbb{R}^2$ is the *initial state*.

For such a system a *critical point* (or *fixed point* or *stationary point*) is a solution of

$$\dot{\bar{x}}(t) = 0$$

In particular, for system (3.1.4), there is a unique critical point given by $(0, 0) \in \mathbb{R}^2$.

The main interest in studying systems (3.1.2) is twofold:

1. A large number of dynamic processes in applied sciences are governed by such systems.
2. The qualitative behavior of their solutions can be illustrated through the geometry in the x_1x_2-plane, known as *phase space*. Any solution of differential system (3.1.2) can be regarded as parametric curve given by $\bar{x}(t) = (x_1(t), x_2(t))$, called *trajectory* or *orbit*. A set of trajectories starting from different initial conditions is called the *state portrait*.

The nature of the solutions of (3.1.4) depends on the eigenvalues of the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix},$$

rather than the roots of the equation

$$\lambda^2 - (a_{1,1} + a_{2,2})\lambda + a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 0$$

given by

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{\sqrt{\tau^2 - 4\Delta}}{2}$$

where

$$\tau = tr(A)$$

$$\Delta = det(A)$$

Theorem 3.1.1 *[1, Theorem 26.1] For the differential system 3.1.4, let λ_1, λ_2 be the eigenvalues of the matrix A. Then the behavior of its trajectories near the critical point $\bar{0} = (0,0) \in \mathbb{R}^2$ is as follows:*

- *λ_1, λ_2 real, distinct and with the same sign:*
  - *both negative , $\bar{0}$ is a stable node,*
  - *both positive, $\bar{0}$ is an unstable node;*
- *λ_1, λ_2 real, distinct and of opposite sign, $\bar{0}$ is a saddle point (unstable);*
- *λ_1, λ_2 real and equal:*
  - *negative, $\bar{0}$ is stable node,*
  - *positive, $\bar{0}$ is an unstable node;*
- *λ_1, λ_2 complex:*
  - *purely imaginary, $\bar{0}$ is a centre;*
  - *with non zero real part:*
    * *with negative real part, $\bar{0}$ is a stable focus,*
    * *with positive real part, $\bar{0}$ is an unstable focus.*

3.2 Historical Background.

In 1988, the Harward mathematician Steven H. Strogatz initiated (unconsciously) a new trend: the study of dibatted romantic relationships via dynamic systems. The initial purpose was that of introducing his student to the study of dynamical systems, describing a model that ”relates mathematics to a topic that is already on the minds of many college students: the time evolution of a love affair between two people”. To this aim, he based the romantic affair on the story of two well known Shakespearean characters, Romeo and Juliet. Nevertheless, in the considered case, it is not their families that keep them apart, but Romeo's fickleness. With this term is meant: the more Juliet loves Romeo, the more he wants to walk away, while if she shows herself detached, he is attracted by her; conversely, Juliet warms up if he loves her and grows cold when he hates her.

The governing equations proposed by Strogatz, appeared for the first time in [79], are based on simple harmonic oscillator system and are given by:

$$\begin{cases} \frac{dR(t)}{dt} = -aJ(t), \\ \frac{dJ(t)}{dt} = bR(t), \end{cases} \tag{3.2.1}$$

where a, b are positive parameters, R(t) is Romeo's love for Juliet at time t and $J(t)$ is Juliet's love for Romeo at the same time. Positive values of $R(t)(J(t))$ mean love or friendship, negative ones mean hate and disdain, while $r(t) = 0$ ($j(t) = 0$) is a state of apathy, indifference. Hence, for Juliet, Romeo's feelings are source of interest, while, for Romeo, Juliet's feelings constitute a factor of estrangement.

The dynamic matrix associated to (3.2.1) is

$$A_S = \begin{bmatrix} 0 & -a \\ b & 0 \end{bmatrix};$$

with eigenvalues $\lambda_{1,2} = \pm i\sqrt{ab}$. Therefore the unique critical point $(0,0)$ is a centre and the outcome of their affair is a never-ending cycle of love and hate.

In his book [80], Strogatz added a short section, in which he contemplated not only others but also own feelings. Then system (3.2.1) became more general:

$$\begin{cases} \frac{dR(t)}{dt} = aR(t) + bJ(t) \\ \frac{dJ(t)}{dt} = cR(t) + dJ(t) \end{cases} \tag{3.2.2}$$

where the parameters a, b, c, d assume variable sign.

Consequently four romantic styles can be determined for Romeo and Juliet respectively. From the perspective of Romeo, there exist the personalities:

1. *Eager beaver* ($a > 0$ and $b > 0$);
2. *Cautious* (or *secure*, or *synergic*) lover ($a < 0$ and $b > 0$);
3. *Narcissistic nerd* ($a > 0$ and $b < 0$);
4. *Hermit* ($a < 0$ and $b < 0$).

The same styles can be exhibited for Juliet, discussing the parameters c and d.

In 2004 Sprott, in his paper [78], from University of Wisconsin, considered the case in which the couple becomes a love triangle since Romeo has a mistresse, Guinevre. He supposed that Romeo adopted the same romantic styles towards his lovers and they did not know about each other.

The resulting model is

$$\begin{cases} \frac{dR_J(t)}{dt} = aR_J(t) + b(J(t) - G(t)) \\ \frac{dJ(t)}{dt} = cR_J(t) + dJ(t) \\ \frac{dR_G(t)}{dt} = aR_G(t) + b(G(t) - J(t)) \\ \frac{dJ(t)}{dt} = eR_G(t) + fG(t) \end{cases} \tag{3.2.3}$$

with the parameters a, b, c, d, e, f with variable sign.

From the first and third equation of system (3.2.3), it can be noted that Juliet's feelings for Romeo influence his feelings for Guinevre in a way that is exactly opposite to Guinevre's way to affect Romeo's feelings toward Juliet.

Let's look back to Strogatz model (3.2.2). It has been generally acknowledged that the model, even if suggestive, is unrealistic. Indeed it does not explain why two individuals, who are initially completely indifferent, begin to develope a love affair.

With the aim to improve Strogatz's model, in 1998 (see [70]) Sergio Rinaldi introduced the linear system

$$\begin{cases} \dot{x}_1(t) = -\alpha_1 x_1(t) + \beta_1 x_2(t) + \gamma_1 A_2 \\ \dot{x}_2(t) = -\alpha_2 x_2(t) + \beta_2 x_1(t) + \gamma_2 A_1 \end{cases} \tag{3.2.4}$$

with $\alpha_i, \beta_i, \gamma_i, A_i > 0$ for $i = 1, 2$.

It takes into account three aspects of a love affair described by linear functions: the *oblivion* $-\alpha_i x_i(t)$, that gives rise to a loss of interest in partner i; the *return* $\beta_i x_j(t)$, which measures the pleasure of i to being loved; the *instinct* $\gamma_i A_j$, expressing the reaction to the appeal of individual j.

The same processes are considered by Rinaldi in [71] in the model describing the tempestuous love between Petrarca and his platonic dame Laura, given by

$$\begin{cases} \frac{dL(t)}{dt} = -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P \\ \frac{dP(t)}{dt} = -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_L}{1+\delta Z(t)} \\ \frac{dZ(t)}{dt} = -\alpha_3 Z(t) + \beta_3 P(t) \end{cases} \tag{3.2.5}$$

where $P(t)$ and $L(t)$ are measures of Petrarca and Laura's emotions respectively, while $Z(t)$ is a new variable state, meaning poetic inspiration. On closer view, in the second equation of (3.2.5) the instinct function of the poet

$$\beta_2 \frac{A_L}{1 + \delta Z(t)}$$

depends not only upon Laura's appeal component A_L, but also upon his poetic inspiration $Z(t)$. Especially it increases when A_L boosts and decreases when the poetic inspiration intensifies. This would stress the fact that moral tension weakens the most basic instincts.

Compared to the previous models, the main innovative factor of system (3.2.4) is the appeal component A_i, that turns out to be the "driving force that creates order in the community"; where, the concept of order (or stability) is realized when the partner of the most attractive woman is the most attractive man.

Let us put

$$A_R = \begin{bmatrix} -\alpha_1 & \beta_1 \\ \beta_2 & -\alpha_2 \end{bmatrix}$$

the dynamical matrix associated to (3.2.4), and

$$\bar{b} = \begin{bmatrix} \gamma_1 A_2 \\ \gamma_2 A_1 \end{bmatrix}$$

the vector of constant terms.

The characteristic equation associated to A_R is

$$\lambda^2 + (\alpha_1 + \alpha_2)\lambda + (\alpha_1\alpha_2 - \beta_1\beta_2) = 0.$$

If $\beta_1\beta_2 < \alpha_1\alpha_2$ then the (unique) equilibrium solution, designated with $\bar{x}_e \equiv (x_1^e, x_2^e)$, is asymptotically stable.

System (3.2.4) is positive, since the associated dynamical matrix A_R is a Metzler matrix (off-diagonal non-negative entries) and the vector $\bar{b}$ is positive. From the theory of positive systems, the following three properties hold:

$P.1$ If the individuals are non-antagonistic at a given time, then they remain non-antagonistic forever,

$P.2$ The equilibrium $\bar{x}_e$ of system (3.2.4) is strictly positive, that is $x_i^e > 0$ for $i = 1, 2$;

$P.3$ Two individuals completely indifferent to each other when they first meet develop a love story characterized by increasing feelings.

However Rinaldi's model (3.2.4) is still minimal, because

- The influences of the external world are not contemplated, the world is kept frozen;
- The behavioral parameters α_i, β_i, γ_i and the appeals A_i are assumed to be constant, hence the model can be used only to do predictions for short periods of time;
- The mechanisms of synergism and adaptation are considered negligible, i.e. oblivion and return functions depend only upon one state variable.

Following the suggestion of Rinaldi, but including the emotional interaction component, in [73] Satsangi e Sinha propose the dynamical model

$$\begin{cases} \frac{dx_1(t)}{dt} = -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) \\ \frac{dx_2(t)}{dt} = -\alpha_2 x_2(t) + \beta_2 x_1(t) + A_1 + d_2 x_1(t) x_2(t), \end{cases} \tag{3.2.6}$$

in which $x_1(t), x_2(t)$ are a measure of love of first and second individual for their respective partners, and $\alpha_1, \beta_1, A_1, d_1$, $\alpha_2, \beta_2, A_2, d_2$ are positive constants with the following meaning: α_1, α_2 are oblivion parameters, β_1, β_2 are reactiveness coefficient, A_1, A_2 are individual appeals, d_1, d_2 mean synergism and emotional interaction. From model (3.2.6), it can be noted that the emotions are variable and, in addition, individual emotion cannot increase infinitely respect to the other.

Hence, compared to Rinaldi's model, the only new factor introduced here is the quadratic term $d_i x_i(t) x_j(t)$, with the meaning of synergism process. It is indicative of learning and adaptation process, deriving from the knowledge of the partner and the experience of relation, that is, for instance, the learning effect after living together.

Anyway the stability analysis opered by Satsangi and Sinha in [73] is invalidate by a serie of errors.

3.3 Our contributions.

In our study we follow the same modelling approach (3.2.6) by Satsangi and Sinha, with a slight modification. The proposed model is a nonlinear system of two first order differential equations, given by

$$\begin{cases} \dot{x}_1(t) = -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) \\ \dot{x}_2(t) = -\alpha_2 x_2(t) + \beta_2 x_1(t) + A_1 - d_2 x_1(t) x_2(t) \end{cases}, \tag{3.3.1}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, d_1, d_2, A_1, A_2$ are positive constants.

The aim of our contribution is to find suitable conditions in order to get asymptotic stability of the determined stationary points. This fact, from the perspective of a love affair, allows us to establish if, at equilibrium, a romantic relation is characterized by a constant (or almost constant) behavior of the feelings or if it registers a brupt change of the emotions, if the initial emotional state (when they first meet) is slightly perturbed towards the equilibrium state. Here we assume that, when they meet for the first time, they are completely indifferent.

If we put

$$\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$A = \begin{bmatrix} -\alpha_1 & \beta_1 \\ \beta_2 & -\alpha_2 \end{bmatrix}$$

and

$$f(\bar{x}(t)) = \begin{bmatrix} f_1(\bar{x}(t)) \\ f_2(\bar{x}(t)) \end{bmatrix} = \begin{bmatrix} A_2 + d_1 x_1(t) x_2(t) \\ A_1 - d_2 x_1(t) x_2(t) \end{bmatrix},$$

it can be rewritten in the standard form

$$\dot{\bar{x}}(t) = A\bar{x}(t) + f(\bar{x}(t)).$$

Such a system is an autonomous system, because it does not depend on time t.

In this context the state variables $x_1(t)$ and $x_2(t)$ are measures of first and second individual emotions respectively and, for $i = 1, 2$, positive values of $x_i(t)$ mean friendship or love, negative values signify hate and zero value means complete indifference.

Especially, for $i, j = 1, 2$ and $j \neq i$, we have that

1. $-\alpha_i x_i(t)$ denotes the forgetting process, that gives rise to a loss of interest in individual i. So that, in absence of the person j, the feelings of individual i decay esponentially, according to $x_i(t) = x_i(0)e^{-\alpha_i t}$;
2. $\beta_i x_j(t)$ is the reaction function, describing the pleasure of individual i to be loved by partner j;
3. $d_i x_i(t) x_j(t)$ is the synergism function, which describes the emotional interaction process of the couple (i.e. the adaptation process after living together);
4. A_i is a measure of the attractiveness of individual i.

From first equation of (3.3.1) we can observe that synergism component is a source of interest for the emotions of individual 1, similarly to the terms due to return and attractiveness processes; while for individual 2 the emotional interaction function contributes to the decay of love feelings, as it can be noted by equation 2 of (3.3.1). This is the basic difference factor that we have introduced in the original model (3.2.6) of Satsangi and Sinha.

3.3.1 Tools

We recall a number of definitions and results that we need for the stability analysis of the critical points. Consider the general nonlinear autonomous system

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) \tag{3.3.2}$$

For a critical point $\bar{x}_e$, that is every solution of the equation

$$f(\bar{x}_e) = 0, \tag{3.3.3}$$

the following definitions hold:

Definition 3.3.1 *[46] The equilibrium state $\bar{x}_e$ of system* (3.3.2) *is stable if for every given $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if*

$$\|\bar{x}(0) - \bar{x}_e\| < \delta$$

then

$$\|\bar{x}(t) - \bar{x}_e\| < \epsilon$$

for all $t \geq 0$. Otherwise it is unstable.

Definition 3.3.2 *[46] The equilibrium state $\bar{x}_e$ of system* (3.3.2) *is asymptotically stable if it is stable and exists $\delta > 0$ such that if*

$$\|\bar{x}(0) - \bar{x}_e\| < \delta$$

then

$$\lim_{t\to\infty} \|\bar{x}(t) - \bar{x}_e\| = 0.$$

In order to testing the previous stability properties for $\bar{x}_e$, we recall a well known criterion, proposed for the first time by Lyapunov, often cited in litterature as *Lyapunov's indirect method*. This method is based on the linearization of the nonlinear system (3.3.2) in a neighborhood of the considered equilibrium state $\bar{x}_e$, that is

$$\delta\dot{\bar{x}}(t) = J(\bar{x}_e)\delta\bar{x}(t) \tag{3.3.4}$$

where $\delta\bar{x}(t) = \bar{x}(t) - \bar{x}_e$ is a measure of the distance between the perturbed state and the equilibrium state at time t, and

$$J(\bar{x}_e) = \left.\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}\right|_{\bar{x}=\bar{x}_e}$$

is the Jacobian matrix associated to f at $\bar{x} = \bar{x}_e$. System (3.3.4) is called *linearized* system and its dynamic matrix is the Jacobian matrix of f at $\bar{x} = \bar{x}_e$.

For system (3.3.4) can be applied the analysis techniques which hold for linear systems, and the stability results obtained imply the stability of the original non linear system (3.3.2) in a way that is expressed in the following:

Theorem 3.3.3 *[35, Lyapunov's first criterion]*

- *If all eigenvalues of the matrix $J(\bar{x}_e)$ have negative real parts, then the equilibrium state $\bar{x}_e$ is asymptotically stable for the original nonlinear system;*
- *If the matrix $J(\bar{x}_e)$ has one or more eigenvalues with positive real part, the the equilibrium state $\bar{x}_e$ is unstable.*

Remark 3.3.4 *If $J(\bar{x}_e)$ has at least an eigenvalue λ on the imaginary axis ($Re(\lambda) = 0$) and all others are in the left half of the complex plane, then one cannot conclude any type of stability at $\bar{x} = \bar{x}_e$ for the nonlinear original system.*

Therefore stability properties of equilibrium points in a nonlinear system can be analyzed by locating zeros of the characteristic polynomial associated to the linearized system. To this regard, Routh-Hurwitz crietrion allows us to decide if they all lie in the left half-plane:

Theorem 3.3.5 *[17, Thoerem 1.5.1] All characteristic roots of a real polynomial*

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a^n = 0$$

have negative real parts if and only if $D_k > 0$ for $k = 1, \cdots, n$, where

$$D_1 = a_1,$$

$$D_2 = det \begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}$$

$$D_k = det \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ 0 & 0 & a_1 & \cdots & a_{2k-5} \\ \cdots & \cdots & & & \\ 0 & 0 & \cdots & \cdots & a_k \end{bmatrix}$$

with $a_j = 0$ *for all* $j > n$.

Remark 3.3.6 *It follows that, for the case* $n = 2$*, that is*

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

in order to satisfy Routh criterion must be a_1 *and* a_2 *both positive.*

3.3.2 Results.

In order to determine the critical points we give the following:

Proposition 3.3.7 *[2, Proposition 3.1] Let us assume that*

1. $d_2A_2 > d_1A_1$,
2. $\frac{d_2}{d_1} > \frac{\alpha_2\beta_2A_2}{\alpha_1\beta_1A_1}$,
3. $\frac{d_2}{d_1} \neq \frac{\alpha_2}{\beta_1}$,
4. $\frac{d_2}{d_1} \neq \frac{\beta_2}{\alpha_1}$,
5. $-\alpha_2\beta_2 + d_2A_1 = 0$,

then system (3.3.1) *has three couples of critical points.*
Proof. *See page 95.* □

The next purpose is to study asymptotic stability of the determined points.

Let us proceed to apply the Lyapunov indirect method to our case.

The Jacobian matrix associated to (3.3.1) in $\bar{x}_e = (x_1^e, x_2^e)$ turns out to be

$$J(x_e) = \begin{vmatrix} -\alpha_1 + d_1x_2^e & \beta_1 + d_1x_1^e \\ \beta_2 - d_2x_2^e & -\alpha_2 - d_2x_1^e \end{vmatrix}$$

and its corresponding characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.3.5}$$

where

$$a_1 = \alpha_1 + \alpha_2 + d_2x_1^e - d_1x_2^e, \tag{3.3.6}$$

$$a_2 = \alpha_1\alpha_2 - \beta_1\beta_2 - \frac{n}{2}x_1^e - \frac{m}{2}x_2^e. \tag{3.3.7}$$

The following result holds:

Proposition 3.3.8 *[2, Proposition 3.6] Let us assume that* (1), (2), (3), (4), (5) *hold. Then*

1. *The critical point* $(\frac{h+l}{n}, \frac{k+l}{m})$ *is unstable for system* (3.3.1);

2. *If the conditions*

$$\alpha_2 < \beta_1, \tag{3.3.8}$$

$$d_1 < d_2, \tag{3.3.9}$$

$$d_1 A_1 > 4\alpha_1\alpha_2 \tag{3.3.10}$$

$$d_1 A_1 > 4\beta_1\beta_2 \tag{3.3.11}$$

$$d_1 A_1 > 4d_2 A_2 \tag{3.3.12}$$

hold, then the critical point $\big(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2+d_2A_2}{m}\big)$ *is aymptotically stable for system* (3.3.1);

3. *Supposed that* A_1 *and* A_2 *are both negligible compared to the other parameters, then the critical point* $(\frac{h-l}{n}, \frac{k-l}{m})$ *is asymptotically stable for system* (3.3.1).

Proof. *See page 97.* □

Remark 3.3.9 *As we have previously observed, while in system*

$$\begin{cases} \frac{dx_1(t)}{dt} = -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) \\ \frac{dx_2(t)}{dt} = -\alpha_2 x_2(t) + \beta_2 x_1(t) + A_1 + d_2 x_1(t) x_2(t), \end{cases}$$

introduced by Satsangi and Sinha, the synergism component is a source of interest for both individuals 1 *and* 2, *in our model*

$$\begin{cases} \frac{dx_1(t)}{dt} = -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) \\ \frac{dx_2(t)}{dt} = -\alpha_2 x_2(t) + \beta_2 x_1(t) + A_1 - d_2 x_1(t) x_2(t) \end{cases}$$

the same component is source of interest for the first individual, but contribute to the decay of the emotions in the second individual. This is the basic, but not unique, difference factor between the two models.

Indeed, article [73] contains some errors in the determination of solutions at equilibrium and in the study of stability properties of the determined critical points. Concerning the first point, it is not guaranted the existence of solutions at equilibrium, at least without doing further hypothesis. In addition, the same solutions at equilibrium, determined in the article, are not correct.

While, as regards the second point, it can be observed a wrong application of Routh-Hurwitz stability criterion. This last criterion allows us to decide if the roots, of the characteristic polynomial associated to the considered problem, all lye in the left half complex plain. Especially, in case $n = 2$, *corresponding to the polynomial*

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2,$$

the coefficients a_1 *and* a_2 *must both be positive in order to satisfy Routh Hurwitz criterion 3.3.5. Instead, in article [73], Satsangi and Sinha deduce asymptotic stability property for system* (3.2.6) *in its critical points, starting from the fact that* a_1 *and* a_2 *have different sign, under suitable hypothesis. According to the above statement, these conclusions are not correct.*

Remark 3.3.10 *In case* (2) *of the above Proposition, asymptotic stability is achieved if the ratio of appeals* $\frac{A_1}{A_2}$ *is greater than the the quatruple of reciprocal of ratio of synergism coefficients* (3.3.12), *and the geometric mean of* $d_1 A_1$ *is greater than the double of geometric mean of forgetting parameters* (3.3.10) *and reactiveness coefficients* (3.3.11). *Moreover the second individual must be less reticent then the reactiveness of the first one and more synergic compared to individual 1, in order to get asymptotic stability.*

Remark 3.3.11 *We recall that, in Rinaldi's model, the introduction of components* A_1 *and* A_2 *explains why two individuals initially indifferent begin to develop a love affair, while, in system* (3.3.1), *they must be negligible compared to other components in order to get asymptotic stability for the equilibrium state* $\bar{x}_e = (\frac{h-l}{n}, \frac{k-l}{m})$ *in case* (3) *of the previous Proposition. This fact does not seems too strange if one considers the presence of the forth component* d_i *of emotional interaction. This last constitues a measure of the synergism, adaptation and learning effect of the couple, i.e., after living together. Hence the obtained result seems to suggest that the equilibrium of the couple depends further upon the experience of the couple rather than the mutual attractiveness, that plays a crucial role in the early stage, although* A_1 *and* A_2 *must be never null.*

Following a suggestion of Sprott ([78]), we consider the case in which the individuals involved in the romantic relationship have the same behavioral styles (”romantic clones”), e.g.

$$a_1 = a_2, \tag{3.3.13}$$

$$b_1 = b_2, \tag{3.3.14}$$

$$d_1 = d_2, \tag{3.3.15}$$

$$A_1 = A_2, \tag{3.3.16}$$

with $\alpha_1, \beta_1, d_1, A_1 > 0$, then the starting model becomes

$$\begin{cases} \dot{x}_1(t) = -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_1 + d_1 x_1(t) x_2(t) \\ \dot{x}_2(t) = -\alpha_1 x_2(t) + \beta_1 x_1(t) + A_1 - d_1 x_1(t) x_2(t) \end{cases}. \tag{3.3.17}$$

In order to determine the critical points give the following:

Proposition 3.3.12 *[2, Proposition 4.1] Let us suppose that*

(6) $\alpha_1 \neq \beta_1$,

(7) $-\alpha_1\beta_1 + d_1 A_1 = 0$,

then system (3.3.17) *has three couples of critical points.*
Proof. *See page 99.* □

Even in this context the principal tool of our analysis are the previously stated Lyapunov's first criterion and Routh theorem. Thus we must consider the characteristic poynomial associated to the system in each point and study the sign of its coefficients.

To this aim, first of all we introduce the Jacobian matrix associated to problem (3.3.17) at quilibrium state $\bar{x}_e = (x_1^e, x_2^e)$, given by

$$J(\bar{x}_e) = \begin{vmatrix} -\alpha_1 + d_1 x_2^e & \beta_1 + d_1 x_1^e, \\ \beta_1 - d_1 x_2^e & -\alpha_1 - d_1 x_1^e. \end{vmatrix}$$

Thus, the corresponding characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2,$$

with coefficients

$$a_1 = 2\alpha_1 - d_1 x_2^e + d_2 x_1^e,$$

and

$$a_2 = \alpha_1^2 - \beta_1^2 - \frac{m}{2}x_2^e - \frac{n}{2}x_1^e.$$

In order to study stability properties we give the following:

Proposition 3.3.13 *[2, Proposition 4.2] Let us suppose that* (6) *and* (7) *hold, then*

1. *The critical point* $(\frac{h+l}{n}, \frac{k+l}{m})$ *is unstable for system* (3.3.17)*;*
2. *The critical point* $(-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2 + d_1 A_1}{m})$ *is unstable for system* (3.3.17)*;*
3. *Let us assume that*

$$\beta_1 = \frac{d_1 A_1}{\alpha_1}, \tag{3.3.18}$$

$$\alpha_1 > \beta_1, \tag{3.3.19}$$

then the critical point $(\frac{h-l}{n}, \frac{k-l}{m})$ *is asymptotically stable for system* (3.3.17).

Proof. *See page 100.* □

Remark 3.3.14 *In case* (3) *of Proposition* (3.3.13)*, the assumptions for stability suggest that if the individuals are two romantic clones then the stability is guaranteed if the individuals have forgetting process coefficients greater then the reactiveness parameters, condition* (3.3.19)*. In other words, they have the tendency to forget each other rather than to be reactive to partner's love, thus the feelings start to flag. This result leads us to believe that if individuals have the same behavior they can hardly generate instability in the relationship, instead of of what happens between individuals with different. Hence this case could confirm that only "the opposites attact". Instead the condition* (3.3.18) *would mean that the reactiveness increases if the forgetting process decreases et vice versa, while it increases if the appeal and synergism grow up. This last result is in agreement with the fact than* β_1, d_1, A_1 *are sources of interest for the couple instead of* α_1*.*

3.3.3 Proofs.

Proof.(Proposition 3.3.7) We need to solve the system

$$\begin{cases} -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) = 0 \\ -\alpha_2 x_2(t) + \beta_2 x_1(t) + A_1 - d_2 x_1(t) x_2(t) = 0. \end{cases} \tag{3.3.20}$$

Supposed $x_1(t) \neq \frac{-\alpha_2}{d_2}$, we can obtain $x_2(t)$ from the second equation of (3.3.20), thus system (3.3.20) becomes:

$$\begin{cases} -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) = 0, \\ x_2(t) = \frac{\beta_2 x_1(t) + A_1}{\alpha_2 + d_2 x_1(t)}. \end{cases} \tag{3.3.21}$$

Otherwise if $x_1(t) = \frac{-\alpha_2}{d_2}$, for condition (5), the first equation of (3.3.20) is identically satisfied and system (3.3.20) reduces to

$$\begin{cases} x_1(t) = -\frac{\alpha_2}{d_2}, \\ -\alpha_2 x_2(t) + \beta_2 x_1(t) + A_1 - d_2 x_1(t) x_2(t) = 0. \end{cases} \tag{3.3.22}$$

Since, from hypothesis, conditions (1), (2), (3), (4), (5) hold, system (3.3.21) has the following two couples of solutions:

$$\begin{aligned} x_1^1 = \frac{1}{2(\beta_2 d_1 - \alpha_1 d_2)} \big\{ & \alpha_1\alpha_2 - \beta_1\beta_2 - d_1 A_1 - d_2 A_2 \\ & + [\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + d_1^2 A_1^2 + d_2^2 A_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 \\ & -2\alpha_1\alpha_2 d_1 A_1 + 2\alpha_1\alpha_2 d_2 A_2 - 2\beta_1\beta_2 d_1 A_1 + 2\beta_1\beta_2 d_2 A_2 \\ & +2 d_1 d_2 A_1 A_2 + 4\alpha_1\beta_1 d_2 A_1 - 4\alpha_2\beta_2 d_1 {}_{A_2]}^{\frac{1}{2}} \big\} \end{aligned}$$

$$x_2^1 = \frac{\beta_2 x_1 + A_1}{\alpha_2 + d_2 x_1^1}$$

and

$$\begin{aligned} x_1^2 = \frac{1}{2(\beta_2 d_1 - \alpha_1 d_2)} \big\{ & \alpha_1\alpha_2 - \beta_1\beta_2 - d_1 A_1 - d_2 A_2 \\ & - [\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + d_1^2 A_1^2 + d_2^2 A_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 \\ & -2\alpha_1\alpha_2 d_1 A_1 + 2\alpha_1\alpha_2 d_2 A_2 - 2\beta_1\beta_2 d_1 A_1 + 2\beta_1\beta_2 d_2 A_2 \\ & +2 d_1 d_2 A_1 A_2 + 4\alpha_1\beta_1 d_2 A_1 - 4\alpha_2\beta_2 d_1 A_2]^{\frac{1}{2}} \big\}, \end{aligned}$$

$$x_2^2 = \frac{\beta_2 x_1 + A_1}{\alpha_2 + d_2 x_1^2}$$

,

while system (3.3.22) admits the unique solution

$$x_1^3 = -\frac{\alpha_2}{d_2},$$

$$x_2^3 = \frac{\alpha_1\alpha_2 + d_2 A_2}{\alpha_2 d_1 - \beta_1 d_2}.$$

In order to semplify the notation we put

$$h = \alpha_1\alpha_2 - \beta_1\beta_2 - d_1A_1 - d_2A_2,$$

$$\begin{aligned} l = [\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + d_1^2A_1^2 + d_2^2A_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 \\ -2\alpha_1\alpha_2 d_1A_1 + 2\alpha_1\alpha_2 d_2A_2 - 2\beta_1\beta_2 d_1A_1 + 2\beta_1\beta_2 d_2A_2 \\ +2d_1d_2A_1A_2 + 4\alpha_1\beta_1 d_2A_1 - 4\alpha_2\beta_2 d_1A_2]^{\frac{1}{2}}, \end{aligned}$$

$$n = 2(\beta_2 d_1 - \alpha_1 d_2),$$

hence the equilibrium points can be rewritten as

$$\Big(\frac{h+l}{n}, \frac{\beta_2(h+l)+A_1n}{\alpha_2 n + d_2(h+l)}\Big), \tag{3.3.23}$$

$$\Big(\frac{h-l}{n}, \frac{\beta_2(h-l)+A_1n}{\alpha_2 n + d_2(h-l)}\Big), \tag{3.3.24}$$

$$\Big(-\frac{\alpha_2}{d_2}, \frac{\alpha_1\alpha_2 + d_2A_2}{2(\alpha_2 d_1 - \beta_1 d_2)}\Big). \tag{3.3.25}$$

Let us introduce

$$k = \alpha_1\alpha_2 - \beta_1\beta_2 + d_1A_1 + d_2A_2,$$

$$m = 2(\alpha_2 d_1 - \beta_1 d_2).$$

Since it can be proved that

$$\frac{\beta_2(h+l)+A_1n}{\alpha_2 n + d_2(h+l)} = \frac{k+l}{m}$$

and

$$\frac{\beta_2(h-l)+A_1n}{\alpha_2 n + d_2(h-l)} = \frac{k-l}{m},$$

definitively the critical points can be rapresented as

$$\Big(\frac{h+l}{n}, \frac{k+l}{m}\Big),$$

$$\Big(\frac{h-l}{n}, \frac{k-l}{m}\Big).$$

$$\Big(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2 + d_2A_2}{m}\Big).$$

□

Proof.(Proposition 3.3.8)

(1) In the present case, the characteristic polynomial associated to the problem at $\bar{x}_e = (\frac{h+l}{n}, \frac{k+l}{m})$ is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.3.26}$$

with coefficients a_1 and a_2 given by

$$a_1 = \alpha_1 + \alpha_2 + d_2\frac{h+l}{n} - d_1\frac{k+l}{m},$$

$$a_2 = -l.$$

Since $a_2 < 0$, there exists at least a positive root of $P(\lambda)$. This fact implies, by Lyapunov's first criterion, that $(\frac{h+l}{n}, \frac{k+l}{m})$ is an unstable equilibrium state for system (3.3.1).

(2) The considered equilibrium state is $\bar{x}_e = \big(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2+d_2A_2}{m}\big)$, therefore the characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.3.27}$$

with coefficients

$$a_1 = \alpha_1 - 2d_1\frac{\alpha_1\alpha_2 + d_2A_2}{m},$$

$$a_2 = -\beta_1\beta_2 + \frac{n}{2}\frac{\alpha_2}{d_2} - d_2A_2.$$

From hypothesis (3.3.8), (3.3.9), (3.3.10), (3.3.11) and (3.3.12) we get that

$$a_1 = -d_2\frac{\alpha_1\beta_1 + A_2d_1}{\alpha_2d_1 - \beta_1d_2} > 0$$

and

$$a_2 = -\alpha_1\alpha_2 - \beta_1\beta_2 + d_1A_1 - d_2A_2 > 0.$$

Hence the characteristic polynomial $P(\lambda)$ has both the roots with negative real part. From Lyapunov's first criterion, we conclude that the point $\big(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2+d_2A_2}{m}\big)$ is asymptotically stable for system (3.3.1).

(3) As in the above point, we consider the characteristic polynomial associated to system (3.3.1) at $\bar{x}_e = (\frac{h-l}{n}, \frac{k-l}{m})$, that is given by

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 \tag{3.3.28}$$

and it has coefficients

$$a_1 = \alpha_1 + \alpha_2 - d_1\frac{k-l}{m} + d_2\frac{h-l}{n},$$

$$a_2 = l.$$

We readily note that a_2 is positive.

According to the hypothesis for which A_1 and A_2 must be negligible compered to the other behavioral parameters, we can assume that

$$A_1 = \epsilon_1$$

and

$$A_2 = \epsilon_2,$$

with ϵ_1, ϵ_2 both negligible compared to the other parameters, so that

$$l \simeq |\alpha_1\alpha_2 - \beta_1\beta_2|,$$

$$k \simeq \alpha_1\alpha_2 - \beta_1\beta_2,$$

$$h \simeq \alpha_1\alpha_2 - \beta_1\beta_2.$$

Hence we obtain that

$$a_1 \simeq \alpha_1 + \alpha_2 > 0.$$

As in the previous case, we can conclude that the critical point $\bar{x}_e = (\frac{h-l}{n}, \frac{k-l}{m})$ is asymptotically stable for system (3.3.1).

□

Proof.(Proposition 3.3.12)

For the perspective to find critical points we need to solve the system

$$\begin{cases} -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_1 + d_1 x_1(t) x_2(t) = 0 \\ -\alpha_1 x_2(t) + \beta_1 x_1(t) + A_1 - d_1 x_1(t) x_2(t) = 0 \end{cases}. \tag{3.3.29}$$

If we assume that $x_1(t) \neq \frac{-\alpha_1}{d_1}$, we can get $x_2(t)$ from second equation of (3.3.29), thus system (3.3.29) becomes

$$\begin{cases} -\alpha_1 x_1(t) + \beta_1 x_2(t) + A_2 + d_1 x_1(t) x_2(t) = 0 \\ x_2(t) = \frac{\beta_1 x_1(t) + A_1}{\alpha_1 + d_1 x_1(t)} \end{cases}. \tag{3.3.30}$$

Otherwise if $x_1(t) = \frac{-\alpha_1}{d_1}$, since condition (7) hold, the first equation of (3.3.29) is identically satisfied and system (3.3.29) reduces to

$$\begin{cases} x_1(t) = -\frac{\alpha_1}{d_1}, \\ -\alpha_1 x_2(t) + \beta_1 x_1(t) + A_1 - d_1 x_1(t) x_2(t) = 0. \end{cases} \tag{3.3.31}$$

Let us introduce h, k, l, m, n, which are definied as in the previous section, considering the conditions (3.3.13), (3.3.14), (3.3.15), (3.3.16) on the behavioral parameters:

$$h = \alpha_1^2 - \beta_1^2 - 2d_1A_1,$$

$$k = \alpha_1^2 - \beta_1^2 + 2d_1A_1,$$

$$l = \sqrt{(\alpha_1^2 - \beta_1^2)^2 + 4d_1^2A_1^2},$$

$$m = 2(\alpha_1 - \beta_1)d_1,$$

$$n = 2(\beta_1 - \alpha_1)d_1.$$

Since conditions (6) and (7) hold, system (3.3.30) has solutions

$$\left(\frac{h+l}{n}, \frac{k+l}{m}\right),$$

$$\Big(\frac{h-l}{n}, \frac{k-l}{m}\Big),$$

while system (3.3.31) admit solution

$$\Big(-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2 + d_1 A_1}{m}\Big).$$

□

Proof.(Proposition 3.3.13)

(1) If the equilibrium state is $\bar{x}_e = (\frac{h+l}{n}, \frac{k+l}{m})$, then the characteristic polynomial associated to the linearized system of problem (3.3.17) is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.3.32}$$

where

$$a_1 = 2\alpha_1 - d_1\frac{k+l}{m} + d_2\frac{h+l}{n}$$

and

$$a_2 = -l.$$

We readily note that $a_2 < 0$. Therefore there exists at least a positive root of $P(\lambda)$ and, for Lyapunov's first criterion, the point $(\frac{h+l}{n}, \frac{k+l}{m})$ is unstable for system (3.3.17).

(2) If the equilibrium state is $\bar{x}_e = (-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2+d_1A_1}{m})$ then the characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.3.33}$$

where

$$a_1 = -\frac{\alpha_1\beta_1 + d_1 A_1}{\alpha_1 - \beta_1},$$

and

$$a_2 = -\alpha_1^2 - \beta_1^2 + \alpha_1\beta_1 - d_1 A_1.$$

From relation (7) we get that

$$a_2 = -\alpha_1^2 - \beta_1^2 < 0.$$

Hence, as in the previous case, the characteristic polynomial (3.3.33) has at least a positive root and we conclude that the point $(-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2+d_1A_1}{m})$ is unstable for system (3.3.17).

(3) The characteristic polynomial $P(\lambda)$ associated to the critical point $(\frac{h-l}{n}, \frac{k-l}{m})$ is given by

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 \tag{3.3.34}$$

where

$$a_1 = 2\alpha_1 - d_1\frac{k-l}{m} + d_2\frac{h-l}{n}$$

and

$$a_2 = l.$$

We readily see that a_2 is positive. Under hypothesis (3.3.18) and (3.3.19), we obtain that

$$a_2 = l = \alpha_1^2 + \beta_1^2 > 0$$

and

$$a_1 = 2\alpha_1 + \frac{2\beta_1^2}{\alpha_1 - \beta_1} > 0.$$

Hence the characteristic polynomial (3.3.34) has both roots with negative real part. From Lyapunov's first criterion, we can conclude that $(\frac{h-l}{n}, \frac{k-l}{m})$ is asymptotically stable for system (3.3.17).

□

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