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Advanced linear beam models to be exploited in the Implicit Corotational framework and FEM implementation

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Contents

Abstract

Some advanced linear models of beam are proposed with the aim to well represent 3D effects due to complex material behaviours or warping deformations avoiding the use of much more computationally expensive folded plates or 3D-like simulations.

They are potentially very usefull also for the geometrically-nonlinear case due to the recent proposal of the Implicit Corotational Method (or simply ICM), which represents a framework to obtain frame-invariant structurals models for one or two-dimensional fibred continua starting from the corresponding linear solution for the 3D Cauchy continuum, increasing the research interest in this kind of formulations.

At first, a mixed model of laminated beam including in-plane and out-of-plane warping deformations is derived. The linear formulation proposed is obtained on the basis of a semi-analytical approach which solves the Cauchy continuum equations for beam-like bodies under the usual Saint Venánt (SV) loading assumptions, but without introducing any simplifying hypothesis on the stress field. It can be proved as if no end-effects are considered the formulation is coincident with that proposed by Ieşan which, due to the generality of the stress field considered, allows the treatment of beams with nonhomogeneous and anisotropic materials where significant interlaminar tensions could arise.

The 3D stress solution obtained far from the end bases and that due to all the warping effects added are exploited to obtain the cross-section flexibility matrix, while the kinematics of the mixed model is described in a simplyfied way, i.e. in terms of a rigid section motion and the deformative modes of the cross-section considered independently amplified along the longitudianal axis.

The SV stress field for the isotropic and homogeneous case can be easly recovered from the Ieşan solution and enriched with a simplified treatment of nonuniform out plane-warping effects of the cross section, providing a less expensive, but suitable enough, description of beams with both thin-walled or compact section. Also in this case, as advantage of the mixed formulation employed, an essential kinematical description is possible. It is based on the standard rigid section motion and out-of-plane variable warping effects, whose shape over the section is furnished by the SV shears and torsion functions. A FE based on the exact solution of the model is provided and numerical analyses are performed. Comparisons are proposed with respect to 3D or plate-based solutions showing the accuracy of the approach.

Introduction

Beam-like structures are widely used in engineering practice and the improvement of both continuum models and FEM solution procedures for their linear and nonlinear analysis still represents a primary task for researchers.

Saint Venant (SV) rod theory, see [13], is a powerful theoretical basis for deriving beam models to be used in standard 3D frame analysis [20, 15] because it allows an accurate one-dimensional description of the 3D continuum behaviour in terms only of cross-section generalized parameters. Subsequent extensions, like that of Ieşan [13], allows the SV solution also to be exploited for non-isotropic and non-homogeneous materials. In some cases of loading and boundary conditions, for example torsion actions applied to open profiles, the structural behaviour is not correctly described by the SV theory as the end effects due to variable warping along the beam axis could produce important additional normal and shear stresses which are not negligible. Following the pioneering work of Vlasov [23] much researches have been devoted to the formulation of mechanical models capable of describing this phenomenon accurately, focusing their attention on isotropic beams only. The initial Vlasov theory has been notably refined in terms of both the theoretical aspects, for example see [5, 4], and the numerical methods of analysis [22, 1, 14, 6]. The main part of these works are focused on the analysis of thin-walled profiles, while more recent contributions extended the range of application by proposing beam theories suitable for the FEM analysis of one-dimensional structures with generic cross-sections and subjected also to non-uniform shear warping effects [17, 7].

In the present work advanced linear models of beam are proposed with the aim to well represent 3D effects due to complex material behaviours or warping deformations.

At first, the semi-analytical approach used for instance in [12, 18, 24] is exploited to obtain the 3D exact stress solution for a composite beam. The basic idea is that of formulating the Cauchy problem in a manner that allows its analytical solution with respect the beam axial coordinate while a FEM description of the cross-section is required. The approach is potentially very accurate as it does not introduce any simplifying hypothesis on the stress field, completely 3D, and allows all the in-plane and out-of-plane deformative shapes of the cross-section to be exactly calculated.

A mixed one-dimensional model is then derived so to maintain all the details of this exact solution. This goal is reached by a separate description for the 3D stress and displacement fields. In particular, the latter is represented in terms of a rigid section motion and the other section modal shapes indepentently amplyfied along the axial direction of the beam, while stresses are exactly those provided by the solution far from the end bases enriched by the contributions due to all the other warping effects added. The assumed fields are introduced in the Hellinger–Reissner functional to obtain an accurate Ritz-Galerkin approximation of the beam model in terms of generalized static and kinematic quantities.

With respect to the homogeneous and isotropic case, a more simple but accurate enough, linear model of beam is then proposed with the aim to account for the variable out-of-plane warping of the cross-section due to shear and torsion.

Its main feature is that it maintains all the details of the standard SV solution in order to analyze beams with both compact or thin-walled sections. This goal is reached by formulating the model in the mixed format. In particular, the kinematical description maintains, as standard compatible models [7, 1], a rigid section motion and out-of-plane deformations represented by the SV shears and torsion warping functions independently amplified along the beam axis. The stress field is more accurately evaluated as the sum of the exact contribution due to the SV solution and to some further terms due to variable warping. The constitutive laws in terms of generalized quantities so obtained account for all the coupling effects arising from the SV problem.

The warping functions are calculated through a preliminary analysis which involves the solution of 3 Neumann boundary-value problems on the cross-section domain. These problems are solved numerically using a FE approach like those presented in [20, 15], see also [11, 17] for a solution based on the Boundary Element Method.

The description of the stress field induced by the non-uniform warping is obtained on the basis of two distinct approaches which differ in the evaluation of the shear contributions. The first one uses a *Benscoter*-like [4] expression. The second approach, denoted as *Jourawsky*-like, evaluates this contribution through the equilibrium equation in the axial direction, requiring the FEM evaluation of 3 additional warping functions on the cross section domain.

The validation of the proposed model is performed by means of a mixed finite element formulated on the basis of exact shape functions. In particular the static fields interpolation exactly satisfy the homogeneous form of the equilibrium equations adopting an exponential distribution of the bi-moments and bi-shears, while the resultant force and moment are constant and linear respectively, as in standard beam models. The static interpolation also allows the discrete form of the strain energy to be evaluated exactly without using any explicit displacement interpolations. Externally the element exposes kinematical parameters only, thanks to the use of static condensation so reducing the global computational efforts. The finite element proposed has no discretization error. This feature allows us to perform the numerical experimentation focusing attention only on the beam model approximation. The numerical tests presented regard single or framed beam structures and the results obtained are compared with those proposed by other authors or calculated by using shell model analyses.

As a final comment observe how the mixed model adopted here is particularly suitable for the extension to geometrically nonlinear analyses using corotational strategies [10, 3] where displacements and rotations require complex change–of–observer rules on the contrary the stresses are not affected by this change. The use of a formulation valid for generic cross sections, that refers all the variables to the same axis and which is able to detect eventual coupling between torsional and shear warping, is a further advantage especially in the geometrically nonlinear case [21, 3].

Chapter 1

The Ieşan approach

1.1 The Ieşan approach to derive the SV solution

In this section the SV solution is described according to the approach of Ieşan [13] whose main advantage consists in its generality as no simplifying hypothesis on the stress field are introduced. This allows for instance to consider the presence of different fiber materials and fiber orientations which may cause complex 3D stress states (including significant interlaminar stresses) not present in the standard formulation. For a review of the SV problem we refer also to [15, 20].

1.1.1 Preliminaries

Let us consider a cylinder occupying a reference configuration $\mathcal{B}$ of length ℓ confined by the lateral boundary (the so called *mantel*) denoted by $\partial\mathcal{B}$ and two terminal bases Ω_0 and Ω_ℓ on which the external forces are applied.

The cylinder is referred to a Cartesian frame $(\mathcal{O}, s, x_2, x_3)$ with unit vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\boldsymbol{e}_1$ aligned with the cylinder axis. In this system, see Figure 1.1, we denote with $\boldsymbol{X}$ the position of a point P

$$\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{x} \quad \text{with} \quad \begin{cases} \boldsymbol{X}_0 = s\boldsymbol{e}_1 \\ \boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha \end{cases}$$

where $\boldsymbol{X}_0$ represents the position of P with respect to the beam axis, s is an abscissa which identifies the generic cross-section $\Omega[s]$ of the beam and $\boldsymbol{x}$ is the position of P inside $\Omega[s]$. Finally we have assumed the convention of summing on repeated indexes for the Greek letter α that goes from 2 to 3.

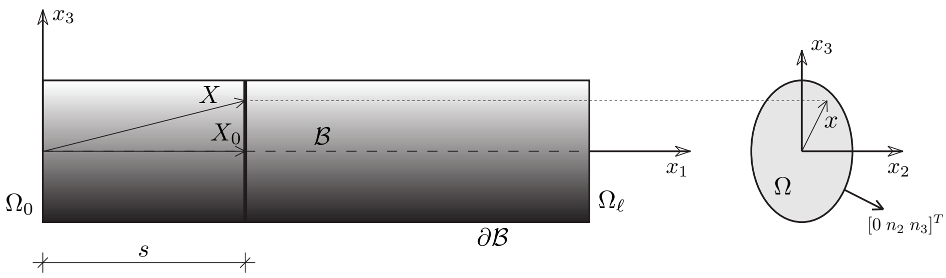

Figure 1.1: The cylindric solid.

Adopting a Voight-like matrix notation, the strain components will be collected in vector $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{g} \end{bmatrix} \tag{1.1a}$$

where $\boldsymbol{e} = \{\epsilon_{11}, \gamma_{12}, \gamma_{13}\}$ collects the terms of the standard SV formulation, while $\boldsymbol{g} = \{\epsilon_{22}, \epsilon_{33}, \gamma_{23}\}$.

Introducing the displacement vector $\boldsymbol{u} = u_k \boldsymbol{e}_k$, the strains-displacements relations of the 3D continuum can be expressed in the following form

$$\boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{u} + \boldsymbol{S}\boldsymbol{u}_{,s} \tag{1.1b}$$

where

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\partial}{\partial x_2} & 0 & 0 \\ \dfrac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial x_3} \\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \end{bmatrix} , \quad \boldsymbol{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eq. (1.1b) allows to express contributions in $\boldsymbol{e}$ and $\boldsymbol{g}$ separately as

$$\boldsymbol{e} = \boldsymbol{D}_e \boldsymbol{u} + \boldsymbol{u}_{,s} \, , \quad \boldsymbol{g} = \boldsymbol{D}_g \boldsymbol{u}, \tag{1.1c}$$

operators $\boldsymbol{D}_e$ and $\boldsymbol{D}_g$ being implicitly defined as blocks of $\boldsymbol{D}$.

With the same notation used in Eq.(1.1a), stresses can collected in vector

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{r} \end{bmatrix} \tag{1.1d}$$

where $\boldsymbol{s} = \{\sigma_{11}, \sigma_{12}, \sigma_{13}\}$ and $\boldsymbol{r} = \{\sigma_{22}, \sigma_{33}, \sigma_{23}\}$.

The equilibrium equations are

$$\begin{aligned} &\boldsymbol{D}^T\boldsymbol{\sigma} + \boldsymbol{S}^T\boldsymbol{\sigma}_{,s} = 0 \quad \text{on} \quad \mathcal{B} \\ &\boldsymbol{N}\boldsymbol{\sigma} = 0 \quad \text{on} \quad \partial\mathcal{B} \end{aligned} \tag{1.1e}$$

where introducing the normal unitary vector to the mantel $\boldsymbol{n} = \{0, n_2, n_3\}$

$$\boldsymbol{N} = \begin{bmatrix} 0 & n_2 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & 0 & 0 & 0 & n_3 & n_2 \end{bmatrix}$$

Finally constitutive equations are

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon} \tag{1.1f}$$

which in the standard case of isotropic material become for instance

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & 0 & 0 & \lambda & \lambda & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \\ \lambda & 0 & 0 & 2\mu+\lambda & \lambda & 0 \\ \lambda & 0 & 0 & \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \end{bmatrix} \tag{1.1g}$$

where μ and λ are the Lamè constants defined as

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)}$$

1.1.2 The generalized SV solution

Following Ieşan the displacement field $\boldsymbol{u}[\boldsymbol{X}]$ is obtained for the case of extension, bending and torsion by integrating along s a particular rigid displacement field, while the solution due to shears can be obtained by a new integration of the displacement field so obtained when the contributions due to extension and torsion are zeroed.

In particular, the initial rigid motion considered is

$$\boldsymbol{u}_{,s} = \boldsymbol{\alpha} + \boldsymbol{\beta} \wedge \boldsymbol{X} \tag{1.2a}$$

with $\boldsymbol{\alpha}$ e $\boldsymbol{\beta}$ linked through the conditions

$$\boldsymbol{\alpha} = \begin{bmatrix} a_E \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} a_T \\ a_{F_3} \\ -a_{F_2} \end{bmatrix}$$

Eq. (1.2a) conveniently can be rewritten in the form

$$\boldsymbol{u}_{,s} = a_E \boldsymbol{u}_{0E} + a_T \boldsymbol{u}_{0T} + a_{F_2}(s\ \boldsymbol{u}_{0F_2} + \boldsymbol{u}_{1F_2}) + a_{F_3}(s\ \boldsymbol{u}_{0F_3} + \boldsymbol{u}_{1F_3}) \tag{1.2b}$$

where

$$\boldsymbol{u}_{0E} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_{0F_2} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_{0F_3} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_{0T} = \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}$$

and

$$\boldsymbol{u}_{1F_2} = \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_{1F_3} = \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix}$$

Note that vectors multiplying coefficients a_β represent a rigid section motion.

The first integration with respect to s gives

$$\begin{aligned} \boldsymbol{u}_1 &= a_E(s\boldsymbol{u}_{0E} + \boldsymbol{u}_{1E}) \\ &+ a_T(s\boldsymbol{u}_{0T} + \boldsymbol{u}_{1T}) \\ &+ a_{F_2}(\frac{1}{2}s^2\boldsymbol{u}_{0F_2} + s\boldsymbol{u}_{1F_2} + \boldsymbol{u}_{2F_2}) \\ &+ a_{F_3}(\frac{1}{2}s^2\boldsymbol{u}_{0F_3} + s\boldsymbol{u}_{1F_3} + \boldsymbol{u}_{2F_3}) \end{aligned} \tag{1.2c}$$

while the second integration furnishes the terms

$$\begin{aligned} \boldsymbol{u}_2 = a_{S_2}(\frac{1}{6}s^3\boldsymbol{u}_{0F_2} + \frac{1}{2}s^2\boldsymbol{u}_{1F_2} + s\boldsymbol{u}_{2F_2} + \boldsymbol{u}_{3F_2}) \\ + a_{S_3}(\frac{1}{6}s^3\boldsymbol{u}_{0F_3} + \frac{1}{2}s^2\boldsymbol{u}_{1F_3} + s\boldsymbol{u}_{2F_3} + \boldsymbol{u}_{3F_3}) \end{aligned} \tag{1.2d}$$

So the final form for the displacement field is

$$\begin{aligned}\boldsymbol{u} &= a_E(s\boldsymbol{u}_{0E} + \boldsymbol{u}_{1E}) \\ &+ a_T(s\boldsymbol{u}_{0T} + \boldsymbol{u}_{1T}) \\ &+ a_{F_2}(\frac{1}{2}s^2\boldsymbol{u}_{0F_2} + s\boldsymbol{u}_{1F_2} + \boldsymbol{u}_{2F_2}) \\ &+ a_{F_3}(\frac{1}{2}s^2\boldsymbol{u}_{0F_3} + s\boldsymbol{u}_{1F_3} + \boldsymbol{u}_{2F_3}) \\ &+ a_{S_2}(\frac{1}{6}s^3\boldsymbol{u}_{0F_2} + \frac{1}{2}s^2\boldsymbol{u}_{1F_2} + s\boldsymbol{u}_{2F_2} + \boldsymbol{u}_{3F_2}) \\ &+ a_{S_3}(\frac{1}{6}s^3\boldsymbol{u}_{0F_3} + \frac{1}{2}s^2\boldsymbol{u}_{1F_3} + s\boldsymbol{u}_{2F_3} + \boldsymbol{u}_{3F_3})\end{aligned} \tag{1.2e}$$

The 6 contributions

$$\boldsymbol{u}_{1E}, \quad \boldsymbol{u}_{1T}, \quad \boldsymbol{u}_{2F_2}, \quad \boldsymbol{u}_{2F_3}, \quad \boldsymbol{u}_{3F_2}, \quad \boldsymbol{u}_{3F_3}$$

depend only on $\boldsymbol{x}$ and appear due to integrations with respect to s.

They represent the in-plane and out-of-plane warping functions and can be evaluated by means of the equilibrium equations (1.1e) once the strain $\boldsymbol{\varepsilon}$ has been obtained from the compatibility condition (1.1b) and the stress $\boldsymbol{\sigma}$ from the elastic constitutive laws (1.1f)

Due to the form assumed for $\boldsymbol{u}$, see Eqs. (1.2e), the equilibrium condition (1.1e) furnishes a series of generalized plane problems defined on the beam section domain $\Omega[s]$ which allow the evaluation of the 18 components of the generalized warping functions (see [13], sections 4.2–4.4). The six constants a_β are finally evaluated in terms of the beam section resultant force $\boldsymbol{N}[s]$ and moment $\boldsymbol{M}[s]$ so defined

$$\boldsymbol{N}[s] = \int_\Omega \boldsymbol{s}\, dA \quad , \quad \boldsymbol{M}[s] = \int_\Omega \boldsymbol{x} \wedge \boldsymbol{s}\, dA. \tag{1.3}$$

where $\boldsymbol{s} = \boldsymbol{S}^T\boldsymbol{\sigma}$ is the traction on the section with positive normal $\boldsymbol{e}_1$.

Denoting as $\boldsymbol{W}_x$ the spin tensor associated to vector $\boldsymbol{x}$ and recalling that for its definition $\boldsymbol{x} \wedge \boldsymbol{s} = \boldsymbol{W}_x\boldsymbol{s}$, the latter of (1.3) can be rewritten also as follows:

$$\boldsymbol{M}[s] = \int_\Omega \boldsymbol{W}_x\boldsymbol{s}\, dA$$

The formulation presented does not involve any a priori assumption and could also be applied to non-homogeneous and non-isotropic materials where the physical intuition of the semi–inverse method, used in the standard derivation of the SV solution, is difficult to apply. The solution is defined apart from a self equilibrated stress state which depends on the exact force distribution on the end bases.

Chapter 2

Semi-analytical formulation for beam-like structures

2.1 A semi analytical formulation for beam-like structures

In this chapter a semi analytical formulation like that proposed in [12, 18, 24] is used to derive a convenient description of slender structures in terms of their axial dimension without loosing the basic three dimensionality of the problem as in the standard SV solution.

A beam description suitable to analyze the largest generality of cross-sections in terms of both composition and shape can be so recovered, as it is potentially possible to consider three dimensional states of stress due to anisotropy and unhomogeneity of the materials as well as end-effects as for example the Vlasov nonuniform warping or local deformation modes of the cross-sections.

The basic idea of the formulation is obtain the equilibrium beam equations exploiting the virtual works principle where a separation of the derivatives with respect to the axial coordinate from the other ones is introduced when expressing the strain-displacement relations. Of a mathematical point of view this allows to separate the solution along the beam axis which is found in a closed form from the the cross-section where a proper FEM discretization is required. This leds to search the eigensolutions of a matrix polynomial when describing the end-effects, while it results quite easy to have the solution far from the end-bases described by a set of linear algebraic equations. It is easy to show as this central solution is equivalent to that of the generalized SV problem as formulated by Ieşan and described in the previous chapter.

2.2 Formulation of the 3D problem

Let consider the same cylinder in figure 1.1 referred to the rectangular Cartesian coordinates system $(\mathcal{O}, s, x_2, x_3)$ with unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ indicated. As in 1.1 $\boldsymbol{e}_1$ runs along the beam axis, while cross-sections are laid in the $x_2 - x_3$ plane. Finally external forces are applied only on the end bases.

The terms in the virtual work equation

$$\delta\mathcal{L}_i - \delta\mathcal{L}_e = 0 \tag{2.1a}$$

can be expressed as follows for the portion of the beam between 2 sections at the infinitesimal distance ds in absence of body forces

$$\delta\mathcal{L}_i = \int_\Omega \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dA\, ds, \quad \delta\mathcal{L}_e = \int_\Omega \left(\delta\boldsymbol{u}^T[s,\boldsymbol{x}]\boldsymbol{s}[s,\boldsymbol{x}] + \delta\boldsymbol{u}^T[s+ds,\boldsymbol{x}]\boldsymbol{s}[s+ds,\boldsymbol{x}]\right) dA \tag{2.1b}$$

$\boldsymbol{s}[s]$ being the tractions on the two sections $s + ds$ and s of normal $\pm\boldsymbol{e}_1$ respectively which can be evaluated as

$$\boldsymbol{s} = \pm\boldsymbol{S}^T\boldsymbol{\sigma} \tag{2.1c}$$

A Taylor expantion of $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ about $s = 0$ gives

$$\mathcal{L}_e = \int_\Omega \left(\delta\boldsymbol{u},_s^T[s,\boldsymbol{x}]\boldsymbol{S}^T\boldsymbol{\sigma}[s,\boldsymbol{x}] + \delta\boldsymbol{u}^T[s,\boldsymbol{x}]\boldsymbol{S}^T\boldsymbol{\sigma},_s[s,\boldsymbol{x}]\right) dA\, ds \tag{2.1d}$$

Exploiting eq. (1.1b), the internal work in (2.1b) becomes

$$\mathcal{L}_i = \int_\Omega (\delta\boldsymbol{u}^T\boldsymbol{D}^T\boldsymbol{\sigma} + \delta\boldsymbol{u},_s^T\boldsymbol{S}^T\boldsymbol{\sigma}) dA\; ds \tag{2.1e}$$

Substituting Eqs. (2.1d) and (2.1e), Eq. (2.1a) becomes

$$\int_\Omega (\delta\boldsymbol{u}^T\boldsymbol{D}^T\boldsymbol{\sigma} - \delta\boldsymbol{u}^T\boldsymbol{S}^T\boldsymbol{\sigma},_s) dA\, ds = 0 \tag{2.1f}$$

Expressing the stresses $\boldsymbol{\sigma}$ through eq. (1.1f) in terms of the strains $\boldsymbol{\varepsilon}$, Eq. (2.1f) becomes

$$\int_\Omega \left\{\delta\boldsymbol{u}^T\boldsymbol{D}^T\boldsymbol{C}\boldsymbol{D}\boldsymbol{u} + \delta\boldsymbol{u}^T(\boldsymbol{D}^T\boldsymbol{C}\boldsymbol{S}\boldsymbol{u},_s - \boldsymbol{S}^T\boldsymbol{C}\boldsymbol{D})\boldsymbol{u},_s - \delta\boldsymbol{u}^T\boldsymbol{S}^T\boldsymbol{C}\boldsymbol{S}\boldsymbol{u},_{ss}\right\} dA = 0 \tag{2.1g}$$

2.3 Solution of the 3D Problem

In order to solve Eq. (2.1g) a proper interpolation of the displacement field $\boldsymbol{u}$ is introduced that can be expressed in a symbolic format as

$$\boldsymbol{u}[s, \boldsymbol{x}] = \boldsymbol{\Phi}[\boldsymbol{x}]\boldsymbol{q}[s] \tag{2.2a}$$

$\boldsymbol{\Phi}[\boldsymbol{x}]$ being the interpolation operator in the cross-section domain and $\boldsymbol{q}[s]$ the nodal values of the unknowns still dependent on s.

Introducing matrix

$$\boldsymbol{\Psi}[\boldsymbol{x}] = \boldsymbol{D}\boldsymbol{\Phi}[\boldsymbol{x}] \tag{2.2b}$$

collecting the derivatives of the shape functions in $\boldsymbol{\Phi}[\boldsymbol{x}]$, Eq. (2.1f) can be rewritten as

$$\boldsymbol{M}\boldsymbol{q}[s],_{ss} - \boldsymbol{W}\boldsymbol{q}[s],_s - \boldsymbol{K}\boldsymbol{q}[s] = \boldsymbol{0} \tag{2.2c}$$

where the terms

$$\begin{aligned} &\boldsymbol{M} = \int_\Omega \boldsymbol{\Phi}^T\boldsymbol{S}^T\boldsymbol{C}\boldsymbol{S}\boldsymbol{\Phi}\, dA, \quad \boldsymbol{W} = \boldsymbol{E} - \boldsymbol{E}^T, \quad \boldsymbol{K} = \int_\Omega \boldsymbol{\Psi}^T\boldsymbol{C}\boldsymbol{\Psi}\, dA, \\ &\boldsymbol{E} = \int_\Omega \boldsymbol{\Psi}^T\boldsymbol{C}\boldsymbol{S}\boldsymbol{\Phi}\, dA \end{aligned} \tag{2.2d}$$

can be calculated by a standard FEM assemblage once a local interpolation for $\boldsymbol{u}$ is defined over a generic element, due to the additivity of integrals.

It is possble to rewrite Eq. (2.2c) to a system of first order differential equations as

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{q},_{ss} \\ \boldsymbol{q},_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{K} \\ \mathbf{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q},_s \\ \boldsymbol{q} \end{bmatrix} \tag{2.3a}$$

So, introducing

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{q},_s \\ \boldsymbol{q} \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \mathbf{I} \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{K} \\ \mathbf{I} & \boldsymbol{0} \end{bmatrix}$$

Eq. (2.3a) is

$$\boldsymbol{\Omega}\boldsymbol{d},_s = \boldsymbol{\Gamma}\boldsymbol{d} \tag{2.3b}$$

where $\boldsymbol{\Omega}$ is a symmetric and posive defined matrix.

The system in (2.3b) can be solved by using the transformation in the Jordan canonical form of the matrix $\boldsymbol{\Omega}^{-1}\boldsymbol{\Gamma}$. As shown by Mielke [16], the Jordan matrix $\boldsymbol{J}$ of the system (2.3b) is characterized by 2 4×4 non-diagonal blocks corresponding to the solution for bendings and shears, 2 2×2 non-diagonal blocks linked to extension and torsion all associated to null eigenvalues in $\boldsymbol{J}$ and by a diagonal part associated to non-null eigenvalues (see Fig. 2.1) whose eigenvectors represent end-effects.

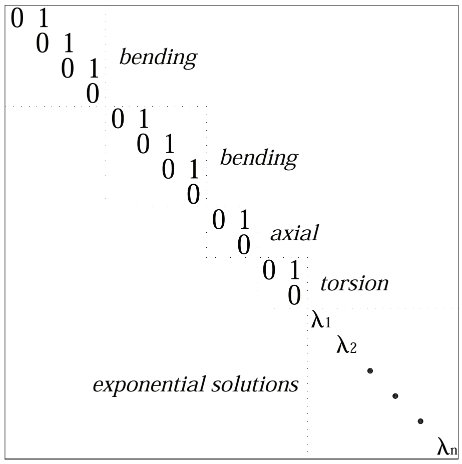

Figure 2.1: Characteristic pattern of the Jordan form of the system.

The solution of the system (2.3b) is then constituted by polynomial contributions which propagate undisturbed along the beam and exponential ones that decay moving away from the first or the end basis respectively. Eigenvalues represent the characteristic decay lenght of the generalized eigenvectors associated so deformative modes linked to the smallest (positive or negative) values of them are more important and will be included in the general beam model that will be derived in the following chapter.

Due to the structure of $\boldsymbol{J}$ extension and torsion should be described by two generalized eigenvectors, while the part of the solution linked to shears and bendings is described in terms of $4 + 4$ generalized eigenvectors.

Practically the central solution due to bendings and shears can be expressed in the form

$$\boldsymbol{q}_\alpha[s] = \begin{bmatrix} \boldsymbol{q}_{0\alpha} & \boldsymbol{q}_{1\alpha} & \boldsymbol{q}_{2\alpha} & \boldsymbol{q}_{3\alpha} \end{bmatrix} \begin{bmatrix} 1 & s & \frac{1}{2}s^2 & \frac{1}{6}s^3 \\ 0 & 1 & s & \frac{1}{2}s^2 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{0\alpha} \\ a_{1\alpha} \\ a_{2\alpha} \\ a_{3\alpha} \end{bmatrix} \tag{2.4a}$$

where $\alpha = F_2$ or F_3, while introducing $\beta = E$ or T we have

$$\boldsymbol{q}_\beta[s] = \begin{bmatrix} \boldsymbol{q}_{0\beta} & \boldsymbol{q}_{1\beta} \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{0\beta} \\ a_{1\beta} \end{bmatrix} \tag{2.4b}$$

for extension and torsion.

Generalized eigenvectors corresponding to the 12 null eigenvalues of $\boldsymbol{J}$ can be calculated introducing a generator vector $\boldsymbol{d}_{0\,k}$ for different Jordan chains k from

$$\boldsymbol{\Gamma d}_{0\,k} = \boldsymbol{0} \tag{2.5a}$$

and then subsequent vectors $\boldsymbol{d}_{i\,k}$ from

$$\boldsymbol{\Gamma d}_{i\,k} = \boldsymbol{\Omega d}_{(i-1)\,k}. \tag{2.5b}$$

Let consider for instant a solution $\boldsymbol{q}_k = \boldsymbol{q}_{0\,k}$ constant with s. In this case $\boldsymbol{d}_{0\,k} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{q}_{0\,k} \end{bmatrix}$. Substituting the assumed solution in Eqs. (2.5a) we obtain

$$\boldsymbol{K q}_{0\,k} = \boldsymbol{0} \tag{2.5c}$$

which means that generator vectors can be obtained as the direction of singularity of $\boldsymbol{K}$ so the rigid translations and the rotation about the beam axis, all constant with s, can be considered. From now on we will denote as $\boldsymbol{q}_{0E}$ and $\boldsymbol{q}_{0T}$ the nodal vectors decribing a rigid displacement of the beam along x_1 a rotation about it, while $\boldsymbol{q}_{0F_2}$ and $\boldsymbol{q}_{0F_3}$ will represent the lateral displacements along x_2 and x_3. Starting from each of the body motions introduced a set of 4 Jordan chains can be generated.

It is easy to demonstrate that a set of vector $\boldsymbol{q}_{1\,k} \dots \boldsymbol{q}_{3\,k}$ calculated as in Eqs. (2.6) are the terms of a Jordan chain whose lead vector is $\boldsymbol{d}_{0\,k}$ and the other terms are $\boldsymbol{d}_{i\,k} = \begin{bmatrix} \boldsymbol{q}_{(i-1)\,k} \\ \boldsymbol{q}_{i\,k} \end{bmatrix}$ with $i = 1$ for extension and torsion ($k = E \,\text{or}\, T$) while it goes from 1 to 3 for the bendig contributions ($k = F_2 \,\text{or}\, F_3$).

$$\begin{cases} \boldsymbol{K q}_{1\,k} = -\boldsymbol{W q}_{0\,k} \\ \boldsymbol{K q}_{2\,k} = \boldsymbol{M q}_{0\,k} - \boldsymbol{W q}_{1\,k} \\ \boldsymbol{K q}_{3\,k} = \boldsymbol{M q}_{1\,k} - \boldsymbol{W q}_{2\,k} \end{cases} \tag{2.6}$$

2.3.1 Meaning of the generalized eigenvectors

Eqs. (2.4a) and (2.4b) can be expanded as follows

$$\begin{aligned}
\boldsymbol{q}_\alpha &= a_{0\alpha}\boldsymbol{q}_{0\alpha} \\
&\quad + a_{1\alpha}(s\boldsymbol{q}_{0\alpha} + \boldsymbol{q}_{1\alpha}) \\
&\quad + a_{2\alpha}(\frac{1}{2}s^2\boldsymbol{q}_{0\alpha} + s\boldsymbol{q}_{1\alpha} + \boldsymbol{q}_{2\alpha}) \\
&\quad + a_{3\alpha}(\frac{1}{6}s^3\boldsymbol{q}_{0\alpha} + \frac{1}{2}s^2\boldsymbol{q}_{1\alpha} + s\boldsymbol{q}_{2\alpha} + \boldsymbol{q}_{3\alpha}), \\
\boldsymbol{q}_\beta &= a_{0\beta}\boldsymbol{q}_{0\beta} \\
&\quad + a_{1\beta}(s\boldsymbol{q}_{0\beta} + \boldsymbol{q}_{1\beta})
\end{aligned} \tag{2.7}$$

where a clear similarity with the displacement field assumed by Ieşan in Eqs. (1.2e) can be seen. A part from the FEM discretization of the solution over the cross-section, Eqs. (2.7) and (1.2e) differ for a rigid body motion represented by the terms $a_{0\alpha}\boldsymbol{q}_{0\alpha}$ and $a_{0\beta}\boldsymbol{q}_{0\beta}$ with $\alpha = F_2$ and F_3 and $\beta = E$ and T which have the clear meaning already discussed, while terms $a_{1\alpha}(\boldsymbol{q}_{0\alpha}s + \boldsymbol{q}_{1\alpha})$ represent a rigid rotations about axes x_3 and x_2 respectively as vectors $\boldsymbol{q}_{1F_2}$ and $\boldsymbol{q}_{1F_3}$ from a comparison with contributions $\boldsymbol{u}_{1F_2} = \{x_2, 0, 0\}$ and $\boldsymbol{u}_{1F_3} = \{x_3, 0, 0\}$ in Eq. (1.2e) are a rotation of the cross-section.

2.3.2 Solvability of the chains

Let introduce now the generalized tensions vector $\boldsymbol{\eta}$ defined by the equivalence

$$\delta\boldsymbol{q}^T\boldsymbol{\eta} = \int_\Omega \delta\boldsymbol{u}^T\boldsymbol{t} = \delta\boldsymbol{q}^T\boldsymbol{E}^T\boldsymbol{q} + \delta\boldsymbol{q}^T\boldsymbol{M}\boldsymbol{q}_{,s}\,. \tag{2.8}$$

All constant vectors $\boldsymbol{q} = \boldsymbol{q}_{0\,j}$ are characterized by null stresses and null generalized tensions $\boldsymbol{\eta}$ which implyes

$$\boldsymbol{E}^T\boldsymbol{q}_{0\,j} = \boldsymbol{0} \tag{2.9}$$

Similary for the two rigid motion defined by the linear solutions $\boldsymbol{q} = \boldsymbol{q}_{0\,\alpha}s + \boldsymbol{q}_{1\,\alpha}$ we have the further condition

$$\boldsymbol{E}^T\boldsymbol{q}_{1\,\alpha} + \boldsymbol{M}\boldsymbol{q}_{0\,\alpha} = \boldsymbol{0} \tag{2.10}$$

From (2.9) it is possible to show that the right-end-side term in the first of Eq. (2.6) has no components along the directions $\boldsymbol{q}_{0\,j}$ of singularity for $\boldsymbol{K}$. Recalling the defintition of the operator $\boldsymbol{W}$ in Eq. (2.2d) we have

$$\boldsymbol{q}_{0\,j}^T \boldsymbol{W} \boldsymbol{q}_{0\,k} = \boldsymbol{q}_{0\,k}^T \boldsymbol{E}^T \boldsymbol{q}_{0\,j} - \boldsymbol{q}_{0\,j}^T \boldsymbol{E}^T \boldsymbol{q}_{0\,k} \tag{2.11a}$$

From Eqs. (2.9) and (2.10) it is also symply to show that

$$\boldsymbol{q}_{0\,j}^T (\boldsymbol{M} \boldsymbol{q}_{0\,\alpha} - \boldsymbol{W} \boldsymbol{q}_{1\,\alpha}) = 0. \tag{2.11b}$$

Always from (2.2d), Eq. (2.11b) becomes

$$\boldsymbol{q}_{0\,j}^T (\boldsymbol{M} \boldsymbol{q}_{0\,\alpha} + \boldsymbol{E}^T \boldsymbol{q}_{1\,\alpha}) - \boldsymbol{q}_{1\,\alpha}^T \boldsymbol{E}^T \boldsymbol{q}_{0\,j} = 0 \tag{2.11c}$$

With respect to the last of (2.6) it is possible to demonstrate the orthogonality between the known term and the only directions $\boldsymbol{q}_{0\,\alpha}$.

We have

$$\boldsymbol{q}_{0\,j}^T (\boldsymbol{M} \boldsymbol{q}_{1\,\alpha} - \boldsymbol{W} \boldsymbol{q}_{2\,\alpha}) = \boldsymbol{q}_{0\,j}^T \boldsymbol{M} \boldsymbol{q}_{1\,\alpha} - \boldsymbol{q}_{1\,j}^T \boldsymbol{K} \boldsymbol{q}_{2\,\alpha} \tag{2.11d}$$

recalling the first of (2.6).

Substituting the second Eq. in (2.6), Eq. (2.11d) becomes

$$\boldsymbol{q}_{0\,j}^T (\boldsymbol{M} \boldsymbol{q}_{1\,\alpha} - \boldsymbol{W} \boldsymbol{q}_{2\,\alpha}) = \boldsymbol{q}_{0\,j}^T \boldsymbol{M} \boldsymbol{q}_{1\,\alpha} - \boldsymbol{q}_{1\,j}^T (\boldsymbol{M} \boldsymbol{q}_{0\,\alpha} - \boldsymbol{W} \boldsymbol{q}_{1\,\alpha}) \tag{2.11e}$$

which expressing $\boldsymbol{W}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{E}^T$ (see Eq. (2.2d)) and recalling Eq. (2.10) furnishes

$$\boldsymbol{q}_{0\,j}^T (\boldsymbol{M} \boldsymbol{q}_{1\,\alpha} - \boldsymbol{W} \boldsymbol{q}_{2\,\alpha}) = \boldsymbol{q}_{1\,\alpha}^T (\boldsymbol{M} \boldsymbol{q}_{0\,j} + \boldsymbol{E}^T \boldsymbol{q}_{1,\,j}) \tag{2.11f}$$

that clearly is zero only when $j = \alpha$.

2.4 Strain Energy

Letting

$$\boldsymbol{L}_\varepsilon[\boldsymbol{x}] = \Big[\boldsymbol{S} \boldsymbol{\Phi}[\boldsymbol{x}], \quad \boldsymbol{\Psi}[\boldsymbol{x}] \Big]$$

strains can be written as $\boldsymbol{\varepsilon} = \boldsymbol{L}_\epsilon[\boldsymbol{x}] \boldsymbol{d}[s]$ and the strain energy density becomes

$$\Phi = \frac{1}{2} \int_\Omega \boldsymbol{\varepsilon}^T \boldsymbol{C} \boldsymbol{\varepsilon} dA = \frac{1}{2} \boldsymbol{d}^T \boldsymbol{C}_d \boldsymbol{d} \tag{2.12}$$

where

$$\boldsymbol{C}_d = \int_\Omega \boldsymbol{L}_\varepsilon^T \boldsymbol{C} \boldsymbol{L}_\varepsilon \, dA = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{E}^T \\ \boldsymbol{E} & \boldsymbol{K} \end{bmatrix}$$

For now, only the terms due to the central solution of Eq. (2.3b) are considered.

The terms $\boldsymbol{q}_k = a_{0\,k}\boldsymbol{q}_{0\,k}$ and $\boldsymbol{q}_\alpha = a_{1\,\alpha}(\boldsymbol{q}_{0\,\alpha}s + \boldsymbol{q}_{1\,\alpha})$ are rigid motions. Vectors

$$\boldsymbol{d}_k = a_{0\,k} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{q}_{0\,k} \end{bmatrix} = \boldsymbol{d}_{0\,k} \tag{2.13a}$$

and

$$\boldsymbol{d}_\alpha = a_{1\,\alpha} \begin{bmatrix} \boldsymbol{q}_{0\,\alpha} \\ \boldsymbol{q}_{0\,\alpha}s + \boldsymbol{q}_{1\,\alpha} \end{bmatrix} = a_{1\,\alpha}(s\boldsymbol{d}_{0\,\alpha} + \boldsymbol{d}_{1\,\alpha}) \tag{2.13b}$$

therefore produce no strain energy whcih implyes

$$\begin{cases} \boldsymbol{C}_d \boldsymbol{d}_{0\,k} = \boldsymbol{0} \quad \forall k \\ \boldsymbol{C}_d(s\boldsymbol{d}_{0\,\alpha} + \boldsymbol{d}_{1\,\alpha}) = \boldsymbol{0} \Rightarrow \boldsymbol{C}_d \boldsymbol{d}_{1\,\alpha} = \boldsymbol{0} \quad \alpha = F_1, F_2 \end{cases} \tag{2.13c}$$

A part from the rigid motion in (2.7), extension and torsion will contribute with the displacement

$$\boldsymbol{q}_\beta(s) = a_{1\,\beta}\left(s\boldsymbol{q}_{0\,\beta} + \boldsymbol{q}_{1\,\beta}\right) \tag{2.14a}$$

or in terms of vector $\boldsymbol{d}$

$$\boldsymbol{d}_\beta(s) = a_{1\,\beta} \begin{bmatrix} \boldsymbol{q}_{0\,\beta} \\ \boldsymbol{q}_{1\,\beta} + s\boldsymbol{q}_{0\,\beta} \end{bmatrix} = a_{1\,\beta}(s\boldsymbol{d}_{0\,\beta} + \boldsymbol{d}_{1\,\beta}) \tag{2.14b}$$

where, recalling Eq. (2.13c), only teh term $a_{1\,\beta}\boldsymbol{d}_{1\,\beta}$ will be included to the strain energy.

From (2.7) we have for bending

$$\begin{aligned} \boldsymbol{q}_\alpha[s] = {} & a_{3\,\alpha}\left(\frac{1}{6}s^3\boldsymbol{q}_{0\,\alpha} + \frac{1}{2}s^2\boldsymbol{q}_{1\,\alpha} + s\boldsymbol{q}_{2\,\alpha} + \boldsymbol{q}_{3\,\alpha}\right) \\ & + a_{2\,\alpha}\left(\frac{1}{2}s^2\boldsymbol{q}_{0\,\alpha} + s\boldsymbol{q}_{1\,\alpha} + \boldsymbol{q}_{2\,\alpha}\right) \end{aligned} \tag{2.15a}$$

and for $\boldsymbol{d}_\alpha[s]$

$$\begin{aligned}\boldsymbol{d}_\alpha[s] &= a_{3\,\alpha}\begin{bmatrix}\frac{1}{2}s^2\boldsymbol{q}_{0\,\alpha}+s\boldsymbol{q}_{1\,\alpha}+\boldsymbol{q}_{2\,\alpha}\\ \frac{1}{6}s^3\boldsymbol{q}_{0\,\alpha}+\frac{1}{2}s^2\boldsymbol{q}_{1\,\alpha}+s\boldsymbol{q}_{2\,\alpha}+\boldsymbol{q}_{3\,\alpha}\end{bmatrix}+a_{2\,\alpha}\begin{bmatrix}s\boldsymbol{q}_{0\,\alpha}+\boldsymbol{q}_{1\,\alpha}\\ \frac{1}{2}s^2\boldsymbol{q}_{0\,\alpha}+s\boldsymbol{q}_{1\,\alpha}+\boldsymbol{q}_{2\,\alpha}\end{bmatrix}\\ &= a_{3\,\alpha}(\frac{1}{6}s^3\boldsymbol{d}_{0\,\alpha}+\frac{1}{2}s^2\boldsymbol{d}_{1\,\alpha}+s\boldsymbol{d}_{2\,\alpha}+\boldsymbol{d}_{3\,\alpha})\\ &+ a_{2\,\alpha}(\frac{1}{2}s^2\boldsymbol{d}_{0\,\alpha}+s\boldsymbol{d}_{1\,\alpha}+\boldsymbol{d}_{2\,\alpha})\end{aligned} \tag{2.15b}$$

that recalling Eqs. in (2.13c) gives as only contributions

$$a_{3\,\alpha}(s\boldsymbol{d}_{2\,\alpha}+\boldsymbol{d}_{3\,\alpha})+a_{2\,\alpha}(\boldsymbol{d}_{2\,\alpha})$$

Introducing quantities

$$\begin{aligned}\boldsymbol{Q}_d &= \begin{bmatrix}\boldsymbol{d}_{1\,E} & \boldsymbol{d}_{2\,F_2} & \boldsymbol{d}_{2\,F_3} & \boldsymbol{d}_{1\,T} & \boldsymbol{d}_{3\,F_2} & \boldsymbol{d}_{3\,F_3}\end{bmatrix}\\ \boldsymbol{a} &= \begin{bmatrix}a_E & a_{F_2} & a_{F_3} & a_T & a_{S_2} & a_{S_3}\end{bmatrix}\end{aligned} \tag{2.16}$$

with

$$\begin{cases}a_E = a_{1\,E},\ a_T = a_{1\,T},\\ a_{F_2} = a_{2\,F_2} + s\,a_{3\,F_2},\ a_{S_2} = a_{3\,F_2},\\ a_{F_3} = a_{2\,F_3} + s\,a_{3\,F_3},\ a_{S_3} = a_{3\,F_3},\end{cases}$$

the more compact expression of $\boldsymbol{d}[s]$ is obtained

$$\boldsymbol{d}[s] = \boldsymbol{Q}_d\boldsymbol{a}[s] \tag{2.17}$$

The strain energy in (2.12) is then

$$\Phi = \frac{1}{2}\boldsymbol{d}[s]^T\boldsymbol{C}_d\boldsymbol{d}[s] = \frac{1}{2}\boldsymbol{a}^T\boldsymbol{C}_a\boldsymbol{a} \tag{2.18}$$

where

$$\boldsymbol{C}_a[s] = \boldsymbol{Q}_d^T\boldsymbol{C}_d\boldsymbol{Q}_d$$

Modal amplitudes in $\boldsymbol{a}$ can be expressed in terms of the cross-section resultants. The constitutive laws allow to evaluate stresses as

$$\boldsymbol{\sigma} = \boldsymbol{L}_\sigma[\boldsymbol{x}]\boldsymbol{d}$$

where $\boldsymbol{L}_\sigma[\boldsymbol{x}] = \boldsymbol{C}\boldsymbol{L}_\varepsilon[\boldsymbol{x}]$. Eqs. (1.3) give then

$$\boldsymbol{N} = \int_\Omega\{\boldsymbol{S}^T\boldsymbol{L}_\sigma dA\}\,\boldsymbol{d},\quad \boldsymbol{M} = \int_\Omega\{\boldsymbol{W}_x\boldsymbol{S}^T\boldsymbol{L}_\sigma dA\}\,\boldsymbol{d} \tag{2.19}$$

that introducing $\boldsymbol{t}_p = \{\boldsymbol{N}, \boldsymbol{M}\}$ and matrix

$$\boldsymbol{L}_p = \begin{bmatrix} \int_\Omega \boldsymbol{S}^T \boldsymbol{L}_\sigma dA \\ \int_\Omega \boldsymbol{W}_x \boldsymbol{S}^T \boldsymbol{L}_\sigma dA \end{bmatrix} = \begin{bmatrix} \int_\Omega \boldsymbol{S}^T \boldsymbol{C}\boldsymbol{S}\boldsymbol{\Phi} dA & \int_\Omega \boldsymbol{S}^T \boldsymbol{C}\boldsymbol{\Psi} dA \\ \int_\Omega \boldsymbol{W}_x \boldsymbol{S}^T \boldsymbol{C}\boldsymbol{S}\boldsymbol{\Phi} dA & \int_\Omega \boldsymbol{W}_x \boldsymbol{S}^T \boldsymbol{C}\boldsymbol{\Psi} dA \end{bmatrix}$$

can be rewritten as

$$\boldsymbol{t}_p = \boldsymbol{L}_p \boldsymbol{d} \tag{2.20}$$

Recalling (2.14b) and (2.15b) and that the following condition holds with respect to the rigid part of the motion in (2.13a) and (2.13b)

$$\begin{cases} \boldsymbol{L}_p \boldsymbol{d}_{0\,k} = \boldsymbol{0} \quad \forall k \\ \boldsymbol{L}_p (s\boldsymbol{d}_{0\,\alpha} + \boldsymbol{d}_{1\,\alpha}) = \boldsymbol{0} \Rightarrow \boldsymbol{L}_p \boldsymbol{d}_{1\,\alpha} = \boldsymbol{0} \quad \alpha = F_1, F_2 \end{cases} \tag{2.21}$$

vector $\boldsymbol{d}$ can be evaluated as in (2.17) and Eq. (2.20) becomes

$$\boldsymbol{t}_p = \boldsymbol{Q}_a \boldsymbol{a}, \quad \boldsymbol{Q}_a = \boldsymbol{L}_p \boldsymbol{Q}_d \tag{2.22}$$

Finally, introducing $\boldsymbol{Q}_t = \boldsymbol{Q}_a^{-1}$ the complementary form of the strain energy in (2.18) becomes

$$\psi[s] = \frac{1}{2} \boldsymbol{t}_p[s]^T \boldsymbol{H} \boldsymbol{t}_p[s] \tag{2.23}$$

where $\boldsymbol{H} = \boldsymbol{Q}_t^T \boldsymbol{C}_a \boldsymbol{Q}_t$ is the flexibility matrix of the cross-section.

Chapter 3

A generalized beam model

3.1 A generalized mixed beam model

In this chapter a beam model suitable to analyze the largest generality of cross-sections in terms of both composition and shape is proposed. The basic idea is to reuse the central solution of the generalized SV problem described in the previous chapter enriched with some of the modal shapes corresponding to the non null eigenvalues of the problem (2.3b), independently amplified along the beam axis through variable warping descriptors.

A mixed approach will be followed with an essential kinematical description based on a rigid section motion and the additional in-plane or out-of-plane modal shapes of the cross section, while the stress field due to both the central solution and the additional contributions will be exactly taken into account. However, this is sufficient to recover the exact expression of the internal work in terms of generalized quantities and the costitutive law between generalized strains and stresses.

3.2 The displacement field

The displacement field is expressed as:

$$\boldsymbol{u}[\boldsymbol{X}] = \boldsymbol{u}_0[s] + \boldsymbol{\varphi}[s] \wedge \boldsymbol{x} + \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{\mu}[s] \tag{3.1}$$

where $\boldsymbol{u}_0[s]$ and $\boldsymbol{\varphi}[s]$ are the mean translation and rotation of the section, while $\boldsymbol{\mu}[s]$ give the variability, along the beam axis, of the warping vectors

$\bar{\boldsymbol{\omega}}_i[\boldsymbol{x}] = \boldsymbol{\Phi}[\boldsymbol{x}]\bar{\boldsymbol{q}}_{\omega i}$ collected in the modal matrix $\boldsymbol{A}_\omega$ as

$$\boldsymbol{A}_\omega[\boldsymbol{x}] = \left[\bar{\boldsymbol{\omega}}_1[\boldsymbol{x}] \quad ... \quad \bar{\boldsymbol{\omega}}_n[\boldsymbol{x}]\right] \tag{3.2}$$

Warping modes are normalized according to the following conditions:

$$\int_\Omega \bar{\boldsymbol{\omega}}_k[\boldsymbol{x}]\, dA = \int_\Omega \boldsymbol{W}_x \bar{\boldsymbol{\omega}}_k[\boldsymbol{x}]\, dA = \mathbf{0}\,. \tag{3.3}$$

that is we scale the original modes $\boldsymbol{\omega}_k$ as follows

$$\bar{\boldsymbol{\omega}}_k[\boldsymbol{x}] = \boldsymbol{\omega}_k[\boldsymbol{x}] - \boldsymbol{k}_k - \boldsymbol{W}_x \boldsymbol{c}_k$$

where constants in $\boldsymbol{k}_k$ and in $\boldsymbol{c}_k$ are calculated from Eq. (4.10) as follows

$$\begin{bmatrix} A\boldsymbol{I} & \int_\Omega \boldsymbol{W}_x \\ \int_\Omega \boldsymbol{W}_x & \int_\Omega \boldsymbol{W}_x \boldsymbol{W}_x \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_k \\ \boldsymbol{c}_k \end{bmatrix} = \begin{bmatrix} \int_\Omega \boldsymbol{\omega}_k \\ \int_\Omega \boldsymbol{W}_x \boldsymbol{\omega}_k \end{bmatrix}$$

If the origin of the system $\mathcal{O}$ is established at the geometrical centroid of the cross-section and axes x_2 and x_3 coincide with its principal directions, we have $\int_\Omega \boldsymbol{W}_x = \mathbf{0}$ and $\int_\Omega \boldsymbol{W}_x \boldsymbol{W}_x = -\mathrm{diag}[J_o,\, J_{22},\, J_{33}]$ where $J_{33} = \int_\Omega x_2^2$, $J_{22} = \int_\Omega x_3^2$ and $J_o = J_{22} + J_{33}$.

From the displacements in (3.1), exploiting (1.1b), the following expression for the strains collected in $\boldsymbol{e}$ and $\boldsymbol{g}$ is obtained

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{\varepsilon}_L[s] + \boldsymbol{\chi}_L[s] \wedge \boldsymbol{x} + \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{\mu}_{,s}[s] + \boldsymbol{D}_{e\omega}[\boldsymbol{x}]\boldsymbol{\mu}[s] \\ \boldsymbol{g} &= \boldsymbol{D}_{g\omega}[\boldsymbol{x}]\boldsymbol{\mu}[s] \end{aligned} \tag{3.4}$$

In Eq.(3.4) $\boldsymbol{\varepsilon}_L[s]$ and $\boldsymbol{\chi}_L[s]$ are generalized strain parameters so defined

$$\boldsymbol{\varepsilon}_L[s] = \boldsymbol{u}_{0,s}[s] + \boldsymbol{e}_1 \wedge \boldsymbol{\varphi}[s], \quad \boldsymbol{\chi}_L[s] = \boldsymbol{\varphi}_{,s}[s] \tag{3.5}$$

while $\boldsymbol{D}_{e\omega}[\boldsymbol{x}] = \boldsymbol{D}_e \boldsymbol{A}_\omega[\boldsymbol{x}]$ and $\boldsymbol{D}_{g\omega}[\boldsymbol{x}] = \boldsymbol{D}_g \boldsymbol{A}_\omega[\boldsymbol{x}]$.

The beam internal work is

$$\mathcal{W}[s] = \int_\ell \int_\Omega \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dAds = \int_\ell \int_\Omega \left\{\boldsymbol{s}^T \boldsymbol{e} + \boldsymbol{r}^T \boldsymbol{g}\right\} dAds \tag{3.6}$$

that, substituting the strain expression of the Eqs.(3.4), becomes

$$\mathcal{W}[s] = \int_\ell \{\boldsymbol{N}[s] \cdot \boldsymbol{\varepsilon}_L[s] + \boldsymbol{M}[s] \cdot \boldsymbol{\chi}_L[s] + \boldsymbol{B}[s] \cdot \boldsymbol{\mu}_{,s}[s] + \boldsymbol{T}[s] \cdot \boldsymbol{\mu}[s]\}\, ds \tag{3.7}$$

where the resultant force $\boldsymbol{N}[s]$ and moment $\boldsymbol{M}[s]$ are the same of Eq. (1.3) while the generalized actions due to the non-uniform warping i.e. the *bi-moment* $\boldsymbol{B}[s]$ and the *bi-shear* $\boldsymbol{T}[s]$ are so defined

$$\boldsymbol{B}[s] = \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{s}\, dA, \quad \boldsymbol{T}[s] = \int_\Omega \left\{ \boldsymbol{D}_{e\omega}^T \boldsymbol{s} + \boldsymbol{D}_{g\omega}^T \boldsymbol{r} \right\} dA \tag{3.8}$$

independently on the stress description. Recalling Eq. (1.1b) which defines operators $\boldsymbol{D}_e$ and $\boldsymbol{D}_g$ the latter of Eq. (3.8) can be rewritten in a more compact format as

$$\boldsymbol{T}[s] = \int_\Omega \boldsymbol{D}_\omega^T \boldsymbol{\sigma}\, dA$$

having introduced $\boldsymbol{D}_\omega = \boldsymbol{D}\boldsymbol{A}_\omega = \boldsymbol{\Psi}[\bar{\boldsymbol{q}}_{\omega\,1} \dots \bar{\boldsymbol{q}}_{\omega\,n}]$.

Static assumptions

Stresses $\boldsymbol{\sigma}$ are independently described as a part due to the central solution of Eq. (2.3b) called *primary stress* and denoted as $\boldsymbol{\sigma}_p$ and an additional term $\boldsymbol{\sigma}_\omega$ due to the warping modes variable with s added to this solution and called *secondary stress*

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_p + \boldsymbol{\sigma}_\omega \tag{3.9}$$

In particular the following distribution on the section is adopted

$$\boldsymbol{\sigma} = \boldsymbol{L}_\sigma[\boldsymbol{x}]\boldsymbol{d}[s] = \boldsymbol{L}_\sigma[\boldsymbol{x}](\boldsymbol{Q}_p\boldsymbol{a}_p[s] + \boldsymbol{Q}_\omega\boldsymbol{a}_\omega[s]) \tag{3.10}$$

where $\boldsymbol{a}_p[s] = \{a_1 \dots a_6\}$ and $\boldsymbol{a}_\omega[s] = \{a_{\omega\,1} \dots a_{\omega\,n},\, a_{\omega\,(n+1)} \dots a_{\omega\,(n+n)}\}$ are generalized strain parameters to be defined using Eqs. (1.3) and (3.8), while modal matrices $\boldsymbol{Q}_p$ and $\boldsymbol{Q}_\omega$ collect the modes due to the central solution (2.16) and the contributions due to the warping eigenvectors. The latter is defined as follows

$$\boldsymbol{Q}_\omega = \left[\boldsymbol{d}_{\omega\,1} \quad \dots \quad \boldsymbol{d}_{\omega\,n}, \boldsymbol{d}_{\omega\,(n+1)} \quad \dots \quad \boldsymbol{d}_{\omega\,(n+n)}\right] \tag{3.11}$$

with

$$\boldsymbol{d}_{\omega\,i} = \begin{bmatrix} \bar{\boldsymbol{q}}_{\omega\,i} \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{d}_{\omega\,(n+i)} = \begin{bmatrix} \boldsymbol{0} \\ \bar{\boldsymbol{q}}_{\omega\,i} \end{bmatrix}$$

The definition of the resultant force and moment $\boldsymbol{N}$ and $\boldsymbol{M}$ leds to Eq. (2.19) once more, while with respect to the bi-moment $\boldsymbol{B}$ and the bi-shear $\boldsymbol{T}$ in Eq. (3.8) we obtain

$$\boldsymbol{B} = \int_\Omega \{\boldsymbol{A}_\omega^T \boldsymbol{S}^T \boldsymbol{L}_\sigma dA\}\, \boldsymbol{d}, \quad \boldsymbol{T} = \int_\Omega \{\boldsymbol{D}_\omega^T \boldsymbol{L}_\sigma dA\}\, \boldsymbol{d} \tag{3.12}$$

So introducing $\boldsymbol{t} = \{\boldsymbol{t}_p, \boldsymbol{t}_\omega\}$, where $\boldsymbol{t}_\omega = \{\boldsymbol{B}, \boldsymbol{T}\}$ collects the actions due to warping modes, and matrix

$$\boldsymbol{L}_\omega = \begin{bmatrix} \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{S}^T \boldsymbol{L}_\sigma dA \\ \int_\Omega \boldsymbol{D}_\omega^T \boldsymbol{L}_\sigma dA \end{bmatrix} = \begin{bmatrix} \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{S}^T \boldsymbol{C}\boldsymbol{S}\boldsymbol{\Phi} dA & \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{S}^T \boldsymbol{C}\boldsymbol{\Psi} dA \\ \int_\Omega \boldsymbol{D}_\omega^T \boldsymbol{C}\boldsymbol{S}\boldsymbol{\Phi} dA & \int_\Omega \boldsymbol{D}_\omega^T \boldsymbol{C}\boldsymbol{\Psi} dA \end{bmatrix}$$

it is possible to write

$$\boldsymbol{t} = \boldsymbol{L}\boldsymbol{d} \tag{3.13}$$

where

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_p \\ \boldsymbol{L}_\omega \end{bmatrix}$$

From Eq. (3.13) we obtain then

$$\boldsymbol{t} = \boldsymbol{Q}_{a\,p}\boldsymbol{a}_p + \boldsymbol{Q}_{a\,\omega}\boldsymbol{a}_\omega, \qquad \boldsymbol{Q}_{a\,p} = \boldsymbol{L}\boldsymbol{Q}_p,\ \boldsymbol{Q}_{a\,\omega} = \boldsymbol{L}\boldsymbol{Q}_\omega \tag{3.14}$$

that can be also rearranged in the more compact form

$$\begin{aligned} &\boldsymbol{t} = \boldsymbol{Q}_a \boldsymbol{a} \\ &\boldsymbol{Q}_a = \begin{bmatrix} \boldsymbol{L}\boldsymbol{Q}_p & \boldsymbol{L}\boldsymbol{Q}_\omega \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_p \\ \boldsymbol{a}_\omega \end{bmatrix} \end{aligned} \tag{3.15}$$

Finally, introducing $\boldsymbol{Q}_t = \boldsymbol{Q}_a^{-1}$, as for the central solution studied in the previous chapter, the strain energy in its complementary form (2.18) becomes

$$\psi[s] = \frac{1}{2}\boldsymbol{t}[s]^T \boldsymbol{H}\boldsymbol{t}[s] \tag{3.16}$$

where $\boldsymbol{H} = \boldsymbol{Q}_t^T \boldsymbol{C}_a \boldsymbol{Q}_t$ is the flexibility matrix of the cross-section.

Chapter 4

Isotropic and homogeneous beam with variable warpings

4.1 The SV solution for homogenous cross-section with isotropic material

The SV problem as formulated in chapter 1 can be easly solved in the case of homogenous and isotropic beams. In the following the analytical solution for the in-plane warping functions is reported, while shear and torsion aut-of-plane warpings must be calculated numerically as solution of a set of 3 Neumann boundary-value problems defined on the cross-section.

4.1.1 Homogenous cross-section with isotropic material

From now on we assume the reference system aligned with the principal axes of inertia of the cross-section and its origin located in the centroid.

In the isotropic case the equilibrium equations furnish (see [13], sec.1.7) the following expression for the generalized warping functions,

$$
\begin{aligned}
&\boldsymbol{u}_{1E} = \begin{bmatrix} 0 \\ -\nu x_2 \\ -\nu x_3 \end{bmatrix}, \quad \boldsymbol{u}_{2F_2} = \begin{bmatrix} 0 \\ \nu \dfrac{x_3^2 - x_2^2}{2} \\ -\nu x_2 x_3 \end{bmatrix}, \quad \boldsymbol{u}_{2F_3} = \begin{bmatrix} 0 \\ -\nu x_2 x_3 \\ \nu \dfrac{x_2^2 - x_3^2}{2} \end{bmatrix}, \\
&\boldsymbol{u}_{1T} = \begin{bmatrix} \omega_1[\boldsymbol{x}] \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_{3F_2} = \begin{bmatrix} \omega_3[\boldsymbol{x}] \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_{3F_3} = \begin{bmatrix} \omega_2[\boldsymbol{x}] \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{4.1}
$$

where introducing the vector $\boldsymbol{\omega}[\boldsymbol{x}] = \{\omega_1, \omega_2, \omega_3\}$, the set of PDEs to solve on the cross-section Ω is defined as follows

$$\begin{cases} \boldsymbol{\omega}_{,22} + \boldsymbol{\omega}_{,33} + 2\boldsymbol{x} = \mathbf{0} & \in \Omega \\ n_2 \boldsymbol{\omega}_{,2} + n_3 \boldsymbol{\omega}_{,3} = \boldsymbol{g} & \in \partial\Omega \end{cases} \quad , \quad \boldsymbol{g} = \begin{bmatrix} x_3 n_2 - x_2 n_3, \\ \nu \left(\dfrac{x_2^2 - x_3^2}{2} n_2 + x_2 x_3 n_3 \right) \\ -\nu \left(\dfrac{x_2^2 - x_3^2}{2} n_3 - x_2 x_3 n_2 \right) \end{bmatrix} . \tag{4.2}$$

The stress solution

Exploiting Eq. (1.3) the only nonzero stress components are (see [13, 20, 15])

$$\boldsymbol{s} = \boldsymbol{D}_n[\boldsymbol{x}]\boldsymbol{N}[s] + \boldsymbol{D}_m[\boldsymbol{x}]\boldsymbol{M}[s] \tag{4.3}$$

where

$$\boldsymbol{D}_n = \begin{bmatrix} \dfrac{1}{A} & 0 & 0 \\ 0 & D_{\tau\,22} & D_{\tau\,23} \\ 0 & D_{\tau\,32} & D_{\tau\,33} \end{bmatrix} \quad , \quad \boldsymbol{D}_m = \begin{bmatrix} 0 & \dfrac{x_3}{J_{22}} & -\dfrac{x_2}{J_{33}} \\ D_{\tau\,11} & 0 & 0 \\ D_{\tau\,21} & 0 & 0 \end{bmatrix} \tag{4.4}$$

with

$$\begin{cases} D_{\tau\,11} = \dfrac{1}{J_t}(\omega_{1,2} - x_3), \\ D_{\tau\,21} = \dfrac{1}{J_t}(\omega_{1,3} + x_2), \\ D_{\tau\,22} = \dfrac{c_3}{J_t}(\omega_{1,2} - x_3) + \dfrac{2\omega_{2,2} + \nu(x_3^2 - x_2^2)}{4J_{33}(1+\nu)}, \\ D_{\tau\,23} = -\dfrac{c_2}{J_t}(\omega_{1,2} - x_3) + \dfrac{\omega_{3,2} - \nu x_2 x_3}{2J_{22}(1+\nu)}, \\ D_{\tau\,32} = \dfrac{c_3}{J_t}(\omega_{1,3} + x_2) + \dfrac{\omega_{2,3} - \nu x_2 x_3}{2J_{33}(1+\nu)}, \\ D_{\tau\,33} = -\dfrac{c_2}{J_t}(\omega_{1,3} + x_2) + \dfrac{2\omega_{3,3} + \nu(x_2^2 - x_3^2)}{4\,J_{22}(1+\nu)} \end{cases} \tag{4.5}$$

J_{22} and J_{33} being the principal inertia moments of the cross-section, ν the Poisson ratio, while J_t, c_2 and c_3 are the *torsional inertia* and *shear center coordinates* defined as

$$\begin{aligned} J_t &\equiv \int_\Omega \{(\omega_{1,3} + x_2)x_2 - (\omega_{1,2} - x_3)x_3\}\, dA\,, \\ c_2 &\equiv \frac{1}{2\,J_{22}(1+\nu)} \left\{ \int_\Omega (x_2\omega_{3,3} - x_3\omega_{3,2})dA + \frac{\nu}{2}\int_\Omega x_2(x_2^2 + x_3^2) \right\}, \\ c_3 &\equiv \frac{1}{2\,J_{33}(1+\nu)} \left\{ \int_\Omega (x_3\omega_{2,2} - x_2\omega_{2,3})dA + \frac{\nu}{2}\int_\Omega x_3(x_2^2 + x_3^2) \right\} \end{aligned} \tag{4.6}$$

4.2 A generalized linear beam model based on an extension of the SV solution

In this section a mixed beam model is obtained on the basis of the Hellinger–Reissner variational principle, introducing an independent description for the 3D stress and displacement fields expressed in terms of generalized quantities varying along s and of an assumed distribution on the beam section domain. They are then introduced in the Hellinger–Reissner functional

$$\Pi_{HR} \equiv \mathcal{W} - \psi - \mathcal{L}_{ext} \tag{4.7}$$

where the complementary strain energy ψ and the internal work $\mathcal{W}$ are

$$\psi = \frac{1}{2}\int_\ell \int_\Omega \boldsymbol{\sigma} : \boldsymbol{C}^{-1}\boldsymbol{\sigma} \, dA \, ds \quad , \quad \mathcal{W} = \int_\ell \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}[\boldsymbol{u}] \, dA \, ds \tag{4.8}$$

while $\mathcal{L}_{ext}$ represents the external work and : is the scalar product. The integration over the cross-section allows Eq.(4.7) to be expressed in terms of generalized stress and displacement parameters alone.

The beam model so obtained can be used for all kinds of cross-sections and is not limited to thin-walled ones. The Vlasov and SV solution are recovered as particular cases.

4.2.1 Kinematic assumptions and internal work

The beam kinematics is described as

$$\boldsymbol{u}[\boldsymbol{X}] = \boldsymbol{u}_0[s] + \boldsymbol{\varphi}[s] \wedge \boldsymbol{x} + \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{\mu}[s] \tag{4.9}$$

where $\boldsymbol{u}_0[s]$ and $\boldsymbol{\varphi}[s]$ are the mean translation and rotation of the section while $\boldsymbol{\mu}[s]$ give the variability, along the beam axis, of the SV warping $\bar{\boldsymbol{\omega}}[\boldsymbol{x}]$ collected in $\boldsymbol{A}_\omega[\boldsymbol{X}]$ as

$$\boldsymbol{A}_\omega[\boldsymbol{x}] = \begin{bmatrix} \bar{\omega}_1 & \bar{\omega}_2 & \bar{\omega}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Warpings $\bar{\boldsymbol{\omega}}[\boldsymbol{x}]$ are evaluated as in Eq.(4.2) and normalized according to the following conditions

$$\int_\Omega \boldsymbol{A}_\omega[\boldsymbol{x}] \, dA = \int_\Omega \boldsymbol{W}_x \boldsymbol{A}_\omega[\boldsymbol{x}] \, dA = \boldsymbol{0} \tag{4.10}$$

The in–plane deformation due to Poisson effects provided by the SV solution can be neglected without drawbacks.

From the kinematics assumed in Eq.(4.9) we have the following expression for the strains

$$\boldsymbol{e} = \boldsymbol{\varepsilon}_L[s] + \boldsymbol{\chi}_L[s] \wedge \boldsymbol{x} + \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{\mu}_{,s}[s] + \boldsymbol{D}_{e\omega}[\boldsymbol{x}]\boldsymbol{\mu}[s] \tag{4.11}$$

where $\boldsymbol{\varepsilon}_L = \{\varepsilon_{L1}, \varepsilon_{L2}, \varepsilon_{L3}\}$ and $\boldsymbol{\chi}_L = \{\chi_{L1}, \chi_{L2}, \chi_{L3}\}$ are the generalized strain parameters defined as

$$\boldsymbol{\varepsilon}_L[s] = \boldsymbol{u}_{0,s}[s] + \boldsymbol{e}_1 \wedge \boldsymbol{\varphi}[s] \quad , \quad \boldsymbol{\chi}_L = \boldsymbol{\varphi}[s]_{,s} \tag{4.12}$$

and

$$\boldsymbol{D}_{e\omega}[\boldsymbol{x}] = \begin{bmatrix} 0 & 0 & 0 \\ \bar{\omega}_{1,2} & \bar{\omega}_{2,2} & \bar{\omega}_{3,2} \\ \bar{\omega}_{1,3} & \bar{\omega}_{2,3} & \bar{\omega}_{3,3} \end{bmatrix}$$

The strain-displacement relationship (4.11) allows the evaluation of the internal work $\mathcal{W}$ in terms of the generalized actions on the section:

$$\mathcal{W} = \int_\ell \int_\Omega \boldsymbol{s} \cdot \boldsymbol{e} dA ds = \int_\ell (\boldsymbol{N}[s] \cdot \boldsymbol{\varepsilon}_L[s] + \boldsymbol{M}[s] \cdot \boldsymbol{\chi}_L[s] + \boldsymbol{B}[s] \cdot \boldsymbol{\mu}_{,s}[s] + \boldsymbol{T}[s] \cdot \boldsymbol{\mu}[s]) ds \tag{4.13}$$

where the resultant force $\boldsymbol{N}[s]$ and moment $\boldsymbol{M}[s]$ are defined in Eq. (1.3) while

$$\boldsymbol{B}[s] = \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{s} \, dA \quad \text{and} \quad \boldsymbol{T}[s] = \int_\Omega \boldsymbol{D}_{e\omega}^T \boldsymbol{s} \, dA \tag{4.14}$$

are the new generalized actions, due to the non-uniform warping, called from now on *bi-moment* and *bi-shear* respectively.

4.2.2 Static assumptions

Stresses $\boldsymbol{\sigma}$ are independently interpolated and are obtained by adding to a part called *primary stress* and denoted as $\boldsymbol{\sigma}_{sv}$ an additional term $\boldsymbol{\sigma}_\omega$ called *secondary stress* due to the variable warping:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sv} + \boldsymbol{\sigma}_\omega.$$

We assume that the significant components of $\boldsymbol{\sigma}$ with respect to $\{\boldsymbol{e}_1, \ \boldsymbol{e}_2, \ \boldsymbol{e}_3\}$ are σ_{11}, σ_{12} and σ_{13} collected in $\boldsymbol{s}$ as in the SV case. In particular the following distribution on the section is adopted for the primary terms

$$\boldsymbol{s}_{sv} = \boldsymbol{D}_n[\boldsymbol{x}]\boldsymbol{a}_n[s] + \boldsymbol{D}_m[\boldsymbol{x}]\boldsymbol{a}_m[s] \tag{4.15}$$

where $\boldsymbol{a}_n$ and $\boldsymbol{a}_m$ are the generalized stress parameters to be defined by the equivalence condition with the section stress resultants, while $\boldsymbol{D}_n$ and $\boldsymbol{D}_m$ are reported in Eqs. (4.4),(4.5).

The stress assumption in Eq. (4.15) accounts for all the correct coupling effects contained in the SV solution which, in this way, is described exactly in the proposed formulation while it is missed in other similar models, see for example [7, 1], formulated on the basis of simplified kinematic assumptions.

Two different approaches will be considered and compared for the evaluation of the shear stress components of $\boldsymbol{\sigma}_\omega$.

4.2.3 Some properties of the SV stress interpolation

From Eq.(1.3) the following properties can be derived

$$\int_\Omega \boldsymbol{D}_n \, dA = \boldsymbol{I}, \int_\Omega \boldsymbol{D}_m \, dA = \boldsymbol{0} \tag{4.16a}$$

and

$$\int_\Omega \boldsymbol{W}_x \boldsymbol{D}_n \, dA = \boldsymbol{0}, \int_\Omega \boldsymbol{W}_x \boldsymbol{D}_m \, dA = \boldsymbol{I} \tag{4.16b}$$

Furthermore from (4.10) and (4.4) and exploiting the properties coming from the solution of problems (4.2) and the use of a principal reference system we have

$$\begin{aligned} \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{D}_n \, dA &= \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{D}_m \, dA = \boldsymbol{0} \\ \int_\Omega \boldsymbol{D}_{e\omega}^T \boldsymbol{D}_n \, dA &= \int_\Omega \boldsymbol{D}_{e\omega}^T \boldsymbol{D}_m \, dA = \boldsymbol{0} \end{aligned} \tag{4.16c}$$

The Benscoter-type stress distribution

The first static description evaluates the additional stress term $\boldsymbol{\sigma}^\omega$ on the basis of the displacement assumption (4.9) allowing the warping function to vary with s in a way similar to that initially proposed by Benscoter in [4] and used also in [1, 7]. The following expression is then adopted

$$\boldsymbol{s}_\omega = \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{a}_b[s] + \boldsymbol{D}_{e\omega}[\boldsymbol{x}]\boldsymbol{a}_t[s] \tag{4.17}$$

where $\boldsymbol{a}_b[s]$ and $\boldsymbol{a}_t[s]$ are the generalized stress components due to warping Equation (4.17) allows the total stress to be written as

$$\boldsymbol{s} = \boldsymbol{D}_n[\boldsymbol{x}]\boldsymbol{a}_n[s] + \boldsymbol{D}_m[\boldsymbol{x}]\boldsymbol{a}_m[s] + \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{a}_b[s] + \boldsymbol{D}_{e\omega}[\boldsymbol{x}]\boldsymbol{a}_t[s]\,. \tag{4.18}$$

The final expression of the stress distribution in terms of the section stress resultants $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{B}$, $\boldsymbol{T}$ is determined Eqs. (1.3) and (4.14). Substituting Eq. (4.18) in Eq. (1.3) and exploting (4.16) and Eqs. (4.10), the following expressions for $\boldsymbol{a}_n[s]$ and $\boldsymbol{a}_m[s]$ can be obtained

$$\boldsymbol{a}_n[s] = \boldsymbol{N}[s] - \boldsymbol{\Omega}_n\, \boldsymbol{a}_t[s] \quad , \quad \boldsymbol{a}_m[s] = \boldsymbol{M}[s] - \boldsymbol{\Omega}_m\, \boldsymbol{a}_t[s] \tag{4.19}$$

where

$$\boldsymbol{\Omega}_n = \int_\Omega \boldsymbol{D}_{e\omega}\, dA, \;\; \boldsymbol{\Omega}_m = \int_\Omega \boldsymbol{W}_x \boldsymbol{D}_{e\omega}\, dA$$

From the definition of $\boldsymbol{B}$ and $\boldsymbol{T}$ in Eq.(4.14) and recalling the properties in Eqs.(4.10) and (4.16), we obtain

$$\boldsymbol{a}_b[s] = \boldsymbol{H}_{BB}\boldsymbol{B}[s], \quad \boldsymbol{a}_t[s] = \boldsymbol{F}_T\boldsymbol{T}[s] \tag{4.20}$$

where

$$\boldsymbol{H}_{BB}^{-1} = \boldsymbol{J}_\omega = \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{A}_\omega\, dA\,, \quad \boldsymbol{F}_T^{-1} = \int_\Omega \boldsymbol{D}_{e\omega}^T \boldsymbol{D}_{e\omega} dA \tag{4.21}$$

and the further property

$$\int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{D}_{e\omega}\, dA = \int_\Omega \boldsymbol{D}_{e\omega}^T \boldsymbol{A}_\omega\, dA = \boldsymbol{0}. \tag{4.22}$$

is exploited.

Finally introducing vector $\boldsymbol{t}[s] = \{\boldsymbol{N}[s], \boldsymbol{M}[s], \boldsymbol{B}[s], \boldsymbol{T}[s]\}$ the expression of the stress field Eq. (4.18) can be rearranged as

$$\boldsymbol{s} = \boldsymbol{D}[\boldsymbol{x}]\boldsymbol{t}[s] \tag{4.23}$$

where

$$\boldsymbol{D}[\boldsymbol{x}] = \Big[\boldsymbol{D}_n[\boldsymbol{x}],\, \boldsymbol{D}_m[\boldsymbol{x}],\, \boldsymbol{D}_b[\boldsymbol{x}],\, \boldsymbol{D}_t[\boldsymbol{x}]\Big]$$

with

$$\boldsymbol{D}_b[\boldsymbol{x}] = \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{H}_{BB}, \;\; \boldsymbol{D}_t[\boldsymbol{x}] = (\boldsymbol{D}_{e\omega}[\boldsymbol{x}] - \boldsymbol{D}_n[\boldsymbol{x}]\boldsymbol{\Omega}_n - \boldsymbol{D}_m[\boldsymbol{x}]\boldsymbol{\Omega}_m)\boldsymbol{F}_T$$

4.2.4 The Jourawsky-type stress interpolation

The stress in Eq. (4.18) does not verify the equilibrium equations of the 3D problem. Alternatively the following form for the secondary stress field can be assumed

$$\boldsymbol{s}_\omega = \boldsymbol{A}_\omega[\boldsymbol{x}]\boldsymbol{a}_\omega[s] + \boldsymbol{D}_\psi[\boldsymbol{x}]\boldsymbol{a}_{\omega,s}[s] \tag{4.24}$$

with

$$\boldsymbol{D}_\psi[\boldsymbol{x}] = \begin{bmatrix} 0 & 0 & 0 \\ \psi_{1,2} & \psi_{2,2} & \psi_{3,2} \\ \psi_{1,3} & \psi_{2,3} & \psi_{3,3} \end{bmatrix}$$

The *secondary warping function* $\boldsymbol{\psi}$ is evaluated from the equilibrium equation in the axial direction, under the usual SV assumption about the loads. The following 2D problem is obtained

$$\begin{cases} \boldsymbol{\psi},_{22} + \boldsymbol{\psi},_{33} + \bar{\boldsymbol{\omega}} & = \boldsymbol{0} \quad \in \Omega \\ \boldsymbol{\psi},_2 n_2 + \boldsymbol{\psi},_3 n_3 & = \boldsymbol{0} \quad \in \partial\Omega \end{cases} \tag{4.25}$$

Also in this case the coefficients $\boldsymbol{a}_n[s]$, $\boldsymbol{a}_m[s]$, $\boldsymbol{a}_\omega[s]$ and $\boldsymbol{a}_{\omega},_s[s]$ can be evaluated through Eq. (1.3) and (4.14) in terms of the stress resultants. The equivalence with the bi-moment gives the same expression for $\boldsymbol{a}_\omega$ reported in Eq. (4.20) exploiting Eqs. in (4.16) and the new condition

$$\int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{D}_\psi = \boldsymbol{0}, \tag{4.26}$$

while recalling Eq. (4.22) from bi-shear equivalence we have

$$\boldsymbol{a}_{\omega},_s[s] = \boldsymbol{H}_{BB}\boldsymbol{T}[s] \tag{4.27}$$

for which now

$$\boldsymbol{J}_\omega = \int_\Omega \boldsymbol{A}_\omega^T \boldsymbol{A}_\omega \, dA = \int_\Omega \boldsymbol{D}_{e\omega}^T \boldsymbol{D}_\psi \, dA \tag{4.28}$$

Quantities $\boldsymbol{n}$ and $\boldsymbol{m}$ are calculated using properties (4.16) and the new conditions

$$\int_\Omega \boldsymbol{D}_\psi \, dA = \boldsymbol{0}, \quad \int_\Omega \boldsymbol{W}_x \boldsymbol{D}_\psi = -(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1)\boldsymbol{J}_\omega \tag{4.29}$$

which allow us to obtain

$$\boldsymbol{a}_n[s] = \boldsymbol{N}[s], \quad \boldsymbol{a}_m[s] = \boldsymbol{M}[s] + T_1[s]\boldsymbol{e}_1$$

The stress field is then defined as in Eq. (4.23) with the only difference that the matrix $\boldsymbol{D}_t$ is now

$$\boldsymbol{D}_t[\boldsymbol{x}] = \boldsymbol{D}_\psi[\boldsymbol{x}]\boldsymbol{H}_{BB} + \boldsymbol{D}_m[\boldsymbol{x}](\boldsymbol{e}_1 \otimes \boldsymbol{e}_1)$$

4.2.5 The Hellinger-Reissner functional

Introducing $\boldsymbol{A} = \text{diag}\{1/E,\ 1/G,\ 1/G\}$ with E and G the Young and the shear modula of the material, for both the stress descriptions the complementary energy ψ can be formulated as

$$\psi = \frac{1}{2}\int_\ell \int_\Omega \boldsymbol{s} \cdot \boldsymbol{A}\boldsymbol{s}\, dA ds = \frac{1}{2}\int_\ell \boldsymbol{t}^T \boldsymbol{H}\boldsymbol{t}\, ds \tag{4.30}$$

where

$$\boldsymbol{H} = \int_\Omega \boldsymbol{D}^T \boldsymbol{A}\boldsymbol{D} = \begin{bmatrix} \boldsymbol{H}_{NN} & \boldsymbol{H}_{NM} & \boldsymbol{0} & \boldsymbol{H}_{NT} \\ \boldsymbol{H}_{NM}^T & \boldsymbol{H}_{MM} & \boldsymbol{0} & \boldsymbol{H}_{MT} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}_{BB} & \boldsymbol{0} \\ \boldsymbol{H}_{NT}^T & \boldsymbol{H}_{MT}^T & \boldsymbol{0} & \boldsymbol{H}_{TT} \end{bmatrix} \tag{4.31}$$

assumes the meaning of flexibility matrix relative to the cross-section.

For the sequel, the complementary energy in Eq. (4.30) can be conveniently rearranged in the following form

$$\psi = \frac{1}{2E}\int_\ell \boldsymbol{t}_n^T \boldsymbol{H}_n \boldsymbol{t}_n\, ds + \frac{1}{2G}\int_\ell \boldsymbol{t}_s^T \boldsymbol{H}_s \boldsymbol{t}_s\, ds$$

where

$$\boldsymbol{t}_n = \begin{bmatrix} \boldsymbol{t}_\sigma \\ \boldsymbol{B} \end{bmatrix}, \quad \boldsymbol{t}_s = \begin{bmatrix} \boldsymbol{t}_\tau \\ \boldsymbol{T} \end{bmatrix}$$

with $\boldsymbol{t}_\sigma = \{N_1,\ M_2,\ M_3\}$ and $\boldsymbol{t}_\tau = \{M_1,\ N_2,\ N_3\}$. Matrices

$$\boldsymbol{H}_n = \begin{bmatrix} \boldsymbol{H}_{\sigma\sigma} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{BB} \end{bmatrix}, \quad \boldsymbol{H}_s = \begin{bmatrix} \boldsymbol{H}_{\tau\tau} & \boldsymbol{H}_{\tau T} \\ \boldsymbol{H}_{\tau T}^T & \boldsymbol{H}_{TT} \end{bmatrix}$$

are implicitly defined by Eq. (4.31).

In this way it is possible to decouple the normal and shear parts of the model and exploit the fact that $\boldsymbol{H}_{\sigma\sigma}$ is a diagonal matrix defined as $\boldsymbol{H}_{\sigma\sigma} = \text{diag}[\frac{1}{A},\ \frac{1}{J_{22}},\ \frac{1}{J_{33}}]$.

With this notation, the internal work $\mathcal{W}$ in Eq.(4.13) can be expressed in compact form as

$$\mathcal{W} = \int_\ell (\boldsymbol{t}_n[s]^T \boldsymbol{\varrho}_n[s] + \boldsymbol{t}_s[s]^T \boldsymbol{\varrho}_s[s])\, ds \tag{4.32}$$

where

$$\boldsymbol{\varrho}_n = \begin{bmatrix} \boldsymbol{\varrho}_\sigma \\ \boldsymbol{\mu}_{,s} \end{bmatrix}, \quad \boldsymbol{\varrho}_s = \begin{bmatrix} \boldsymbol{\varrho}_\tau \\ \boldsymbol{\mu} \end{bmatrix}$$

and $\boldsymbol{\varrho}_\sigma = \{\varepsilon_{L1},\ \chi_{L2},\ \chi_{L3}\}$, $\boldsymbol{\varrho}_\tau = \{\chi_{L1},\ \varepsilon_{L2},\ \varepsilon_{L3}\}$

The model is completed by the definition of the external work. In particular the contribution due to the body force $\boldsymbol{b}[\boldsymbol{X}]$, under the displacement assumption (4.9), becomes

$$\mathcal{L}_{ext} = \int_\ell (\boldsymbol{p}[s]\cdot\boldsymbol{u}_0[s] + \boldsymbol{m}[s]\cdot\boldsymbol{\phi}[s] + \boldsymbol{q}[s]\cdot\boldsymbol{\mu}[s])\, ds \tag{4.33}$$

where

$$\boldsymbol{p}[s] = \int_\Omega \boldsymbol{b}[\boldsymbol{X}]dA\,,\quad \boldsymbol{m}[s] = \int_\Omega \boldsymbol{x}\wedge\boldsymbol{b}[\boldsymbol{X}]dA\,,\quad \boldsymbol{q}[s] = \int_\Omega b_1[\boldsymbol{X}]\bar{\boldsymbol{\omega}}[x_2,x_3]dA\,. \tag{4.34}$$

The one-dimensional beam equations

From the stationary condition of Π_{HR} with respect to the displacement parameters, we obtain the *equilibrium equations* for the one-dimensional model

$$\begin{cases} \boldsymbol{N}_{,s} + \boldsymbol{p}[s] = \boldsymbol{0}\,, \\ \boldsymbol{M}_{,s} + \boldsymbol{e}_1\wedge\boldsymbol{N} + \boldsymbol{m}[s] = \boldsymbol{0}\,, \\ \boldsymbol{B}_{,s} - \boldsymbol{T} + \boldsymbol{q}[s] = \boldsymbol{0}\,. \end{cases} \tag{4.35}$$

while from the stationary condition with respect to stress parameters we obtain *the constitutive laws*:

$$\boldsymbol{\varrho}_n = \frac{1}{E}\boldsymbol{H}_n\boldsymbol{t}_n \quad , \quad \boldsymbol{\varrho}_s = \frac{1}{G}\boldsymbol{H}_s\boldsymbol{t}_s \tag{4.36}$$

By inverting Eqs.(4.36) we obtain

$$\begin{bmatrix}\boldsymbol{t}_\sigma\\ \boldsymbol{B}\end{bmatrix} = \begin{bmatrix}\boldsymbol{K}_{\sigma\sigma} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}_{BB}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varrho}_\sigma\\ \boldsymbol{\mu}_{,s}\end{bmatrix} \quad , \quad \begin{bmatrix}\boldsymbol{t}_\tau\\ \boldsymbol{T}\end{bmatrix} = \begin{bmatrix}\boldsymbol{K}_{\tau\tau} & \boldsymbol{K}_{\tau T}\\ \boldsymbol{K}_{\tau T}^T & \boldsymbol{K}_{TT}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varrho}_\tau\\ \boldsymbol{\mu}\end{bmatrix}. \tag{4.37}$$

Eqs. (4.36) and (4.37) could also be transformed in terms of generalized stress variables $\boldsymbol{N}$ and $\boldsymbol{M}$.

Chapter 5

A mixed FE based on exact shape function

5.1 The finite element interpolation

A finite element based on a mixed interpolation of stresses and displacements is presented. The shape functions used for the generalized stresses are exact in the case of zero body forces and could be easily extended exploiting a particular solution which account for different load cases. The finite element so evaluated is then exact but also simple and suitable for general use and allows the validation of the beam models previously presented avoiding errors due to the domain discretization.

5.1.1 Stress interpolation

The following interpolations for $\boldsymbol{M}$ and $\boldsymbol{N}$, satisfying the homogeneous form of (4.35), are adopted

$$\boldsymbol{N} = \frac{1}{\ell}\begin{bmatrix} n_a \\ -m_{e3} \\ m_{e2} \end{bmatrix}, \quad \boldsymbol{M}[s] = \frac{1}{2}\begin{bmatrix} m_{s1} \\ m_{s2} \\ m_{s3} \end{bmatrix} + r_1[s]\begin{bmatrix} 0 \\ m_{e2} \\ m_{e3} \end{bmatrix}$$

where $r_1[s] = s/\ell - 1/2$ while $m_{sk} = M_k[0] + M_k[\ell]$ and $m_{ek} = M_k[\ell] - M_k[0]$.

Collecting the stress parameters in the vector

$$\boldsymbol{\beta}_S = \begin{bmatrix} n_a & m_{s1} & m_{s2} & m_{s3} & m_{e2} & m_{e3} \end{bmatrix}^T$$

the previous interpolations can be rearranged as follows

$$\boldsymbol{t}_\sigma = \boldsymbol{N}_\sigma[s]\boldsymbol{\beta}_S\,, \quad \boldsymbol{t}_\tau = \boldsymbol{N}_\tau\boldsymbol{\beta}_S \tag{5.1}$$

where

$$\boldsymbol{N}_\sigma[s] = \begin{bmatrix} \frac{1}{\ell} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & r_1[s] & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & r_1[s] \end{bmatrix}, \quad \boldsymbol{N}_\tau = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\ell} \\ 0 & 0 & 0 & 0 & \frac{1}{\ell} & 0 \end{bmatrix}.$$

To obtain the exact shape functions for $\boldsymbol{B}$ and $\boldsymbol{T}$ we need to solve the *bi-moment equilibrium equation* $\boldsymbol{B}_{,s} = \boldsymbol{T}$ that, once the constitutive laws (4.37) have been substituted, becomes

$$\boldsymbol{K}_{BB}\boldsymbol{\mu}_{,ss} - \boldsymbol{K}_\mu\boldsymbol{\mu} = \boldsymbol{K}_{\tau T}^T\boldsymbol{\varrho}_c \quad , \quad \begin{cases} \boldsymbol{K}_\mu = (\boldsymbol{K}_{TT} - \boldsymbol{K}_{\tau T}^T\boldsymbol{K}_{\tau\tau}^{-1}\boldsymbol{K}_{\tau T}) \\ \boldsymbol{\varrho}_c = \boldsymbol{K}_{\tau\tau}^{-1}\boldsymbol{t}_\tau \end{cases} \tag{5.2}$$

where $\boldsymbol{\varrho}_c$ is constant with s and $\boldsymbol{K}_\mu$ is a positive definite symmetric matrix.

The solution of the differential problem (5.2) is obtained, apart for an inessential constant, from its homogeneous form which transforms into the following generalized eigenvalues problem and a set of ODEs

$$\begin{cases} f_{i,ss} - \lambda_i^2 f_i = 0\,, \\ (\boldsymbol{K}_\mu - \lambda_i^2\boldsymbol{K}_{BB})\boldsymbol{q}_i = \boldsymbol{0} \end{cases}. \tag{5.3}$$

once $\boldsymbol{\mu}$ is expressed as

$$\boldsymbol{\mu} = \sum_{i=1}^{3} f_i[s]\boldsymbol{q}_i = \boldsymbol{Q}\boldsymbol{f}[s] \tag{5.4}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \end{bmatrix}, \quad \boldsymbol{f}[s] = \begin{bmatrix} f_1[s] \\ f_2[s] \\ f_3[s] \end{bmatrix}. \tag{5.5}$$

Using the normalization condition $\boldsymbol{q}_i^T\boldsymbol{K}_{BB}\boldsymbol{q}_k = \delta_{ik}$ the following relations holds

$$\boldsymbol{Q}^T\boldsymbol{K}_{BB}\boldsymbol{Q} = \mathbf{I}\,, \quad \boldsymbol{Q}^T\boldsymbol{K}_\mu\boldsymbol{Q} = \boldsymbol{\Lambda}\,, \quad \boldsymbol{\Lambda} = \mathrm{diag}[\lambda_1^2, \lambda_2^2, \lambda_3^2] \tag{5.6}$$

where $\mathbf{I}$ is the identity matrix and λ_i are real and positive due to the properties of $\boldsymbol{K}_{BB}$ and $\boldsymbol{K}_\mu$.

From Eqs.(5.4) and (4.37) we obtain

$$\boldsymbol{B} = \sum_{i=1}^{3}(c_i \cosh(\lambda_i s) + d_i \sinh(\lambda_i s))\boldsymbol{K}_{BB}\boldsymbol{q}_i$$

while the bi-shear is recovered as $\boldsymbol{T} = \boldsymbol{B}_{,s}$. The coefficients c_i and d_i are expressed in terms of the end section values of $\boldsymbol{B}$. Collecting the discrete parameters in vector

$$\boldsymbol{\beta}_\omega = \begin{bmatrix} B_{s1} & B_{s2} & B_{s3} & B_{e1} & B_{e2} & B_{e3} \end{bmatrix}^T \quad , \quad \begin{cases} B_{sk} = \dfrac{B_k[\ell] + B_k[0]}{\ell} \\ B_{ek} = \dfrac{B_k[\ell] - B_k[0]}{\ell} \end{cases}.$$

the interpolation of $\boldsymbol{B}$ and $\boldsymbol{T}$ becomes

$$\boldsymbol{B} = \boldsymbol{K}_{BB}\boldsymbol{Q}\boldsymbol{A}[s]\boldsymbol{Q}_e\boldsymbol{\beta}_\omega \quad , \quad \boldsymbol{T} = \boldsymbol{K}_{BB}\boldsymbol{Q}\boldsymbol{A}[s]_{,s}\boldsymbol{Q}_e\boldsymbol{\beta}_\omega \tag{5.7}$$

where $\boldsymbol{A}[s] = \boldsymbol{N}_B[s]\boldsymbol{N}_e$ and

$$\boldsymbol{Q}_e = \begin{bmatrix} \boldsymbol{Q}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}^T \end{bmatrix} \quad , \quad \boldsymbol{N}_e = \ell \begin{bmatrix} \boldsymbol{N}_B[\ell] + \boldsymbol{N}_B[0] \\ \boldsymbol{N}_B[\ell] - \boldsymbol{N}_B[0] \end{bmatrix}^{-1} \quad , \quad \boldsymbol{N}_B = \begin{bmatrix} \boldsymbol{N}_c & \boldsymbol{N}_s \end{bmatrix}$$

while $\boldsymbol{N}_c = \mathrm{diag}[\cosh(\lambda_k s)]$ and $\boldsymbol{N}_s = \mathrm{diag}[\sin(\lambda_k s)]$.

5.1.2 The discrete form of the internal work

The discrete form of the internal work $\mathcal{W}$ in Eq. (4.32) is obtained without using any explicit interpolation for the displacements. In particular, after by parts integration, the following expression is obtained

$$\mathcal{W} = \boldsymbol{\beta}_S^T\boldsymbol{\phi}_S + \boldsymbol{\beta}_\omega^T\boldsymbol{\phi}_\omega \tag{5.8}$$

where the kinematical parameters, collected in the vectors

$$\boldsymbol{\phi}_S = \begin{bmatrix} \phi_{r1} & \phi_{s1} & \phi_{s2} & \phi_{s3} & \phi_{e2} & \phi_{e3} \end{bmatrix}^T \quad , \quad \boldsymbol{\phi}_\omega = \begin{bmatrix} \boldsymbol{\mu}_s & \boldsymbol{\mu}_e \end{bmatrix}^T$$

are defined as

$$\phi_{e2} = \frac{1}{2}(\varphi_2[0] + \varphi_2[\ell]) - \phi_{r3}\,, \quad \phi_{e3} = \frac{1}{2}(\varphi_3[0] + \varphi_3[\ell]) + \phi_{r2}$$

$$\phi_{sk} = \frac{1}{2}(\varphi_k[\ell] - \varphi_k[0])\,, \quad \phi_{rk} = \frac{1}{\ell}(u_k[\ell] - u_k[0])$$

and

$$\mu_{sk} = \frac{\mu_k[\ell] - \mu_k[0]}{2}\ell \quad , \quad \mu_{ek} = \frac{\mu_k[\ell] + \mu_k[0]}{2}\ell\,.$$

5.1.3 The discrete form of the complementary strain energy

The discrete form of the complementary energy will be obtained by summing the contributions of the normal and tangential stresses

$$\psi = \psi_\sigma + \psi_\tau \,. \tag{5.9}$$

The first term is defined as

$$\psi_\sigma := \frac{1}{2E}\int_\ell (\boldsymbol{t}_n^T \boldsymbol{H}_n \boldsymbol{t}_n) ds \tag{5.10}$$

which, on the basis of Eqs.(5.1),(5.6),(5.7) and (4.37), becomes

$$\psi_\sigma := \frac{1}{2}\left(\boldsymbol{\beta}_\omega^T \boldsymbol{H}_{BB}^e \boldsymbol{\beta}_\omega + \boldsymbol{\beta}_S^T \boldsymbol{H}_{\sigma\sigma}^e \boldsymbol{\beta}_S\right)$$

where

$$\boldsymbol{H}_{BB}^e = \boldsymbol{Q}_e^T \int_\ell \boldsymbol{A}^T \boldsymbol{A}\, ds\, \boldsymbol{Q}_e \,, \quad \boldsymbol{H}_{\sigma\sigma}^e = \frac{1}{E}\int_\ell \boldsymbol{N}_\sigma[s]^T \boldsymbol{H}_{\sigma\sigma} \boldsymbol{N}_\sigma[s]\, ds \,. \tag{5.11}$$

In a similar way the tangential part is defined as

$$\psi_\tau := \frac{1}{2G}\int_0^\ell (\boldsymbol{t}_s^T \boldsymbol{H}_s \boldsymbol{t}_s) ds \,.$$

A by block inversion of the constitutive law (4.37) allows us to device the relation $\boldsymbol{H}_{TT} = G\boldsymbol{K}_\mu^{-1}$, which together with the eigenvalues property in (5.6) gives the equation

$$\boldsymbol{Q}^T \boldsymbol{K}_{BB} \boldsymbol{H}_{TT} \boldsymbol{K}_{BB} \boldsymbol{Q} = G\boldsymbol{\Lambda}^{-1} \,.$$

usefull for rewriting ψ_τ as

$$\psi_\tau = \frac{1}{2}\begin{bmatrix}\boldsymbol{\beta}_S \\ \boldsymbol{\beta}_\omega\end{bmatrix}^T \begin{bmatrix}\boldsymbol{H}_{\tau\tau}^e & \boldsymbol{H}_{\tau T}^e \\ (\boldsymbol{H}_{\tau T}^e)^T & \boldsymbol{H}_{TT}^e\end{bmatrix} \begin{bmatrix}\boldsymbol{\beta}_S \\ \boldsymbol{\beta}_\omega\end{bmatrix}$$

once the stress interpolations (5.1),(5.7) have been substituted. The introduced discrete operators are

$$\boldsymbol{H}_{\tau\tau}^e = \frac{\ell}{G}\boldsymbol{N}_\tau^T \boldsymbol{H}_{\tau\tau} \boldsymbol{N}_\tau \,, \quad \boldsymbol{H}_{\tau T}^e = \frac{1}{G}\boldsymbol{N}_\tau^T \boldsymbol{H}_{\tau T} \boldsymbol{K}_{BB} \boldsymbol{Q} \int_0^\ell \boldsymbol{A}_{,s}\, ds\, \boldsymbol{Q}_e$$

$$\boldsymbol{H}_{TT}^e = \boldsymbol{Q}_e^T \int_0^\ell \boldsymbol{A}_{,s}^T \boldsymbol{\Lambda}^{-1} \boldsymbol{A}_{,s}\, ds\, \boldsymbol{Q}_e$$

Finally the total complementary strain energy of the element can be expressed as follows

$$\psi = \frac{1}{2} \begin{bmatrix} \boldsymbol{\beta}_S \\ \boldsymbol{\beta}_\omega \end{bmatrix}^T \begin{bmatrix} \boldsymbol{H}^e_{SS} & \boldsymbol{H}^e_{S\omega} \\ (\boldsymbol{H}^e_{S\omega})^T & \boldsymbol{H}^e_{\omega\omega} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_S \\ \boldsymbol{\beta}_\omega \end{bmatrix} \tag{5.12}$$

where

$$\boldsymbol{H}^e_{SS} = \boldsymbol{H}^e_{\tau\tau} + \boldsymbol{H}^e_{\sigma\sigma}\,, \quad \boldsymbol{H}^e_{\omega\omega} = \boldsymbol{H}^e_{TT} + \boldsymbol{H}^e_{BB}\,, \quad \boldsymbol{H}^e_{S\omega} = \boldsymbol{H}^e_{\tau T}\,.$$

5.1.4 Local to global contribution

As is standard in mixed formulation the solution is obtained using a pseudo–compatible format (see for example [8, 9]) in which the stresses, that do not require interelement continuity, are locally solved as a function of the element displacements.

The assemblage requires that the kinemathical element variables be express in terms of the global ones by using standard change of reference rules which involve the nodal displacements and rotations while the warping parameters are directly identified with the global ones.

5.2 Numerical examples

In order to validate our proposal a series of numerical tests regarding single beams and frames have been performed. Single beams have been considered mainly to compare the accuracy of the Bensoter–type and Jouwrasky–type stress descriptions. Frames have allowed us to validate the model in more complex cases requiring the connections of more finite elements.

A preliminary *cross section analysis* is performed to solve the 2D PDEs problems (4.2) and (4.25), required for the evaluation of the cross section flexibility matrices, and to calculate the eigensolution of the problem in Eq. (5.5). These computations are performed by means of the general purpose FEM code COMSOL using quadrangular 9 nodes isoparametric FEs. The frame analysis is performed by using a C++ code specifically implemented for this purpose. Thanks to the exact mixed interpolation all the tests are performed by using only one finite element for each member.

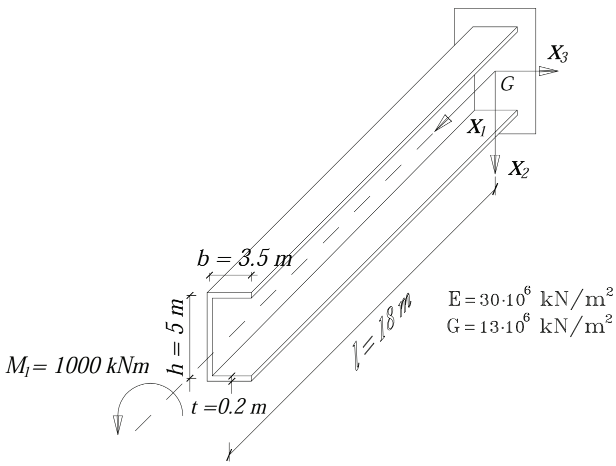

Figure 5.1: Channel-shaped shear wall core with an end torque.

5.2.1 Shear wall core

The first test regards the shear wall core shown in Figure 5.1 fully restrained at one end and subjected to a torque M_1 at the free end. This test, analyzed by many other authors [22, 14, 1, 7], is very sensitive to an accurate evaluation of the secondary shear stress and for this reason is particularly suitable to verify the performance of the Benscoter–type and Jouwrasky–type stress descriptions. In Table 5.1 the values of the torsional rotation at the free end are compared with reference solutions in the literature and by standard Vlasov and SV solutions. The latter is obtained with the model in [10] performing only the linear step of the asymptotic analysis. The sensitivity in the evaluation of the secondary shear stresses is highlighted by the Vlasov solution in which lacks this effect.

The results obtained by the two new proposals improve that of the Vlasov theory but only the Jourawsky-type model gives an accurate prediction of the free end rotation. The different evaluation of the tangential stress is highlighted in Figure 5.3 where the significant components of $\boldsymbol{\sigma}$ are reported. Note how the small difference with respect [22, 14] is related to the use of the sectorial area theory to calculate the cross-section mechanical properties as confirmed by the result labeled "Jourawsky-sect." obtained with the Jourawsky–type stress evaluation but describing the cross section in terms of

	mesh	warp. constr.	warp. free
Benscoter-type	1 FE	4.1395	43.3729
Jourawsky-type	1 FE	4.2114	43.3729
Jourawsky-sect.	1 FE	4.2366	43.2692
SV [10]	1 FE	-	43.3727
Vlasov [23]	-	4.0806	-
Back et al. [1]	6 FE	4.210	-
Tralli [22]	6 FE	4.2389	-
Kim et al. [14]	1 FE	4.236	-

Table 5.1: C section: torsional angle $\varphi_1(\times 10^{-3} rad)$ at the free end.

its midline. Finally note the great difference with respect to the SV rotation that however is exactly reproduced by both our models when the warping is free.

Figure 5.2 reports the shapes of the $\bar{\boldsymbol{\omega}}$ and $\boldsymbol{\psi}$ functions evaluated through the COMSOL code.

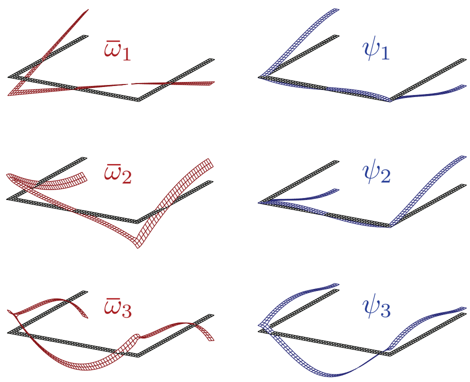

Figure 5.2: C section: $\bar{\omega}$ functions and ψ functions.

5.2.2 Box-shaped beam

The second test regards the beam depicted in Figure 5.4 and proposed in [19] with the aim of emphasising the different behaviour of closed section. Two torsional moments are applied along the beam and the supports constrain the torsional rotation while allow the warping of the section. Table 5.2 furnishes

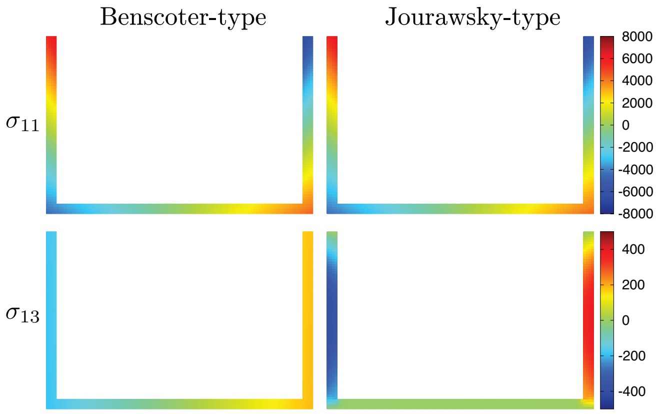

Figure 5.3: C section: significant stress components at the cantilevered end.

	mesh	$x_1 = 2$	$x_1 = 7.5$
Benscoter-type	5 FE	4.0114	-7.0662
Jourawsky-type	5 FE	4.0843	-7.1563
SV [10]	5 FE	4.1039	-7.1876
Murin [19]	5 FE	4.12	-7.21

Table 5.2: Box-shaped beam: torsional angle $\varphi_1(\times 10^{-6} rad)$ at significant sections.

the values of the torsional angle evaluated at the loaded sections. It can be observed how the free–to–warp condition and the shape of the section lead to small differences with the standard SV solution. In Fig. 5.5 the values of B_1 obtained with the Jourawsky–type model are plotted. In Fig. 5.7 the significant stress components for section at $x = 2m$ are shown, note how, also if the two beam models proposed evaluate the same displacements, the stress responses are completely different. Also in this case the warping functions are plotted in Fig. 5.6.

5.2.3 L-shaped frame

In order to consider assemblages of more beams, the L-shaped frame shown in Figure 5.8 has been analyzed. The frame is formed by two orthogonal members subjected to a torsional moment in the mid-span of the column.

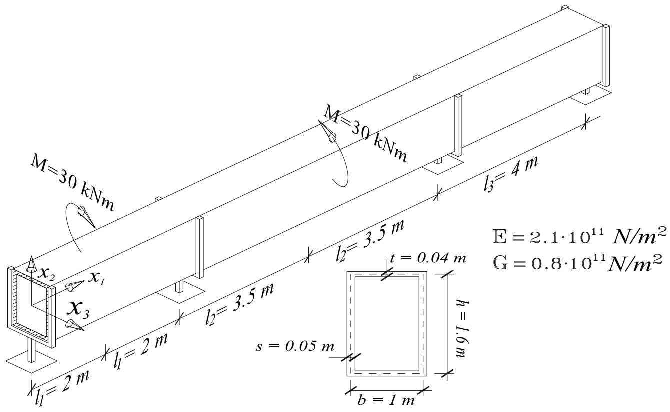

Figure 5.4: Box-shaped beam.

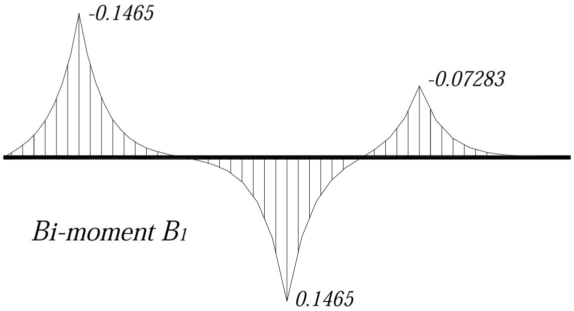

Figure 5.5: Box-shaped beam: bimoment distribution along the beam.

The constraint conditions are completely fixed in points A and D, while in B only the displacement in the x_3 direction is prevented. With respect to warping in joint B three different types of connections are considered [2, 21], *flange continuity*, *diagonal stiffened joint* and *diagonal/box stiffened joint*, which are simulated by applying different continuity conditions on the warping variables, i.e. continuity of the warping parameters for the first two cases and imposing zero warping parameters in the third one.

Figures 5.9 and 5.10 report the computed values of the torsional rotation of the two members forming the frame at different sections. The values obtained with the Jourawsky-type model (continuous lines in the Figures) are compared with those obtained with the ABAQUS commercial code by using a 3D shell model (discrete points in the Figures) with *S4R elements* (see

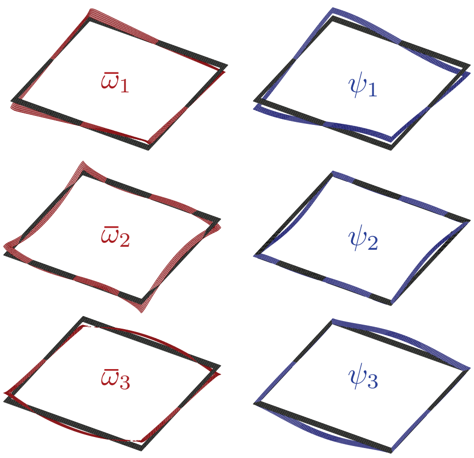

Figure 5.6: Box-shaped beam: $\bar{\omega}$ functions and ψ functions.

Figs. 5.11 and 5.12 for the meshes employed), no important differences can be observed. The results are also in good agreement with analyses presented in [2].

Fig. 5.11 shows the deformed configurations for the different joint considered. In 5.12 the joint region is more deeply showed.

5.2.4 Space frame

The last test is the symmetric space frame shown in Figure 5.13. The structure consists of two portal frames joined through a transverse beam and loaded with a vertical load P applied at the top of each column and at the center of the beam CH. As in [2] the column bases are fixed, nodes B, D, G, I cannot translate along the x_1 and x_3 axes, while in point M only the u_1 displacement component is prevented. Also in this case the previously described *diagonal/box stiffened joint* and *diagonal stiffened joint* conditions on the warping parameters at nodes B, D, G and I are investigated.

Figure 5.14 shows the bi-moment along the axial direction of the elements calculated with the Jourawsky-type model.

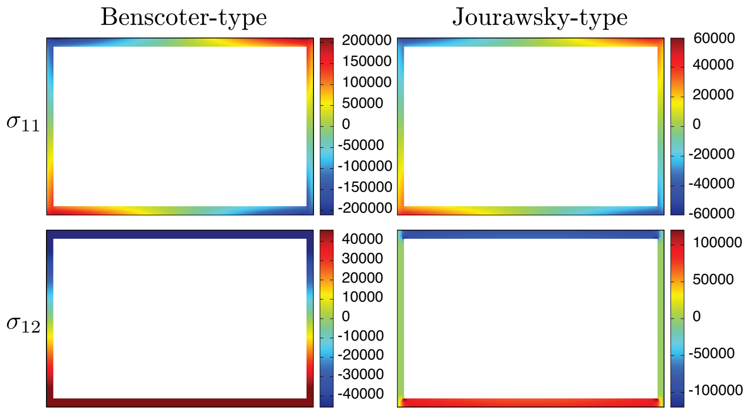

Figure 5.7: Box-shaped beam: significant stress components.

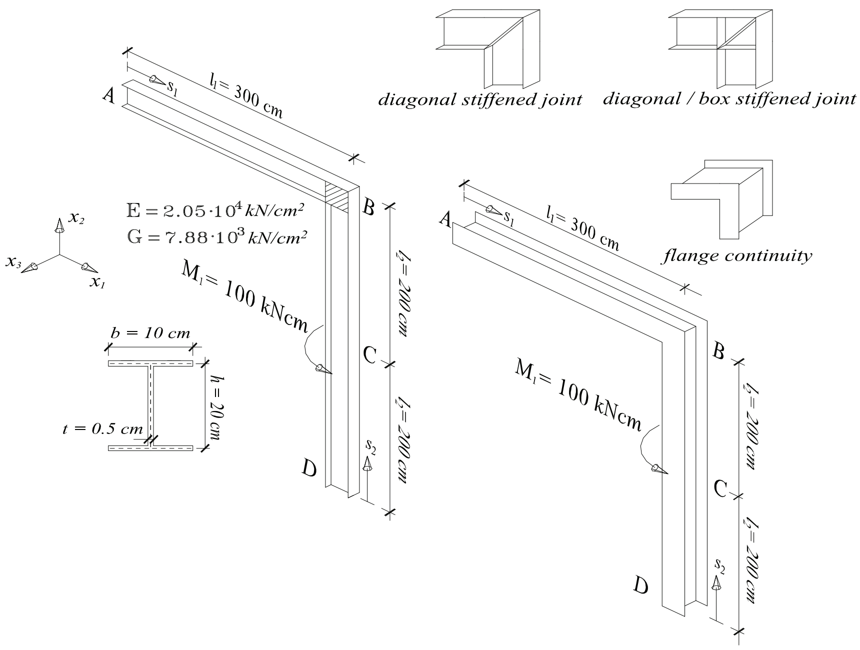

Figure 5.8: L-shaped frame.

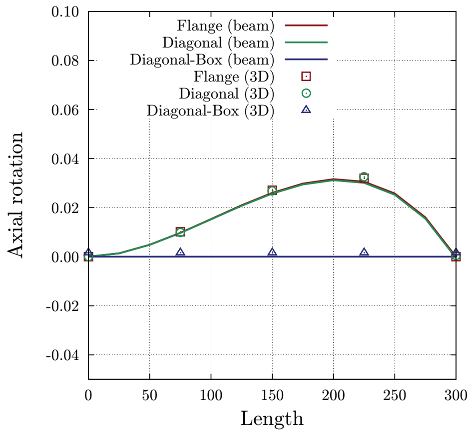

Figure 5.9: L-shaped frame: torsional rotation, $\varphi_1(rad)$, along the horizontal member.

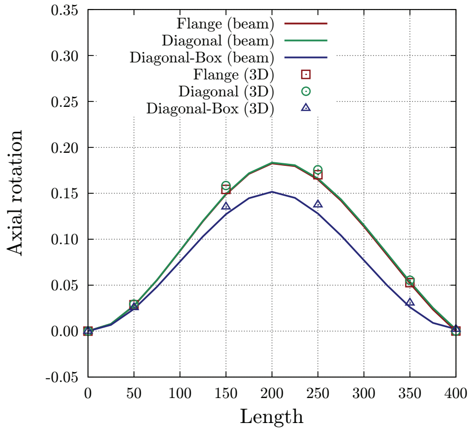

Figure 5.10: L-shaped frame: torsional rotation, $\varphi_2(rad)$, along the vertical member.

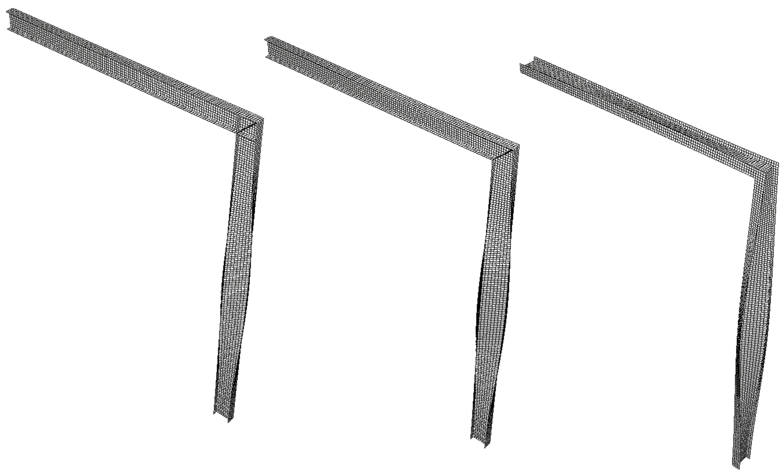

Figure 5.11: Deformed configuration frame.

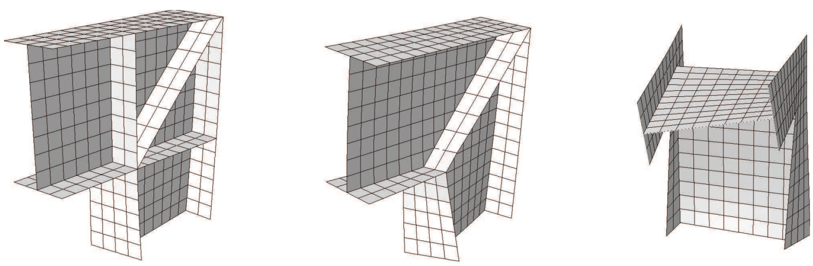

Figure 5.12: Deformed configuration at joints.

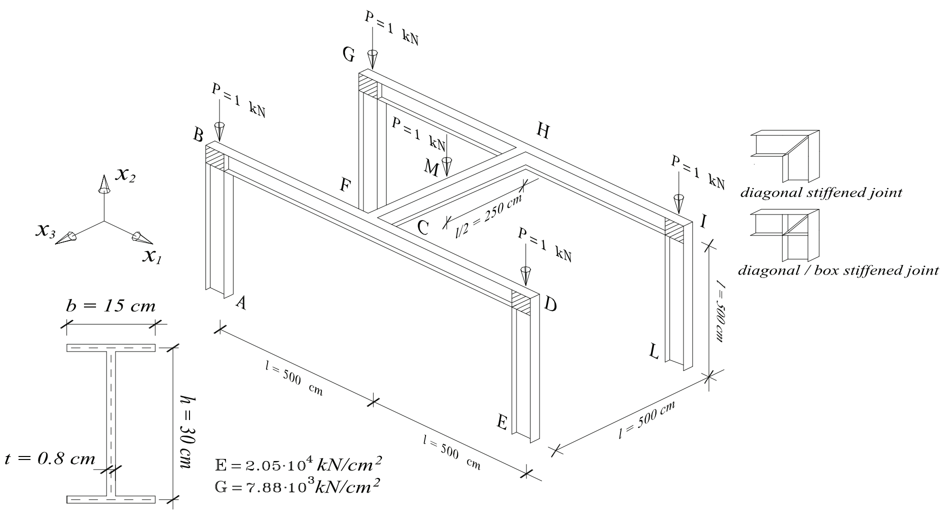

Figure 5.13: Space frame.

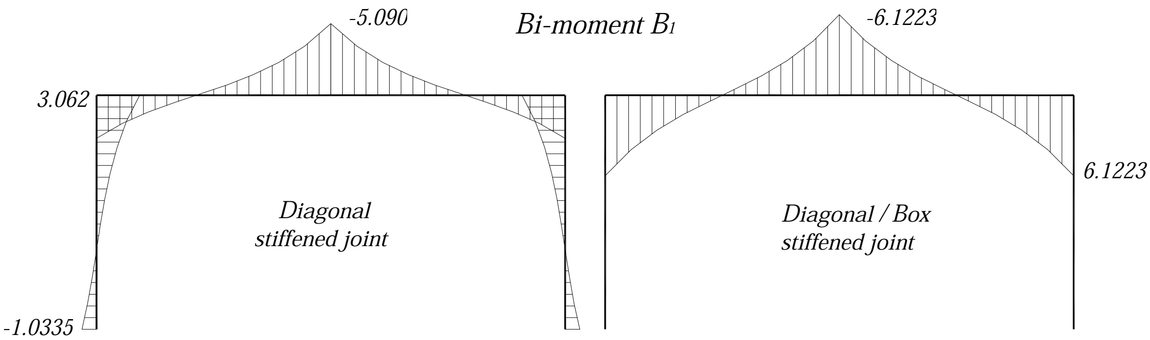

Figure 5.14: Space frame: bimoment distribution for the portal frame ABDE.

Chapter 6

Conclusions

In this work a general beam model suitable to represent 3D effects due to complex material behaviours or warping deformations is presented. It is derived using a mixed approach with a simplified kinematical description in terms of a rigid motion of the cross sections and the contribution due to in-plane or out-of-plane variable warpings effects, while stresses are exactly evaluated. The starting point of the formulation is a semi-analytical approach which allows to solve the Cauchy problem under the standard SV loading conditions without introducing simplifying hypothesis on the stress field. It is exploited to obtain the warping modes shape over the cross-section and to evaluate the stress field due both to these contributions and to what we call central solution. The relations with respect to the Ieşan solution of the generalized SV problem for composite beams are discussed.

The SV stress field for the isotropic and homogeneous case can be easly recovered as a particular case and a simplified linear model for beams with variable out–of–plane warpings due to shear and torsion has been derived. With respect to other proposals, derived only from kinematical hypotheses, the mixed formulation allows to include the SV solution coherently making the analysis of any kind of cross sections, compact or thin–walled possible. The only extra cost consists in the numerical solution of some Neumann boundary-value problems on the cross-section domain.

The model proposed has been implemented on the basis of an exact mixed finite element which makes possible to focus attention on the accuracy of the continuum formulation only, avoiding any disturbing effect due to the discretization. Each numerical test presented is then exactly solved with respect to the beam model equations, except for the errors due to the FEM evaluation of the warping functions.

Two different stress descriptions have been presented and compared. Numerical results show their accuracy with a better behaviour for the formulation denoted as Jourawsky-type.

The extension of the linear beam model proposed to the geometrically nonlinear analysis using corotational strategies appears simple thank to the mixed description adopted.

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