Università della Calabria
Department of Mathematics and Computer Science

Ph.D. in Mathematics and Computer Science
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PH.D. THESIS

On Iterative Methods for Quasi-Nonexpansive Mappings

Scientific Disciplinary Sector: MAT/05 - Mathematical Analysis

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Sommario

Lo scopo di questa tesi è quello di risolvere alcuni problemi di approssimazioni di punti fissi di mappe non lineari e approssimazioni di soluzioni di disequazioni variazionali, illustrati in una ricca raccolta e collezione di significativi risultati sui metodi iterativi introdotti. Numerosi problemi in molte aree della matematica possono essere riformulati come un problema di punto fisso di una mappa non lineare, definita su un sottoinsieme non vuoto di uno spazio di Banach X in se stesso, e, dunque, si riducono a trovare le soluzioni della seguente equazione

$$x = Tx, \quad x \in X.$$

Vi presenteremo il nostro contributo ai seguenti problemi proposti.
Problem 1:
Non sappiamo se è possibile ottenere un Teorema di convergenza forte per un metodo di tipo Halpern per mappe nonspreading, vedi [40]
Abbiamo dato una risposta parziale in F. Cianciaruso, G. Marino, A. Rugiano, B. Scardamaglia, *On Strong convergence of Halpern's method using averaged type mappings*, J. Appl. Math., (2014), Art. ID 473243, 11 pages.
Nel Capitolo 2, discuteremo in modo accurato i nostri risultati.
Problem 2: *Non sappiamo se è possibile ottenere un Teorema di convergenza forte per un metodo di tipo viscoso per approssimare punti fissi comuni di due mappe.*
Abbiamo presentato una risposta parziale in F. Cianciaruso, G. Marino, A. Rugiano, B. Scardamaglia, *On strong convergence of viscosity type method using averaged type mappings*, Journal of Nonlinear and Convex Analysis no. 8, vol. 16, (2015), 1619-1640.
Riporteremo i nostri risultati nel Capitolo 3.
Problem 3: *È di notevole interessante capire per quali classi di mappe è possibile ottenere un Teorema di convergenza forte per un metodo di tipo Halpern, senza introdurre le mappe di tipo average.*
Abbiamo discusso una risposta parziale in J. Garcia Falset, E. LLorens Fuster, G. Marino, A. Rugiano, *On strong convergence of Halpern's method for quasi-nonexpansive mappings in Hilbert Spaces*, submitted for publication.
Il Capitolo 4 è dedicato all'analisi dei rusultati ottenuti.
Problem 4: *Non sappiamo se è possibile ottenere un Teorema di convergenza forte per il metodo iterativo introdotto da Iemoto and Takahashi in [31].*
Abbiamo illustrato una risposta parziale in A. Rugiano, B. Scardamaglia, S. Wang, *Hybrid iterative algorithms for a finite family of nonexpansive mappings and for a countable family of nonspreading mappings in Hilbert spaces*, to appear in Journal of Nonlinear and Convex

Analysis.
Analizzeremo in dettaglio i nostri risultati nel Capitolo 5.

Contents

Introduction

Many of the most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations (nonlinear integral equations, boundary value problems for nonlinear ordinary equations or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations). They can be formulated, in terms of finding the fixed points of a given nonlinear mapping T defined on a nonempty set of a Banach space X into itself, by the following equation

$$x = Tx, \quad x \in X. \tag{1}$$

The solutions of the equation (1) are called *fixed points of* T. We examine the conditions under which the nonlinear equation (1) may be solved by successive approximations (the so-called Picard sequence)

$$x_{n+1} = Tx_n = T^n x_0, \quad x_0 \in X, \quad n = 0, 1, 2... \tag{2}$$

The method of successive approximations, introduced by Liouville in 1837 and systematically developed by Picard in 1890, culminated in the famous formulation known as the Banach Contraction Principle. We recall that a mapping T is said to be a contraction in a metric space (X, d), if there exists $k \in [0, 1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y). \tag{3}$$

As a matter of fact, the Banach Contraction Principle states that *whenever T is a contraction defined on a complete metric space (X, d), T has a unique fixed point and, for any arbitrary choice of initial point x_0 in X, the sequence of Picard iterates (2) strongly converges to the fixed point of T*. The Banach fixed-point Theorem is the primary theoretical instrument for creating unified overview of the entirety of iteration methods. The great significance of Banach Contraction Principle derives from the fact this Theorem contains some important elements for the theoretical and practical treatment of mathematical equations:

(i) existence of a solution;

(ii) uniqueness of the solution;

(iii) existence of a convergent approximation method;

(iv) a priori error estimates;

(v) a posteriori error estimates;

(vi) estimates on rate of convergence;

(vii) stability of the approximation method.

Essentially, the questions about the existence, uniqueness and approximation of fixed points provide three significant aspects in the so-called metric fixed points theory.
Since the sixties, the study of the class of *nonexpansive mappings*, that is obtained for $k = 1$ in (3), constitutes one of the major and most active research areas of nonlinear analysis.

Unlike in the case of the Banach Contraction Principle, trivial examples show that the sequence of successive approximations, defined as in (2) and starting from a point $x_0 \in C$, where C is a nonempty closed convex and bounded subset of a real Banach space and $T : C \to C$ a nonexpansive mapping (even with a unique fixed point), may fail to converge to the fixed point.

It suffices, for example, to take for T, a rotation of the unit ball in the plane around the origin of coordinates.

Therefore, if T is a nonexpansive mapping then we must assume additional conditions on T and (or) the underlying space to ensure the existence of fixed points. On the other hand, it is important not only to show the existence of fixed points of such mappings, but also to develop systematic techniques for the approximation of fixed points. Approximation of fixed points of nonexpansive mappings is a notable topic in nonlinear operator theory and its applications; in particular, in image recovery and signal processing and in transition operators for initial valued problems of differential. It has been investigated by many researchers, indeed, in literature, we can find a large number of iterative processes. We especially recall two interesting iterative schemes: Mann iteration process and Halpern method. The sequence generates by

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \tag{4}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $[0, 1]$ satisfying appropriate conditions and T is a self-mapping of a closed convex subset C of a Hilbert space H, is generally known as Mann iteration; actually, in the Mann's paper [44] the process defined as in (4) did not appear. In 1953, he introduced an infinite triangular matric $A = (a_{n,k})_{n,k\in\mathbb{N}}$

$$\begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ a_{n,1} & a_{n,2} & ... & a_{n,n} & 0 & 0 \\ ... & ... & ... & ... & ... & ... \end{pmatrix} \tag{5}$$

whose elements satisfy the following properties:

(A1) $a_{n,k} \geq 0$, for all $n, k \in \mathbb{N}$;

(A2) $a_{n,k} = 0$, for all $k > n$;

(A3) $\sum_{k=1}^{n} a_{n,k} = 1$ for all $n \in \mathbb{N}$.

Suppose that E is a linear space, C is a convex subset of E, $T : C \to C$ a mapping. Starting with an arbitrary element x_1 in E, Mann defined the following iteration process

$$\begin{cases} x_{n+1} = Tv_n, \\ v_n = \sum_{k=1}^{n} a_{n,k} x_k, \quad n = 1, 2, ... \end{cases} \tag{6}$$

This process is determined by the initial point x_1, the matrix A and the transformation T. It can be denoted briefly by $M(x_1, A, T)$ and can be regarded as a generalized iteration process because when A is the identity matrix I, $M(x_1, I, T)$ is just the Picard iteration (2). Mann proved that *in case E is a Banach space and the domain of a continuous self-mapping T is a closed convex subset C of E, then the convergence of either $(x_n)_{n\in\mathbb{N}}$ or $(v_n)_{n\in\mathbb{N}}$ to a point y implies the convergence of the other to y and also implies $Ty = y$.* As particular case of the iteration process $M(x_1, A, T)$, he considered the Cesaro matrix given by

$$\begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ \frac{1}{n} & \frac{1}{n} & ... & \frac{1}{n} & ... \\ ... & ... & ... & ... & ... \end{pmatrix} \tag{7}$$

and the process (6) becomes

$$v_{n+1} = \left(1 - \frac{1}{n+1}\right) v_n + \frac{1}{n+1} Tv_n.$$

Thirteen years after, Dotson [22] defined a *normal Mann process* as a Mann process $M(x_1, A, T)$ provided the matrix A satisfies (A1), (A2), (A3) and

(A4) $\lim_{n\to\infty} a_{n,k} = 0$ for all $k \in \mathbb{N}$;

(A5) $a_{n+1,k} = (1 - a_{n+1,n+1})a_{n,k}$, $k = 1, 2, ..., n$ and $n = 1, 2, ...$;

(A6) either $a_{n,n} = 1$ for all n, or $a_{n,n} < 1$ for all $n > 1$.

He showed that the sequence $(v_n)_{n\in\mathbb{N}}$ in a normal Mann process $M(x_1, A, T)$ satisfies

$$v_{n+1} = (1 - t_n)v_n + t_n Tv_n, \quad \forall n \geq 1, \tag{8}$$

where $t_n = a_{n+1,n+1}$.
The problem concerning approximation of fixed points of nonexpansive mappings via Mann's algorithm has intensively been studied recently in literature. It is worth mentioning Reich's result: *in a uniformly convex Banach space with Fréchet differentiable norm, if T is a nonexpansive mapping with a fixed point and if the the control sequence $(\alpha_n)_{n\in\mathbb{N}}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the Mann iteration process converges weakly to a fixed point of T*(see [53]). However, this convergence is, in general, not strong (see the counterexample in [24]). In order to get strong convergence, many authors modified Mann iterations for nonexpansive

mappings. In this direction, Halpern [28], in 1967, showed that *in the case* C *is a bounded, closed and convex subset of a Hilbert space* H *and* $T : C \to C$ *is a nonexpansive mapping, for any initialization* $x_1 \in C$ *and for any anchor* $u \in C$*, the sequence* $(x_n)_{n\in\mathbb{N}}$ *in* C *defined by*

$$x_{n+1} = n^{-\theta}u + (1 - n^{-\theta})Tx_n, \quad \forall n \geq 1,$$

where $\theta \in (0, 1)$*, converges strongly to the element of* $Fix(T)$ *nearest to* u*.*

An iteration method with recursion formula of the form

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \tag{9}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $[0, 1]$, is referred to as a *Halpern type iteration method.* There is an extensive literature regarding the convergence analysis of (9) with several types of operator T in the setting of Hilbert spaces and Banach spaces.

An iterative scheme executes steps in iterations and his aim is to generate a sequence, starting from given initial approximations, of successive iterations to approximate solutions for a class of problems. Although nonexpansive mappings are perhaps one of the most significant class of mappings in the so-called metric fixed points theory, there exist other fascinating classes of mappings related to the iterative algorithms that we will analyze in the next Chapters. For instance, let C be a closed convex subset of a Hilbert space,

- *Nonspreading mappings* because to approximate fixed points of such mappings it is equivalent to approximate zeros of monotone operators in Banach spaces. This last problem is closely related to minimization problems, saddle point problems, equilibrium problems and others. In order to approximate the solution to this problem, various types of iterative schemes have been proposed.

- *L-hybrid mappings*, where $L \geq 0$ because for particular choices of L we obtain several important well-known classes of nonlinear mappings. Precisely, $T : C \to H$ is said L-hybrid, $L \geq 0$, signified as $T \in \mathcal{H}_L$, if, for all $x, y \in C$,

  $$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle,$$

  and

  - $\mathcal{H}_0$ is the class of the nonexpansive mappings;
  - $\mathcal{H}_2$ is the class of the nonspreading mappings;
  - $\mathcal{H}_1$ is the class of the hybrid mappings.

- *Quasi-nonexpansive mappings* since the lack of continuity makes difficult to prove asymptotical stability of iterates. Many recent papers which concern the approximation of fixed points of quasi-nonexpansive mappings assume some other property in order to bypass the lack of continuity.
  We recall that $T : C \to H$ with $Fix(T) = \{z \in C : Tz = z\} \neq \emptyset$ is said to be quasi-nonexpansive if, $\forall x \in C, \forall p \in Fix(T)$,

  $$\|Tx - p\| \leq \|x - p\|.$$

- *Strongly quasi-nonexpansive mappings* since this class includes the interesting class of *averaged type mappings* $T_\delta = (1-\delta)I + \delta T$, where $\delta \in (0,1)$, whenever T is quasi-nonexpansive. Averaged type mappings plays a regularizing role in the convergence analysis of iterative algorithms.
  A mapping $T : C \to C$ is said strongly quasi-nonexpansive if $Fix(T) \neq \emptyset$, S is quasi-nonexpansive and $x_n - Tx_n \to 0$ whenever (x_n) is a bounded sequence such that $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in Fix(T)$.

It is very interesting and quite significant to establish also the strong convergence or weak convergence results on the iteration schemes for finding a common fixed point of finitely many nonexpansive mappings $T_1, T_2, ..., T_N$ of C into itself. Finding an optimal point in the intersection of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. To our knowledge there are some different kinds of approach to this approximation problem. For instance, in 1981, Kuhfittig, starting with a finite family of nonexpansive mappings, introduced a process which involves the convex combinations between every map with the identity map using a fixed appropriate coefficient. The resulting mapping is a nonexpansive and its fixed points coincide with the common fixed points of the family. This approach has opened a wide area of research followed by many mathematicians that study iterative methods approximating common fixed points of nonlinear mappings.

The interactions between fixed points theory and variational inequality problems are certainly not recent and it is natural to consider a unified approach to these two different problems. The concept of variational inequalities plays an important role in various kinds of problems in pure and applied sciences. For instance, several physical problems such as the theories of lubrications, filtration and flows, moving boundary problems can be reduced to variational inequality problems. In literature, many significant contributions concern the approximation of solutions of variational inequalities are reached, in which variational inequality problems are studied by extragradient methods based on Korpelevich's method, subgradient extragradient methods and other iterative-type methods. This strategy suggests, hence, for solving some certain variational inequalities an iterative algorithm must be used. In this direction, in the next Chapters, in the introduced convergence results it was showed the deep connections between the approximation of fixed points of a nonlinear mapping and the approximation of solutions of variational inequalities defined on the set of fixed point of the involved mapping.

The aim of this Thesis is to solve some proposed problems, related to approximation of fixed points of a nonlinear mapping or common fixed points of families of mappings and approximation of solutions of variational inequalities, illustrated in an unified treatment and collection of the main ideas, concepts and important results on our iterative methods.
Next, we present our contribution to the following proposed problems.
Problem 1:
We do not know whether a strong convergence Theorem of Halpern's type for nonspreading mappings holds or not, see [40] .

We give a partial answer in F. Cianciaruso, G. Marino, A. Rugiano, B. Scardamaglia, *On Strong convergence of Halpern's method using averaged type mappings*, J. Appl. Math., (2014), Art. ID 473243, 11 pages.
In Chapter 2, we discuss our results.
Problem 2: *One can ask if a strong convergence Theorem for a viscosity type method to approximate common fixed points of two mappings holds.*
We present a partial answer in F. Cianciaruso, G. Marino, A. Rugiano, B. Scardamaglia, *On strong convergence of viscosity type method using averaged type mappings*, Journal of Nonlinear and Convex Analysis no. 8, vol. 16, (2015), 1619-1640.
For a deep understanding, we discuss our results in Chapter 3.
Problem 3: *It is interesting to understand for which classes of mappings a strong convergence Theorem for a Halpern's type method holds or not, without the introduction of averaged type mappings.*
We discuss a partial answer in J. Garcia Falset, E. LLorens Fuster, G. Marino, A. Rugiano, *On strong convergence of Halpern's method for quasi-nonexpansive mappings in Hilbert Spaces*, submitted for publication.
We show our results in Chapter 4.
Problem 4: *One can ask if a strong convergence Theorem for the iterative scheme introduced by Iemoto and Takahashi in [31] holds.*
We illustrate a partial answer in A. Rugiano, B. Scardamaglia, S. Wang, *Hybrid iterative algorithms for a finite family of nonexpansive mappings and for a countable family of nonspreading mappings in Hilbert spaces*, to appear in Journal of Nonlinear and Convex Analysis.
We analyze our results in Chapter 5.

Chapter 1

Preliminaries

The aim of this Chapter is to introduce some basic concepts, notations and tools of fixed points theory. In spite of their elementary character, the given results have a number of significant applications in the next Chapters. Moreover, we give a substantial number of useful properties of mappings related to the studied algorithms.
Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\| \cdot \|$. In the sequel, we denote by $Fix(T)$ the set of fixed points of T, that is, $Fix(T) = \{z \in C : Tz = z\}$.

1.0.1 Fundamentals

As is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the nicest geometric properties. First, we list below fundamental inequalities in a Hilbert space, which will be used in the next results.

Lemma 1.0.1. *Let H be a real Hilbert space. There hold the following inequalities*

(i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$,
for all $x, y \in H$ and for all $t \in [0,1]$.

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$,
for all $x, y \in H$.

(iii) $2\langle x-y, z-w \rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$,
for all $x, y, z, w \in H$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such P_C mapping is called the metric projection of H onto C.
The following Lemma [64] characterizes the projection P_C.

Lemma 1.0.2. *Let C be a closed and convex subset of a real Hilbert space and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$; then $z = P_C x$ if and only if the following inequality holds:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{1.1}$$

Proof. Assume that $z = P_C x$, x given in H. We have, for all $y \in C$ and $t \in [0, 1]$,

$$(1-t)z + ty \in C,$$

and thus

$$\|x - z\| \leq \|x - (1-t)z - ty\| = \|x - z - t(y - z)\|.$$

Therefore

$$\|x - z\|^2 \leq \|x - z\|^2 - 2t\langle x - z, y - z\rangle + t^2\|y - z\|^2,$$

which implies

$$2\langle x - z, y - z\rangle \leq t\|y - z\|^2, \quad \forall t \in (0, 1].$$

As $t \to 0$, we obtain (1.1).
Conversely, assume that z satisfies (1.1). Then, we obtain

$$\langle x - z, y - x\rangle + \langle x - z, x - z\rangle \leq 0,$$

hence

$$\|z - x\|^2 \leq \langle x - z, x - y\rangle \leq \|x - z\|\|x - y\|.$$

This implies

$$\|x - z\| \leq \|x - y\|.$$

□

The availability of the inner product, the characterization of Projection P_C onto a closed convex subset of a Hilbert space discussed in Lemma 1.0.2, the inequalities in Lemma 1.0.1 are some of the geometric properties that characterize inner product spaces and also make certain problems posed in Hilbert spaces more manageable than those in general Banach spaces.

We recall some basic definitions concerning the convergence of a sequence.

Definition 1.0.3. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space H is said to be convergent to x if

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

In this case, we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

Definition 1.0.4. A sequence $(x_n) \rightharpoonup x$ in a Hilbert space H is said to be weakly convergent to x if and only if

$$\langle x_n, z\rangle \to \langle x, z\rangle$$

for all $z \in H$. In this case, we write $x_n \rightharpoonup x$ or $w\text{-}\lim\limits_{n\to\infty} x_n = x$.

We denote by $\omega_w(x_n) = \{x : \quad \exists x_{n_j} \rightharpoonup x\}$ denotes the ω-limit set of $(x_n)_{n\in\mathbb{N}}$.

Next, we list some basic facts for weak and strong convergence in Hilbert space without proof. Some of the following properties are not true or they have a more complicated proof in a general Banach space.

Remark 1.0.5. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space H.

1. The limit of convergent sequence is unique.

2. The weak limit x of a convergent sequence $(x_n)_{n\in\mathbb{N}}$ is unique.

3. Strong convergence implies weak convergence with the same limit. The converse is not generally true. If the Hilbert space is finite-dimensional, then the concepts of weak convergence and strong convergence are the same.

4. Recall that a Hilbert space has the Kadec-Klee property, denoted by KK, i.e. $x_n \to x$, whenever $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$.

5. Every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space has a weakly convergent subsequence.

6. Let H be a Hilbert space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Then $(x_n)_{n\in\mathbb{N}}$ is weakly convergent if and only if each weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$ has the same weak limit, that is, for any $x \in H$,

$$x_n \rightharpoonup x \quad \Leftrightarrow \quad (x_{n_i} \rightharpoonup y \Rightarrow x = y).$$

The Opial condition plays an important role in convergence of sequences and in the study of the demiclosedness principle of nonlinear mappings.

Definition 1.0.6. A Banach space E is said to satisfy Opial's condition if for all sequences $(x_n)_{n\in\mathbb{N}}$ in E converge weakly to $x_0 \in E$, then the following inequality holds, for all $x \neq x_0$,

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|. \tag{1.2}$$

Every Hilbert space satisfies Opial's condition. Indeed,

Lemma 1.0.7. *[50] If in a Hilbert space H the sequence $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to x_0, then for any $x \neq x_0$,*

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|. \tag{1.3}$$

Proof. Since every weakly convergent sequence is bounded, both limits in (1.3) are finite. Thus, in order to prove the inequality (1.3), it suffices to observe that

$$\begin{aligned} \|x_n - x\|^2 &= \|x_n + x_0 - x_0 - x\|^2 \\ &= \|x_n - x_0\|^2 + \|x_0 - x\|^2 + 2\langle x_n - x_0, x_0 - x\rangle. \end{aligned} \tag{1.4}$$

Taking the lim inf as $n \to \infty$ in (1.4) and using $x_n \rightharpoonup x_0$ and $x \neq x_0$, we get the inequality (1.3). □

Remark 1.0.8. We observe that thanks to the steps in the previous proof, (1.3) is equivalent to the analogous condition obtained by replacing lim inf by lim sup.

The Demiclosedness Principle states:

Definition 1.0.9. [50] *Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that $Fix(T) \neq \emptyset$. The mapping $I - T$ is said demiclosed at* 0 *if for every sequence $(x_n)_{n\in\mathbb{N}}$ weakly convergent to $p \in H$ such that $x_n - Tx_n \to 0$, it follows that $p \in Fix(T)$.*

In the following we give some nice properties for real sequences. Let us to start with the next Lemma proved by H.K Xu in [69]. It is frequently used throughout the proofs of the next Chapters, in order to prove the strong convergence of the sequence defined by an iterative method. For completeness, we include the proof.

Lemma 1.0.10. *[69] Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers satisfying the condition*

$$x_{n+1} \leq (1-\alpha_n)x_n + \beta_n\alpha_n \quad n \in \mathbb{N}, \tag{1.5}$$

where $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ are sequences of real numbers such that:

(i) $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$; or

(ii') $\sum_{n=1}^{\infty} \alpha_n\beta_n$ is convergent.

Then, $\lim_{n\to\infty} x_n = 0$.

Proof. First, we assume that (i) and (ii) hold. For any $\epsilon > 0$, let $N \geq 1$ be an integer big enough so that

$$\beta_n \leq \epsilon, \quad \forall n > N.$$

It follows from (1.5) that, for $n > N$,

$$\begin{aligned} x_{n+1} &\leq (1-\alpha_n)x_n + \epsilon\alpha_n \\ &\leq (1-\alpha_n)(1-\alpha_{n-1})x_{n-1} + \epsilon(1-(1-\alpha_n)(1-\alpha_{n-1})). \end{aligned}$$

By induction, we have

$$x_{n+1} \leq \prod_{j=N}^{n}(1-\alpha_j)x_N + \epsilon\left[1 - \prod_{j=N}^{n}(1-\alpha_j)\right], \quad \forall n > N. \tag{1.6}$$

Taking the lim sup as $n \to \infty$ in the inequality (1.6) and using the condition (i), we conclude that

$$\limsup_{n\to\infty} x_{n+1} \leq \epsilon.$$

Next, assume that (i) and (ii') hold. Then, repeatedly using (1.5), we get that, for all $n > m$,

$$x_{n+1} \leq \prod_{j=m}^{n}(1-\alpha_j)x_m + \sum_{j=m}^{n}\alpha_j\beta_j. \tag{1.7}$$

Taking in (1.7), first $n \to \infty$ and then $m \to \infty$, we deduce

$$\limsup_{n\to\infty} x_{n+1} \leq 0.$$

□

The following Lemma, showed by P.E. Maingé in [42], is of fundamental importance for the techniques of analysis developed in the next Chapters.

Lemma 1.0.11. *[42] Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_j})_{j\in\mathbb{N}}$ of $(\gamma_n)_{n\in\mathbb{N}}$ such that*

$$\gamma_{n_j} < \gamma_{n_j+1}, \quad \forall j \in \mathbb{N}. \tag{1.8}$$

Consider the sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ defined by

$$\tau(n) := \max\{k \leq n : \quad \gamma_k < \gamma_{k+1}\}. \tag{1.9}$$

Then $(\tau(n))_{n\in\mathbb{N}}$ is a nondecreasing sequence for all $n \geq n_0$, verifying

$$\lim_{n\to\infty} \tau(n) = \infty, \tag{1.10}$$

then, $\forall\, n \geq n_0$, the following two estimates hold

(i) $\gamma_{\tau(n)} < \gamma_{\tau(n)+1}$;

(ii) $\gamma_n < \gamma_{\tau(n)+1}$.

The interactions between fixed point theory and calculus of variations are certainly not recent. In this direction, Moudafi [47] in 2000 and Marino and Xu [46] in 2006 showed deep connections between the approximation of fixed points for nonlinear mappings and the approximation of solutions of variational inequalities. Hence, it is natural to consider a unified approach to these two different problems. Applications of variational inequalities span as diverse disciplines as differential equations, time-optimal control, optimization, mathematical programming, mechanics, finance and so on.
Let $D : H \to H$ be a nonlinear operator and let C be a nonempty closed and convex subset of H. The variational inequality (VIP) is formulated as finding a point $p \in C$ such that

$$\langle Dp, p - y\rangle \leq 0, \quad \forall y \in C. \tag{1.11}$$

Variational inequalities were initially studied by Stampacchia [36] and there after the problem of existence and uniqueness of solutions of (1.11) has been widely investigated by many authors in different disciplines as partial differential equations, optimal control, optimization, mechanics and finance. It is known that the problem (1.11) admits a unique solution if D is β-strongly monotone, i.e.

$$\langle Dx - Dy, x - y\rangle \geq \beta\|x - y\|^2 \quad \forall\, x, y \in H$$

and if D is ρ-Lipschitz continuous operator, i.e.

$$\|Dx - Dy\| \leq \rho\|x - y\|, \quad \forall\, x, y \in H.$$

Further, the (VIP) (1.11) is equivalent to the fixed-point equation

$$p = P_C(p - \mu Dp), \tag{1.12}$$

where P_C is the (nearest point) projection from H onto C and $\mu > 0$ is an arbitrarily fixed constant. Hence, fixed-point methods approximate a solution of the (VIP) if D satisfies some conditions and $\mu > 0$ is chosen appropriately. For instance, if D is a strongly monotone and Lipschitzian mapping on C, and $\mu > 0$ is small enough, then the mapping $D_\mu x := P_C(I - \mu D)x$ is a contraction. Indeed, for $x, y \in C$, we have

$$\begin{aligned}
\|P_C(I - \mu D)x - P_C(I - \mu D)y\|^2 &\leq \|(I - \mu D)x - (I - \mu D)y\|^2 \\
&= \|x - y\|^2 - 2\mu\langle x - y, Dx - Dy\rangle + \mu^2\|Dx - Dy\|^2 \\
&\leq \left(1 - \mu\left(2\beta - \frac{\mu}{\rho^2}\right)\right)\|x - y\|^2.
\end{aligned}$$

Then, for $0 < \mu < 2\frac{\beta}{\rho^2}$, D_μ is a contraction. In addition, the Banach contraction principle guarantees that the Picard iterates converge strongly to the unique solution of the (1.11).

We will state definitions in Banach space, useful in the next results. Let E be a Banach space and let E^* be the dual space of E. Let $D(T) \subset E$ be the domain of T.

We use the pairing between elements of a Banach space E and elements of its dual space E^*, i.e. $\langle x, f\rangle$, to denote $f(x), x \in E, f \in E^*$; and as a suitable analogue of the inner product in Hilbert spaces.

Definition 1.0.12. A mapping $T : D(T) \subset E \to E^*$ is said to be monotone if for all $x, y \in D(T)$

$$\langle x - y, Tx - Ty\rangle \geq 0. \tag{1.13}$$

A monotone operator $T : E \to E^*$ is also said to be maximal monotone if $T = S$ whenever $S : E \to E^*$ is a monotone operator such that $T \subset S$.

Remark 1.0.13. If $D(T) \subset \mathbb{R}$ and $T : D(T) \to \mathbb{R}$, then (1.13) means that T is monotone increasing.

The next class of operators was introduced independently in 1967 by Browder [8] and Kato [34]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces.

Definition 1.0.14. A mapping $T : D(T) \to E$ is said to be accretive iff, for all $x, y \in D(T)$, there exists $j \in J(x - y)$ such that

$$\langle Tx - Ty, j\rangle \geq 0,$$

where J denotes the normalized duality mapping, for $x \in E$, $J(x) = \{j \in E^* : \quad \langle x, j\rangle = \|x\|^2 = \|j\|^2\}$.

Notice that if $E = H$, then $E = E^* = H$, the class of monotone and accretive operators defined in E coincide.

In the sequel, we discuss nice properties and some existence fixed points theorems for the mappings related to the studied algorithms. First, we will present the nice class of quasi-nonexpansive mappings.

1.0.2 Quasi-nonexpansive mappings

The notion of quasi-nonexpansivity was introduced, in our knowledge, by Tricomi [66] in 1916 for a real function f defined on a finite or infinite interval (a, b) with the values in the same interval. He proved that the sequence $(x_k)_{k\in\mathbb{N}}$ generated by the simple iteration $x_{k+1} = f(x_k)$, x_0 is given in (a, b), converges to a fixed point of f provided that f is continuous and strictly quasi-nonexpansive on (a, b). The importance of the concept of quasi-nonexpansivity for the computation of fixed points in more general cases had been emphasized by many authors (see Diaz and Metcalf [21], K. Wongchan and S. Saejung [68]). If $T : C \to H$ is a mapping with $Fix(T) \neq \emptyset$, then T is said to be quasi-nonexpansive if, $\forall x \in C, \forall p \in Fix(T)$,

$$\|Tx - p\| \leq \|x - p\|. \tag{1.14}$$

The choice of considering quasi-nonexpansive mappings turns out to be quite reasonable since these operators are widely used in the subgradient projection techniques to solve nonsmooth convex constrained problems (see Maingé [43]). The lack of continuity of a quasi-nonexpansive mapping makes difficult to prove asymptotical stability of iterates of T. This fact is one of the raison to study the convergence analysis of some well-known iterative methods. Many recent papers which concern the approximation to fixed points of quasi-nonexpansive mappings assume some other property in order to bypass the lack of continuity. Some other approaches use a convex combination between the mapping and another opportune mapping, e.g. identity mapping, in order to obtain better regularizing properties.

Notice that there exist continuous nonlinear quasi-nonexpansive mappings.

Example 1.0.15. *[52] The mapping $T : [-1, 1] \to [-1, 1]$, defined by*

$$T(x) = \begin{cases} \frac{x}{2}\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is a continuous quasi-nonexpansive mapping on $[-1, 1]$.

Example 1.0.16. *[29] Let $H = \mathbb{R}$ with the absolute-value norm $|\cdot|$ and $C = [0, \infty)$. Define $T : C \to C$ by*

$$Tx = \frac{x^2 + 2}{x + 1}, \quad \forall x \in C.$$

Obviously, $Fix(T) = \{2\}$. Moreover, T is a continuous quasi-nonexpansive mapping on C.

The following result, concerning the set of fixed points of a quasi-nonexpansive mapping, was showed by W. G. Jr. Dotson in [23] and was also obtained by S.Itoh and W.Takahashi in [33].

Theorem 1.0.17. *If C is a closed convex subset of a strictly convex normed linear space, and $T : C \to C$ is quasi-nonexpansive, then $Fix(T) := \{p : \quad p \in C, \quad Tp = p\}$ is a nonempty closed convex set on which T is continuous.*

1.0.3 Firmly nonexpansive mappings

Bruck [12] introduced the class of firmly nonexpansive mappings in the setting of Banach spaces. We recall the definition in a Hilbert space.

Definition 1.0.18. A mapping $T : C \to C$ is said to be firmly nonexpansive if, for all $x, y \in C$ and $t \geq 0$, there holds the inequality

$$\|(1-t)(Tx - Ty) + t(x - y)\| \geq \|Tx - Ty\|.$$

Firmly nonexpansive mappings are fascinating because they have several remarkable properties which are not shared by all nonexpansive mappings. Moreover, they have interesting equivalent formulations and connections with accretive operators and maximal monotone operators. Indeed,

(a) Let E be a Banach space and let C be a nonempty closed convex subset of E. It is known that T is firmly nonexpansive if and only if there exists an accretive operator $A : E \to E$ such that $D(A) \subset C \subset R(I + A)$ and $Tx = (I + A)^{-1}x$ for all $x \in C$. In this case, $Fix(T) = A^{-1}0$.

(b) The problem of finding zero points of maximal monotone operators in Hilbert spaces is reduced to the fixed point problem for firmly nonexpansive mappings. Let H be a Hilbert space and let $A : H \to H$ be a maximal monotone operator. Further, the maximal monotonicity of A implies that $R(I + rA) = H$. Then, for each $r > 0$, the resolvent J_r of A is defined by $J_r x = (I + rA)^{-1}x$ for all $x \in H$. It is well-known that J_r is a single-valued firmly nonexpansive mapping, that is, for all $x, y \in H$

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle.$$

Therefore the problem: finding $u \in H$ such that $0 \in Au$, where A is maximal monotone operator is reduced to a fixed point problem for a firmly nonexpansive mapping in the setting of Hilbert spaces. Hence, it holds that $Fix(J_r) = A^{-1}0$.

(c) Let $T : C \to C$ a mapping in a Hilbert space. There are equivalent formulations.

1. T is firmly nonexpansive, that is, $\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in C$;
2. for all $x, y \in C$, the convex function $\phi_{x,y} : [0, 1] \to \mathbb{R}$ defined by $\phi_{x,y}(t) = \|(1 - t)(x - y) + t(Tx - Ty)\|$ is nonincreasing on $[0, 1]$;
3. the mapping $(2T - I)$ is nonexpansive;
4. the mapping $T = \frac{1}{2}(I + N)$, where N is nonexpansive;
5. $\|Tx - Ty\| \leq \phi_{x,y}(t)$ for all $t \in [0, 1]$ and for all $x, y \in C$;

6. the mapping $I - T$ is firmly nonexpansive;
7. for all $x, y \in C$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$;
8. One has, for all $x, y \in C$, $2\|Tx - Ty\|^2 + \|Tx - x\|^2 + \|Ty - y\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$.

Important classes of mappings contain the firmly nonexpansive mappings : the averaged mappings, the nonexpansive mappings, the nonspreading mappings, the L-hybrid mappings, the strongly nonexpansive mappings

1.0.4 Averaged type mappings

We shall introduce auxiliary mappings T_δ which play a regularizing role in the convergence analysis of iterative algorithms. If T is a nonexpansive mapping of C into itself, **the averaged mapping** is defined as follows, (see [14]),

$$T_\delta = (1 - \delta)I + \delta T = I - \delta(I - T), \tag{1.15}$$

where $\delta \in (0, 1)$. Thus firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged. Byrne [14] and successively Moudafi [49] proved some key properties for averaged mappings.

Proposition 1.0.19.
- *The convex combination of an averaged mapping and a nonexpansive mapping is an averged mapping.*
- *The composite of finitely many averaged mappings is averaged.*
- *If the mappings $(T_i)_{i=1}^N$ are averaged and have a nonempty common fixed point, then $\bigcap_{i=1}^N Fix(T_i) = Fix(T_1 T_2 ... T_N)$.*
- *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
- *T is averaged if, and only if, its complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$.*

The nonexpansive averaged mapping regularizes a nonexpansive mapping according to the celebrated Schaefer's result [56].

Theorem 1.0.20. *Any orbit $(T_\delta^k)_{k \in \mathbb{N}}$ of a nonexpansive average mapping $T_\delta = (1 - \delta)I + \delta T$ converges weakly to a fixed point of T whenever such points exist.*

In the sequel we define **the averaged type mapping** as follows

$$T_\delta = (1 - \delta)I + \delta T\,, \tag{1.16}$$

where $\delta \in (0, 1)$.

In the following Lemma we collect some proprieties for the averaged type mapping T_δ.

Proposition 1.0.21. *Let $T : C \to C$ be a quasi-nonexpansive mapping on $C \subset H$, and for $\delta \in (0, 1)$, $T_\delta := (1 - \delta)I + \delta T$. Then:*

(1) $Fix(T) = Fix(T_\delta)$*;*

(2) T_δ *is a quasi-nonexpansive mapping;*

(3) Let $p \in Fix(T)$*. Then,* T_δ *is a quasi-firmly nonexpansive type mapping, that is, for every* $x \in C$

$$\|T_\delta x - p\|^2 \leq \|x - p\|^2 - (1-\delta)\|x - T_\delta x\|^2. \tag{1.17}$$

(4) [68] T *is a quasi-nonexpansive mapping if and only if, for all* $x \in C, p \in Fix(T)$*,*

$$\langle x - T_\delta x, x - p\rangle \geq \frac{\delta}{2}\|x - Tx\|^2. \tag{1.18}$$

Proof. It easy to check (1) and (2).
(3)
Let $p \in Fix(T)$, for all $x \in C$, we calculate

$$\begin{array}{rcl} \|T_\delta x - p\|^2 & = & \|(1-\delta)(x-p) + \delta(Tx - p)\|^2 \\ & = & (1-\delta)\|x-p\|^2 + \delta\|Tx - p\|^2 - \delta(1-\delta)\|Tx - x\|^2 \\ (T \text{ quasi-nonexp.}) & \leq & \|x-p\|^2 - \dfrac{(1-\delta)}{\delta}\|x - T_\delta x\|^2 \\ (\delta < 1) & \leq & \|x-p\|^2 - (1-\delta)\|x - T_\delta x\|^2. \end{array}$$

(4)
For all $x \in C, p \in Fix(T)$, we compute,

$$\begin{array}{rcl} \|T_\delta x - p\|^2 & = & \|T_\delta - x + x - p\|^2 \\ & = & \|T_\delta x - x\|^2 + 2\langle T_\delta x - x, x - p\rangle + \|x-p\|^2 \\ & = & \delta^2\|Tx - x\|^2 + 2\langle T_\delta x - x, x-p\rangle + \|x-p\|^2. \end{array}$$

On the other hand,

$$\begin{array}{rcl} \|T_\delta x - p\|^2 & = & \|(1-\delta)(x-p) + \delta(Tx-p)\|^2 \\ (\text{ by Lemma 1.0.1}) & = & (1-\delta)\|x-p\|^2 + \delta\|Tx-p\|^2 - \delta(1-\delta)\|x - Tx\|^2. \end{array}$$

Hence,

$$\begin{array}{rcl} 2\langle x - T_\delta x, x - p\rangle & = & \delta\left(\|x-p\|^2 - \|Tx-p\|^2\right)^2 + \delta\|Tx - x\|^2 \\ (\text{ by } T \text{ nonexpansive}) & \geq & \delta\|Tx - x\|^2. \end{array}$$

□

1.0.5 Nonexpansive mappings

Nonexpansive mappings $T : C \to H$ are those which have Lipschitz's constant equal to 1, i.e. if for all $x, y \in C$, the following inequality holds,

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1.19}$$

Thus this class of mappings includes the contractions, the strictly contractive mappings, isometries (including the identity). A wide variety of problems can be solved by finding fixed points of nonexpansive mappings and the study of iterative methods for approximating fixed points of a nonexpansive mappings T has yielded a lot of relevant progresses. Nonexpansive mappings are perhaps one of the most fascinating topics in the so-called metric fixed points theory, mainly for the following three reasons:

(1) Nonexpansive mapping are an obvious generalization of the contraction mappings. The Banach Contraction principle appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation.

Theorem 1.0.22. *Let X be a complete metric space and let $T : X \to X$ be a contraction. Then, T has a unique fixed point in X and, for any $x_0 \in X$, Picard iteration $\{T^n x_0\}$ converges strongly to the unique fixed point of T.*

The simplicity of its proof and the possibility of attaining the fixed point by using successive approximations let this theorem become a very useful tool in analysis and its applications. There is a great number of generalizations of the Banach contraction principle in the literature. Unlike in the case of the Banach Contraction Principle, trivial examples show that the sequence of successive approximations, defined as

$$x_0 \in C, \quad x_{n+1} = Tx_n, \quad n \geq 0,$$

where C is a nonempty closed convex and bounded subset of a real Banach space and $T : C \to C$ a nonexpansive mapping (even with a unique fixed point), may fail to converge to the fixed point. It suffices, for example, to take for T, a rotation of the unit ball in the plane around the origin of coordinates. More precisely, we have the following example.

Example 1.0.23. *Let $B = \{x \in \mathbb{R}^2 : \quad \|x\| \leq 1\}$ and let T denote an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as the only fixed point. Therefor, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by*

$$x_{n+1} = Tx_n, \quad x_0 = (1, 0) \in B, \quad n \geq 0,$$

does not converge to zero.

(2) Nonexpansive maps are strictly connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.

(3) The study of nonexpansive mappings has been motivated by the study of monotone and accretive operators, two classes of operators which arise naturally in the theory of differential equations. The fixed point problem for nonexpansive mappings in Hilbert spaces is connected with the problem of finding zero points of maximal monotone operators in the spaces, i.e. finding a point $u \in H$ satisfying

$$0 \in Au, \tag{1.20}$$

where $A : H \to 2^H$ is a maximal monotone operator. This problem is related to convex optimization problems, minimax problems, variational inequality problems, equilibrium problems, and so on.

Next, we discuss two important connections between the classes of nonexpansive and of accretive mappings which give rise to a strong connection between the fixed point theory of nonexpansive mappings and the mapping theory of accretive maps.

(i) Let $T : D(T) \to H$ be a nonexpansive mapping. Then, the mapping U defined as $U = I - T$ is accretive.
In fact, for $x, y \in D(T)$, we have

$$\begin{aligned}\langle Tx - Ty, x - y\rangle &= \langle x - y - (Ux - Uy), x - y\rangle \\ &= \|x - y\|^2 - \langle Ux - Uy, x - y\rangle \\ &\geq \|x - y\|^2 - \|Ux - Uy\|\|x - y\| \geq 0.\end{aligned}$$

It is well known that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0, \tag{1.21}$$

where A is an accretive map on an appropriate Banach space, introduced independently in 1967 by Browder and Kato. Interest in such mappings stems mainly from their connection with the existence theory for nonlinear equations of evolution in Banach spaces. We remark that in the evolution equation (1.21), if u is independent of t, then $\frac{du}{dt} = 0$ and the equation reduces to $Au = 0$ whose solutions correspond to the equilibrium points of the system described by the equation.

(ii) If $\{T(t), \quad t \geq 0\}$ is a semigroup of (nonlinear) mappings of a Banach space E into E with infinitesimal generator S, then all the mappings $T(t)$ are nonexpansive if and only if $(-S)$ is accretive

In the following Proposition we discuss a connection between nonexpansive mappings and firmly nonexpansive mappings.

Proposition 1.0.24. *Let H a Hilbert space and C a nonempty subset of H. Then, every firmly nonexpansive mapping $T : C \to H$ is a nonexpansive mapping.*

Proof. We compute, for all $x, y \in C$,

$$\begin{aligned}
\|Tx - Ty\|^2 &\leq \langle x - y, Tx - Ty\rangle \\
&\Leftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), Tx - Ty\rangle \\
&\Rightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\
&\Leftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2,
\end{aligned}$$

which implies that T is nonexpansive. □

The converse is not true.

Remark 1.0.25. In a Hilbert space $Tx = -x$ is clearly a nonexpansive mapping, but T is not firmly nonexpansive.

The following result summarizes some significant properties of $I - T$ if T is a nonexpansive mapping ([14],[10]).

Lemma 1.0.26. *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be nonexpansive. Then:*

(i) $I - T : C \to H$ is $\frac{1}{2}$-inverse strongly monotone, i.e.,

$$\frac{1}{2}\|(I - T)x - (I - T)y\|^2 \leq \langle x - y, (I - T)x - (I - T)y\rangle,$$

for all $x, y \in C$;

(ii) moreover, if $Fix(T) \neq \emptyset$, $I - T$ is demiclosed at 0, i.e. for every sequence $(x_n)_{n\in\mathbb{N}}$ weakly convergent to p such that $x_n - Tx_n \to 0$ as $n \to \infty$, it follows $p \in Fix(T)$;

Proof. (i) For all $x, y \in H$,

$$\begin{aligned}
\|x - y\|^2 &\geq \|Tx - Ty\|^2 = \|(I - T)x - (I - T)y - (x - y)\|^2 \\
&= \|(I - T)x - (I - T)y\|^2 - 2\langle x - y, (I - T)x - (I - T)y\rangle + \|x - y\|^2.
\end{aligned}$$

Hence, we get

$$\langle x - y, (I - T)x - (I - T)y\rangle \geq \frac{1}{2}\|(I - T)x - (I - T)y\|^2. \tag{1.22}$$

□

The following are interesting problems related to the study of nonexpansive mappings:

- Finding assumptions on the structure of the space and (or) restriction on the mapping to assure the existence of at least one fixed point;
- the structure of the fixed points sets;
- the behavior of the iterates of the mapping;
- the approximation of fixed points.

Next, we will give some fundamental existence Theorems for nonexpansive mappings. First, we illustrate some examples to show that nonexpansive mappings may fail to have fixed points in general Banach spaces.

Example 1.0.27. *Let X be a Banach space and $T: X \to X$ a translation mapping defined by $Tx = x + a, \quad a \neq 0$. Then T is nonexpansive and a fixed point free mapping.*

Example 1.0.28. *(Sadovski) (see, i.e. [1]) Let c_0 be the Banach space of null sequences and $C = \{x \in c_0 : \quad \|x\| \leq 1\}$, the unit closed ball in c_0. Define a mapping $T: C \to C$ by $T(x_1, x_2, ..., x_i, ...) = (1, x_1, x_2, x_3, ...)$. It is obvious that T is nonexpansive on the closed convex bounded set C and $x = (1, 1, 1, ...)$ is a fixed point of T. But $(1, 1, 1, ...)$ does not belong to c_0.*

Theorem 1.0.29. *[7][27](Browder's Theorem and Göhde's Theorem) Let X be a uniformly convex Banach space and C a nonempty closed convex bounded subset of X. Then every nonexpansive mapping $T: C \to C$ has a fixed point in C.*

Corollary 1.0.30. *[6] (Browder's Theorem) Let H be a Hilbert space and C a nonempty closed convex bounded subset of H. Then every nonexpansive mapping $T: C \to C$ has a fixed point in C.*

The following result is widely regarded as the fundamental existence theorem for nonexpansive mappings.

Theorem 1.0.31. *[37](Kirk's Fixed Point Theorem) Let X be a Banach space and C a nonempty weakly compact convex subset of X with normal structure. Then every nonexpansive mapping $T: C \to C$ has a fixed point.*

We will discuss the structure of the set of fixed points of nonexpansive mappings. In the following example is shown that the set of fixed points can not be convex in a general Banach space.

Example 1.0.32. *[12] Let $X = \mathbb{R}^2$ be a Banach space with maximum norm defined by $\|(a,b)\|_\infty = \max\{|a|, |b|\}$ for all $x = (a,b) \in \mathbb{R}^2$. Let $T: X \to X$ be a mapping defined by $T(a,b) = (a, |a|)$ for all $(a,b) \in \mathbb{R}^2$. Then T is nonexpansive and $Fix(T) = \{(a,b) : \quad b = |a|\}$. However, $Fix(T)$ is not convex.*

Proposition 1.0.33. *[26] Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T: C \to X$ a nonexpansive mapping. Then $Fix(T)$ is closed and convex.*

1.0.6 Nonspreading mappings

Let E be a smooth, strictly convex and riflexive Banach space, let J the duality mapping of E and let C be a nonempty closed convex subset of E. Kohsaka and Takahashi, see [38], introduced the concept of nonspreading mappings. $T: C \to C$ is said nonspreading if, for all $x, y \in C$,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x),$$

where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$. If E is a Hilbert space H, we know that $\phi(x,y) = \|x-y\|^2$, for all $x, y \in H$. Consequently, a nonspreading mapping $T : C \to C$ in a Hilbert space is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C. \tag{1.23}$$

Let us give an example of a nonspreading mapping.

Example 1.0.34. *[20] Let T a mapping defined on $[0,3]$ by*

$$\begin{cases} 0 & x \neq 3, \\ 2 & x = 3. \end{cases}$$

It is easy to see that T is nonspreading. Indeed, if $x = 3$ and $y \neq 3$, we have

$$2\|Tx - Ty\|^2 = 8 < 9 = \|Ty - x\|^2.$$

In the other case, we get

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

Remark 1.0.35. A nonexpansive mapping can not be a nonspreading mapping and a nonspreading can not be nonexpansive. We list some examples.

1. $Tx = 1 - x$, for all $x \in [0,1]$, is a nonexpansive mapping (see [20]) but it is easy to check that T is not a nonspreading mapping, e.g. if we set $x = 0, y = 1$;

2. The following is an example of a nonspreading mapping which is not nonexpansive (see Igarashi, Takahashi and Tanaka [32]). Let H a Hilbert space, set $E = \{x \in H : \ \|x\| \leq 1\}$, $D = \{x \in H : \ \|x\| \leq 2\}$ and $C = \{x \in H : \ \|x\| \leq 3\}$. Let $S : C \to C$ be a mapping defined as follows:
$$\begin{cases} 0 & x \in D \\ P_E(x) & x \in C \backslash D, \end{cases} \tag{1.24}$$
where P_E is the metric projection of H onto E. Then, S is not a nonexpansive mapping but it is a nonspreading mapping.

Nonspreading mappings are deduced from firmly nonexpansive mappings in a Hilbert space.

Proposition 1.0.36. *Let H a Hilbert space and C a nonempty subset of H.Then, every firmly nonexpansive mapping $T : C \to H$ is nonspreading.*

Proof. W have, for all $x, y \in C$,

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \langle x - y, Tx - Ty\rangle \\ &\Leftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), Tx - Ty\rangle \\ &\Leftrightarrow 0 \leq 2\langle x - Tx, Tx - Ty\rangle + 2\langle Ty - y, Tx - Ty\rangle \\ &\Rightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\ &\Leftrightarrow 0 \leq \|x - Ty\|^2 - 2\|Tx - Ty\|^2 + \|Tx - y\|^2 \\ &\Leftrightarrow 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned}$$

□

Since a mapping is firmly nonexpansive if and only if it is the resolvent of a monotone operator, it follows that to approximate fixed points of nonspreading mappings it is equivalent to approximating zeros of monotone operators in Banach spaces.

The following Lemma is a characterization for nonspreading mappings, see [31].

Lemma 1.0.37. *Let C be a nonempty closed subset of a Hilbert space H. Then a mapping $T : C \to C$ is nonspreading if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{1.25}$$

Iemoto and Takahashi showed the demiclosedness of $I - S$ at 0 and a suitable property of $I - S$.

Lemma 1.0.38. *[31] Let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonspreading mapping such that $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed at* 0.

Lemma 1.0.39. *[31] Let C be a nonempty, closed and convex subset of H. Let $S : C \to C$ be a nonspreading mapping. Then*

$$\|(I-T)x - (I-T)y\|^2 \leq \langle x - y, (I-T)x - (I-T)y\rangle + \frac{1}{2}\Big(\|x - Tx\|^2 + \|y - Ty\|^2\Big),$$

for all $x, y \in C$.

Following the same line in [51], we propose a property for the averaged type mapping of a nonspreading mapping.

Proposition 1.0.40. *[17] Let C be a nonempty closed and convex subset of H and let $T : C \to C$ be a nonspreading mapping such that $Fix(T)$ is nonempty. Then the averaged type mapping*

$$T_\delta = (1 - \delta)I + \delta T, \tag{1.26}$$

is quasi-firmly type nonexpansive mapping with coefficient $k = (1 - \delta) \in (0, 1)$.

Proof. We obtain

$$\begin{array}{rcl}
\|T_\delta x - T_\delta y\|^2 & = & \|(1-\delta)(x-y) + \delta(Tx - Ty)\|^2 \\
\text{(by Lemma 1.0.1)} & = & (1-\delta)\|x-y\|^2 + \delta\|Tx - Ty\|^2 \\
 & - & \delta(1-\delta)\|(x - Tx) - (y - Ty)\|^2 \\
 & \leq & (1-\delta)\|x-y\|^2 + \delta\Big[\|x-y\|^2 + 2\langle x - Tx, y - Ty\rangle\Big] \\
 & - & \delta(1-\delta)\|(x - Tx) - (y - Ty)\|^2 \\
 & = & \|x-y\|^2 + \dfrac{2}{\delta}\langle\delta(x - Tx), \delta(y - Ty)\rangle \\
 & - & \dfrac{1-\delta}{\delta}\|\delta(x - Tx) - \delta(y - Ty)\|^2
\end{array}$$

$$
\begin{aligned}
\text{(by (1.26))} \quad &= \quad \|x-y\|^2 + \frac{2}{\delta}\langle x - T_\delta x, y - T_\delta y\rangle \\
&- \quad \frac{1-\delta}{\delta}\|(x - T_\delta x) - (y - T_\delta y)\|^2 \\
&\leq \quad \|x-y\|^2 + \frac{2}{\delta}\langle x - T_\delta x, y - T_\delta y\rangle \\
&- \quad (1-\delta)\|(x - T_\delta x) - (y - T_\delta y)\|^2 .
\end{aligned}
$$

Hence, we have

$$
\|T_\delta x - T_\delta y\|^2 \leq \|x-y\|^2 + \frac{2}{\delta}\langle x - T_\delta x, y - T_\delta y\rangle - (1-\delta)\|(x - T_\delta x) - (y - T_\delta y)\|^2. \quad (1.27)
$$

In particular, choosing in (1.27) $y = p \in Fix(T) = Fix(T_\delta)$ we obtain, for all $x \in C$,

$$
\|T_\delta x - p\|^2 \leq \|x-p\|^2 - (1-\delta)\|x - T_\delta x\|^2 . \quad (1.28)
$$

□

The following Lemma shows that the limit of family of nonspreadings is still nonspreading.

Lemma 1.0.41. *[54] Let H be a Hilbert space and C be a nonempty subset of H. Let $(T_n)_{n\in\mathbb{N}}$ be a family of nonspreading mappings of C into itself and assume that $\lim_{n\to\infty} T_n x$ exists for each $x \in C$. Define the mapping $T : C \to C$ by $Tx = \lim_{n\to\infty} T_n x$. Then the mapping T is a nonspreading mapping.*

Next, we list fixed point theorems for nonspreading mappings in a Banach space, given by F. Kohsaka and W. Takahashi, using some techniques developed by W. Takahashi in [63], [64].

Theorem 1.0.42. *[38] Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself. Then the following are equivalent:*

- *There exists $x \in C$ such that $(T^n x)$ is bounded;*
- *$Fix(T)$ is nonempty.*

Corollary 1.0.43. *Every bounded closed convex subset of a smooth, strictly convex and reflexive Banach space has the fixed point property for nonspreading self mappings.*

Proposition 1.0.44. *[38] Let E be a smooth and strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself. Then $Fix(T)$ is closed and convex.*

We recall the concept of Chatterjea mapping which will be used in next Chapters.

Remark 1.0.45. Recently, T. Suzuki [62] introduced the concept of Chatterjea mapping. Let T be a mapping on a subset C of a Banach space E and let η be a continuous strictly increasing function from $[0, \infty)$ into itself with $\eta(0) = 0$. Then, T is called a Chatterjea mapping with respect to η if

$$
2\eta(\|Tx - Ty\|) \leq \eta(\|Tx - y\|) + \eta(\|x - Ty\|), \quad \forall x, y \in C. \quad (1.29)
$$

It is easy to check that a nonspreading mapping on a subset C of a Hilbert space is a Chatterjea mapping with respect to the function $t \mapsto t^2$.

1.0.7 L-hybrid mappings

K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi [2] first introduced the class of L-hybrid mappings in Hilbert spaces. Let $T : C \to H$ be a mapping and $L \geq 0$ a nonnegative number. T is said L-hybrid, signified as $T \in \mathcal{H}_L$, if, for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle, \tag{1.30}$$

or equivalently

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\left(1 - \frac{L}{2}\right)\langle x - Tx, y - Ty\rangle. \tag{1.31}$$

Notice that for particular choices of L we obtain several important classes of nonlinear mappings. In fact

- $\mathcal{H}_0$ is the class of the nonexpansive mappings;
- $\mathcal{H}_2$ is the class of the nonspreading mappings;
- $\mathcal{H}_1$ is the class of the hybrid mappings.

Moreover, we have

(i) if $Fix(T) \neq \emptyset$, each L-hybrid mapping is quasi-nonexpansive mapping (see [2]), i.e.

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C \quad \text{and} \quad p \in Fix(T);$$

(ii) T is L-hybrid if and only if for all $x, y \in C$

$$\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Ty\|^2 + 2\langle(1 - \frac{L}{2})x + \frac{L}{2}Tx - Ty, Ty - y\rangle,$$

(see, [2]). In fact, for all $x, y \in C$ we have

$$\begin{aligned}\|Tx - Ty\|^2 &\leq \|x - Ty + Ty - y\|^2 - L\langle x - Tx, Ty - y\rangle \\ &= \|x - Ty\|^2 + \|Ty - y\|^2 + 2\langle(1 - \frac{L}{2})x + \frac{L}{2}Tx - Ty, Ty - y\rangle.\end{aligned}$$

(iii) If $T \in \mathcal{H}_\lambda$, then $T_\delta := (1 - \delta)I + \delta T$ belongs to $\mathcal{H}_{\frac{L}{\delta}}$ for $\delta > 0$ (see [4], Lemma 2.1).

Next, we provide some examples of L-hybrid mappings.

Example 1.0.46. *[2]*

- *Let C a nonempty subset of H and $T : C \to H$ a mapping such that, for all $x, y \in C$,*

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2.$$

Then, T is $\frac{2}{3}$-hybrid.

- *The following is an example of L-hybrid mapping which is not continuous in the case $L \in (0,2]$. Let $\alpha := \frac{\left(1-\frac{L}{2}\right)\frac{L}{2}+\sqrt{L}}{1-\left(1-\frac{L}{2}\right)^2}$, $A := \{x \in H: \quad \|x\| \leq 1\}$, $B := \{x \in H: \quad \|x\| \leq \alpha\}$. Let $T: H \to H$ a mapping defined by*

$$T(x) = \begin{cases} P_A(x) = \frac{x}{\|x\|}, & x \in H \setminus B, \\ 0, & x \in B. \end{cases}$$

Then, T is L-hybrid.

Next, we give some other nice facts related to L-hybrid mappings.

Proposition 1.0.47. *(a) If T is firmly nonexpansive. Then T is L-hybrid, for all $0 \leq L \leq 2$. In particular, If T is a contraction with $\alpha \in (0,1)$, then T is L-hybrid for all $L \in [0,2]$.*

(b) For any $L > 0$ fixed, there is $\alpha \in (-1,1)$ such that the contraction $Tx = \alpha x$ is not L-hybrid mapping.

Proof. (a)
Let T be a firmly nonexpansive mapping, $0 \leq L \leq 2$. Then, for all $x, y \in H$, we get

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x-y\|^2 - \|x - Tx - (y - Ty)\|^2 \\ &\leq \|x-y\|^2 - \frac{L}{2}\|x - Tx - (y - Ty)\|^2 \\ &= \|x-y\|^2 - \frac{L}{2}(\|x - Tx\|^2 + \|y - Ty\|^2) + L\langle x - Tx, y - Ty\rangle \\ &\leq \|x-y\|^2 + L\langle x - Tx, y - Ty\rangle. \end{aligned}$$

(b)
Let $L > 0$ be fixed. We set $x = x_0, y = -x_0$ with $x_0 \neq 0$, in

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + L\langle x - Tx, y - Ty\rangle, \quad x, y \in C,$$

and we obtain

$$\|Tx - Ty\|^2 = 4\alpha^2\|x_0\|^2 \leq 4\|x_0\|^2 - L(1-\alpha)^2\|x_0\|^2,$$

hence

$$\begin{aligned} 4\alpha^2 \leq 4 - L(1-\alpha)^2 &\Leftrightarrow L(1-\alpha)^2 \leq 4(1-\alpha^2) \\ &\Leftrightarrow L(1-\alpha) \leq 4(1+\alpha) \\ &\Leftrightarrow L - 4 \leq \alpha(L+4) \\ &\Leftrightarrow \alpha \geq \frac{L-4}{L+4}. \end{aligned}$$

Notice that $-1 < \frac{L-4}{L+4} < 1$. For $\alpha < \frac{L-4}{L+4}$, $Tx = \alpha x$ is not L-hybrid. Observe that if $L < 4$, then $\alpha < 0$. Consequently, (a) and (b) are not in contradiction. $\square$

In the next Lemma we prove a suitable property for $I - T$.

Lemma 1.0.48. *[18] Let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a L-hybrid mapping. Then*

$$
\begin{aligned}
\|(I-T)x-(I-T)y\|^2 \;\leq\;& \langle x-y,(I-T)x-(I-T)y\rangle + \frac{1}{2}\Bigg(\|x-Tx\|^2+\|y-Ty\|^2 \\
&- 2\left(1-\frac{L}{2}\right)\langle x-Tx,y-Ty\rangle\Bigg),
\end{aligned}
$$

for all $x, y \in C$.

Proof. Put $A = I - T$. For all $x, y \in C$ we have

$$
\begin{aligned}
\|Ax-Ay\|^2 &= \langle Ax-Ay,Ax-Ay\rangle \\
&= \langle (x-y)-(Tx-Ty),Ax-Ay\rangle \\
&= \langle x-y,Ax-Ay\rangle - \langle Tx-Ty,Ax-Ay\rangle. \qquad (1.32)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
2\langle Tx-Ty,Ax-Ay\rangle &= 2\langle Tx-Ty,(x-y)-(Tx-Ty)\rangle \\
&= 2\langle Tx-Ty,x-y\rangle - 2\|Tx-Ty\|^2 \\
\text{(by Lemma (1.0.1), (iii))} \quad &\geq \|x-Ty\|^2+\|y-Tx\|^2-\|x-Tx\|^2-\|y-Ty\|^2 \\
\text{(by (1.31))} \quad &- \left(\|x-Ty\|^2+\|y-Tx\|^2-2(1-\frac{L}{2})\langle x-Tx,y-Ty\rangle\right) \\
&= -\|x-Tx\|^2-\|Ty-y\|^2+2(1-\frac{L}{2})\langle x-Tx,y-Ty\rangle \\
&= -\|Ax\|^2-\|Ay\|^2+2(1-\frac{L}{2})\langle x-Tx,y-Ty\rangle. \qquad (1.33)
\end{aligned}
$$

Finally, from (1.32) and (1.33), we can conclude

$$
\begin{aligned}
\|(I-T)x-(I-T)y\|^2 \;\leq\;& \langle x-y,(I-T)x-(I-T)y\rangle + \frac{1}{2}\Bigg(\|x-Tx\|^2+\|y-Ty\|^2 \\
&- 2(1-\frac{L}{2})\langle x-Tx,y-Ty\rangle\Bigg).
\end{aligned}
$$

In particular, if $Fix(T) \neq \emptyset$, we choose $y = p \in Fix(T) = Fix(T_\delta)$ and we obtain, for all $x \in C$,

$$
\frac{1}{2}\|x-Tx\|^2 \leq \langle x-p,x-Tx\rangle. \qquad (1.34)
$$

□

Lemma 1.0.49. *[2]Let H be a Hilbert space, C a nonempty closed convex subset of H, $\lambda \in \mathbb{R}$, $T : C \to H$ a L-hybrid mapping. Then, $I - T$ is demiclosed at 0.*

1.0.8 Strongly quasi-nonexpansive mappings

The concept of strongly nonexpansive mappings was introduced by Bruck and Reich in 1977 [13] in the setting of a Banach space E.

Definition 1.0.50. A mapping $T : D(T) \to E$ is said strongly nonexpansive if T is nonexpansive and whenever $(x_n - y_n)$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_{\|} \to 0$, it follows that $(x_n - y_n) - (Tx_n - Ty_n) \to 0$.

Since the identity mapping is certainly strongly nonexpansive, the following result shows that an averaged mapping defined in a uniformly convex Banach space is strongly nonexpansive, [13].

Proposition 1.0.51. *[13] If E is uniformly convex Banach space, $T_0 : D(T_0) \to E$ is strongly nonexpansive, $T_1 : D(T_1) \to E$ is nonexpansive, and $0 < c < 1$, then $S = (1-c)T_0 + cT_1$ is strongly nonexpansive.*

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ belong to $D(S) = D(T_0) \cap D(T_1)$, $(x_n - y_n)_{n\in\mathbb{N}}$ bounded and $\|x_n - y_n\| - \|Sx_n - Sy_n\| \to 0$. Since

$$0 \le (1-c)(\|x_n-y_n\|-\|T_0x_n-T_0y_n\|)+c(\|x_n-y_n\|-\|T_1x_n-T_1y_n\|) \le \|x_n-y_n\|-\|Sx_n-Sy_n\| \to 0,$$

it follows $\|x_n - y_n\| - \|T_ix_n - T_iy_n\| \to 0$ for $i = 0, 1$. We assume that $\|x_n - y_n\| \to d$. Then $\|T_1x_n - T_1y_n\| \to d$ and $\|Sx_n - Sy_n\| \to d$. Since T_0 is strongly nonexpansive, we get $x_n - y_n - (T_0x_n - T_0y_n) \to 0$. Therefore,

$$\|(1-c)(x_n - y_n) + c(T_1x_n - T_1y_n)\| \to d.$$

The uniform convexity of E implies that $x_n - y_n - (T_1x_n - T_1y_n) \to 0$. Finally, $x_n - y_n - (Sx_n - Sy_n) \to 0$. □

In the following Proposition we present the connection between a firmly nonexpansive mapping and a strongly nonexpansive mapping.

Proposition 1.0.52. *[13] If E is uniformly convex Banach space, then every firmly nonexpansive mapping is strongly nonexpansive.*

Remark 1.0.53.

Proposition 1.0.51 and Proposition 1.0.52 are false without the uniform convexity assumption.

In the next Proposition we give some properties related to composition of strongly nonexpansive mappings shown in [13].

Proposition 1.0.54.
- *The class of strongly nonexpansive mappings is closed under composition. Precisely, if $S : D(S) \to E$ and $T : D(T) \to E$ are strongly nonexpansive, then ST is also strongly nonexpansive.*
- *If $\{T_i : 1 \le i \le k\}$ are strongly nonexpansive and $\cap\{Fix(T_i) : 1 \le i \le k\} \ne \emptyset$, then $\cap\{Fix(T_i) : 1 \le i \le k\} = Fix(T_kT_{k-1}...T_1)$.*

Saejung [55] in 2010 introduced the concept of strong quasi-nonexpansivity in the setting of Hilbert spaces:

Definition 1.0.55. Let C be a closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is said strongly quasi-nonexpansive if $Fix(T) \neq \emptyset$, T is quasi-nonexpansive and $x_n - Tx_n \to 0$ whenever (x_n) is a bounded sequence such that $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in Fix(T)$.

Proposition 1.0.56. *In a Hilbert space, every quasi-firmly type nonexpansive mapping $T : C \to C$ with $Fix(T) \neq \emptyset$ is a strongly quasi-nonexpansive mapping.*

Proof. We recall that $T : C \to C$ with $Fix(T) \neq \emptyset$ is said to be quasi-firmly type nonexpansive mapping [59] if, for all $x \in C, p \in Fix(T)$, there exists $k \in (0, \infty)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 - k\|x - Tx\|^2. \tag{1.35}$$

Clearly T is quasi-nonexpansive mapping. Suppose that $(x_n)_{n\in\mathbb{N}}$ is bounded and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in Fix(T)$. From (1.35) it follows that

$$k\|Tx_n - x_n\|^2 \leq \|x_n - p\|^2 - \|Tx_n - p\|^2 \to 0.$$

□

Following the same idea in [13], we get:

Proposition 1.0.57. *Let $T : C \to H$ be a quasi-nonexpansive mapping, where C is a closed convex subset of H. Then the averaged type mapping $T_\delta = (1 - \delta)I + \delta T$ is a strongly quasi-nonexpansive mapping.*

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in C such that $\|x_n - p\| - \|T_\delta x_n - p\| \to 0$, as $n \to \infty$, for some $p \in Fix(T)$. We recall that if T ia a quasi-nonexpansive mapping, then T_δ is a quasi-firmly type nonexpansive mapping (see Proposition 1.0.21, (3)). Hence, we get

$$\|T_\delta x_n - p\|^2 \leq \|x_n - p\|^2 - (1 - \delta)\|x_n - T_\delta x_n\|^2 \tag{1.36}$$

which implies

$$(1 - \delta)\|x_n - T_\delta x_n\|^2 \leq \|x_n - p\|^2 - \|T_\delta x_n - p\|^2.$$

Then, it follows that

$$\|x_n - T_\delta x_n\| \to 0.$$

□

There is a strongly quasi-nonexpansive mapping T such that T is not of the form $(1-\delta)I+\delta T$ where $\delta \in (0, \frac{1}{2})$ and T is a quasinonexpansive mapping. This means that the class of strongly quasi-nonexpansive maggings is wider than the averaged type mappings.

Example 1.0.58. *Let $A = \{(x,x) : \quad x \in \mathbb{R}\}$. It is clear that A is a closed and convex subset of $\mathbb{R}^2$. Notice that $S = P_A$ is a strongly quasi-nonexpansive mapping and $(0,0) \in Fix(S)$. Suppose that $S = (1-\delta)I + \delta T$ where $\delta \in (0, \frac{1}{2})$ and T is a quasi-nonexpansive mapping. Then, by Lemma 1.0.21, we have*

$$\langle (2,0) - S(2,0), (2,0) - (0,0) \rangle \geq \frac{\delta}{2} \|(2,0) - T(2,0)\|^2 = \frac{1}{2\delta} \|(2,0) - S(2,0)\|^2.$$

From $S(2,0) = (1,1)$, we have

$$2 = \langle (2,0) - (1,1), (2,0) \rangle \geq \frac{1}{2\delta} \|(2,0) - (1,1)\|^2 = \frac{1}{\delta}.$$

Thus $\delta \geq \frac{1}{2}$, which is a contradiction.

By the quasi-nonexpansivity of $L-$hybrid mappings, it follows that the assumption that S is a strongly quasi-nonexpansive mapping instead of averaged type nonspreading (or also averaged type $L-$hybrid) mapping, is a weaker assumption.

Of course, one can ask if some important $L-$hybrid mappings, as nonspreading mappings, or nonexpansive mappings, are already strongly quasi-nonexpansive. This is not always true.

Proposition 1.0.59. *[25] There exist nonexpansive mappings that are not strongly quasi-nonexpansive. Moreover, there exist nonspreading mappings that are not strongly quasi-nonexpansive.*

Proof. Let $T : H \to H$ defined by $Tx = -x$. Then T is nonexpansive but not strongly quasi-nonexpansive.

Moreover, let $X = A \cup B \cup C \subset H$, where

$$A = \{x \in H : \|x\| \leq 1\}; B = \{x \in H : 1 < \|x\| < 2\}; C = \{x \in H : 2 \leq \|x\| \leq 3\}.$$

Let $S : X \to X$ defined by

$$Sx := \begin{cases} x, & \text{if } x \in A, \\ \dfrac{x}{\|x\|}, & \text{if } x \in B, \\ 0, & \text{if } x \in C. \end{cases} \tag{1.37}$$

It easy to check that S is a nonspreading mapping, distinguishing three cases ($x \in A$, $y \in B$), ($x \in A$, $y \in C$), ($x \in B$, $y \in C$). However, S is not strongly quasi-nonexpansive, in fact, if take x_0 with $\|x_0\| = 1$, then $x_0 \in Fix(S)$. Moreover, we define $z_n = \left(2 + \frac{1}{n}\right) x_0$. Then $Sz_n = 0$ and $\|z_n - x_0\| - \|Sz_n - x_0\| = 1 + \frac{1}{n} - 1 \to 0$ but $Sz_n - z_n = (2 + \frac{1}{n})x_0 \to 2x_0$. □

Conversely, we have the following proposition.

Proposition 1.0.60. *[25] There exist strongly quasi-nonexpansive mappings that are not L-hybrid mappings for any L (and hence, there are averaged type mappings S_δ with S not $L-$hybrid).*

Proof. Let $H = \mathbb{R}$. We consider

$$\begin{aligned} T(0) &= 0, \\ T(n) &= -n + \tfrac{1}{n+1}, \quad \forall n \in \mathbb{N}, n > 0, \\ T(-n) &= n - \tfrac{1}{n+1}, \quad \forall n \in \mathbb{N}, n > 0. \end{aligned}$$

Then, we define in linear way T on each interval $[n, n+1]$, $n \in \mathbb{Z}$.
One can see easily that $Fix(T) = 0$ and T is strongly quasi-nonexpansive.
Moreover, from the fact that for large $n \in \mathbb{Z}$, T is defined *almost* as $-I$, one can prove that can not be $L-$hybrid for any $L \geq 0$. □

Chapter 2

Halpern's method for nonexpansive mappings and nonspreading mappings

In this Chapter, we will discuss the convergence of the following algorithm, introduced in [17], defined as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n S_\delta x_n + (1-\beta_n) T_\delta x_n], \end{cases} \tag{2.1}$$

where $T : C \to C$ is a nonexpansive mapping, $S : C \to C$ is a nonspreading mapping, $T_\delta, S_\delta : C \to C$ are the rispectively averaged type mappings, $(\beta_n)_{n\in\mathbb{N}}$ denotes a real sequence in $[0,1]$ and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence in $(0,1)$. The starting point will be Iemoto and Takahashi's Theorem to weakly approximate common fixed points of T a nonexpansive mapping and of S a nonspreading mapping. Their iterative scheme is defined as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n S x_n + (1-\beta_n) T x_n], \end{cases}$$

2.1 Introduction

Halpern [28], in 1967, was the first who considered the following explicit method in the framework of Hilbert spaces:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n, \quad \forall n \geq 1 \tag{2.2}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ and $u \in C$ is a fixed anchor.
Halpern proved in [28] the following Theorem on the convergence of (2.2) for a particular choice of $(\alpha_n)_{n\in\mathbb{N}}$.

Theorem 2.1.1. *Let C be a bounded, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. For any initialization $x_1 \in C$ and anchor $u \in C$,* *define a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* C *by*

$$x_{n+1} = n^{-\theta}u + (1-n^{-\theta})Tx_n, \quad \forall n \geq 1,$$

where $\theta \in (0,1)$. *Then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to the element of* $Fix(T)$ *nearest to* u.

He also point out that the control conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

are necessary for the convergence of (2.2) to a fixed point of T.
Many mathematicians have investigated and extended Halpern's iteration to certain Banach spaces and they replaced the sequence $(n^{-\theta})_{n\in\mathbb{N}}$, $\theta \in (0,1)$, with the sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ satisfying some conditions.
Next, we will discuss the question if the control conditions $(C1)$ and $(C2)$ are sufficient or not for the convergence of (2.2) to a fixed point of T. Lions [41] investigated the problem and established a strong convergence theorem under the control conditions (C1), (C2) and (C3): $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_{n+1}^2} = 0$. In 1980, Reich [53] showed that the result of Halpern remains true when E is uniformly smooth. However, both Lions 'and Reich's control conditions excluded the natural choice of $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$. In 1992, Wittmann proved strong convergence of the Halpern iteration ([67], Theorem 2) under the control conditions (C1), (C2) and (C4): $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ in a Hilbert space. The main advantage of Wittmann's convergence theorem lies in the fact that it allows $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$ to be a candidate. In 1997, Shioji and Takahashi [58] also extended Wittmann's result to a real Banach space with a uniformly Gâteaux differentiable norm. In 2002, Xu [69] improved the result of Lion twofold. First, he weakened the condition (C3) by removing the square in the denominator,that is, he replaced condition (C3) with (C5): $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 1$, so that the canonical choice of $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$ is possible. Secondly, he showed the strong convergence of the scheme in the framework of real uniformly smooth Banach spaces.

In the setting of Banach spaces, Song and Chai [59], under the same conditions $(C1)$ and $(C2)$, presented a strong convergence result for the Halpern's iteration (2.2), where T is a firmly type nonexpansive mapping defined on a Banach space with a uniformly Gâteaux differentiable norm. Firmly type nonexpansive mappings could be looked upon as an important subclass of nonexpansive mappings. Let T be a mapping with domain $D(T)$. T is said to be firmly type nonexpansive [59] if for all $x, y \in D(T)$, there exists $k \in (0, +\infty)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - k\|(x-Tx)-(y-Ty)\|^2. \tag{2.3}$$

Saejung [55] proved the strong convergence of the Halpern's iterations (2.2) for strongly nonexpansive mappings in a Banach space E such that one of the following conditions is satisfied:

- E is uniformly smooth;
- E is reflexive, strictly convex with a uniformly Gateâux differentiable norm.

We recall that every firmly type nonexpansive mapping is a strongly nonexpansive. C.E. Chidume and C.O. Chidume [16] and Suzuki [61], independently, obtained that the conditions $(C1)$ and $(C2)$ are sufficient to assure strong convergence to a fixed point of T, introducing the concept of auxiliary mappings. Precisely, they studied the convergence of an iterative scheme, defined as in (2.2), where the mapping T is replaced by the averaged type mapping T_δ, i.e. with the mapping:

$$T_\delta = (1-\delta)I + \delta T, \quad \delta \in (0,1). \tag{2.4}$$

On the other hand, Osilike and Isiogugu [51] studied the Halpern's method type related to the averaged of a $k-$strictly pseudononspreading mapping. We recall that $T : C \to C$ is said to be $k-$strictly pseudononspreading, if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in D(T),$$

which are a more general class of the nonspreading mappings.
Moudafi considered the following iterative procedure for two nonexpansive mappings T_1, T_2 of C into itself:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n T_1 x_n + (1-\beta_n)T_2 x_n], \quad \forall n \in \mathbb{N} \end{cases}$$

$T_1, T_2 : C \to C$ are nonexpansive mappings, $Fix(T_1) \neq \emptyset$, $Fix(T_2) \neq \emptyset$, $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are sequences in $(0,1)$. The Iemoto and Takahashi's Theorem [31] will be the starting point for the Main Theorem in the next Section. The following iterative scheme weakly approximates common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space using Moudafi's iterative scheme [47].

Theorem 2.1.2. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Assume that $Fix(S) \cap Fix(T) \neq \emptyset$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:*

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n S x_n + (1-\beta_n)T x_n], \end{cases} \tag{2.5}$$

for all $n \in \mathbb{N}$, where $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} \subset [0,1]$. Then, the following hold:

(i) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $p \in Fix(S)$;

(ii) If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $p \in Fix(T)$;

(iii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $p \in Fix(S) \cap Fix(T)$.

2.2 Main Theorem

To overcome the weak convergence of the sequence (x_n) defined by the iterative process (2.5), in [17] we modified the process in a Halpern's type method which converges to different fixed points, under certain appropriate conditions imposed on control coefficients $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$. Specifically, our attention will be focused on the following Halpern's type algorithm for a nonexpansive mapping T and a nonspreading mapping S:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n], \quad n \in \mathbb{N}. \end{cases} \tag{2.6}$$

We will discuss the computation of fixed points of averaged type mappings with interesting properties. Indeed, the auxiliary mappings explored combined with the Haplern's type method can be regarded as a regularizing process which induces the convergence in norm of the sequence $(x_n)_{n\in\mathbb{N}}$ to fixed points of mappings. Another advantage of this method is that allows to select a particular fixed point of involved mapping which satisfies a variational inequality. Motivated by these facts, we propose an analysis of the previous approximation method in the framework of Hilbert spaces.

We denote by $O(1)$ is any bounded real sequence. Before starting the main result in [17], we establish the boundedness of $(x_n)_{n\in\mathbb{N}}$.

Lemma 2.2.1. *[17] Let $T : C \to C$ be a nonexpansive mapping, $S : C \to C$ be a nonspreading mapping, $T_\delta, S_\delta : C \to C$ the rispectively averaged type mappings such that $Fix(S)\cap Fix(T) \neq \emptyset$. Moreover, $(\beta_n)_{n\in\mathbb{N}}$ denotes a real sequence in $[0,1]$ and $U_n : C \to C$ denotes the convex combination of T_δ and S_δ, i.e.*

$$U_n = \beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n, \quad n \in \mathbb{N}.$$

Further, we assume that $(\alpha_n)_{n\in\mathbb{N}}$ a sequence in $(0,1)$ a real sequence such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = 0$. Let $u \in C$ be an anchor and let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) U_n x_n.$$

Then,

1. *U_n is quasi-nonexpansive for all $n \in \mathbb{N}$.*
2. *$(x_n)_{n\in\mathbb{N}}, (Sx_n)_{n\in\mathbb{N}}, (Tx_n)_{n\in\mathbb{N}}, (S_\delta x_n)_{n\in\mathbb{N}}, (T_\delta x_n)_{n\in\mathbb{N}}, (U_n x_n)_{n\in\mathbb{N}}$ are bounded sequences.*

Proof. (1) Any convex combination of quasi-nonexpansive mappings is quasi-nonexpansive too. So is every U_n, since T_δ and S_δ are quasi-nonexpansive.
(2) The boundedness of $(x_n)_{n\in\mathbb{N}}$ follows by the fact that U_n is quasi-nonexpansive. In fact, let $q \in Fix(T) \cap Fix(S)$. Then

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n(u-q) + (1-\alpha_n)(U_n x_n - q)\| \\ &\leq \alpha_n\|u-q\| + (1-\alpha_n)\|U_n x_n - q\| \\ &\leq \alpha_n\|u-q\| + (1-\alpha_n)\|x_n - q\| \end{aligned} \tag{2.7}$$

Since

$$\|x_1 - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\},$$

and by induction we assume that

$$\|x_n - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\},$$

then

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|u - q\| + (1 - \alpha_n) \max\{\|u - q\|, \|x_1 - q\|\} \\ &\leq \alpha_n \max\{\|u - q\|, \|x_1 - q\|\} + (1 - \alpha_n) \max\{\|u - q\|, \|x_1 - q\|\} \\ &= \max\{\|u - q\|, \|x_1 - q\|\}. \end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is bounded. The boundedness of the other sequences follows by boundedness of $(x_n)_{n\in\mathbb{N}}$ and by the quasi-nonexpansivity of involved mappings. □

The next Lemma will be a pertinent tool in the proof of the Main Theorem.

Lemma 2.2.2. *Let C a nonempty closed and convex subspace of H, $u \in C$ fixed, T a nonexpansive mapping from C into itself and S a nonspreading mapping from C into itself such that $Fix(T) \cap Fix(S) \neq \emptyset$. Consider a bounded sequence $(y_n)_{n\in\mathbb{N}} \subset C$. Then:*

(1) If $\|y_n - Ty_n\| \to 0$, as $n \to \infty$, then

$$\limsup_{n\to\infty} \langle u - \bar{p}, y_n - \bar{p} \rangle \leq 0,$$

where $\bar{p} = P_{Fix(T)}u$ is the unique point in $Fix(T)$ that satisfies the variational inequality

$$\langle u - \bar{p}, x - \bar{p} \rangle \leq 0, \quad \forall x \in Fix(T). \tag{2.8}$$

(2) If $\|y_n - Sy_n\| \to 0$, as $n \to \infty$, then

$$\limsup_{n\to\infty} \langle u - \hat{p}, y_n - \hat{p} \rangle \leq 0,$$

where $\hat{p} = P_{Fix(S)}u$ is the unique point in $Fix(S)$ that satisfies the variational inequality

$$\langle u - \hat{p}, x - \hat{p} \rangle \leq 0, \quad \forall x \in Fix(S). \tag{2.9}$$

(3) If $\|y_n - Sy_n\| \to 0$ and $\|y_n - Ty_n\| \to 0$, as $n \to \infty$, then

$$\limsup_{n\to\infty} \langle u - p_0, y_n - p_0 \rangle \leq 0,$$

where $p_0 = P_{Fix(T)\cap Fix(S)}u$ is the unique point in $Fix(T) \cap Fix(S)$ that satisfies the variational inequality

$$\langle u - p_0, x - p_0 \rangle \leq 0, \quad \forall x \in Fix(T) \cap Fix(S). \tag{2.10}$$

Proof. (1) Let $\bar{p}$ satisfying (3.6). Let $(y_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(y_n)_{n\in\mathbb{N}}$ for wich

$$\limsup_{n\to\infty} \langle \bar{p}-u, y_n-\bar{p}\rangle = \lim_{k\to\infty} \langle \bar{p}-u, y_{n_k}-\bar{p}\rangle .$$

Select a subsequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_{k_j}} \rightharpoonup v$ (this is possible by boundedness of $(y_n)_{n\in\mathbb{N}}$). By the hypothesis $\|y_n - Ty_n\| \to 0$, as $n\to\infty$, and by demiclosedness of $I-T$ at 0 we have $v \in Fix(T)$, and

$$\limsup_{n\to\infty} \langle \bar{p}-u, y_n-\bar{p}\rangle = \lim_{j\to\infty} \left\langle \bar{p}-u, y_{n_{k_j}}-\bar{p}\right\rangle = \langle \bar{p}-u, v-\bar{p}\rangle ,$$

so the claim follows by (2.8).
(2) Is the same of (1) since S is also demiclosed at 0.
(3) Select a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \langle p_0-u, y_n-p_0\rangle = \lim_{k\to\infty} \langle p_0-u, y_{n_k}-p_0\rangle ,$$

where p_0 satisfies (2.10). Now select a subsequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_{k_j}} \rightharpoonup w$. Then by demiclosedness of $I-T$ and $I-S$ at 0, and by the hypotheses $\|y_n - Ty_n\| \to 0$ and $\|y_n - Sy_n\| \to 0$, as $n\to\infty$, we obtain that $w = Tw = Sw$, i.e. $w \in Fix(T)\cap Fix(S)$. So,

$$\limsup_{n\to\infty} \langle p_0-u, y_n-p_0\rangle = \lim_{j\to\infty} \left\langle p_0-u, y_{n_{k_j}}-p_0\right\rangle = \langle p_0-u, w-p_0\rangle ,$$

so the claim follows by (2.10). □

Next, we are in position to claim the main convergence Result.

Theorem 2.2.3. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a nonspreading mapping such that $Fix(S)\cap Fix(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings. Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ is a real sequence in $(0,1)$ satisfying the conditions:*

C1) $\lim_{n\to\infty} \alpha_n = 0,$

C2) $\sum_{n=1}^{\infty} \alpha_n = \infty.$

If $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ and $(x_n)_{n\in\mathbb{N}}$ defined as in (2.6). Then, the following hold:

(i) If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $\overline{p} = P_{Fix(T)}u$ which is the unique solution in $Fix(T)$ of the variational inequality $\langle u-\overline{p}, x-\overline{p}\rangle \le 0$, for all $x \in Fix(T)$.

(ii) If $\sum_{n=1}^{\infty}\beta_n < \infty$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $\widehat{p} = P_{Fix(S)}u$ which is the unique solution in $Fix(S)$ of the variational inequality $\langle u-\widehat{p}, x-\widehat{p}\rangle \le 0$, for all $x \in Fix(S)$.

(iii) If $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$*, then* $(x_n)_{n\in\mathbb{N}}$ *strongly converges to* $p_0 = P_{Fix(T)\cap Fix(S)}u$ *which is the unique solution in* $Fix(T)\cap Fix(S)$ *of the variational inequality* $\langle u - p_0, x - p_0\rangle \le 0$*, for all* $x \in Fix(T)\cap Fix(S)$*.*

Proof. of (i)
We rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\delta x_n + (1-\beta_n)E_n, \tag{2.11}$$

where $(E_n)_{n\in\mathbb{N}}$ is a sequence defined by $E_n = (1-\alpha_n)(S_\delta x_n - T_\delta x_n)$. From Lemma 2.2.1 (E_n) is a bounded sequence and hence for every positive integer n, $\|E_n\| \le O(1)$.
We begin to prove that $\lim_{n\to\infty} \|x_n - T_\delta x_n\| = 0$.
Let $\overline{p} \in Fix(T) = Fix(T_\delta)$ the unique solution in $Fix(T)$ of the variational inequality

$$\langle u - \overline{p}, x - \overline{p}\rangle \le 0, \quad \forall x \in Fix(T).$$

We have

$$\begin{array}{rcl}
\|x_{n+1} - \overline{p}\|^2 & = & \|\alpha_n u + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta T x_n + (1-\beta_n)E_n - \overline{p}\|^2 \\
& = & \|[(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}] + [\alpha_n(u - x_n) + (1-\beta_n)E_n]\|^2 \\
\text{(by Lemma 1.0.1)} & \le & \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
& + & 2\langle \alpha_n(u - x_n) + (1-\beta_n)E_n, x_{n+1} - \overline{p}\rangle \\
& \le & \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
& + & 2\alpha_n\langle u - x_n, x_{n+1} - \overline{p}\rangle + 2(1-\beta_n)\langle E_n, x_{n+1} - \overline{p}\rangle \\
& \le & (1-\alpha_n)^2\delta^2\|Tx_n - x_n\|^2 + \|x_n - \overline{p}\|^2 \\
& - & 2(1-\alpha_n)\delta\langle x_n - \overline{p}, x_n - Tx_n\rangle \\
& + & 2\alpha_n\|u - x_n\|\|x_{n+1} - \overline{p}\| + 2(1-\beta_n)\|E_n\|\|x_{n+1} - \overline{p}\| \\
& = & (1-\alpha_n)^2\delta^2\|x_n - Tx_n\|^2 + \|x_n - \overline{p}\|^2 \\
((I-T)\overline{p} = 0) & - & 2(1-\alpha_n)\delta\langle x_n - \overline{p}, (I-T)x_n - (I-T)\overline{p}\rangle \\
& + & \alpha_n O(1) + (1-\beta_n)O(1) \\
\text{(by Lemma 1.0.26)} & \le & \|x_n - \overline{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Tx_n\|^2 \\
& - & (1-\alpha_n)\delta\|(I-T)x_n - (I-T)\overline{p}\|^2 + \alpha_n O(1) + (1-\beta_n)O(1) \\
& = & \|x_n - \overline{p}\|^2 - (1-\alpha_n)\delta[1 - \delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \\
& + & \alpha_n O(1) + (1-\beta_n)O(1)
\end{array}$$

and hence

$$\begin{array}{rcl}
0 \le (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 & \le & \|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 + \\
& + & \alpha_n O(1) + (1-\beta_n)O(1).
\end{array} \tag{2.12}$$

We turn our attention on the sequence $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$.
The rest of the proof will be divided into two parts.

Case A. $\|x_{n+1} - \overline{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \overline{p}\| < \|x_{n_k+1} - \overline{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing. In this situation, $\lim_{n\to\infty} \|x_n - \overline{p}\|^2$ exists. From (3.10), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2\Big) \\ &\leq \limsup_{n\to\infty}\Big(\|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 \\ &+ \alpha_n O(1) + (1-\beta_n)O(1)\Big) = 0, \end{aligned}$$

then, we can conclude that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0, \tag{2.13}$$

and

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{2.14}$$

By Lemma 2.2.2, it follows thus

$$\limsup_{n\to\infty}\langle u - \overline{p}, x_n - \overline{p}\rangle \leq 0. \tag{2.15}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p}$.
We compute

$$\begin{aligned} \|x_{n+1} - \overline{p}\|^2 &= \|\alpha_n(u-\overline{p}) + (1-\alpha_n)(T_\delta x_n - \overline{p}) + (1-\beta_n)E_n\|^2 \\ \text{(by Lem. 1.0.1)} &\leq \|\alpha_n(u-\overline{p}) + (1-\alpha_n)(T_\delta x_n - \overline{p})\|^2 \\ &+ 2(1-\beta_n)\langle E_n, x_{n+1} - \overline{p}\rangle \\ &\leq \alpha_n^2\|u-\overline{p}\|^2 + (1-\alpha_n)^2\|T_\delta x_n - \overline{p}\|^2 \\ &+ 2\alpha_n(1-\alpha_n)\langle u-\overline{p}, T_\delta x_n - \overline{p}\rangle + (1-\beta_n)O(1) \\ (T_\delta \text{ nonexp.}) &\leq (1-\alpha_n)\|x_n - \overline{p}\|^2 + \alpha_n^2 O(1) \\ &+ 2\alpha_n(1-\alpha_n)\langle u-\overline{p}, T_\delta x_n - x_n\rangle \\ &+ 2\alpha_n(1-\alpha_n)\langle u-\overline{p}, x_n - \overline{p}\rangle + (1-\beta_n)O(1) \\ &\leq (1-\alpha_n)\|x_n - \overline{p}\|^2 + \alpha_n^2 O(1) + \alpha_n O(1)\|T_\delta x_n - x_n\| \\ &+ 2\alpha_n(1-\alpha_n)\langle u-\overline{p}, x_n - \overline{p}\rangle + (1-\beta_n)O(1) \end{aligned}$$

If we put $\sigma_n = \alpha_n O(1) + O(1)\|T_\delta x_n - x_n\| + 2(1-\alpha_n)\langle u - \overline{p}, x_n - \overline{p}\rangle$ and $\theta_n = (1-\beta_n)O(1)$, we have

$$\|x_{n+1} - \overline{p}\|^2 \leq (1-\alpha_n)\|x_n - \overline{p}\|^2 + \alpha_n\sigma_n + \theta_n.$$

Hence, from assumption $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, from (3.11) and $\limsup_{n\to\infty}\langle u - \overline{p}, x_n - \overline{p}\rangle \leq 0$ we can apply Xu's Lemma 1.0.10.

Case B. Suppose that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \overline{p}\| < \|x_{n_k+1} - \overline{p}\| \text{ for all } k \in \mathbb{N}.$$

In this situation, we consider the sequence of indices $(\tau(n))_{n\in\mathbb{N}}$ as defined in Maingé Lemma 1.0.11. It follows that there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(a) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;

(b) $\lim_{n\to\infty} \tau(n) = \infty$;

(c) $\|x_{\tau(n)} - \overline{p}\| < \|x_{\tau(n)+1} - \overline{p}\|$;

(d) $\|x_n - \overline{p}\| < \|x_{\tau(n)+1} - \overline{p}\|$.

Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\|\Big) \\
&\leq \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\|\Big) \\
&\leq \limsup_{n\to\infty}\Big(\|x_{n+1} - \overline{p}\| - \|x_n - \overline{p}\|\Big) \\
\text{(by (3.9))} \quad &= \limsup_{n\to\infty}\Big(\|\alpha_n(u - T_\delta x_n) + T_\delta x_n - \overline{p} \\
&+ (1-\beta_n)E_n\| - \|x_n - \overline{p}\|\Big) \\
(T_\delta \text{ nonexpansive}) \quad &\leq \limsup_{n\to\infty}\Big(\alpha_n O(1) + \|x_n - \overline{p}\| + (1-\beta_n)O(1) - \|x_n - \overline{p}\|\Big) = 0,
\end{aligned}$$

hence

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\|\Big) = 0. \tag{2.16}$$

By (3.10), we have

$$\begin{aligned}
0 &\leq (1-\alpha_{\tau(n)})\delta[1 - \delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\
&\leq \|x_{\tau(n)} - \overline{p}\|^2 - \|x_{\tau(n)+1} - \overline{p}\|^2 + \alpha_n O(1) + (1-\beta_{\tau(n)})O(1),
\end{aligned}$$

from (3.13), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$ we get

$$\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0. \tag{2.17}$$

By Lemma 2.2.2 and (3.14) we have

$$\limsup_{n\to\infty} \langle u - \overline{p}, x_{\tau(n)} - \overline{p}\rangle \leq 0. \tag{2.18}$$

Finally, we show that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p}$.
As in the Case $A.$, we can obtain

$$\lim_{n\to\infty} \|x_{\tau(n)} - \overline{p}\| = 0;$$

then, from property (d) of Maingé Lemma 1.0.11 and (3.13) we can conclude

$$\lim_{n\to\infty} \|x_n - \overline{p}\| = 0.$$

□

Proof. of (ii)
Now, we rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n u + (1-\alpha_n)S_\delta x_n + \beta_n E_n, \tag{2.19}$$

where (E_n) is a sequence defined by $E_n = (1-\alpha_n)(T_\delta x_n - S_\delta x_n)$. From Lemma 2.2.1 (E_n) is a bounded sequence and hence for every positive integer n, $\|E_n\| \leq O(E_n)$.
From Lemma **??** (E_n) is a bounded sequence and hence for every positive integer n, $\|E_n\| \leq O(E_n)$.
We begin to prove that $\lim_{n\to\infty} \|x_n - S_\delta x_n\| = 0$.

Let $\widehat{p} \in Fix(S) = Fix(S_\delta)$ the unique solution in $Fix(S)$ of the variational inequality $\langle u - \widehat{p}, x - \widehat{p}\rangle \leq 0$, for all $x \in Fix(S)$. We compute

$$\begin{aligned}
\|x_{n+1} - \widehat{p}\|^2 &= \|\alpha_n u + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta S x_n + \beta_n E_n - \widehat{p}\|^2 \\
&= \|[(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}] + [\alpha_n(u - x_n) + \beta_n E_n]\|^2 \\
\text{(by Lemma 1.0.1)} \quad &\leq \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
&+ 2\langle \alpha_n(u - x_n) + \beta_n E_n, x_{n+1} - \widehat{p}\rangle \\
&\leq \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
&+ 2\alpha_n\langle u - x_n, x_{n+1} - \widehat{p}\rangle + 2\beta_n\langle E_n, x_n - \widehat{p}\rangle \\
&\leq (1-\alpha_n)^2\delta^2\|Sx_n - x_n\|^2 + \|x_n - \widehat{p}\|^2 \\
&- 2(1-\alpha_n)\delta\langle x_n - \widehat{p}, x_n - Sx_n\rangle
\end{aligned}$$

$$
\begin{array}{rcl}
& + & \alpha_n O(1) + \beta_n O(1) \\
& = & (1-\alpha_n)^2\delta^2\|x_n - Sx_n\|^2 + \|x_n - \widehat{p}\|^2 \\
((I-S)\widehat{p} = 0) & - & 2(1-\alpha_n)\delta\langle x_n - \widehat{p}, (I-S)x_n - (I-S)\widehat{p}\rangle \\
& + & \alpha_n O(1) + \beta_n O(1) \\
(\text{ by Lemma 1.0.39 }) & \leq & \|x_n - \widehat{p}\|^2 \\
& + & (1-\alpha_n)^2\delta^2\|x_n - Sx_n\|^2 \\
& - & 2(1-\alpha_n)\delta\Big[\|(I-S)x_n - (I-S)\widehat{p}\|^2 \\
& - & \frac{1}{2}\Big(\|x_n - Sx_n\|^2 + \|p - Sp\|^2\Big)\Big] \\
& + & \alpha_n O(1) + \beta_n O(1) \\
& \leq & \|x_n - \widehat{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Sx_n\|^2 \\
& - & (1-\alpha_n)\delta\|x_n - Sx_n\|^2 + \alpha_n O(1) + \beta_n O(1) \\
& = & \|x_n - \widehat{p}\|^2 - (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 \\
& + & \alpha_n O(1) + \beta_n O(1)
\end{array}
$$

and hence

$$
\begin{array}{rcl}
0 \leq (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 & \leq & \|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 + \\
& + & \alpha_n O(1) + \beta_n O(1).
\end{array} \tag{2.20}
$$

Again, we turn our attention on the sequence $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$. We consider the following two cases.

Case A. $\|x_{n+1} - \widehat{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$
\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.
$$

Case A. Since $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - \widehat{p}\|^2$ exists. From (3.10), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, we have

$$
\begin{array}{rcl}
0 & \leq & \limsup_{n\to\infty}\Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2\Big) \\
& \leq & \limsup_{n\to\infty}\Big(\|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 \\
& + & \alpha_n O(1) + \beta_n O(1)\Big) = 0,
\end{array}
$$

hence

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{2.21}$$

By Lemma 2.2.2, it follows thus

$$\limsup_{n\to\infty} \langle u - \widehat{p}, x_n - \widehat{p}\rangle \leq 0.$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\widehat{p}$.

We compute

$$\begin{array}{rcl} \|x_{n+1} - \widehat{p}\|^2 & = & \|\alpha_n(u - \widehat{p}) + (1-\alpha_n)(S_\delta x_n - \widehat{p}) + \beta_n E_n\|^2 \\ \text{(by Lemma 1.0.1)} & \leq & \|\alpha_n(u - \widehat{p}) + (1-\alpha_n)(S_\delta x_n - \widehat{p})\|^2 \\ & + & 2\beta_n\langle E_n, x_{n+1} - \widehat{p}\rangle \\ & \leq & \alpha_n^2\|u - \widehat{p}\|^2 + (1-\alpha_n)^2\|S_\delta x_n - \widehat{p}\|^2 \\ & + & 2\alpha_n(1-\alpha_n)\langle u - \widehat{p}, S_\delta x_n - \widehat{p}\rangle + \beta_n O(1) \\ (S_\delta \text{ quasi-nonexp.}) & \leq & (1-\alpha_n)\|x_n - \widehat{p}\|^2 + \alpha_n^2 O(1) \\ & + & 2\alpha_n(1-\alpha_n)\langle u - \widehat{p}, S_\delta x_n - x_n\rangle \\ & + & 2\alpha_n(1-\alpha_n)\langle u - \widehat{p}, x_n - \widehat{p}\rangle + \beta_n O(1) \\ & \leq & (1-\alpha_n)\|x_n - \widehat{p}\|^2 + \alpha_n^2 O(1) + \alpha_n O(1)\|S_\delta x_n - x_n\| \\ & + & 2\alpha_n(1-\alpha_n)\langle u - \widehat{p}, x_n - \widehat{p}\rangle + \beta_n O(1) \end{array}$$

If we put $\sigma_n = \alpha_n O(1) + O(1)\|S_\delta x_n - x_n\| + 2(1-\alpha_n)\langle u - \widehat{p}, x_n - \widehat{p}\rangle$ and $\theta_n = \beta_n O(1)$, we have

$$\|x_{n+1} - \widehat{p}\|^2 \leq (1-\alpha_n)\|x_n - \widehat{p}\|^2 + \alpha_n\sigma_n + \theta_n.$$

So, from assumption $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, from (3.17) and $\limsup_{n\to\infty}\langle \widehat{p} - u, x_n - \widehat{p}\rangle \geq 0$ we can apply Xu's Lemma 1.0.10.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.$$

Then by Maingé Lemma there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(a) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;

(b) $\lim_{n\to\infty} \tau(n) = \infty$;

(c) $\|x_{\tau(n)} - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$;

(d) $\|x_n - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$.

Consequently,

$$\begin{array}{rcl} 0 & \leq & \liminf\limits_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big) \\ & \leq & \limsup\limits_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big) \\ & \leq & \limsup\limits_{n\to\infty}\Big(\|x_{n+1}-\widehat{p}\|-\|x_n-\widehat{p}\|\Big) \\ \text{(by (3.15))} & = & \limsup\limits_{n\to\infty}\Big(\|\alpha_n(u-S_\delta x_n)+S_\delta x_n-\widehat{p}+\beta_n E_n\|-\|x_n-\widehat{p}\|\Big) \\ (S_\delta \text{ quasi-nonexpansive}) & \leq & \limsup\limits_{n\to\infty}\Big(\alpha_n O(1)+\|x_n-\widehat{p}\| \\ & + & \beta_n O(1)-\|x_n-\widehat{p}\|\Big)=0, \end{array}$$

hence

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big)=0. \tag{2.22}$$

By (3.16), we obtain

$$\begin{array}{rcl} 0 & \leq & (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)}-Sx_{\tau(n)}\|^2 \\ & \leq & \|x_{\tau(n)}-\widehat{p}\|^2-\|x_{\tau(n)+1}-\widehat{p}\|^2+\alpha_n O(1)+\beta_{\tau(n)}O(1), \end{array}$$

from (3.19), $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum\limits_{n=1}^{\infty}\beta_n<\infty$ we get

$$\lim_{n\to\infty}\|x_{\tau(n)}-Sx_{\tau(n)}\|=0. \tag{2.23}$$

By Lemma 2.2.2 and (3.20) we get

$$\limsup_{n\to\infty}\langle u-\widehat{p},x_{\tau(n)-\widehat{p}}\rangle\leq 0.$$

Finally, we show that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p}$.

As in the Case A., we obtain

$$\lim_{n\to\infty}\|x_{\tau(n)}-\widehat{p}\|=0,$$

then, from property (d) of Maingé Lemma and (3.19) we can conclude

$$\lim_{n\to\infty}\|x_n-\widehat{p}\|=0.$$

□

Proof. *of (iii)* We recall that the sequence $(x_{n+1})_{n\in\mathbb{N}}$ is defined as

$$x_{n+1} = \alpha_n u + (1-\alpha_n)U_n x_n, \tag{2.24}$$

where $U_n = \beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n$.
We first show that $\lim_{n\to\infty} \|x_n - U_n x_n\| = 0$.

Let $p_0 \in Fix(T) \cap Fix(S)$ is the unique solution of the variational inequality $\langle u-p_0, x-p_0\rangle \leq 0$, for all $x \in Fix(T) \cap Fix(S)$. We compute

$$\begin{aligned}
\|U_n x_n - p_0\|^2 &= \|\beta_n(T_\delta x_n - p_0) + (1-\beta_n)(S_\delta x_n - p_0)\|^2 \\
\text{(by Lemma 1.0.1)} &= \beta_n\|T_\delta x_n - p_0\|^2 + (1-\beta_n)\|S_\delta x_n - p_0\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\
\text{(T_δ nonexp. and by (1.28))} &\leq \beta_n\|x_n - p_0\|^2 + (1-\beta_n)\|x_n - p_0\|^2 \\
&- (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\
&= \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2.
\end{aligned}$$

So, we get

$$\|U_n x_n - p_0\|^2 \leq \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 - \beta_n(1-\beta_n)\|T_\delta x_n - S - \delta x_n\|^2. \tag{2.25}$$

We have

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &= \|U_n x_n - p_0 + \alpha_n(u - U_n x_n)\|^2 \\
&\leq \|U_n x_n - p_0\|^2 + \alpha_n(\alpha_n\|u - U_n x_n\|^2 + 2\|U_n x_n - p_0\|\|u - U_n x_n\|) \\
&\leq \|U_n x_n - p_0\|^2 + \alpha_n O(1) \\
\text{(by (3.22))} &\leq \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 + \alpha_n O(1),
\end{aligned} \tag{2.26}$$

From (3.23), we derive

$$(1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \tag{2.27}$$

and

$$\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \tag{2.28}$$

Next, also we consider two cases.

Case A. $\|x_{n+1} - \widehat{p_0}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p_0}\| < \|x_{n_k+1} - \widehat{p_0}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - p_0\|^2$ exists. From (3.24), $\lim_{n\to\infty} \alpha_n = 0$ and since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ we conclude

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{2.29}$$

Furthermore, from (3.25) we have

$$\lim_{n\to\infty} \|S_\delta x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|Sx_n - Tx_n\| = 0; \tag{2.30}$$

since

$$\|x_n - Tx_n\| \leq \|x_n - Sx_n\| + \|Sx_n - Tx_n\|,$$

by (3.26) and (3.27) we obtain

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{2.31}$$

Then, since

$$\|U_n x_n - x_n\| \leq \beta_n\|T_\delta x_n - x_n\| + (1-\beta_n)\|S_\delta x_n - x_n\|,$$

by (3.26) and (3.28) we get

$$\lim_{n\to\infty} \|x_n - U_n x_n\| = 0. \tag{2.32}$$

By Lemma 2.2.2, we have

$$\limsup_{n\to\infty} \langle u - p_0, x_n - p_0\rangle \leq 0.$$

Finally, $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

We compute

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &\leq \|(1-\alpha_n)(U_n x_n - p_0) + \alpha_n(u - p_0)\| \\
&= (1-\alpha_n)^2\|U_n x_n - p_0\|^2 + \alpha_n^2\|u - p_0\|^2 \\
&+ 2\alpha_n(1-\alpha_n)\langle U_n x_n - p_0, u - p_0\rangle \\
(U_n \text{ quasi-nonexp.}) &\leq (1-\alpha_n)^2\|x_n - p_0\|^2 + \alpha_n^2 O(1) \\
&+ 2\alpha_n(1-\alpha_n)\langle U_n x_n - x_n, u - p_0\rangle \\
&+ 2\alpha_n(1-\alpha_n)\langle x_n - p_0, u - p_0\rangle \\
&\leq (1-\alpha_n)\|x_n - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1)\|U_n x_n - x_n\| \\
&+ 2\alpha_n(1-\alpha_n)\langle x_n - p_0, u - p_0\rangle
\end{aligned}$$

If we set $\sigma_n = \alpha_n O(1) + O(1)\|U_n x_n - x_n\| + 2(1-\alpha_n)\langle x_n - p_0, u - p_0\rangle$ we have

$$\|x_{n+1} - p_0\|^2 \leq (1-\alpha_n)\|x_n - p_0\|^2 + \alpha_n\sigma_n.$$

From $\sum_{n=1}^{\infty} \alpha_n = \infty$, (2.32) and $\limsup_{n\to\infty}\langle u - p_0, x_n - p_0\rangle \leq 0$ we conclude that $(x_n)_{n\in\mathbb{N}}$ strongly converges to p_0.

Case B. $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ does not be definitively nonincreasing. This means that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

In this situation, we consider the sequence of indices $(\tau(n))_{n\in\mathbb{N}}$ as defined Maingé Lemma 1.0.11. It yields that there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies some properties defined previous.
Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\Big)\\
&\leq \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\Big)\\
&\leq \limsup_{n\to\infty}\Big(\|x_{n+1} - p_0\| - \|x_n - p_0\|\Big)\\
\text{(by (3.21))} \quad &= \limsup_{n\to\infty}\Big(\|\alpha_n(u - p_0) + (1-\alpha_n)(U_n x_n - p_0)\| - \|x_n - p_0\|\Big)\\
(U_n \text{ quasi-nonexp.}) \quad &\leq \limsup_{n\to\infty}\Big(\alpha_n O(1) + \|x_n - p_0\| - \|x_n - p_0\|\Big) = 0,
\end{aligned}$$

hence

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\Big) = 0. \tag{2.33}$$

By (3.24)we obtain

$$(1-\beta_{\tau(n)})(1-\delta)\|x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)}O(1), \tag{2.34}$$

and by(3.25) we have

$$\beta_{\tau(n)}(1-\beta_{\tau(n)})\|T_\delta x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)}O(1), \tag{2.35}$$

As in the Case A., we get

(a) $\lim_{n\to\infty}\|x_{\tau(n)} - Sx_{\tau(n)}\| = 0,$

(b) $\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0,$

(c) $\lim_{n\to\infty}\|x_{\tau(n)} - U_{\tau(n)}x_{\tau(n)}\| = 0.$

By Lemma 2.2.2, (a) and (b) we have

$$\limsup_{n\to\infty}\langle u - p_0, x_{\tau(n)} - p_0\rangle \leq 0. \tag{2.36}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

As in the Case A., using (c), the assumption $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (3.34) we can apply Xu's Lemma 1.0.10 and conclude

$$\lim_{n\to\infty} \|x_{\tau(n)} - \widehat{p}\| = 0,$$

then, from property (d) of Maingé Lemma and (3.31) we can derive

$$\lim_{n\to\infty} \|x_n - p_0\| = 0.$$

□

Remark 2.2.4. Finally, some conclusions:

(A) The introduction of the auxiliary mappings T_δ, S_δ in the main Theorem allows us to obtain strong convergence results for Halpern's type for nonspreading mappings and nonexpansive mappings. Hence, in [17] we resolved in the affirmative way the open problem raised by Kurokawa and Takahashi in their final remark in [40], in the case that S_δ is the averaged type mapping of a nonspreading mapping and $\beta_n = 0$.

(B) Observe that in the proof of (i) we used:

- some properties related to the mapping $T : C \to C$:
  - the quasi-nonexpansivity of T;
  - the demiclosedness Principle;
  - $I - T : C \to H$ is $\frac{1}{2}$-inverse strongly monotone, i.e., for all $x, y \in C$,

$$\frac{1}{2}\|(I-T)x - (I-T)y\|^2 \le \langle x - y, (I-T)x - (I-T)y\rangle;$$

- the quasi-nonexpansivity of S.

In the proof of (ii) we needed

- some properties connected to the mapping S:
  - the quasi-nonexpansivity of S;
  - the demiclosedness Principle;
  - $I - S : C \to C$ satisfies, for all $x, y \in C$,

$$\|(I-T)x-(I-T)y\|^2 \le \langle x-y, (I-T)x-(I-T)y\rangle + \frac{1}{2}\Big(\|x-Tx\|^2 + \|y-Ty\|^2\Big);$$

- the quasi-nonexpansivity of T.

In the last case (iii) we used the properties of both mappings.

(C) In order to prove

$$\|x_n - Tx_n\| \to 0,\ \|x_n - Sx_n\| \to 0,\ \|x_n - U_n x_n\| \to 0,$$

Maingé 's Lemma was a considerable tool to solve the gap in [59]. Since every firmly type nonexpansive mapping is strongly nonexpansive, it follows by S. Saejung's Theorem [55] that Song and Chai's result remains true.

Chapter 3

On strong convergence of viscosity type method using averaged type mappings

We will propose the following viscosity type algorithm, introduced in [18],

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n(I - \mu_n D)x_n + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n], \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $T_\delta = (1-\delta)I + \delta T$, $S_\delta = (1-\delta)I + \delta S$, $\delta \in (0,1)$ are the averaged type mappings of T a nonexpansive mapping and of S a L-hybrid mapping, D is a β-strongly monotone and a ρ-Lipschitzian operator.
The starting point for the Main Theorem of this Chapter will be the Main Result of Chapter 2.

3.1 Introduction

We will discuss the special aspect of the iterative scheme (3.1). Motivated by the fact that $B_n := I - \mu_n D$ is a contraction for all $n \in \mathbb{N}$, we call the iterative scheme (3.1) viscosity type method. An appropriate setting for the operator D or for the control sequences makes the algorithm (3.1) close to well known iterative method. In fact,

- if $\mu_n = \mu$ and $D = \frac{I-u}{\mu}$, where u is the constant contraction, we obtain the algorithm proposed in [17];
- if we consider $\mu_n = \mu$ and $D = \frac{I-f}{\mu}$, where f is a contraction, we have a viscosity method

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n]. \tag{3.2}$$

  For $\beta_n = 1$ this scheme (3.2) was proposed by A. Moudafi in [47]. He proved the strong convergence of algorithm (3.2) to the unique solution $x^* \in C$ of the variational inequality

$$\langle (I-f)x^*, x^* - x\rangle \le 0, \quad \forall x \in C,$$

where f is a contraction, in Hilbert spaces.
In [70] Xu extended Moudafi's results in a uniformly smooth Banach space;

- for $D = (I - \gamma f)$, where f is a contraction with coefficient α and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, we have

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n)[\beta_n T_\delta x_n + (1 - \beta_n) S_\delta x_n]$$

  if we set $y_n = \mu_n \gamma f(x_n) + (1 - \mu_n) x_n$.

If $A : H \to H$ is a strongly positive operator, i.e. there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H,$$

and f is a contraction, then $D = A - \gamma f$ is a strongly monotone operator (see [46]). Next, we will discuss connections between approximation of fixed points of nonexpansive mappings, solutions of variational inequalities and minimization problems. In recent years, many significant contributions concern the approximation of solutions of variational inequality are reached, in which variational inequality problems are studied by extragradient methods based on Korpelevich's method, subgradient extragradient methods and other iterative-type methods. Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle,$$

where C is the fixed point set of a nonexpansive mapping T on H, b is a given point in H and A is strongly positive. In [19], [46], [48], the authors consider iterative methods approximating a fixed point of nonexpansive mappings that is also the unique solution of the variational inequality problem

$$\langle (A - \gamma f) x^*, x^* - x \rangle \leq 0, \quad \forall x \in C. \tag{3.3}$$

(3.3) is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf, i.e. $h'(x) = \gamma f(x)$ for $x \in H$.

3.2 Main Theorem

In this Section we propose a viscosity approximation method which generate sequences that strongly converge to particular fixed-point of a given nonexpansive self-mapping or of a given L-hybrid mapping, under suitable assumptions on coefficients.

In the sequel, we denote by $O(1)$ any bounded real sequence (so, for example, $O(1)+O(1) = O(1)$).
Let us to start with the two following Lemmata. The first one ensures the boundedness of (x_n) defined by (3.1).

Lemma 3.2.1. *Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a L-hybrid mapping such that $Fix(S) \cap Fix(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings, i.e.*

$$T_\delta = (1-\delta)I + \delta T, \; S_\delta = (1-\delta)I + \delta S, \, \delta \in (0,1).$$

Let $D : H \to H$ be a β-strongly monotone and ρ-Lipschitzian operator. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0,\mu)$, $\mu < \frac{2\beta}{\rho^2}$, and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$x_{n+1} = \alpha_n(1-\mu_n D)x_n + (1-\alpha_n)U_n x_n,$$

where

$$U_n = \beta_n T_\delta + (1-\beta_n)S_\delta, \quad \forall n \in \mathbb{N}.$$

Then

1. *U_n is quasi-nonexpansive for all $n \in \mathbb{N}$.*
2. *$(x_n)_{n\in\mathbb{N}}, (Sx_n)_{n\in\mathbb{N}}, (Tx_n)_{n\in\mathbb{N}}, (S_\delta x_n)_{n\in\mathbb{N}}, (T_\delta x_n)_{n\in\mathbb{N}}, (U_n x_n)_{n\in\mathbb{N}}$ are bounded sequences.*

Proof. (1)
To simplify the notation, we set

$$B_n := I - \mu_n D \tag{3.4}$$

and $\tau := \frac{(2\beta - \mu\delta^2)}{2}$. It is known that B_n is a contraction [71], that is,

$$\|B_n x - B_n y\| \leq (1-\mu_n\tau)\|x-y\|, \forall x, y \in H.$$

We observe that U_n is quasi-nonexpansive, for all $n \in \mathbb{N}$, since T_δ is a nonexpansive mapping and S_δ is a quasi-nonexpansive mapping.
(2)
First we prove that $(x_n)_{n\in\mathbb{N}}$ is bounded. For $q \in Fix(T) \cap Fix(S)$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(B_n x_n - q) + (1-\alpha_n)(U_n x_n - q)\| \\
&\leq \alpha_n\|B_n x_n - q\| + (1-\alpha_n)\|U_n x_n - q\| \\
(U_n \text{ quasi-nonexpansive}) &\leq \alpha_n\|B_n x_n - q\| + (1-\alpha_n)\|x_n - q\| \\
&\leq \alpha_n\|B_n x_n - B_n q\| + \alpha_n\|B_n q - q\| + (1-\alpha_n)\|x_n - q\| \\
(B_n \text{ contraction}) &\leq \alpha_n(1-\mu_n\tau)\|x_n - q\| + \alpha_n\|B_n q - q\| + (1-\alpha_n)\|x_n - q\| \\
&= (1-\alpha_n\mu_n\tau)\|x_n - q\| + \alpha_n\|B_n q - q\| \\
(\text{by } (3.4)) &= (1-\alpha_n\mu_n\tau)\|x_n - q\| + \alpha_n\mu_n\tau\frac{\|Dq\|}{\tau}
\end{aligned} \tag{3.5}$$

Since

$$\|x_1 - q\| \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\},$$

and by induction we assume that

$$\|x_n - q\| \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\},$$

then

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} + \alpha_n\mu_n\tau\frac{\|Dq\|}{\tau} \\
&\leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} + \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} \\
&= 2\max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\}.
\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is bounded. Consequently, $(T_\delta x_n)_{n\in\mathbb{N}}$, $(S_\delta x_n)_{n\in\mathbb{N}}$, $(U_n x_n)_{n\in\mathbb{N}}$ and $(B_n x_n)_{n\in\mathbb{N}}$ are bounded as well.

□

Lemma 3.2.2. *Let C be a nonempty closed and convex subspace of H, let $D : H \to H$ be a β-strongly monotone and ρ-Lipschitzian operator.*

(i) Let V be a nonlinear mapping from C into itself such that $I - V$ is demiclosed at 0 and $Fix(V) \neq \emptyset$. Consider a bounded sequence $(y_n)_{n\in\mathbb{N}} \subset C$ such that $\|y_n - Vy_n\| \to 0$, as $n \to \infty$, then:

$$\limsup_{n\to\infty} \langle D\bar{p}, y_n - \bar{p}\rangle \leq 0,$$

where $\bar{p}$ is the unique point in $Fix(V)$ that satisfies the variational inequality

$$\langle D\bar{p}, x - \bar{p}\rangle \leq 0, \quad \forall x \in Fix(V). \tag{3.6}$$

(ii) Let V, W be a nonlinear mappings from C into itself such that $I - V$ and $I - W$ are demiclosed at 0 and $Fix(V) \cap Fix(W) \neq \emptyset$. Consider a bounded sequence $(y_n)_{n\in\mathbb{N}} \subset C$ such that $\|y_n - Vy_n\| \to 0$ and $\|y_n - Wy_n\| \to 0$, as $n \to \infty$, then:

$$\limsup_{n\to\infty} \langle Dp_0, y_n - p_0\rangle \leq 0,$$

where p_0 is the unique point in $Fix(V) \cap Fix(W)$ that satisfies the variational inequality

$$\langle Dp_0, x - p_0\rangle \leq 0, \quad \forall x \in Fix(V) \cap Fix(W). \tag{3.7}$$

Proof. (i)
Let $\bar{p}$ satisfying (3.6). Let $(y_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(y_n)_{n\in\mathbb{N}}$ for which

$$\limsup_{n\to\infty} \langle D\bar{p}, y_n - \bar{p}\rangle = \lim_{k\to\infty} \langle D\bar{p}, y_{n_k} - \bar{p}\rangle .$$

Select a subsequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_{k_j}} \rightharpoonup v$ (this is possible by boundedness of $(y_n)_{n\in\mathbb{N}}$). By the hypothesis $\|y_n - Vy_n\| \to 0$, as $n \to \infty$, and by demiclosedness of $I - V$ at 0 we have $v \in Fix(V)$, and

$$\limsup_{n\to\infty} \langle D\bar{p}, y_n - \bar{p}\rangle = \lim_{j\to\infty} \left\langle D\bar{p}, y_{n_{k_j}} - \bar{p}\right\rangle = \langle D\bar{p}, v - \bar{p}\rangle ,$$

so the claim follows by (3.6).
(ii)
Select a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \langle Dp_0, y_n - p_0\rangle = \lim_{k\to\infty} \langle Dp_0, y_{n_k} - p_0\rangle,$$

where p_0 satisfies (3.7). Now select a subsequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_{k_j}} \rightharpoonup w$. Then by demiclosedness of $I-V$ and $I-W$ at 0, and by the hypotheses $\|y_n - Vy_n\| \to 0$ and $\|y_n - Wy_n\| \to 0$, as $n\to\infty$, we obtain that $w = Vw = Ww$, i.e. $w \in Fix(V)\cap Fix(W)$. So,

$$\limsup_{n\to\infty} \langle Dp_0, y_n - p_0\rangle = \lim_{j\to\infty} \left\langle Dp_0, y_{n_{k_j}} - p_0\right\rangle = \langle Dp_0, w - p_0\rangle,$$

so the claim follows by (3.7). □

Next, we will present the main Result in [18].

Theorem 3.2.3. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a L-hybrid mapping such that $Fix(S)\cap Fix(T) \neq \emptyset$. Let $D : H \to H$ be a β-strongly monotone and ρ-Lipschitzian operator. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0,\mu)$, $\mu < \frac{2\beta}{\rho^2}$, and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$, satisfying the conditions:*

1. $\lim_{n\to\infty} \alpha_n = 0,$
2. $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0,$
3. $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty.$

If $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$, we define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n(I-\mu_n D)x_n + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n], \quad n\in\mathbb{N}. \end{cases} \tag{3.8}$$

Then, the following hold:

(i) *If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p} \in Fix(T)$ which is the unique solution in $Fix(T)$ of the variational inequality $\langle D\overline{p}, x - \overline{p}\rangle \le 0$, for all $x \in Fix(T)$.*

(ii) *If $\sum_{n=1}^{\infty}\beta_n < \infty$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\widehat{p} \in Fix(S)$ which is the unique solution in $Fix(S)$ of the variational inequality $\langle D\widehat{p}, x - \widehat{p}\rangle \le 0$, for all $x \in Fix(S)$.*

(iii) *If $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $p_0 \in Fix(T)\cap Fix(S)$ which is the unique solution in $Fix(T)\cap Fix(S)$ of the variational inequality $\langle Dp_0, x - p_0 - \rangle \le 0$, for all $x \in Fix(T)\cap Fix(S)$.*

Proof. Proof. of (i)
We rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n)T_\delta x_n + (1-\beta_n)E_n, \tag{3.9}$$

where $(E_n)_{n\in\mathbb{N}}$ is the sequence defined by $E_n = (1-\alpha_n)(S_\delta x_n - T_\delta x_n)$. From Lemma 3.2.1 (E_n) is a bounded sequence and hence for every positive integer n, $\|E_n\| \le O(1)$.
We begin to prove that $\lim_{n\to\infty} \|x_n - T_\delta x_n\| = 0$.
Let $\overline{p}$ the unique solution in $Fix(T) = Fix(T_\delta)$ of the variational inequality

$$\langle D\overline{p}, x - \overline{p}\rangle \le 0, \quad \forall x \in Fix(T).$$

We have

$$\begin{aligned}
\|x_{n+1} - \overline{p}\|^2 &= \|\alpha_n B_n x_n + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta T x_n + (1-\beta_n)E_n - \overline{p}\|^2 \\
&= \|[(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}] + [\alpha_n(B_n x_n - x_n) + (1-\beta_n)E_n]\|^2 \\
\text{(by Lemma 1.0.1)} \quad &\le \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
&+ 2\langle \alpha_n(B_n x_n - x_n) + (1-\beta_n)E_n, x_{n+1} - \overline{p}\rangle \\
&= \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
&+ 2\alpha_n\langle B_n x_n - x_n, x_{n+1} - \overline{p}\rangle + 2(1-\beta_n)\langle E_n, x_{n+1} - \overline{p}\rangle \\
&\le (1-\alpha_n)^2\delta^2\|Tx_n - x_n\|^2 + \|x_n - \overline{p}\|^2 \\
&- 2(1-\alpha_n)\delta\langle x_n - \overline{p}, x_n - Tx_n\rangle \\
&+ 2\alpha_n\|B_n x_n - x_n\|\|x_{n+1} - \overline{p}\| + 2(1-\beta_n)\|E_n\|\|x_{n+1} - \overline{p}\| \\
\text{(by (1.0.39))} \quad &\le \|x_n - \overline{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Tx_n\|^2 \\
&- (1-\alpha_n)\delta\|x_n - Tx_n\|^2 + \alpha_n O(1) + (1-\beta_n)O(1) \\
&= \|x_n - \overline{p}\|^2 - (1-\alpha_n)\delta[1 - \delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \\
&+ \alpha_n O(1) + (1-\beta_n)O(1)
\end{aligned}$$

and hence

$$\begin{aligned}
0 \le (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 &\le \|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 + \\
&+ \alpha_n O(1) + (1-\beta_n)O(1).
\end{aligned} \tag{3.10}$$

We turn our attention on the monotonicity of the sequence $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$.
Next, the proof will be divided in two cases.

Case A. $\|x_{n+1} - \overline{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \overline{p}\| < \|x_{n_k+1} - \overline{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - \overline{p}\|^2$ exists. From (3.10), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \Big) \\ &\leq \limsup_{n\to\infty} \Big(\|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 \\ &+ \alpha_n O(1) + (1-\beta_n)O(1) \Big) = 0, \end{aligned}$$

so, we can conclude that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{3.11}$$

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and from $I - T$ is demiclosed at 0, we can use Lemma 3.2.2 (i), so we get

$$\limsup_{n\to\infty} \langle D\overline{p}, x_n - \overline{p}\rangle \leq 0. \tag{3.12}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p}$.
We compute

$$\begin{aligned} \|x_{n+1} - \overline{p}\|^2 &\leq \|\alpha_n(B_n x_n - \overline{p}) + (1-\alpha_n)(T_\delta x_n - \overline{p}) + (1-\beta_n)E_n\|^2 \\ \text{(by Lemma 1.0.1)} &\leq \|(1-\alpha_n)(T_\delta x_n - \overline{p})\|^2 \\ &+ 2\langle (1-\beta_n)E_n + \alpha_n(B_n x_n - \overline{p}), x_{n+1} - \overline{p}\rangle \\ &\leq (1-\alpha_n)^2\|T_\delta x_n - \overline{p}\|^2 + 2\alpha_n\langle B_n x_n - \overline{p}, x_{n+1} - \overline{p}\rangle \\ &+ (1-\beta_n)O(1) \\ (T_\delta \text{ nonexpansive}) &\leq (1-\alpha_n)^2\|x_n - \overline{p}\|^2 + 2\alpha_n\langle B_n x_n - B_n\overline{p}, x_{n+1} - \overline{p}\rangle \\ &+ 2\alpha_n\langle B_n\overline{p} - \overline{p}, x_{n+1} - \overline{p}\rangle + (1-\beta_n)O(1) \\ (B_n \text{ contraction}) &\leq (1-\alpha_n)^2\|x_n - \overline{p}\|^2 \\ &+ 2\alpha_n(1-\mu_n\tau)\|x_n - \overline{p}\|\|x_{n+1} - \overline{p}\| \\ &+ 2\alpha_n\langle B_n\overline{p} - \overline{p}, x_{n+1} - \overline{p}\rangle + (1-\beta_n)O(1) \\ (B_n := (1-\mu_n D)) &\leq (1-\alpha_n)^2\|x_n - \overline{p}\|^2 \\ &+ \alpha_n(1-\mu_n\tau)\Big[\|x_n - \overline{p}\|^2 + \|x_{n+1} - \overline{p}\|^2\Big] \\ &- 2\alpha_n\mu_n\langle D\overline{p}, x_{n+1} - \overline{p}\rangle + (1-\beta_n)O(1), \end{aligned}$$

Then it follows that

$$\|x_{n+1} - \overline{p}\|^2 \leq \frac{1-(1+\mu_n\tau)\alpha_n + \alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}\|x_n - \overline{p}\|^2$$

$$
\begin{aligned}
& - \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle D\overline{p}, x_{n+1}-\overline{p}\rangle \\
& + \frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1) \\
& \leq \frac{1-(1+\mu_n\tau)\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\|x_n-\overline{p}\|^2 \\
& - \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle D\overline{p}, x_{n+1}-\overline{p}\rangle \\
& + \frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1) + \frac{\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}O(1) \\
& \leq \left(1-\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\right)\|x_n-\overline{p}\|^2 \\
& + \frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\left[-\frac{1}{\tau}\langle D\overline{p}, x_{n+1}-\overline{p}\rangle + \frac{\alpha_n}{2\mu_n\tau}O(1)\right] \\
& + \frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1)
\end{aligned}
$$

Notice that by

$$\lim_{n\to\infty}\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}=0,$$

it follows that

$$0<\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}<1, \quad definitively.$$

Moreover, using $\sum_{n=1}^{\infty}\alpha_n\mu_n=\infty$, $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, (3.12) and $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n}=0$, we can apply Lemma 1.0.10 and conclude that

$$\lim_{n\to\infty}\|x_{n+1}-\overline{p}\|=0.$$

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k}-\overline{p}\|<\|x_{n_k+1}-\overline{p}\| \text{ for all } k\in\mathbb{N}.$$

Considering $(\tau(n))_{n\in\mathbb{N}}$ defined as in Maingé Lemma 1.0.11, it immediately follows that there exists a sequence of integers that it satisfies

(a) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;

(b) $\lim_{n\to\infty}\tau(n)=\infty$;

(c) $\|x_{\tau(n)}-\overline{p}\|<\|x_{\tau(n)+1}-\overline{p}\|$;

(d) $\|x_n-\overline{p}\|<\|x_{\tau(n)+1}-\overline{p}\|$.

Consequently,

$$\begin{array}{rcl} 0 & \leq & \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1}-\overline{p}\|-\|x_{\tau(n)}-\overline{p}\|\Big) \\ & \leq & \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1}-\overline{p}\|-\|x_{\tau(n)}-\overline{p}\|\Big) \\ & \leq & \limsup_{n\to\infty}\Big(\|x_{n+1}-\overline{p}\|-\|x_{n}-\overline{p}\|\Big) \\ \text{(by (3.9))} & = & \limsup_{n\to\infty}\Big(\|\alpha_n(B_nx_n-T_\delta x_n)+T_\delta x_n-\overline{p}+(1-\beta_n)E_n\|-\|x_n-\overline{p}\|\Big) \\ (T_\delta \text{ nonexpansive}) & \leq & \limsup_{n\to\infty}\Big(\alpha_nO(1)+\|x_n-\overline{p}\|+(1-\beta_n)O(1)-\|x_n-\overline{p}\|\Big)=0, \end{array}$$

so

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1}-\overline{p}\|-\|x_{\tau(n)}-\overline{p}\|\Big)=0. \tag{3.13}$$

By (3.10), we have

$$\begin{array}{rcl} 0 & \leq & (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)}-Tx_{\tau(n)}\|^2 \\ & \leq & \|x_{\tau(n)}-\overline{p}\|^2-\|x_{\tau(n)+1}-\overline{p}\|^2+\alpha_nO(1)+(1-\beta_{\tau(n)})O(1), \end{array}$$

from (3.13), $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$ we get

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0. \tag{3.14}$$

By (3.14), as in the Case A, we have

$$\limsup_{n\to\infty}\langle D\overline{p},\overline{p}-x_{\tau(n)}\rangle\leq 0$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)}-\overline{p}\|=0;$$

then, in the light of property (d) of Maingé Lemma 1.0.11 and (3.13) we conclude that

$$\lim_{n\to\infty}\|x_n-\overline{p}\|=0.$$

$\square$

Proof. of (ii)
Now, we rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1}=\alpha_nB_nx_n+(1-\alpha_n)S_\delta x_n+\beta_nE_n, \tag{3.15}$$

where $(E_n)_{n\in\mathbb{N}}$ is the sequence defined by $E_n = (1-\alpha_n)(T_\delta x_n - S_\delta x_n)$. From Lemma 3.2.1 (E_n) is a bounded sequence and hence for every positive integer n, $\|E_n\| \leq O(1)$.

We begin to prove that $\lim_{n\to\infty} \|x_n - S_\delta x_n\| = 0$.

Let $\widehat{p}$ the unique solution in $Fix(S) = Fix(S_\delta)$ of the variational inequality $\langle D\widehat{p}, x-\widehat{p}\rangle \leq 0$, for all $x \in Fix(S)$. We have

$$\begin{array}{rcl}
\|x_{n+1}-\widehat{p}\|^2 & = & \|\alpha_n B_n x_n + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta S x_n + \beta_n E_n - \widehat{p}\|^2 \\
& = & \|[(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}] + [\alpha_n(B_n x_n - x_n) + \beta_n E_n]\|^2 \\
\text{(by Lemma 1.0.1)} & \leq & \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
& + & 2\langle \alpha_n(B_n x_n - x_n) + \beta_n E_n, x_{n+1} - \widehat{p}\rangle \\
& \leq & \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
& + & 2\alpha_n\langle B_n x_n - x_n, x_{n+1} - \widehat{p}\rangle + 2\beta_n\langle E_n, x_n - \widehat{p}\rangle \\
& \leq & (1-\alpha_n)^2\delta^2\|Sx_n - x_n\|^2 + \|x_n - \widehat{p}\|^2 \\
& - & 2(1-\alpha_n)\delta\langle x_n - \widehat{p}, x_n - Sx_n\rangle \\
& + & \alpha_n O(1) + \beta_n O(1) \\
\text{(by (1.34))} & \leq & \|x_n - \widehat{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Sx_n\|^2 \\
& - & (1-\alpha_n)\delta\|x_n - Sx_n\|^2 + \alpha_n O(1) + \beta_n O(1) \\
& = & \|x_n - \widehat{p}\|^2 - (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 \\
& + & \alpha_n O(1) + \beta_n O(1)
\end{array}$$

and hence

$$\begin{array}{rcl}
0 \leq (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 & \leq & \|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 + \\
& + & \alpha_n O(1) + \beta_n O(1).
\end{array} \tag{3.16}$$

To prove that $\|x_n - Sx_n\| \to 0$, we consider the following two cases regarding the monotonicity of the sequence $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$.

Case A. $\|x_{n+1} - \widehat{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Assume that $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, that is, $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ $\|x_{n+1} - \widehat{p}\| \leq \|x_n - \widehat{p}\|$. It follows then $\lim_{n\to\infty} \|x_n - \widehat{p}\|^2$ exists. From (3.10),

$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2\Big) \\ &\leq \limsup_{n\to\infty}\Big(\|x_n-\widehat{p}\|^2 - \|x_{n+1}-\widehat{p}\|^2 \\ &+ \alpha_n O(1) + \beta_n O(1)\Big) = 0, \end{aligned}$$

hence

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{3.17}$$

Since $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and from $I-S$ is demiclosed at 0, we can use Lemma 3.2.2 (i) and we have

$$\limsup_{n\to\infty} \langle D\widehat{p}, x_n - \widehat{p}\rangle \leq 0. \tag{3.18}$$

Finally, we can prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\widehat{p}$ as in the proof (i).

So, using $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, (3.18) and $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0$, we can apply Lemma 1.0.10 and conclude that

$$\lim_{n\to\infty} \|x_{n+1} - \widehat{p}\| = 0.$$

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k\in\mathbb{N}.$$

In this case, it follows from Maingé Lemma that there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(a) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;

(b) $\lim_{n\to\infty} \tau(n) = \infty$;

(c) $\|x_{\tau(n)} - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$;

(d) $\|x_n - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$.

Consequently,

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\| - \|x_{\tau(n)}-\widehat{p}\|\Big) \\ &\leq \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\| - \|x_{\tau(n)}-\widehat{p}\|\Big) \end{aligned}$$

$$
\begin{array}{rcl}
 & \leq & \limsup_{n\to\infty}\Big(\|x_{n+1}-\widehat{p}\|-\|x_n-\widehat{p}\|\Big) \\
\text{(by (3.15))} & = & \limsup_{n\to\infty}\Big(\|\alpha_n(B_nx_n-S_\delta x_n)+S_\delta x_n-\widehat{p}+\beta_nE_n\|-\|x_n-\widehat{p}\|\Big) \\
(S_\delta\text{ quasi-nonexpansive}) & \leq & \limsup_{n\to\infty}\Big(\alpha_nO(1)+\|x_n-\widehat{p}\|+\beta_nO(1)-\|x_n-\widehat{p}\|\Big)=0,
\end{array}
$$

hence

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1}-\widehat{p}\|-\|x_{\tau(n)}-\widehat{p}\|\Big)=0. \tag{3.19}$$

By (3.16), we obtain

$$
\begin{array}{rcl}
0 & \leq & (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)}-Sx_{\tau(n)}\|^2 \\
 & \leq & \|x_{\tau(n)}-\widehat{p}\|^2-\|x_{\tau(n)+1}-\widehat{p}\|^2+\alpha_nO(1)+\beta_{\tau(n)}O(1),
\end{array}
$$

from (3.19), $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\beta_n<\infty$ we get

$$\lim_{n\to\infty}\|x_{\tau(n)}-Sx_{\tau(n)}\|=0. \tag{3.20}$$

By(3.20), as in the Case A, we get

$$\limsup_{n\to\infty}\langle D\widehat{p},x_{\tau(n)}-\widehat{p}x_{\tau(n)}\rangle\leq 0,$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)}-\widehat{p}\|=0,$$

then, from property (d) of Maingé Lemma and (3.19) it follows that

$$\lim_{n\to\infty}\|x_n-\widehat{p}\|=0.$$

□

Proof. *of (iii)* We recall that the sequence $(x_{n+1})_{n\in\mathbb{N}}$ is defined as

$$x_{n+1}=\alpha_nB_nx_n+(1-\alpha_n)U_nx_n, \tag{3.21}$$

where $U_n=\beta_nT_\delta x_n+(1-\beta_n)S_\delta x_n$ for all $n\in\mathbb{N}$.
We first show that $\lim_{n\to\infty}\|x_n-T_\delta x_n\|=0$ and $\lim_{n\to\infty}\|x_n-S_\delta x_n\|=0$.

Let $p_0\in Fix(T)\cap Fix(S)$ is the unique solution of the variational inequality $\langle Dp_0,x-p_0\rangle\leq 0$, for all $x\in Fix(T)\cap Fix(S)$. We compute

$$
\begin{array}{rcl}
\|U_nx_n-p_0\|^2 & = & \|\beta_n(T_\delta x_n-p_0)+(1-\beta_n)(S_\delta x_n-p_0)\|^2 \\
\text{(by Lemma 1.0.1)} & = & \beta_n\|T_\delta x_n-p_0\|^2+(1-\beta_n)\|S_\delta x_n-p_0\|^2
\end{array}
$$

$$\begin{aligned}
& - && \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\
(T_\delta \text{ nonexp. and by Prop. 1.0.21, (3)}) & \leq && \beta_n\|x_n - p_0\|^2 + (1-\beta_n)\|x_n - p_0\|^2 \\
& - && (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
& - && \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\
& = && \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
& - && \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2.
\end{aligned}$$

So, we get

$$\|U_n x_n - p_0\|^2 \leq \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 - \beta_n(1-\beta_n)\|T_\delta x_n - S - \delta x_n\|^2. \quad (3.22)$$

We have

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 & = && \|U_n x_n - p_0 + \alpha_n(B_n x_n - U_n x_n)\|^2 \\
& \leq && \|U_n x_n - p_0\|^2 + \alpha_n(\alpha_n\|B_n x_n - U_n x_n\|^2 + 2\|U_n x_n - p_0\|\|B_n x_n - U_n x_n\|) \\
& \leq && \|U_n x_n - p_0\|^2 + \alpha_n O(1) \\
(\text{by } (3.22)) & \leq && \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
& - && \beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 + \alpha_n O(1), \quad (3.23)
\end{aligned}$$

From (3.23), we derive

$$(1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \quad (3.24)$$

and

$$\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \quad (3.25)$$

Now, also we consider two cases.

Case A. $\|x_{n+1} - p_0\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

Case A. Suppose that $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ is definitively nonincreasing. It implies that $\lim_{n\to\infty} \|x_n - p_0\|^2$ exists. From (3.24), $\lim_{n\to\infty} \alpha_n = 0$ and since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ we conclude

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \quad (3.26)$$

Furthermore, from (3.25) we have

$$\lim_{n\to\infty} \|S_\delta x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|Sx_n - Tx_n\| = 0; \quad (3.27)$$

since

$$\|x_n - Tx_n\| \leq \|x_n - Sx_n\| + \|Sx_n - Tx_n\|,$$

by (3.26) and (3.27) we obtain

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{3.28}$$

By (3.28) and (3.26) and by the demiclosedness of $I - T$ at 0 and of $I - S$ at 0, we can conclude using Lemma 3.2.2 (ii)

$$\limsup_{n\to\infty} \langle Dp_0, x_n - p_0\rangle \leq 0. \tag{3.29}$$

Finally, $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

We compute

$$\begin{array}{rcl}
\|x_{n+1} - p_0\|^2 & = & \|\alpha_n(B_n x_n - p_0) + (1-\alpha_n)(U_n x_n - p_0)\|^2 \\
\text{(by Lemma 1.0.1)} & \leq & (1-\alpha_n)^2\|U_n x_n - p_0\|^2 \\
& + & 2\alpha_n\langle B_n x_n - p_0, x_{n+1} - p_0\rangle \\
& = & (1-\alpha_n)^2\|U_n x_n - p_0\|^2 \\
& + & 2\alpha_n\langle B_n x_n - B_n p_0, x_{n+1} - p_0\rangle + 2\alpha_n\langle B_n p_0 - p_0, x_{n+1} - p_0\rangle \\
(U_n \text{ quasi-nonexpansive}) & \leq & (1-\alpha_n)^2\|x_n - p_0\|^2 \\
(B_n \text{ contraction}) & + & 2\alpha_n(1-\mu_n\tau)\|x_n - p_0\|\|x_{n+1} - p_0\| \\
& + & 2\alpha_n\langle B_n p_0 - p_0, x_{n+1} - p_0\rangle \\
(B_n := (1-\mu_n D)) & \leq & (1-\alpha_n)^2\|x_n - p_0\|^2 + \alpha_n(1-\mu_n\tau)(\|x_n - p_0\|^2 + \|x_{n+1} - p_0\|^2) \\
& - & 2\alpha_n\mu_n\langle Dp_0, x_{n+1} - p_0\rangle
\end{array}$$

Then, it follows that

$$\begin{array}{rcl}
\|x_{n+1} - p_0\|^2 & \leq & \dfrac{1-(1+\mu_n\tau)\alpha_n + \alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}\|x_n - p_0\|^2 \\
& - & \dfrac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle Dp_0, x_{n+1} - p_0\rangle \\
& \leq & \dfrac{1-(1+\mu_n\tau)\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\|x_n - p_0\|^2 \\
& + & \dfrac{\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}O(1) - \dfrac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle Dp_0, x_{n+1} - p_0\rangle \\
& \leq & \left(1 - \dfrac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\right)\|x_n - p_0\|^2 \\
& + & \dfrac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\left[-\dfrac{1}{\tau}\langle Dp_0, x_{n+1} - p_0\rangle + \dfrac{\alpha_n}{2\mu_n\tau}O(1)\right].
\end{array} \tag{3.30}$$

Using $\sum_{n=1}^{\infty} \alpha_n \mu_n = \infty$, (3.29) and $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0$, we can apply Lemma 1.0.10 and conclude that

$$\lim_{n\to\infty} \|x_{n+1} - p_0\| = 0.$$

So, $(x_n)_{n\in\mathbb{N}}$ converges strongly to a fixed point of $Fix(T) \cap Fix(S)$.

Case B. $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ does not be definitively nonincreasing. This means that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

Then by Maingé Lemma 1.0.11 there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies some properties defined previous.
Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty} \left(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\| \right) \\
&\leq \limsup_{n\to\infty} \left(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\| \right) \\
&\leq \limsup_{n\to\infty} \left(\|x_{n+1} - p_0\| - \|x_n - p_0\| \right) \\
\text{(by (3.21))} \quad &= \limsup_{n\to\infty} \left(\|\alpha_n(B_n x_n - p_0) + (1-\alpha_n)(U_n x_n - p_0)\| - \|x_n - p_0\| \right) \\
(U_n \text{ quasi-nonexpansive}) \quad &\leq \limsup_{n\to\infty} \left(\alpha_n O(1) + \|x_n - p_0\| - \|x_n - p_0\| \right) = 0,
\end{aligned}$$

hence

$$\lim_{n\to\infty} \left(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\| \right) = 0. \tag{3.31}$$

By (3.24) we get

$$(1-\beta_{\tau(n)})(1-\delta)\|x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)} O(1), \tag{3.32}$$

and by(3.25) we have

$$\beta_{\tau(n)}(1-\beta_{\tau(n)})\|T_\delta x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)} O(1), \tag{3.33}$$

As in the Case A., we get

(a) $\lim_{n\to\infty} \|x_{\tau(n)} - S x_{\tau(n)}\| = 0$,

(b) $\lim_{n\to\infty} \|x_{\tau(n)} - T x_{\tau(n)}\| = 0$.

By (a) and (b), as in the Case A, we have

$$\limsup_{n\to\infty}\langle Dp_0, p_0 - x_{\tau(n)}\rangle \le 0. \tag{3.34}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

As in the Case A., using $\sum_{n=1}^{\infty}\alpha_n\mu_n = \infty$, $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n} = 0$, and (3.34) we can apply Xu's Lemma 1.0.10 and we yield that

$$\lim_{n\to\infty}\|x_{\tau(n)} - \widehat{p}\| = 0,$$

then, from property (d) of Maingé Lemma and (3.31) we can derive

$$\lim_{n\to\infty}\|x_n - p_0\| = 0.$$

□

□

We summarize the motivations, the aim and the novelty of the Theorem 3.2.3.

Remark 3.2.4.
- We show that there exist some examples of sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ satisfying assumption in Theorem 3.2.3. Let

$$\alpha_n = \frac{1}{n^{\frac{2}{3}}}, \quad \mu_n = \frac{\beta}{\rho^2 n^{\frac{1}{3}}}, \quad \forall n\in\mathbb{N},$$

then it is easy to check that they satisfy the conditions:

1. $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$,
2. $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0,\mu)$, $\mu < \frac{2\beta}{\rho^2}$,
3. $\lim_{n\to\infty}\alpha_n = 0$,
4. $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n} = 0$,
5. $\sum_{n=1}^{\infty}\alpha_n\mu_n = \infty$.

- We can not apply Suzuki's result, showed in [60], to our Theorem 2.2.3 in Chapter 2. We recall that Suzuki's result states that Halpern's type convergence theorems imply Moudafi's viscosity approximations. On the other hand, our goal in [18] is to replace the fixed anchor u in(2.6) with $B_n := I - \mu_n D$, that is a contraction for all $n\in\mathbb{N}$, and to combine some tools used in the proof of main Theorem in [17] with regularizing properties of auxiliary mappings T_δ, S_δ. In light of the above facts, we also obtain an unified approach to these two different problems: approximation of fixed points for nonlinear mappings and approximation of solutions of variational inequalities.

- We remark that (i) and (ii) of Theorem 3.2.3 actually hold for a wide class of nonlinear mappings. In fact, in (i) it is possible to substitute a L-hybrid mapping S with a quasi-nonexpansive mapping because in the first proof we used only that fact S is quasi-nonexpansive mapping. For the same reason, in (ii) it is possible to replace a nonexpansive mapping T with a quasi-nonexpansive mapping. In the last case (iii) the properties of both mappings are required.

- In (i) and (ii), some properties related to $I - T$ and $I - S$ are fundamental: the demiclosedness Principle and the inequalities (1.0.39) and (1.34). In the last case (iii), it is useful the inequality (3) in Proposition 1.0.21.

- Observe that if T is quasi-nonexpansive, then $Fix(T)$ is closed convex [33]; hence the variational inequality problem , for all $x \in C$,

$$\langle Dp, x - p \rangle \geq 0, \tag{3.35}$$

  admits a unique solution $p \in Fix(T)$ if D is β-strongly monotone and a ρ-Lipschitzian operator. Notice that (3.35) represents quite a general formulation of variational inequalities. Depending on the choice of controll coefficients (α_n) and (β_n), we showed the strong convergence of the iterative method (3.8) to the unique solution of a variational inequality (3.35) on the set of common fixed points of T and S or on the fixed points of one of them. Consequently, for these reasons, it was assumed that the operator $D : H \to H$ β-strongly monotone and a ρ-Lipschitzian operator.

Chapter 4

On strong convergence of Halpern's method for quasi-nonexpansive mappings in Hilbert Spaces

In this Chapter, we will present two iterative schemes introduced in [25]. The first algorithm strongly approximates common fixed points of a nonexpansive mapping $T : C \to C$ and a strongly quasi-nonexpansive mappings $S : C \to C$,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T x_n + (1-\beta_n) S x_n], \quad n \geq 1. \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequences in $(0,1)$, $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ and u a fixed anchor in C. The second result involves the averaged type mappings of two quasi-nonexpansive mappings $T, S : C \to C$,

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n]. \end{cases}$$

Both results show as the same algorithm converges to different points, depending on the assumptions of the coefficients. In both Theorems we will show the deep connections between the approximation of fixed points for nonlinear mappings and the approximation of solutions of variational inequalities. Moreover, a numerical example of the iterative scheme is given.

4.1 Main result

Concerning the Halpern's method to approximate a fixed point of $T : C \to C$

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n, \quad \forall n \geq 1, \tag{4.1}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is sequence in $[0,1]$ and $u \in C$ fixed; it is interesting to know for which classes of mappings a strong convergence Theorem of Halpern's type holds or not. Saejung [55] provided that the strongly quasi-nonexpansive mappings together just conditions $(C1)$ and $(C2)$ are sufficient for the strong convergence of Halpern's iteration to a fixed point of T. Moreover, he applied his result for finding zeros of an accretive operator. In the literature, iterative Halpern's schemes for two mappings have been widely studied and extended. In this direction, in [17] we obtained a strong convergence result for a Halpern's type method using the averaged type mappings S_δ, T_δ,

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n]. \end{cases} \tag{4.2}$$

to approximate common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space.

The aim of this Section is to discuss an significant improvement of main Theorem in [17] without using averaged type mappings and to establish whether a strong convergence Theorem for Halpern's type method which involves a nonexpansive mapping and a strongly quasi-nonexpansive mapping holds. The idea for the next process was proposed in [46, 30, 68].

Theorem 4.1.1. *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a strongly quasi-nonexpansive mapping such that $I - S$ is demiclosed in 0. Assume that $Fix(T) \cap Fix(S) \neq \emptyset$. Let u be a fixed anchor in C. Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ is a real sequence in $(0,1)$ satisfying the conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $(\beta_n)_n$ is a sequence in $[0,1]$, we define a sequence $(x_n)_{n\in\mathbb{N}}$

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T x_n + (1-\beta_n) S x_n], \quad n \geq 1. \end{cases}$$

Then,

(1) If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, then (x_n) strongly converges to $\bar{p} \in Fix(T)$ that is the unique point in $Fix(T)$ that solves the variational inequality

$$\langle \bar{p} - u, x - \bar{p} \rangle \geq 0, \qquad \forall x \in Fix(T), \tag{4.3}$$

(2) If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\frac{\beta_n}{\alpha_n} \to 0$ then $(x_n)_n$ converges strongly to $\tilde{p} \in Fix(S)$ that is the unique solution in $Fix(S)$ of the variational inequality

$$\langle \tilde{p} - u, x - \tilde{p} \rangle \geq 0, \qquad \forall x \in Fix(S). \tag{4.4}$$

(3) If $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then (x_n) strongly converges to $p_0 \in Fix(T) \cap Fix(S)$ is the unique solution in $Fix(T) \cap Fix(S)$ of the variational inequality

$$\langle p_0 - u, x - p_0 \rangle \geq 0, \qquad \forall x \in Fix(T) \cap Fix(S). \tag{4.5}$$

Proof.

In the sequel, we denote by $O(1)$ any bounded real sequence (so, for example, $O(1)+O(1) = O(1)$).

First of all, we check that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence.

Indeed, let $(U_n)_{n\in\mathbb{N}}$ a sequence defined by $U_n = \beta_n T + (1-\beta_n)S$ and $z \in Fix(T) \cap Fix(S)$. Then,

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n u + (1-\alpha_n)U_n x_n - z\| \\
&= \|(1-\alpha_n)(U_n x_n - z) + \alpha_n(u - z)\| \\
&\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\|u - z\| \\
\text{(by convexity)} &\leq \max\{\|x_n - z\|, \|u - z\|\} \\
\text{(from induction)} &\leq \max\{\|x_1 - z\|, \|u - z\|\}.
\end{aligned}$$

Then, (x_n) is bounded.

Moreover

$$x_{n+1} - U_n x_n = \alpha_n(u - U_n x_n) \to 0, \qquad n \to \infty. \tag{4.6}$$

since $\alpha_n \to 0$.

Proof of (1). The the key will be to prove that

$$x_{n+1} - x_n \to 0. \tag{4.7}$$

In order to show (4.7), we calculate

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)(U_n x_n - U_{n-1}x_{n-1}) - (\alpha_n - \alpha_{n-1})U_{n-1}x_{n-1} \\
&+ (\alpha_n - \alpha_{n-1})u)\| \\
&= \|(1-\alpha_n)(U_n x_n - U_{n-1}x_{n-1}) + (\alpha_{n-1} - \alpha_n)(U_{n-1}x_{n-1} - u)\| \\
&= \|(\alpha_{n-1} - \alpha_n)(U_{n-1}x_{n-1} - u) \\
&+ (1-\alpha_n)[\beta_n T x_n + (1-\beta_n)S x_n - \beta_{n-1}T x_{n-1} - (1-\beta_{n-1})S x_{n-1}]\| \\
&= \|(\alpha_{n-1} - \alpha_n)(U_{n-1}x_{n-1} - u) \\
&+ (1-\alpha_n)[\beta_n(T x_n - T x_{n-1}) + (1-\beta_n)(S x_n - S x_{n-1}) \\
&+ (\beta_n - \beta_{n-1})T x_{n-1} + (\beta_{n-1} - \beta_n)S x_{n-1}]\| \\
\text{(by nonexpansivity of } T) &\leq |\alpha_{n-1} - \alpha_n|O(1) \\
&+ (1-\alpha_n)[\beta_n\|x_n - x_{n-1}\| + (1-\beta_n)O(1) + |\beta_n - \beta_{n-1}|O(1)] \\
&= (1 - s_n)\|x_n - x_{n-1}\| + \gamma_n,
\end{aligned}$$

We set $s_n = 1 - \beta_n + \alpha_n\beta_n \geq \alpha_n O(1)$ and

$$\gamma_n = |\alpha_{n-1} - \alpha_n|O(1) + [(1-\alpha_n)(1-\beta_n) + |\beta_n - \beta_{n-1}|]O(1).$$

Thanks to hypotheses on $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$, it yields that $s_n \to 0$, $\sum_{n=1}^{\infty} s_n = +\infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$.

This is sufficient, by Xu's Lemma, to conclude $x_{n+1} - x_n \to 0$.

Since $x_n - U_n x_n = (x_n - x_{n+1}) - (x_{n+1} - U_n x_n)$, from (4.7) and (4.6) it follows immediately

$$x_n - U_n x_n \to 0. \tag{4.8}$$

Moreover,

$$\begin{aligned} \|x_n - U_n x_n\| &= \|x_n - \beta_n T x_n - (1-\beta_n) S x_n\| \\ &\geq \|x_n - \beta_n T x_n\| - (1-\beta_n)\|S x_n\|, \end{aligned}$$

hence

$$\|x_n - \beta_n T x_n\| \leq \|x_n - U_n x_n\| + (1-\beta_n) O(1).$$

From (4.8) and assumption $\sum_n (1-\beta_n) < \infty$, we have

$$x_n - \beta_n T x_n \to 0.$$

Furthermore, we deduce

$$x_n - \beta_n T x_n - (1-\beta_n) T x_n = x_n - T x_n \to 0. \tag{4.9}$$

Consequently, since T is nonexpansive and any weak limit of (x_n) is in $Fix(T)$, the Principle of Demiclosedness is satisfied. Next, we will show that $x_n \to \bar{p}$, where $\bar{p}$ is the unique solution in $Fix(T)$ of the variational inequality (4.3). First, we shall prove that

$$\limsup_{n\to\infty} \langle U_n x_n - \bar{p}, u - \bar{p}\rangle \leq 0. \tag{4.10}$$

Indeed, let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \langle x_n - \bar{p}, u - \bar{p}\rangle = \lim_{k\to\infty} \langle x_{n_k} - \bar{p}, u - \bar{p}\rangle,$$

and $x_{n_k} \rightharpoonup z$. Then $z \in Fix(T)$ and from (4.11),

$$\limsup_{n\to\infty} \langle x_n - \bar{p}, u - \bar{p}\rangle = \langle z - \bar{p}, u - \bar{p}\rangle, \tag{4.11}$$

and this is nonpositive by definition of $\bar{p}$.
We have,

$$\begin{aligned} \limsup_{n\to\infty} \langle U_n x_n - \bar{p}, u - \bar{p}\rangle &= \limsup_{n\to\infty} [\langle x_n - \bar{p}, u - \bar{p}\rangle \\ &+ \langle U_n x_n - x_n, u - \bar{p}\rangle] \\ \text{(from (4.8))} \quad &= \limsup_{n\to\infty} \langle x_n - \bar{p}, u - \bar{p}\rangle \\ \text{(by (4.11))} \quad &\leq 0. \end{aligned}$$

Finally,

$$
\begin{aligned}
\|x_{n+1}-\bar{p}\| &= \|(1-\alpha_n)(U_nx_n-\bar{p})+\alpha_n(u-\bar{p})\|^2 \\
&= (1-\alpha_n)^2\|U_nx_n-\bar{p}\|^2+\alpha_n^2\|u-\bar{p}\|^2 \\
&+ 2\alpha_n\langle(1-\alpha_n)(U_nx_n-\bar{p}),u-\bar{p}\rangle \\
&= (1-\alpha_n)^2\|\beta_n(Tx_n-\bar{p})+(1-\beta_n)(Sx_n-\bar{p})\|^2+\alpha_n^2\|u-\bar{p}\|^2 \\
&+ 2\alpha_n\langle U_nx_n-\bar{p},u-\bar{p}\rangle-2\alpha_n^2\langle U_nx_n-\bar{p},u-\bar{p}\rangle \\
\text{(by } T \text{ quasi-nonexp.)} &\leq (1-\alpha_n)^2[\beta_n\|x_n-\bar{p}\|+(1-\beta_n)O(1)]^2+\alpha_n^2O(1) \\
&+ 2\alpha_n\langle U_nx_n-\bar{p},u-\bar{p}\rangle+2\alpha_n^2O(1) \\
&\leq (1-\alpha_n)^2\|x_n-\bar{p}\|^2+(1-\beta_n)O(1)+\alpha_n^2O(1) \\
&+ 2\alpha_n\langle U_nx_n-\bar{p},u-\bar{p}\rangle \\
&\leq (1-\alpha_n)^2\|x_n-\bar{p}\|^2+(1-\beta_n)O(1)+\alpha_n^2O(1) \\
&+ 2\alpha_n\langle U_nx_n-\bar{p}+u-\bar{p}\rangle \\
&= (1-2\alpha_n)\|x_n-\bar{p}\|^2+(1-\beta_n)O(1)+\alpha_n^2O(1) \\
&+ 2\alpha_n\langle U_nx_n-\bar{p}+u-\bar{p}\rangle \\
&= (1-s_n)\|x_n-\bar{p}\|^2+s_n\sigma_n+\gamma_n.
\end{aligned}
$$

We set

$$
\begin{aligned}
s_n &= 2\alpha_n, \\
\sigma_n &= \frac{\alpha_n}{2}O(1)+\langle U_nx_n-\bar{p},u-\bar{p}\rangle, \gamma_n=(1-\beta_n)O(1).
\end{aligned}
$$

Using the hypotheses on control coefficients $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ and (4.10) we can apply the Xu's Lemma and we obtain $x_n \to \bar{p}$. □

Proof of 2. Let $\tilde{p}$ be the unique solution of variational inequality (4.4).
We want to show that $x_n \to \tilde{p}$. We compute,

$$
\begin{aligned}
\|x_{n+1}-\tilde{p}\|^2 &= \|\alpha_nu+(1-\alpha_n)U_nx_n-\tilde{p}+\alpha_n\tilde{p}-\alpha_n\tilde{p}\|^2 \\
&= \|\alpha_n(u-\tilde{p})+(1-\alpha_n)(U_nx_n-\tilde{p})\|^2 \\
\text{(by Lemma 1.0.1, (ii))} &\leq (1-\alpha_n)^2\|U_nx_n-\tilde{p}\|^2 \\
&+ 2\alpha_n\langle u-\tilde{p},x_{n+1}-\tilde{p}\rangle \\
&= (1-\alpha_n)^2\|\beta_n(Tx_n-\tilde{p})+(1-\beta_n)(Sx_n-\tilde{p})\|^2 \\
&+ 2\alpha_n\langle u-\tilde{p},x_{n+1}-\tilde{p}\rangle \\
\text{(by } T,S \text{ quasi-nonexp.)} &\leq (1-\alpha_n)^2\|x_n-\tilde{p}\|^2 \\
&+ \beta_nO(1)+2\alpha_n\langle x_{n+1}-\tilde{p},u-\tilde{p}\rangle. \qquad (4.12)
\end{aligned}
$$

At this point we distinguish two cases.

Alternative 1. $(\|x_n-\tilde{p}\|)$ is eventually not increasing, hence

$$\|x_{n+1}-\tilde{p}\| \leq \|x_n-\tilde{p}\|, \quad \forall n \geq N.$$

Putting $s_n = 2\alpha_n$, $\sigma_n = \langle x_{n+1} - \tilde{p}, u - \tilde{p}\rangle + \frac{1}{2}\alpha_n O(1)$, $\gamma_n = \beta_n O(1)$, we can rewrite (4.12) as

$$\|x_{n+1} - \tilde{p}\|^2 \leq (1 - s_n)\|x_n - \tilde{p}\|^2 + s_n\sigma_n + \gamma_n. \tag{4.13}$$

We will apply the Xu's Lemma, if we are able to show that

$$\limsup_{n\to\infty}\langle x_{n+1} - \tilde{p}, u - \tilde{p}\rangle \leq 0. \tag{4.14}$$

Note that until now we have not used the hypothesis of strong quasi-nonexpansivity of S. Since $(\|x_n - \tilde{p}\|)_{n\in\mathbb{N}}$ is definitively not increasing, there exists the $\lim_{n\to\infty}\|x_n - \tilde{p}\|$. Then

$$\begin{array}{rcl} 0 & = & \lim_{n\to\infty}(\|x_{n+1} - \tilde{p}\| - \|x_n - \tilde{p}\|) \\ & \leq & \liminf_{n\to\infty}(\alpha_n\|u - \tilde{p}\| + (1-\alpha_n)\|U_n x_n - \tilde{p}\| - \|x_n - \tilde{p}\|) \\ (\text{since } \alpha_n \to 0) & = & \liminf_{n\to\infty}(\|U_n x_n - \tilde{p}\| - \|x_n - \tilde{p}\|) \\ & = & \liminf_{n\to\infty}(\|\beta_n(Tx_n - \tilde{p}) + (1-\beta_n)(Sx_n - \tilde{p})\| - \|x_n - \tilde{p}\|) \\ (\text{by hypothesis } \beta_n \to 0) & = & \liminf_{n\to\infty}(\|Sx_n - \tilde{p}\| - \|x_n - \tilde{p}\|) \\ (\text{by quasi-nonexpansivity of } S) & \leq & \limsup_{n\to\infty}(\|x_n - \tilde{p}\| - \|x_n - \tilde{p}\|) = 0. \end{array} \tag{4.15}$$

Thus,

$$\lim_{n\to\infty}(\|Sx_n - \tilde{p}\| - \|x_n - \tilde{p}\|) = 0.$$

From the strong quasi-nonexpansivity of S, we deduce

$$Sx_n - x_n \to 0. \tag{4.16}$$

At this point, by using the demiclosedness of $(I - S)$ at 0, we can proceed as in the Proof of (1) to show (4.14).
The statement is proved when the Alternative 1 holds.

Alternative 2. Suppose that $(\|x_n - \tilde{p}\|)_{n\in\mathbb{N}}$ is not definitively not increasing, i.e. there exists a subsequence $(\|x_{n_j} - \tilde{p}\|)_{j\in\mathbb{N}}$ such that

$$\|x_{n_j} - \tilde{p}\| < \|x_{n_j+1} - \tilde{p}\|, \qquad \forall j \in \mathbb{N}. \tag{4.17}$$

In this situation, we consider the sequence of indices $(\tau(n))_{n\in\mathbb{N}}$ as defined in Maingé's Lemma. It follows that there exists an increasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying

$$\lim_n \tau(n) = +\infty, \quad \|x_{\tau(n)} - \tilde{p}\| < \|x_{\tau(n)+1} - \tilde{p}\|, \tag{4.18}$$

$$\|x_n - \tilde{p}\| < \|x_{\tau(n)} - \tilde{p}\|, \qquad \forall n \geq n_0. \tag{4.19}$$

Then

$$0 \leq \liminf_{n\to\infty}(\|x_{\tau(n)+1} - \tilde{p}\| - \|x_{\tau(n)} - \tilde{p}\|).$$

In the previous steps in order to prove (4.16), we replace n with $\tau(n)$ and we obtain

$$\lim_{n\to\infty}(\|Sx_{\tau(n)} - \tilde{p}\| - \|x_{\tau(n)} - \tilde{p}\|) = 0.$$

By the strong quasi-nonexpansivity, it yields

$$Sx_{\tau(n)} - x_{\tau(n)} \to 0, \tag{4.20}$$

and from the demiclosedness of $I - S$ in 0, we deduce as above,

$$\limsup_{n\to\infty}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle \leq 0. \tag{4.21}$$

Incidentally, we observe that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| = \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\alpha_{\tau(n)})\beta_{\tau(n)}O(1) + (1-\alpha_{\tau(n)})\|Sx_{\tau(n)} - x_{\tau(n)}\|.$$

In addition, from (4.20) it follows also $x_{\tau(n)+1} - x_{\tau(n)} \to 0$.
We replace in (4.12) n with $\tau(n)$ and we get

$$\begin{aligned} \|x_{\tau(n)+1} - \tilde{p}\| &\leq (1-\alpha_{\tau(n)})^2\|x_{\tau(n)} - \tilde{p}\|^2 + 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \beta_{\tau(n)}O(1) \\ \text{(by property (4.18))} &\leq (1-\alpha_{\tau(n)})^2\|x_{\tau(n)+1} - \tilde{p}\|^2 + 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \beta_{\tau(n)}O(1), \end{aligned}$$

consequently

$$\begin{aligned} 2\alpha_{\tau(n)}\|x_{\tau(n)+1} - \tilde{p}\|^2 &\leq (\alpha_{\tau(n)})^2\|x_{\tau(n)+1} - \tilde{p}\|^2 + \\ &+ 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \beta_{\tau(n)}O(1), \end{aligned}$$

and dividing by $\alpha_{\tau(n)}$, we have

$$\begin{aligned} 0 &\leq 2\|x_{\tau(n)+1} - \tilde{p}\|^2 \leq \alpha_{\tau(n)}\|x_{\tau(n)+1} - \tilde{p}\|^2 \\ &+ 2\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}O(1). \end{aligned}$$

Taking the limsup and recalling the assumption $\frac{\beta_n}{\alpha_n} \to 0$ and (4.21), we obtain

$$\lim_{n\to\infty}\|x_{\tau(n)} - \tilde{p}\| \leq \lim_{n\to\infty}\|x_{\tau(n)+1} - \tilde{p}\| = 0.$$

The (4.19) ensures that $x_n \to \tilde{p}$. □

Proof of 3. Let p_0 the unique point in $Fix(S) \cap F$ that satisfies the variational inequality (4.5). Then

$$\begin{aligned} \|U_nx_n - p_0\|^2 &= \|\beta_n(Tx_n - p_0) + (1-\beta_n)(Sx_n - p_0)\|^2 \\ \text{(by Lemma 1.0.1, (i))} &= \beta_n\|Tx_n - p_0\|^2 + (1-\beta_n)\|Sx_n - p_0\|^2 \\ &- \beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 \\ &\leq \|x_n - p_0\|^2 - \beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \|x_{n+1} - p_0\|^2 &= \|\alpha_n(u - p_0) + (1-\alpha_n)(U_nx_n - p_0)\|^2 \\ &\leq (1-\alpha_n)^2\|U_nx_n - p_0\|^2 + \alpha_n^2O(1) + \alpha_nO(1) \\ \text{(by (4.22))} &\leq \|x_n - p_0\|^2 - \beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 + \alpha_n^2O(1) \\ &+ \alpha_nO(1). \end{aligned} \tag{4.23}$$

At this point, the proof will be divided in two parts.

Alternative 1. $(\|x_n - p_0\|)_{n\to\infty}$ is eventually not increasing,

$$\|x_{n+1} - p_0\| \leq \|x_n - p_0\|, \quad \forall n \geq N.$$

Then there exists $\lim_{n\to\infty} \|x_n - \tilde{p}\|$. Further, (4.23) furnishes

$$\beta_n(1-\beta_n)\|Tx_n - Sx_n\| \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1)$$

and hence, by assumption $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, we deduce

$$Tx_n - Sx_n \to 0. \tag{4.24}$$

Moreover, we compute

$$\begin{array}{rcl}
0 & = & \lim_{n\to\infty}(\|x_{n+1} - p_0\| - \|x_n - p_0\|) \\
& \leq & \liminf_{n\to\infty}(\alpha_n\|u - p_0\| + (1-\alpha_n)\|U_n x_n - p_0\| - \|x_n - p_0\|) \\
(\text{ since } \alpha_n \to 0) & = & \liminf_{n\to\infty}(\|U_n x_n - p_0\| - \|x_n - p_0\|) \\
& = & \liminf_{n\to\infty}(\|\beta_n(Tx_n - p_0) \\
& + & (1-\beta_n)(Sx_n - p_0)\| - \|x_n - p_0\|) \\
& \leq & \liminf_{n\to\infty}(\beta_n(\|Tx_n - p_0\| - \|x_n - p_0\|) \\
& + & (1-\beta_n)(\|Sx_n - p_0\| - \|x_n - p_0\|)) \\
& \leq & \limsup_{n\to\infty}(\beta_n(\|Tx_n - p_0\| - \|x_n - p_0\|) \\
& + & (1-\beta_n)(\|Sx_n - p_0\| - \|x_n - p_0\|)) \\
(\ T, S \text{ quasi-nonep.}) & \leq & 0.
\end{array}$$

Then, we deduce

$$\lim_{n\to\infty}(\beta_n(\|Tx_n - p_0\| - \|x_n - p_0\|) + (1-\beta_n)(\|Sx_n - p_0\| - \|x_n - p_0\|)) = 0.$$

Since both the addends are non positive and the limit of the sum is zero, it follows that

$$\lim_{n\to\infty}\beta_n(\|Tx_n - p_0\| - \|x_n - p_0\|) = \lim_{n\to\infty}(1-\beta_n)(\|Sx_n - p_0\| - \|x_n - p_0\|) = 0. \tag{4.25}$$

It implies, using hypothesis $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, that

$$\lim_{n\to\infty}(\|Tx_n - p_0\| - \|x_n - p_0\|) = \lim_{n\to\infty}(\|Sx_n - p_0\| - \|x_n - p_0\|) = 0. \tag{4.26}$$

From strong quasi-nonexpansivity of S it follows

$$Sx_n - x_n \to 0. \tag{4.27}$$

Since $x_n - Tx_n = x_n - Sx_n + Sx_n - Tx_n$, by (4.24) and (4.27), we have

$$x_n - Tx_n \to 0. \tag{4.28}$$

Next, we will show that

$$\limsup_{n\to\infty} \langle x_n - p_0, u - p_0\rangle \leq 0. \tag{4.29}$$

Indeed, select a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup z$ and

$$\limsup_{n\to\infty} \langle x_n - p_0, u - p_0\rangle = \lim_{k\to\infty} \langle x_{n_k} - p_0, u - p_0\rangle = \langle z - p_0, u - p_0\rangle.$$

But by the demiclosedness of both $I - T$ and $I - S$ at 0 and by (4.27) and (4.28), one deduces that $z \in Fix(S) \cap Fix(T)$, and by definition of p_0, (4.29) is obtained. Moreover, from (4.27) and (4.28), we have

$$U_n x_n - x_n \to 0. \tag{4.30}$$

Finally we are able to show $x_n \to p_0$. Indeed,

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &= \|(1-\alpha_n)(U_n x_n - p_0) + \alpha_n(u - p_0)\|^2 = \\
&= (1-\alpha_n)^2\|U_n x_n - p_0\|^2 + \alpha_n^2\|u - p_0\|^2 \\
&+ 2\alpha_n\langle (1-\alpha_n)(U_n x_n - p_0), u - p_0\rangle \\
(\ U_n \text{ quasi-nonexp.}) &\leq (1-\alpha_n)^2\|x_n - p_0\|^2 + \alpha_n^2[\|u - p_0\|^2 - 2\langle U_n x_n - p_0, u - p_0\rangle] \\
&+ 2\alpha_n\langle U_n x_n - x_n, u - p_0\rangle + 2\alpha_n\langle x_n - p_0, u - p_0\rangle \\
&\leq (1-\alpha_n)\|x_n - p_0\|^2 + \alpha_n^2 O(1) + 2\alpha_n\langle U_n x_n - x_n, u - p_0\rangle \\
&+ 2\alpha_n\langle x_n - p_0, u - p_0\rangle.
\end{aligned}$$

Put

$$\sigma_n = 2\langle U_n x_n - x_n, u - p_0\rangle + 2\langle x_n - p_0, u - p_0\rangle + \alpha_n O(1), \tag{4.31}$$

the thesis follows again by Xu's Lemma, reminding (4.29) and (4.30).

Alternative 2. $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ is not eventually not increasing, i.e. there exists a subsequence $(\|x_{n_j} - p_0\|)_{j\in\mathbb{N}}$ such that $\|x_{n_j} - p_0\| < \|x_{n_{j+1}} - p_0\|$, $\forall j \in \mathbb{N}$.

From Maingé's Lemma it follows that there exists an increasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying (4.18) and (4.19).

Then

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|) \\
&\leq \limsup_{n\to\infty}(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|) \\
&\leq \limsup_{n\to\infty}(\|x_{n+1} - p_0\| - \|x_n - p_0\|) \\
&\leq \limsup_{n\to\infty}(\|(1-\alpha_n)(U_n x_n - p_0) + \alpha_n(u - p_0)\| - \|x_n - p_0\|) \\
(\text{by } U_n \text{ quasi-nonexp.}) &\leq \limsup_{n\to\infty}((1-\alpha_n)\|x_n - p_0\| + \alpha_n O(1) - \|x_n - p_0\|) \\
(\text{by } \alpha_n \to 0) &= 0.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}(\|x_{\tau(n)+1}-p_0\|-\|x_{\tau(n)}-p_0\|)=0. \tag{4.32}$$

Now, retracing the same inequalities to obtain (4.27), we can replace n with $\tau(n)$ and we have

$$Sx_{\tau(n)}-x_{\tau(n)}\to 0. \tag{4.33}$$

Moreover, we can rewrite (4.23) as

$$0\le \beta_{\tau(n)}(1-\beta_{\tau(n)})\|Tx_{\tau(n)}-Sx_{\tau(n)}\|^2\le \|x_{\tau(n)}-p_0\|^2-\|x_{\tau(n)+1}-p_0\|^2+\alpha_n O(1),$$

from (4.32) and by the hypothesis $\liminf_n \beta_n(1-\beta_n)>0$,

$$Tx_{\tau(n)}-Sx_{\tau(n)}\to 0. \tag{4.34}$$

Further, since

$$x_{\tau(n)}-Tx_{\tau(n)}=x_{\tau(n)}-Sx_{\tau(n)}+Sx_{\tau(n)}-Tx_{\tau(n)},$$

by (4.33) and (4.34), we get

$$x_{\tau(n)}-Tx_{\tau(n)}\to 0, \tag{4.35}$$

and

$$x_{\tau(n)}-U_{\tau(n)}x_{\tau(n)}\to 0. \tag{4.36}$$

Following the same line to prove (4.29), we replace n with $\tau(n)$ and we obtain

$$\limsup_{n\to\infty}\langle x_{\tau(n)}-p_0,u-p_0\rangle\le 0. \tag{4.37}$$

We compute,

$$\begin{array}{rcl}\|x_{\tau(n)+1}-p_0\| & \le & (1-\alpha_{\tau(n)})^2\|x_{\tau(n)}-p_0\|^2+\alpha_{\tau(n)}^2 O(1)\\ & + & 2\alpha_{\tau(n)}\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)},u-p_0\rangle\\ & + & 2\alpha_{\tau(n)}\langle x_{\tau(n)}-p_0,u-p_0\rangle\\ \text{(by property (4.18))} & \le & (1-\alpha_{\tau(n)})^2\|x_{\tau(n)+1}-p_0\|^2+\alpha_{\tau(n)}^2 O(1)\\ & + & 2\alpha_{\tau(n)}\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)},u-p_0\rangle\\ & + & 2\alpha_{\tau(n)}\langle x_{\tau(n)}-p_0,u-p_0\rangle,\end{array}$$

consequently

$$\begin{array}{rcl}2\alpha_{\tau(n)}\|x_{\tau(n)+1}-p_0\|^2 & \le & \alpha_{\tau(n)}^2 O(1)\\ & + & 2\alpha_{\tau(n)}\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)},u-p_0\rangle\\ & + & 2\alpha_{\tau(n)}\langle x_{\tau(n)}-p_0,u-p_0\rangle,\end{array}$$

dividing by $\alpha_{\tau(n)}$, we get

$$\begin{array}{rcl}0 & \le & 2\|x_{\tau(n)+1}-p_0\|^2\le \alpha_{\tau(n)}O(1)\\ & + & 2\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)},u-p_0\rangle+2\langle x_{\tau(n)}-p_0,u-p_0\rangle.\end{array}$$

Taking the limsup and recalling the hypothesis (4.36) and (4.37), we obtain

$$\lim_{n\to\infty} \|x_{\tau(n)} - p_0\| \le \lim_{n\to\infty} \|x_{\tau(n)+1} - p_0\| = 0.$$

Once again by (4.19) we deduce $x_n \to p_0$. □

□

Remark 4.1.2. We recall that in Proposition 1.0.57 we discussed the connection between an averaged type mapping of quasi-nonexpansive mapping and a strongly quasi-nonexpansive mapping. Consequently, we can obtain the same thesis of Theorem 4.1.1 for two quasi-nonexpansive mappings under the assumptions:

- $T, S : C \to C$ be two quasi-nonexpansive mappings such that $I - T, I - S$ are demiclosed at 0;
- $Fix(S) \cap Fix(T) \neq \emptyset$;
- same hypotheses on the coefficients $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$;

for a sequence (x_n) generated by the Algorithm (4.38) in which appear the averaged type mappings T_δ, S_δ,

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n], \end{cases} \tag{4.38}$$

Instead of the $\frac{1}{2}$-inverse strong monotone property of the mapping $I - T$ when T is a nonexpansive mapping, it was useful the following inequality established in [68]

$$\langle x - T_\delta x, x - p\rangle \ge \frac{\delta}{2}\|x - Tx\|^2, \quad \forall x \in C, p \in Fix(T). \tag{4.39}$$

It should be noted that the hypotheses of quasi-nonexpansivity and demiclosedness on the mappings involved in our algorithm (4.38) are independent .

Remark 4.1.3. In the literature, there exist some interesting mappings T which are quasi-nonexpansive and such that $I - T$ are demiclosed at 0. Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H and $T : C \to C$ such that $Fix(T) \neq \emptyset$. Next, we list some examples of such mappings.

1. T nonexpansive mapping ([10]);
2. T nonspreading mapping ([31]);
3. T Chatterjea mapping ([62]);
4. T L-hybrid mapping ([31]).

Further, there exist mappings T such that $I - T$ are demiclosed at 0 but not necessarily quasi-nonexpansive:

(a) [73] T continuous pseudocontractive mapping, i.e. if $\forall x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2;$$

(b) [9] T k-strictly pseudononspreading mapping, i.e. if there exists $k \in [0, 1)$ such that for all $x, y \in D(T)$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Next, we will give an example of quasi-nonexpansive mapping which does not satisfy the Demiclosedness Principle.

Example 4.1.4. *Let*

$$B^+_{\ell^2} = \{(x_i)_{i\in\mathbb{N}} : \quad \sum_{i=1}^{\infty} x_i^2 \leq 1, \quad x_i \geq 0, i = 1, 2, ...\},$$

be the part of the unit ball of ℓ_2 *contained in the positive cone.*
Let $T : B^+_{\ell^2} \to B^+_{\ell^2}$ *be defined by*

$$Tx = \begin{cases} 0_{\ell^2} & 0 \leq \|x\| \leq \frac{1}{2} \\ \frac{x}{2\|x\|} & \frac{1}{2} < \|x\| \leq 1. \end{cases}$$

We have $Fix(T) = \{0_{\ell^2}\}$. *It is obvious that* T *is a quasi-nonexpansive mapping. We want to prove that* $I - T$ *is not demiclosed at* 0_{ℓ^2}.
It is easy to check that if $1 \geq \|x_n\| > \frac{1}{2}$ *and* $\lim_{n\to\infty} \|x_n\| = \dfrac{1}{2}$, *then* (x_n) *is an almost fixed point sequence for* T *in* $B^+_{l^2}$. *Indeed,*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \left\| \frac{x_n}{2\|x_n\|} - x_n \right\| = \lim_{n\to\infty} \left| 1 - \frac{1}{2\|x_n\|} \right| \|x_n\| = 0; \qquad (4.40)$$

Thus, if we consider the sequence $(x_n)_{n\in\mathbb{N}}$ *defined by* $x_n = \frac{e_1}{3} + e_n \frac{\sqrt{5n^2+36n+36}}{6n}$, $\quad (n \geq 2)$, *then*

$$\begin{aligned} \|x_n\|^2 &= \frac{1}{9} + \left[\frac{5n^2 + 36n + 36}{36n^2}\right] = \frac{4n^2 + 5n^2 + 36n + 36}{36n^2} \\ &= \frac{(3n + 6)^2}{36n^2} = \left(\frac{1}{2} + \frac{1}{n}\right)^2, \quad n \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2},$$

and hence $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.
On the other hand, we have $x_n \rightharpoonup \frac{e_1}{3}$, *but* $T\left(\frac{e_1}{3}\right) = 0_{\ell^2} \neq \frac{e_1}{3}$.

4.1.1 Numerical Example

Next, we illustrate the result (4.1.3) with a numerical example given in [25].

Example 4.1.5. *Let $T : B^+_{\ell^2} \to B^+_{\ell^2}$ be defined by $T(x) = (x_1 - x_1^3, x_2 - x_2^3, ...)$. Then, $Fix(T) = \{0_{\ell^2}\}$.*
We denote by $P_{B^+[0,\frac{1}{2}]}$ the metric projection from ℓ^2 onto

$$B^+[0, \frac{1}{2}] := \left\{(x_i) \in B^+_{\ell^2} : \quad \|x\| \leq \frac{1}{2}\right\}.$$

Let $S : B^+_{\ell^2} \to B^+_{\ell^2}$ defined by $S(x) = P_{B^+[0,\frac{1}{2}]}x$. It is obvious that $Fix(S) = B^+[0, \frac{1}{2}]$. Notice that $Fix(T) \cap Fix(S) = \{0_{l^2}\}$.
T is a quasi-nonexpansive mapping, for all $x \in B^+_{l^2}$,

$$\|Tx - 0_{\ell^2}\|^2 = \sum_{n=1}^{\infty}(x_n - x_n^3)^2 = \sum_{n=1}^{\infty} x_n^2(1 - x_n^2)^2 \leq \|x - 0_{\ell^2}\|^2.$$

To show that T is a pseudo-contractive mapping we will see that $I - T$ is accretive. If $x \in B^+_{\ell^2}$, we denote by $x^3 = (x_1^3, x_2^3, \ldots) \in B^+_{\ell^2}$. For all $x, y \in B^+_{\ell^2}$, we claim that

$$\langle x - y, x^3 - y^3\rangle \geq 0.$$

Indeed, notice that

$$\langle x - y, x^3 - y^3\rangle = \sum_{i=1}^{\infty}(x_i - y_i)(x_i^3 - y_i^3), \tag{4.41}$$

is a series of positive terms, since $(x_i - y_i)(x_i^3 - y_i^3) \geq 0$ for every $i \in \mathbb{N}$. It is easy to check that T is continuous. Therefore, from [73], $I - T$ is demiclosed at 0_{ℓ_2}.

On the other hand, T is not a nonexpansive mapping. Indeed, for $x = (1, 0, \ldots), y = (\frac{3}{4}, 0, \ldots) \in B^+_{l^2}$, one has

$$\|Tx - Ty\| = \frac{3}{4} - \left(\frac{3}{4}\right)^3 = \frac{21}{64} > \|x - y\| = \frac{1}{4}.$$

It is well known that S is a nonexpansive mapping, and hence $I - S$ is demiclosed at 0 (see[10]).

We recall that $x = (x_1, x_2, ...) \in B^+_{\ell^2}$ and for $k \in \{2, 3\}$, $x^k = (x_1^k, x_2^k, \ldots) \in B^+_{\ell^2}$. Let $T_\delta(x) = x - \delta x^3$ and $S_\delta(x) = (1 - \delta)x + \delta P_{B^+[0,\frac{1}{2}]}x$.
Notice that, if $x \in B^+[0, \frac{1}{2}]$, we get $S_\delta x = x$.

In the sequel we will consider two cases:

- *in the first one we choose x_1, u in $\{(x_i) \in B^+_{\ell^2} : \quad \|x\| > \frac{1}{2}\}$;*
- *in the second case we set $x_1, u \in B^+[0, \frac{1}{2}]$.*

Case 1. (i)

We set $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} = \left(1 - \frac{1}{(n+1)^2}\right)_{n\in\mathbb{N}}$, $x_1 = e_1$, $u = e_2$ *and* $\delta = \frac{1}{3}$.

We consider the sequence $(x_{n+1})_{n\in\mathbb{N}}$ *where* $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ *and*

$$\begin{aligned} x_{n+1} &= \tfrac{1}{n+1}e_2 \\ &+ \left(1 - \tfrac{1}{n+1}\right)\left[\left(1 - \tfrac{1}{(n+1)^2}\right)\left(x_n - \tfrac{1}{3}{x_n}^3\right) + \tfrac{1}{(n+1)^2}\left(\tfrac{2}{3}x_n + \tfrac{1}{3}P_{B^+[0,\frac{1}{2}]}x_n\right)\right]. \end{aligned}$$

From remark 4.1.2, *we know that this algorithm converges to* $\overline{p} = \{0_{\ell^2}\} = P_{Fix(T)}u$.

First, we calculate components of each (x_i) *and* $\|x_i - \overline{p}\|$ *for* $i = 1, 2, 3, ..., 10^7$.

i	x_i^1	x_i^2	x_i^3	...	$\|x_i\|$
$i = 1$	1	0	0	...	1
$i = 2$	$3.541666666666666 \cdot 10^{-1}$	$5 \cdot 10^{-1}$	0	...	$6.127 \cdot 10^{-1}$
$i = 3$	$2.257270516465794 \cdot 10^{-1}$	$6.397040059395266 \cdot 10^{-1}$	0	...	$6.783 \cdot 10^{-1}$
...	...	...	...	...	...
$i = 10^7$	$6.363036990406555 \cdot 10^{-8}$	$6.684363438458816 \cdot 10^{-3}$	0	...	$6.684 \cdot 10^{-3}$
...	...	...	...	...	...

(ii)

We set $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} = \left(\frac{1}{8^n}\right)_{n\in\mathbb{N}}$, $x_1 = e_1$, $u = e_2$ *and* $\delta = \frac{1}{3}$.

We consider the sequence $(x_{n+1})_{n\in\mathbb{N}}$ *where* $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ *and*

$$\begin{aligned} x_{n+1} &= \tfrac{1}{n+1}e_2 + \\ &\left(1 - \tfrac{1}{n+1}\right)\left[\tfrac{1}{8^n}\left(x_n - \tfrac{1}{3}{x_n}^3\right) + \left(1 - \tfrac{1}{8^n}\right)\left(\tfrac{2}{3}x_n + \tfrac{1}{3}P_{B^+[0,\frac{1}{2}]}x_n\right)\right]. \end{aligned}$$

From remark 4.1.2, *we know that this algorithm converges to*

$$\widehat{p} = P_{Fix(S)}u = (0, \frac{1}{2}, 0, ...).$$

In this case, we calculate the components of each (x_i) *and the norm of* $x_i - \widehat{p}$, *for* $i = 1, 2, 3, ..., 10^7$.

Let $\widehat{p} = (0, \frac{1}{2}, 0, 0, ...)$.

i	x_i^1	x_i^2	x_i^3	...	$\|x_i - \widehat{p}\|$
$i = 1$	1	0	0	...	$\sqrt{\frac{5}{4}}$
$i = 2$	$4.062500000000000 \cdot 10^{-1}$	$5.000000000000000 \cdot 10^{-1}$	0	...	$4.062500000000000 \cdot 10^{-1}$
$i = 3$	$2.507044117060245 \cdot 10^{-1}$	$6.417451076516054 \cdot 10^{-1}$	0	...	$2.880006555409710 \cdot 10^{-1}$
...	...	...	...	...	...
$i = 100$	$4.599359984722416 \cdot 10^{-4}$	$5.149997813245735 \cdot 10^{-1}$	0	...	$1.500683114144068 \cdot 10^{-2}$
...	...	...	...	...	...
$i = 10^7$	$4.691347264050784 \cdot 10^{-14}$	$5.000001500000000 \cdot 10^{-1}$	0	...	$1.499999999765651 \cdot 10^{-7}$
...	...	...	...	...	...

(iii)

We set $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} = \left(\frac{9n}{10n+1}\right)_{n\in\mathbb{N}}$, $x_1 = e_1$, $u = e_2$ *and* $\delta = \frac{1}{3}$.

We consider the sequence $(x_{n+1})_{n\in\mathbb{N}}$ *where* $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ *and*

$$\begin{aligned} x_{n+1} &= \tfrac{1}{n+1}e_2 + \\ &\left(1 - \tfrac{1}{n+1}\right)\left[\left(\tfrac{9n}{10n+1}\right)\left(x_n - \tfrac{1}{3}{x_n}^3\right) + \left(\tfrac{n+1}{10n+1}\right)\left(\tfrac{2}{3}x_n + \tfrac{1}{3}P_{B^+[0,\frac{1}{2}]}x_n\right)\right]. \end{aligned}$$

From remark 4.1.2, we know that this algorithm converges to $p_0 = \{0_{\ell^2}\} = P_{Fix(T)\cap Fix(S)}u$. *In this case, we calculate components of each* (x_i) *and* $\|x_i\|$ *for* $i = 1, 2, 3, ..., 10^7$.

i	x_i^1	x_i^2	x_i^3	...	$\|x_i\|$
$i = 1$	1	0	0	...	1
$i = 2$	$3.484848484848485 \cdot 10^{-1}$	$5.0 \cdot 10^{-1}$	0	...	$6.094601624581445 \cdot 10^{-1}$
$i = 3$	$2.222752307203779 \cdot 10^{-1}$	$6.400063201074601 \cdot 10^{-1}$	0	...	$6.775059910947576 \cdot 10^{-1}$
...	...	...	...	...	...
$i = 100$	$41.332411716134864 \cdot 10^{-2}$	$, 3.446163391574661 \cdot 10^{-1}$	0	...	$3.448738223066852 \cdot 10^{-1}$
...	...	...	...	...	...
$i = 10^7$	$2.357010822671114 \cdot 10^{-1}$	$2.357040822488564 \cdot 10^{-1}$	0	...	$3.333337885223508 \cdot 10^{-1}$
...	...	...	...	...	...

Case 2.

If we choose $x_1, u \in B^+[0, \frac{1}{2}]$, *let* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence in* $(0, 1)$ *and let* $(\beta_n)_{n\in\mathbb{N}}$ *be a sequence in* $[0, 1]$, *we derive by induction that*

$$0 \leq \|x_n\| \leq \frac{1}{2}, \quad \forall n \in \mathbb{N}. \tag{4.42}$$

For $n = 1$, *we have* $\|x_1\| \leq \frac{1}{2}$. *We assume that* (4.42) *is true for some* $k \in \mathbb{N}$ *and we prove that* (4.42) *is obtained for some* $k + 1 \in \mathbb{N}$.

$$\begin{aligned}
\|x_{k+1}\| &= \left\|\alpha_k u + (1-\alpha_k)\left[\beta_k\left(x_k - \frac{1}{3}x_k^3\right) + (1-\beta_k)\left(\frac{2}{3}x_k + \frac{1}{3}Proj_{B[0,\frac{1}{2}]}x_k\right)\right]\right\| \\
&\leq \alpha_k\|u\| + (1-\alpha_k)\left\|\beta_k x_k\left(1 - \frac{1}{3}x_k^2\right) + (1-\beta_k)x_k\right\| \\
&\leq \alpha_k\frac{1}{2} + (1-\alpha_k)\left[\beta_k\left\|x_k\left(1 - \frac{1}{3}x_k^2\right)\right\| + (1-\beta_k)\|x_k\|\right] \\
&\leq \alpha_k\frac{1}{2} + (1-\alpha_k)\left[\beta_k\frac{1}{2} + (1-\beta_k)\frac{1}{2}\right] \\
&= \frac{1}{2}
\end{aligned}$$

Moreover, $S_\delta x_n = x_n$, *for all* $n \in \mathbb{N}$.

Hence, the algorithm (4.2) *becomes,*

$$\begin{aligned}
x_{n+1} &= \alpha_n u + (1-\alpha_n)[\beta_n(x_n - \frac{1}{3}x_n^3) + (1-\beta_n)x_n] \\
&= \alpha_n u + (1-\alpha_n)[x_n - \frac{1}{3}\beta_n x_n^3], \quad \forall n \in \mathbb{N}.
\end{aligned}$$

We fix $u = (0, \frac{1}{4}, 0, ...)$ *and* $x_1 = (\frac{1}{2}, 0, 0, ...)$ *in the following three cases.*

(i)

We set $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}} = \left(1 - \frac{1}{(n+1)^2}\right)_{n\in\mathbb{N}}$.

We consider the sequence $(x_{n+1})_{n\in\mathbb{N}}$ *where* $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ *and*

$$x_{n+1} = \frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right)\left[x_n - \frac{1}{3}\left(1 - \frac{1}{(n+1)^2}\right)x_n^3\right]. \tag{4.43}$$

From remark 4.1.2, *we know that the algorithm (4.43) converges to* $\overline{p} = \{0_{\ell^2}\} = P_{Fix(T)}u$.

i	$\|x_i\|$
$i=1$	$\frac{1}{2}$
$i=2$	$2.656250000000000 \cdot 10^{-1}$
$i=3$	$2.264401183959056 \cdot 10^{-1}$
$i=4$	$2.184937817972105 \cdot 10^{-1}$
...	...
$i=100$	$1.576038253078700 \cdot 10^{-1}$
...	...
$i=10^7$	$4.201325133422668 \cdot 10^{-3}$
...	...

(ii)

We set $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}} = \left(\frac{1}{8^n}\right)_{n\in\mathbb{N}}$.

We consider the sequence $(x_{n+1})_{n\in\mathbb{N}}$ *where* $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ *and*

$$x_{n+1} = \frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right)\left[x_n - \frac{1}{3}\left(\frac{1}{8^n}\right)x_n^3\right]. \tag{4.44}$$

From remark 4.1.2, *we know that the algorithm (4.44) converges to* $\widehat{p} = P_{Fix(S)}u = u = (0, \frac{1}{4}, 0, 0, ...)$.

i	$\|x_i - \widehat{p}\|$
$i=1$	$\sqrt{\frac{5}{16}}$
$i=2$	$2.771817063781347 \cdot 10^{-1}$
$i=3$	$1.847439390850734 \cdot 10^{-1}$
$i=4$	$1.385570207701869 \cdot 10^{-1}$
...	...
$i=100$	$5.542286740277723 \cdot 10^{-3}$
...	...
$i=10^7$	$5.542290840699581 \cdot 10^{-8}$
...	...

(iii)

We set $(\alpha_n)_{n\in\mathbb{N}} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}} = \left(\frac{9n}{10n+1}\right)_{n\in\mathbb{N}}$.

We consider the sequence $(x_{n+1})_{n\in\mathbb{N}}$ *where* $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ *and*

$$x_{n+1} = \frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right)\left[x_n - \frac{1}{3}\left(\frac{9n}{10n+1}\right)x_n^3\right]. \tag{4.45}$$

From remark 4.1.2, *we know that the algorithm (4.45) converges to* $p_0 = \{0_{\ell^2}\} = P_{Fix(T)\cap Fix(S)}u$.

i	$\|x_i\|$
$i=1$	$\frac{1}{2}$
$i=2$	$2.643725028211783 \cdot 10^{-1}$
$i=3$	$2.258999829506434 \cdot 10^{-1}$
$i=4$	$2.182873807654092 \cdot 10^{-1}$
...	...
$i=100$	$1.616140813223594 \cdot 10^{-1}$
...	...
$i=10^7$	$4.350910515547696 \cdot 10^{-3}$
...	...

Remark 4.1.6. We recall that we have studied two cases:

- in the first one we choose x_1, u in $\left\{(x_i) \in B^+_{\ell^2} : \quad \|x\| > \frac{1}{2}\right\}$;
- in the second case we set $x_1, u \in B^+[0, \frac{1}{2}]$.

In the Example, in Case 1 and Case 2, we have showed that choosing different coefficients $(\beta_n)_{n\in\mathbb{N}}$ our algorithm converges to the fixed point of T in (i), to the fixed point of S in (ii) and to common fixed point of two mappings in (iii). In the Case 1, using Matlab Code we have obtained that the speed of convergence of our iterative scheme to $\{0_{\ell^2}\}$ in (i) is faster than that one in (iii).
Moreover, in (ii) the sequence goes fast to $\widehat{p} = P_{Fix(S)}u = (0, \frac{1}{2}, 0, ...)$. If we compare the Case 1 and Case 2 of the previous Example, we observe that in (iii) of Case 2 the sequence (x_n) converges to $\{0_{\ell^2}\}$ faster than the sequence (x_n), which also converges to $\{0_{\ell^2}\}$, considered in (iii) of Case 1.

Next, we give the Matlab code for the example 4.1.5, i).

format long e
$n = 10000000;$
$x = zeros(n, 2);$
$x(1, :) = [1, 0];$
$E = ones(n, 1);$
$u = [0, 1];$
$for \quad i = 1 : n$

$$
\begin{aligned}
x(i+1,:) \;&=\; \left(\frac{1}{1+i}\right) * u \\
&\quad + \left(1-\frac{1}{i+1}\right) * \left(\left(1-\frac{1}{(i+1)^2}\right) * \left(x(i,:) - \frac{1}{3} * x(i,:).^3\right)\right. \\
&+\; \left.\frac{1}{(1+i)^2} * \left(\frac{2}{3} * x(i,:) + \frac{x(i,:)}{6 * E(i)}\right)\right);
\end{aligned}
$$

$E(i+1) = norm(x(i+1, :), 2);$
end
disp(' ')
disp('sequences')
disp(x)
disp('norm')
disp(E)

4.2 Conclusions

In light of the above facts, two algorithms have been proposed. Both schemes show the deep connections between approximation of fixed points for nonlinear mappings and approximation of solutions of variational inequalities using a Halpern's type method. In the first scheme, we obtain strong convergence for a Halpern's method in the case of a nonexpansive mapping and a strongly quasi-nonexpansive mapping, without using the regularizing properties of averaged type mappings. In the second Halpern's type method, we consider averaged type mappings of two quasi-nonexpansive mappings and we get same convergence results, under same assumptions on coefficients. Moreover, MATLAB programming language has been used to obtain computational results presented in the example to illustrate the convergence of the iterative scheme to different fixed points, choosing suitable coefficients in every case.

Chapter 5

Hybrid iterative algorithms for a finite family of nonexpansive mappings and for a countable family of nonspreading mappings in Hilbert spaces

In this Chapter, the aim is to improve and extend results showed by Iemoto and Takahashi [31]. They proved a weak convergence theorem to approximate common fixed points of nonexpansive mapping T and nonspreading mapping S in a Hilbert space. Motivated by the research in this direction, we propose a modification of Iemoto and Takahashi's iterative method (2.5), introducing two hybrid iterative algorithms, to obtain strong convergence instead of weak convergence. Further, the next results hold for a finite family of nonexpansive mappings $(S_j)_{j=1}^N : C \to C$ and for a countable family of nonspreading mappings $(T_i)_{i=1}^\infty : C \to C$. Precisely, we will define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_i x_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \le \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ are sequence in $[0,1]$, W_n is generated by $S_1, ..., S_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, as in the scheme (5.1). The second hybrid algorithm is defined as follows:

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + (1-\beta_n)T_n x_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ are sequence in $[0,1]$, W_n is generated by $S_1, ..., S_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, as in the scheme (5.1).

5.1 Introduction

Next, some preliminary facts are described.

The problem of finding a point in the intersection of a convex set $C = \cap_{i\in I} C_i$ is of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such point. It is well known that, in the setting of Hilbert spaces using the metric projection this problem becomes equivalent approximating a common fixed point of the family of firmly nonexpansive mappings $\{P_{C_i} : \quad i \in I\}$.

To our knowledge there are some different kinds of approach to this approximation problem. In 1981 Kuhfittig [39] starting with a finite family of nonexpansive mappings, introduced a procedure which employs the convex combinations between each map with the identity map using a fixed appropriate coefficient. The resulting mapping is a nonexpansive and its fixed points coincide with the common fixed points of the family. This approach has opened a wide area of research followed by many mathematicians that study iterative methods approximating common fixed points of nonlinear mappings. It is very interesting and quite significant to establish the strong convergence or weak convergence results on the iteration schemes for finding a common fixed point of finite family of nonexpansive mappings $T_1, T_2, ..., T_N$ of C into itself. Our trick to obtain convergence results is to use the concept of W_n-mappings, [64], [65]. Indeed, it is one of the main tools in studying convergence of iterative methods to approach a common fixed point of nonlinear mappings, see [5], [19], [35], [72]. Atsushiba and Takahashi, [5], defined a mapping W_n as follows:

Definition 5.1.1. Let C be a nonempty convex subset of a Hilbert space and let $\lambda_{n,1}, ..., \lambda_{n,N} \in (0,1], n \in \mathbb{N}$. Let $(T_i)_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself. We define the mappings $U_{n,1}, ..., U_{n,N}$, for each n, by

$$\begin{aligned} U_{n,1} &:= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\ U_{n,2} &:= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})I, \\ &. \\ &. \end{aligned}$$

.

$$\begin{aligned} U_{n,N-1} &:= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n &:= U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1-\lambda_{n,N})I. \end{aligned} \tag{5.1}$$

Such a mapping W_n is called the W_n-mapping generated by $T_1, ..., T_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$.

The next properties of W_n and W will be crucial in the proof of the Theorems (5.2.1) and (5.2.2).

Lemma 5.1.2. *([5])Let C be a nonempty closed convex set of strictly convex Banach space. Let $T_1, ..., T_N$ be nonexpansive mappings of C into itself and let $\lambda_1, ..., \lambda_N$ be a real numbers such that $0 \leq \lambda \leq 1$ for every $i = 1, ..., N$. Let W be the W-mapping, i.e., the mapping of C into itself generated, following the previous scheme* (5.1)*, by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then, $U_1, ..., U_{N-1}$ e W are also nonexpansive.*

Lemma 5.1.3. *([5])Let C be a nonempty closed convex set of strictly convex Banach space. Let $T_1, ..., T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be a real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let W be the $W-$mapping of C into itself generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(W) = \bigcap\limits_{i=1}^{N} Fix(T_i)$.*

Lemma 5.1.4. *([19])Let C be a nonempty closed convex set of strictly convex Banach space. Let $T_1, ..., T_N$ be nonexpansive mappings of C into itself and $(\lambda_{n,i})_{i=1}^N$ be a sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i (i = 1, ..., N)$. Moreover for every $n \in N$, let W and W_n, generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$ and $T_1, ..., T_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, respectively. Then for every $x \in C$, it follows that*

$$\lim_{n\to\infty} \|W_n x - Wx\| = 0.$$

Convergence theorems have also been obtained for countable infinite families of nonexpansive mappings. In the paper [3] for approximating a common fixed points for a countable infinite family of nonexpansive maps was introduced the concept of $AKTT$-condition. Let E be a Banach space and K be a nonempty closed convex subset of E. Let $(T_n)_{n\in\mathbb{N}} : K \to K$ be a family of mappings. Then $(T_n)_{n\in\mathbb{N}}$ is said to satisfy the , see [3], if for each bounded subset B of K, one has

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : \quad z \in B\} < \infty. \tag{5.2}$$

The following is an important result on a family of mappings $(T_n)_{n\in\mathbb{N}}$ satisfying the $AKTT$-condition.

Lemma 5.1.5. *Let K be a nonempty and closed subset of Banach space E, and let $(T_n)_{n\in\mathbb{N}}$ be a family of mappings of K into itself which satisfies the $AKTT$-condition* (5.2)*. Then, for each $x \in K$, $(T_n x)_{n\in\mathbb{N}}$ converges strongly to a point in K. Moreover, let the mapping $T : K \to K$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in K,$$

then, for each bounded subset B of K,

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : \quad z \in B\} = 0.$$

If a family of mappings $(T_n)_{n\in\mathbb{N}}$ satisfy the $AKTT$-condition, then it is unnecessary that $Fix(T) = \bigcap_{i=1}^{\infty} Fix(T_i)$.

Example 5.1.6. *[54] Let $E = \mathbb{R}$ and $K = [0, 2]$. Let $(T_n)_{n\in\mathbb{N}} : K \to K$ a family of mapping defined by*

$$T_1 x = 0, \quad T_n x = \frac{1}{n}(1 + x), \quad n \geq 2.$$

We compute

$$\begin{aligned}
\sum_{n=1}^{\infty} \sup\{|T_{n+1}x - T_n x| : \quad x \in K\} &= \sup\{|T_2 x - T_1 x| : \quad x \in K\} \\
&+ \sum_{n=2}^{\infty} \sup\{|T_{n+1}x - T_n x| : \quad x \in K\} \\
&= \sup\{\frac{1}{2}(1 + x) : \quad x \in K\} \\
&+ \sum_{n=2}^{\infty} \sup\{|\frac{-1}{n(n+1)}(1 + x)| : \quad x \in K\} \\
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{3}{n(n+1)} < \infty
\end{aligned}$$

Then $(T_n)_{n\in\mathbb{N}}$ satisfy the $AKTT$-condition (5.2).
For each $x \in K$, $\lim_{n\to\infty} T_n x = 0$.
If we define $T : K \to K$ as follows: $Tx = \lim_{n\to\infty} T_n x$, we get $Tx = 0$, for all $x \in K$.
We conclude $Fix(T) \neq \bigcap_{n=1}^{\infty} Fix(T_n)$.

Definition 5.1.7. [54] $((T_n)_{n\in\mathbb{N}}, T)$ satisfy the $AKTT$-condition (5.2) if $(T_n)_{n\in\mathbb{N}}$ satisfy the $AKTT$-condition with $Fix(T) = \bigcap_{i=1}^{\infty} Fix(T_i)$.

In general, one can not apply (5.2) directly for a sequence of nonexpansive mappings. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. However, in [3] the authors generated a sequence $(T_n)_{n\in\mathbb{N}}$ of nonexpansive mappings, satisfying condition (5.2), by using convex combination of a general sequence $(S_k)_{k\in\mathbb{N}}$ of nonexpansive mappings with a common fixed points, as follows

$$T_n x = \sum_{k=1}^{n} \beta_n^k S_k x, \quad for \quad x \in C, n \in \mathbb{N},$$

where (β_n^k) be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ satisfying some conditions. Their idea was a consequence of the Theorem proved by Bruck [11].

Theorem 5.1.8. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $(S_k)_{k\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into E and $(\beta_k)_{k\in\mathbb{N}}$ a sequence of positive real numbers such that $\sum_{k=1}^{\infty}\beta_k = 1$. If $\cap_{k=1}^{\infty}Fix(S_k) \neq \emptyset$, then the mapping $T = \sum_{k=1}^{\infty}\beta_k S_k$ is well defined and $Fix(T) = \cap_{k=1}^{\infty}Fix(S_k)$.*

5.2 Results: two hybrid iterative schemes

In this Section the purpose will be to give strong convergence results for two hybrid iterative algorithms. Note that the algorithms are called the hybrid algorithms, due to the fact that each iterate x_{n+1} is obtained by projecting x onto the intersection of the suitably constructed closed convex sets C_n. We suggest and analyze convergence of algorithms which improve and generalize the Iemoto and Takahashi's Theorem, in fact, in [54],

- we obtained strong convergence instead of weak convergence (improvement);
- we considered a finite family of nonexpansive mappings and for a countable family of nonspreading mappings (generalization).

We shall show the strong convergence of the first algorithm.

Theorem 5.2.1. *[54] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(S_j)_{j=1}^{N} : C \to C$ be a finite family of nonexpansive mappings and let $(T_i)_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $F = [\bigcap_{j=1}^{N} Fix(S_j)] \cap [\bigcap_{i=1}^{\infty} Fix(T_i)] \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ are sequences in $[0,1]$, W_n is generated by $S_1, ..., S_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, as in the scheme (5.1). Assume that β_n is strictly decreasing and $\beta_0 = 1$. Then, the following hold:

i) If $\liminf_{n\to\infty}\alpha_n > 0$ and $\lim_{n\to\infty}\beta_n = 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$;

ii) If $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in F$.

Proof. Since C_n is closed and convex, hence D_n is closed and convex. In order to show that $F \subset D_n$ for all $n \in \mathbb{N}$, we will prove that $F \subset C_n$ for all $n \in \mathbb{N}$. Note that $\beta_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i) = 1$. By the nonexpansivity of W_n and T_i, for $i = 1, ..., N$, we have, for each $p \in F$,

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n] \\
&- \alpha_n(\beta_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i))p + \alpha_n p - p\| \\
&= \|(1-\alpha_n)(x_n - p) + \alpha_n[\beta_n(W_n x_n - p) + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - p)]\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|\beta_n(W_n x_n - p) + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - p)\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[\beta_n\|W_n x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - p\|] \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[\beta_n\|x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|x_n - p\|] \\
&= \|x_n - p\|. \qquad (5.3)
\end{aligned}$$

This implies that $p \in C_n, \quad \forall n \in \mathbb{N}$. Hence, C_n is nonempty for all $n \in \mathbb{N}$. This implies that $(x_n)_{n\in\mathbb{N}}$ is well-defined From $x_{n+1} = P_{D_n}x$, we have

$$\|x_{n+1} - x\| \leq \|v - x\|, \quad \forall v \in D_n, \quad \forall n \in \mathbb{N}.$$

Since $P_F x \in F \subset D_n$,

$$\|x_{n+1} - x\| \leq \|P_F x - x\|, \quad \forall n \in \mathbb{N}. \qquad (5.4)$$

Consequently, $(x_n)_{n\in\mathbb{N}}$ is bounded and from (5.3), also, $(y_n)_{n\in\mathbb{N}}$ is bounded.
Since $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$,

$$x_{n+2} = P_{D_{n+1}}x \in D_{n+1} \subset D_n, \quad \forall n \in \mathbb{N}.$$

This implies that

$$\|x_{n+1} - x\| \leq \|x_{n+2} - x\|, \quad \forall n \in \mathbb{N}. \qquad (5.5)$$

It follows from (5.4) and (5.5) that the limit of $(x_n - x)_{n\in\mathbb{N}}$ exists.
From $x_{m+1} = P_{D_m}x \in D_m \subset D_n$ for all $m \geq n$ and by Lemma (1.0.2), we deduce

$$\langle x_{n+1} - x, x_{m+1} - x_{n+1}\rangle \geq 0, \quad \forall m \geq n. \qquad (5.6)$$

From (5.6), we derive

$$\|x_{m+1} - x_{n+1}\|^2$$

$$
\begin{aligned}
&= \|x_{m+1} - x - (x_{n+1} - x)\|^2 \\
&= \|x_{m+1} - x\|^2 + \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x\rangle \\
&= \|x_{m+1} - x\|^2 + \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x_{n+1} + x_{n+1} - x\rangle \\
&= \|x_{m+1} - x\|^2 - \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x_{n+1}\rangle \\
&\le \|x_{m+1} - x\|^2 - \|x_{n+1} - x\|^2. \qquad (5.7)
\end{aligned}
$$

Since the limit of $\|x_n - x\|$ exists, we have

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0.$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $q \in C$ such that

$$x_n \to q, \quad as \quad n \to \infty. \qquad (5.8)$$

Putting $m = n + 1$ in (5.7),

$$\lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = 0,$$

i.e.,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (5.9)$$

From the fact that $x_{n+1} = P_{D_n}x \in D_n \subset C_n$, we have

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| \to 0, \quad as \quad n \to \infty,$$

and we derive

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad as \quad n \to \infty. \qquad (5.10)$$

It yields, from (5.8) and (5.10), that

$$\lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|y_n - x_n + x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = \|q - p\|, \quad \forall p \in F. \qquad (5.11)$$

We prove (i). We compute

$$
\begin{aligned}
y_n &= (1 - \alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n] \pm \alpha_n(1 - \beta_n)x_n \\
&= (1 - \alpha_n)x_n + \alpha_n[\beta_n W_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n)] + \alpha_n(1 - \beta_n)x_n \\
&= (1 - \alpha_n\beta_n)x_n + \alpha_n\beta_n W_n x_n + \alpha_n \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n),
\end{aligned}
$$

we deduce

$$
\begin{aligned}
\alpha_n \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n) &= y_n - (1 - \alpha_n\beta_n)x_n - \alpha_n\beta_n W_n x_n \qquad (5.12) \\
&= (1 - \alpha_n\beta_n)(y_n - x_n) + \alpha_n\beta_n(y_n - W_n x_n).
\end{aligned}
$$

For all $p \in \bigcap_{i=1}^{N} Fix(T_i)$, by the quasi-nonexpansivity of T_i, for $i = 1, ..., N$,

$$\begin{aligned} \|x_n - p\|^2 &\geq \|T_i x_n - T_i p\|^2 = \|T_i x_n - p\|^2 = \|T_i x_n - x_n + (x_n - p)\|^2 \\ &= \|T_i x_n - x_n\|^2 + \|x_n - p\|^2 + 2\langle T_i x_n - x_n, x_n - p\rangle, \end{aligned}$$

and hence, we have

$$\|T_i x_n - x_n\|^2 \leq 2\langle x_n - T_i x_n, x_n - p\rangle, \quad \forall i \in \mathbb{N}. \tag{5.13}$$

Since $(\beta_n)_{n\in\mathbb{N}}$ is strictly decreasing and from (5.13) and (5.12), we obtain

$$\begin{aligned} \alpha_n(\beta_{i-1} - \beta_i)\|T_i x_n - x_n\|^2 &\leq \alpha_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\|T_i x_n - x_n\|^2 \\ &\leq 2\alpha_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\langle x_n - T_i x_n, x_n - p\rangle \\ &= 2\langle (1 - \alpha_n\beta_n)(y_n - x_n) + \alpha_n\beta_n(y_n - W_n x_n), x_n - p\rangle, \end{aligned}$$

so,

$$\begin{aligned} \|T_i x_n - x_n\|^2 &\leq \frac{2}{\alpha_n(\beta_{i-1} - \beta_i)}[(1 - \alpha_n\beta_n)\langle y_n - x_n, x_n - T_i p\rangle \\ &+ \alpha_n\beta_n\langle y_n - W_n x_n, x_n - p\rangle], \quad \forall i \in \mathbb{N}. \end{aligned} \tag{5.14}$$

Since $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\lim_{n\to\infty} \beta_n = 0$, it follows, from (5.10) and (5.14),

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{5.15}$$

From (5.8) and (5.15) and since $(T_i)_{i=1}^{\infty}$ are nonspreading,

$$\|T_i q - T_i x_n\|^2 \leq \|x_n - q\|^2 + 2\langle q - T_i q, x_n - T_i x_n\rangle, \quad \forall i \in \mathbb{N},$$

it implies that

$$\lim_{n\to\infty} \|T_i q - T_i x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{5.16}$$

Hence, we conclude

$$0 \leq \|q - T_i q\| \leq \|q - x_n\| + \|x_n - T_i x_n\| + \|T_i x_n - T_i q\|, \quad \forall i \in \mathbb{N}, \tag{5.17}$$

i.e. $\|q - T_i q\| = 0$. We can conclude that

$$q \in \bigcap_{i=1}^{\infty} Fix(T_i).$$

Next, we prove (ii). We need to show that

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0.$$

For every p, using again $\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)+\beta_n=1$, we compute

$$\begin{aligned}
\|y_n-p\|^2 &= \|(1-\alpha_n\beta_n)x_n+\alpha_n\beta_n W_n x_n+\alpha_n\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)(T_i x_n-x_n)\\
&- (\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)+\beta_n)p\|^2\\
&= \|(\beta_n+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)-\alpha_n\beta_n)x_n+\beta_n(\alpha_n W_n x_n-p)\\
&+ \alpha_n\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)(T_i x_n-x_n)-\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)p\|^2\\
&= \|\beta_n[(1-\alpha_n)x_n+\alpha_n W_n x_n-p]+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)[(1-\alpha_n)x_n+\alpha_n T_i x_n-p]\|^2\\
&\leq \beta_n\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|[(1-\alpha_n)x_n+\alpha_n T_i x_n\\
&- (1-\alpha_n+\alpha_n)p]\|^2\\
&\leq \beta_n\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)[(1-\alpha_n)\|x_n-p\|^2\\
&+ \alpha_n\|T_i x_n-p\|^2]\\
&\leq \beta_n\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|x_n-p\|^2\\
&\leq \beta_n\|(1-\alpha_n)x_n+\alpha_n W_n x_n-(1-\alpha_n+\alpha_n)p\|^2+(1-\beta_n)\|x_n-p\|^2 \qquad (5.18)\\
&\leq \beta_n(1-\alpha_n)\|x_n-p\|^2+\alpha_n\beta_n\|W_n x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2 \qquad (5.19)\\
&\leq \|x_n-p\|^2.
\end{aligned}$$

Since, by (5.18), we have

$$\|y_n-p\|^2-\|x_n-p\|^2\leq \beta_n\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2-\beta_n\|x_n-p\|^2,$$

then, from (5.19), we obtain

$$\begin{aligned}
0 &\leq \|x_n-p\|^2-\beta_n\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2-(1-\beta_n)\|x_n-p\|^2\\
&= \beta_n[\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2] \qquad (5.20)\\
&\leq \|x_n-p\|^2-\|y_n-p\|^2\to 0.
\end{aligned}$$

Since $\liminf_{n\to\infty}\beta_n>0$, we get, from (5.20),

$$\lim_{n\to\infty}(\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_n W_n x_n-p\|^2)=0. \qquad (5.21)$$

By (5.21) and (5.14)

$$\|(1-\alpha_n)x_n+\alpha_nW_nx_n-p\|^2=(1-\alpha_n)\|x_n-p\|^2+\alpha_n\|W_nx_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-W_nx_n\|^2,$$

we conclude

$$\begin{aligned}\alpha_n(1-\alpha_n)\|x_n-W_nx_n\|^2 &= (\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nW_nx_n-p\|^2)\\ &- \alpha_n\|x_n-p\|^2+\alpha_n\|W_nx_n-p\|^2\\ &\leq (\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nW_nx_n-p\|^2)\\ &- \alpha_n\|x_n-p\|^2+\alpha_n\|x_n-p\|^2\\ &= \|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nW_nx_n-p\|^2\to 0.\end{aligned}$$

Since $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we have

$$\lim_{n\to\infty}\|x_n-W_nx_n\|=0. \tag{5.22}$$

Since $\|y_n-x_n\|\to 0$, as $n\to\infty$, and for (5.22), we have

$$\lim_{n\to\infty}\|y_n-W_nx_n\|=0.$$

Since $x_n\to q\in\bigcap_{i=1}^{\infty}Fix(T_i)$, as $n\to\infty$, and for (5.22), we derive

$$\|q-W_nq\|\leq\|q-x_n\|+\|x_n-W_nx_n\|+\|W_nx_n-W_nq\|\leq 2\|q-x_n\|+\|x_n-W_nx_n\|,$$

which implies that

$$q\in Fix(W_n),$$

and we conclude that $q\in F$. □

Next, we propose the second strong convergence result [54], under the assumption that $((T_n)_{n\in\mathbb{N}},T)$ satisfies the $AKTT$-condition.

Theorem 5.2.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $(S_j)_{j=1}^N : C\to C$ be finite family of nonexpansive mappings and let $(T_i)_{i=1}^{\infty}: C\to C$ be a countable family of nonspreading mappings such that $F=[\bigcap_{j=1}^{N}Fix(S_j)]\cap[\bigcap_{i=1}^{\infty}Fix(T_i)]\neq\emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1=x\in C \quad chosen \quad arbitrarily,\\ y_n=(1-\alpha_n)x_n+\alpha_n[\beta_nW_nx_n+(1-\beta_n)T_nx_n],\\ C_n=\{v\in C: \quad \|y_n-v\|\leq\|x_n-v\|\},\\ D_n=\bigcap_{j=1}^{n}C_j,\\ x_{n+1}=P_{D_n}x, \quad n\in\mathbb{N},\end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}},(\beta_n)_{n\in\mathbb{N}}\subset[0,1]$, W_n is generated by $S_1,...,S_N$ and $\lambda_{n,1},...,\lambda_{n,N}$, as in the scheme (5.1). Assume that either $((T_n)_{n\in\mathbb{N}},T)$ satisfies the $AKTT$-condition. Then the following hold:

i) If $\liminf_{n\to\infty} \alpha_n > 0$ *and* $\lim_{n\to\infty} \beta_n = 0$*, then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$*;*

ii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n > 0$*, then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $q \in F$*.*

Proof. Following the same steps in the proof of Theorem (5.2.1), we get that $(x_n)_{n\in\mathbb{N}}$ is bounded, converges strongly to some $q \in C$ and

$$\|x_n - y_n\| \to 0, \quad as \quad n \to \infty.$$

We first prove (i). We compute

$$y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + (1-\beta_n)T_n x_n] \pm \alpha_n\beta_n x_n,$$

$$y_n - x_n = \alpha_n(1-\beta_n)(T_n x_n - x_n) + \alpha_n\beta_n(W_n x_n - x_n),$$

$$T_n x_n - x_n = \frac{1}{\alpha_n(1-\beta_n)}(y_n - x_n) - \frac{\beta_n}{1-\beta_n}(W_n x_n - x_n)$$

hence

$$\|T_n x_n - x_n\| \le \frac{1}{\alpha_n(1-\beta_n)}\|y_n - x_n\| + \frac{\beta_n}{1-\beta_n}\|W_n x_n - x_n\|.$$

Since $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, we obtain

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{5.23}$$

Furthermore, by lemma (5.1.5) and (5.23) we have

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\le \|x_n - T_n x_n\| + \sup\{\|T_n z - Tz\| : z \in (x_n)_{n\in\mathbb{N}}\},\end{aligned}$$

and we derive

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{5.24}$$

Since each T_n is a nonspreading mapping, lemma (1.0.41) shows that T is a nonspreading mapping verifying the following inequality

$$\|Tq - Tx_n\|^2 \le \|x_n - q\|^2 + 2\langle q - Tq, x_n - Tx_n\rangle, \quad \forall n \in \mathbb{N},$$

consequently

$$\lim_{n\to\infty} \|Tq - Tx_n\| = 0. \tag{5.25}$$

From (5.23) and (5.25) we get

$$\|q - Tq\| \le \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\|, \tag{5.26}$$

this shows that $q \in Fix(T)$. Since $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the $AKTT$-condition, it yields that $q \in \bigcap_{i=1}^{\infty} Fix(T_i) = Fix(T)$. This completes (i).

Next, we show (ii). Using a similar process as in the proof of Theorem (5.2.1) and by (5.24),(5.25),(5.26), we have that

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0, \ \lim_{n\to\infty} \|x_n - T_n x_n\| = 0,$$

$$\lim_{n\to\infty} \|Tx_n - Tq\| = 0, \ \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Finally, from

$$\|q - W_n q\| \le \|q - x_n\| + \|x_n - W_n x_n\| + \|W_n x_n - W_n q\| \le 2\|x_n - q\| + \|x_n - W_n x_n\|,$$

we get

$$\lim_{n\to\infty} \|q - W_n\| = 0$$

i.e. $q \in Fix(W_n)$. Moreover,

$$\|q - Tq\| \le \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\|,$$

where the last terms go to zero as n go to infinity, so $q \in Fix(T)$. Whereas $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the $AKTT$-condition, $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$. Then, we conclude that $q \in F$. We conclude that (ii) holds. □

Next, as consequence we have some Corollaries. Putting $T_i = T$ for all $i \in \mathbb{N}$ in Theorem (5.2.1) and Theorem (5.2.2), we obtain the following.

Corollary 5.2.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(S_j)_{j=1}^N : C \to C$ be a finite family of nonexpansive mappings and let $T : C \to C$ be a nonspreading mapping such that $F = [\bigcap_{j=1}^{N} Fix(S_j)] \cap Fix(T) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n W_n x_n + (1-\beta_n)Tx_n], \\ C_n = \{v \in C : \quad \|y_n - v\| \le \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n} x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$, W_n is generated by $S_1, ..., S_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$, as in the scheme (5.1). Then, the following hold:

i) If $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in Fix(T)$;

ii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in F$.

Putting $S_j = I$, for $j = 1, ..., N$, in Theorem (5.2.1) and Theorem (5.2.2), we have the following.

Corollary 5.2.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(T_i)_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1 - \alpha_n(1 - \beta_n))x_n + \alpha_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i x_n, \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n} x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0, 1]$. Assume that β_n is strictly decreasing and $\beta_0 = 1$. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

Corollary 5.2.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $(T_i)_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in C \quad chosen \quad arbitrarily, \\ y_n = (1 - \alpha_n(1 - \beta_n))x_n + \alpha_n(1 - \beta_n) T_n x_n, \\ C_n = \{v \in C : \quad \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n} x, \quad n \in \mathbb{N}, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0, 1]$. Assume that $((T_n)_{n\in\mathbb{N}}, T)$ satisfies the AKTT-condition. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $q \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

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