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Surfaces of general type with $K^2 = 3$, $p_g = 2$, whose canonical system induces a genus 2 fibration

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Introduction

This thesis studies a family of surfaces with $p_g = 2, K^2 = 3$, whose canonical system induces a genus 2 fibration.

Surfaces with these invariants verify the equality $K^2 = 2\chi - 3$; surfaces with $K^2 = 2\chi - 2$ and nontrivial torsion were studied by C. Ciliberto and M. Mendes Lopes, while M. Murakami studied surfaces with $K^2 = 2\chi - 1$, finding a bound for the order of the possible torsion groups and recently he studied the case when the torsion group is $\mathbb{Z}_2$. The existence of surfaces with $p_g = 2$, $K^2 = 3$ has been proved by U. Persson, while M. Reid determined the canonical ring of surfaces with these invariants, that are universal cover of Godeaux surfaces with torsion group $\mathbb{Z}_3$; the surfaces studied by Reid have a birational bicanonical map.

Chapter 1 is devoted to recalling well known results about surfaces of general type, canonical maps and double coverings used in the following of the thesis.

Chapter 2 is devoted to general results about surfaces with our invariants; in particular we start by partitioning the class of minimal surfaces with $p_g = 2$, $K^2 = 3$ in three families as regards the fixed and mobile parts of the canonical system. Named $|M|$ the free component and Z the fixed component of $|K|$, we obtain the following possibilities:

(I) $Z \neq 0$, $M^2 = 0$, $MZ = 2$; these equalities imply that $|M|$ is a free pencil with general element of genus 2;

(II) $Z \neq 0$, $M^2 = 1$, $MZ = 2$; in this case the general curve C in $|M|$ has genus 3;

(III) $Z = 0$, so the general C in $|M|$ is a smooth curve of genus 4.

We prove the triviality of the torsion group and determine the direct image of the relative canonical system.

In chapter 3 our attention focuses on the first case. Since the canonical map of a curve of genus 2 is a double cover of $\mathbb{P}^1$ branched in 6 points, we use the geometric approach developed by Horikawa, that studied genus 2 fibrations $f : S \to B$ through a finite double cover $\phi : Y \to \mathbb{P}$ of a $\mathbb{P}^1$ bundle $\mathbb{P}$ over B, where Y is birational to S.

We determine the branch locus of ϕ (in our case $\mathbb{P}$ is the Segre-Hirzebruch surface Σ_2), proving its existence. The principal result of the thesis is the following theorem:

Theorem. *Let S be a surface with $K^2 = 3$, $p_g = 2$. If we call $|M|$ the mobile part of the canonical system and Z the fixed part, if $M^2 = 0$ and $MZ = 2$ then S is birationally equivalent to a double cover of Σ_2. The branch locus B is linearly equivalent to $6\Delta_\infty + 14\Gamma$ or $6\Delta_\infty + 16\Gamma$; in the first case B has the following form:*

1a. B contains a fibre Γ and $B_0 = B - \Gamma$ has two triple points in Γ; each of these points has 2 or 3 infinitely near triple points;

2a. B contains a fibre Γ and $B_0 = B - \Gamma$ has a triple point in Γ, that has 5 or 6 infinitely near triple points;

3a. B contains a fibre Γ and $B_0 = B - \Gamma$ has two simple or 2- fold triple points (that may be infinitely near) in Γ and two 2-fold or 3-fold triple points (that may be infinitely near) in another fibre.

4a. B contains a fibre Γ and $B_0 = B - \Gamma$ has two 2-fold or 3-fold triple points (that may be infinitely near) and a quadruple point p on Γ that, after a blowing up in p, results in a double point on the proper transform of Γ, in another fibre.

If B is linearly equivalent to $6\Delta_\infty + 16\Gamma$, then it has the following form:

1b. *B contains three fibres Γ_1, Γ_2 and Γ_3 and $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has two triple points (that may be infinitely near) on each fibre;*

2b. *B contains three fibres Γ_1, Γ_2 and Γ_3 and $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has two triple points (that may be infinitely near) on Γ_1, two triple points (that may be infinitely near) on Γ_2, a quadruple point on Γ_3, that, after a blowing up, results in a double point on the proper transform of the fibre.*

3b. *B contains three fibres Γ_1, Γ_2 and Γ_3 and $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has a quadruple point on Γ_1, a quadruple point on Γ_2 and two triple points (that may be infinitely near) on Γ_3; after a blowing up, each quadruple point results in a double point on the proper transform of the fibre;*

4b. *B contains three fibres Γ_1, Γ_2 and Γ_3 and on each fibre $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has a quadruple point that, after a blowing up, results in a double point on the proper transform of the fibre.*

Finally we observe that all the configurations of the branch locus can be obtained by degeneration from one of them and that the studied surfaces form an irreducible subvariety of a component of the moduli space, but they do not an irreducible component.

Chapter 1

Some general results

1.1 Basic facts

Let us recall some definitions and results we will use later on. By *surface* we will mean a smooth projective surface over the field $\mathbb{C}$ of complex numbers. The sheaf Ω^2_S of regular 2-forms on S will be named *canonical sheaf* and will be indicated with ω_S, while K_S will be a *canonical divisor*, i.e. a divisor such that $\mathcal{O}_S(K_S) = \omega_S$. By *curve* we shall mean an effective divisor on S.

Let C be a curve on S.

Definition. The *arithmetic genus* of C is $p_a(C) = 1 - \chi(\mathcal{O}_C)$.

As C is Gorenstein, there exists a canonical sheaf ω_C which is invertible; by adjunction

$$\omega_C = \omega_S \otimes_{\mathcal{O}_S} \mathcal{O}_S(C) \otimes_{\mathcal{O}_S} \mathcal{O}_C.$$

Since $h^0(C, \mathcal{O}_C) \geq 1$, $p_a(C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C) \leq h^1(\mathcal{O}_C)$.

Given an invertible sheaf $\mathscr{L}$ on a surface S, we define the Euler-Poincaré characteristic of the sheaf $\mathscr{L}$ as $\chi(\mathscr{L}) = h^0(\mathscr{L}) - h^1(\mathscr{L}) + h^2(\mathscr{L})$.

Two fundamental birational invariants of a surface S are the geometric genus of S, $p_g(S) := h^0(\omega_S)$, and the irregularity of S, $q(S) := h^1(\omega_S)$.

Theorem 1.1.1 (Serre duality). *Let X be a smooth projective variety of dimension n, and $\mathscr{L}$ an invertible sheaf on X. Then, for $0 \le i \le n$ there are natural isomorphisms*

$$H^i(X, \mathscr{L}) \cong H^{n-i}(S, \omega_S \otimes \mathscr{L}^{-1})^\vee.$$

In particular, $\chi(\mathscr{L}) = \chi(\omega_X \otimes \mathscr{L}^{-1})$.

Using Serre duality, we have that if S is a smooth projective surface, $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$.

Let C be a smooth curve, Ω^1_C the sheaf of regular 1-forms on C, then $\Omega^1_C = \omega_C$; we define the *geometric genus* $g(C)$ as $\dim H^0(\Omega^1_C)$. By Serre duality, $H^1(\mathcal{O}_C) = H^0(\Omega^1_C)^\vee$, so $p_a(C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C) = h^1(\mathcal{O}_C) = h^0(\omega^1_C) = g(C)$.

Theorem 1.1.2 (Riemann-Roch on a surface). *Let S be a surface, let D be a divisor on S. Then*

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{D(D-K)}{2}.$$

Theorem 1.1.3 (Noether's formula). *Let S be a surface. Then*

$$\chi(\mathcal{O}_S) = \frac{K_S^2 + \chi_{top}(S)}{12},$$

where $\chi_{top}(S) = \sum_i^4 (-1)^i \dim H^i(S, \mathbb{C})$.

Theorem 1.1.4 (Riemann-Roch on a divisor D on S). *Let $\mathscr{L}$ be an invertible sheaf on a divisor D on a surface S. Then*

$$\chi(\mathcal{O}_D) = -\frac{1}{2}(D^2 + DK_S),$$

$$\chi(\mathcal{O}_D(\mathscr{L})) = \chi(\mathcal{O}_D) + \deg_D \mathscr{L}.$$

Since $\chi(\mathcal{O}_D) = 1 - p_a(D)$, we have

Theorem 1.1.5 (Genus formula). *Let S be a surface and let $D \subset S$ be a curve. Then*

$$p_a(D) = 1 + \frac{D(D+K)}{2}.$$

For any $n > 0$ we define the n-th *plurigenus* of a smooth surface S to be $P_m(S) := h^0(\omega_S^{\otimes m})$.

Kodaira dimension $\kappa(S)$ is defined as the transcendence degree over $\mathbb{C}$ of the graded ring

$$R(S) := \bigoplus_{n\geq 0} H^0(S, \omega_S^{\otimes n})$$

minus 1. $R(S)$, which is called *canonical ring*, and $\kappa(S)$ are birational invariants. $\kappa(S)$ can be expressed as the largest dimension of the image of S in $\mathbb{P}^N$ by the rational map given by the linear system $|nK_S|$ for some $n \geq 1$, or $\kappa(S) = -1$ if $|nK_S| = \emptyset$ for all $n \geq 1$. S is called of general type if $\kappa(S) = 2$.

Theorem 1.1.6 (Hodge Index Theorem). *Let H be an ample divisor on the surface X, and suppose that D is a divisor not numerically equivalent to zero, $D \not\equiv 0$, with $D \cdot H = 0$. Then $D^2 < 0$.*

Definition. A smooth surface S is called *minimal* (or a *minimal model*) if it does not contain any exceptional curve E of the first kind (i.e. $E \cong \mathbb{P}^1, E^2 = -1$).

Every surface can be obtained by a minimal one (its minimal model) after a finite sequence of blowing ups of smooth points; moreover this model is unique if $\kappa(S) \geq 0$ (see [BHPVdV04]). Thus, every birational class of surfaces of general type contains exactly one minimal surface, and one classifies surfaces of general type by studying their minimal models.

The following results are true for minimal surfaces of general type:

- $K_S^2 > 0$;
- $\chi(\mathcal{O}_S) > 0$, then $p_g(S) \geq q(S)$;
- $P_m = \chi(\mathcal{O}) + \binom{m}{2} K^2$ for $m \geq 2$ (cf.[Bom73]);
- $K^2 \geq 2p_g - 4$ if K^2 is even,
  $K^2 \geq 2p_g - 3$ if K^2 is odd (Noether's inequality, cf. [BHPVdV04]);

- If S is an irregular surface, then $K_S^2 \geq 2p_g(S)$ (Debarre's inequality, cf. [Deb82]).

Further, for every surface of general type S the inequality $K^2 \leq 3\chi_{top}(\mathcal{O})$ holds; equivalently $K^2 \leq 9\chi(\mathcal{O})$ (Miyaoka's inequality).

Theorem 1.1.7 (Bombieri). *Let S be a surface of general type with $q(S) = 0$, let m be the order of its torsion group. Then*

$$p_g(S) \leq \frac{1}{2}K_S^2 + \frac{3}{m} - 1$$

and

$$p_g(S) \leq \frac{1}{2}K_S^2 - 1$$

if there exists a finite non ramified abelian cover of S with irregularity major then or equal to 1.

Proof. Cf. [Bom73]. □

1.2 n-canonical maps

An important tool for the study of the surfaces of general type are the n-canonical maps. We recall that the canonical ring of a surface S is defined as $R(S) = \bigoplus_{n=0}^{\infty} H^0(S, \mathcal{O}_S(nK_S))$; it is a graded ring, whose n-th graded part is $R_n(S) = H^0(S, \mathcal{O}_S(nK))$. It was proved by Mumford that $R(S)$ is a finitely generated noetherian ring ([Mum62]). Let us define $R^{[n]}(S) := \bigoplus_{m=0}^{\infty} R_n(S)^m, X(S) := \mathbb{P}(R(S))$, $X^{[n]}(S) = \mathbb{P}(R^{[n]}(S))$. $X(S)$ is the *canonical model* of S and $X^{[n]}(S)$ is the *n-canonical image* of S.

The inclusion $R^{[n]}(S) \to R(S)$ induces a rational map $\psi_n : X \to X^{[n]}$, and on the other hand we have also the rational map $\phi_n := \phi_{|nK|} : S \to \mathbb{P}(H^0(S, \mathcal{O}_S(nK))^*)$, whose image is $X^{[n]}(S)$. ψ_n and ϕ_n are called the *n-canonical map* of $X(S)$ and S, respectively.

The following theorems give us important results about the structure of the n-canonical map.

Theorem 1.2.1. *Let $p_g \geq 1$. Then*

- *if $n \geq 4$, ϕ_n is a morphism, i.e. $|nK|$ is base point free;*
- *ϕ_3 is a morphism unless $K^2 = 1$;*
- *ϕ_2 is a morphism, i.e. $|2K|$ is base point free.*

Proof. Cf. [Bom73] and [Cil97] □

Remark 1.2.2. There exist surfaces with $K^2 = 1$ and such that ϕ_3 is not a morphism (an example is due to Enriques).

Theorem 1.2.3. *Let $p_g \geq 1$. Then*

- *if $n \geq 5$ the map ψ_n is a morphism;*
- *if $K^2 \geq 2$ the map ψ_4 is an isomorphism;*
- *if $n \geq 3$, then the map ψ_n is birational unless $K^2 = 1$, $p_g = 2$, $n = 3, 4$ or $K^2 = 2$, $p_g = 3$, $n = 3$;*
- *if $K^2 \geq 10$ and $p_g \geq 6$ then ψ_2 is birational unless there is a morphism $f : S \to B$ where B is a smooth curve and the general fibre F of f is a smooth connected curves of genus two.*

Proof. Cf. [Bom73] □

By the theorems on pluricanonical maps, minimal surfaces S of general type with fixed invariants are birationally mapped to normal surfaces X in a fixed projective space of dimension $P_5(S) - 1$. X is uniquely determined, is called the *canonical model* of S and is obtained contracting to points all the -2-curves on S (curves $E \cong \mathbb{P}^1$, $E^2 = -2$).

When there is a morphism $f : S \to B$ where B is a smooth curve and the general fibre F of f is a smooth connected curves of genus two, we will say that this is the *standard case* for the non birationality of the bicanonical map.

The result *(iv)* in the precedent theorem can be refined. In fact:

Theorem 1.2.4. *If S is a minimal surface of general type with non-birational bicanonical map presenting the non-standard case, and if $p_g \geq 3$, then $p_g \leq 6$ and*

- *if $p_g \geq 4$, then $q = 0$ and either $p_g = 6$, $K^2 = 8, 9$ or $p_g = 5$, $K^2 = 7, 8$ or $p_g = 4$, $K^2 = 6, 7, 8$;*
- *if $p_g = 3$ and $q = 0$ then $5 \leq K^2 \leq 7$;*
- *if $p_g = 3$ and $q > 0$, then $q = 3$ and S is the symmetric product of a smooth curve of genus 3.*

Proof. Cf. [Cil97] □

Theorem 1.2.5. *Let S be a minimal surface of general type. Then $|2K_S|$ is not composed with a pencil (i.e. the image of S via its bicanonical map is a surface).*

Proof. Cf. [Xia85a]. □

1.3 Double coverings

We will briefly recall some results about double coverings of surfaces and the canonical resolution for singularities on double coverings.

Let X be a normal surface and Y a smooth surface, both connected, and let $f : X \to Y$ be a double covering, with branch locus $B \subset Y$ and ramification locus $R \subset X$. We have that $P \in \mathrm{Sing}(X) \Leftrightarrow f(P) \in \mathrm{Sing}(B)$, where $\mathrm{Sing}(X)$ (resp. $\mathrm{Sing}(B)$) is the set of the singular points in X (resp. B). We have also that $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}$, where $\mathcal{L}$ is such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$. Moreover, if X is smooth, $f^*B \cdot f^*C = 2B \cdot C$ and $K_X = f^*(K + F) = f^*(K_Y) + R$, where $\mathcal{O}_Y(F) = \mathcal{L}$; if C is an irreducible component of B, then $f^*C = 2D$ and $D^2 = \frac{1}{2}C^2$; if C is an irreducible rational curve, disjoint from B, then $f^*C = D_1 + D_2$ and $D_1^2 = D_2^2 = C^2$. There is a natural map $f_* : \mathrm{Div}\, X \to \mathrm{Div}\, Y$ which is defined on an irreducible reduced divisor D as follows:

$$f_*D = rf(D),$$

where $1 \leq r = \deg[\mathbb{C}(D) : \mathbb{C}(f(D))] \leq 2$. We have that $f^*D \cdot f^*D = 2D^2$, f_*f^* is multiplication by 2 in $\operatorname{Div} Y$ and $f^*f_*D = D + \sigma(D)$, where σ is the involution of X induced by $f : X \to Y$. Further, we have the projection formula $f_*(D \cdot f^*D) = f_*D \cdot F$. Note that $f_*(D)$ is a divisor, whereas $f_*\mathcal{O}_X(D)$ is a locally free sheaf of rank 2.

If $f : X \to Y$ is a double covering and X, Y are smooth we have the following formulas:

$$\begin{aligned}
\chi(\mathcal{O}_X) &= 2\chi(\mathcal{O}_Y) + \frac{1}{2}F(F + K_Y) \\
p_g(X) &= p_g(Y) + h^0(Y, K_Y \otimes F) \\
K_X^2 &= 2K_Y^2 + 4F \cdot K_Y + 2F^2 \\
\chi_{top}(X) &= 2\chi_{top}(Y) + 2F \cdot K_Y + 4F^2.
\end{aligned}$$

Let B have at most simple singularities, i.e. A-D-E singularities, that have the following explicit equations:

$$\begin{array}{llcll}
A_n & (n \geq 1) & : & z^2 + x^2 + y^{n+1} & = 0 \\
D_n & (n \geq 4) & : & z^2 + y(x^2 + y^{n-2}) & = 0 \\
E_6 & : & & z^2 + x^3 + y^4 & = 0 \\
E_7 & : & & z^2 + x(x^2 + y^3) & = 0 \\
E_8 & : & & z^2 + x^3 + y^5 & = 0,
\end{array}$$

(here the covering is $(z, x, y) \mapsto (x, y)$ and the singularity lies in the origin). Then the same formulas for χ, p_g, K_X^2, χ_{top} hold.

Now suppose B reduced and let $\{y_1, \ldots, y_r\}$ be the set of singular points in B. We want to describe the canonical resolution of Horikawa. If m_j is the multiplicity of $y_j \in B$, $\sigma_1 : Y_1 \to Y$ is the composition of the blowing ups applied in all singularities $y_1, \ldots, y_r$ in B and E_j is the exceptional curve over y_j, then $\sigma^*(B) = \overline{B} + \sum m_j E_j$ is the total transform of B. Let X_1 be the normalization of $X \times_Y Y_1$: X_1 is a double covering of Y_1, ramified over

$$B_1 = \overline{B} + \sum_{m_j\, odd} E_j = \sigma^*B - 2\sum \left[\frac{m_j}{2}\right] E_j.$$

Unless B_1 is nonsingular we repeat the construction. Since the new ramification curves $B_1, B_2, ...$ are contained in the total transforms of B, after finitely many step (locally with respect to Y), we arrive at a ramification curve B_k with at worst nodes. So all singularities of B_k have multiplicities $m_j = 2$, hence $B_{k+1} = \overline{B}_k$ is a nonsingular curve, and X_{k+1} is a resolution of the singularities of X. The effects of the resolution of the singularities on the invariants are:

$$\chi(\mathcal{O}_{X_k}) = 2\chi(\mathcal{O}_Y) + \frac{1}{2}F(K_Y + F) - \sum_{j=1}^{r} \frac{1}{2}\left[\frac{m_j}{2}\right](\left[\frac{m_j}{2}\right] - 1);$$

$$K_{X_k}^2 = 2(K_Y + F)^2 - 2\sum_{j=1}^{r}(\left[\frac{m_j}{2}\right] - 1)^2.$$

Note that, in general, the canonical resolution is not a minimal resolution.

1.4 Some results on fibrations

Let us recall the definition of fibration:

Definition. A fibration $f : S \to B$ is a surjective proper morphism with connected fibres, where S is a smooth projective surface and B is a smooth curve.

We have that f is flat (see e.g. [Fis76], p. 154, or [Har77], III.9.7, p. 257).Let $f : S \to B$ be a fibration. $P \in B$ is a critical value of f if and only if the fibre f^*P is a singular fibre. So almost all fibres are nonsingular.

Definition. If $f^*(P) = F = \sum n_i C_i$ and $n = m.c.d.(n_i) \geq 1$, where the C_i are irreducible reduced curves, then we will say that F is a *multiple fibre* with multiplicity n.

Lemma 1.4.1. *If f is a fibration and F is a fibre of f, then $\mathcal{O}_F(F) = \mathcal{O}_F$. If $F = nD$ is a fibre of multiplicity n then $\mathcal{O}_F(D)$ and $\mathcal{O}_D(D)$ are two n-torsion sheaf.*

Proof. Cf. [BHPVdV04]. □

Remark 1.4.2. Since $\mathcal{O}_F(F)$ is trivial, we have $\omega_{F_P} = \mathcal{O}_{F_P}(K_S + F_P) = \mathcal{O}_{F_P}(K_S)$.

Lemma 1.4.3 (Zariski). *Let $F_P = \sum n_i C_i$ be a fibre of the fibration $f : S \to B$, with $n_i > 0$, C_i irreducible. Then*

- *$C_i F_P = 0$ for all i;*
- *if $D = \sum_{i=0}^{i=k} m_i C_i$, $m_i \in \mathbb{Z}$, then $D^2 \leq 0$;*
- *$D^2 = 0$ if and only if $D = rF_P$, with $r \in \mathbb{Q}$.*

Proof. Cf. [BHPVdV04]. □

We can prove that, if f is a fibration and F_P is the fibre over the point $P \in B$, then $h^0(F_P, \mathcal{O}_{F_P}) = 1$ for all P. Using Serre duality and the fact that $p_a(F_P)$ (and so $\chi(\mathcal{O}_{F_P})$) does not depend from P, we obtain that the genus of the fibre F_P is constant as P varies in B.

Definition. f is a fibration of genus g if g is the genus of its fibres.

Proposition 1.4.4. *If $f : S \to B$ is a fibration (not necessarily connected), then the sheaves $f_*\mathcal{O}_S$ and $R^1 f_*\mathcal{O}_S$ are locally free and have the base-change property.*

Proof. Cf. [BHPVdV04]. □

Proposition 1.4.5. *If $f : S \to B$ is a fibration and F_{gen} is a nonsingular fibre, then*

- *$\chi_{top}(F_P) \geq \chi_{top}(F_{gen})$ for each F_P;*
- *if S is compact then*

$$\chi_{top}(S) = \chi_{top}(F_{gen})\chi_{top}(B) + \sum_{P \in B} (\chi_{top}(F_P) - \chi_{top}(F_{gen})).$$

Proof. Cf. [BHPVdV04]. □

Definition. We say that a fibration is relatively minimal if it has no (-1)-rational curve in any of its fibres.

Let $f : S \to B$ be a fibration of genus $g \geq 1$ not relatively minimal. We name $\sigma_1 : S_1 \to S$ the contraction of one (-1)-curve and$f_1 = \sigma_1 \circ f : S_1 \to B$ the induced fibration. There exists a succession of contractions $\sigma_i : S_i \to S_{i-1}$ and of induced fibrations $f_i = S_i \to B$, $i \in 1, \ldots, n$ such that f_n is a minimal fibration. Moreover if S contains two (-1)-curve C_1, C_2 in a fibre F, with $C_1 \cap C_2 \neq \emptyset$, then $(C_1 + C_2)^2 \geq 0$ and $(C_1 + C_2)^2 = 0$ only if $C_1 C_2 = 1$. From Zariski lemma it follows that $F = n(C_1 + C_2)$ and so, if $\tilde{F}$ is a general fibre, then $K_S \tilde{F} = nK_S(C_1 + C_2) = -2n < 0$, so the general fibre is rational. We can enunciate the following proposition:

Proposition 1.4.6. *If the genus of its general fibre is strictly positive, then $S \to B$ factors through a unique relatively minimal fibration.*

Definition. Let $f_1 : S_1 \to B$, $f_2 : S_2 \to B$ be two fibrations. f_1 and f_2 are birationally (or biregularly) equivalent if there exist a surface S and two birational (or biregular) maps $h_i : S \to S_i$, $i = 1, 2$, such that $f_1 \circ h_1 = f_2 \circ h_2$.

Definition. $f : S \to B$ is a trivial fibration if it is biregular to the fibration $pr_B : F \times B \to B$. f is isotrivial (or of constant moduli) if all the smooth fibres of f are isomorphic, that is if there exist a Galois covering $\tau : B' \to B$ such that the induced fibration $S' \to B'$ is trivial.

Definition. Let $f : S \to B$ be a fibration. A section of f is a map $\sigma : B \to S$ such that $f \circ \sigma = id_B$.

Definition. The line bundle $\omega_{S|B} = \omega_S \otimes f^*(\omega_B^\vee)$ on S is called *dualizing sheaf of f* or *relative canonical sheaf*, while $K_{S|B}$ will be a *relative canonical divisor*, i.e. a divisor such that $\mathcal{O}_S(K_{S|B}) = \omega_{S|B}$

Let us remark that ${\omega_{S|B}}_{|F_P}$ is isomorphic to ω_{F_P} for each P.

Other sheaves that one associates to a relatively minimal fibration f are the sheaves $V_n := f_*(\omega_{S|B}^{\otimes n})$, that are vector bundles.

Theorem 1.4.7. *The vector bundles V_n are* semipositive, *i.e., every locally free quotient of it has nonnegative degree.*

Proof. Cf. [Fuj78a], [Fuj78b]. □

Theorem 1.4.8. *If $n \geq 2$ the vector bundle V_n is ample unless f has constant moduli.*

Proof. Cf. [EV90] □

Proposition 1.4.9. *If $f : S \to B$ is a fibration, then the sheaves $f_*\omega_{S|B}$ and $R^1 f_*\omega_{S|B}$ are locally free on B and have the base-change property. $f_*\omega_{S|B}$ has rank g and the map $f_*\omega_{S|B} \otimes k(P) \to H^0(F_P, \omega_{F_P})$ is an isomorphism for each P (let us remember that $k(P) = \mathcal{O}_{B,P}/\mathfrak{m}_{B,P}$).*

Proof. Cf. [BHPVdV04] □

Theorem 1.4.10 (Relative duality theorem). *If $f : S \to B$ is a fibration and $\mathscr{F}$ is a locally free $\mathcal{O}_S$-module, then the relative duality morphism $f_*(\mathscr{F}^\vee \otimes \omega_{S|B}) \to (R^1 f_*\mathscr{F})^\vee$ is bijective.*

Proof. Cf. [BHPVdV04] □

Theorem 1.4.11. *If $f : S \to B$ is a relatively minimal fibration of genus $g > 0$ then $\deg f_*\omega_{S|B} \geq 0$. Moreover $\deg f_*\omega_{S|B} = 0$ if and only if*

- *f is locally trivial or*
- *$g = 1$ and the unique singular fibres are multiple fibres supported on a smooth elliptic curve.*

Proof. Cf. [BHPVdV04] □

We introduce the following invariants of the fibration f:

$$K_f^2 := K_{S|B}^2 = K_S^2 - 8(g-1)(b-1),$$

that is the self intersection of the relative canonical divisor, and

$$\chi_f := \chi(\omega_{S|B}) = \chi\mathcal{O}_S - (g-1)(b-1),$$

the Euler characteristic of the relative canonical divisor.

Since $R^1 f_*\omega_{S|B} = \mathcal{O}_B$ by relative duality, and $R^1 f_*\omega_{S|B}^{\otimes n} = 0$ for all $n \geq 2$ by the assumption of relative minimality, one can compute the Euler characteristic of V_n by Riemann-Roch, and consequently its degree:

$$\deg V_n = \chi_f + \frac{n(n-1)}{2}K_f^2.$$

Theorem 1.4.12 (Arakelov). *Let S be a smooth minimal surface and $f : S \to B$ be a fibration of genus $g \geq 2$. We have that*

- $K_f^2 \geq 0$ *and* $K_{S|B} \cdot C \geq 0$*;*
- *if f is not isotrivial then $K_{S|B}C > 0$ except if C is a (-2)-rational curve contained in a fibre.*

Proof. Cf. [Bea82]. □

Corollary 1.4.13 (Arakelov, Beauville). *Let S be a smooth minimal surface and $f : S \to B$ be a fibration of genus $g \geq 2$, $b = g(B)$. Then*

- $K_S^2 \geq 8(b-1)(g-1)$*, and if we have equality f has constant moduli;*
- $\chi_{top}(S) \geq 4(b-1)(g-1)$*, and if we have equality f is smooth;*
- $\chi_f \geq 0$*, i.e. $\chi(\mathcal{O}_S) \geq (b-1)(g-1)$ and if we have equality f is an étale bundle (i.e. there exists an étale cover $B' \to B$ such that the pullback is a product $B' \times F$).*

Proof. Cf. [Bea82]. □

Chapter 2

Surfaces with $K^2 = 3$, $p_g = 2$

Throughout this chapter, a surface will mean a minimal surface, unless differently specified.

We will deal with surfaces with invariants $K^2 = 3$, $p_g = 2$. Previous examples of these surfaces are due to M. Reid and to U. Persson. In [Rei78] M. Reid calculated the canonical ring of the universal cover X of a $\mathbb{Z}_3$ Godeaux surface, i.e. of a minimal surface of general type with $K^2 = 1$, $p_g = 0$ and torsion $\mathbb{Z}_3$, while in [Per81] Persson obtains a surface with given invariants as the resolution of a double cover of Σ_1 branched along a sextic with three infinitely near triple points and no other essential singularity.

2.1 Generalities about surfaces with $p_g = 2$, $K_S^2 = 3$

Let us consider the canonical system $|K_S|$. If we name $|M|$ its mobile part and Z its fixed component, since $p_g = 2$, we have that $|M|$ has dimension equal to 1 and, as S is regular, the general element of $|M|$ is irreducible and smooth by Bertini's theorem. Let us recall that $MZ \geq 2$ since K is 2-connected. There are the following possibilities:

(I) $Z \neq 0$, $M^2 = 0$, $MZ = 2$, which imply that $|M|$ is a free pencil. Since $Z^2 = K_S^2 - 2MZ - M^2$, $Z^2 = -1$ and since $K^2 = (M + Z)K = M^2 + MZ + KZ$, $KM = 2$, $g(C) = 2$ (where C is a general curve in $|M|$) and the map induced by the bicanonical system is not birational;

(II) $Z \neq 0$, $M^2 = 1$, $MZ = 2$, which imply $Z^2 = -2$. Since $KZ = 0$, Z is a sum of (-2)-curves, and $KM = 3$, so the general C in $|M|$ is a curve of genus 3; since $M^2 = 1$, $|M|$ has one fixed point;

(III) $Z = 0$, then the general element C in $|M|$ is a genus 4 smooth curve ($g(C) = 1 + C^2$) and there exists a divisor D on C with $K_C \equiv 2D$; since $M^2 = 3$, $|M|$ has three fixed points.

The case $MZ = 3$ cannot verify, since it would imply $M^2 = KZ = 0$, so $MZ + Z^2 = 0$ and $Z^2 = -3$, a contradiction with $Z^2 \equiv KZ \bmod 2$.

We refer to surfaces of type (I), (II), (III) in accordance with the canonical system is as at point (I), (II), (III) of the previous list.

Debarre's inequality implies that surfaces with invariants $K^2 = 3$, $p_g = 2$ are regular and so $\chi(\mathcal{O}_S) = 3$. Then these surfaces satisfy the equality $K_S^2 = 2\chi(\mathcal{O}_S) - 3$. A minimal surface with $K_S^2 = 2\chi(\mathcal{O}_S) - 3$ is regular, too. In fact $K_S^2 = 2p_g + 2 - 2q - 3 = 2p_g - 2q - 1 < 2p_g$, and again by Debarre's inequality, this implies that the surface is regular. So $2\chi(\mathcal{O}_S) - 3 = 2p_g - 1$.

Lemma 2.1.1. *Let S be a surface with $K_S^2 = 2\chi(\mathcal{O}_S) - 3$. Then the torsion group of S is trivial.*

Proof. Since $K_S^2 = 2\chi(\mathcal{O}_S) - 3 = 2p_g(S) - 1$, using theorem 14 in [Bom73], we obtain that $\frac{K_S^2 + 1}{2} \leq \frac{K_S^2}{2} + \frac{3}{m} - 1$, that is $m \leq 2$. Let us suppose that $m = 2$, and so that there exists a divisor η in S such that $2\eta \sim 0$, $\eta \not\sim 0$. It is possible to construct the étale Galois cover associated to η, which we will name $\pi : \tilde{S} \to S$. Then we can state the following equalities

$$K_{\tilde{S}}^2 = 2K_S^2,$$

$$\chi(\mathcal{O}_{\tilde{S}}) = 2\chi(\mathcal{O}_S),$$

that imply

$$2p_g(S) - 1 = 2\chi(\mathcal{O}_S) - 3 = \chi(\mathcal{O}_{\tilde{S}}) - 3 = p_g(\tilde{S}) - q(\tilde{S}) - 2$$

and so

$$K_{\tilde{S}}^2 = 2K_S^2 = 2(2p_g(S) - 1) = 2(p_g(\tilde{S}) - q(\tilde{S}) - 2) \leq 2p_g(\tilde{S}) - 4.$$

Since $\tilde{S}$ is minimal ($K_{\tilde{S}} = \pi^*(K_S + \eta)$ is nef), using Noether inequality we obtain $K_{\tilde{S}}^2 = 2p_g(\tilde{S}) - 4$, $q(\tilde{S}) = 0$. So $\tilde{S}$ is a Horikawa surface and contains a free genus two pencil. Then there exists a morphism $f : \tilde{S} \to \mathbb{P}^1$ with general fibre $\tilde{F}$ of genus two. Let us consider the restriction of π to $\tilde{F}$, $\pi_{|_{\tilde{F}}} : \tilde{F} \to \pi(\tilde{F}) = F$. $\pi_{|_{\tilde{F}}}$ is a map of degree 1 or 2. If $\deg \pi_{|_{\tilde{F}}}$ was 2, $\pi_{|_{\tilde{F}}}$ would be an étale double cover and so we would have $2g(\tilde{F}) - 2 = 2(2g(F) - 2)$ and from this $2 = 4(g(F) - 1)$, that is absurd. So $\pi_{|_{\tilde{F}}}$ is an isomorphism and $|F|$ is a genus two pencil on S.

Now let us prove that this implies that $|K_S + \eta|$ is composed with this pencil. Let us suppose that this is not true, i.e. that $|K_X + \eta|$ is not composed with $|F|$. Using the exact sequence

$$0 \to \mathcal{O}_S(K_S + \eta - F) \to \mathcal{O}_S(K_S + \eta) \to \mathcal{O}_F((K_S + \eta)_{|_F}) \to 0$$

we obtain the cohomology sequence

$$0 \to H^0(K_S + \eta - F) \to H^0(K_S + \eta) \xrightarrow{r} H^0((K_S + \eta)_{|_F}).$$

From this we deduce that $\ker r = H^0(K_S + \eta - F)$ and $\dim \operatorname{Im} r \geq 2$. Since $2p_g(S)+1 = h^0(\tilde{S}, K_{\tilde{S}}) = h^0(S, K_S)+h^0(K_S+\eta)$ we have $p_g(S) = h^0(S, K_S+\eta)-1$, so $\dim \ker r = h^0(K_S + \eta - F) \leq p_g - 1$. Riemann-Roch theorem gives that $h^0(F - \eta) = h^2(K_S + \eta - F) = p_g(S) + h^1(K_S + \eta - F) - h^0(K_S + \eta - F)$, and so $h^0(F - \eta) \geq p_g(S) + h^1(K_S + \eta - F) + 1 - p_g(S) = 1 + h^1(K + \eta - F) \geq 1$. We can infer that there exists $\overline{F} \equiv F$ with $\overline{F} \not\sim F$ and $\overline{F}$ effective, so that $\overline{F}$ is a sum of submultiple of curves of the pencil. Since a genus two pencil has not multiple fibres, $\overline{F}$ is a fibre and $\overline{F} \sim F$, since $q(S) = 0$. This contradiction proves that $|K_S + \eta|$ is composed with the pencil.

So we have that $|K_S + \eta| = |p_g(S)F| + Z$. Now $K_S^2 = p_g F K_S + Z K_S = 2p_g + K_S Z$, that implies $K_S Z = K_S^2 - 2p_g(S) = -1$, in contradiction with the fact that K_S is nef. This proves that $m = 1$ $\square$

Remark 2.1.2. The proposition implies that there do not exist non ramified abelian covers of S.

Let $\widetilde{S}$ be the surface obtained blowing up the fixed points in $|M|$. Let us name $|\widetilde{M}|$ the linear system on $\widetilde{S}$ induced by $|M|$. We can consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{S} & \overset{\Phi}{\dashrightarrow} & \mathbb{P}(f_*\omega_{\widetilde{S}|\mathbb{P}^1}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

where f is the morphism induced by $|\widetilde{M}|$.

Lemma 2.1.3.

$$f_*\omega_{\widetilde{S}|\mathbb{P}^1} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) & \textit{case (I);} \\ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) & \textit{case (II);} \\ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) & \textit{case (III).} \end{cases}$$

Proof. Consider the first case. $|M|$ has no fixed points, so $\widetilde{S}$ coincides with S. $f_*\omega_{S|\mathbb{P}^1}$ is a locally free sheaf of rank $g(F)$, where F is a general fibre of f. Since $g(F) = 2$, we have that rk $f_*\omega_{S|\mathbb{P}^1} = 2$, and so $f_*\omega_{S|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$. Moreover, since $\deg \omega_{S|\mathbb{P}^1} \geq 0$, if E is a subbundle of $f_*\omega_{S|\mathbb{P}^1}$, then $\deg \bigwedge^k E \geq 0$, and so $a \geq 0$, $b \geq 0$.

Since $\omega_S = \omega_{S|\mathbb{P}^1} \otimes f^*\omega_{\mathbb{P}^1} = \omega_{S|\mathbb{P}^1} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-2)$, $p_g = h^0(S, \omega_S) = h^0(\mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(b-2)) = 2$ we have the following cases: $(a, b) = (2, 2)$ or $(a, b) = (a, 3)$ with $a < 2$. Now let us consider $H^0(\omega_S \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1))$: we have $H^0(\omega_S \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)) =$

$H^0(S, \omega_S \otimes \mathcal{O}_S(-M)) = H^0(S, M + Z - M) = \mathbb{C}$, but we have also $H^0(\omega_S \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(a-3) \oplus \mathcal{O}_{\mathbb{P}^1}(b-3))$. This implies $0 \leq a < 2$ and $b = 3$.

We have to verify that $a = 1$. Now, $f_*\omega_{S|\mathbb{P}^1} = f_*(\omega_S \otimes f^*\mathcal{O}_{\mathbb{P}^1}(2)) = f_*(\mathcal{O}(3M + Z))$. From the exact sequence

$$0 \to \mathcal{O}(-M) \to \mathcal{O}_S \to \mathcal{O}_M \to 0 \tag{2.1.1}$$

we deduce that $h^1(-M) = 0$. If we tensor the same sequence by $\omega_S \otimes \mathcal{O}_S(M)$, we will obtain $h^0(M + K) = 4$. Using this result and tensoring 2.1.1 by $\omega_S \otimes \mathcal{O}_S(2M)$ we obtain $h^0(2M+K) = h^0(3M+Z) = 6$. But this implies that the only possibility we have is that $f_*\omega_{S|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$.

Now consider the second case: $|M|$ has one fixed point since $M^2 = 1$, so we have to blow up S one time. $f_*\omega_{\widetilde{S}|\mathbb{P}^1}$ is a locally free sheaf of rank 3 on $\mathbb{P}^1$, so we can write $f_*\omega_{\widetilde{S}|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$. As before $a \geq 0$, $b \geq 0$, $c \geq 0$.

Since $\omega_{\widetilde{S}} = \omega_{\widetilde{S}|\mathbb{P}^1} \otimes f^*\omega_{\mathbb{P}^1} = \omega_{\widetilde{S}|\mathbb{P}^1} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-2)$, $p_g = h^0(\widetilde{S}, \omega_{\widetilde{S}}) = h^0(\mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(b-2) \oplus \mathcal{O}_{\mathbb{P}^1}(c-2)) = 2$, we have the following cases: $(a, b, c) = (a, 2, 2)$ with $a < 2$ or $(a, b, c) = (a, b, 3)$ with $a < 2, b < 2$. Let us consider $H^0(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1))$: we have $h^0(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = h^0(\widetilde{S}, \omega_{\widetilde{S}} \otimes \mathcal{O}_{\widetilde{S}}(-\widetilde{M})) = 1$, but we have also $H^0(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(a-3) \oplus \mathcal{O}_{\mathbb{P}^1}(b-3)) \oplus \mathcal{O}_{\mathbb{P}^1}(c-3))$. This implies $0 \leq a < 2, 0 \leq b < 2$, $c = 3$.

We have to verify that $a = b = 1$. Now, $f_*\omega_{\widetilde{S}|\mathbb{P}^1} = f_*(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(2)) = f_*(\mathcal{O}(K_{\widetilde{S}} + 2\widetilde{M}))$. From the exact sequence

$$0 \to \mathcal{O}(-\widetilde{M}) \to \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{M}} \to 0 \tag{2.1.2}$$

we deduce that $h^1(-\widetilde{M}) = 0$. If we tensor the same sequence by $\omega_{\widetilde{S}} \otimes \mathcal{O}_{\widetilde{S}}(\widetilde{M})$, we obtain $h^0(\widetilde{M} + K_{\widetilde{S}}) = 5$. Using this result and tensoring 2.1.2 by $\omega_{\widetilde{S}} \otimes \mathcal{O}(2\widetilde{M})$ we obtain $h^0(2\widetilde{M} + K_{\widetilde{S}}) = 8$. But this implies that the only possibility we have is that $f_*\omega_{\widetilde{S}|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$.

Finally, we consider the third case. Since $M^2 = 3$, $|M|$ has three base points, so we have to blow up S three times. Now $f_*\omega_{\widetilde{S}|\mathbb{P}^1}$ is a locally free sheaf of rank 4 on $\mathbb{P}^1$ and we can write $f_*\omega_{\widetilde{S}|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ with $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$.

Since $p_g = h^0(\mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(b-2) \oplus \mathcal{O}_{\mathbb{P}^1}(c-2), \oplus \mathcal{O}_{\mathbb{P}^1}(d-2)) = 2$ we have the following cases: $(a, b, c, d) = (a, b, 2, 2)$ with $a < 2$, $b < 2$ or $(a, b, c, d) = (a, b, c, 3)$ with $a < 2, b < 2, c < 2$. Let us consider $h^0(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1))$: we have $h^0(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = h^0(\widetilde{S}, K_{\widetilde{S}} - \widetilde{M}) = 1$, but we have also $H^0(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(a-3) \oplus \mathcal{O}_{\mathbb{P}^1}(b-3)) \oplus \mathcal{O}_{\mathbb{P}^1}(c-3) \oplus \mathcal{O}_{\mathbb{P}^1}(d-3))$. This implies $0 \leq a < 2, 0 \leq b < 2, 0 \leq c < 2$ and $d = 3$.

We have to verify that $a = b = c = 1$. Now, $f_*\omega_{\widetilde{S}|\mathbb{P}^1} = f_*(\omega_{\widetilde{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(2)) = f_*(\mathcal{O}(K_{\widetilde{S}} + 2\widetilde{M}))$. From the exact sequence

$$0 \to \mathcal{O}(-\widetilde{M}) \to \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{M}} \to 0 \tag{2.1.3}$$

we deduce that $h^1(-\widetilde{M}) = 0$. If we tensor the same sequence by $\mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{M})$, we obtain $h^0(K_{\widetilde{S}} + \widetilde{M}) = 6$. Using this result and tensoring 2.1.3 by $\mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} + 2\widetilde{M})$ we obtain $h^0(K_{\widetilde{S}} + 2\widetilde{M}) = 10$. But this implies that the only possibility we have is that $f_*\omega_{\widetilde{S}|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$. □

If we have a surface of type (II), we can blow up the fixed point in $|M|$ and name $\widetilde{S}$ the surface we obtain and $|\widetilde{M}|$ the linear system on $\widetilde{S}$ induced by $|M|$. $|\widetilde{M}|$ is a pencil of genus 3 on $\widetilde{S}$, so we can consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{S} & \overset{\Phi}{\dashrightarrow} & \mathbb{P}(f_*\omega_{\widetilde{S}|\mathbb{P}^1}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

where f is the morphism induced by $|\widetilde{M}|$. If $\widetilde{M}$ is hyperelliptic then Φ is a rational morphism of degree 2, while if $\widetilde{M}$ is not hyperelliptic then Φ is birational. The image of the general fibre of f in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ is a quartic curve in $\mathbb{P}^2$ if Φ is birational or a conic in $\mathbb{P}^2$ if Φ has degree 2.

If we have a surfaces of type (III), we can blow up the three fixed point in $|M|$ and name $\widetilde{S}$ the surface we obtain and $|\widetilde{M}|$ the linear system on $\widetilde{S}$ induced by

$|M|$. $|\widetilde{M}|$ is a pencil of genus 4 on $\widetilde{S}$, so we can consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{S} & \overset{\Phi}{\dashrightarrow} & \mathbb{P}(f_*\omega_{\widetilde{S}|\mathbb{P}^1}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

where f is the morphism induced by $|\widetilde{M}|$. If the general divisor in $|\widetilde{M}|$ is hyperelliptic then Φ is a rational morphism of degree 2, otherwise Φ is birational. If Φ is birational, the image of the general fibre of f in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(3))$ is a curve of genus 4 in $\mathbb{P}^3$, so the ideal of C is generated by quadratic equations, unless it is trigonal or isomorphic to a smooth plane quintic, in which cases the ideal is generated by quadratic and cubic equations. If Φ has degree 2 the image of the general fibre of f in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(3))$ is a rational normal curve of degree 3 in $\mathbb{P}^3$.

Now let Σ be the image of the bicanonical map $\phi_{|2K|}: S \to \mathbb{P}^5$. We have that Σ is a surface since $|2K_S|$ has not base points (cf. results of P. Francia in [Fra91]) and ϕ_{2K_S} is generically finite (cf. [Xia85a]). In [Cil83] Ciliberto proved the following lemma:

Lemma 2.1.4. *If $p_g(S) \geq 2$ and $q(S) = 0$ then the bicanonical map $\phi_{|2K|}$ of S is birational or of degree 2.*

Since $(2K_S)^2 = \deg\phi_{|2K|}\deg\Sigma$ and $\deg\Sigma \geq \operatorname{codim}\Sigma + 1$, we have the following possibilities:

- $\deg\phi = 2$, $\deg\Sigma = 6$;
- $\deg\phi = 1$, $\deg\Sigma = 12$, that implies that S has not pencils of genus two .

2.2 Reid's example

We said that in [Rei78] M. Reid constructed a family of surfaces of general type with $p_g = 2$, $K_S^2 = 3$. He considered the projective variety $\overline{F} = \mathbb{P}(R)$ associated to the graded ring $R = k[x_1, x_2, y_0, y_1, y_2]$, with $\deg x_i = 1$, $\deg y_i = 2$. The elements of degree 2 $x_1^2, x_2^2, x_1x_2, y_0, y_1, y_2$ define an embedding of $\overline{F}$ in $\mathbb{P}^5$ as a quadric of rank 3. The natural desingularisation F of $\overline{F}$ is the rational scroll $F = \mathbb{P}(E)$, $F \xrightarrow{\pi} \mathbb{P}^1$, where E is the rank 4 vector bundle $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Let A be a fibre of $F \xrightarrow{\pi} \mathbb{P}^1$ and B the unique section of E, so that $\mathcal{O}_F(B)$ is the tautological bundle of F. Let Q, C be two irreducible divisors with $Q \in |6A+2B|$, $C \in |6A+3B|$ such that

(1) Q contains three lines $\mathbb{P}^1 \times p_i$, with $p_i \in \mathbb{P}^2$ non collinear points, and is nonsingular along them;

(2) C contains three lines $\mathbb{P}^1 \times p_i$, and contains three fibres Q_i of $Q \to \mathbb{P}^1$;

(3) $Q \cap C = \widetilde{Y} + \sum_i Q_i$, with $\widetilde{Y}$ a surfaces which is nonsingular along $\mathbb{P}^1 \times p_i$, and has only rational double points elsewhere;

(4) B touches $\widetilde{Y}$ along the three lines $\mathbb{P}^1 \times p_i = l_i$.

Now, since $K_F = -8A - 4B$, using the adjunction formula, we obtain $K_Q = \mathcal{O}_Q(-2A-2B)$ and $K_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(A+B)$. But $\mathcal{O}_{\widetilde{Y}}(B) = 2\sum_i l_i$; since each $l_i^2 < 0$ (it contracts under $\phi : F \to \overline{F}$) it follows that $l_i^2 = -1$. If we contract these lines we obtain a surface Y with invariants $p_g = 2$, $K_Y^2 = 3$.

The surface Y so obtained is a surface with canonical system without fixed component. In fact, since $K_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(A+B)$ and $B = 2\sum_i l_i$, if we contract l_1, l_2, l_3, we will obtain $K_Y = \mathcal{O}_Y(\overline{A})$ (where $\overline{A} = \sigma(A)$ and $\sigma : \widetilde{Y} \to Y$ is the contraction of l_1, l_2, l_3), so K_Y has not fixed components but has three fixed points.

Moreover Reid proves that $|2K_Y|$ is without fixed points and defines a birational morphism $\phi_{2K_Y} : Y \to \bar{F} \subset \mathbb{P}^5$.

These observations allow us to conclude that the surface described by Reid is of type (III).

2.3 Persson's example

Persson constructs a family of sextic branch curves of Σ_N with many infinitely near triple points and applies this construction to the exhibition of a family of genus two fibrations.

In particular this construction gives a divisor $B \sim 6\Delta_0 + 4\Gamma$ in Σ_1 (where Δ_0 is the section with self-intersection 1 and Γ is the fibre) with exactly three infinitely near triple points and no other essential singularities.

The resolution of the double covering of Σ_1 along B is a surface S with $p_g = 2$, $K^2 = 3$. The canonical map of S is composed with a pencil of genus two, so S is of type (I).

Chapter 3

Surfaces of type (I)

3.1 Classification

We focus on the case $M^2 = 0$, $MZ = 2$. As we said, the general element in $|M|$ is a genus two curve, while $p_a(Z) = 1$. We name f the map $S \to \mathbb{P}^1$ induced by the linear system $|M|$.

We consider now the projective bundle $\mathbb{P}(R^1 f_* \mathcal{O}_S) = \mathbb{P}((f_* \omega_{S|\mathbb{P}^1})^\vee) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ and the relative canonical map $\Phi : S \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, defined by the linear system $|K_S + nM|$, with n sufficiently big.

Φ can be constructed as follows: for every integer n let us set

$$b(n) = h^0(K + nM) - h^0(K + (n-1)M);$$

from the exact sequence

$$0 \to \mathcal{O}_S(K + nM) \to \mathcal{O}_S(K + (n+1)M) \to \mathcal{O}_M(K + (n+1)M) \to 0$$

we have that $b(n) \leq b(n+1) \leq 2$, and, since $h^1(-M) = 0$, $h^0(K + M) = p_g + 2$. This implies that $b(1) = 2$. Now set

$$n_j = min\{n | b(n) \geq j\},$$

$j = 1, 2$ and $d = n_2 - n_1$. Since $p_g = 2$ and $h^0(K - 2M) = h^0(Z - M) = 0$, from the exact sequence

$$0 \to \mathcal{O}_S(K + nM) \to \mathcal{O}_S(K + (n+1)M) \to \mathcal{O}_M(K + (n+1)M) \to 0,$$

we get that $b(n) = 0$ for $n \leq -2$; for $n = -1$ we get

$$0 \to \mathcal{O}_S(K - 2M) \to \mathcal{O}_S(K - M) \to \mathcal{O}_M(K - M) \to 0,$$

so $b(-1) = 1$; for $n = 0$ we get

$$0 \to \mathcal{O}_S(K - M) \to \mathcal{O}_S(K) \to \mathcal{O}_M(K) \to 0,$$

so $b(0) = 1$; for $n = 1$ we get

$$0 \to \mathcal{O}_S(K) \to \mathcal{O}_S(K + M) \to \mathcal{O}_M(K + M) \to 0,$$

so $b(1) = 2$; we can conclude that $d = 2$. From the definition of n_1 and n_2 it follows that we can find $\omega_i \in H^0(S, \mathcal{O}(K+n_iM))$ so that their restrictions to $C = f^{-1}(p), p$ point in $\mathbb{P}^1$, make a basis of $H^0(C, K_C)$. The pair ω_1, ω_2 defines a generically surjective rational map $\Phi : S \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$.

Let us name Σ_2 the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, let q be the morphism $\Sigma_2 \to \mathbb{P}^1$, Δ_∞ the section of q such that $\Delta_\infty^2 = -2$, Δ_0 the section of q such that $\Delta_0^2 = 2$ and Γ the fibre of q. $\operatorname{Pic}\Sigma_2$ is generated by Δ_0 and Γ and $\Delta_\infty = \Delta_0 - 2\Gamma$. Since the general fibre of f has genus two, Φ is a degree two rational map. We have the following commutative diagram:

$$\begin{array}{ccc} S & \overset{\Phi}{\dashrightarrow} & \Sigma_2 \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathbb{P}^1 & \end{array}$$

Remark 3.1.1. The rational map $S \dashrightarrow \mathbb{P}(H^0(K_S + nM))$, induced by the linear system $|K_S + nM|$, can be factorized as follows

$$\begin{array}{ccc} S & \overset{\phi_{|K_S+nM|}}{\dashrightarrow} & \mathbb{P}(H^0(K_S + nM)) \\ & \searrow \quad \nearrow{\scriptstyle \phi_{|\mathcal{O}_{\Sigma_2}(1)|}} & \\ & \Sigma_2 = \mathbb{P}(f_*K_S + \mathfrak{n}) & \end{array}$$

where $\mathfrak{n}$ is a degree n divisor in $\mathbb{P}^1$, $\mathbb{P}(H^0(K_S + nM)) = \mathbb{P}(H^0(\mathbb{P}^1, f_*K_S + \mathfrak{n}))$, $\mathbb{P}(f_*K_S + \mathfrak{n}) \cong \mathbb{P}(f_*\omega_{S|\mathbb{P}^1})$. $\phi_{|\mathcal{O}_{\Sigma_2}(1)|}$ is an embedding since $\mathcal{O}_{\Sigma_2}(1)$ is very ample for $\mathfrak{n}$ sufficiently ample.

We name $\Phi' : \tilde{S} \to S$ be the composition of a finite number of blowing ups such that $\Phi \circ \Phi'$ extends to a regular morphism $\tilde{\Phi} : \tilde{S} \to \Sigma_2$. Let R (respectively B) be the ramification divisor (respectively branch divisor) of $\tilde{\Phi}$. There exists a divisor $F \in \mathrm{Pic}(\Sigma_2)$ such that $2F = B$. Let S' be the double cover of Σ_2 in F with branch divisor B. Using the theory of double covers one can prove that, given $B \in \mathrm{Pic}(\Sigma_2)$, there exists a birational morphism from $\tilde{S}$ to S'.

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\Phi'} & S \\
\downarrow & \searrow^{\tilde{\Phi}} & \downarrow \\
S' & \longrightarrow & \Sigma_2
\end{array}
$$

Since $\widetilde{\Phi}$ induces a double cover from $\Phi'^*(M)$ to $\mathbb{P}^1$, using Hurwitz formula we obtain $B \cdot \Gamma = 6$, and so $B = 6\Delta_0 + q^*\alpha = 6\Delta_\infty + q^*\beta$, for suitable $\alpha, \beta \in \mathrm{Pic}(\mathbb{P}^1)$.

Lemma 3.1.2. *After applying a finite number of elementary transformations to the triple* (Σ_2, B, F) *we obtain another triple* (Σ_d, B_d, F_d) *such that* B_d *has not singular points with multiplicity greater than three.*

Proof. Cf. [Hor77]. □

Let x_1 be a triple point of B_d and let $\Sigma'_d \to \Sigma_d$ the blowing up in x_1. If the proper transform B' of B_d has a triple point x_2 on x_1, we can blow up Σ'_d in x_2 and we can go on in this way until we obtain a finite sequence $x_1, \ldots, x_k$ of triple points on x_1.

Definition. x_1 is a *k-fold triple point.* If $k = 1$ the triple point is said simple, otherwise $x_1, \ldots, x_k$ are said *infinitely near.*

Lemma 3.1.3. *After applying a finite number of elementary transformations to the triple* (Σ_d, B_d, F_d) *we obtain another triple* $(\Sigma_{d_1}, B_{d_1}, F_{d_1})$ *such that* B_{d_1} *has only the following singularities, defined when* $k \geq 1$*:*

(0) *a double point or a simple triple point;*

(I_k) *a fibre* Γ *plus two triple points on it (hence these are quadruple points of* B_{d_1}*); each of these triple points is* $(2k-1)$*-fold or* $2k$*-fold;*

(II_k) *two triple points on a fibre, each of these is* $2k$*-fold or* $(2k+1)$*-fold;*

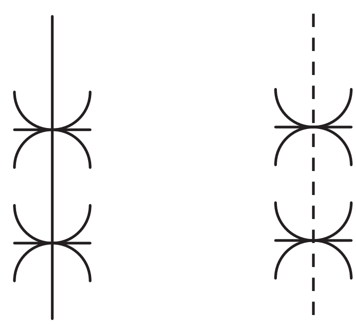


Figure 3.1: Singularities of type (I_k) and (II_k)

(III_k) *a fibre* Γ *plus a* $(4k-2)$ *or a* $(4k-1)$*-fold triple point on it which has a contact of order 6 with* Γ*;*

(IV_k) *A* $4k$ *or* $(4k+1)$*-fold triple point* x *which has a contact of order 6 with the fibre through* x*;*

(V) *a fibre* Γ *plus a quadruple point* x *on* Γ*, which after a blowing up in* x *results in a double point on the proper transform of* Γ*.*

Proof. Cf. [Hor77]. □

Blowing up the k-fold triple points or the quadruple points, after a finite number of blowing ups we obtain a triple $(\hat{\Sigma}_{d_1}, \hat{F}_{d_1}, \hat{B}_{d_1})$ such that $\hat{B}_{d_1}$ has not infinitely near triple points nor quadruple points. Let $\pi : \hat{\Sigma}_{d_1} \to \Sigma_2$ be the composition of these blowing ups, $\phi : \hat{\Sigma}_{d_1} \to \mathbb{P}^1$ the natural projection, $\hat{S}_{d_1}$ the double cover of $\hat{\Sigma}_{d_1}$ in $\hat{F}_{d_1}$ with branch divisor $\hat{B}_{d_1}$. The only singularities of $\hat{S}_{d_1}$ are rational double points.

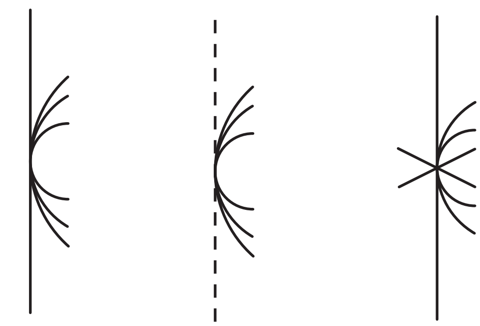


Figure 3.2: Singularities of type (III_k), of type (IV_k) and of type (V)

Let $\check{S} \to \hat{S}_{d_1}$ be the minimal resolution of $\hat{S}_{d_1}$, double cover of $\hat{\Sigma}_{d_1}$. Let $\check{f}$ be the projection $\check{S} \to \mathbb{P}^1$. Since the fibres of $f : S \to \mathbb{P}^1$ do not contain exceptional curves, there exists a unique birational morphism $\check{S} \to S$ compatible with $\check{f}$ and f.

Lemma 3.1.4. *Let $K_{\mathbb{P}^1}$ be the canonical bundle of $\mathbb{P}^1$ and let us write F_d in the form*

$$3\Delta_\infty + \phi^*(-K_{\mathbb{P}^1} + \mathfrak{f} + 2p)$$

with $\mathfrak{f}$ suitable line bundle on $\mathbb{P}^1$, $p \in \mathbb{P}^1$. Then the canonical bundle $K_{\check{S}}$ on $\check{S}$ is induced by

$$\pi^*(\Delta_\infty + \phi^*\mathfrak{f}) - \sum \mathscr{G}_i,$$

where the sum is extended to all the singular fibres Γ_i, except those of type (0). $\mathscr{G}_i$ are suitable linear combinations of the components of the pullbacks on $\hat{\Sigma}_d$ of the singular fibres.

Proof. This lemma is a consequence of the fact that if $r : X \to Y$ is a double covering with branch divisor $B = 2F$ then $K_X = r^*(K_Y + F)$, and if $\sigma : Y' \to Y$ is a blowing up in a point x with exceptional curve E then $K_{Y'} = \sigma^* K_Y + E$. $\square$

Let $\{\Gamma_1, \ldots \Gamma_n\}$ be the set of all the singular fibres, except those of type (0), and let p_i be $\phi(\Gamma_i)$. Let us set $\beta_i = k$ if Γ_i is of type (I_k), (II_k), (III_k) or (IV_k) and $\beta_i = 1$ if Γ_i is of type (V). Then let us define $\mathfrak{c} = \sum \beta_i p_i$ on $\mathbb{P}^1$.

Lemma 3.1.5. *The canonical bundle $K_{\check{S}}$ of $\check{S}$ is induced by*

$$\pi^*(\Delta_\infty + \phi^*(\mathfrak{f} - \mathfrak{c})) + \sum \mathscr{F}_i,$$

where the sum is extended to all the singular fibres Γ_i, except those of type (0)*. For every singular fibre Γ_i, $\mathscr{F}_i$ is a suitable linear combination of the components of the pullbacks on $\hat{\Sigma}_d$ of the singular fibres.*

Proof. This lemma is a consequence of the previous one and of the definition of $\mathfrak{c}$. □

Theorem 3.1.6. *Let $\mathfrak{m}$ be a sufficiently ample divisor on $\mathbb{P}^1$. Then the linear system $|K + f^*\mathfrak{m}|$ defines a rational map of degree 2 onto the $\mathbb{P}^1-$bundle Σ_d (that can be identified with Σ_2) over $\mathbb{P}^1$. Its branch locus B_d (that can be identified with B) has only those singularities of types* (0), (I_k), (II_k), (III_k), (IV_k) *and* (V) *which are listed in lemma 3.1.3.*

Proof. Cf. [Hor77]. □

The exceptional curves we have to contract to obtain S from $\check{S}$ are components of the pull-backs on $\check{S}$ of the singular fibres.

Let us name $\nu(*)$ the number of fibres of type $*$. We have the following

Theorem 3.1.7.

$$K_S^2 = 2p_a - 4 + \sum_k \{(2k-1)(\nu(I_k) + \nu(III_k)) + 2k(\nu(II_k) + \nu(IV_k))\} + \nu(V).$$

Proof. The equality is deduced from the following equalities:

$$p_a(\check{S}) = p_g - q = \deg(2\mathfrak{f} - 2\mathfrak{c}),$$

$$K_{\check{S}}^2 = 2\deg(2\mathfrak{f} - 2\mathfrak{c}) - 4,$$

proved using lemma 6 in [Hor73] and the fact that each fibre of type (I_k) or (III_k) induces $2k-1$ exceptional curves, each fibre of type (II_k) or (IV_k) induces $2k$ exceptional curves, each fibre of type (V) induce an exceptional curve. □

If $K_S^2 = 3$, $p_g = 2$ the theorem implies

$$\sum_k \{(2k-1)(\nu(I_k)+\nu(III_k)) + 2k(\nu(II_k)+\nu(IV_k))\} + \nu(V) = 3. \qquad (3.1.1)$$

We have the following possibilities:

1. a fibre of type (I_2);
2. a fibre of type (III_2);
3. a fibre of type (I_1), a fibre of type (II_1);
4. a fibre of type (I_1), a fibre of type (IV_1);
5. a fibre of type (III_1), a fibre of type (II_1);
6. a fibre of type (III_1), a fibre of type (IV_1);
7. a fibre of type (V), a fibre of type (II_1);
8. a fibre of type (V), a fibre of type (IV_1);
9. three fibres of type (I_1);
10. two fibres of type (I_1), a fibre of type (III_1);
11. two fibres of type (I_1), a fibre of type (V);
12. a fibre of type (I_1), two fibres of type (III_1);
13. a fibre of type (I_1), a fibre of type (III_1), a fibre of type (V);
14. a fibre of type (I_1), two fibres of type (V);
15. three fibres of type (III_1);
16. two fibres of type (III_1), a fibre of type (V);
17. a fibre of type (III_1); two fibres of type (V);

18. three fibres of type (V).

So in the first eight cases we have $\deg \mathfrak{c} = 2$, while in the remaining $\deg \mathfrak{c} = 3$; since $2 = p_g - q = \deg(2\mathfrak{f} - 2\mathfrak{c})$ we have $\deg \mathfrak{f} = \deg \mathfrak{c} + 1$, that implies that $\deg \mathfrak{f} = 3$ if $\deg \mathfrak{c} = 2$ and $\deg \mathfrak{f} = 4$ if $\deg \mathfrak{c} = 3$.

We can state:

Theorem 3.1.8. *Let S be a surface with $K^2 = 3$, $p_g = 2$. If we call $|M|$ the mobile part of the canonical system and Z the fixed part, if $M^2 = 0$ and $MZ = 2$ then S is birationally equivalent to a double cover of Σ_2. If $\deg \mathfrak{c} = 2$ the branch locus B is linearly equivalent to $6\Delta_\infty + 14\Gamma$ and has the following form:*

1a. B contains a fibre Γ and $B_0 = B - \Gamma$ has two triple points in Γ; each of these points has 2 or 3 infinitely near triple points;

2a. B contains a fibre Γ and $B_0 = B - \Gamma$ has a triple point in Γ, that has 5 or 6 infinitely near triple points;

3a. B contains a fibre Γ and $B_0 = B - \Gamma$ has two simple or $2-$ fold triple points (that may be infinitely near) in Γ and two 2-fold or 3-fold triple points (that may be infinitely near) in another fibre.

4a. B contains a fibre Γ and $B_0 = B - \Gamma$ has two 2-fold or 3-fold triple points (that may be infinitely near) and a quadruple point p on Γ that, after a blowing up in p, results in a double point on the proper transform of Γ, in another fibre.

If $\deg \mathfrak{c} = 3$ the branch locus B is linearly equivalent to $6\Delta_\infty + 16\Gamma$ and has the following form:

1b. B contains three fibres Γ_1, Γ_2 and Γ_3 and $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has two triple points (that may be infinitely near) on each fibre;

2b. B contains three fibres Γ_1, Γ_2 and Γ_3 and $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has two triple points (that may be infinitely near) on Γ_1, two triple points (that may be infinitely near) on Γ_2, a quadruple point on Γ_3, that, after a blowing up, results in a double point on the proper transform of the fibre.

3b. B contains three fibres Γ_1, Γ_2 and Γ_3 and $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has a quadruple point on Γ_1, a quadruple point on Γ_2 and two triple points (that may be infinitely near) on Γ_3; after a blowing up, each quadruple point results in a double point on the proper transform of the fibre;

4b. B contains three fibres Γ_1, Γ_2 and Γ_3 and on each fibre $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3$ has a quadruple point that, after a blowing up, results in a double point on the proper transform of the fibre.

3.2 Existence of the branch locus

Now we want to show that the surfaces we have described in the theorem 3.1.8 really exist, i.e. we are interested in finding a smooth divisor in $|\tilde{B}_0|$, where $\tilde{B}_0$ is the strict transform of B_0. This will ensure us the existence of divisors on Σ_2 that have only the singularities that we have described in the theorem. In what follows we will use the same name to indicate the exceptional curves and their strict transforms.

Remark 3.2.1. For each case, for simplicity, we prove the existence of the branch locus with as "best" as possible singularities, e.g. in case 1 we prove the existence of a branch locus with a singular fibre of type (I_2), supposing that the singular points are 3-fold triple points (and not 4-fold triple points), so we are interested in finding the general divisor for each configuration.

Case 1. B is linearly equivalent to $6\Delta_\infty + 14\Gamma$ and contains one fibre Γ_1 of type (I_2), so $B_0 = B - \Gamma_1 \sim 6\Delta_\infty + 13\Gamma$. An easy calculation shows that dim $|6\Delta_\infty +$

$13\Gamma| = 55$. We name p_1, q_1 the singular points on Γ_1, $\pi_1 : \Sigma_2' \to \Sigma_2$ the composition of the blowing ups in p_1, q_1 and E_1, F_1 the related exceptional curves. On E_1 and F_1 there are two triple points that we name, respectively, p_1' and q_1'. Let $\pi_2 : \Sigma_2'' \to \Sigma_2'$ be the composition of the blowing ups in p_1', q_1', and let E_1', F_1' be the corresponding exceptional curves. We name $\pi_3 : \Sigma_2''' \to \Sigma_2''$ the composition of the blowing ups in the quadruple points $p_1'' \in E_1'$, $q_1'' \in F_1'$ (cf. figure 3.3[1]).

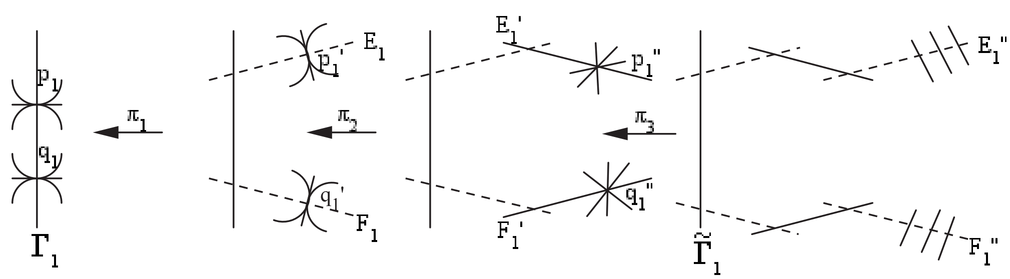


Figure 3.3: Blowing up of a singularity of type I_2 with two 3-fold triple points

Since every triple point imposes six conditions and since we can move p_1, q_1, p_1', q_1', p_1'', q_1'' on a line (the fibre or the exceptional curves) and Γ_1 on Σ_2, we have that the described singularities impose 29 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

The strict transform of B_0 is $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E_1' - 9E_1'' - 3F_1 - 6F_1' - 9F_1''$. We consider a section $\Delta_1 \in |\Delta_0|$ containing p_1 and the infinitely near points p_1' and p_1''. If we name $\tilde{\Delta}_1$ its strict transform we have $\tilde{\Delta}_1 = \pi^*\Delta_1 - E_1 - 2E_1' - 3E_1''$; analogously, if $\Delta_2 \in |\Delta_0|$ is a section containing q_1 and the infinitely near points q_1' and q_1'', $\tilde{\Delta}_2 = \pi^*\Delta_2 - F_1 - 2F_1' - 3F_1''$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - E_1' - E_1'' - F_1 - F_1' - F_1''$. Then $|\tilde{B}_0|$ contains $3\tilde{\Delta}_1 + 3\tilde{\Delta}_2 + |\pi^*\Gamma|$ and $9\tilde{\Gamma}_1 + 6E_1 + 3E_1' + 6F_1 + 3F_1' + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$. Since these two subsystems have not common components and the fixed loci of these two subsystems do not intersect,

[1]Now and in what follows the continuous line indicates the components of the branch locus in the canonical resolution, the dashed line indicates curves that are not component of the branch locus

$|\tilde{B}_0|$ has not base points, and, by Bertini theorem, its general element is smooth. This means that there exists a divisor in $|B_0|$ that has exactly the singularities we described.

Case 2. B is linearly equivalent to $6\Delta_\infty + 14\Gamma$ has one fibre Γ_1 of type (III_2). We name p_1 the singular point on Γ_1 and $\pi_1 : \Sigma_2' \to \Sigma_2$ the blowing up in p_1 with exceptional divisor E_1. On E_1 there is one triple point that we name p_1' and that we have to blow up. After other five blowing ups (that we name $\pi_2, \pi_3, \ldots, \pi_6$) we obtain a surface $\hat{\Sigma}_2$ such that the divisor induced by B has no singularity. We will name $\pi : \hat{\Sigma}_2 \to \Sigma_2$ the composition of $\pi_1, \pi_2, \ldots, \pi_6$.

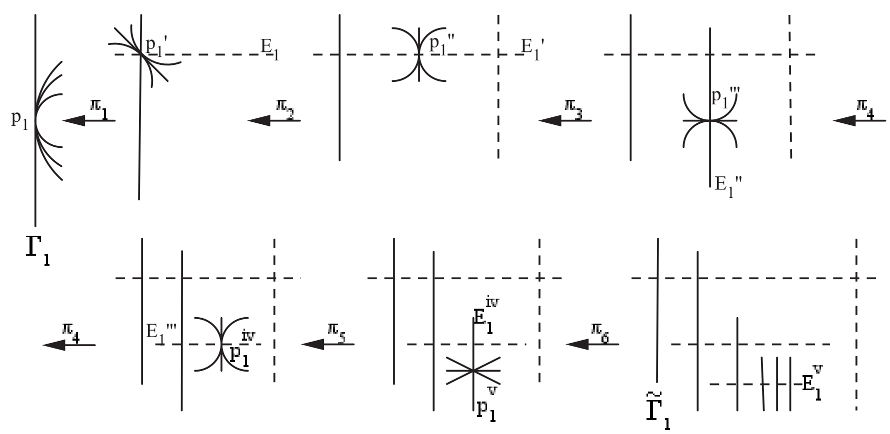


Figure 3.4: Blowing up of a singularity of type (III_2) with a 6-fold triple point

Since every triple point imposes six condition and since we can move p_1, p_1''', p_1^{iv}, p_1^v on a line (the fibre or the exceptional curves) and Γ_1 on Σ_2 we have that the described singularities impose 30 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

The strict transform $\tilde{B}_0$ of B_0 is linearly equivalent to $\pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E_1' - 9E_1'' - 12E_1''' - 15E_1^{iv} - 18E_1^v$. Let us consider a section $D \in |2\Delta_0|$ containing p_1 and the infinitely near points $p_1', p_1'', \ldots, p_1^v$; $\tilde{D} = \pi^*D - E_1 - 2E_1' - 3E_1'' - 4E_1''' - 5E_1^{iv} - 6E^v$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E_1' - 2E_1'' - 2E_1''' - 2E_1^{iv} - 2E_1^v$. Then $|\tilde{B}_0|$ contains $3\tilde{D} + |\pi^*\Gamma|$ and $9\tilde{\Gamma}_1 + 6E_1 + 12E_1' + 9E_1'' + 6E_1''' + 3E_1^{iv} + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$.

As before the fixed components of these two subsystems do not intersect, so $|\tilde{B}_0|$ does not have base points, and by Bertini theorem, its general element is smooth.

Case 3. The branch locus has one fibre of type (I_1) and one of type (II_1). Let us name them, respectively, Γ_1, Γ_2, while p_1, q_1 are the singular points on Γ_1, and p_2 and q_2 are the singular points on Γ_2. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, q_1, p_2, q_2 and E_1, F_1, E_2, F_2 are the related exceptional curves. On E_2 and F_2 there are two singular points (p_2' and q_2'). $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in p_2', q_2', and E_2', F_2' are the corresponding exceptional curves.

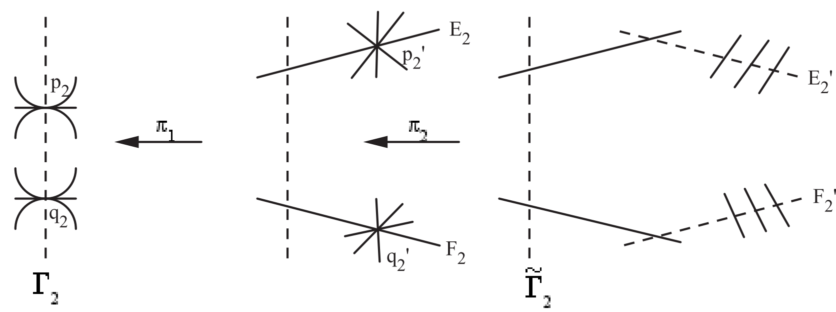


Figure 3.5: Blowing up of a singularity of type (II_1) with a 3-fold triple point

Since every triple point imposes six conditions and since we can move p_1, q_1, p_2, q_2, p_2', q_2' on a line (the fibres or the exceptional curves) and Γ_1, Γ_2 on Σ_2 we have that the described singularities impose 28 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 6E_2' - 3F_2 - 6F_2'$. We can consider a section $\Delta_1 \in |\Delta_0|$ containing p_1, p_2, p_2' and a section $\Delta_2 \in |\Delta_0|$ containing q_1, q_2, q_2'. Then $\tilde{\Delta}_1 = \pi^*\Delta_1 - E_1 - E_2 - 2E_2'$ and $\tilde{\Delta}_2 = \pi^*\Delta_2 - F_1 - F_2 - 2F_2'$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - F_1$ and $\tilde{\Gamma}_2 = \pi^*\Gamma_1 - E_2 - E_2' - F_2 - F_2'$. So $3\tilde{\Delta}_1 + 3\tilde{\Delta}_2 + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 6\tilde{\Gamma}_2 + 3E_2 + 3F_2 + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts do not intersect. This means that the general element in

$|\tilde{B}_0|$ is smooth.

Case 4. The branch locus has one fibre of type (I_1) and one of type (IV_1). We name them, respectively, Γ_1, Γ_2, while p_1, q_1 are the singular points on Γ_1, and p_2 is the singular point on Γ_2.

Let $\pi_1 : \Sigma_2' \to \Sigma_2$ be the composition of the blowing ups in p_1, q_1, p_2 and let E_1, F_1, E_2 be the related exceptional curves. On E_2 there is a quadruple points that we name p_2'. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the blowing up in p_2', and E_2' is the corresponding exceptional curve. To resolve the singularity we need other two blowing ups in two points that we name p_2'' and p_2''', while we name the related exceptional curves E_2'', E_2'''.

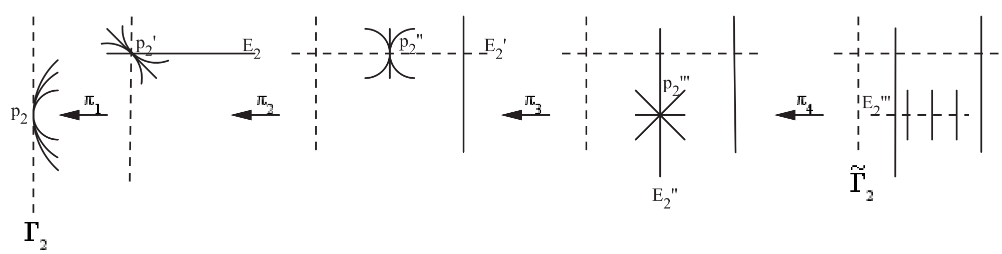


Figure 3.6: Blowing up of a singularity of type (IV_1) with a 4-fold triple point

Since every triple point imposes six conditions and since we can move p_1, q_1, p_2, p_2'', p_2''' on a line (the fibres or the exceptional curves) and Γ_1 and Γ_2 on Σ_2 we have that the described singularities impose 29 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

$\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 6E_2' - 9E_2'' - 12E_2'''$. We can consider a section $D \in |2\Delta_0|$ that contains $p_1, q_1, p_2, p_2', p_2'', p_2'''$. Then $\tilde{D} = \pi^*D - E_1 - F_1 - E_2 - 2E_2' - 3E_2'' - 4E_2'''$, $\tilde{\Gamma}_1 = \pi^*\Gamma - E_1 - F_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E_2' - 2E_2'' - 2E_2'''$. This implies that $3\tilde{D} + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 6\tilde{\Gamma}_2 + 3E_2 + 6E_2' + 3E_2'' + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $\tilde{B}_0$. Since their fixed parts do not intersect, the general element in $|\tilde{B}_0|$ is smooth.

Case 5. The branch locus has one fibre of type (III_1) and one of type (II_1), that we name, respectively, Γ_1, Γ_2, while p_1 is the singular point on Γ_1, and p_2 and q_2 are the singular points on Γ_2. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, p_2, q_2 and E_1, E_2, F_2 are the related exceptional curves. On E_1 there is another singular point (p_1') and on E_2 and F_2 there are two singular points (p_2' and q_2'). $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in p_1', p_2', q_2', and E_1', E_2', F_2' are the corresponding exceptional curves.

Since every triple point imposes six conditions and since we can move p_1, p_2, q_2, p_2', q_2' p_2'', q_2'' on a line (the fibres or the exceptional curves) and Γ_1 and Γ_2 on Σ_2 we have that the described singularities impose 29 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E_1' - 3E_2 - 6E_2' - 3F_2 - 6F_2'$. We consider a section $\Delta_1 \in |\Delta_0|$ containing p_1, p_2, p_2' and a section $\Delta_2 \in |\Delta_0|$ containing p_1, q_2, q_2'. We have that $\tilde{\Delta}_1 = \pi^*\Delta_1 - E_1 - E_1' - E_2 - 2E_2'$, $\tilde{\Delta}_2 = \pi^*\Delta_2 - E_1 - E_1' - F_2 - 2F_2'$, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E_1'$ and $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - E_2' - F_2 - F_2'$. So $3\tilde{\Delta}_1 + 3\tilde{\Delta}_2 + 3E_1 + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 6\tilde{\Gamma}_2 + 3E_2 + 3F_2 + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed part do not intersect, so the general divisor in $|\tilde{B}_0|$ is smooth.

Case 6. The branch locus has one fibre of type (III_1) and one of type (IV_1), that we name, respectively, Γ_1 and Γ_2, while p_1 is the singular point on Γ_1, and p_2 is the singular point on Γ_2. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1 and p_2, with exceptional curves E_1 and E_2. On E_1 there is a singular point that we name p_1' and on E_2 there is a singular point that we name p_2'. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the blowing up in p_1', p_2', and E_1', E_2' are the corresponding exceptional curves. To resolve the singularity we need other two blowing ups in $p_2'' \in E_2'$ and in $p_2''' \in E_2''$, where E_2'' is the exceptional curve obtained blowing up p_2' and E_2''' is the exceptional curve obtained blowing up p_2''.

Since every triple point imposes six conditions and since we can move p_1, p_2, p_2''

on a line (the fibres or the exceptional curves) and Γ_1, Γ_2 on Σ_2 we have that the described singularities impose 30 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

Now $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E_1' - 3E_2 - 6E_2' - 9E_2'' - 12E_2'''$. We consider $D \in |2\Delta_0|$ containing $p_1, p_1', p_2, p_2', p_2'', p_2'''$; then $\tilde{D} = \pi^* D - E_1 - 2E_1' - E_2 - 2E_2' - 3E_2'' - 4E_2'''$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E_1'$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E_2' - 2E_2'' - 2E_2'''$. So $3\tilde{D} + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 6\tilde{\Gamma}_2 + 3E_2 + 6E_2' + 3E_2'' + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed part do not intersect.

Case 7. The branch locus has one fibre of type (V) and one of type (II_1), that we name, respectively, Γ_1, Γ_2, while we name p_1 the singular point on Γ_1, and p_2 and q_2 the singular points on Γ_2. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, p_2, q_2 and E_1, E_2, F_2 are the related exceptional curves. On E_1, E_2 and F_2 there are three singular points that we name, respectively, p_1', p_2' and q_2'. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in p_1', p_2', q_2', and E_1', E_2', F_2' are the corresponding exceptional curves.

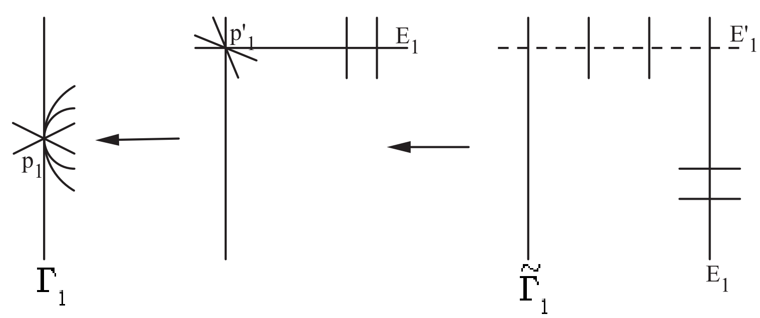


Figure 3.7: Blowing up of a singularity of type (V)

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions and since we can move p_1, p_2, q_2, p_2', q_2' on a line (the fibres or the exceptional curves) and Γ_1 and Γ_2 on Σ_2 we have that the described singularities impose 30 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

Now $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 4E_1 - 6E_1' - 3E_2 - 6E_2' - 3F_2 - 6F_2'$. We can consider a divisor $D_1 \in |2\Delta_0|$ containing $p_1, p_1', p_2, p_2', q_2, q_2'$ and a divisor $D_2 \in |2\Delta_0|$ containing p_2, p_2', q_2, q_2' and having a double point in p_1; then $\tilde{D}_1 = \pi^* D_1 - E_1 - 2E_1' - E_2 - 2E_2' - F_2 - 2F_2'$ and $\tilde{D}_2 = \pi^* D_1 - 2E_1 - 2E_1' - E_2 - 2E_2' - F_2 - 2F_2'$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E_1'$ and $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - E_2' - F_2 - F_2'$. So $2\tilde{D}_1 + \tilde{D}_2 + |\pi^*\Gamma|$ and $4\tilde{\Gamma}_1 + 6\tilde{\Gamma}_2 + 2E_1' + 3E_2 + 3F_2 + 6\pi^*\Delta_\infty + |3\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts intersect in $\tilde{D}_1 \cap E_1'$, but, since we can move D_1 so that $\tilde{B}_0$ has not fixed points, the general element in $|\tilde{B}_0|$ is smooth.

Case 8. The branch locus has one fibre of type (V) and one of type (IV_1). We name them, respectively, Γ_1, Γ_2, while p_1 is the singular point on Γ_1 and p_2 is the singular point on Γ_2. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, p_2 and E_1, E_2 are the related exceptional curves. On E_1 and E_2 there are two singular points that we name, respectively, p_1' and p_2'. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the blowing up in p_1', p_2', and E_1', E_2' are the corresponding exceptional curves. To resolve the singularity we need other two blowing ups in two points that we name p_2'', p_2''', while we name the exceptional curves E_2'', E_2'''.

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions and since we can move p_1, p_2, p_2'', p_2''' on a line (the fibres or the exceptional curves) and Γ_1 and Γ_2 on Σ_2 we have that the described singularities impose 31 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

Now $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 4E_1 - 6E_1' - 3E_2 - 6E_2' - 9E_2'' - 12E_2'''$. We can consider a divisor $D_1 \in |2\Delta_0|$ containing $p_1, p_1', p_2, p_2', p_2'', p_2'''$ and a divisor $D_2 \in |2\Delta_0|$ containing p_2, p_2', p_2'', p_2''' and having a double point in p_1; then $\tilde{D}_1 = \pi^* D_1 - E_1 - 2E_1' - E_2 - 2E_2' - 3E_2'' - 4E_2'''$ and $\tilde{D}_2 = \pi^* D_1 - 2E_1 - 2E_1' - E_2 - 2E_2' - 3E_2'' - 4E_2'''$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E_1'$ and $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E_2' - 2E_2'' - 2E_2'''$. As before $2\tilde{D}_1 + \tilde{D}_2 + |\pi^*\Gamma|$ and $4\tilde{\Gamma}_1 + 6\tilde{\Gamma}_2 + 2E_1' + 3E_2 + 6E_2' + 3E_2'' + 6\pi^*\Delta_\infty + |3\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts intersect in $\tilde{D}_1 \cap E_1'$. Since we can move D_1 in the linear system made up of the divisors in $|2\Delta_0|$ containing $p_1, p'_1, p_2, p'_2, p''_2, p'''_2$, we can say that the general element in $|\tilde{B}_0|$ is smooth.

Case 9. This is the first case with $\deg \mathfrak{c} = 3$. B has three singular fibres $\Gamma_1, \Gamma_2, \Gamma_3$ of type (I_1), with singular points, respectively, p_1 and q_1, p_2 and q_2, p_3 and q_3. $\pi_1 : \Sigma'_2 \to \Sigma_2$ is the composition of the blowing ups in the six points and $E_1, F_1, E_2, F_2, E_3, F_3$ are the corresponding exceptional curves.

Now $B \sim 6\Delta_\infty + 16\Gamma$, so, as in the previous cases, $B_0 = B - \Gamma_1 - \Gamma_2 - \Gamma_3 \sim 6\Delta_\infty + 13\Gamma$.

Since every triple point imposes six conditions and since we can move p_1, q_1, p_2, q_2, p_3, q_3 on a line (the fibres) and Γ_1, Γ_2 and Γ_3 on Σ_2 we have that the described singularities impose 27 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 3F_2 - 3E_3 - 3F_3$. Let Δ_1 be a section in $|\Delta_0|$ containing p_1, p_2, p_3 and let Δ_2 be a section in $|\Delta_0|$ containing q_1, q_2, q_3. We have that $\tilde{\Delta}_1 = \pi^*\Delta_1 - E_1 - F_1$, $\tilde{\Delta}_2 = \pi^*\Delta_2 - E_2 - F_2$, $\tilde{\Delta}_3 = \pi^*\Delta_3 - E_3 - F_3$. So $3\tilde{\Delta}_1 + 3\tilde{\Delta}_2 + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 3\tilde{\Gamma}_3 + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts do not intersect, so the general element in $|\tilde{B}_0|$ is smooth.

Case 10. The branch locus has two fibres of type (I_1) and one of type (III_1). As usual, we name them, respectively, Γ_1, Γ_2, Γ_3, while p_1, q_1, p_2, q_2 are the singular points on Γ_1 and Γ_2, respectively, and p_3 is the singular point on Γ_3. $\pi_1 : \Sigma'_2 \to \Sigma_2$ is the composition of the blowing ups in p_1, q_1, p_2, q_2, p_3 and E_1, F_1, E_2, F_2, E_3 are the related exceptional curves, p'_3 is the singular point on E_3, $\pi_2 : \Sigma''_2 \to \Sigma'_2$ is the composition of the blowing up in p'_3, and E'_3 is the corresponding exceptional curve.

Since every triple point imposes six conditions and since we can move p_1, q_1, p_2, q_2, p_3 on a line and Γ_1, Γ_2 and Γ_3 on Σ_2 we have that the described singularities impose 28 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 3F_2 - 3E_3 - 6E_3'$. If D is a section in $|2\Delta_0|$ containing $p_1, q_1, p_2, q_2, p_3, p_3'$, then we have that $\tilde{D} = \pi^*\Delta_1 - E_1 - F_1 - E_2 - F_2 - E_3 - 2E_3'$; moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - F_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_1 - E_2 - F_2$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_1 - E_3 - 2E_3'$. So we have that $3\tilde{D} + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 3\tilde{\Gamma}_3 + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$; their fixed parts do not intersect in $\tilde{\Gamma}_3 \cap E_3$, so the general element in $|\tilde{B}_0|$ is smooth.

Case 11. The branch locus has two fibres of type (I_1) (Γ_1 with singular points p_1, q_1 and Γ_2 with singular points p_2, q_2) and one of type (V) (Γ_3 with singular point p_3). $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, q_1, p_2, q_2, p_3 and E_1, F_1, E_2, F_2, E_3 are the related exceptional curves; p_3' is the other singular point on E_3. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the blowing up in p_3' and E_3' is the corresponding exceptional curve.

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions and since we can move p_1, q_1, p_2, q_2, p_3 on a line (the fibres) and Γ_1, Γ_2, Γ_3 on Σ_2 we have that the described singularities impose 29 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

Now we have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 3F_2 - 4E_3 - 6E_3'$. We can consider a divisor D_1 in $|2\Delta_0|$ containing $p_1, q_1, p_2, q_2, p_3, p_3'$ and a divisor D_2 in $|2\Delta_0|$ containing p_1, q_1, p_2, q_2 and having a double point on p_3. We have that $\tilde{D}_1 = \pi^*D_1 - E_1 - F_1 - E_2 - F_2 - E_3 - 2E_3'$, $\tilde{D}_2 = \pi^*D_1 - E_1 - F_1 - E_2 - F_2 - 2E_3 - 2E_3'$ and, moreover, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - F_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - F_2$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_2 - E_3 - 2E_3'$. So $2\tilde{D}_1 + \tilde{D}_2 + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 4\tilde{\Gamma}_3 + 2E_3' + 6\pi^*\Delta_\infty + |3\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts intersect in $\tilde{D} \cap E_3'$. Since we can move D_1 in the linear system of section in $|2\Delta_0|$ containing $p_1, q_1, p_2, q_2, p_3, p_3'$, we can say that the general element in $|\tilde{B}_0|$ is smooth.

Case 12. In this case the branch locus has one fibre of type (I_1) and two fibres of type (III_1). As before, we name the fibres, respectively, Γ_1, Γ_2, Γ_3, and we name p_1, q_1 the singular points on Γ_1 and p_2 and p_3 are the singular points on Γ_2 and Γ_3, respectively. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, q_1, p_2, p_3 and E_1, F_1, E_2, E_3 are the related exceptional curves. On E_2 and E_3 there are other singular points that we name p_2' and p_3'. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing up in p_2', p_3', and E_2', E_3'.

Since every triple point imposes six conditions and since we can move p_1, q_1, p_2, p_3 on a line (the fibres or the exceptional curves) and Γ_1, Γ_2 and Γ_3 on Σ_2, we have that the described singularities impose 29 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 6E_2' - 3E_3 - 6E_3'$. Let $D \in |2\Delta_0|$ be a section containing $p_1, q_1, p_2, p_2', p_3, p_3'$; we have that $\tilde{D} = \pi^*D - E_1 - F_1 - E_2 - 2E_2' - E_3 - 2E_3'$ and, moreover, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - F_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_1 - E_2 - 2E_2'$, $\tilde{\Gamma}_3 = \pi^*\Gamma_1 - E_3 - 2E_3'$. So we have that $3\tilde{D} + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 3\tilde{\Gamma}_3 + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$, and since their fixed parts do not intersect, the general element in $|\tilde{B}_0|$ is smooth.

Case 13. The branch locus has one fibre of type (I_1), one of type (III_1) and one of type (V). We name them, respectively, Γ_1, Γ_2, Γ_3, while p_1, q_1 are the singular points on Γ_1, p_2 is the singular point on Γ_2 and p_3 is the singular point on Γ_3. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, q_1, p_2, p_3 and E_1, F_1, E_2, E_3 are the related exceptional curves. On E_2 and E_3 there are other singular points that we name, respectively, p_2' and p_3'. $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in p_2', p_3', E_2' and E_3' are the corresponding exceptional curves.

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions and since we can move p_1, q_1, p_2 , p_3 on a line (the fibres or the exceptional curves) and Γ_1, Γ_2 and Γ_3 on Σ_2 we have that the described singularities impose 30 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 3F_1 - 3E_2 - 6E_2' - 4E_3 - 6E_3'$. Let us consider a divisor $D_1 \in |2\Delta_0|$ containing $p_1, q_1, p_2, p_2', p_3, p_3'$ and a divisor

$D_2 \in |2\Delta_0|$ containing p_1, q_1, p_2, p_2' and having a double point in p_3; then $\tilde{D}_1 = \pi^* D_1 - E_1 - F_1 - E_2 - 2E_2' - E_3 - 2E_3'$ and $\tilde{D}_3 = \pi^* D_1 - E_1 - F_1 - E_2 - 2E_2' - 2E_3 - 2E_3'$. Moreover $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - F_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E_2'$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_2 - E_3 - 2E_3'$. So $2\tilde{D}_1 + \tilde{D}_2 + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 4\tilde{\Gamma}_3 + 2E_3' + 6\pi^*\Delta_\infty + |3\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts intersect in $\tilde{D}_1 \cap E_3'$, and since we can move D_1, the general element in $|\tilde{B}_0|$ is smooth.

Case 14. The branch locus has one fibre of type (I_1) (Γ_1 with singular points p_1 and q_1) and two fibres of type (V) (Γ_2 with singular point p_2 and Γ_3 with singular point p_3). $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, q_1, p_2 and p_3, E_1, F_1, E_2 and E_3 are the corresponding exceptional curves; p_2' (resp. p_3') is the singular point on E_2 (resp. E_3). $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in p_2', p_3' and E_2', E_3' are the related exceptional curves.

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions, and since we can move p_1, q_1, p_2, p_3 on a line (the fibres or the exceptional curves) and Γ_1, Γ_2, Γ_3 on Σ_2 we have that the described singularities impose 31 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

Now we have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 4E_1 - 6E_1' - 4E_2 - 6E_2 - 3E_3 - 3F_3$. We consider a divisor D_1 in $|2\Delta_0|$ containing $p_1, q_1, p_2, p_2', p_3, p_3'$, a divisor D_2 in $|2\Delta_0|$ containing p_1, q_1, p_2, p_2' and having a double point on p_2 and a divisor D_3 in $|2\Delta_0|$ containing p_1, q_1, p_2, p_2' and having a double point on p_3. We have that $\tilde{D}_1 = \pi^* D_1 - E_1 - F_1 - E_2 - 2E_2' - E_3 - 2E_3'$, $\tilde{D}_2 = \pi^* D_2 - E_1 - F_1 - 2E_2 - 2E_2' - E_3 - 2E_3'$, $\tilde{D}_3 = \pi^* D_1 - E_1 - F_1 - E_2 - 2E_2' - 2E_3 - 2E_3'$ and, moreover, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - F_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E_2'$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_3 - E_3 - 2E_3'$. So $\tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 + |\pi^*\Gamma|$ and $4\tilde{\Gamma}_1 + 4\tilde{\Gamma}_2 + 3\tilde{\Gamma}_3 + 2E_2' + 2E_3' + 6\pi^*\Delta_\infty + |2\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ intersecting in $\tilde{D}_1 \cap E_2'$, $\tilde{D}_1 \cap E_3'$, $\tilde{D}_2 \cap E_3'$, $\tilde{D}_3 \cap E_2'$.

Since we can move D_1, D_2, D_3, we can say that $|\tilde{B}_0|$ has no fixed points.

Case 15. B has three singular fibres $\Gamma_1, \Gamma_2, \Gamma_3$ of type (III_1). We name these points p_1, p_2, p_3. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in the three points, E_1, E_2, E_3 are the related exceptional curves, $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in the triple points p_1', p_2', p_3' on E_1, E_2, E_3 and E_1', E_2', E_3' are the corresponding exceptional curves.

Since every triple point imposes six conditions and since we can move p_1, p_2, p_3 on a line (the fibres or the exceptional curves) and Γ_1, Γ_2 and Γ_3 on Σ_2 we have that the described singularities impose 30 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E_1' - 3E_2 - 6E_2' - 3E_3 - 6E_3'$. Let D be a section in $|2\Delta_0|$ containing $p_1, p_1', p_2, p_2', p_3, p_3'$; we have that $\tilde{D} = \pi^*D - E_1 - 2E_1' - E_2 - 2E_2' - E_3 - 2E_3'$, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1' - 2E_1''$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_1' - 2E_1''$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_3 - E_1' - 2E_1''$. So $3\tilde{D} + \pi^*|\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 3\tilde{\Gamma}_3 + 6\pi^*\Delta_\infty + |4\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ whose fixed parts do not intersect, and the general element in $|\tilde{B}_0|$ is smooth.

Case 16. The branch locus has two fibres of type (III_1) (we name them Γ_1, Γ_2) with singular points p_1, p_2, and one fibre of type (V) (we name it Γ_3) with singular point p_3. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in p_1, p_2, p_3 and E_1, E_2, E_3 are the corresponding exceptional curves. The singular point on E_1 (resp. E_2, E_3) is p_1' (resp. p_2', p_3'). $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in p_1', p_2', p_3' and E_1', E_2', E_3' are the related exceptional curves.

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions, and since we can move p_1, p_2 p_3 on a line (the fibres or the exceptional curves) and Γ_1, Γ_2, Γ_3 on Σ_2 we have that the described singularities impose 31 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E_1' - 3E_2 - 6E_2' - 4E_3 - 6E_3'$. We can consider a divisor D_1 in $|2\Delta_0|$ containing $p_1, p_1', p_2, p_2', p_3, p_3'$ and a divisor D_2 in $|2\Delta_0|$ containing p_1, p_1', p_2, p_2' and having a double point on p_3. We have that

$\tilde{D}_1 = \pi^* D_1 - E_1 - 2E'_1 - E_2 - 2E'_2 - E_3 - 2E'_3$, $\tilde{D}_2 = \pi^* D_1 - E_1 - 2E'_1 - E_2 - 2E'_2 - 2E_3 - 2E'_3$ and, moreover, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E'_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E'_2$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_2 - E_3 - 2E'_3$. So $2\tilde{D}_1 + \tilde{D}_2 + |\pi^*\Gamma|$ and $3\tilde{\Gamma}_1 + 3\tilde{\Gamma}_2 + 4\tilde{\Gamma}_3 + 2E'_3 + 6\pi^*\Delta_\infty + |3\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ that intersect in $\tilde{D}_1 \cap E'_1$. Since we can move D_1 we have that the general element in $|\tilde{B}_0|$ is smooth.

Case 17. In the fifteenth case the branch locus has one fibre of type (III_1) (Γ_1 with singular point p_1) and two fibres of type (V) (Γ_2 with singular point p_2 and Γ_3 with singular point p_3). $\pi_1 : \Sigma'_2 \to \Sigma_2$ is the composition of the blowing ups in p_1, p_2, p_3, E_1, E_2, E_3 are the corresponding exceptional curves. On each of them there is another singular point, that we name, respectively, p'_1, p'_2, p'_3. $\pi_2 : \Sigma''_2 \to \Sigma'_2$ is the composition of the blowing ups in p'_1, p'_2, p'_3 and E'_1, E'_2, E'_3 are the related exceptional curves.

Since every quadruple point imposes ten conditions, every triple point imposes six conditions, every double point imposes three conditions and since we can move p_1, p_2, p_3 on a line (the fibres or the exceptional curves) and Γ_1, Γ_2, Γ_3 on Σ_2 we have that the described singularities impose 32 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

Now we have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 3E_1 - 6E'_1 - 4E_2 - 6E_2 - 4E_3 - 6E'_3$. We can consider a divisor D_1 in $|2\Delta_0|$ containing $p_1, p'_1, p_2, p'_2, p_3, p'_3$, a divisor D_2 in $|2\Delta_0|$ containing p_1, p'_1, p_3, p'_3 and having a double point on p_2 and a divisor D_3 in $|2\Delta_0|$ containing p_1, p'_1, p_2, p'_2 and having a double point on p_3. We have that $\tilde{D}_1 = \pi^* D_1 - E_1 - 2E'_1 - E_2 - 2E'_2 - E_3 - F_3$, $\tilde{D}_2 = \pi^* D_2 - E_1 - 2E'_1 - 2E_2 - 2E'_2 - E_3 - F_3$, $\tilde{D}_3 = \pi^* D_1 - E_1 - 2E'_1 - E_2 - 2E'_2 - 2E_3 - F_3$ and, moreover, $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1 - 2E'_1$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_2 - 2E'_2$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_2 - E_3 - 2E'_3$. So $\tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 + |\pi^*\Gamma|$ and $4\tilde{\Gamma}_1 + 4\tilde{\Gamma}_2 + 3\tilde{\Gamma}_3 + 2E'_2 + 2E'_3 + 6\pi^*\Delta_\infty + |2\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$ intersecting in $\tilde{D}_1 \cap E'_2, \tilde{D}_1 \cap E'_3, \tilde{D}_2 \cap E'_3, \tilde{D}_3 \cap E'_2$. Since we can move D_1, D_2, D_3 we can say that $|\tilde{B}_0|$ has no fixed points.

Case 18. Now B has three singular fibres $\Gamma_1, \Gamma_2, \Gamma_3$ of type (V) with singular points p_1, p_2, p_3 respectively. $\pi_1 : \Sigma_2' \to \Sigma_2$ is the composition of the blowing ups in the three points, E_1, E_2, E_3 are the related exceptional curves, $\pi_2 : \Sigma_2'' \to \Sigma_2'$ is the composition of the blowing ups in the singular points p_1', p_2', p_3' on E_1, E_2, E_3, E_1', E_2', E_3' are the related exceptional curves.

Since every quadruple point imposes ten conditions, since every double point imposes three conditions and since we can move p_1, p_2, p_3 on a line (the fibres) and Γ_1, Γ_2, Γ_3 on Σ_2 we have that the described singularities impose 33 conditions to the linear system $|6\Delta_\infty + 13\Gamma|$.

We have that $\tilde{B}_0 \sim \pi^*(6\Delta_0 + \Gamma) - 4E_1 - 6E_1' - 4E_2 - 6E_2' - 4E_3 - 6E_3'$. Let D_1 be a section in $|2\Delta_0|$ containing p_2, p_2', p_3, p_3' and having a double point in p_1, then $\tilde{D}_1 = \pi^* D - 2E_1 - 2E_1' - E_2 - 2E_2' - E_3 - 2E_3'$; analogously, if D_2 is a section in $|2\Delta_0|$ containing p_1, p_1', p_3, p_3' and having a double point in p_2 and D_3 is a section in $|2\Delta_0|$ containing p_1, p_1', p_2, p_2' and having a double point in p_3 we have that $\tilde{D}_2 = \pi^* D - E_1 - 2E_1' - 2E_2 - 2E_2' - E_3 - 2E_3'$ and $\tilde{D}_3 = \pi^* D - E_1 - 2E_1' - E_2 - 2E_2' - 2E_3 - 2E_3'$; moreover we have that $\tilde{\Gamma}_1 = \pi^*\Gamma_1 - E_1' - 2E_1''$, $\tilde{\Gamma}_2 = \pi^*\Gamma_2 - E_1' - 2E_1''$ and $\tilde{\Gamma}_3 = \pi^*\Gamma_3 - E_1' - 2E_1''$. So $\tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 + \pi^*|\Gamma|$ and $4\tilde{\Gamma}_1 + 4\tilde{\Gamma}_2 + 4\tilde{\Gamma}_3 + 2E_1' + 2E_2' + 2E_3' + 6\pi^*\Delta_\infty + |\pi^*\Gamma|$ are two subsystems of $|\tilde{B}_0|$, whose fixed parts intersect in $\tilde{D}_1 \cap E_2'$, $\tilde{D}_1 \cap E_3'$, $\tilde{D}_2 \cap E_1'$, $\tilde{D}_2 \cap E_3'$, $\tilde{D}_3 \cap E_1'$, $\tilde{D}_3 \cap E_2'$. Since we can move D_1, D_2, D_3 we can say that $|\tilde{B}_0|$ has no fixed points.

3.3 The number of moduli

In this section we want to determine if minimal surfaces with $p_g = 2$, $K^2 = 3$ of type (I) induce an irreducible component or a subvariety of a component in the moduli space, and determine its dimension.

As it is well known, there exists a quasi-projective coarse moduli space $\mathfrak{M}$ for minimal surfaces of general type modulo birational equivalence (cf. [Gie77]). We know, by the results of E. Bombieri, (cf. [Bom73]), that the surfaces of general type with given numerical invariants χ, K^2 belong to a finite number of families. We introduce the number $m(S)$, the number of moduli of S, that is, by definition, the dimension of $\mathfrak{M}$ at the point $[S]$, representing the isomorphism class of S in $\mathfrak{M}$. By the results of Kuranishi, locally around the point $[S]$, $\mathfrak{M}$ is analytically isomorphic to the quotient of the base B of the Kuranishi family by the finite group $\mathrm{Aut}(S)$. Hence, if $\mathcal{T}_S$ is the tangent sheaf of S,

$$\dim H^1(S, \mathcal{T}_S) \geq m(S) \geq \dim H^1(S, \mathcal{T}_S) - \dim H^2(S, \mathcal{T}_S) = 10\chi - 2K^2.$$

In particular, we have that

Theorem 3.3.1. *Let $[S]$ be a generic point of the moduli space $\mathfrak{M}$. Then $m(S) \geq 10\chi(S) - 2K_S^2$.*

Now we want to count the number of parameters on which each family of double covers depend:

Lemma 3.3.2. *Let S be a minimal surface with $\chi = 3$, $K^2 = 3$ of type (I), then the values of $m(S)$ are those specified in the following table:*

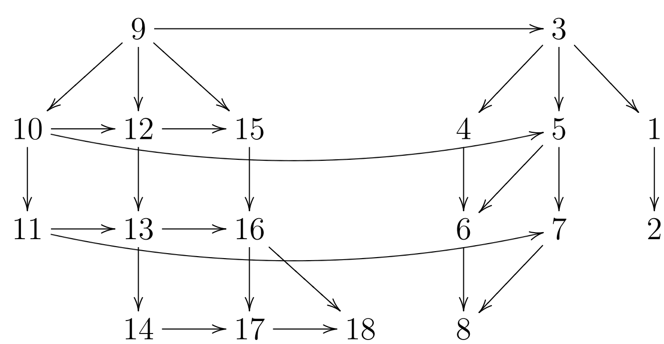
$B \sim 6\Delta_\infty + 14\Gamma$	*Case 1*	$m(S) = 19$	$B \sim 6\Delta_\infty + 16\Gamma$	*Case 9*	$m(S) = 21$
	Case 2	$m(S) = 18$		*Case 10*	$m(S) = 20$
	Case 3	$m(S) = 20$		*Case 11*	$m(S) = 19$
	Case 4	$m(S) = 19$		*Case 12*	$m(S) = 19$
	Case 5	$m(S) = 19$		*Case 13*	$m(S) = 18$
	Case 6	$m(S) = 18$		*Case 14*	$m(S) = 17$
	Case 7	$m(S) = 18$		*Case 15*	$m(S) = 18$
	Case 8	$m(S) = 17$		*Case 16*	$m(S) = 17$
				Case 17	$m(S) = 16$
				Case 18	$m(S) = 15$

Proof. The number of parameters is obtained subtracting to the dimension of $|6\Delta_0 + \Gamma|$ the number of the conditions which the singularities impose to the branch curve B (which we counted in section 3.2) and the dimension of $\mathrm{Aut}(\Sigma_2)$, since we have a natural action of $\mathrm{Aut}(\Sigma_2)$ on the parameters. □

In our case, $10\chi(S) - 2K_S^2 = 24$, and since we always get that our families have dimension smaller that 24, we can conclude that they do not give rise to an irreducible component of $\mathfrak{M}$.

Theorem 3.3.3. *The closure in moduli of the surfaces presenting case 9 is a proper irreducible subvariety of dimension 21 of the moduli space.*

Proof. We have the following degenerations: a fibre of type (I_1) degenerates to a fibre of type (III_1), a fibre of type (III_1) degenerates to a fibre of type (V) (see [Hor78]), a fibre of type (II_1) degenerates to a fibre of type (IV_1), a fibre of type (I_2) degenerates to a fibre of type (III_2). We have also that two fibres Γ_1 and Γ_2 of type (I_1) degenerate to a fibre Γ_3 of type (II_1), letting Γ_1 and Γ_2 come together to Γ_3. The same is true if we have a fibre of type (I_1) and a fibre of type (II_1), that come together to a fibre of type (I_2). We will motivate these degenerations in the following section. Hence we have the following diagram of specialization in moduli:



□

3.4 Collision of points

In this section we recall and develop a result from [CM05].

We want to analyze degenerations of schemes defined by two triple points on a surface X.

Let X be a surface, $f : \mathcal{X} = X \times \Delta \to \Delta$ a trivial family, C a very ample a divisor in X; $f^*C = \mathcal{C} \subset \mathcal{X}$ is an irreducible surface. We want to analyze the divisors in $|C|$ when the fat points degenerate in a general way to a point p on X. We will denote with $\mathcal{X}_t, \mathcal{C}_t, \ldots$ the fibre of a family $\mathcal{X}, \mathcal{C}, \ldots$.

Collision of two simple triple points. We suppose that $\mathcal{C}$ contains two curves s_1, s_2 which determine, via f, two triple points for $\mathcal{C}_t \cong C$, $t \neq 0$, and which are such that they intersect in $p \in \mathcal{X}_0$, where $\mathcal{X}_0$ is the central fibre. Blow up the point $(p, \{0\})$ in the central fibre to a plane $E \cong \mathbb{P}^2$. Then the new central fibre consists of two surfaces, the blowing up X' of X at p and the plane E, meeting along a smooth rational curve R. R is the exceptional curve in X' and is a line in E, since, named $\pi : \mathcal{X}' \to \mathcal{X}$ the blowing up, we have that $\mathcal{X}'_0 = E \cup X'$ and $0 = R \cdot \mathcal{X}'_t = R \cdot \mathcal{X}'_0 = E \cdot X' \cdot \mathcal{X}'_0 = R^2_{X'} + R^2_E$, so $R^2_E = -R^2_{X'} = 1$. The assumption that the points are degenerating generally means that the two limit points are two general points on E.

If we denote by $\mathcal{C}$ also the pullback bundle on $\mathcal{X}'$ and by C the pullback bundle on X', one limiting line bundle on the central fibre is the bundle which is C on X' and is trivial on E. Other limiting line bundles to C on the general fibre are obtained by twisting this basic limiting line bundle by the divisor $-tE$, for any integer t; this gives the limiting line bundle which is $C - tR$ con X', and $\mathcal{O}(t)$ on the plane E. Any general curve in C on the general X with multiplicity 3 at the two points will degenerate to a curve in the central fibre, which must be a union of two curves, one in X' and one in E.

We choose t so that it is minimal with respect to the property that $\mathcal{O}(t)$ has a nonzero section with multiplicity 3 at the two general limit points on E; this gives rise to the lowest multiplicity of the limit curve on X'. So $t = 3$, since the divisor with minimal degree that has two triple points at the two general limit points on

E is three times the line L joining the two points of multiplicity 3. Hence we can consider $\mathcal{C}' = \mathcal{C} - 3E$; we have that $\mathcal{C}' \cap E = 3L$ and $\mathcal{C}' \cap R = 3L \cap R = 3q$, where $q = L \cap R$. So, we have that if two triple points come together generically, the limit point is a triple point.

Now let us blow up $\mathcal{X}'$ along $\tilde{s}_1$, $\tilde{s}_2$ (that are the strict transforms of s_1, s_2): we name $\pi_2 : \mathcal{X}'' \to \mathcal{X}'$ the map, F_1, F_2 the exceptional divisors (which induce the exceptional curves $F_{1,t}, F_{2,t}$ in the fibre $\mathcal{X}''_t$). Note that the strict transform E' of E is isomorphic to the blow up of E in $p_1 = \tilde{s}_1 \cap E$ and $p_2 = \tilde{s}_2 \cap E$, so L', the strict transform of L, is a (-1) curve on E' and $\mathcal{C}'' = \pi_1^*\mathcal{C}' - 3F_1 - 3F_2 = \mathcal{C} - 3E' - 3F_1 - 3F_2$, hence $\mathcal{C}'' \cdot L' = -3{L'_{E'}}^2 - 3F_1 \cdot L' - 3F_2 \cdot L' = -3$.

Now let $\pi_3 : \mathcal{X}''' \to \mathcal{X}''$ be the blow up along L'. This will create the exceptional divisor $\Theta \subset \mathcal{X}'''$ which will be a ruled surface; L' becomes a section of the ruling on Θ. For $t \neq 0$, $\mathcal{X}'''_t = \mathcal{X}''_t$, but the central fibre consists now of three components, Θ, the proper transforms X'' of X' and E'. Note that X'' is isomorphic to the blow up of X' at the point on R where L' intersect R. We will denote by T the exceptional curve on X''; T is a fibre of the ruling of Θ. Let α be $mult_{L'}\mathcal{C}''$; then, if we denote Θ_λ a fibre of Θ, $\mathcal{C}'''_{|\Theta} = \pi_3^*\mathcal{C}''_{|\pi_3^*L'} - \alpha\Theta_{|\Theta} = -3\Theta_\lambda + \alpha L' + \alpha T = (\alpha - 3)\Theta_\lambda + \alpha L'$. This divisor is effective if and only if $\alpha \geq 3$, but, since $\mathcal{C}'' \cap E' = 3L'$, we have also that $\alpha \leq 3$, so $\alpha = 3$ and L' is contained in $\mathcal{C}''$ with multiplicity 3. Hence the curves of X'' must have a point of multiplicity three at the point of intersection $L' \cap R$. This means that on X we have a point of multiplicity 3 with an infinitely near point of multiplicity 3.

If $X = \Sigma_2$ we can suppose that $s_i : \Delta \to \Delta \times \Sigma_2$, so that $s_i(t) \in \{t\} \times \Gamma$, hence $s_1, s_2 \subset \Delta \times \Gamma$. We have that $\Gamma \cap E = \mathbb{P}(T_p(0,\Gamma))$ and $T_p(0,\Gamma) \subset T_p(\Delta \times \Gamma)$, but $T_p(\Delta \times \Gamma) = < T_p s_1, T_p s_2 >$, and $L = \mathbb{P} < T_p s_1, T_p s_2 >$, so $\Gamma \cap E = \Gamma \cap L$, and we can say that if two triple points come together along a fibre, their limit is tangent to the fibre.

Summarizing, if two simple triple points in Σ_2 come together, their limit is a 2-fold triple point and if they come together along a fibre, their limit is tangent to the fibre.

Collision of two 2-fold triple points. Now s_1, s_2 are two curves of 2-fold triple points on $\mathcal{C}$.

The construction is analogous to the previous: we blow up the point $(p, \{0\})$ to a plane $E \cong \mathbb{P}^2$ in the central fibre, the blow up X' of X at p and the plane E meet along a smooth rational curve R, which is the exceptional curve in X' and is a line in E.

We name $\mathcal{C}$ the pullback bundle on $\mathcal{X}'$; it has 2-fold triple points in $\tilde{s}_1$ and $\tilde{s}_2$. The curve of minimal degree in $\mathbb{P}^2$ with two 2-fold triple points is again three times the line L containing these points. By the generality of $\mathcal{C}$, we can suppose that $\mathcal{C} \cap E = 3L$, so $\mathcal{C}' \sim \mathcal{C} - 3E$ and $\mathcal{C}' \cap R = 3L \cap R = 3q$.

Let $\pi_2 : \mathcal{X}'' \to \mathcal{X}'$ be the blowing up of $\mathcal{X}'$ along $\tilde{s}_1$ and $\tilde{s}_2$, let F_1 and F_2 be the exceptional divisors, let γ_1, γ_2 be the strict transforms of $\tilde{s}_1, \tilde{s}_2$, let E' be the blowing up of E in $p_1 = \tilde{s}_1 \cap E$ and $p_2 = \tilde{s}_2 \cap E$ and L' the strict transform of L. $L'^2_{E'} = -1$ and $\mathcal{C}'' = \mathcal{C} - 3E' - 3F_1 - 3F_2$, so $\mathcal{C}'' \cdot L' = -3$.

Now we blow up $\mathcal{X}''$ along L' ($\pi_3 : \mathcal{X}''' \to \mathcal{X}'$ is the blow up and Θ is the new exceptional divisor, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the strict transforms of γ_1 and γ_2). We have that $\mathcal{X}'''_0 = E' + \Theta + X''$, ${L'_{E'}}^2 = -1$, ${L'_\Theta}^2 = 0$, ${T_{X''}}^2 = -1$, $T^2_\Theta = 0$ (where $T = \Theta \cap X''$), so $\Theta = \mathbb{P}^1 \times \mathbb{P}^1$. Let us set $q_1 = \tilde{\gamma}_1 \cap \Theta$, $q_2 = \tilde{\gamma}_2 \cap \Theta$, $\alpha = mult_{L'}\mathcal{C}''$. As before $\mathcal{C}'''_{|\Theta} = -3\Theta_\lambda + \alpha L' + \alpha\Theta_\lambda = (\alpha - 3)\Theta_\lambda + \alpha L'$ (Θ_λ is a fibre of Θ). $\alpha = 3$, hence $\mathcal{C}''$ has a point of multiplicity 3 in q and, coming back to $\mathcal{C}$, $\mathcal{C}$ has at least a 2-fold triple point in p.

Now, since $\mathcal{O}_\Theta(\Theta)_{|L'} = \mathcal{O}_\Theta(-L' - T)_{|L'} = \mathcal{O}_{L'}(-1)$ we can contract Θ on $\Theta \cap X'$ through a map $\pi_4 : \mathcal{X}''' \to \mathcal{Y}$; we name $\mathcal{S} = \pi_4(\mathcal{C}''')$. Through π_4, E' induces $\tilde{E} = \mathbb{P}^1 \times \mathbb{P}^1$ (the images $\tilde{F}_{1,0}, \tilde{F}_{2,0}$ of $F_{1,0}, F_{2,0}$ are such that $\tilde{F}^2_{1,0} = \tilde{F}^2_{2,0} = 0$) and we have that $R_{|\tilde{E}} \sim (1,1)$ ($R^2_{\tilde{E}} = 2$), so R is a section of the two projection morphisms from $\tilde{E}$ to $\mathbb{P}^1$. Moreover, $\mathcal{O}_{\tilde{E}}(\tilde{E})_{|\tilde{F}_{i,0}} = \mathcal{O}_{\tilde{E}}(-R)_{|\tilde{F}_{i,0}} = \mathcal{O}_{\tilde{F}_{i,0}}(-1)$, so we can contract $\tilde{E}$ on R, contracting one of the rulings $\tilde{F}_{1,0}$, $\tilde{F}_{2,0}$. We name $\pi_5 : \mathcal{Y} \to \mathcal{Y}_1$ the contraction and $\mathcal{S}_1 = \pi_5(\mathcal{S})$. $\mathcal{S}_1$ has two curves of triple points that intersect T in two points r_1, r_2 that are imagine of $(\tilde{\gamma}_1 \cup \tilde{\gamma}_2) \cap \Theta$ in the contraction of Θ on T. These two points must coincide, in fact, if $r_1 \neq r_2$, we would have $\deg \mathcal{C}'''_{|T} \geq 6$, but $X' \cap \mathcal{C}''$ has in q a triple point, so $\mathcal{C}'''_{|X''} = \pi_3^*(\mathcal{C}''_{|X'}) - 3T$, that implies that $\mathcal{C}'''_{|X''}$ does not contain T and intersects it with multiplicity exactly equal to three; the contractions do not have effects on $\mathcal{C}'''_{|X''}$, $\mathcal{S}_{|X''}$, $\mathcal{S}_{1|X''}$, so $r_1 = r_2 = r$ and by the result in the previous paragraph we have that $\mathcal{C}'''_{|X''} = \mathcal{S}_{|X''} = \mathcal{S}_{1|X''}$ has a 2-fold triple point in r. We can conclude that two 2-fold triple points that come together generically, come together to a 4-fold triple point.

As in the previous case, if $X = \Sigma_2$ and two 2-fold triple points come together along the fibre their limit point is tangent to the fibre.

Now we want to prove that if the two 2-fold triple points come together along a fibre Γ_1 and $C = 6\Delta_0 + \Gamma_1$, their limit has a contact of order 6 with the fibre. To do this, we can consider $\mathcal{C}'' \cdot \tilde{\Gamma}_1$, where $\tilde{\Gamma}_1$ is the strict transform of Γ_1: we have that $\mathcal{C}'' \cdot \tilde{\Gamma}_1 = (\mathcal{C} - 3E' - 3F_1 - 3F_2) \cdot \tilde{\Gamma}_1 = 3$, so we are done.

Summarizing, if two 2-fold triple points come together along the fibre their limit point is a 4-fold triple point tangent to the fibre with a contact of order 6.

Collision of two 3-fold triple points. The argument is the same as the previous paragraph, but now, when we blow up $\tilde{s}_1$ and $\tilde{s}_2$, we obtain two 2-fold triple points on the general fibre $\mathcal{X}''_t$, so, using the result of the previous paragraph, r is a 4-fold triple point for $\mathcal{C}'''$, and p is a 6-fold triple point.

More generally, the argument of the previous paragraph can be generalized to two k-fold triple points that come together to a $2k$-fold triple point.

Using similar arguments one can prove that an m-fold triple point and an n-fold triple point come together to a $(m+n)$-fold triple point.

Remark 3.4.1. The previous degenerations can be deduced from remark 4.15 in [CP06]. E.g., we can see that a fibre of type (I_1) degenerates to a fibre of type (III_1) simply letting λ go to 0.

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