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On irregular surfaces of general type with $K^2 = 2\chi + 1$ and $p_g = 2$

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Introduzione

This thesis is devoted to one of the classic topics about algebraic surfaces: the classification of irregular surface of general type and the analysis their moduli space.

To a minimal surface of general type S we associates the following numerical invariants:

- the self intersection of the canonical class K_S^2;
- the geometric genus $p_g := h^0(\omega_S)$
- the irregularity $q := h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$.

A surface S is called irregular if $q > 0$. By a theorem of Gieseker the coarse moduli space $\mathcal{M}_{a,b}$ corresponding to minimal surfaces with $K_S^2 = a$ and $p_g = b$ is a quasi projective scheme, and it has finitely many irreducible components.

The above invariants determine the other classical invariants:

- the holomorphic Euler–Poincarè characteristic $\chi(S) := \chi(\mathcal{O}_S) = 1 - q + p_g$;
- the second Chern class $c_2(S)$ of the tangent bundle which is equal to the topological Euler characteristic $e(S)$ of S.

The classical question that naturally rises at this point is the so–called geographical question, i.e., for which values of $a,\ b$ is $\mathcal{M}_{a,b}$ nonempty? The answer to this question is obviously non trivial.

There exists the following inequalities holding among the invariants of minimal surfaces of general type:

- $K_S^2, \chi \geq 1$;
- $K_S^2 \geq 2p_g - 4$ (Noether's inequality);
- if S is an irregular surface, then $K_S^2 \geq 2p_g$ (Debarre's inequality);
- $K_S^2 \leq 9\chi\mathcal{O}_S$ (Miyaoka–Yau inequality).

Thus $\chi = 1$ is the lowest possible value for a surface of general type. By the Miyaoka–Yau inequality, we have that $K_S^2 \leq 9$, hence by the Debarre's inequality we get $q = p_g \leq 4$. All known results about the classification of such surfaces are listed in [MePa, Section 2.5 a].

If $K_S^2 = 2\chi$, we have that necessarily $q = 1$. Since in this case $f : S \longrightarrow \mathrm{Alb}(S)$ is a genus 2 fibration, by using the fact that all fibres are 2–connected, the classification was completed by Catanese for $K^2 = 2$, and by Horikawa in [Hor3] in the general case.

Catanese and Ciliberto in [CaCi1] and [CaCi2] studied the case $K^2 = 2\chi + 1$, with $\chi = 1$. So in this case, by the above inequalities we get that the surfaces have the following numerical invariants:

$$K_S^2 = 3 \text{ and } p_g = q = 1.$$

The classification of such surfaces was completed by Catanese and Pignatelli in [CaPi]. The main tool for this classification is the structure theorem for genus 2 fibration, which is proved in the same work.

For $\chi \geq 2$ the situation is far more complicated and not yet studied. We consider in this thesis the case $\chi = 2$. So our surfaces have the following numerical characters

$$K_S^2 = 5,\ p_g = 2,\ q = 1.$$

By a theorem of Horikawa, which affirms that for an irregular minimal surface of general type with $2\chi \leq K^2 \leq \frac{8}{3}\chi$, the Albanese map

$$f : S \longrightarrow \mathrm{Alb}(S)$$

induces a connected fibration of curves of genus 2 over a smooth curve of genus q, we have that in the considered case a fibration $f : S \longrightarrow B$ over an elliptic curve B and with fibres of genus 2.

So we can use the results of Horikawa–Xiao and most of all those of Catanese–Pignatelli to face the challenge to completely classify all surface with the above numerical invariants. Their approach is of algebraic nature and in particular is based on a new method for studying genus 2 fibration, basically giving generators and relations of their relative canonical algebra, seen as a sheaf of algebras over the base curve B.

Our main results are as follows. First at all we studied the various possibilities for the 2–rank bundle $f_*\omega_S$. We have that $f_*\omega_S$ can be decomposable or indecomposable. In the first case the usual invariant e, associated to $f_*\omega_S$ by Xiao in [Xia1] can be equal to 0 or 2. We prove that the case $e = 2$ does not occur.

Subsequently we study the case $e = 0$ with $f_*\omega_S$ decomposable. In such case we divide the problem in various subcases. For each such subcase we study the corresponding subspace of the moduli space $\mathcal{M}$ of surfaces with $K^2 = 5$, $p_g = 2$ e $q = 1$.

By using the following formula:

$$\dim \mathcal{M} \geq 10\chi - 2K^2 + p_g = 12$$

we can consider only the strata of dimension greater than or equal to 12.

We proved that almost all the strata has dimension ≤ 11, so they don't give components of the moduli space.

The most important result is that, for the so-called strata VII, we have the following theorem.

Theorem 0.0.1.

(i) $\mathcal{M}_{VII,\mathrm{gen}}$ *is non-empty and of dimension* 12*;*

(ii) $\mathcal{M}_{VII2}$ *is non-empty and of dimension* 13*.*

For the most difficult case of $f_*\omega_S$ indecomposable, the results are promising but still partial.

Notazioni

We work over the complex number field $\mathbb{C}$. All surfaces are projective algebraic and, unless otherwise specified, smooth. We do not distinguish between line bundles and divisors on a smooth surface. If C and D are divisors on a surface S, $C \cdot D$ denotes the intersection number of C and D, and C^2 is the self–intersection of the divisor C. Furthermore $\equiv$ denotes linear equivalence and $\sim$ denotes algebraic equivalence. For a Gorenstein projective variety X, ω_X is the canonical sheaf of X. A divisor in the linear system $|\omega_X|$ is called a canonical divisor and it is denoted by K_X. If $\mathcal{F}$ is a coherent sheaf on X then we denote $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$, $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$, $\chi(\mathcal{F}) = \sum_{i=0}^{\dim X}(-1)^i h^i(\mathcal{F})$. As usual we denote $p_g(S)) = h^0(K_S)$ the geometric genus, $q = h^1(\mathcal{O}_S)$ the irregularity and $\chi(S) = 1 - q(S) + p_g(S)$ the Euler characteristic of the structure sheaf of S.

Chapter 1

Preliminaries

1.1 Surfaces of General Type

Let S be a surface, i.e. a smooth projective surface and let D be a divisor on S. We associate to D the graded ring:

$$R(S,D) := \bigoplus_{0 \leq m \leq \infty} H^0(S, \mathcal{O}_S(mD))$$

We note that the subspace $R(S,D)_0$ of the homogeneous elements of degree zero, equals the base field $\mathbb{C}$. To the ring $R(S,D)$, we associate the subfield $Q(S,D) := \{f/g | f,g \in R(S,D)_m,\ m > 0\}$ of $\mathbb{C}(S)$.

Proposition 1.1.1. *Let S be a smooth projective surface and D a divisor on S. Then $Q(S,D)$ is a finitely generated field extension of $\mathbb{C}$ and is algebraically closed in $\mathbb{C}(S)$. In particular its transcendence degree is finite and at most equal to the dimension of S (cf. [And1]).*

Definition 1.1.2. *Let S be a smooth projective surface and D a divisor on S. Then we define the Kodaira–Iitaka dimension of D as:*

(*i*) $\mathrm{Kod}(D) := \mathrm{tr.deg}_{\mathbb{C}}\, Q(S, D)$ *if* $R(S, D) \neq \mathbb{C}$;

(*ii*) $\mathrm{Kod}(D) := -\infty$ *if* $R(S, D) = \mathbb{C}$ (*or equivalently if* $Q(S, D) = 0$).

If $D = K_S$, *the graded ring* $K(S) := R(S, K) = \bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(mK_S)$ *is called the canonical ring of* S *and the Kodaira–Iitaka dimension of* K_S *is called the Kodaira dimension of* S.

Remark 1.1.3. *The canonical ring* $R(S)$ *of* S, *the plurigenus* $P_m(S)$ *and* $h^0(S, \Omega^1_S)$ *are birational invariants, so* $\mathrm{Kod}(S)$ *is also a birational invariant.*

We have the following result:

Theorem 1.1.4. *Let* S *be a minimal surface. The following three conditions are equivalent:*

(*i*) $\mathrm{Kod}(S) = 2$;

(*ii*) $K_S^2 > 0$ *and* K_S *is nef;*

(*iii*) *there exists an integer* n_0 *such that for any* $n \geq n_0$ *the* n*–canonical map* φ_{nK} *is birational to its image.*

If these conditions hold, then S is called a surface of general type.

1.2 Fibrations

The purpose in this section is to give an introduction to the theory of fibrations of algebraic surfaces to curves. We will collect here some results.

Definition 1.2.1. *Let* S *be a smooth projective surface and* B *a smooth projective curve. A fibration* $f : S \longrightarrow B$ *is a surjective morphism with connected fibres. The fibration is said to be relatively minimal if* $f : S \longrightarrow B$ *has no rational smooth* *curves of self intersection -1 in any of its fibres. Relatively minimal models always exist.*

We denote by b the genus of the curve B, and by g the genus of a general fibre F. Notice that, for $g \geq 1$, the fibration f is relatively minimal if and only if the canonical divisor K_S is f–nef, i.e. $K_S \cdot C \geq 0$ for every irreducible curve C contained in a fibre of f. In the case $g \geq 1$ the relatively minimal model of f is unique.

Proposition 1.2.2. *Let $f : X \longrightarrow Y$ a proper morphism of algebraic varieties with Y normal. If f has connected fibres then $f_*\mathcal{O}_X = \mathcal{O}_Y$*

This result is a consequence of the Zariski's Main Theorem via Stein Factorization (see [Har, Chapter III, Corollary 11.5]). We will get that for a fibration $f : S \longrightarrow B$

$$f_*\mathcal{O}_S = \mathcal{O}_B.$$

Notice that a fibration $f : S \longrightarrow B$ is a flat morphism ([Har, 9.7.1]). We need some Lemmas about fibrations.

Lemma 1.2.3. (Zariski's Lemma) *Let $f : S \longrightarrow B$ be a fibration and $F_b = \sum n_i C_i$, $n_i > 0$, C_i irreducible, be a fibre of f. Then we have:*

(i) $C_i F_b = 0$ for all i;

(ii) If $D = m_i C_i$, $m_i \in \mathbb{Z}$, then $D^2 \leq 0$;

(iii) $D^2 = 0$ holds in (ii) if and only if $pD = qF_b$, with $p, q \in \mathbb{Z}$, $p \neq 0$.

Definition 1.2.4. *A singular fibre $F_b = \sum n_i C_i$ is called a multiple fibre of multiplicity n if $n = \gcd\{n_i\} > 1$.*

In such case, $F_b = nF$, with F an effective divisor on S. $F_b^2 = 0$ implies that $F^2 = 0$. Furthermore F is 1–connected: let

$$F = F_1 + F_2, \quad F_1 > 0, \quad F_2 > 0$$

be a nontrivial decomposition of F. Since, by Zariski's lemma, $F_1^2 < 0$ and $F_2^2 < 0$ we get $F_1 \cdot F_2 \geq 1$, by using the equality $0 = F^2 = F_1^2 + F_2^2 + 2F_1 \cdot F_2$.

Lemma 1.2.5. *Let $F_b = nF$, $n > 1$ be a multiple fibre. Then $\mathcal{O}_S(F)$ and $\mathcal{O}_F(F)$ are both torsion line bundles of order n (cf. [BHPV, Chapter III, Lemma 8.3]).*

Lemma 1.2.6. *Let $f : S \longrightarrow B$ be a fibration. Then $h^0(F_b, \mathcal{O}_{F_b})$ is independent of $b \in B$. Since the general fibre is connected and smooth, $h^0(F_b, \mathcal{O}_{F_b}) = 1$ for all $b \in B$.*

Proof. Suppose that for some $b \in B$ we have $h^0(F_b, \mathcal{O}) > 1$. We get that F_b is not 1–connected by Ramanujam's Lemma (cf. [BHPV, Chapter II, Lemma 12.3]). Then, as we have noticed before, F_b is a multiple fibre, i.e. $F_b = nF$, $n > 1$, with F 1–connected. Consider now, for $1 \leq m \leq n-1$, the decomposition sequence

$$0 \longrightarrow \mathcal{O}_F(-mF) \longrightarrow \mathcal{O}_{(m+1)F} \longrightarrow \mathcal{O}_{mF} \longrightarrow 0.$$

Now, we have that if F is 1–connected, then Ramanujam's Lemma implies that $h^0(F, \mathcal{O}_F(-mF)) \leq 1$ and $h^0(F, \mathcal{O}_F(-mF)) = 1$ if and only if $\mathcal{O}_F(-mF) = \mathcal{O}_F$. Thus $h^0(F, \mathcal{O}_F(-mF)) = 0$, since the torsion bundles $\mathcal{O}_F(-mF)$ are nontrivial for $1 \leq m \leq n-1$. By induction, we get

$$h^0(F, \mathcal{O}_{F_b}) = h^0(F, \mathcal{O}_{nF}) \leq h^0(F, \mathcal{O}_F)$$

with $h^0(F, \mathcal{O}_F) = 1$, since F is 1–connected. So we have $h^0(F_b, \mathcal{O}_{F_b}) = 1$, for all $b \in B$. Since the Euler characteristic $\chi(F_b, \mathcal{O}_{f_B})$ is independent of b, we get that also $h^1(F_b, \mathcal{O}_{F_b})$ is independent of $b \in B$. $\square$

Let $f : S \longrightarrow B$ be a relatively minimal fibration.

Definition 1.2.7. *The line bundle* $\omega_{S|B} := K_S \otimes f^*(K_B^{-1})$ *on* S *is called the dualizing sheaf of* f.

Since the normal bundle $\mathcal{O}_{F_b}(F_b)$ of any fibre F_b is trivial, we have

$$\omega_{S|B}|_{F_b} = \mathcal{O}_S(K_S) \otimes f^*(K_B^{-1})|_{F_b} \cong \mathcal{O}_S(K_S)|_{F_b} \cong \mathcal{O}_S(K_S + F_b)|_{F_b} = \omega_{F_b},$$

for every $b \in B$.

Recall now two important results, one on the cohomology and base change, another on relative duality.

Theorem 1.2.8. *Let* $f : X \longrightarrow Y$ *a proper morphism of algebraic varieties,* $X_y = f^{-1}(y)$ *the fibre over* y*. If* $\mathcal{E}$ *is a coherent sheaf on* X*, which is flat over* Y*, we have:*

(i) The Euler characteristic $\chi(X_y, \mathcal{E}|_{X_y})$ *is constant;*

(ii) $h^q(X_y, \mathcal{E}|_{X_y})$ *is an upper semicontinuous function of* y*, for all* $q \geq 0$*;*

(iii) If $h^q(X_y, \mathcal{E}|_{X_y})$ *is constant, then* $R^q f_*(\mathcal{E})$ *is locally free;*

(iv) If $h^q(X_y, \mathcal{E}|_{X_y})$ *is constant, then the "base change morphism"*

$$R^q f_*(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{O}_y/\mathfrak{m} \longrightarrow H^q(X_y, \mathcal{E}|_{X_y})$$

is an isomorphism.

Theorem 1.2.9. (Relative Duality Theorem) *If* $f : S \longrightarrow B$ *is a fibration and* $\mathcal{E}$ *a locally free* $\mathcal{O}_S$*–sheaf, then we have that the (duality) morphism*

$$f_*(\mathcal{E}^\vee \otimes \omega_{S|B}) \longrightarrow (R^1 f_* \mathcal{E})^\vee$$

is an isomorphism

In particular we get

$$(R^1 f_* \omega_S)^\vee \cong f_*(f^* \omega_B^{-1}) \cong \omega_B^{-1}$$

i.e.

$$R^1 f_* \omega_S \cong \omega_B,$$

equivalently

$$R^1 f_*(\omega_S \otimes f^* \omega_B^{-1}) \cong R^1 f_* \omega_{S|B} \cong \mathcal{O}_B.$$

Remark 1.2.10. *If B has genus 1, we have*

$$R^1 f_* \omega_S \cong \mathcal{O}_B. \tag{1.1}$$

Since $\chi(F_b, \mathcal{O}_{F_b}) = h^0(F_b, \mathcal{O}_{F_b}) - h^1(F_b, \mathcal{O}_{F_b})$ is constant for a fibration $f : S \longrightarrow B$, and $h^0(F_b, \mathcal{O}_{F_b}) = 1$ for all $b \in B$, we obtain (using the duality on F_b)

$$h^1(F_b, \omega_{F_b}) = h^0(F_b, \mathcal{O}_{F_b}) = 1$$

and

$$h^0(F_b, \omega_{F_b}) = h^1(F_b, \mathcal{O}_{F_b}) = g$$

Furthermore, if the fibration is relatively minimal, $g \geq 2$, then $\deg \omega_{F_b} > 0$, for all $b \in B$. Thus

$$h^1(F_b, \omega_{F_b}^{\otimes n}) = h^1(F_b, \omega_{F_b}^{\otimes(1-n)}) = 0, \qquad \text{for } n \geq 2$$

In conclusion we have:

Theorem 1.2.11. *If $f : S \longrightarrow B$ is a relatively minimal fibration, then:*

(i) $f_* \omega_{S|B}$ is locally free of rank g;

(ii) $f_* \omega_{S|B}^{\otimes n}$ is locally free of rank $(2n-1)(g-1)$;

(iii) $R^1 f_* \omega_{S|B}^{\otimes n} = 0$ *for* $n \geq 2$ *when* $g \geq 2$.

Let $f : S \longrightarrow B$ be a relatively minimal fibration with S of general type. Since $R^1 f_* \omega_{S|B} = \mathcal{O}_B$ and $R^1 f_* \omega_{S|B}^{\otimes n} = 0$ for $n \geq 2$ we can compute the Euler characteristic of $f_* \omega_{S|B}^{\otimes n} = 0$ by Riemann–Roch and conseguently its degree.

We introduce now the following invariants of f:

- The self intersection of the relative dualizing sheaf:

$$K_{S|B}^2 := \omega_{S|B}^2 = K_S^2 - 8(b-1)(g-1); \tag{1.2}$$

- the Euler characteristic of the relative dualizing sheaf:

$$\chi_{S|B} := \chi(\mathcal{O}_S) - (b-1)(g-1). \tag{1.3}$$

It follows by Riemann–Roch that for $n \geq 1$:

$$\chi(f_* \omega_{S|B}^{\otimes n}) = \chi(\omega_{S|B}^{\otimes n}) = \frac{1}{2} n(n-1) K_{S|B}^2 + 2\chi(f_* \omega_{S|B}^{\otimes n}) \chi(\mathcal{O}_B) + \chi_{S|B},$$

$$\deg(f_* \omega_{S|B}^{\otimes n}) = \frac{1}{2} n(n-1) K_{S|B}^2 + \chi_{S|B}.$$

For simplicity, we define $\mathsf{V}_n := f_* \omega_{S|B}^{\otimes n}$. The vector bundles V_n have very nice properties.

Theorem 1.2.12. (Fujita) *The vector bundles* V_n *are semipositive, i.e. every locally free quotient of it has nonnegative degree. Precisely,* $\mathsf{V}_1 = \mathcal{O}_B^{q-b} \oplus \mathsf{A} \oplus (\bigoplus_i M_i)$ *where* A *is an ample bundle, each* M_i *is an indecomposable and stable of degree* 0 *with* $h^0(M_i) = 0$. *If* $\operatorname{rank} M_i = 1$, *then* M_i *is a torsion line bundle.(for this last observation see [Zuc])*

Fujita's theorem shows that

$$\deg \mathsf{V}_n = \frac{1}{2}n(n-1)K^2_{S|B} + \chi_{S|B} \geq 0$$

The Arakelov inequality

$$K^2_{S|B} = K^2_S - 8(b-1)(g-1) \geq 0 \tag{1.4}$$

follows as corollary, together with the inequality

$$\chi_{S|B} = \chi(\mathcal{O}_S) - (b-1)(g-1) \geq 0 \tag{1.5}$$

which is note as the Beauville's inequality. For $n \geq 2$ we have:

Theorem 1.2.13. (Esnault, Viehweg) *For any $n \geq 2$ the vector bundle V_n is ample unless f has constant moduli, which means that all the smooth fibres are isomorphic.*

We now restrict to fibrations $f : S \longrightarrow B$, where S is a minimal surface of general type with the general fibre F of genus 2.

Remark 1.2.14. *A fibration $f : S \longrightarrow B$ with the general fibre of genus 2 has not multiple fibres. For that, since $F^2 = 0$ and $K_S \cdot F = 2$, for a multiple fibre $F = nF'$, $n \geq 2$, we would get*

$$2 = n(K_S \cdot F'), \text{ then } K_S \cdot F' = \frac{2}{n};$$

since $K_S \cdot F'$ is even, we have a contradiction.

Another property of a fibration with fibre of genus 2 (or more generally hyperelliptic fibres) is that f is not smooth. The relative canonical map of a fibration of genus $g \geq 2$ is a generically finite rational map of degree 2,

$$S \dashrightarrow \Sigma \subseteq \mathbb{P}(f_*\omega_{S|B}),$$

where $\Sigma \subseteq \mathbb{P}(f_*\omega_{S|B}) = \mathbb{P}(\mathsf{V}_1)$ is birationally equivalent to a ruled surface over B. Then S has a birational involution σ which restricts to the hyperelliptic involution of F. Since S is minimal, σ act biregularly on the fibration, i.e.

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & S \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

$\sigma^2 = \mathrm{id}$, $\sigma \neq \mathrm{id}$ and $f \circ \sigma = f$. We recall the well–known procedure that associates to $f : S \longrightarrow B$ a double cover of a (relatively) minimal fibration.

Chapter 2

Double Covers and Genus 2 Fibrations

2.1 Double Covers

Definition 2.1.1. *A cover is a finite surjective morphism $f : X \longrightarrow Y$ between algebraic irreducible varieties*

A cover is said flat if the morphism f is flat. Recall that:

Proposition 2.1.2. *A finite morphism $f : X \longrightarrow Y$ is flat if and only if $f_*\mathcal{O}_X$ is locally free on Y. (see [Mum, p. 43])*

A useful criterion for flatness is the following:

Proposition 2.1.3. *Let $f : X \longrightarrow Y$ be a finite morphism. Suppose Y is a nonsingular variety. Then f is flat if and only if X is a Cohen–Macaulay variety. (cf. [Eis])*

We are interested to double covers, i.e. such that $\deg f = [K(X) : f^*K(Y)] =$

2. If $f : X \longrightarrow Y$ is a surjective morphism between surfaces, in general, f is not finite.

In such situation we use Stein factorization in order to get a finite morphism. In fact we have the following:

Theorem 2.1.4. (Stein factorization) *Let $f : X \longrightarrow Y$ be a surjective morphism between algebraic surfaces. We suppose X normal and Y nonsingular. Then f factors:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z \arrow[ur, "h"'] & \end{tikzcd}$$

where h is a double cover, g is a birational morphism with $g_*\mathcal{O}_X = \mathcal{O}_Z$. In particular g has connected fibre and Z is a normal surface.

By Proposition 2.1.3 we get that, being Z a Cohen–Macaulay variety, h is a flat morphism and $h_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_Y$–module of rank 2. Actually Z is the normalization of Y in the field $K(X)$.

Then the natural injection $0 \longrightarrow \mathcal{O}_Y \longrightarrow h_*\mathcal{O}_Z$ has an invertible cokernel:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow h_*\mathcal{O}_Z \longrightarrow \mathcal{O}_Y(-\delta) \longrightarrow 0 \tag{2.1}$$

with $\delta \in \mathrm{Pic}(Y)$.

Working locally, we can see that the branch locus B of f (and of h) is a reduced divisor linearly equivalent to 2δ. The surface Z is nothing but $\mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-\delta))$. Z is smooth if and only if B is a smooth divisor. So if $B \equiv 2\delta$ is singular, then also Z is singular.

The singularities of Z can be resolved by the canonical resolution. (see [Hor1]). Set $Y_0 = Y$ and $B_0 = B$. Let y_1 be a singular point of B of multiplicity m_1. Let

$\sigma_1 : Y_1 \longrightarrow Y_0$ be the blowup of y_1 with exceptional curve E_1. Then $B_1 = \sigma_1^* B_0 - 2[\frac{m_1}{2}]E_1$ is a reduced curve linearly equivalent to $2\delta_1$, where $\delta_1 = \sigma^* S - [\frac{m_1}{2}]E_1$; $[\frac{m_1}{2}]$ is the greatest integer less than or equal to $\frac{m_1}{2}$. Therefore there exists a double cover $Z_1 \longrightarrow Y_1$ branched along B_1 and a birational morphism $Z_1 \longrightarrow Z$. If Z_1 is singular we repeat this construction. After finitely many steps we arrive at a ramification divisor B_d smooth, and hence Z_d is smooth. $Z^* := Z_d$ is called the canonical resolution of Z. Generally Z^* is not the minimal resolution of Z. We have the

Theorem 2.1.5. *Let $h : Z \longrightarrow Y$ be a double cover with Z normal and Y non-singular, ramified over the reduced divisor $B \subset Y$. Let δ be the line bundle on Y, satisfying $B \equiv 2\delta$ such that*

$$Z = \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-\delta))$$

Consider the canonical resolution

$$\begin{tikzcd} Z^* = Z_d \arrow[r] \arrow[d, "h^*"'] & Z_{d-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Z_1 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ Y_d \arrow[r, "\sigma_d"'] & Y_{d-1} \arrow[r, "\sigma_{d-1}"'] & \cdots \arrow[r] & Y_1 \arrow[r, "\sigma_1"'] & Y_0 := Y \end{tikzcd}$$

Let $\sigma = \sigma_1 \circ \ldots \circ \sigma_d$, $\pi : Z^* \longrightarrow Z$ the induced birational morphism. Then there exists an effective divisor $E \geq 0$ on Z^*, with Supp(E) *contained in the union of the exceptional curves for π such that*

$$K_{Z^*} = (h \circ \pi)^*(K_Y + \delta) \otimes \mathcal{O}_{Z^*}(-E)$$

Furthemore, $E \equiv 0$ if and only if the singularities of B (hence of Z) are simple, i.e. B has no singular point of multiplicity greater than 3, and any triple point P of B decomposes into singularities of multiplicities less than or equal to 2 on the *proper transform of B after one blowup with center P. In such case $(E=0)$, the canonical resolution is the minimal resolution. The numerical characters of Z^* are the following:*

$$\chi(Z^*, \mathcal{O}_Z^*) = \frac{1}{2}(K_Y+\delta)\delta + 2\chi(Y, \mathcal{O}_Y) - \sum \frac{1}{2}\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2} - 1\right]\right)$$

$$K_{Z^*}^2 = 2(K_Y+\delta)^2 - 2\sum\left(\left[\frac{m_i}{2}\right] - 1\right)^2$$

where m_i $(i = 1, ..., d)$ denotes the multiplicity of B_{i-1} at the center of the blowup y_i which appears in the construction of Z^*.

2.2 Fibrations of Genus 2

Let $f : S \longrightarrow B$ be a fibration with fibres of genus 2. We often call a such fibration a genus 2 fibration. Let σ be the biregular involution on S. The fixed locus of σ is the union of a smooth reduced curve R and finitely many isolated points $p_1, \ldots, p_\varepsilon$. Let $\varrho : \widehat{S} \longrightarrow S$ be the blow–up of the isolated points of σ, $E_i = \varrho^{-1}(p_i)$ the exceptional curves. The involution σ induces an involution $\widehat{\sigma}$ on $\widehat{S}$, which has as fixed locus the smooth curve $\widehat{R} = \varrho^* R + \sum_{i=1}^{\varepsilon} E_i$. Hence the quotient $\widehat{W} := \widehat{S}/ < \widehat{\sigma} >$ is a smooth surface, and the projection morphism $\widehat{\varrho} : \widehat{S} \longrightarrow \widehat{W}$ is a flat double cover branched along the smooth reduced curve $\widehat{C} = \widehat{\varrho}(\widehat{R}) = \widehat{\varrho}_*(\widehat{R})$.

There exists a line bundle $\widehat{\Delta} \in \mathrm{Pic}(\widehat{W})$ such that $\widehat{C} \in |2\widehat{\Delta}|$. Then $\widehat{S}$ is isomorphic to the double cover of $\widehat{W}$ constructed in the total space of the line bundle $\widehat{\Delta}$: if $p : \widehat{\Delta} \longrightarrow \widehat{W}$ is the bundle projection, then

$$\widehat{S} = (p^*s - t^n = 0) \subset \widehat{\Delta}$$

where $t \in H^0(\widehat{\Delta}, p^*\widehat{\Delta})$ is the tautological section, and s is a section in $H^0(\widehat{W}, \mathcal{O}_{\widehat{\Delta}}(2\widehat{\Delta}))$ such that $\mathrm{div}(s) = \widehat{C}$. Since $\widehat{W}$ has a natural fibration over B, we can make a relatively minimal model $\pi : W \longrightarrow B$. If the genus h of the fibres of $\pi : W \longrightarrow B$ is ≥ 1, then W is unique.

If the genus h is equal to 0, then a relatively minimal model is not unique and we can move from a model to another via elementary transformations. We have a commutative diagram

$$\begin{tikzcd} \widehat{S} \arrow[r] \arrow[d, "\varrho"'] & \widehat{W} \arrow[d, "\psi"] \\ S \arrow[dr, "f"'] & W \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Let C be the direct image $\psi_*\widehat{C}$ of the branch locus $\widehat{C}$ in W. Then C is an even reduced divisor, i.e. $C = 2\mathsf{L}$, with L a line bundle on W.

Hence we have a double cover $S' \longrightarrow W$, with S' minimal, but not necessarily smooth. So S' is birational to S. By construction, S' is a divisor in a smooth 3–fold (the total space of the line bundle L),which is smooth over B, so $f' : S' \longrightarrow B$ admits an invertible relative dualizing sheaf, which is induced by $\omega_W + \mathsf{L}$. The singularities of S' can be resolved in a natural way performing the canonical resolution:

$$\begin{tikzcd} S_n \arrow[r, "\sigma_n"] \arrow[d] & S_{n-1} \arrow[r, "\sigma_{n-1}"] \arrow[d] & \cdots \arrow[r] & S_1 \arrow[r, "\sigma_1"] \arrow[d] & S_0 \arrow[d] \\ W_n \arrow[r, "\tau_n"'] & W_{n-1} \arrow[r, "\tau_{n-1}"'] & \cdots \arrow[r] & W_1 \arrow[r, "\tau_1"'] & W_0 \end{tikzcd}$$

such that the branch locus C_n of $S_n \longrightarrow W_n$ is smooth.

We know that each morphism $S_j \longrightarrow W_j$ is the double cover with branch locus $C_j := \tau_j^*(C_{j-1}) - 2[\frac{m_{j-1}}{2}]E_j$, where as usual E_j is the exceptional divisor of τ_j, m_{j-1}

is the multiplicity of the blown–up point. If we choose such n minimal, then we can prove that S_n is isomorphic to $\widehat{S}$. A proof of this fact can be find in [Bau] (see theorem 3.43).

Let $f_j : S_j \longrightarrow B$ and $f' : S' \longrightarrow B$ be the induced fibrations. We can calculate the invariants of $f' : S' \longrightarrow B$ and $f_j : S_j \longrightarrow B$:

$$(\omega_{f_n} \cdot \omega_{f_n}) = (\omega_{f'} \cdot \omega_{f'}) - 2\sum_{i=0}^{n} \left(\left[\frac{m_i}{2} \right] - 1 \right)^2$$

and

$$\deg(f_{n*}\omega_{f_n}) = \deg(f'_*\omega_{f'}) - \frac{1}{2}\sum_{i=0}^{n} \left[\frac{m_i}{2} \right] \left(\left[\frac{m_i}{2} \right] - 1 \right)$$

Suppose that the sequence $S_n \longrightarrow \ldots \longrightarrow S_1 \longrightarrow S'$ is minimal. Since S_n is smooth, $f : S \longrightarrow B$ is relatively minimal and the induced birational map $S_n = \widehat{S} \dashrightarrow S$ is a regular map. Therefore

$$(\omega_f \cdot \omega_f) = (\omega_{S|B})^2 = (\omega_{f_n} \cdot \omega_{f_n})^2 + \varepsilon$$

where ε is the number of blow–ups that make up $\varrho : \widehat{S} \longrightarrow S$. We get the following identity:

$$(\omega_{S|B})^2 = \frac{4(g-1)}{g} \deg(f_*\omega S|B) + \frac{2}{g}\sum_{i=0}^{n} \left(\left[\frac{m_i}{2} \right] - 1 \right) \left(g - \left[\frac{m_i}{2} \right] \right) + \varepsilon \tag{2.2}$$

Therefore, if $g = 2$, we get:

$$\omega^2_{S|B} = 2 \deg f_*\omega_{S|B} + \sum_{i=1}^{k} \left(\left[\frac{m_i}{2} \right] - 1 \right) \left(2 - \left[\frac{m_i}{2} \right] \right) + \varepsilon \tag{2.3}$$

Consider the even reduced divisor C as sum of irreducible vertical components and irreducible horizontal components, i.e.

$$C = C_v + C_h \tag{2.4}$$

where C_v is the sum of all irreducible components D of C such that $\pi(D) =$ point, while C_h is the sum of the irreducible components of C which go onto B.

Then it is possible to show that we can choose W such that the singularities of C_h are at most of order $g+1$ and C^2 is the smallest among all such choices. Therefore as C is reduced, the singularities of C are at most $g+2$, and if p is a singular point of order $g+2$, C contains the fibre of π passing through p (see [Xia2] for the details).

Then, in the case $g = 2$, we obtain that we can choose the ruled surface $W \xrightarrow{\pi} B$ such that, for all i, $([\frac{m_i}{2}]-1)(2-[\frac{m_i}{2}]) = 0$. Then $\omega^2_{S|B} = 2\deg(f_*\omega_{S|B})+\varepsilon$.

Equivalently,

$$K_S^2 = 2\deg(f_*\omega_{S|B}) + 8(b-1) + \varepsilon \tag{2.5}$$

where b is the genus of B.

From now on we consider fibrations $f : S \longrightarrow B$ with general fibre of genus 2. We have seen that, the genus formula,

$$2\pi(F) - 2 = \frac{F^2 + F \cdot K}{2} \tag{2.6}$$

implies that S has not multiple fibres, and so all the fibres are 1–connected.

We will consider the relative canonical algebra in order to give the structure theorem, proved by Catanese and Pignatelli in [CaPi] for fibrations of genus 2.

This approach uses the geometry of the bicanonical map of a 1–connected divisor of genus 2, which is a morphism generically of degree 2 onto a plane curve Q which may be reducible or nonreduced.

The above approach was that of Horikawa.

We saw that the ruled surface $\pi : W \longrightarrow B$ is not uniquely determined if $b := \text{genus}(B) \geq 1$. In case $b = 0$, Horikawa proved that W is canonically determined and is isomorphic to $\mathbb{P}(f_*\omega_{S|B})$ (cf. Hor2 th.1). The proof is based on the isomorphism (2) of that paper and on the assertum that for a sufficiently ample divisor L on B, we have

$$\begin{aligned}\mathbb{P} := \mathbb{P}(f_*\omega_S \otimes \mathsf{L}) \hookrightarrow \mathbb{P}(\omega_S + f * \mathsf{L}) &= \mathbb{P}(H^0(\omega_S + f^*\mathsf{L})) = \\ &= \mathbb{P}(H^0(f_*\omega_S + \mathsf{L})) = \\ &= \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))).\end{aligned}$$

In the approach of Catanese and Pignatelli, there is a unique birational model X of S, which admits a double cover $\psi : X \longrightarrow \mathcal{C}$, where $\mathcal{C}$ is a conic bundle over B, the branch divisor Δ has only simple singularities and X is the relative canonical model of f. X is obtained contracting the (-2)–curves D, (i.e. $K_{S|B} \cdot D = 0$) contained in the fibres to singularities which are then rational double points.

In order to have a better understanding of X, we consider the relative canonical algebra $R(f)$

$$R(f) := \bigoplus_{n=0}^{\infty} f_*\omega_{S|B}^{\otimes n} = \bigoplus_{n=0}^{\infty} \mathsf{V}_n \tag{2.7}$$

where we have put $\mathsf{V}_n = f_*\omega_{S|B}^{\otimes n}$. By base change, its stalk at $p \in B$ is an $\mathcal{O}_{B,p}$–algebra whose reduction modulo $\mathfrak{m}_p$ is the canonical K–algebra

$$R(F_p) = \bigoplus_{n=0}^{\infty} H^0(F_p, \omega_{F_p}^{\otimes n}) \tag{2.8}$$

where $F_p = f^{-1}(p)$ is the scheme theoretic fibre of f and $\omega_{F_p} = \omega_{S|B}|_{F_p}$.

We have natural homomorphism induced by multiplication:

$$\mu_{m,n} : \mathsf{V}_m \otimes \mathsf{V}_n \longrightarrow \mathsf{V}_{m+n} \tag{2.9}$$

and

$$\sigma_n : S^n(\mathsf{V}_1) := \mathrm{Sym}^n(\mathsf{V}_1) = S^n(f_*\omega_{S|B}) \longrightarrow \mathsf{V}_n = f_*\omega_{S|B}^{\otimes n} \tag{2.10}$$

If there are no multiple fibres, the relative canonical algebra is generated by elements of degree ≤ 3. Since for $g = 2$, there are no multiple fibres, the canonical algebra $R(f)$ is generated in degree ≤ 3. The hyperelliptic involution $\sigma : S \longrightarrow S$

acts linearly on the space of sections $\Gamma(U, \omega_{S|B}^{\otimes})$, where U is open and σ–invariant. Then $\Gamma(U, \omega_{S|B}^{\otimes n})$ splits as the direct sum of the invariant and the antinvariant spaces of sections. We obtain the decomposition:

$$\mathsf{V}_n = \mathsf{V}_n^+ \oplus \mathsf{V}_n^- = f_*(\omega_{S|B}^{\otimes n})^+ \oplus f_*(\omega_{S|B}^{\otimes n})^- \tag{2.11}$$

Therefore we get that the canonical algebra splits as

$$R(f) = R(f)^+ \oplus R(f)^- \tag{2.12}$$

where $R(f)^+ = \bigoplus_{n-1}^{\infty} \mathsf{V}_n^+$, $R(f)^- = \bigoplus_{n-1}^{\infty} \mathsf{V}_n^-$.

Since genus$(F) = g = 2$, we have:

$$\mathsf{V}_1^- = \mathsf{V}_1, \quad \mathsf{V}_1^+ = (0) \tag{2.13}$$

and the sheaf homomorphisms σ_n are injective. In particular, for $n = 2$, we get the important sheaf exact sequence:

$$0 \longrightarrow \mathrm{S}^2\,\mathsf{V}_1 \longrightarrow \mathsf{V}_2 \longrightarrow \mathcal{T}_2 \longrightarrow 0$$

where $\mathcal{T}_2 := \operatorname{coker} \sigma_2$.

Now we want to give another proof of the formula

Proposition 2.2.1.

$$\begin{aligned}\omega_S^2 &= 2\chi(\mathcal{O}_S) - 6\chi(\mathcal{O}_B) + \text{lenght}(\operatorname{coker}(S^2 V_1 \xrightarrow{\sigma_2} V_2)) \\ &= 2 \deg f_*\omega_{S|B} + \deg \mathcal{T}_2 \ (:= \text{lenght}(\operatorname{coker}(S^2 V_1 \xrightarrow{\sigma_2} V_2)).\end{aligned}$$

Proof.

Since $S^2\mathsf{V}_1 \xrightarrow{\sigma_2} \mathsf{V}_2$ is injective, we have

$$\deg \mathcal{T}_2 = \chi(\mathsf{V}_2) - \chi(\mathrm{S}^2\,\mathsf{V}_1).$$

We have

$$\chi(B, \mathsf{V}_2) - \chi(B, \mathrm{S}^2\,\mathsf{V}_1) = \deg \mathcal{T}_2$$

By Riemann–Roch on B:

$$\chi(B, \mathrm{S}^2\,\mathsf{V}_1) = \deg \mathrm{S}^2\,\mathsf{V}_1 - 2K(\mathrm{S}^2\,\mathsf{V}_1)(b-1) == 3\deg \mathsf{V}_1 - 3(b-1) == 3\deg \mathsf{V}_1 + 3\chi(\mathcal{O}_B) =$$

Now, by using the Leray's spectral sequence, we have

$$\begin{aligned}\chi(\mathsf{V}_1) &= h^0(f_*\omega_{S|B}) - h^1(f_*\omega_{S|B}) = \\ &= h^0(f_*\omega_{S|B} - [h^1(\omega_{S|B}) - h^0(R^1 f_*\omega_{S|B})] = \\ &= h^0(\omega_{S|B}) - h^1(\omega_{S|B}) + h^2(\omega_{S|B}) - h^2(\omega_{S|B}) + h^0(\mathcal{O}_B),\end{aligned}$$

since $\mathcal{O}_B = R^1 f_*\omega_{S|B}$.

Then, by Riemann–Roch on S, we have

$$\begin{aligned}\chi(\mathsf{V}_1) &= \chi(\omega_{S|B}) + 1 - b = \\ &= \chi(\mathcal{O}_S) + \frac{1}{2}(K_S - f^*K_B)(f^*K_B) + \chi(\mathcal{O}_B) = \\ &= \chi(\mathcal{O}_S) + 2\chi(\mathcal{O}_B) + \chi(\mathcal{O}_B) = \\ &= \chi(\mathcal{O}_S) + 3\chi(\mathcal{O}_B).\end{aligned}$$

Similarly

$$\chi(\mathsf{V}_2) = \chi(\mathcal{O}_S) + K_S^2 + 12\chi(\mathcal{O}_B)$$

Then

$$\begin{aligned}\chi(\mathsf{V}_2) - \chi(\mathrm{S}^2\,\mathsf{V}_1) &= \chi(\mathcal{O}_S) + K_S^2 + 12\chi(\mathcal{O}_B) - 3\deg \mathsf{V}_1 - 3\chi(\mathcal{O}_B) = \\ &= K_S^2 + \chi(\mathcal{O}_S) + 9\chi(\mathcal{O}_B) - 3\deg \mathsf{V}_1.\end{aligned}$$

By Riemann–Roch on B we get

$$\chi(\mathsf{V}_1) = \deg \mathsf{V}_1 + 2(1-b) = \deg \mathsf{V}_1 + 2\chi(\mathcal{O}_B).$$

Then

$$\begin{aligned}\deg \mathsf{V}_1 &= \chi(\mathsf{V}_1) - 2\chi(\mathcal{O}_B) = \\ &= \chi(\mathcal{O}_S) + 3\chi(\mathcal{O}_B) - 2\chi(\mathcal{O}_B) = \\ &= \chi(\mathcal{O}_S) + \chi(\mathcal{O}_B).\end{aligned}$$

In conclusion

$$\chi(\mathsf{V}_2) - \chi(\mathrm{S}^2\,\mathsf{V}_1) = K_S^2 + \chi(\mathcal{O}_S) + 9\chi(\mathcal{O}_B) - 3(\chi(\mathcal{O}_S) + \chi(\mathcal{O}_B)) =$$
$$= K_S^2 - 2\chi(\mathcal{O}_S) + 6\chi(\mathcal{O}_B),$$

so

$$K_S^2 = 2\chi(\mathcal{O}_S) - 6\chi(\mathcal{O}_B) + \deg \mathcal{T}_2$$

□

In their paper Catanese and Pignatelli use the graded canonical ring, $R(f) = \bigoplus_{n\geq 0} H^0(F, \omega_F^{\otimes n})$ of a curve F of genus 2. We now recall this result (see [Men]).

Theorem 2.2.2. *Let F be a fibre of a genus 2 fibration $f : S \longrightarrow B$. Then either F is honestly hyperelliptic, i.e. the graded ring $R(f)$ is isomorphic to*

$$\mathbb{C}[x_0, x_1, z]/(z^2 - g_6(x_0, x_1)) \tag{2.14}$$

where $\deg x_0 = \deg x_1 = 1$, $\deg z = 3$, $\deg g_6 = 6$, *or the fibre F is not 2–connected and the graded ring $R(f)$ is isomorphic to*

$$\mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6) \tag{2.15}$$

where $\deg x_0 = \deg x_1 = 1$, $\deg y = 2$, $\deg z = 3$ *and*

$$Q_2 := x_0^2 - \lambda x_0 x_1$$
$$Q_6 := z^2 - y^3 - x_1^2(\alpha_0 y^2 + \alpha_1 x_1^4)$$

The first case is the one where the fibres are 2–connected

Using this result, they prove that the sheaf $\mathcal{T}_2 := \operatorname{coker}(\sigma_2 : \mathrm{S}^2\,\mathsf{V}_1 \longrightarrow \mathsf{V}_2)$ is isomorphic to the structure sheaf of an effective divisor $\mathcal{T}$ on B, supported on the points of B corresponding to the fibres of $f : S \longrightarrow B$ which are not 2–connected.

Consider now the exact sequence

$$0 \longrightarrow \mathrm{S}^2\,\mathsf{V}_1 \longrightarrow \mathsf{V}_2 \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow 0$$

We have the following natural map induced by σ_2:

$$q : \mathbb{P}(\mathsf{V}_2) \dashrightarrow \mathbb{P}(\mathrm{S}^2\,\mathsf{V}_1)$$

which is birational, and the Veronese embedding

$$\nu_2 : \mathbb{P}(\mathsf{V}_1) \hookrightarrow \mathbb{P}(\mathrm{S}^2\,\mathsf{V}_1).$$

Then the composition

$$\nu := q^{-1} \circ \nu_2 : \mathbb{P}(\mathsf{V}_1) \hookrightarrow \mathbb{P}(\mathsf{V}_2)$$

can be considered as the relative 2–Veronese map.

If we consider the pluricanonical relative maps

$$\varphi_1 : S \dashrightarrow \mathbb{P}(f_*\omega_{S|B}) = \mathbb{P}(\mathsf{V}_1)$$

$$\varphi_2 : S \dashrightarrow \mathbb{P}(f_*\omega_{S|B}) = \mathbb{P}(\mathsf{V}_2)$$

we have that φ_1 is a rational map generically of degree 2, since F is hyperelliptic, while φ_2 is a morphism of degree 2, since every fibre F is 1–connected and then $|\omega_F^{\otimes 2}|$ is a free linear system.

The diagram

$$\begin{tikzcd} S \arrow[rr, "\varphi_2"] \arrow[dr, dashed, "\varphi_1"'] & & \mathbb{P}(\mathsf{V}_2) \\ & \mathbb{P}(\mathsf{V}_1) \arrow[ur, dashed, "\nu"'] & \end{tikzcd}$$

is commutative, i.e. $\nu \circ \varphi_1 = \varphi_2$ as rational maps. The image of φ_2 is a conic bundle $\mathcal{C}$ over B.

The structure theorem of Catanese and Pignatelli proves that to reconstruct the pair (S, f) one only needs to know σ_2, which gives at once the conic bundle $\mathcal{C}$ and the isolated branch points of φ_2, and the divisorial part Δ of the branch locus of φ_2.

Furthermore, it gives a concrete recipe to construct all possible pairs (σ_2, Δ).

We now introduce the five fundamental ingredients $(B, \mathsf{V}_1, \mathcal{T}, \xi, w)$. Their order is important since each ingredient is given in a space which depends on the previous introduced ingredients:

1. B, any smooth curve;

2. V_1, any rank 2 vector bundle over B;

3. $\mathcal{T}$, any effective divisor on B;

4. ξ, any extension class

$$\xi \in \operatorname{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_{\mathcal{T}}, \mathrm{S}^2(\mathsf{V}_1)/\operatorname{Aut}_{\mathcal{O}_B}(\mathcal{O}_{\mathcal{T}})$$

   such that the extension V_2 given by ξ,

$$0 \longrightarrow \mathrm{S}^2\,\mathsf{V}_1 \longrightarrow \mathsf{V}_2 \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow 0$$

   is a vector bundle;

5. w, a non trivial element of

$$\operatorname{Hom}((\det \mathsf{V}_1 \otimes \mathcal{O}_B(\mathcal{T}))^2, \mathcal{A}_6)/\mathbb{C}^*,$$

   where $\mathcal{A}_6$ is a vector bundle determined by ξ in the following way:

   let $\sigma_2 : \mathrm{S}^2\,\mathsf{V}_1 \longrightarrow \mathsf{V}_2$ be an injective homomorphism whose cokernel is $\mathcal{O}_{\mathcal{T}}$. The $\mathcal{A}_6$ is a vector bundle

$$(\operatorname{coker} L_3) \otimes (\det \mathsf{V}_1 \otimes \mathcal{O}_B(\mathcal{T}))^{-2} \tag{2.16}$$

where the map $\mathsf{L}_3 : (\det \mathsf{V}_1)^2 \otimes \mathsf{V}_2 \longrightarrow \mathrm{S}^3(\mathsf{V}_2)$ is the one induced by σ_2 as follows. Consider the map η in the natural exact sequence

$$0 \longrightarrow \det(\mathsf{V}_1)^2 \xrightarrow{\eta} \mathrm{S}^2(\mathrm{S}^2(\mathsf{V}_1)) \longrightarrow S^4(\mathsf{V}_1) \longrightarrow 0$$

given locally, if x_0 and x_1 are locally generators of V_1, by

$$\eta((x_0 \wedge x_1)^{\otimes 2}) = (x_0)^2(x_1)^2 - (x_0x_1)^2.$$

$\mathcal{A}_6$ is then the cokernel of the composition of the maps

$$\det(\mathsf{V}_1)^2 \overset{\eta\otimes \mathrm{id}_{\mathsf{V}_2}}{\longrightarrow} \mathrm{S}^2(\mathrm{S}^2(\mathsf{V}_1)) \otimes \mathsf{V}_2 \overset{\mathrm{S}^2(\sigma_2)\otimes \mathrm{id}_{\mathsf{V}_2}}{\longrightarrow} \mathrm{S}^2(\mathsf{V}_2) \otimes \mathsf{V}_2 \xrightarrow{\mu_{2,1}} \mathrm{S}^3(\mathsf{V}_2)$$

Putting L_3 for $(\mu_{2,1})\circ(\mathrm{S}^2(\sigma_2) \otimes \mathrm{id}_{\mathsf{V}_2})\circ(\eta \otimes \mathrm{id}_{\mathsf{V}_2})$, we obtain that $\mathcal{A}_6$ fits in the following exact sequence:

$$0 \longrightarrow \det(\mathsf{V}_1)^2 \otimes \mathsf{V}_2 \xrightarrow{\mathsf{L}_3} \mathrm{S}^3(\mathsf{V}_2) \longrightarrow \mathcal{A}_6 \longrightarrow 0$$

These five ingredients is required to satisfy some open conditions:

(i) The conic bundle $\mathcal{C}$ coming from the first 4 ingredients, has only Rational Double Points as singularities;

(ii) Let Δ be the divisor defined by w in $\mathcal{C}$. Then Δ has only simple singularities.

Now the map σ_2 on the points of Supp$(\mathcal{T})$ defines a rank 2 matrix, whose image defines a pencil of lines in the corresponding $\mathbb{P}^2$, thus having a base point. Denote by $\mathcal{P}$ the union of such base points;

(iii) Then we impose that Δ does not contains any point of the set $\mathcal{P}$.

If the 5–tuple $(B, \mathsf{V}_1, \mathcal{T}, \xi, w)$ satisfies the above conditions, (i), (ii) and (iii), we say that it is an admissible genus two 5–tuple.

Then the structure theorem they obtain (for genus 2 fibration) is the following:

Theorem 2.2.3. *Let $f : S \longrightarrow B$ be a relatively minimal genus two fibration. Then the associated 5–tuple $(B, V_1 := f_*\omega_{S|B}, \mathcal{T}, \xi, w)$ is admissible.*

Vice versa every admissible genus two 5–tuple $(B, V_1, \mathcal{T}, \xi, w)$ determines a sheaf of algebras $\mathcal{R}$ over B whose relative projective spectrum X is the relative canonical model of a relatively minimal genus two fibration $f : S \longrightarrow B$ having the above as associated 5–tuple.

Moreover, the surface S has the following invariants:

$$\begin{aligned} \chi(\mathcal{O}_S) &= \deg V_1 + (b-1), \\ K_S^2 &= 2\deg V_1 + 8(b-1) + \deg(\mathcal{T}). \end{aligned} \tag{2.17}$$

2.3 Ruled Surfaces

In this section we will recall basic facts about ruled surfaces.

A surfaces is birationally ruled if it is birationally isomorphic to $\mathbb{C} \times \mathbb{P}^1$, where C is a smooth curve.

A (geometrically) ruled surface is a surface S, together with a smooth surjective morphism $\pi : S \longrightarrow C$ to a smooth curve C such that the fibre F_x is isomorphic to $\mathbb{P}^1$, for every point $x \in C$.

It is a classical result of Noether and Enriques that $\pi : S \longrightarrow C$ is a $\mathbb{P}^1$–bundle over C.

Theorem 2.3.1. (Noether–Enriques) *Suppose $\pi : S \longrightarrow C$ is a smooth surjective map such that $F_x := \pi^{-1}(x) \cong \mathbb{P}^1$, for every $x \in C$. Then, for any $x \in C$, there exists a Zariski open set $U \subset C$, containing x, and a commutative diagram:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^1 \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

So by the Noether–Enriques theorem, a geometrically ruled surface is locally trivial in the Zariski topology.

Thus these projective bundles are classified by

$$H^1(C, \mathcal{PGL}(2,\mathbb{C}))$$

where $\mathcal{PGL}(2,\mathbb{C})$ is the sheaf of germs of regular maps from C into $\mathrm{PGL}(2,\mathbb{C})$.

Since C is a curve ($H^2(C, \mathcal{O}_C) = 0$), we have an exact sequence of cohomology sets:

$$\mathrm{Pic}(C) \xrightarrow{\sigma} H^1(C, \mathcal{GL}(2,\mathbb{C}) \xrightarrow{\varrho} H^1(C, \mathcal{PGL}(2,\mathbb{C}) \longrightarrow 0$$

Now, $H^1(C, \mathcal{GL}(2,\mathbb{C})$ parametrizes rank 2 vector bundles on C, while $H^1(C, \mathcal{PGL}(2,\mathbb{C}))$ parametrizes $\mathbb{P}^1$–bundles over C.

The above exact sequence says that $S \xrightarrow{\pi} C$ is isomorphic to $\mathbb{P}_C(\mathcal{E}) := \mathrm{Proj} \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(\mathcal{E})$ for some rank 2 locally free sheaf (vector bundle) $\mathcal{E}$ over C. The bundles $\mathbb{P}_C(\mathcal{E})$ and $\mathbb{P}_C(\mathcal{E}')$ are isomorphic over C if and only if there exists an invertible sheaf (line bundle) $\mathcal{L}$ on C such that

$$\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}.$$

From the trivialization of $\pi : S \longrightarrow C$ over an open Zariski set $U \subset C$:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^1 \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

we get a rational section $s : U \longrightarrow S$, i.e. $\pi \circ s = \mathrm{id}_U$, but since C is a smooth complete curve, s extends to a regular map from C to S, which is necessarily a section.

Let $D := s(C) \subset S$ be the image of s. Then D is a divisor on S and $D \cdot F_x = 1$ for every fibre F_x of π. This implies (using base change) that

$$\mathcal{E} := \pi_* \mathcal{O}_S(D)$$

is a locally free sheaf of rank $2 = h^0(F_x, \mathcal{O}_{F_x}(1))$. The surface S is isomorphic just to $\mathbb{P}_C(\mathcal{E})$ over C.

Proposition 2.3.2. *Let $\pi : S \longrightarrow B$ be a ruled surface, let $D \subset S$ be a section and let F be a fibre. Then*

$$\operatorname{Pic}(S) \cong \pi^* \operatorname{Pic}(C) \oplus \mathbb{Z},$$

where $\mathbb{Z}$ is generated by the class of D. Also

$$\operatorname{Num} S \cong \mathbb{Z} \oplus \mathbb{Z}$$

with D and F as generators, $D \cdot F = 1$, $F^2 = 0$ and $D^2 = \deg \mathcal{E}$.

Let $\pi : S \longrightarrow C$ be a ruled surface. Then it is possible to write

$$S \cong \mathbb{P}_C(\mathcal{E})$$

where $\mathcal{E}$ is a locally free sheaf on C with the property that $H^0(\mathcal{E}) \neq 0$, but for all $\mathcal{L} \in \operatorname{Pic}(C)$ with $\deg \mathcal{L} < 0$, we have $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ In this case the degree $-e := \deg \mathcal{E}$ of $\mathcal{E}$ is an invariant of S.

Furthermore in this case there is a section $\sigma_0 : C \longrightarrow S$ with image C_0 such that

$$\mathcal{O}_S(C_0) \cong \mathcal{O}_{\mathbb{P}(E)}(1) (= \mathcal{O}_S(1)) \tag{2.18}$$

where $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the Serre tautological sheaf on $\mathbb{P}(E)$.

If $\mathcal{E}$ has the above properties, we say that $\mathcal{E}$ is normalized.

We put $\mathfrak{e} := \bigwedge^2 \mathcal{E}$ as divisor on C, so that $e := -\deg \mathfrak{e}$.

Lemma 2.3.3. *Let $S = \mathbb{P}_C(\mathcal{E})$, with $\mathcal{E}$ normalized. Then the canonical divisor K_S of S is given by*

$$K_S \equiv -2C_0 + \pi^*(K_C + \mathfrak{e}) \tag{2.19}$$

where K_C is the canonical divisor on C.

For numerical equivalence, we have

$$K_S \cong -2C_0 + (2g - 2 - e)\mathfrak{f} \tag{2.20}$$

where g is the genus of C and $\mathfrak{f}$ is the numerical class of the fibres. In particular

$$K_S^2 = 8(1 - g) \tag{2.21}$$

Remark 2.3.4. *If $C = \mathbb{P}^1$, then by a theorem of Grothendieck every vector bundle over B is isomorphic to a direct sum of line bundles. So in this case every ruled surface over $\mathbb{P}^1$ is of the form $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. If we choose $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ normalized, then necessarily $n \leq 0$.*

With regarg to the possible values of e, we have the following theorem:

Theorem 2.3.5. *Let S be a ruled surface over the curve C of genus g, determined by a normalized locally free sheaf $\mathcal{E}$. Then;*

(a) If $\mathcal{E}$ is decomposable, i.e. $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$, with $\mathcal{L}_1$, $\mathcal{L}_2 \in \operatorname{Pic}(C)$, then $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic}(C)$, with $\deg \mathcal{L} \leq 0$. Therefore $e \geq 0$. All values $e \geq 0$ are possible.

(b) If $\mathcal{E}$ is indecomposable, then

$$-g \leq e \leq 2g - 2 \tag{2.22}$$

([[Har]], V.2)

Remark 2.3.6. *If $\mathcal{E}$ is indecomposable, to the section $C_0 \hookrightarrow S$ corresponds a nontrivial extension of vector bundles:*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{2.23}$$

for some $\mathcal{L} \in \operatorname{Pic}(C)$. It corresponds to a nonzero element

$$\xi \in \operatorname{Ext}^1(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^\vee)$$

In particular $H^1(C, \mathcal{L}^\vee) \neq 0$. If $g = 1$, $H^1(C, \mathcal{L}^\vee) \cong H^0(C, \mathcal{L})^\vee$ since $\omega_C = \mathcal{O}_C$. Now $H^0(C, \mathcal{L})^\vee \neq 0$ implies that $\deg \mathcal{L} \geq 0$, and if $\deg \mathcal{L} = 0$, $\mathcal{L}$ is not of nontrivial torsion.

Theorem 2.3.7. *Let C be an elliptic curve and let S be a ruled surface on C corresponding to an indecomposable (normalized) sheaf $\mathcal{E}$. Then $e = 0$ or $e = -1$, and there is exactly one such ruled surface for each of these two values of e. Precisely, for $e = 0$, $\mathcal{E}$ is given by a nontrivial extension*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}(2,0) \longrightarrow \mathcal{O}_C \longrightarrow 0$$

For $e = -1$, $\mathcal{E}$ is given by a nontrivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}_u(2,0) \longrightarrow \mathcal{O}_C(u) \longrightarrow 0$$

where u is a point on C.

In general, given a point u over an elliptic curve C and integers r, d, with $r > 0$, $(r, d) = 1$, Atiyah in [[Ati]] proved that there exists a unique indecomposable vector bundle $E_u(r, d)$ of rank r on C with

$$\det E_u(r, d) = \mathcal{O}_C(u)^{\otimes d} = \mathcal{O}_C(d \cdot u).$$

In the same paper, Atiyah proved that there exists a unique indecomposable vector bundle $E(r,0)$ over C of rank r and degree 0 with $H^0(E(r,0)) \neq 0$. Moreover there is an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}(r,0) \longrightarrow \mathcal{E}(r-1,0) \longrightarrow 0 \tag{2.24}$$

Furthermore if $\mathcal{E}$ is an indecomposable vector bundle of degree 0 and rank r, we have

$$E \cong \mathcal{L} \otimes E(r,0) \tag{2.25}$$

where $\mathcal{L} \in \operatorname{Pic}(C)$ with $\deg \mathcal{L} = 0$, unique up to isomorphism and such that

$$\mathcal{L} \cong \det E \tag{2.26}$$

Remark 2.3.8. *Suppose that C is an elliptic curve. Then the symmetric product $S^n(C)$ of C is isomorphic as $\mathbb{P}^{n-1}$–bundle over C to the projective bundle $\mathbb{P}(E_u(n, n-1))$.*

Chapter 3

Surfaces with $p_g = 2$, $q = 1$ and $K^2 = 5$

3.1 The Invariant e

In this chapter we consider the fibration

$$f : S \longrightarrow B$$

where $b := \text{genus}(B) = 1$, $K_S^2 = 5$, $q = b = 1$, induced by the Albanese map of S (see [Hor3]).

To f we associate the following diagram

$$\begin{tikzcd} S \arrow[r, dashed, "\psi"] \arrow[dr, "f"'] & \mathbf{Proj}(f_*\omega_S) \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

We have that

1. Every fibre F of f is 1–connected (since f has not multiple fibres);

2. ψ is a morphism on any 2–connected fibre;

3. There is one 2–disconnected fibre, since

$$\deg \mathcal{T} = K_S^2 - 2\deg E + 8(1-b) = 1$$

where $E := f_*\omega_{S|B} = f_*\omega_S$, and $\mathcal{T}$ is defined by

$$0 \longrightarrow S^2 f_*\omega_{S|B} \longrightarrow f_*\omega_{S|B}^{\otimes 2} \longrightarrow \mathcal{T} \longrightarrow 0$$

Consider the 'Horikawa diagram'

$$\begin{tikzcd} & S^* \arrow[dl, "\varrho"'] \arrow[dr, "h"] & \\ \widehat{S} \arrow[d, "\sigma"'] & & \widetilde{S} \arrow[d, "g"] \\ S \arrow[rr, dashed, "\psi"'] & & W := \mathbf{Proj}(f_*\omega_S) \end{tikzcd}$$

where $\widehat{S} \overset{\sigma}{\dashrightarrow} S$ is the resolution of indeterminates of ψ, S^* ($\cong \widehat{S}$, as proved in [Bau]) is the Horikawa resolution of the branch locus B of ψ (or equivalently of $\psi\circ\sigma$). The branch locus is linearly equivalently to

$$B \equiv 6D - b\Gamma \qquad F = B/2$$

$$\chi(W) = 1 - q - p_g = 0$$

and

$$K_W \equiv -2D + 2F$$

Now $2 = \chi = \frac{1}{3}(3D - \frac{b}{2}F)(D + \frac{4-b}{2}\Gamma)$?

$$\Rightarrow b = 2, \text{ and so } B \equiv 6D - 2\Gamma$$

The 2–disconnected fibre F is of type I, III_1, V in the classification of Horikawa.

Consider again the rank 2–bundle $E = f_*\omega_{S|B}$. We have $\mathrm{rank}(E) = 2$, $\deg(E) = 2$.

Let $E_1 \subset E$ be the line subbundle of maximal degree.

$E_2 := E/E_1$ is torsion free, so E_2 is a line bundle. For this consider the following exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \xrightarrow{\pi} E_2 \oplus T \longrightarrow 0$$

We have $E_1 \subset \pi^{-1}(T)$. If $E_1 \subsetneq \pi^{-1}(T)$, then $\deg(E_1) < \deg(T)$, absurd. Then $T = 0$.

Let $e := \deg(E_1) - \deg(E_2)$. Fujita's theorem $\Rightarrow \deg(E_2) \geq 0$.

By a theorem of Xiao (see [Xia1, p. 71] we have that either $e = 0$ or $e = 2$.

3.1.1 The Case $e = 2$

In this case we get that:

$$\deg(E_1) = 2,\ \deg(E_2) = 0$$

Considering the long cohomology sequence associated to the exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \xrightarrow{\pi} E_2 \longrightarrow 0,$$

i.e.

$$0 \longrightarrow H^0(E_1) \longrightarrow H^0(E) \longrightarrow H^0(E_2) \longrightarrow H^1(E_1) \longrightarrow \dots$$

We have $H^1(E_1) = H^0(E_1^\vee)^\vee = 0$ because $E_1^\vee$ has degree -2. Now $h^0(E_1) = 2 = h^0(E)$, then $H^0(E_2) = 0$, i.e. E_2 is a torsion line bundle, $E_2 = \mathcal{O}_B(\eta)$.

Since $f_*H^0(E_1) = H^0(\omega_S)$, we get that the canonical map factors through f,

$$\varphi_{K_S} = \varphi_{|E_1|} \circ f.$$

Now $|E_1| = g_2^1$ without base points.

Then φ_{K_S} is a morphism, i.e.

$$|K_S| = |M| + Z$$

where $M = \mathsf{F}_1 + \mathsf{F}_2$, Z is the fixed component and $M^2 = 0$.

Thus we have

$$\mathrm{Ext}^1(E_2, E_1) = H^0(E_2 \otimes E_1^{\vee})^{\vee} = 0$$

so

$$f_*\omega_S = \mathcal{O}_B(P+Q) \oplus \mathcal{O}_B(\eta),\ \text{with } \eta \not\equiv 0,\ \eta^k \equiv 0,\ k \geq 2 \text{ and } P+Q \in |E_1|$$

The Horikawa diagram becomes

$$\begin{tikzcd} & S^* \arrow[dl, "\varrho"'] \arrow[dr, "h"] & \\ \widehat{S} \arrow[d, "\sigma"'] & & \widetilde{S} \arrow[d, "g"] \\ S \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(\mathcal{O}(P+Q)) \ \oplus \mathcal{O}(\eta =: \mathbb{P} \end{tikzcd}$$

We have $K_{\widehat{S}}^2 = K_{S^*} = 4$, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\widehat{S}}) = 2$, $K_{\mathbb{P}} = -2H + 2\Gamma$, and since $H^2 = \deg E = 2$, $K_{\mathbb{P}}^2 = 4H^2 - 8 = 0$.

Moreover $\chi(\mathcal{O}_{\mathbb{P}}) = 1 - q + p_g = 0$

$m_2 = 4 \Rightarrow [\frac{m_2}{2}] = 2$ and $m_1 = 4 \Rightarrow [\frac{m_1}{2}] = 2$.

We get

$$2 = \chi(\mathcal{O}_{\widehat{S}}) = 2\chi(\mathcal{O}_{\mathbb{P}}) + \frac{1}{2}(3H + b\Gamma)(-2H + 2\Gamma) + \frac{1}{2}(BH + b\Gamma)^2 - 2 = 2b + 4 \Rightarrow b = 1.$$

The branch locus is algebraically equivalent to $6H - 3\Gamma$.

$B = B^* + \Gamma_o \Rightarrow B^* \equiv 6H - 3\Gamma$.

B^* has 2 ordinary triple points.

$$\begin{aligned}
H^0(6H-3\Gamma_o) &= H^0(\mathrm{Sym}^6(\mathcal{O}(P+Q)\oplus\mathcal{O}(\eta))\otimes\mathcal{O}(-3o)) = \\
&= H^0(\mathcal{O}(P+Q)^6\otimes\mathcal{O}(-3o)\oplus H^0(\mathcal{O}(P+Q)^5\otimes\mathcal{O}(\eta-3o) \\
&\oplus H^0(\mathcal{O}(P+Q)^4\otimes\mathcal{O}(2\eta-3o)\oplus H^0(\mathcal{O}(P+Q)^3\otimes\mathcal{O}(3\eta-3o) \\
&\oplus H^0(\mathcal{O}(P+Q)^2\otimes\mathcal{O}(4\eta-3o) \\
&\oplus H^0(\mathcal{O}(P+Q)\otimes\mathcal{O}(5\eta-3o)\otimes H^0(6\eta-3o).
\end{aligned}$$

The dimensions of each terms of such decomposition are

$$9,\ 7,\ 5,\ 3,\ 1,\ 0,\ 0$$

So $h^0(6H-3\Gamma_o)=9+7+5+3+1+0+0=25$.

The coordinates along the fibre Γ of $\mathbb{P}$ are

$$\begin{aligned}
x_0 &\in H^0(\mathcal{O}(1)\otimes\pi^*\mathcal{O}(-P-Q) \\
x_1 &\in H^0(\mathcal{O}(1)\otimes\pi^*\mathcal{O}(-\eta)
\end{aligned}$$

with $x_0^i x_1^j \in H^0(\mathcal{O}(6)\otimes\mathcal{O}(-iP-iq-j\eta))$ if $i+j=6$.

$$\sum \psi_{ij}x_0^i x_1^j \in H^0(\mathcal{O}(6)\otimes\pi^*\mathcal{O}(-3o)) \Rightarrow \psi_{ij}\in H^0(\mathcal{O}(ip+j+iq-3o-3o))$$

$$h^0(ip+iq+j\eta-3o)=2i-3 \tag{3.1}$$

Obviously $2i-3>0 \Leftrightarrow i\geq 2$. Then $\psi_{06},\ \psi_{15}=0$.

$C\in|6H-o|$. Equation of C:

$$\psi_{60}x_0^6+\psi_{51}x_0^5x_1+\psi_{42}x_0^4x_1^2+\psi_{33}x_0^3x_1^3+\psi_{24}x_0^2x_1^4 = x_0^2Q_4(x_0,x_1). \tag{3.2}$$

The branch locus should have at least a double component: this is a contradiction.

$$D = \mathrm{div}(x_0) \equiv H - 2\Gamma,\ \ B^{\#} \cdot D = -3 \Rightarrow B^{\#} = (B^{\#})' + DB^{\#} \cdot D = -1 \Rightarrow$$

$$\Rightarrow B^{\#} = (B^{\#})'' + 2D$$

Corollary 3.1.1. *The case $e = 2$ does not occur.*

3.1.2 The Case $e = 0$

By the previous subsection, we have that $e = 0$. Thus we have

$$\deg(E_1) = 1 = \deg(E_2)$$

so

$$E_1 = \mathcal{O}(P),\, E_2 = \mathcal{O}(Q)$$

If $P \neq Q$, then

$$\mathrm{Ext}^1(\mathcal{O}(Q), \mathcal{O}(P)) = H^0(\mathcal{O}(Q - P)) = 0$$

Otherwise if $P = Q$

$$\mathrm{Ext}^1(\mathcal{O}(P), \mathcal{O}(P)) = \mathbb{C}.$$

By tensoring the following exact sequence:

$$0 \longrightarrow \mathcal{O}(P) \longrightarrow f_*\omega_S \longrightarrow \mathcal{O}(P) \longrightarrow 0$$

with $\mathcal{O}(-P)$, we obtain:

$$0 \longrightarrow \mathcal{O}_B \longrightarrow f_*\omega_S(-P) \longrightarrow \mathcal{O}_B \longrightarrow 0$$

Then $f_*\omega_S(-P)$ is normalized.

If $f_*\omega_S(-P)$ is decomposable, then

$$f_*\omega_S(-P) = \mathcal{O}_B \oplus \mathcal{O}_B(-\eta).$$

But $\bigwedge^2(f_*\omega_S(-P)) = \mathcal{O}_B \Rightarrow \eta \equiv 0$, i.e.

$$f_*\omega_S(-P) = \mathcal{O}_B^2$$

Note that $H^0(f_*\omega_S \otimes \mathcal{O}(-P)) = H^0(K_S - \mathsf{F}_P) = \mathbb{C}^2 \Rightarrow |K_S| = |M| + \mathsf{F}_P$.

If $f_*\omega_S(-P)$ is indecomposable, then

$$f_*\omega_S(-P) = \mathsf{F}_2.$$

Choose coordinates

$$x_0 \in H^0(\mathcal{O}(1) \otimes \pi^*\mathcal{O}(-P))$$
$$x_1 \in H^0(\mathcal{O}(1) \otimes \pi^*\mathcal{O}(-Q))$$

We have

$$H^0(\mathcal{O}(1) \otimes \pi^*\mathcal{O}(-P)) = H^0(\mathcal{O}_B) \oplus H^0(\mathcal{O}_B(Q-P))$$

$$h^0(H - \Gamma_P) = \begin{cases} 1 & \text{if } P \neq Q \\ 2 & \text{if } P = Q \end{cases} \tag{3.3}$$

If $P = Q$ we can choose x_0, x_1 independent sections. Then

$x_0^6 \in H^0(6H - 6\Gamma_P)$;

$x_0^5 x_1 \in H^0(6H - 5\Gamma_P - \Gamma_Q)$;

$x_0^4 x_1^2 \in H^0(6H - 4\Gamma_P - 2\Gamma_Q)$;

$x_0^3 x_1^3 \in H^0(6H - 3\Gamma_P - 3\Gamma_Q)$;

$x_0^2 x_1^4 \in H^0(6H - 2\Gamma_P - 4\Gamma_Q)$;

$x_0 x_1^5 \in H^0(6H - \Gamma_P - 5\Gamma_Q)$;

$x_1^6 \in H^0(6H - 6\Gamma_Q)$.

$$\sum_{i+j=6} \psi_{ij} x_0^i x_1^j \in H^0(6H - 3\Gamma)$$

$\psi_{60} \in H^0(5P - Q - P_1)$;

$\psi_{51} \in H^0(4P - P_1)$;

$\psi_{42} \in H^0(3P + Q - P_1)$;

$\psi_{33} \in H^0(2P + 2Q - P_1)$;

$\psi_{24} \in H^0(P + 3Q - P_1)$;

$\psi_{15} \in H^0(4Q - P_1)$;

$\psi_{06} \in H^0(5Q - P - P_1)$.

All groups $H^0((i-1)P + (j-1)Q - P_1)$, with $i, j = 0, \ldots, 6$ $i + j = 6$ has dimension $= 3$

$$[1, 0], \quad [0, 1] \in \Gamma_{P_1}. \tag{3.4}$$

Set $x = x_1/x_0$, $y = x_0/x_1$, and consider a coordinate t near P_1 Then

$$\psi_{60} + \psi_{51}x + \psi_{42}x^2 + \psi_{33}x^3 + \psi_{24}x^4 + \psi_{15}x^5 + \psi_{60}x^6 = 0$$

Condition on ψ_{60}: to find a point o such that $5P - Q - P_1 \sim 3o$. $|5P - Q - P_1|$ is very ample, then it is equivalently to find a point o that is a flex in the immersion associated to $|5P - Q - P_1|$.

Condition on ψ_{06}. As before $5P - Q - P_1 \sim 3o \Rightarrow 6P \sim 6Q$. If $P = Q$, it is obvious. If $P \neq Q$, then $P - Q$ must have torsion order 2, 3 or 6.

3.2 Sezione 3.2

We recall the following result:

Proposition 3.2.1. *Let $u \in E$, and set $\mathsf{W} := E_u(2,\, 1)$. Then we have*

$$\mathrm{S}^2\mathsf{W} = \bigoplus_{i=1}^{3} \mathsf{L}_i(u), \quad \mathrm{S}^3\mathsf{W} = \mathsf{W}(u) \oplus \mathsf{W}(u),$$

where the L_i are the three non-trivial 2-torsion line bundles on E.

Proof. If $u = o$, see [Ati, p. 438-439]. Now the general case follows, since $E_u(2,\, 1) = E_o(2,\, 1) \otimes \mathcal{L}$, where $\mathcal{L}$ is any line bundle on E such that $\mathcal{L}^2 = \mathcal{O}_B(u - o)$. □

Consider the exact sequence

$$0 \longrightarrow (\det \mathsf{V}_1)^2 \otimes \mathsf{V}_2 \xrightarrow{i_3} \mathrm{S}^3\mathsf{V}_2 \longrightarrow \mathsf{A}_6 \longrightarrow 0. \tag{3.5}$$

Now let

$$0 \longrightarrow \mathsf{G}_1 \longrightarrow \mathsf{G}_2 \longrightarrow \widetilde{\mathsf{A}}_6 \longrightarrow 0 \tag{3.6}$$

be the exact sequence obtained by twisting (3.5) by $(\det \mathsf{V}_1 \otimes \mathcal{O}_B(\tau))^{-2}$.

Lemma 3.2.2. *We have*

$$h^0(\widetilde{\mathsf{A}}_6) \leq h^0(\mathsf{G}_2) - h^0(\mathsf{G}_1) + h^1(\mathsf{G}_1).$$

Proof. By (3.6), we obtain

$$0 \longrightarrow H^0(\mathsf{G}_1) \longrightarrow H^0(\mathsf{G}_2) \longrightarrow H^0(\widetilde{\mathsf{A}}_6) \xrightarrow{\delta} H^1(\mathsf{G}_1) \longrightarrow \operatorname{coker}(\delta) \longrightarrow 0,$$

that is

$$h^0(\widetilde{\mathsf{A}}_6) = h^0(\mathsf{G}_2) - h^0(\mathsf{G}_1) + h^1(\mathsf{G}_1) - \dim \operatorname{coker}(\delta).$$

$\square$

3.3 The sheaf $\mathsf{V}_2 = f_*\omega^2_{S|B}$

3.3.1 The case where V_1 is decomposable

In this case $\mathrm{S}^2\mathsf{V}_1 = \bigoplus_{i=1}^3 \mathsf{P}_i$, where $\mathsf{P}_1 = \mathcal{O}_B(2o)$, $\mathsf{P}_2 = \mathcal{O}_B(3o-p)$, $\mathsf{P}_3 = \mathcal{O}_B(4o-2p)$. Fix a section $f_0 \in H^0(\mathcal{O}_B(\tau))\setminus\{0\}$; applying the functor $\mathrm{Hom}(-,\ \mathrm{S}^2\mathsf{V}_1)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_B(o-\tau) \stackrel{(-f_0)}{\longrightarrow} \mathcal{O}_B(o) \longrightarrow \mathcal{O}_\tau \longrightarrow 0$$

we obtain

$$\mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau,\ \mathrm{S}^2\mathsf{V}_1) = \bigoplus_{i=1}^3 \frac{H^0(P_i(\tau-o))}{H^0(P_i(-o))} \cong \mathbb{C}^3. \tag{3.7}$$

Hence an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau,\ \mathrm{S}^2\mathsf{V}_1)$ is given by a triple $(\bar{f}_1,\ \bar{f}_2,\ \bar{f}_3)$, with $f_i \in H^0(P_i(\tau-o))$. Arguing as in [CaPi, p. 1032], this implies that V_2 is the cokernel of the short exact sequence

$$0 \longrightarrow \mathcal{O}_B(o-\tau) \longrightarrow \mathcal{O}_B(o) \oplus \bigoplus_{i=1}^3 \mathsf{P}_i \longrightarrow \mathsf{V}_2 \longrightarrow 0, \tag{3.8}$$

where the injective map is induced by ${}^t(f_0, f_1, f_2, f_3)$.

Notice that V_2 is a vector bundle if and only if $f_1,\ f_2,\ f_3$ do not vanish simultaneously in τ, that is if and only if $\xi = (\bar{f}_1,\ \bar{f}_2,\ \bar{f}_3)$ is not the trivial extension class. Let m be the cardinality of the set $\{i \mid \bar{f}_i = 0\}$; hence $0 \leq m \leq 2$. Now we give the description of V_2 in the different cases.

Proposition 3.3.1. *Assume* $\mathsf{V}_1 = \mathcal{O}_B(o) \oplus \mathcal{O}_B(2o - p)$.

- *If* $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$ *then there are precisely the following possibilities.*

(I) $m = 0,\ \mathsf{V}_2(-2o) = \mathsf{E}_{3o-3p+\tau}(3, 1)$

(IIa) $m = 1,\ \mathsf{V}_2(-2o) = \mathsf{E}_{3o-3p+\tau}(2, 1) \oplus \mathcal{O}_B$

(IIb) $m = 1,\ \mathsf{V}_2(-2o) = \mathsf{E}_{3o-3p+\tau}(2, 1) \oplus \mathcal{O}_B(o - p)$

(IIc) $m = 1,\ \mathsf{V}_2(-2o) = \mathsf{E}_{3o-3p+\tau}(2, 1) \oplus \mathcal{O}_B(2o - 2p)$

(IIIa) $m = 2,\ \mathsf{V}_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B(o - p) \oplus \mathcal{O}_B(2o - 2p + \tau)$

(IIIb) $m = 2,\ \mathsf{V}_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B(o - p + \tau) \oplus \mathcal{O}_B(2o - 2p)$

(IIIc) $m = 2,\ \mathsf{V}_2(-2o) = \mathcal{O}_B(\tau) \oplus \mathcal{O}_B(o - p) \oplus \mathcal{O}_B(2o - 2p)$

- *If* $\mathcal{O}_B(2o - 2p) = \mathcal{O}_B$ *and* $o \neq p$ *then there are precisely the following possibilities.*

(IV) $m = 0,\ \mathsf{V}_2(-2o) = \mathsf{F}_2 \oplus \mathcal{O}_B(o - p + \tau)$

(Va) $m = 1,\ \mathsf{V}_2(-2o) = \mathsf{E}_{o-p+\tau}(2, 1) \oplus \mathcal{O}_B$

(Vb) $m = 1,\ \mathsf{V}_2(-2o) = \mathsf{E}_{o-p+\tau}(2, 1) \oplus \mathcal{O}_B(o - p)$

(VIa) $m = 2,\ \mathsf{V}_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B(o - p + \tau) \oplus \mathcal{O}_B$

(VIb) $m = 2,\ \mathsf{V}_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B(o - p) \oplus \mathcal{O}_B(\tau)$

- *Finally, if* $o = p$ *then there is exactly one possibility.*

(VII) $0 \leq m \leq 2$, $\mathsf{V}_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$.

Proof. We only consider the case $\mathcal{O}_B(2o-2p) \neq 0$; the remaining two are similar and they are left to the reader. Let $\mathsf{L} \in \mathrm{Pic}^0(B)$; twisting the exact sequence (3.8) by $\mathsf{L}(-2o)$ we obtain

$$0 \longrightarrow \mathsf{L}(-o-\tau) \longrightarrow \mathsf{L}(-o) \oplus \mathsf{L} \oplus \mathsf{L}(o-p) \oplus \mathsf{L}(2o-2p) \longrightarrow \mathsf{V}_2(-2o) \longrightarrow 0, \quad (3.9)$$

and this in turn induces a linear map in cohomology

$$\alpha \colon H^1(\mathsf{L}(-o-\tau)) \longrightarrow H^1(\mathsf{L}(-o) \oplus \mathsf{L} \oplus \mathsf{L}(o-p) \oplus \mathsf{L}(2o-2p)).$$

Now there are several possibilities.

Assume $\mathsf{L} \notin \{\mathcal{O}_B, \mathcal{O}_B(p-o), \mathcal{O}_B(2p-2o)\}$. Then $\ker(\alpha) \cong \mathbb{C}$, hence $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 0$.

Assume $\mathsf{L} = \mathcal{O}_B$. If $\bar{f}_1 \neq 0$ then α is an isomorphism, and we have $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 0$; if $\bar{f}_1 = 0$ then $\ker(\alpha) \cong \mathbb{C}$, and we have $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 1$.

Assume $\mathsf{L} = \mathcal{O}_B(p-o)$. If $\bar{f}_2 \neq 0$ then α is an isomorphism, and we have $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 0$; if $\bar{f}_2 = 0$ then $\ker(\alpha) \cong \mathbb{C}$, and we have $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 1$.

Assume $\mathsf{L} = \mathcal{O}_B(2p-2o)$. If $\bar{f}_3 \neq 0$ then α is an isomorphism, and we have $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 0$; if $\bar{f}_3 = 0$ then $\ker(\alpha) \cong \mathbb{C}$, and we have $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 1$.

Therefore $\mathsf{V}_2(-2o)$ is a vector bundle of rank 3 and determinant $\mathcal{O}_B(3o-3p+\tau)$ such that there exist exactly m line bundles $\mathsf{L} \in \mathrm{Pic}^0(B)$ with the property

$h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) \neq 0$.

If $\mathsf{V}_2(-2o)$ is indecomposable, then $\mathsf{V}_2(-2o) = \mathsf{E}_{3o-3p+\tau}$ by Atiyah's classification; this sheaf has always trivial first cohomology group when twisted with any degree 0 line bundle; hence $m = 0$ and we are in case (I).

Assume now that $\mathsf{V}_2(-2o)$ is the direct sum of three line bundles, that is $\mathsf{V}_2 = \mathsf{L}_1 \oplus \mathsf{L}_2 \oplus \mathsf{L}_3$.

Since $h^1(\mathsf{V}_2(-2o) \otimes \mathsf{L}) = 0$ for a general $\mathsf{L} \in \mathrm{Pic}^0(B)$, it follows $\deg \mathsf{L}_i \geq 0$ for all $1 \leq i \leq 3$. On the other hand, since $\deg \mathsf{V}_2(-2o) = 1$ we see that exactly two summands have degree 0. Therefore it is clear that $m = 2$; more precisely, if $\bar{f}_1 = \bar{f}_2 = 0$ we are in case (IIIa), if $\bar{f}_1 = \bar{f}_3 = 0$ we are in case (IIIb), if $\bar{f}_2 = \bar{f}_3 = 0$ we are in case (IIIc).

Finally, let us assume $\mathsf{V}_2(-2o) = \mathsf{W} \oplus \mathsf{L}$, where W is indecomposable of rank 2 and L is a line bundle; as before, we must have $\deg \mathsf{L} \geq 0$. Let us exclude first the case $\deg \mathsf{W} = 0$, $\deg \mathsf{L} = 1$. If $\deg \mathsf{W} = 0$ by Atiyah's classification there exists exactly one line bundle $\mathsf{F} \in \mathrm{Pic}^0(B)$ such that $h^1(\mathsf{W} \otimes \mathsf{F}) \neq 0$. Hence $m = 1$; but if $\bar{f}_i = 0$ then P_i is a direct summand of $\mathsf{V}_2(-2o)$, a contradiction. Hence we obtain $\deg \mathsf{W} = 1$, $\deg \mathsf{L} = 0$.

It follows that every twist of W by a degree 0 line bundle has trivial cohomology, hence the cohomology of $\mathsf{V}_2(-2o)$ jumps if and only if we tensor it by L^{-1}. Therefore $m = 1$, that is exactly one of the $\bar{f}_i$ vanishes.

More precisely, if $\bar{f}_1 = 0$ we are in case (IIa), if $\bar{f}_2 = 0$ we are in case (IIb) and if $\bar{f}_3 = 0$ we are in case (IIc).

$\square$

3.4 The moduli space

Let $\mathcal{M}$ be the moduli space of minimal surfaces of general type S with $p_g(S) = 2$, $q(S) = 1$, $K_S^2 = 5$. We write $\mathcal{M} = \mathcal{M}' \cup \mathcal{M}''$, where $\mathcal{M}'$ corresponds to surfaces such that V_1 is decomposable and $\mathcal{M}''$ corresponds to surfaces such that V_1 is indecomposable.

Let us start by studying $\mathcal{M}'$.

Definition 3.4.1. *We stratify* $\mathcal{M}'$ *as*

$$\mathcal{M}' = \mathcal{M}_{\mathrm{I}} \cup \mathcal{M}_{\mathrm{IIa}} \cup \cdots \cup \mathcal{M}_{\mathrm{VII}}$$

according to the decomposition type for $\mathsf{V}_2 = f_* \omega^2_{S|B}$*, as in Proposition* 3.3.1*.*

3.4.1 The stratum $\mathcal{M}_{\mathrm{I}}$

Proposition 3.4.2. *The stratum* $\mathcal{M}_I$ *is either empty or it has dimension* 13*.*

Proof. Set $\mathsf{W} := \mathsf{E}_{3o-3p+\tau}(3,\,1)$; then we have a short exact sequence

$$0 \longrightarrow \mathsf{W}(2o-2\tau) \longrightarrow \mathrm{S}^3\mathsf{W}(2p-2\tau) \longrightarrow \widetilde{\mathsf{A}}_6 \longrightarrow 0.$$

By [CaCi2, Section 1] we obtain

$$h^0(\mathsf{W}(2o-2\tau)) = 1, \quad h^1(\mathsf{W}(2o-2\tau)) = 0, \quad h^0(\mathrm{S}^3\mathsf{W}(2p-2\tau)) = 10,$$

hence $h^0(\widetilde{\mathsf{A}}_6) = 9$.

We have 1 parameter for B, 1 parameter for p, 2 parameters for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$. Therefore either $\mathcal{M}_{\mathrm{I}}$ is empty or it has dimension 13.

$\square$

3.4.2 The stratum $\mathcal{M}_{\mathbf{IIa}}$

Proposition 3.4.3. *The stratum $\mathcal{M}_{\mathrm{IIa}}$ is either empty or it has dimension* 12.

Proof. Set $\mathsf{W} = \mathsf{E}_{3o-3p+\tau}$; then $\mathsf{V}_2(-2o) = \mathsf{W} \oplus \mathcal{O}_B$ and twisting the exact sequence (3.5) by $\mathcal{O}_B(-6o)$ we obtain

$$0 \longrightarrow \mathsf{W} \oplus \mathcal{O}_B \xrightarrow{i_3} \big(\mathrm{S}^3\mathsf{W} \oplus \mathrm{S}^2\mathsf{W}\big) \oplus (\mathsf{W} \oplus \mathcal{O}_B) \longrightarrow \mathsf{A}_6(-6o) \longrightarrow 0. \tag{3.10}$$

Arguing as in [CaPi, Lemma 6.14], we see that the second component of the map i_3 is actually the identity, hence the exact sequence (3.10) splits, giving

$$\widetilde{\mathsf{A}}_6 = \mathsf{A}_6(-6o + 2p - 2\tau) = (\mathrm{S}^3\mathsf{W} \oplus \mathrm{S}^2\mathsf{W})(2p - 2\tau).$$

By Proposition 3.2.1 this in turn implies

$$\widetilde{\mathsf{A}}_6 = \left(\mathsf{W} \oplus \mathsf{W} \oplus \bigoplus_{i=1}^{3} \mathsf{L}_i\right)(3o - p - \tau),$$

hence $h^0(\widetilde{\mathsf{A}}_6) = 9$. We have 1 parameter for B, 1 parameters for p, 1 parameter for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$. Therefore $\mathcal{M}_{\mathrm{IIa}}$ is either empty or it has dimension 11.

$\square$

3.4.3 The strata $\mathcal{M}_{\mathbf{IIb}}$, $\mathcal{M}_{\mathbf{IIc}}$

Proposition 3.4.4. *The dimension of the strata* $\mathcal{M}_{\mathrm{IIb}}$, $\mathcal{M}_{\mathrm{IIc}}$ *is at most* 12.

Proof. In order to give an unified treatment of these strata, set

$$\mathsf{W} := \mathsf{E}_{3o-3p+\tau}(2,1), \quad \mathsf{L} := \begin{cases} \mathcal{O}_B(o-p) & \text{in case (IIb);} \\ \mathcal{O}_B(2o-2p) & \text{in case (IIc).} \end{cases}$$

Then $\mathsf{V}_2(-2o) = \mathsf{W} \oplus \mathsf{L}$ and twisting the exact sequence (3.5) by $\mathcal{O}_B(-6o)$ we obtain

$$0 \longrightarrow \mathsf{W} \oplus \mathsf{L} \xrightarrow{i_3} \mathrm{S}^3\mathsf{W} \oplus (\mathrm{S}^2\mathsf{W} \otimes \mathsf{L}) \oplus (\mathsf{W} \otimes \mathsf{L}^2) \oplus \mathsf{L}^3 \longrightarrow \mathsf{A}_6(-6o) \longrightarrow 0. \quad (3.11)$$

Hence $\widetilde{\mathsf{A}}_6 = \mathsf{A}_6(-6o + 2p - 2\tau)$ fits into the short exact sequence

$$0 \longrightarrow \mathsf{G}_1 \longrightarrow \mathsf{G}_2 \longrightarrow \widetilde{\mathsf{A}}_6 \longrightarrow 0, \quad (3.12)$$

where

$$\mathsf{G}_1 = (\mathsf{W} \oplus \mathsf{L})(2p - 2\tau), \quad \mathsf{G}_2 = \big(\mathrm{S}^3\mathsf{W}_2 \oplus (\mathrm{S}^2\mathsf{W} \otimes \mathsf{L}) \oplus (\mathsf{W} \otimes \mathsf{L}^2) \oplus \mathsf{L}^3\big)(2p - 2\tau).$$

There are several possibilities.

Case (i). $\mathsf{L}(2p - 2\tau) \neq \mathcal{O}_B$, $\mathsf{L}^3(2p - 2\tau) \neq \mathcal{O}_B$. In this case

$$h^0(\mathsf{G}_1) = 1, \quad h^1(\mathsf{G}_1) = 0, \quad h^0(\mathsf{G}_2) = 10,$$

hence $h^0(\widetilde{\mathsf{A}}_6) = 9$. We have 1 parameter for B, 1 parameter for p, 1 parameter for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$.

Case (ii). $\mathsf{L}(2p-2\tau) \neq \mathcal{O}_B$, $\mathsf{L}^3(2p-2\tau) = \mathcal{O}_B$. In this case

$$h^0(\mathsf{G}_1) = 1, \quad h^1(\mathsf{G}_1) = 0, \quad h^0(\mathsf{G}_2) = 11,$$

hence $h^0(\widetilde{\mathsf{A}}_6) = 10$. We have 1 parameter for B, 1 parameter for p, 1 parameter for ξ, no parameters for τ and 9 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$.

Case (iii). $\mathsf{L}(2p-2\tau) = \mathcal{O}_B$. Since $\mathsf{L}^2 \neq \mathcal{O}_B$, this implies $\mathsf{L}^3(2p-2\tau) \neq \mathcal{O}_B$. We have

$$h^0(\mathsf{G}_1) = 2, \quad h^1(\mathsf{G}_1) = 1, \quad h^0(\mathsf{G}_2) = 10,$$

hence $h^0(\widetilde{\mathsf{A}}_6) \leq 9$ by Lemma 3.2.2. We have 1 parameter for B, 1 parameter for p, 1 parameter for ξ, no parameters for τ and at most 8 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$.

Summing up, we conclude that the dimension of the strata $\mathcal{M}_{\mathrm{IIb}}$, $\mathcal{M}_{\mathrm{IIc}}$ is at most 12.

□

3.4.4 The stratum $\mathcal{M}_{\mathrm{IIIa}}$

Proposition 3.4.5. *The stratum $\mathcal{M}_{\mathrm{IIIa}}$ is either empty or it has dimension* 11.

Proof. In case (IIIa) the linear map σ_2 has the form

$$\sigma_2\colon \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o-p) \oplus \mathcal{O}_B(4o-2p) \longrightarrow \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o-p) \oplus \mathcal{O}_B(4o-2p+\tau).$$

Take global coordinates $x_0,\ x_1$ on the fibres of V_1 and $y_0',\ y_1',\ y_2'$ on the fibres of V_2; with respect to this coordinates, σ_2 can be represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & f_0 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = y_0' + a y_2' \\ \sigma_2(x_0 x_1) = y_1' + b y_2' \\ \sigma_2(x_1^2) = f_0 y_2', \end{cases} \tag{3.13}$$

where $a \in H^0(\mathcal{O}_B(2o-2p+\tau))$, $b \in H^0(\mathcal{O}_B(o-p+\tau))$. By applying the linear change of coordinates

$$y_0 := y_0' + a y_2', \quad y_1 := y_1' + b y_2', \quad y_2 := y_2'$$

we see that σ_2 can be written in the diagonal form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f_0 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = y_0 \\ \sigma_2(x_0 x_1) = y_1 \\ \sigma_2(x_1^2) = f_0 y_2. \end{cases} \tag{3.14}$$

Hence the map $i_3 \colon (\det \mathsf{V}_1)^2 \otimes \mathsf{V}_2 \longrightarrow \mathrm{S}^3\mathsf{V}_2$ is locally defined as follows:

$$\begin{cases} i_3((x_0 \wedge x_1)^{\otimes 2} \otimes y_0) = f_0 y_0^2 y_2 - y_0 y_1^2 \\ i_3((x_0 \wedge x_1)^{\otimes 2} \otimes y_1) = f_0 y_0 y_1 y_2 - y_1^3 \\ i_3((x_0 \wedge x_1)^{\otimes 2} \otimes y_2) = f_0 y_0 y_2^2 - y_1^2 y_2. \end{cases}$$

Therefore the matrix representing i_3 is given, in suitable coordinates, by the transpose of

$$\begin{pmatrix} 1 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that $\mathsf{A}_6 = \operatorname{coker} i_3$ is isomorphic to

$$\begin{aligned} &\mathcal{O}_B(6o) \oplus \mathcal{O}_B(7o-p) \oplus \mathcal{O}_B(8o-2p+\tau) \oplus \mathcal{O}_B(9o-3p+\tau) \\ &\oplus \mathcal{O}_B(10o-4p+2\tau) \oplus \mathcal{O}_B(11o-5p+2\tau) \oplus \mathcal{O}_B(12o-6p+3\tau), \end{aligned} \tag{3.15}$$

so we obtain

$$h^0(\widetilde{\mathsf{A}}_6) = h^0(\mathsf{A}_6(-6o+2p-2\tau)) = \begin{cases} 10 & \text{if either } \mathcal{O}(2p-2\tau) = \mathcal{O}_B \text{ or } \mathcal{O}_B(o+p-2\tau) = \mathcal{O}_B; \\ 9 & \text{otherwise.} \end{cases}$$

Summing up, if either $\mathcal{O}(2p-2\tau) = \mathcal{O}_B$ or $\mathcal{O}_B(o+p-2\tau) = \mathcal{O}_B$ we have 1 parameter for B, 1 parameter for p, no parameters for τ and ξ and 9 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$; otherwise we have 1 parameter for B, 1 parameter for p, 1 parameter for τ, no parameters for ξ and 8 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$. In all cases the construction depends on 11 parameters, hence either $\mathcal{M}_{\mathrm{IIIa}}$ is empty or it has dimension 11.

□

Remark 3.4.6. *Equations* (3.4.6) *show that relative conic $\mathcal{C} \subset \mathbb{P}(\mathsf{V}_2)$ is defined by $y_1^2 - f_0 y_0 y_2 = 0$. Since the coefficient of the monomial y_1^2 is a non-zero constant, the same argument of* [Pig, Lemma 3.5] *shows that in this case the exact sequence* (3.5) *actually splits.*

3.4.5 The stratum $\mathcal{M}_{\mathrm{IIIb}}$

Proposition 3.4.7. *The stratum $\mathcal{M}_{\mathrm{IIIb}}$ has dimension at most* 11.

Proof. Take global coordinates as before so that the linear map

$$\sigma_2\colon \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o-p) \oplus \mathcal{O}_B(4o-2p) \longrightarrow \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o-p+\tau) \oplus \mathcal{O}_B(4o-2p)$$

can be represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & f_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = y_0 \\ \sigma_2(x_0x_1) = f_0y_1 \\ \sigma_2(x_1^2) = y_2. \end{cases}$$

Arguing as in the previous case we obtain

$$\begin{aligned} \widetilde{\mathsf{A}}_6 =& \mathcal{O}_B(2p-2\tau) \oplus \mathcal{O}_B(6o-4p-2\tau) \oplus \mathcal{O}_B(o+p-\tau) \oplus \mathcal{O}_B(5o-3p-\tau) \\ & \oplus \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o-p+\tau) \oplus \mathcal{O}_B(4o-2p). \end{aligned}$$

Hence we have $H^0(\widetilde{\mathsf{A}}_6) \leq 11$ and equality holds if and only if $\mathcal{O}_B(6o-6p) = \mathcal{O}_B(2p-2\tau) = \mathcal{O}_B$. Write $\mathcal{M}_{\mathrm{IIIb}} = \bigcup_{p\in B} \mathcal{M}_{\mathrm{IIIb}}(p)$. Counting parameters as before, we conclude that $\mathcal{M}_{\mathrm{IIIb}}(p)$ has dimension at most 11; moreover the points p such that $\mathcal{M}_{\mathrm{IIIb}}(p)$ has dimension 11 form a finite set. Therefore the stratum $\mathcal{M}_{\mathrm{IIIb}}$ has dimension at most 11.

□

3.4.6 The stratum $\mathcal{M}_{\mathrm{IIIc}}$

Proposition 3.4.8. *The stratum $\mathcal{M}_{\mathrm{IIIc}}$ is either empty or it has dimension* 11.

Proof. We can take global coordinates so that the linear map

$$\sigma_2\colon \mathcal{O}_B(2o)\oplus\mathcal{O}_B(3o-p)\oplus\mathcal{O}_B(4o-2p) \longrightarrow \mathcal{O}_B(2o)\oplus\mathcal{O}_B(3o-p)\oplus\mathcal{O}_B(4o-2p+\tau)$$

can be represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f_0 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = y_0 \\ \sigma_2(x_0x_1) = y_1 \\ \sigma_2(x_1^2) = f_0y_2. \end{cases}$$

The rest of the proof is exactly as in case (IIIa), so it is left to the reader. □

3.4.7 The stratum $\mathcal{M}_{\mathbf{IV}}$

Proposition 3.4.9. *The stratum* $\mathcal{M}_{\mathrm{IV}}$ *has dimension at most* 12.

Proof. Set $\mathsf{L} := \mathcal{O}(o - p + \tau)$. Then twisting the exact sequence (3.5) by $\mathcal{O}_B(-6o)$ we obtain

$$0 \longrightarrow \mathsf{F}_2 \oplus \mathsf{L} \longrightarrow \mathrm{S}^3\mathsf{F}_2 \oplus (\mathrm{S}^2\mathsf{F}_2 \otimes \mathsf{L}) \oplus (\mathsf{F}_2 \otimes \mathsf{L}^2) \oplus \mathsf{L}^3 \longrightarrow \mathsf{A}_6(-6o) \longrightarrow 0. \quad (3.16)$$

Hence $\widetilde{\mathsf{A}}_6 = \mathsf{A}_6(-6o + 2p - 2\tau)$ fits into the short exact sequence

$$0 \longrightarrow \mathsf{G}_1 \longrightarrow \mathsf{G}_2 \longrightarrow \widetilde{\mathsf{A}}_6 \longrightarrow 0, \quad (3.17)$$

where

$$\mathsf{G}_1 = (\mathsf{F}_2 \oplus \mathsf{L})(2p - 2\tau), \quad \mathsf{G}_2 = \big(\mathrm{S}^3\mathsf{F}_2 \oplus (\mathrm{S}^2\mathsf{F}_2 \otimes \mathsf{L}) \oplus (\mathsf{F}_2 \otimes \mathsf{L}^2) \oplus \mathsf{L}^3\big)(2p - 2\tau).$$

By [Ati, Theorem 9] we have

$$\mathrm{S}^2\mathsf{F}_2 = \mathsf{F}_3, \quad \mathrm{S}^3\mathsf{F}_2 = \mathsf{F}_4.$$

There are two possibilities.

Case (i). $\mathcal{O}_B(2p - 2\tau) \neq \mathcal{O}_B$. In this case

$$h^0(\mathsf{G}_1) = 1, \quad h^1(\mathsf{G}_1) = 0, \quad h^0(\mathsf{G}_2) = 10.$$

Therefore $h^0(\widetilde{\mathsf{A}}_6) = h^0(\mathsf{G}_2) - h^0(\mathsf{G}_1) = 9$. We have 1 parameter for B, 2 parameters for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}(H^0(\widetilde{\mathsf{A}}_6))$.

Case (ii). $\mathcal{O}_B(2p-2\tau) \neq \mathcal{O}_B$. In this case

$$h^0(\mathsf{G}_1) = 1, \quad h^1(\mathsf{G}_1) = 0, \quad h^0(\mathsf{G}_2) = 11,$$

hence $h^0(\widetilde{\mathsf{A}}_6) \leq 10$ by Lemma 3.2.2. We have 1 parameter for B, 2 parameters for ξ, no parameters for τ and at most 9 parameters from $\mathbb{P}(H^0(\widetilde{\mathsf{A}}_6))$.

Summing up, we conclude that $\mathcal{M}_{\mathrm{IV}}$ has dimension at most 12.

□

3.4.8 The strata $\mathcal{M}_{\mathbf{Va}}$ and $\mathcal{M}_{\mathbf{Vb}}$

Proposition 3.4.10. *The strata $\mathcal{M}_{\mathrm{Va}}$, $\mathcal{M}_{\mathrm{Vb}}$ are either empty or they have dimension* 11*.*

Proof. In order to give an unified treatment of these strata, set

$$\mathsf{W} := E_{o-p+\tau}(2,1), \quad \mathsf{L} := \begin{cases} \mathcal{O}_B & \text{in case (Va);} \\ \mathcal{O}_B(o-p) & \text{in case (Vb).} \end{cases}$$

Then $\mathsf{V}_2(-2o) = \mathsf{W} \oplus \mathsf{L}$ and twisting the exact sequence (3.5) by $\mathcal{O}_B(-6o)$ we obtain, since $\mathsf{L}^2 = \mathcal{O}_B$,

$$0 \longrightarrow \mathsf{W} \oplus \mathsf{L} \xrightarrow{i_3} \big(\mathrm{S}^3\mathsf{W} \oplus (\mathrm{S}^2\mathsf{W} \otimes \mathsf{L})\big) \oplus (\mathsf{W} \oplus \mathsf{L}) \longrightarrow \mathsf{A}_6(-6o) \longrightarrow 0. \qquad (3.18)$$

Arguing as in [CaPi, Lemma 6.14], we see that the second component of the map i_3 is actually the identity, hence the exact sequence (3.18) splits, giving

$$\widetilde{\mathsf{A}}_6 = \mathsf{A}_6(-6o + 2p - 2\tau) = \big(\mathrm{S}^3\mathsf{W} \oplus (\mathrm{S}^2\mathsf{W} \otimes \mathsf{L})\big)(2p - 2\tau).$$

By Proposition 3.2.1 this in turn implies

$$\widetilde{\mathsf{A}}_6 = \left(\mathsf{W} \oplus \mathsf{W} \oplus \bigoplus_{i=1}^{3}(\mathsf{L}_i \otimes \mathsf{L})\right)(o + p - \tau),$$

hence $h^0(\widetilde{\mathsf{A}}_6) = 9$. We have 1 parameter for B, no parameters for p, 1 parameter for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}(H^0(\widetilde{\mathsf{A}}_6))$. Therefore $\mathcal{M}_{\mathrm{Va}}$ and $\mathcal{M}_{\mathrm{Vb}}$ are either empty or they have dimension 11.

□

3.4.9 The stratum $\mathcal{M}_{\mathbf{VIa}}$

Proposition 3.4.11. $\mathcal{M}_{\mathrm{VIa}}$ *has dimension at most* 11.

Proof. In case (VIa) we have $o \neq p$ but $\mathcal{O}_B(2o - 2p) = \mathcal{O}_B$; moreover $\bar{f}_1 = \bar{f}_3 = 0$. Hence the linear map σ_2 has the form

$$\sigma_2 \colon \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o - p) \oplus \mathcal{O}_B(2o) \longrightarrow \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o - p + \tau) \oplus \mathcal{O}_B(2o).$$

Take global coordinates $x_0,\ x_1$ on the fibres of V_1 and $y_0',\ y_1',\ y_2'$ on the fibres of V_2; with respect to this coordinates, σ_2 can be represented by the matrix

$$\begin{pmatrix} a_1 & 0 & a_2 \\ 0 & f_0 & 0 \\ b_1 & 0 & b_2 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = a_1 y_0' + b_1 y_2' \\ \sigma_2(x_0 x_1) = f_0 y_1' \\ \sigma_2(x_1^2) = a_2 y_0' + b_2 y_2', \end{cases} \qquad (3.19)$$

where $a_1,\ a_2,\ b_1,\ b_2 \in \mathbb{C}$. Moreover, since the rank of σ_2 drops exactly at the point τ, it follows $a_1b_2 - a_2b_1 \neq 0$. Therefore, by using the change of coordinates

$$y_0 := a_1y_0' + b_1y_2', \quad y_1 := y_1', \quad y_2 := a_2y_0' + b_2y_2'$$

we see that σ_2 can be written in the diagonal form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & f_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = y_0 \\ \sigma_2(x_0x_1) = f_0y_1 \\ \sigma_2(x_1^2) = y_2. \end{cases} \tag{3.20}$$

Arguing as in the previous cases we obtain

$$\widetilde{\mathsf{A}}_6 = \mathcal{O}_B(2o - 2\tau)^2 \oplus \mathcal{O}_B(3o - p - \tau)^2 \oplus \mathcal{O}_B(2o)^2 \oplus \mathcal{O}_B(5o - 3p + \tau).$$

If $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$ we have 1 parameter for B, no parameters for ξ and τ and $h^0(\widetilde{\mathsf{A}}_6) = 11$. If $\mathcal{O}_B(2o - 2\tau) \neq \mathcal{O}_B$ we have 1 parameter for B, no parameters for ξ, 1 parameter for τ and $h^0(\widetilde{\mathsf{A}}_6) = 9$. It follows that $\mathcal{M}_{\mathrm{VIa}}$ has dimension at most 11.

□

3.4.10 The stratum $\mathcal{M}_{\mathbf{VIb}}$

Proposition 3.4.12. $\mathcal{M}_{\mathrm{VIb}}$ *is either empty or it has dimension* 10.

Proof. In case (VIb) we have $o \neq p$ but $\mathcal{O}_B(2o - 2p) = \mathcal{O}_B$; moreover $\bar{f}_1 = \bar{f}_2 = 0$. Hence the linear map σ_2 has the form

$$\sigma_2 \colon \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o - p) \oplus \mathcal{O}_B(2o) \longrightarrow \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o - p) \oplus \mathcal{O}_B(2o + \tau).$$

Take global coordinates $x_0,\ x_1$ on the fibres of V_1 and $y_0,\ y_1,\ y_2$ on the fibres of V_2; with respect to this coordinates, σ_2 can be represented by the matrix

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ \lambda f_0 & d & \mu f_0 \end{pmatrix}, \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = ay_0 + \lambda f_0 y_2 \\ \sigma_2(x_0 x_1) = cy_1 + dy_2 \\ \sigma_2(x_1^2) = by_0 + \mu f_0 y_2, \end{cases} \tag{3.21}$$

where $a,\ b,\ c,\ \lambda,\ \mu \in \mathbb{C}$ and $d \in H^0(\mathcal{O}_B(p - o + \tau))$.

Therefore the equation of the relative conic $\mathcal{C} \subset \mathbb{P}(\mathsf{V}_2)$ is

$$(cy_1 + dy_2)^2 - (ay_0 + \lambda f_0 y_2)(by_0 + \mu f_0 y_2) = 0.$$

Moreover, since the rank of σ_2 drops exactly at the point τ, it follows $c \neq 0$. This means that the coefficient of the term y_1^2 is a non-zero constant, hence the exact sequence (3.5) splits (see Remark 3.4.6). Therefore we obtain

$$\begin{aligned} \widetilde{\mathsf{A}}_6 = {} & \mathcal{O}_B(2o - 2\tau) \oplus \mathcal{O}_B(3o - p - 2\tau) \oplus \mathcal{O}_B(2o - \tau) \\ & \oplus \mathcal{O}_B(3o - p - \tau) \oplus \mathcal{O}_B(2o) \oplus \mathcal{O}_B(3o - p) \oplus \mathcal{O}_B(2o + \tau), \end{aligned}$$

so

$$h^0(\widetilde{\mathsf{A}}_6) = \begin{cases} 10 & \text{if either } \mathcal{O}(2o - 2\tau) = \mathcal{O}_B \text{ or } \mathcal{O}_B(3o - p - 2\tau) = \mathcal{O}_B; \\ 9 & \text{otherwise.} \end{cases}$$

Counting parameters as in the previous cases, we find that $\mathcal{M}_{\mathrm{VIb}}$ is either empty or it has dimension 10.

$\square$

3.4.11 The stratum $\mathcal{M}_{\mathbf{VII}}$

We write $\mathcal{M}_{\mathrm{VII}} = \mathcal{M}_{\mathrm{VII,\,gen}} \cup \mathcal{M}_{\mathrm{VII},2}$, where $\mathcal{M}_{\mathrm{VII},\,2}$ consists of surfaces with $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$ and $\mathcal{M}_{\mathrm{VII,\,gen}}$ is the rest.

Proposition 3.4.13. *(i)* $\mathcal{M}_{\mathrm{VII,\,gen}}$ *is nonempty, of dimension* 12.

(ii) $\mathcal{M}_{\mathrm{VII,\,2}}$ *is nonempty, of dimension* 13.

Proof. In case (VII) we have $o = p$, hence the linear map σ_2 has the form

$$\sigma_2 \colon \mathcal{O}_B(2o)^3 \longrightarrow \mathcal{O}_B(2o)^2 \oplus \mathcal{O}_B(2o+\tau).$$

Recall that for the general σ_2 we have $\bar{f}_i \neq 0$ for all $i \in \{1,\ 2\ ,3\}$.

Take global coordinates $x_0,\ x_1$ on the fibres of V_1 and $y_0,\ y_1,\ y_2$ on the fibres of V_2; with respect to this coordinates, σ_2 can be represented by the matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ \lambda f_0 & \mu f_0 & \gamma f_0 \end{pmatrix} \quad \text{that is} \quad \begin{cases} \sigma_2(x_0^2) = ay_0 + dy_1 + \lambda f_0 y_2 \\ \sigma_2(x_0x_1) = by_0 + ey_1 + \mu f_0 y_2 \\ \sigma_2(x_1^2) = cy_0 + fy_1 + \gamma f_0 y_2, \end{cases} \tag{3.22}$$

where $a,\ b,\ c,\ d,\ e,\ f,\ \lambda,\ \mu,\ \gamma \in \mathbb{C}$.

Moreover, since the rank of σ_2 drops exactly at the point τ, it follows

$$\begin{vmatrix} a & b & c \\ d & e & f \\ \lambda & \mu & \gamma \end{vmatrix} \neq 0.$$

Therefore the equation of the relative conic $\mathcal{C} \subset \mathbb{P}(\mathsf{V}_2)$ is

$$(by_0 + ey_1 + \mu f_0 y_2)^2 - (ay_0 + dy_1 + \lambda f_0 y_2)(cy_0 + fy_1 + \gamma f_0 y_2) = 0$$

Up to a linear change of coordinates we can assume $e^2 - df \neq 0$; this means that the coefficient of the term y_1^2 is a non-zero constant, hence the exact sequence (3.5) splits (see Remark 3.4.6). Therefore we obtain

$$\widetilde{\mathsf{A}}_6 = \mathcal{O}_B(2o-2\tau)^2 \oplus \mathcal{O}_B(2o-\tau)^2 \oplus \mathcal{O}_B(2o)^2 \oplus \mathcal{O}_B(2o+\tau),$$

so

$$h^0(\widetilde{\mathsf{A}}_6) = \begin{cases} 11 & \text{if } \mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B; \\ 9 & \text{otherwise.} \end{cases}$$

It is now easy to compute the number of parameters. If $\mathcal{O}_B(2o-2\tau) = \mathcal{O}_B$ we have 1 parameter for B, 2 parameters for ξ and 10 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$; otherwise we have 1 parameter for B, 2 parameters for ξ, 1 parameter from τ and 8 parameters from $\mathbb{P}H^0(\widetilde{\mathsf{A}}_6)$.

It remains to show that both $\mathcal{M}_{\text{VII, gen}}$ and $\mathcal{M}_{\text{VII, 2}}$ are non-empty.

Choose $a = c = e = \mu = 0$, $b = d = f = \lambda = 1$, $\gamma = -1$, so that the equation of $\mathcal{C}$ becomes

$$y_0^2 - y_1^2 + f_0^2 y_2^2 = 0.$$

Notice that this conic bundle has a unique singular point, namely the point with homogeneous coordinates $[0 : 0 : 1]$ lying on the fibre over τ.

Since (3.5) splits, the relative cubic given by the corresponding section of $H^0(\widetilde{A}_6)$ is cut by a relative cubic $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}(\mathsf{V}_2)}(3) - \pi^*\mathcal{O}_B(4o + 2\tau)|$; let us write the equation of $\mathcal{G}$ as

$$\sum_{i+j+k=3} a_{ijk}\ y_0^i y_i^j y_2^k = 0, \tag{3.23}$$

where $a_{ijk} \in \pi^*\mathcal{O}_B(2o + (k-2)\tau)$.

If $\mathcal{O}_B(2o - 2\tau) \neq \mathcal{O}_B$ then all the coefficients of $\mathcal{G}$ are generically non-zero; one easily checks that in this case the linear system $|\mathcal{G}|$ in $\mathbb{P}(\mathsf{V}_2)$ is base-point free, hence the linear system $|\Delta|$ in $\mathcal{C}$ is base-point free too; by Bertini theorem, we conclude that $\mathcal{M}_{\text{VII, gen}}$ is nonempty.

If $\mathcal{O}_B(2o-2\tau)=\mathcal{O}_B$, then $a_{300}=a_{210}=a_{120}=a_{030}=0$. So the relative cubic $\mathcal{G}$ splits as $\mathcal{G}=\mathcal{H}\cup\mathcal{G}'$, where $\mathcal{H}$ is the relative hyperplane $y_2=0$ and $\mathcal{G}'$ is the relative conic

$$a_{201}y_0^2+a_{111}y_0y_1+a_{102}y_0y_2+a_{021}y_1^2+a_{012}y_1y_2+a_{003}y_2^2=0.$$

Consequently, the curve Δ splits as $\Delta=\mathcal{H}_{\mathcal{C}}\cup\Delta'$, where $\mathcal{H}_{\mathcal{C}}=\mathcal{H}\cap\mathcal{C}$ and $\Delta'=\mathcal{G}'\cap\mathcal{C}$.

The sections $a_{201},\ a_{021},\ a_{111}$ all vanish at the same point, namely the unique point $q\in B$ such that $\mathcal{O}_B(2o-\tau)=\mathcal{O}_B(q)$; notice that $q\neq\tau$. Hence the base locus of $|\mathcal{G}'|$ is the line $y_2=0$ in the fibre $\pi^{-1}(q)$, and this in turn implies that the base locus of $|\Delta'|$ in $\mathcal{C}$ are the two points $P_1=[1:\ 1:\ 0]$ and $P_1=[1:\ -1:\ 0]$ on the fibre of $\mathcal{C}$ over q. Now let us make a general choice of the coefficients in (3.23). Then the curve Δ does not contain the unique singular point of $\mathcal{C}$; moreover, a standard local computation together with Bertini theorem show that

- Δ' is smooth;
- Δ' and $\mathcal{H}_{\mathcal{C}}$ intersect transversally at P_1 and P_2.

This implies that $\mathcal{M}_{\mathrm{VII},\,2}$ is nonempty.

$\square$

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