UNIVERSITA' DELLA CALABRIA

DOTTORATO DI RICERCA IN MATEMATICA E INFORMATICA

XX Ciclo

Tesi di Dottorato (S.S.D. MAT/07 Fisica Matematica)

Outflanking quantum complementarity: Detection of incompatible properties in double-slit experiments

Candidata

dott.ssa Angela Sestito

Supervisore

Prof. Giuseppe Nisticò

Coordinatore

Prof.ssa Annamaria Canino

A.A. 2006-2007

Contents

List of Figures

Introduction

The modern scientific age begins in XVI century when N. Copernicus suggests that it is the Sun, rather than Earth, the center of the solar system; it is the first time that observations and experiments take the place of religious or philosophical dogma in order to explain natural phenomena. In XVII century I. Newton expressed the laws of nature in quantitative form and used mathematics to deduce the motion of physical systems: to each physical system a certain number of dynamical variables are associated, possessing, at each instant, a *well-defined* value; the specification of this set of values defines the dynamical state of the system at any instant. The time evolution of the system is postulated to be completely determined by the equations of motion if the state at a given initial time is known. This mathematical approach culminates in XIX century when J.C. Maxwell shows that all phenomena about electricity and magnetism can be deduced from four equations (Maxwell's equations, 1855). Now it seems that every phenomenon is governed by well known equations: matter, made up of localizable particle, is subject to Newton's laws; radiation obeys to Maxwell's laws and, in contrast to matter, it exhibits a wave-like behaviour which manifest itself through phenomena of interference and diffraction. At the end of XIX century, efforts of physicists had the purpose, at first, to make a precise analysis of the microscopic structure of matter and, then, to determine the interaction of particles with the electromagnetic field; after the discovery of the electron (J.J. Thomson, 1897), a complete theory of such an interaction is established in the electron theory of Lorentz [1]. From now on, decisive progresses (study of Brownian motion, determination of Avogadro's number, measurement of elementary electromagnetic charge by Millikan, Geiger counter, etc.) confirm the existence of atoms and molecules. On the other hand, new advances in spectral analysis of radiation are established by the discovery of x-rays (Röntgen, 1895). However, as studies in matter and radiation becomes more and more precise, the comparison of the prediction of the theory with the experimental results reveals a deep disagreement. As a matter of fact, classical theory is not able to justify the energy distribution in the continuous spectrum of black-body radiation (a detailed study is in [2]) arising from experimental observations. The study of such a phenomenon led Planck (1900) to give up the classical laws of interaction between matter and radiation, and to postulate that they cannot exchange energy with each other in continuous manner but by discrete quantities, called *quanta*; moreover, he showed that the quantum of energy must be proportional to the frequency ν of the radiation, $\epsilon = h\nu$, where h is known as *Planck's constant.* Taking an opposite view, the study of the photoelectric effect led Einstein (1905) to postulate that light radiation itself consists of a beam of particles, *light quanta* or *photon* of energy $E = h\nu$ and velocity c, hypothesis confirmed by the Compton effect (1924). From experimental results, it turns out that light exhibits both particle and wave-like behaviour, depending upon the phenomenon under consideration. The acceptance of wave-particle duality induced Bohr to develop his idea of complementarity: the two descriptions are mutually exclusive but necessary for an exhaustive treatment of a quantum physical system; moreover, this kind of duality arises from the indeterminateness of the concept of *observation*: it is impossible to separate the object to be observed from the effect of the measuring apparatus that must be expressed in classical terms; furthermore, the observation depends upon the chosen experimental set-up (for a detailed description of the conceptual development of quantum theory see [3]).

The physical situation better describing wave-particle duality is double-slit experiment; it is an ideal experiment, i.e. simple in principle and disregarding the difficulties of actually doing it, whose set-up consists of two screens placed in front of each other; the first is endowed with two slits, labeled *slit 1* and *slit 2*; a beam of electrons passes through the first screen and then they impinge on the second one. Let us suppose that the intensity of the beam is very low, so that electrons arrive one at a time; we are interested in the distribution of the electrons on the final screen. If we close slit 2 and measure the distribution D_1 of the electrons on the final screen, with only slit 1 open, we find the curve in figure 1.a; similarly, if slit 1 is closed and we measure the distribution D_2 of the electrons with only slit 2 open, we find curve in figure 1.b. If both slits are open, we are lead to suppose that, since electrons behave as particles, each of them either goes through slit 1 *or* it goes through slit 2; assuming such a proposition, we expect that the total distribution D_{12} of the electrons on the final screen is given by the sum of the previous distributions, $D_{12} = D_1 + D_2$ (figure 1.c); however, by direct measurement, the observed distribution D (figure 1.d) is not in agreement with the expected D_{12}. It is in this sense that that electrons behave sometimes like particles and sometimes like wave.

This kind of experiment makes evident that a description of physical reality

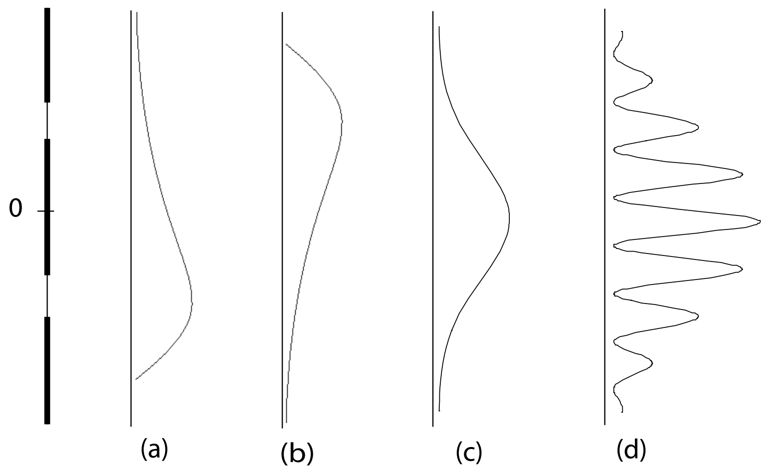

Figure 1. Classical reasoning: a) slit 2 closed; b) slit 1 closed; c) both slits open

in classical term is not possible; a new and shaking formulation is needed. The problem of its construction, in the first twenty-five years of XX century, culminates between 1923 and 1927 with more than one formulation of Quantum Theory; the *wave mechanics*, proposed by Schrödinger and the *matrix mechanics* of Heisenberg are the main ones; however, the more rigorous and elegant formulation, presented for the first time in 1927, is due to Von Neumann. He develops an axiomatic theory of abstract Hilbert spaces which makes possible to reformulate a statistical foundation of quantum mechanics. In such a manner, *observables* and *expectation values*, represented in mathematical terms by self-adjoint operators on Hilbert spaces and measurable functions respectively, take the place of the dynamical variables of classical physics and its well-defined values.

A deep abstraction characterizes quantum theory, that, this notwithstanding, provides the key for the explanation of the unsolved phenomena, in perfect agreement with experience. Coherently with the theory, no contradiction arises from double-slit experiment: there are some properties that make sense for a system, and some others that do not; in presence of the interference pattern (figure 1.c) the property of the passage through which slit (that from now on we shall indicate as *WS property*) does not make sense and, similarly, the knowledge of which slit the particle passes through makes the interference pattern without sense. Moreover, the point of impact on the final screen and WS property are complementary observables so that, according to the formal definition of complementarity provided in the eighties by Busch and Lahti, there exists no measurement apparatus providing knowledge about both of them. Though no *direct* measurement is allowed, several devices are conceived over the years providing *indirect* knowledge of WS property; among them, the experiment, conceived in 1991 by Englert, Scully and Walther (ESW) [4], provides WS knowledge by exploiting recent advances in quantum optics. The experiment works as follows: instead of measuring WS property, represented by the projection operator E, a different property, represented by T, correlated with it and compatible with the measurement of the final impact point, is measured; in such a manner, inferences about WS property E can be made by means of the outcome of T and no interference appears, of course. An alternative set-up of ESW experiment makes possible to indirectly ascertain a different property L, incompatible with WS one, by means of a second supplementary measurement V, restoring interference pattern for those particles which turn out to have this alternative property.

In this work we face the question of ascertaining together *more* incompatible properties by exploiting the notion of detector, within the framework of double-slit experiment. Namely, we shall show that WS knowledge can be ascertained together with an incompatible property G, by means of two supplementary measurement T (for E) and Y (for G); in such a manner, three incompatible properties, E, G and the final impact point, can be inferred together by means of two simultaneous measurements, T and Y. This kind of detection is made possible by the fact that besides the position of the centre-of-mass, our system possesses further degrees of freedom. As a consequence, the Hilbert space describing the entire system can be decomposed as $\mathcal{H}_I \otimes \mathcal{H}_{II}$, where $\mathcal{H}_I$ is the Hilbert space used to represent the position observable, and $\mathcal{H}_{II}$ is the Hilbert space used to represent the observable arising from the further degrees of freedom. The detection of Which Slit property E is obtained by measuring an observable represented by a particular projection operator T acting on $\mathcal{H}_{II}$, which is correlated, in the particular quantum state of the system, with which-slit property, so that this last can be inferred from the outcome of T. This kind of detection was studied from a mathematical point of view in [5] and [6], where the interpretative questions were analyzed. The possibility of detecting an incompatible property G is provided by the existence of an observable represented by another projection operator Y acting on $\mathcal{H}_{II}$, but which can be measured together with T. A systematic investigation establishes that the existence of such an observable (projection operator) depends on the dimension of space $\mathcal{H}_I$. In particular, no solution exists if $dim(\mathcal{H}_I) < 4$; they exist but are correlated if $dim(\mathcal{H}_I) = 4$ [7]; they exist, also non correlated for $dim(\mathcal{H}_I) = 6$ [8]. We notice that our approach to the problem of inferring the outcomes of more incompatible properties is not the only one. In [9], Vaidman and his co-workers (VAA) describe a procedure allowing to make inferences about the three cartesian spin-components of a spin-$\frac{1}{2}$ particle. However, this kind of inferences are of a quite different nature with respect to the detections we are dealing with in the present work; moreover, VAA claim that, coherently with their method, it is not possible to produces inferences about more than three observables. For this reason, the question whether two mutually incompatible properties, G and L, both incompatible with Which Slit property E, can be detected, together with the measurement of the final impact point (four incompatible properties), is investigated. In particular, we show that such a question has an affirmative answer; as in the previous case, the existence of solutions depends on dimension of space $\mathcal{H}_I$; we find a particular solution for $dim(\mathcal{H}_I) = 10$, nevertheless, in such a case the properties L and G turn out to be correlated.

The content of this thesis is organized in three chapters.

In chapter 1 Von Neumann theory is presented, in a modern form; once introduced the axiomatic basis (sect. 1.1), an essential tool in order to investigate the properties of self-adjoint operators, hence of observables, is the spectral representation theorem (sec 1.2); density operators are the mathematical representation of the expectation values (sect. 1.3); for these reasons either observables and expectation values are strictly linked to projection operators, which represent the particular class of 0-1 observables, interpreted as properties that the physical system may or may not possesses (sect. 1.4 and 1.5). Taking into account the system of axioms of the Von Neumann approach, several consequences follow: since the algebraic structure of self-adjoint operators is not commutative, then it is not possible to measure on an individual specimen of the physical system (we shall say *simultaneously*) all the observables but only those corresponding to commutative self-adjoint operators (sect. 1.6). Hence, in a natural manner, the concept of complementarity is introduced, following the approach proposed by Busch and Lahti (sect. 1.8). This chapter has an introductory character; its role is to illustrate the notions involved in the thesis and to establish the notation. It could be skipped by the reader well aware of Quantum Mechanics.

In chapter 2 we examine the double-slit experiment, according to the quantum theory; after a detailed mathematical description, we can see that empirical results are perfectly in agreement with the quantum theoretical predictions (sect. 2.2); moreover, now we are able to solve the interpretative question, remained open, about fig.1: WS property cannot be measured together with the final impact point because they are complementary observables (sect. 2.3); indirect WS knowledge can be obtained if, instead of measuring it, a different property T is measured, compatible with the measurement of the final impact point and correlated with WS property; this is the idea beyond ESW experiment (sect. 2.4).

In chapter 3 the question of ascertaining together more incompatible properties is investigated, in the framework of the double-slit experiment, by means of the notion of detector. At first, an ideal experiment where a property L, incompatible with WS property, is detected together with the measurement of the final impact point and together with the detection of WS property is presented; however in the situation envisaged (similar to ESW experiment) the two detections turn out to be correlated (sect. 3.2). Then, a systematic investigation of the problem of ascertaining together WS property, an incompatible one G and the final impact point (three incompatible properties) is carried out (sect. 3.3). We have to stress the difference between the nature of inferences provided by our method of attaining indirect knowledge with respect to other ones. In particular, we refer to VAA procedure (sect. 3.4); since it is inadequate for the detection of more than three observables, we treat the case of four incompatible properties in the same kind of ideal experiment described in sect. 3.3, showing that it has affirmative answer (sect. 3.5).

Chapter 1

Formulation and Interpretation of Quantum Theory

Since the birth of Quantum Mechanics, several formulations of the theory were developed; the main ones are the *wave mechanics*, proposed by Erwin Schrödinger [10, 11, 12], and the *matrix mechanics* of Werner Heisenberg [13]; whereas the starting point of this last is a critical analysis of the old theories, Schrödinger's formulation originates from the works of L. de Broglie on matter waves [14]. As pointed out by Schrödinger himself [11], although apparently different, these are two equivalent formulations of a more general theory whose formalism is due to Dirac [15]. However, the more rigorous and elegant formulation of Quantum Theory is due to Von Neumann [16]; the basic idea is to found the theory on a system of axioms, so that every valid assertion must be proved as consequence of these assumptions. A deep abstraction arises because it is formulated by means of the Hilbert space theory. We will see, in fact, that measurable physical magnitudes are represented by self-adjoint operators on a suitable Hilbert space. A strict link exists between Quantum

Mechanics and the spectral theory; a particularly important role is played by the spectral theorem for self-adjoint operators, according to which self-adjoint operators can be decomposed in terms of projections operators.

In spite of its deep abstraction, this theory turns out to agree with empirical results. It is introduced here in a modern and suitable version.

1.1 Von Neumann Theory

The basic concepts of Von Neumann theory are essentially two:

Observables. By observable we mean any physical magnitude measurable, by means of a concrete apparatus, on individual specimens of the physical system under investigation, and having real numbers as outcomes; position, momentum along a direction, energy are examples of observables. Let $\mathcal{O}$ be the set of all observables.

Expectation Value of observables or R-*functions.* By expectation value we mean a function $Ev : \mathcal{O} \to \mathbb{R}$ assigning a numerical value $Ev(\mathcal{R})$ to every observable $\mathcal{R}$, to be interpreted as the expectation value of the observable $\mathcal{R}$: thus any expectation value Ev must be referred to a statistical ensemble of physical systems. The following physical interpretation of the notion of expectation value is adopted: let N be an ensemble of individual physical system corresponding to an expectation value Ev; the measurement of every observable $\mathcal{R}$ on the sub-ensemble N_1, made up of n_1 physical systems, provides n_1 outcomes a_1,$a_2, \ldots$,a_{n_1}, whose mean value (statistical concept) $\langle\mathcal{R}\rangle_{N_1} = \sum_{i=1}^{n_1} a_i/n_1$ converges to the expectation value $Ev(\mathcal{R})$ (probabilistic concept) as $n_1 \to \infty$.

For a physical system, more than one expectation value exists: different expectation values correspond to different ways of preparing - in a laboratory or by a natural process - ensembles of physical systems. As noticed by Ballentine [17], the ensembles at issue are different from those used in classical statistical mechanics, where an ensemble is a virtual set of copies of the physical system used for calculations whose results are interpreted as a measurement on a single system. In Quantum Theory, the ensemble is the set of all systems which have undergone the same preparation technique, generally by interaction with a suitable apparatus; these physical systems are similar in their properties, but not in all of them: thus, in general, the theory does not make predictions about a single measurement, but rather it establishes the probability of each possible outcome; then, in general, its prediction can be verified by repeating the preparation and the measurement many times, and then constructing the statistical distribution of the results.

Further assumptions, induced by the interpretation of the concepts of observable and expectation value, are the following:

- Given an observable $\mathcal{R}$, we can introduce the set $\widetilde{\sigma}_{\mathcal{R}}$ of its possible outcomes (said physical spectrum); in correspondence with each function $f : \widetilde{\sigma}_{\mathcal{R}} \to \mathbb{R}$, there is another observable, denoted by $f(\mathcal{R})$, whose outcomes are obtained by applying the function f to the outcomes of $\mathcal{R}$; we notice that, if f is injective, $\mathcal{R}$ and $f(\mathcal{R})$ measure the same magnitude, by using two different scales.

- The set of $\mathcal{R}$-functions is endowed with a convex operation: if $\{Ev_1, Ev_2, \ldots\}$ is a countable set of $\mathcal{R}$-functions, then $\sum_i \alpha_i Ev_i$, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$, is a $\mathcal{R}$-function too, corresponding to the ensemble obtained by making a mixture with statistical weights α_i of the statistical ensembles of physical systems corresponding to the Ev_i.

Von Neumann theory is based on the following axioms:

Axiom 1. *Given a physical system, a complex and separable Hilbert space* $\mathcal{H}$ *can be associated to it such that to every observable* $\mathcal{R}$ *there corresponds one, and only one, self-adjoint operator* R *of* $\mathcal{H}$*. The correspondence is bijective.*

Axiom 2. *Let* $\mathcal{R}$ *be an observable and* $f : \widetilde{\sigma}_{\mathcal{R}} \to \mathbb{R}$ *a measurable function; if* $\mathcal{R}$ *is represented by the operator* R*, then* $f(\mathcal{R})$ *is represented by the operator* $f(R) = \int_{\mathbf{R}} f(\lambda) dE^R(\lambda)$*, where* $E^R(\lambda)$ *denote the spectral resolution of* R*.*

Axiom 3. *Let* $\mathcal{R}$*,* $\mathcal{S}$*, ..., be observables represented by the operators* R*,* S*, ..., then there exists another observable, denoted by* $\mathcal{R} + \mathcal{S} + \ldots$*, which is represented by the operator* $R + S + \ldots$*.*

Axiom 4. *If* $\mathcal{R}$ *is a non-negative quantity, then* $Ev(\mathcal{R}) \geq 0$*.*

Axiom 5. *For every* $\mathcal{R}$*-function* Ev*,* $Ev(a\mathcal{R}+b\mathcal{S}+\ldots) = aEv(\mathcal{R})+bEv(\mathcal{S})+ \ldots$*, where* $\mathcal{R}$*,* $\mathcal{S}$*, ... are observables and* a*,* b*, ... are numbers.*

Axiom 1 states that observables of a physical system are represented, in mathematical framework, by self-adjoint operators. Axioms 2-5 establish a strong link between the algebraic structure of the set of observables and that of the self-adjoint operators of $\mathcal{H}$.

1.2 Spectral Representation

Because of the correspondence between observables and self-adjoint operators, it is useful to investigate the properties of such operators on a Hilbert space.

According to the spectral representation theorem [16], each self-adjoint operator can be analyzed in term of more simple operators (projection operators):

Theorem 1 (Spectral representation). *For every self-adjoint operator R with spectrum $\sigma(R)$ there exists a unique family of projections $\{E^R(\lambda)|\lambda \in \sigma(R)\}$ (spectral resolution), increasing in λ with $E^R(-\infty) = 0$ and $E^R(\infty) = 1$, such that $R = \int_{-\infty}^{+\infty} \lambda dE^R(\lambda)$.*

The integral is to be understood in the sense that $\forall \psi \in D_R$,

$$\langle \psi \mid R\psi \rangle = \int_{-\infty}^{+\infty} \lambda d\langle \psi \mid E_R(\lambda)\psi \rangle$$

according to the Lebesgue-Stieltjes integral. In particular:

- if E^R is non-increasing in λ_0, i.e. if $\delta > 0$ exists such that $E^R_\lambda = E^R_{\lambda_0} \quad \forall \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, then $\lambda_0 \notin \sigma(R)$ or, equivalently, $\lambda_0 \in \rho(R)$ ($\rho(R)$ is the resolvent set);

- if E^R is increasing and continuous in λ_0, then λ_0 is in the continuous spectrum;

- if E^R is increasing and not continuous in λ_0, then λ_0 is in the point spectrum, i.e. λ_0 is an eigenvalue;

1.3 States and density operators

The mathematical representation of the expectation values makes use of the *density operators*:

Definition 1. *A linear operator $\rho : \mathcal{H} \to \mathcal{H}$ is said to be a* density operator *if the following three conditions hold:*

1. *ρ is self-adjoint*

2. *ρ is positive, i.e.* $\langle\psi|\rho\psi\rangle \geq 0 \quad \forall\psi \in \mathcal{H}$

3. $Tr(\rho) = 1$

The following theorem characterizes density operators:

Theorem 2. *Every density operator has a purely discrete spectrum. Conversely, if P_1,P_2,... are projection operators of rank 1 and if α_1,α_2,... are positive numbers such that $\sum_i \alpha_i = 1$, the operator $\rho = \sum_i \alpha_i P_i$ exists and defines a density operator.*

The theorem in this form is in [18](p. 87) (see also [16](p. 328)); it ensures that eigenvalues suffice in order to construct the identity resolution of a density operator. Von Neumann proved the following theorem

Theorem 3. *For every $\mathcal{R}$-function Ev satisfying axioms 0-4 there is a density operator ρ, such that $Ev(\mathcal{R}) = Tr(\rho R)$.*

Notice that, taking into account theorem 2, the following equality holds

$$Ev(\mathcal{R}) = \sum_i \alpha_i Tr(|\psi_i\rangle\langle\psi_i|R) = \sum_i \alpha_i \langle\psi_i|R\psi_i\rangle.$$

The operator ρ is identified as the *state* of the system; as pointed out in [17, 19], whereas in classical physics the concept of state is used to refer to the coordinates and momenta of an individual system, a quantum state represents an ensemble of similarly prepared systems; it is the mathematical representation of the distribution of the results of a certain state preparation procedure.

The set of all state operators form a convex set. From axioms 0-4, the spectral representation theorem 1 and theorem 3, it follows that:

Proposition 1. *For every $\mathcal{R}$-function Ev there exists ρ such that*

$$Ev(\mathcal{R}) = \int_{\sigma(R)} \lambda Tr(\rho dE^R(\lambda))$$

As a consequence, $Tr(\rho dE^R(\lambda))$ has to be interpreted as the probability that the outcome of $\mathcal{R}$ is a value in the interval $(\lambda,\lambda + d\lambda)$.

1.4 0-1 observables and projection operators

We call *0-1 observable* any observable $\mathcal{P}$ having 0 and 1 as possible outcomes. For a 0-1 observable $\mathcal{P}$ the expectation value $Ev(\mathcal{P})$ must be interpreted as the probability of occurrence of the outcome 1 for $\mathcal{P}$; indeed, if $p(0)$ and $p(1)$ denote the probabilities of occurrence of 0 and 1, then

$$Ev(\mathcal{P}) = 0 \cdot p(0) + 1 \cdot p(1) = p(1).$$

Fixed an observable $\mathcal{A}$, a 0-1 observable can be constructed in correspondence with every interval. Let $\mathcal{A}$ be any observable and $\varDelta = (a,b]$ an interval; let us consider the characteristic function

$$\chi_\varDelta : \mathbb{R} \to \mathbb{R}\,, \qquad \chi_\varDelta = \begin{cases} 1, & \text{if } x \in \varDelta \\ 0, & \text{if } x \notin \varDelta. \end{cases}$$

Then $\chi_\varDelta(\mathcal{A})$ is a 0-1 observable, assuming value 1 if $a \in \varDelta$ and value 0 otherwise, represented by the self-adjoint operator

$$T_\varDelta = \int \chi_\varDelta(\lambda) dE^A_\lambda = lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b+\epsilon} dE^A_\lambda = E^A_b - E^A_a.$$

$T_\varDelta$ is a projection operator.

Such a kind of observable provides a characterization of the spectrum of an observable. Let $\mathcal{A}$ be an observable, represented by the self-adjoint operator

A; the probability $p(A,\Delta)$ that a measurement of $\mathcal{A}$ provides a result $a \in \Delta$ is given by

$$p(A,\Delta) = Tr(\rho[E_b^A - E_a^A]). \tag{1.1}$$

A real number a is a possible outcome for the observable $\mathcal{A}$ if and only if, fixed $\delta \geq 0$, the probability $p(\mathcal{A},(a-\delta,a+\delta])$ that a measurement of $\mathcal{A}$ gives a result in the neighborhood $(a-\delta,a+\delta]$ of a is not zero. Then, it follows that

Proposition 2. *The real number a is a possible result for the measurement of $\mathcal{A}$ if and only if $a \in \sigma(A)$.*

Then, by theorem 1 and (1.1) it follows that the physical spectrum of $\mathcal{A}$ coincides with the mathematical spectrum of the operator A.

Let $\mathcal{P}$ be a 0-1-observable and P the associated self-adjoint operator; then proposition 1 implies that $\sigma(P) = \{0,1\}$. A self-adjoint operator P has such a spectrum if and only if $P^2 = P$, i.e. if P is a projection operator; in general, the following theorem holds:

Proposition 3. *If P is the self-adjoint operator associated to a 0-1 observable $\mathcal{P}$, then $\sigma(P) = \{0,1\}$, that is to say P is a projection operator. The converse holds too.*

We notice that 0-1 observables can be interpreted as properties that the system may or may not possess: if the outcome of a measurement is 1 than the system possesses the property, otherwise it does not.

1.5 Pure states

Within the set of all states operator we distinguish the important class of *pure states*.

Definition 2. *A density operator is pure if*

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2, \text{ where } \lambda^2 < \lambda \quad \text{imply} \quad \rho_1 = \rho_2 = \rho$$

A general state which is not pure is called *mixed state.* From a physical point of view, if ρ is pure, then the statistical ensemble described by ρ cannot be obtained as statistical mixture of ensembles described by different $\mathcal{R}$-functions.

By definition, a pure state has the form $\rho = |\psi\rangle\langle\psi|$; the expectation value of the observable $\mathcal{R}$, in this pure state, is

$$Ev(\mathcal{R}) = Tr(|\psi\rangle\langle\psi|R) = \langle\psi|R\psi\rangle.$$

We observe that the state vector is not unique; indeed, any vector of the form $e^{i\alpha}|\psi\rangle$, with $\alpha \in \mathbb{R}$, is physically equivalent.

1.6 Simultaneous measurability

We have seen that the algebraic structure of the set of observables is mirrored in the algebraic structure of self-adjoint operators, which is not commutative; then, whereas in Classical Mechanics all the observables can be measured simultaneously on a physical system, this is not the case in Quantum Mechanics. Such a limitation is the object of the following theorem:

Theorem 4. *Two observables, $\mathcal{A}$ and $\mathcal{B}$ are measurable together if and only if the corresponding self-adjoint operators, A and B, commute i.e.* $[A,B] = AB - BA = \mathbf{0}$.

This theorem proved by Von Neumann ([16], p. 223) at first for operators A and B with pure discrete spectra, is extended to the more general case by means of the following theorems ([16], p. 170).

Theorem 5. *If two self-adjoint operators are commutative, then each function $f(A)$ of A commute with each function $g(B)$ of B.*

Since the hypothesis is always satisfied for $A = B$, two functions of the same operator always commute; the converse is also valid:

Theorem 6. *A and B are two commuting self-adjoint operators if and only if there exist a self-adjoint operator C and two functions, f and g, such that $A = f(C)$ and $B = g(C)$.*

(for a proof of theorem 6 see [20]). Hence the following interpretation can be given to simultaneous measurability: observables $\mathcal{A}$ and $\mathcal{B}$ are measurable together if and only if there exist an observable $\mathcal{C}$ and functions, f and g, such that $\mathcal{A} = f(\mathcal{C})$ and $\mathcal{B} = g(\mathcal{C})$, i.e. two observables $\mathcal{A}$ and $\mathcal{B}$ are simultaneously measurable if there is an arrangement, say $\mathcal{C}$, which measures both $\mathcal{A}$ and $\mathcal{B}$ on the same system.

The theorem can be extended to the case of more than two observables: "without this condition [the commutativity of operator R_1,$R_2, \ldots$], nothing can be said regarding the results of simultaneous measurement of $\mathcal{R}_1$,$\mathcal{R}_2, \ldots$, since simultaneous measurements of these quantities are then in general not possible" ([16], p. 230).

Since we restrict ourselves to the case of projection operators, a characterization can be given to simultaneous measurement by means of the following theorem:

Theorem 7. *Two projection operators A and B are commuting if and only if there exist three mutually orthogonal projection operators A_1, B_1 and K, such that $A = A_1 + K$ and $B = B_1 + K$.*

Then, the following definition can be given:

Definition 3. *Observables $\mathcal{A}$ and $\mathcal{B}$, corresponding to projection operators A and B such that $[A,B] = 0$, are said* compatible.

Examples. Position and momentum are examples of observables; the operator representing the x_i coordinate ($i = 1,2,3$) of position is the multiplication operator

$$(Q_i\psi)\mathbf{x} = x_i\psi(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3\,, \quad \forall \psi \in D_{Q_i}$$

where $D_{Q_i} = \{\psi \in L_2(\mathbb{R}^3) : x_i\psi \in L_2(\mathbb{R}^3)\}$. The operator representing the p_i coordinate ($i = 1,2,3$) of momentum is the derivation operator

$$(P_i\psi)\mathbf{x} = -i\frac{\partial\psi(\mathbf{x})}{\partial x_i} \qquad \forall \mathbf{x} \in \mathbb{R}^3\,, \quad \forall \psi \in D_{P_i}$$

where $D_{P_i} = \{\psi \in L_2(\mathbb{R}^3) : x_i\psi$ absolutely continuous and $\frac{\partial\psi}{\partial x_i} \in L_2(\mathbb{R}^3)\}$.

Now we ask wether position and momentum can be measured together or not on the same physical system; this is possible if and only if the respective operators commute; straightforward calculations show that

- if $i = j$, then $[Q_i,P_i] = i\mathbf{1}$
- if $i \neq j$, then $[Q_i,P_j] = \mathbf{0}$

Hence we may conclude that different components of position and momentum can be measured together, whereas same components cannot.

1.7 Uncertainty relations

In previous section we have dealt with the question of measuring simultaneously two or more observables. Now we are interested in observables, $\mathcal{A}$ and $\mathcal{B}$, represented by non-commuting operators, A and B.

In spite of the hypothesis that A and B do not commute, states ψ may exist in which both quantities have defined values: these are, for instance, the common eigenvectors of both A and B (however, they cannot form a complete orthogonal set since, in this case, A and B would commute). In such a case, indeed, if ψ_1, $\psi_2 \ldots$, are the set of common eigenvectors, we may extend it to a complete orthonormal set by the addition of an orthonormal set ϕ_1, $\phi_2 \ldots$; now, let us introduce the self-adjoint operator $T = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n| + \sum_n \mu_n |\phi_n\rangle\langle\phi_n|$ where λ_1, $\lambda_2 \ldots$, μ_1, $\mu_2 \ldots$ are non-degenerate eigenvalues of T, pairwise different to each other. The measurement of the associated observable $\mathcal{T}$ produces one of the eigenvalues; if a λ_n is found, then we also know the values of $\mathcal{A}$ and $\mathcal{B}$; on the other hand, if a μ_n is the result, nothing can be said about the values of $\mathcal{A}$ and $\mathcal{B}$.

If no common eigenvector exists for A and B, let $\langle A\rangle_\psi = \langle\psi \mid A\psi\rangle$ and $\langle B\rangle_\psi = \langle\psi \mid B\psi\rangle$ be the expectation values of $\mathcal{A}$ and $\mathcal{B}$ respectively; let us define the operators $(\Delta A)^2 = (A - \langle A\rangle_\psi)^2$ and $(\Delta B)^2 = (B - \langle B\rangle_\psi)^2$ and evaluate the variances $(\sigma_A)^2 = \langle\psi \mid (A - \langle A\rangle_\psi)^2\psi\rangle$ and $(\sigma_B)^2 = \langle\psi \mid (B - \langle B\rangle_\psi)^2\psi\rangle$. Straightforward calculations shows that $(\sigma_A)^2 = \| A\psi \|^2 - \langle A\psi,\psi\rangle$ and similarly $(\sigma_B)^2 = \| B\psi \|^2 - \langle B\psi,\psi\rangle$; (σ_A) and (σ_B) cannot be jointly zero, since in this case ψ would be a common eigenvector of A and B. But one of the variances, say (σ_A), can be made arbitrary small by choosing a suitable state vector ψ.

Heisenberg [21] discovered a relation preventing both variances from becoming arbitrary small

$$\sigma_A\sigma_B \geq \frac{1}{4}|\langle\psi|[A,B]\psi\rangle|^2 \tag{1.2}$$

known as *uncertainty relation*. For a proof see [22]; in [16] (p. 233) the same result is established for position and momentum operators (or more generally for canonically conjugate operators) which satisfy $[\sigma_P \sigma_Q \geq \frac{\hbar}{2}]$.

As pointed out by [23], if conditions

1. A and B do not commute,

2. neither A nor B is bounded,

3. A and B have no point spectrum, or, if either of them has nonempty point spectrum, its eigenvectors lie outside the domain of the other.

are all fulfilled, this inequality states that for certain pairs of observables the right-hand side of (1.2) majorizes some positive number, independently of the choice of ψ; otherwise, if at least one condition is not fulfilled, it is always possible to exhibit a state vector ψ which makes the right-hand side arbitrary small or null.

Result (1.2) can be interpreted as follows: if A and B do not commute, observables $\mathcal{A}$ and $\mathcal{B}$ cannot be measured together on the same physical system; however two disjoint sub-ensembles can be extracted from the ensemble corresponding with ψ, one for measuring $\mathcal{A}$ and the other for measuring $\mathcal{B}$. The uncertainty relation states that there exists a lower bound for the product $\sigma_A \sigma_B$ of the variances outcoming from these measurements.

We emphasize that σ_A and σ_B are not errors of measurement; as pointed out by Ballentine [19], the error (or preferably the resolutions) δA and δB of the measuring instruments, are unrelated to σ_A and σ_B, except for the practical requirement that if δA is larger than σ_A then it is not possible to determine σ_B in the experiment, and so it is not possible to test (1.2).

1.8 Complementarity

Complementarity is a characteristic feature of quantum theory, discussed since the birth of Quantum Theory; Bohr's conception of complementarity arises from the acceptance of the wave-particle duality [24, 25]. However, the first definition of complementarity is due to Pauli [26] and it is referred to classical concepts, rather than to modes of description: two observables are complementary if the experimental arrangements which permit their unambiguous definitions are mutually exclusive; a rigorous notion is proposed by Ludwig [27]- [29]. A less strong, but formal and empirically clear definition of complementary physical quantities, due to Lahti, appeared only in 1979 [30]; a systematic analysis is found in the context of the so called *operational* or *convex approach* proposed by Busch and Lahti. A complete description of this approach is found in [31]- [35]; here we only sketch the basic ideas in order to give a formal definition of the notion of complementarity.

1.8.1 The operational approach

Within the operational approach, the description of a physical system is exclusively based on its set of states. The basic assumption is conveniently summarized as follows: the set of states of a physical system is represented by a norm closed generating cone $\mathcal{T}(\mathcal{H})^+$ for a complete base norm space $(\mathcal{T}(\mathcal{H}),\mathcal{T}(\mathcal{H})_1^+)$, where $\mathcal{T}(\mathcal{H})$ is the set of all self-adjoint trace class operators on $\mathcal{H}$, $\mathcal{T}(\mathcal{H})_1^+$ is the set of all density operators (normalized states). Thus $\mathcal{T}(\mathcal{H})_1^+ = \{\alpha \in \mathcal{T}(\mathcal{H})^+ : e(\alpha) = 1\}$, where $e : \mathcal{T}(\mathcal{H}) \to \mathbb{R}$ is the usual trace operator.

Operations. The basic idea of the approach is that any change in the system, like those caused by measurements, can be described by transformations of states of the system, formally by *operations*. An operation when performed on the system, changes a given initial state into a well-defined final state; it does not increase the trace of any state; it acts additively and homogenously on states; it leaves the empty state on its own. Formally, an operation is described by a positive, norm non-decreasing, linear mapping, $\phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$, and the set $O(\mathcal{T})$ of all operations on the state space $\mathcal{T}(\mathcal{H})$ is represented as the set of all positive elements in the unit ball of $\mathcal{L}(\mathcal{T}(\mathcal{H}))$, the space of bounded linear operators on $\mathcal{T}(\mathcal{H})$ equipped with the strong operator topology. The set $O(\mathcal{T})$ of the operations is naturally ordered as follows: for any two operations ϕ_1 and ϕ_2 in $O(\mathcal{T})$, $\phi_1 \leq \phi_2$ if and only if $\phi_1\alpha \leq \phi_2\alpha$ for all $\alpha \in \mathcal{T}(\mathcal{H})^+$; the 0 operation, $0 : \rho \mapsto 0$ is the least element in the partial ordered set $(O(\mathcal{T}), \leq)$, whereas each $\phi \in O(\mathcal{T})$ with the property that $Tr(\phi\rho) = Tr(\rho)$, for all $\rho \in \mathcal{T}(\mathcal{H})_1^+$, is maximal (an example of maximal operation is the identity operation). Finally, *lower bound*$\{\phi_1,\phi_2\} = \{\phi \in O(\mathcal{T}) : \phi \leq \phi_1, \phi \leq \phi_2\}$.

Effects. Any operation ϕ lead to a detectable *effect* when combined with detecting the trace of a state after it has undergone the operation ϕ. Thus with any $\phi \in O(\mathcal{T})$ there is associated its detectable effect, $e \circ \phi$, which is a positive linear functional on $\mathcal{T}(\mathcal{H})$ with $0 \leq e \circ \phi \leq e$; on the other hand, for any positive linear functional a on $\mathcal{T}(\mathcal{H})$ such that $0 \leq a \leq e$ there is an operation ϕ_a whose associated effect is $e \circ \phi_a = a$ [32]. Formally, the set of all effects of the physical system is represented by the set $\mathcal{E}$ of elements a in the dual space $\mathcal{T}^*(\mathcal{H})$, $\mathcal{E} = \{Tr(\phi\cdot) : \phi \in O(\mathcal{T})\}$ which, due to the duality $\mathcal{T}^*(\mathcal{H}) \simeq \mathcal{L}(\mathcal{H})$ can be identified with the set $E(\mathcal{H}) = \{E \in \mathcal{L}(\mathcal{H}) : 0 \leq E \leq I\}$; in particular, the set of extreme effects $Ex(\mathcal{E})$ shall be identified with the set $\mathcal{P}(\mathcal{H})$ of projections on $\mathcal{H}$. To grasp the physical meaning of effects, we shall consider extreme effects; in section 1.4 we noticed that projection operators correspond to properties that the system may or may not possess, hence every projection operator E of $\mathcal{P}(\mathcal{H})$ describes a particular kind of effects, those corresponding to measurements with two possible outcomes, yes and no (or 1 and 0); in such a case, a single change may happens in the measuring apparatus: the system may be found to possess or not possess the property. Calling effect a change in the measuring apparatus, we say that the result of a single measurement is *yes* if the effect occurs (i.e. if the system possesses the property) and *no* if the effect does not occurs (i.e. if the system does not possess the property). For any effect $a \in Ex(\mathcal{E})$ let $a_1 = \{\rho \in \mathcal{T}(\mathcal{H})_1^+ : a(\rho) = 1\}$ be the *yes-domain*; a_1 may or may not be empty: if a_1 is not empty the effect a can be *actualized* or realized, by preparing the system in any of the states in a_1, where the notion of *state preparation* can be described as a constant operation $P_\rho : x \mapsto P_\rho(x) = \rho$. The number $(e \circ \phi)(\rho) = Tr(\phi(\rho))$ represents the probability of occurrence of the effect E, associated to the operation ϕ in the state ρ. Since the set of projection operators is not convex, *mixing* of (extreme) effects would not be allowed; however, the set of all effects $E(\mathcal{H})$, made up of all non-negative, self-adjoint operators, bounded by the unit operator, contains $\mathcal{P}(\mathcal{H})$ as a proper subset; it is also convex, i.e. if E_1, $E_2 \in E(\mathcal{H})$ also $\lambda E_1 + (1-\lambda)E_2 \in E(\mathcal{H})$ for every real number λ, with $0 < \lambda < 1$. In such a case, by repeating N times the preparation procedure, λN times the yes-no measurement of E_1 has been carried out and $(1 - \lambda N)$ times the yes-no measurement of E_2, with respective probabilities of occurrence for E_1 and E_2. The set of effects is naturally ordered as follows: for any $a,b \in \mathcal{E}$, $a \leq b$ if and only if $a\alpha \leq b\alpha$ for any $\alpha \in \mathcal{T}(\mathcal{H})^+$.

There is a many to one correspondence between the sets $O(\mathcal{T})$ of operations and $\mathcal{E}$ of effects, $\Psi : O(\mathcal{T}) \to \mathcal{E}$, $\phi \mapsto \Psi(\phi)$, where $\Psi(\phi)(\rho) = e(\phi\rho)$, $\rho \in \mathcal{T}(\mathcal{H})_1^+$. The set $\Psi^{-1}(a) = \{\phi \in O(\mathcal{T}) : e \circ \phi = a\}$ contains exactly those operations ϕ in $O(\mathcal{T})$ which uniquely define the effect a. A *measurement of an effect* is any operation ϕ in $O(\mathcal{T})$ which provides information on a, i.e. which provides knowledge about the (probability of) occurrence of a, when performed for each system of an ensemble; hence, any $\phi \in \cup(\Psi^{-1}(c) : c \leq a)$ is a measurement of a. A *joint measurement of two effects a and b* is any operation which provides the same kind of information about both the two effects; hence, any operation $\phi \in \cup(\Psi^{-1}(c) : c \leq a, c \leq b)$ is a joint measurement of a and b. This means that, if we have an ensemble of particles prepared by means of a certain preparation procedure, we can attain both the probability of occurrence of the effect a and the probability of occurrence of the effect b, for each system of the ensemble (i.e. simultaneously).

Instruments and obsevables. Characterizing notions of this approach are those of *instruments* and *observables*. An instrument corresponds to an experimental arrangement, defining a family of operations which can be performed on the system with the arrangement; hence we give the following definition.

Definition 4. *An instrument I is a map $I : \mathcal{B}(\mathbb{R}) \to O(\mathcal{T})$ satisfying*

1. $e(I(\mathbb{R})\psi) = e(\psi) \quad \forall \psi \in \mathcal{T}(\mathcal{H})$

2. *for any countable family (X_i) of pairwise disjoint set in $\mathcal{B}(\mathbb{R})$*

$$I(\bigcup X_i) = \sum I(X_i)$$

where the sum converges in the strong operator topology.

To each instrument there is an associated observable corresponding to the family of the detectable effects of the operations performable with the instruments; thus

Definition 5. *An observable $\mathcal{A}$ is defined as an effect-valued measure on the real Borel space with the properties*

1. $\mathcal{A}(\mathbb{R}) = e$*;*

2. *for any countable family (X_i) of pairwise disjoint sets in $\mathcal{B}(\mathbb{R})$*

$$\mathcal{A}(\bigcup X_i) = \sum \mathcal{A}(X_i)$$

   where the sum converge in the weak topology of $\mathcal{T}^*(\mathcal{H})$.

Although the formulation of the operational approach has been given in the extended frame which results from considering the whole set $O(\mathcal{T})$, these definitions apply, in particular, to the Von Neumann formulation of quantum theory where the observable $\mathcal{A}$ is represented by the self-adjoint operator A or, equivalently, by the spectral resolution E^A.

For each observable $\mathcal{A}$ there is associated at least one instrument, $I^A : \mathcal{B}(\mathbb{R}) \to O(\mathcal{T})$, $X \mapsto I^A(X)$, which uniquely defines this observable, through the relation $Tr(I^A(X)\rho) = Tr(\rho A(X))$, for all Borel set X and $\rho \in T(H)_1^+$. Through its operations such an instrument characterizes an experimental arrangement which can be used to measure all the possible values of the observable; actually, for each observable $\mathcal{A}$ there is associated a (unique) family of instruments I_i^A, $i \in I$ a suitable index set, which contains all the possible measurements for $\mathcal{A}$.

1.8.2 Complementarity in the operational approach

Let I_i^A, $i \in I(A)$ and I_j^B, $j \in I(B)$, be any pair of instruments associated with the observables $\mathcal{A}$ and $\mathcal{B}$, respectively. The operations $I_i^A(X)$ and $I_j^B(Y)$, where $X,Y \in \mathcal{B}(\mathbb{R})$, describe two particular measurements of the two properties $A(X)$ and $B(Y)$ of the system. Taking into account the ordering relation of $O(\mathcal{T})$, if there exists a non-zero operation $\phi \in O(\mathcal{T})$ which is contained in the lower bound of $\{I_i^A(X),I_j^B(Y)\}$, this operation, when performed on the system, provides information on both the properties: it is a joint measurement of $\mathcal{A}$ and $\mathcal{B}$. Hence, if we have an ensemble of systems, prepared by means of a certain preparation procedure, for each system we can attain knowledge about the occurrence of both the outcome of $\mathcal{A}$ and the outcome of $\mathcal{B}$. Thus, in a natural manner, we can define the notion of mutually exclusiveness of instruments:

Definition 6. *Two instruments $I_1 : \mathcal{B}(\mathbb{R}) \to \mathcal{O}(\mathcal{T})$ and $I_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{O}(\mathcal{T})$ are mutually exclusive if and only if lower bound $\{I_1(X),I_2(Y)\} = \{0\}$ for any two bounded X and Y in $\mathcal{B}(\mathbb{R})$ for which neither $I_1(X)$ nor $I_2(Y)$ is maximal.*

The restriction to non-maximal operations allows the possibility that also instruments with bounded value set might be mutually exclusive. The existence of mutually exclusive instruments is expression of the fact that in an ensemble, knowledge about both the occurrence of $\mathcal{A}$ and the occurrence of $\mathcal{B}$ is not allowed, in general, on the same system. According to this definition, the formal definition of complementary observables is attained:

Definition 7. *Observables $\mathcal{A}$ and $\mathcal{B}$ are complementary if and only if any two instruments I_i^A, $i \in I(A)$ and I_j^B, $j \in I(B)$ uniquely defining these observables are mutually exclusive.*

An immediate consequence of the above two definitions is

Corollary 1. *Observables $\mathcal{A}$ and $\mathcal{B}$ are complementary if and only if*

$$E^A(X) \wedge E^B(Y) = 0 \qquad \textit{for any bounded } X \textit{ and } Y \textit{ in } \mathcal{B}(\mathbb{R}).$$

This shows that the notion of complementarity expresses the strongest case of incompatibility ($[A,B] \neq 0$): e.g., in the case of purely discrete spectra, incompatibility of $\mathcal{A}$ and $\mathcal{B}$ means that there is not a complete set of eigenvectors, but it does not exclude the existence of common eigenvectors; complementary obsevables do have no common eigenvectors [32]. Then, in the extreme case of incompatibility, which occurs when $[A,B] = ic$ (c is any non-vanishing constant), the two observables turn out to be complementary.

Corresponding to the intuitive idea that mutually exclusive experimental arrangements cannot be applied at the same time, there are no simultaneous measurements of complementary observables $\mathcal{A}$ and $\mathcal{B}$. Then

Corollary 2. *Complementary observables do not admit any simultaneous measurements.*

Examples. If $\mathcal{F}$ is the unitary transformation of $L_2(-\infty,\infty)$ onto itself determined by the Fourier- Plancherel transform

$$\mathcal{F}\psi = lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ikx} dx$$

then $P = \mathcal{F}^{-1} Q \mathcal{F}$ (see [36]), that is Q and P are Fourier- equivalent operators. Because of this unitary equivalence, the corresponding spectral resolutions are unitary equivalent, too:

$$E^P(\mathit{\Delta}) = \mathcal{F}^{-1} E^Q(\mathit{\Delta}) \mathcal{F} \qquad \text{for every Borel-set } \mathit{\Delta}$$

As a consequence, from the Paley-Wiener theorem [37], it follows

Theorem 8. $E^Q(\Delta_1) \wedge E^P(\Delta_2) = 0$ *for all bounded Borel-sets* Δ_1 *and* Δ_2*.*

(see [30] for a proof). Hence position and momentum are complementary observables. The physical meaning of this theorem is that there are no states of a physical system such that both the position and the momentum observables are contained in given finite intervals; that is there is no physical arrangement in which the proposition "the position has a value in $[x_0,x_0 + D_Q]$" and "the momentum has a value in $[p_0,p_0 + D_P]$" can be simultaneously verified [30].

1.8.3 Degrees of commutativity

According to [38, 39], complementarity between two observables can be characterized as a form of non-commutativity of these observables. In order to introduce the notion of degree of commutativity of two observables, the following definition is needed.

Definition 8. *For any pair of projection operators* E *and* F*, we define the projection operator* $com(E,F)$ *as*

$$com(E,F) = (E \wedge F) \vee (E \wedge F^{\perp}) \vee (E^{\perp} \wedge F) \vee (E^{\perp} \wedge F^{\perp})$$

where $E^{\perp} = I - E$*,* $F^{\perp} = I - F$*,* $\vee$ *and* $\wedge$ *denote the meet and the joint in the projection lattice;*

Then,

- $com(E,F) = I$ if and only if E and F commute,
- if $com(E,F) = 0$ they are totally non-commutative and satisfy

$$E \wedge F = E \wedge F^{\perp} = E^{\perp} \wedge F = E^{\perp} \wedge F^{\perp} = 0$$

  i.e. they are in *generic position*;

- if $0 \neq com(E,F) \neq I$, E and F partially commute.

The degree of commutativity of any two observables $\mathcal{A}$ and $\mathcal{B}$ is characterized by the *commutativity operator*:

Definition 9. *Let A and B be self-adjoint operators, the commutativity operator, $com(A,B)$, is the projection operator*

$$com(A,B) = \wedge(com(E^A(X),E^B(Y))) \qquad X,Y \in \mathcal{B}(\mathbb{R})$$

where $E^A(X)$ and $E^B(Y)$ are the spectral projections associated with bounded Borel set X and Y, respectively, on the real line $\mathbb{R}$.

By definition, it is the projection operator associated with the commutativity domain

$$\{\psi \in \mathcal{H} : E^Q(X)E^P(Y)\psi = E^P(Y)E^Q(X)\psi \quad \forall X,Y \in \mathcal{B}(\mathbb{R})\},$$

One of three cases may occurs:

(a.) $com(A,B) = I$ if and only if A and B commute,

(b.) for $com(A,B) = 0$ A and B are totally non-commutative, i.e. they commute with respect to no state ψ in the Hilbert space $\mathcal{H}$;

(c.) if $0 \neq com(A,B) \neq I$, A and B partially commute, i.e. they commute with respect to some states ψ in the Hilbert space $\mathcal{H}$.

By definition 8, the study of the degree of commutativity of any pair of observables $\mathcal{A}$ and $\mathcal{B}$ reduces to the study of their spectral projections.

It is important to remark that totally non-commutativity of A and B does not imply that any pair of their spectral projection operators is totally non-commutative; rather $com(E^A(X),E^B(Y)) = 0$ for some operators of the spectral resolutions. Then, even for totally non-commutative observables some of their values can be measured together.

Totally non commutativity of position and momentum follows not only by Fourier-equivalence but also by complementarity and Heisenberg relation; however, the examination of the degree of commutativity of the spectral projections $E^Q(X)$ and $E^P(Y)$, $X,Y \in \mathcal{B}(\mathbb{R})$ shows that any of the three cases, (a.), (b.) and (c.), occur; in particular, in [38], examples of pairs of position and momentum projection operators, $E^Q(X)$ and $E^P(Y)$ are found for each case: $E^Q(X)$ and $E^P(Y)$ are totally non-commutative for X and Y being half-lines, partially commutative for X and Y being bounded Borel-sets and commutative for (X,Y) being α-periodic, i.e. $(X,Y) = (X + \alpha, Y + \frac{2\pi}{\alpha})$.

1.9 Ordering relation and probabilities

Let us denote by $\mathcal{E}(\mathcal{H})$ the set of all projection operators of $\mathcal{H}$. $\mathcal{E}(\mathcal{H})$ is endowed with the following partial relation:

$$P \leq Q \quad \text{if} \quad \langle \psi \mid P\psi \rangle \leq \langle \psi \mid Q\psi \rangle \quad \forall \psi \in \mathcal{H}$$

or equivalently $P\mathcal{H} \subseteq Q\mathcal{H}$.

Taking into account theorem 3

$$P \leq Q \quad \text{implies} \quad Ev(P) \leq Ev(Q) \quad \forall \mathcal{R} - \text{function} \quad Ev.$$

Since P and Q represent 0-1 observables, then formula $P \leq Q$ may be interpreted as follows

The probability of occurrence of outcome 1 for P is less than or equal to the probability of occurrence of outcome 1 for Q.

We notice that $P \leq Q$ is equivalent to $QP = P$ (see [40], p. 268) furthermore

$PQ = (QP)^* = P^* = P$, hence $PQ = QP = P$. As a consequence,

$$P \leq Q \text{ implies } [P,Q] = 0.$$

Let P and Q be two projection operators which commute with each other so that they can be measured together. Since $[P,Q] = 0$, the operator PQ is the projection $P \wedge Q$; then it makes sense to consider the probability that the outcome 1 for P occurs under the condition that the outcome 1 occurs for Q, that is the conditional probability $P(P \mid Q)$. It is clear that

$$p(P \mid Q) = \frac{Tr(\rho PQ)}{Tr(\rho Q)}. \tag{1.3}$$

When ρ describes the pure expectation value $Ev(R) = \langle \psi \mid R\psi \rangle$, where $\psi \in \mathcal{H}$ and $||\psi|| = 1$, then (1.3) becomes

$$p(P \mid Q) = \frac{\langle \psi \mid PQ\psi \rangle}{\langle \psi \mid Q\psi \rangle}.$$

Therefore, in a simultaneous measurement of P and Q, the outcome 1 of Q implies the outcome 1 of P if and only if $p(P|Q) = 1$, i.e. $PQ\psi = Q\psi$.

As a consequence, if $P \leq Q$, since $P = PQ$, then $p(Q \mid P) = 1$ implies $P\psi = PQ\psi$ for all ψ; in other words

Whenever P and Q are measured together, if the outcome of P is 1, then the outcome of Q is 1 too.

The outcomes of P and Q are *completely correlated*, i.e. the outcome 1 of Q occurs if and only if the outcome of P is 1, when

$$p(P \mid Q) = p(Q \mid P) = 1 \quad \text{or} \quad p(P \mid Q) = p(Q \mid P) = 0,$$

i.e. if and only if $P\psi = Q\psi$. First (resp. second) equation means that the outcome 1 (resp. 0) for P occurs if and only if outcome 1 (resp. 0) occurs for Q.

1.10 Time evolution

The subject of the previous sections was the mathematical representation of the states of a physical system and of the properties that can be measured on it at a fixed time. Now, we deal with the dynamics of the system; for this purpose we have to specify how states and observables evolve in time. There are essentially two equivalent descriptions for time evolution, *Schrödinger picture* and *Heisenberg picture.*

Schrödinger picture. In this framework time evolution of the system is governed by the time evolution of the states, whereas operators representing the observables without explicit dependence on time remain unchanged in time. Evolution, in such a case, is described by a time-dependent unitary transformation applied to the state vector

$$\psi(t) = U(t,t_0)\psi(t_0) \text{ where } U(t,t_0) = e^{-iH(t-t_0)}$$

for the case of a system governed by a time-independent Hamiltonian operator H. Hence, as a function of time, $\psi(t)$ satisfies the differential equation

$$i\frac{d}{dt}\psi(t) = H\psi(t)$$

known as *Schrödinger equation*, which specifies the dynamical evolution in infinitesimal form. This differential equation admits a generalization to the case that the Hamiltonian is time dependent

$$i\frac{d}{dt}\psi(t) = H(t)\psi(t)$$

corresponding to time variation of the external forces. We notice that the just described scheme assigns a time evolution only to states, leaving operators constant in time.

Heisenberg picture. This is an equivalent description, where the time evolution of the system is carried by the time evolution of the operators, keeping states fixed in time, in such a manner that all expectation values are identical with the expectation values calculated in Schrödinger picture; indeed, we may also write

$$\langle\psi_t|A|\psi_t\rangle = \langle U_t\psi_0|A|U_t\psi_0\rangle = \langle\psi_0|{U_t}^{-1}AU_t|\psi_0\rangle = \langle\psi_0|A_t|\psi_0\rangle$$

Then, in Heisenberg picture, the equation for time evolution is

$$\frac{d}{dt}A_t = i[H,A_t] + \left[\frac{\partial A}{\partial t}\right]_t$$

The possibility of going from Schrödinger picture to Heisenberg picture is not generally ensured; we have implicitly used the fact that the dynamics comes out from a unitary time evolution operator.

1.11 Composite systems

Consider two physical systems, S_I and S_{II}, described in the Hilbert spaces $\mathcal{H}_I$ and $\mathcal{H}_{II}$ respectively. The appropriate Hilbert space in order to represent the composite system $S_I + S_{II}$ is the tensor product space $\mathcal{H}_I \otimes \mathcal{H}_{II}$. An exhaustive treatment of tensor product in Hilbert space is found in [41]; here we only report some needed properties.

Definition 10. *[23] An Hilbert space $\mathcal{H}$ is said to be the tensor product of $\mathcal{H}_I$ and $\mathcal{H}_{II}$ if there exists a bilinear mapping $\langle\cdot\otimes\cdot\rangle$ from the cartesian product $\mathcal{H}_I \times \mathcal{H}_{II}$ (i.e. the set of all ordered pairs $\langle|\varphi\rangle,|\psi\rangle$ with $|\varphi\rangle \in \mathcal{H}_I$ and $|\psi\rangle \in \mathcal{H}_{II}$) into $\mathcal{H}$ such that:*

- *the set* $\{|\varphi\rangle \otimes |\psi\rangle : |\varphi\rangle \in \mathcal{H}_I, |\psi\rangle \in \mathcal{H}_{II}\}$ *spans* $\mathcal{H}$

- *the inner product is defined as*

$$\langle((\langle\varphi_1| \otimes \langle\psi_1|)|(|\varphi_2\rangle \otimes |\psi_2\rangle)\rangle_{\mathcal{H}} = \langle\varphi_1|\varphi_2\rangle_{\mathcal{H}_I}\langle\psi_1|\psi_2\rangle_{\mathcal{H}_{II}}$$

for all $|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}_I$ *and* $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}_{II}$

The product $|\varphi\rangle \otimes |\psi\rangle$, often denoted $|\varphi,\psi\rangle$, is known as the *Kronecker product*. The Kronecker product between two vectors of two space of dimensions M and N respectively, is a vector in a space of dimension MN (M and N may be infinite). The tensor product is unique up to unitary equivalence.

Proposition 4. *If* $\{|\varphi_k\rangle\}$ *and* $\{|\psi_l\rangle\}$ *are orthonormal bases for* $\mathcal{H}_I$ *and* $\mathcal{H}_{II}$ *respectively, then* $\{|\varphi_k,\psi_l\rangle\}$ *is an orthormal basis for* $\mathcal{H}_I \otimes \mathcal{H}_{II}$.

(see [42] for a proof). However, a vector in $\mathcal{H}_I \otimes \mathcal{H}_{II}$ is not necessary in the form $|\varphi\rangle \otimes |\psi\rangle$ (where $|\varphi\rangle \in \mathcal{H}_I$ and $|\psi\rangle \in \mathcal{H}_{II}\}$) but it can always be expanded as a linear combination of vectors of this form.

Let A_I a self-adjoint operator of $\mathcal{H}_I$ associated with an observable $\mathcal{A}_I$ of S_I and, similarly, B_{II} of $\mathcal{H}_{II}$ the self-adjoint operator associated with an observable $\mathcal{B}_{II}$ of S_{II}. We can also define a Kronecker product between operators, by the relation

$$(A_I \otimes B_{II})|\varphi,\psi\rangle = (A_I|\varphi\rangle) \otimes (B_{II}|\psi\rangle). \tag{1.4}$$

A_I (observable of $\mathcal{H}_I$), when considered as an obsevable of the composite system, is represented by the operator $A = A_I \otimes \mathbf{1}_{II}$, where $\mathbf{1}_{II}$ is the identity operator of $\mathcal{H}_{II}$; we notice that A_I and A represent the same physical quantity, the only difference is that A_I refers to the physical system S_I itself, whereas A

refers to the physical system S_I as making part of a composite system. Similarly an observable $\mathcal{B}_{II}$ of S_{II} is represented in the tensor product space as $B = \mathbf{1}_I \otimes B_{II}$, where $\mathbf{1}_I$ is the identity operator of $\mathcal{H}_{II}$.

If A_I, C_I are operators of $\mathcal{H}_I$ and B_{II}, D_{II} operators of $\mathcal{H}_{II}$, relation

$$(A_I \otimes B_{II})(C_I \otimes D_{II}) = A_I C_I \otimes B_{II} D_{II}$$

immediately follows from (1.4); as a special case of this result

$$(A_I \otimes \mathbf{1}_{II})(\mathbf{1} \otimes B_{II}) = A_I \otimes B_{II}.$$

We note also that, for self-adjoint operators A_I, B_{II}, relations

$$f(A_I \otimes \mathbf{1}_{II}) = f(A_I) \otimes \mathbf{1}_{II}$$

$$g(\mathbf{1} \otimes B_{II}) = \mathbf{1} \otimes (B_{II})$$

hold, for every pair of functions f and g ([43]).

Physical quantities represented as operators of the form $A_I \otimes B_{II}$ are characterized by the following property about expectation values:

$$Ev(A_I \otimes B_{II}, |\varphi\rangle \otimes |\psi\rangle) = Ev(A_I, |\varphi\rangle) Ev(B_{II}, |\psi\rangle)$$

However there are physical quantities that are not represented by operator in the tensor product form (i.e. the total energy) and in such a case expectation values are not multiplicative [23].

Until now we have considered the composite system $S_I + S_{II}$ arising from two different systems, S_I and S_{II}; however, these properties hold for any operator that represents independent degrees of freedom, even if they are related to the same physical system; an example, for a particle with spin, are position and spin.

Chapter 2

Double-slit experiments

2.1 Introduction

In chapter 1, a formulation of Quantum Theory has been given, which makes evident that certain properties cannot be measured together on a same system, or, even, they are complementary. Now we "examine a phenomenon which is impossible, *absolutely* impossible to explain in any classical way, and which has in it the heart of quantum mechanics" [44]; here, Feynman refers to the duality between corpuscular and wave-like behaviours of physical entities. The most effective example in describing such a phenomenon is the double slit experiment (p. 3); in describing it, by Which Slit (WS) property we mean the property of localization of the particle when it crosses the support of the slits. WS cannot be measured by means of a localization measurement together with the final impact point on the final screen, since the corresponding observables are complementary. Then, several devices have been conceived over the years in order to attain *indirect* information about WS property; in Einstein's recoiling slit this is done by means of the recoil of the first plate suspended to a spring [45, 46]; in Feynman's light microscope, the same purpose is achieved by the scattering of photons [44]. A different experiment has been conceived by Scully, Englert and Walther (ESW) [4]; it makes possible to obtain which-slit information by exploiting recent advances in quantum optics (micro-maser cavities and laser cooling). All these experiments work by measuring a property, different from which-slit one, compatible with the measurement of the final impact point and correlated with WS property in such a manner that inferences about WS can be made by means of the outcome of T.

In section 2 we analyze a toy-model of the double-slit experiment, where the quantum description predicts the appearance of interference fringes, in agreement with empirical results.

In section 3 we show that in the general case a direct measurement of the position of a particle at different times in such kind of experiments is not possible, since the corresponding observables are complementary.

In section 4 another kind of double-slit experiment, proposed by ESW, is analyzed; for each particle, it makes possible to ascertain *indirectly* the (not directly measured) property of the passage through the slits, by means of the outcome of a suitable supplementary property; in so doing, no interference appears. A different arrangement of ESW experiment makes possible to ascertain a property, incompatible with WS property, by means of a second supplementary property; the particles which possesses this last property give rise to interference, however, information about which slit it passed through is lost.

2.2 A typical double-slit experiment

In this section a quantum description of the double-slit experiment is presented. In so doing we restrict ourselves to a "toy-model", which effectively shows the quantum features of the involved physical phenomena, without a cumbersome formalism.

The physical system considered in the following sub-sections is an "elementary" particle, (i.e. its internal structure and its size are not relevant in order to describe its motion). In our toy-model, we consider only one spatial dimension, corresponding to the position of the particle along the screen supporting the slits (see fig. 2.1). The Hilbert space for quantum description of such a

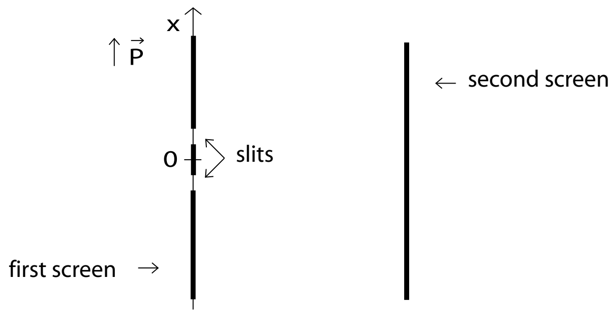

Figure 2.1. Double-slit experimental set-up

particle is $L_2(\mathbb{R})$; position and momentum operators defined in section 1.6 can be adopted here, with the only remark that $\psi \in L_2(\mathbb{R})$ rather than $L_2(\mathbb{R}^3)$, so that the index i can be omitted.

The expectation value for Q is

$$\langle Q \rangle = \langle \psi | Q\psi \rangle = \int \overline{\psi(x)} x \psi(x) dx = \int x |\psi(x)|^2 dx$$

where $|\psi(x)|^2$ is interpreted as the probability density of position, i.e. if ψ represents the wave function, $|\psi(x)|^2 dx$ represents the probability that the position has a value in the interval $[x,x+dx]$. A similar interpretation in the case of the momentum operator P is not immediate:

$$\langle P \rangle = \langle \psi | P\psi \rangle = -i \int \overline{\psi(x)} \frac{d\psi(x)}{dx} dx = ?$$

but we can recover it by means of the Fourier transform.

The following theorem states a condition of existence and furthermore a property of the Fourier transform:

Theorem 9 *(PLANCHEREL). If $\varphi(x) \in L_2(\mathbb{R})$, then the function defined by*

$$g(k) = (\mathcal{F}\varphi)(k) = lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ikx} dx,$$

known as Fourier transform of $\varphi(x)$, exists for every $k \in \mathbb{R}$ and belongs to $L_2(\mathbb{R})$. Furthermore,

$$\int_{-\infty}^{\infty} |g(k)|^2 dk = \int_{-\infty}^{\infty} |\varphi(x)|^2 dx \tag{2.1}$$

and

$$f(x) = lim_{b\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} g(k) e^{ikx} dk, \text{ a.e. on } \mathbb{R}.$$

For a proof see [47] (the operator of Fourier-Plancherel, introduced in section 1.8, is the operator $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ that $\varphi \mapsto \mathcal{F}\varphi$). Hence, the expectation value for P is

$$\langle P \rangle = \langle \psi | P\psi \rangle = \langle \mathcal{F}\psi | \mathcal{F}(P\psi) \rangle;$$

defining $\hat{\psi} = \mathcal{F}\psi$ and taking into account that the *Fourier transform* $\hat{P}$ of the derivation operator $P = -i\frac{\partial}{\partial x}$ is the multiplication operator, then

$$\langle P \rangle = \langle \hat{\psi} \mid \hat{P}\hat{\psi} \rangle = \int k |\hat{\psi}(k)|^2 dk$$

in such a manner that $\hat{\rho}(k) = |\hat{\psi}(k)|^2$ represents the probability density for the wave number $k = p/m$, where p is the momentum along the direction x:

$|\hat{\psi}(k)|^2 dk$ is the probability that a measurement of the momentum provides an outcome in the interval $[k,(k+dk)]$.

Equipped with this formalism, we shall study the toy-model of double slit experiment.

2.2.1 A one-slit experiment

Let us consider the passage of an ensemble of free particles (similarly prepared), one at once, through a screen endowed with one slit; suppose that the motion is in the direction perpendicular to the screen supporting the slit, with constant velocity **v**; a second screen which can detect the arrival of the particles (for example, a photographic plate) is placed in front of the first screen. These particles determine on the final screen a statistical distribution and actually performed experiments make evident a diffraction pattern (see fig. 2.2). We

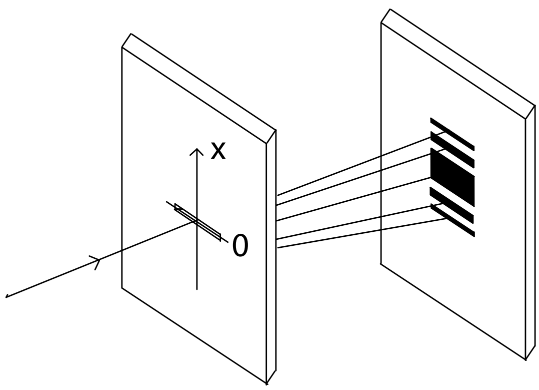

Figure 2.2. Diffraction experimental set-up

shall see that a quantum description of the experiment is in agreement with observed result.

Let the origin of the frame of reference be the center of the slit and take the x-direction on the screen; the position of the particle is described by the L_2-function

$$\psi(x) = \begin{cases} 0 & \text{if } |x| > a/2 \\ c & \text{if } |x| \leq a/2; \end{cases}$$

Coherently with the fact that the particle is somewhere, by imposing that $||\psi|| = 1$, the value of the constant c can be determined; it turns out $c = 1/\sqrt{a}$. Taking into account theorem 9, the spatial distribution of the particles on the final screen can be analyzed by calculating the momentum probability distribution.

The momentum distribution is provided by the Fourier transform of $\psi(x)$,

$$\stackrel{\wedge}{\psi}(k) = \frac{1}{k}\sqrt{\frac{2\pi}{a}}\sin\left(\frac{ak}{2}\right).$$

Theoretical description of the phenomenon (fig. 2.3.c), agrees with the dif-

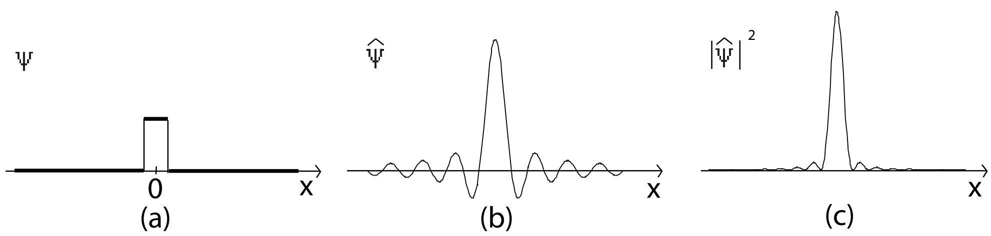

Figure 2.3. Diffraction: graphs of $\psi(x)$ (a), $\stackrel{\wedge}{\psi}(k)$ (b) and $|\stackrel{\wedge}{\psi}(k)|^2$ (c)

fraction pattern (fig. 2.2) actually observed in experiments.

2.2.2 A double-slit experiment

Let us consider now a similar example where, instead of one slit, are present two slits (see fig. 2.4) of width a, at a distance $d >> a$. We are interested with

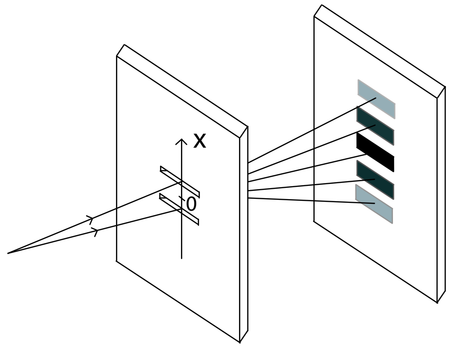

Figure 2.4. Interference experimental set-up

the final distribution of the particles. Position is described by the L_2-function

$$\psi(x) = \begin{cases} c & \text{if } x \in \left[-\frac{d}{2}-\frac{a}{2}, -\frac{d}{2}+\frac{a}{2}\right] \cup \left[\frac{d}{2}-\frac{a}{2}, \frac{d}{2}+\frac{a}{2}\right] \\ 0 & \text{elsewhere;} \end{cases}$$

where $c = 1/\sqrt{2a}$ (by imposing that $||\psi|| = 1$). Momentum distribution is described by its Fourier transform

$$\overset{\wedge}{\psi}(k) = \frac{1}{k}\sqrt{\frac{4}{a\pi}} \sin\left(\frac{ak}{2}\right)\cos\left(\frac{dk}{2}\right).$$

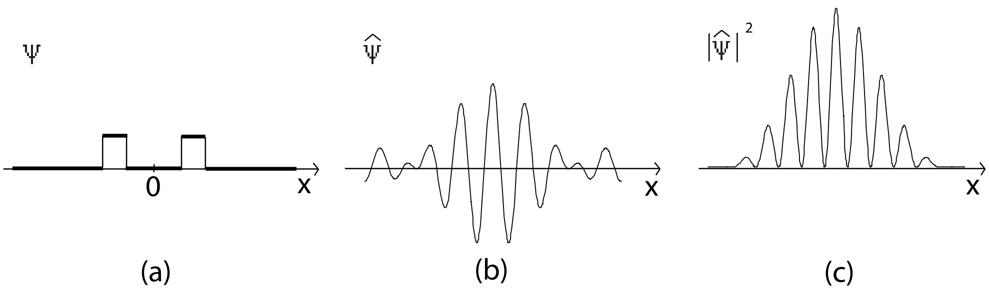

Figure 2.5. Interference: graphs of $\psi(x)$ (a), $\overset{\wedge}{\psi}(k)$ (b)and $|\overset{\wedge}{\psi}(k)|^2$ (c)

Once more, theoretical results (fig. 2.5.c) are in agreement with the interference pattern actually observed (fig.2.4).

2.3 Complementary observables in double-slit experiment

At the time t_1, when the particle crosses the slits support, let E and $E' = \mathbf{1} - E$ be the mutually orthogonal projection operators corresponding to the events "the particle is localized in slit 1" and the "particle is localized in slit 2" respectively; for any interval Δ on the final screen, let $F(\Delta,t_1)$ be the projection operator corresponding to the event "the particle is localized in a point within Δ" and $F'(\Delta,t_1) = \mathbf{1} - F(\Delta,t_1)$ its negation. Since E, E', $F(\Delta,t_1)$ and $F'(\Delta,t_1)$ are projection operators relative to t_1, they are mutually commutative (according to the fact that, if the particle is located in one of the slits, certainly it is not on the final screen).

Let t_2 be the time of the impact on the final screen and $F(\Delta) = F(\Delta,t_2)$; quantum theory prescribes that the point of impact on the final screen can be measured together with the property of the passage through slit 1 or 2 if and only if the respective operators, relative to t_2, would commute, i.e. $[F(\Delta),E] = 0$; we will see that this is not the case. Taking into account the time evolution of the operators, we have

$$F(\Delta) = U_t^{-1} F(\Delta,t_1) U_t = e^{iH(t_2-t_1)} F(\Delta,t_1) e^{-iH(t_2-t_1)}$$

Hence, the measurement of the point of impact on the final screen, $F(\Delta)$, together with WS property E, corresponds to the measurement of position observales Q^{t_1} and Q^{t_2}. Since in the passage of the particle from the first screen to the second one, the assumption that the particle is free holds, its Hamiltonian operator is given by $H = P^2/2m$, hence $U^t = e^{-i\frac{P^2 t}{2m}}$; as a consequence Heisenberg equation gets

$$\frac{d}{dt}Q^t = i[H,Q^t] = \frac{i}{2m}[P^2,Q^t] = \frac{P}{m}$$

Then $Q^{t_2} = Q^{t_1} + i\frac{P}{m}(t_2 - t_1)$, which implies

$$[Q^{t_1},Q^{t_2}] = -i\frac{t_2 - t_1}{m}.$$

We notice that both E (E') and $F(\Delta)$ are projection operators involving bounded Borel-sets (both the slits and the interval Δ are bounded). Moreover, by means of Stone- Von Neumann theorem ([20], a simple proof is in [48]), Q^{t_1} and Q^{t_2} are unitary equivalent to P and Q respectively; then, according to the operational approach of Bush and Lahti presented in section (1.8), position observables at different times, Q^{t_1} and Q^{t_2}, are complementary observables. Hence, it is not generally possible to measure WS property together with the final impact point.

2.3.1 Interference excludes WS property

Theoretical results are in agreement with the interference pattern actually observed but not with classical reasoning: if only one of the slits is open, then the momentum distribution is described by a diffraction pattern (see fig. 2.2); hence, classically, we expect that if both slits are open then the final pattern is the sum of two such patterns (see fig. 2.6.b) rather than the interference pattern (see fig. 2.6.c) actually observed. This is a consequence of an erroneous application of "classical" probability theory in Quantum Mechanics: at time t_1, corresponding to the passage of the particle through the support of

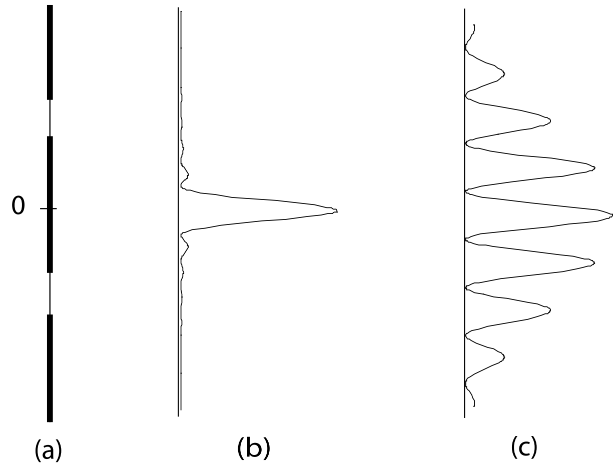

Figure 2.6. Misleading reasoning: (a) screen, (b) expected pattern, (c) observed pattern

the slits, according to quantum theory, $p(E) = \langle\Psi|E\Psi\rangle$ and $p(E') = \langle\Psi|E'\Psi\rangle$ are the probabilities for the passage through slit 1 or 2 respectively. Moreover, since E and E' are mutually orthogonal projections, $E \perp E'$, the following assumption would seem natural:

(C) At a time $t > t_1$ the particle possesses either the property "at time t_1 the particle passed through slit 1" or the property "at time t_1 the particle passed through slit 2", with probability $p(E)$ and $p(E')$ respectively.

However, the acceptance of (C) has some consequences, contradicting the results of a quantum treatment, as we shall see below.

Condition (C) should entails that a conditional probability of every event F given E, $p(F|E)$, should exist also in the case $[F,E] \neq 0$. Taking into account [23, 49], such a generalized conditional probability, that maintains all the characterizing features of a conditional probability, exists in the non-commutative case and it is unique. Namely, if for every projection operator F there should be a conditional probabilities $p(F|E)$ such that

(i.) if $F = \sum_i F_i$, where $F_i \perp F_j$ for $i \neq j$, then $p(F|E) = \sum_i p(F_i|E)$

(ii.) $F \leq E$ implies $p(F|E) = \langle \Psi | F\Psi \rangle$.

then, according to a theorem proved by Cassinelli and Zanghì [49],

$$p(F|E) = \frac{\langle \Psi | EFE\Psi \rangle}{\langle \Psi | E\Psi \rangle} = \langle \frac{\Psi_1}{\|\Psi_1\|} | F \frac{\Psi_1}{\|\Psi_1\|} \rangle \tag{2.2}$$

where $\Psi_1 = E\Psi$.

In the case under consideration $F = F(\Delta)$ and

$$p(F|E) = \frac{\langle \Psi | EFE\Psi \rangle}{\langle \Psi | E\Psi \rangle} = \langle \frac{\Psi_1}{\|\Psi_1\|} | F \frac{\Psi_1}{\|\Psi_1\|} \rangle, \qquad \Psi_1 = E\Psi$$

$$p(F|E') = \frac{\langle \Psi | E'FE'\Psi \rangle}{\langle \Psi | E'\Psi \rangle} = \langle \frac{\Psi_2}{\|\Psi_2\|} | F \frac{\Psi_2}{\|\Psi_2\|} \rangle, \qquad \Psi_2 = E'\Psi$$

where $\frac{\Psi_i}{\|\Psi_i\|}$, $i = 1,2$, is the state vector representing the particle coming from slit i. Then, assumption (C) implies that the probability that the particle hits the final screen in a point within Δ should be

$$\begin{aligned} p_C(F) &= p(E) \langle \tfrac{\Psi_1}{\|\Psi_1\|} | F \tfrac{\Psi_1}{\|\Psi_1\|} \rangle + p(E) \langle \tfrac{\Psi_2}{\|\Psi_2\|} | F \tfrac{\Psi_2}{\|\Psi_2\|} \rangle \\ &= \langle \Psi_1 | F\Psi_1 \rangle + \langle \Psi_2 | F\Psi_2 \rangle \end{aligned} \tag{2.3}$$

However, the correct quantum prediction for this probability is

$$p(F) = \langle \Psi | F\Psi \rangle = \langle \Psi_1 | F\Psi_1 \rangle + \langle \Psi_2 | F\Psi_2 \rangle + 2Re(\langle \Psi_1 | F\Psi_2 \rangle) \tag{2.4}$$

The final interference term is responsible for the fact that the probability of finding the particle within Δ is not the sum of the probabilities that one would have for each particle separately, as implied by assumption (C): a contradiction arises between the quantum prediction (2.4) and prediction (2.3) implied by assumption (C). Hence, interference denies the possibility of assigning the property of the passage through either slit 1 or 2 to the particle.

We have to stress that no contradiction follows from these facts: in Heisenberg interpretation of Quantum Mechanics some properties do not make sense for a system; in presence of the interference pattern the property of the passage through which slit does not make sense.

2.4 ESW double-slit experiments

We have seen in the previous section that WS property and the measurement of the position of the impact on the final screen cannot be measured together by means of a localization measurement, because the corresponding observables are complementary. Here we analyze an experiment allowing *indirect* knowledge of WS property: instead of it, a different property T is measured, compatible with the measurement of the final impact point, and correlated with WS property. We are in presence of an apparatus allowing the knowledge of WS property, hence, coherently with what we said at the end of last section, no interference can appear. Similarly, a different set-up of the same experiment makes possible to ascertain a property L, incompatible with E; particles possessing such a property give rise to interference, but nothing can be said about WS property for them.

In the rest of this section we describe the experiment proposed by Englert, Scully and Walther and formalize it in mathematical terms; the notion of *detector* is introduced and then the presence of interference in both the arrangements of the experiment is investigated.

2.4.1 The experimental set-up

The physical system is an atom in a long-lived Rydberg state, i.e. Rubidium in state $63p_{3/2}$, prepared to travel towards the two slits but not elsewhere. Before entering the slits the atom travels through one of two resonant cavities, placed in front to each slit, and tuned to a micro-wave frequency able to provoke decay of the atom and hence the emission of a photon. Further suitable devices make possible to ascertain which cavity the photon is in and so through which slit the atom passes before hitting the final screen, where no interference pattern appears.

A different set-up of the same experiment is arranged, with the two micro-maser cavities separated by two shutters; in such a manner radiation will be constrained to remain either in the upper or in the lower cavity, when the shutters are closed; whereas radiation is allowed to interact with a particular photo-detector wall, when shutters are open and, in this way, it is absorbed. As a consequence, which-slit information is "erased" but interference pattern is regained for the detected particles.

Let us introduce the formalism for describing these two different double-slit experiments. The system is a localizable particle, let $\mathcal{H}_I$ be the Hilbert space describing the centre-of-mass motion of the particle, and $\mathcal{H}_{II}$ the Hilbert space for the further degrees of freedom, concerning the two cavities, labeled as $\overset{\wedge}{1}$ and $\overset{\wedge}{2}$; the complete Hilbert space is $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$. In general, we denote a linear operator acting on Hilbert space $\mathcal{H}_I$ ($\mathcal{H}_{II}$), hence representing an observable $\mathcal{A}$ of $\mathcal{H}_I$ ($\mathcal{H}_{II}$), by index I (II), as A_I (A_{II}); the same observable, when considered on the whole Hilbert space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$, is represented by the operator $A = A_I \otimes \mathbf{1}_{II}$ ($A = \mathbf{1}_I \otimes A_{II}$).

For the first arrangement, we introduce the events "a photon is revealed in cavity $\hat{1}$" and "a photon is revealed in cavity $\hat{2}$", at time t_0 when the particle passes through the cavities, which are represented respectively by projection operators $T_{II} = |1\rangle\langle 1|$ and $T'_{II} = |2\rangle\langle 2|$ of $\mathcal{H}_{II}$, where $|1\rangle$ and $|2\rangle$ are state vectors of $\mathcal{H}_{II}$ such that $T_{II}|1\rangle = |1\rangle$ and $T_{II}|2\rangle = |2\rangle$; since we have supposed that the particles are prepared to travel towards the two slits but not elsewhere, we may assume that $T'_{II} = \mathbf{1}_{II} - T_{II}$ without losing generality. Let E_I (E'_I) be the projection operator representing the event "the particle is localized in slit 1 (2) at time t_1, when the particle passes through the first screen"; given any interval Δ on the final screen, let $F_I(\Delta)$ be the projection operator representing the event "the particle is localized in a point within Δ at the time t_2 of the impact on the final screen" and $F'_I(\Delta)$ its negation. Let ψ_1 and ψ_2 be state vectors of $\mathcal{H}_I$ respectively localized in slit 1 and 2 at the time of the passage through the slits, hence such that $E_I\psi_1 = \psi_1$ and $E_I\psi_2 = 0$; moreover, we require to ψ_i, $i = 1{,}2$, that $Re(\langle\psi_1|F\psi_2\rangle) \neq 0$ (this can be done because $[E{,}F(\Delta)] \neq 0$, as we saw in section 2.3). The complete state vector of the particle is $\Psi = \frac{1}{\sqrt{2}}(\psi_1 \otimes |1\rangle + \psi_2 \otimes |2\rangle)$.

Let us describe now the second arrangement. At time t_0, when the particle passes through the cavities, the events " the photon is detected by the photo-detector wall" and its negation are represented by projection operators $V = \mathbf{1} \otimes |+\rangle\langle +|$ and $V' = \mathbf{1} \otimes |-\rangle\langle -|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ (resp. $|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$) is the state of the photon when a (resp. no) photocount is observed in the detector wall. Let ψ_+ and ψ_- be state vectors of $\mathcal{H}_I$ defined as $\psi_+ = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ and $\psi_- = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$, and L, L' the corresponding properties $L = |\psi_+\rangle\langle\psi_+| \otimes \mathbf{1}_{II}$, $L' = |\psi_-\rangle\langle\psi_-| \otimes \mathbf{1}_{II}$.

2.4.2 Indirect knowledge by detectors

Equation $[E,F(\mathit{\Delta})] \neq 0$ denies the possibility of measuring both WS property and the point of impact on the final screen. However, we shall see that in some situations indirect knowledge about them can be obtained by means of suitable devices called *detectors*.

In ESW experimental situation, it is easily shown that, in correspondence with the state vector $\mathit{\Psi}$ describing the system, projection operators T and E satisfy equations $[T,E] = 0$ and $T\mathit{\Psi} = E\mathit{\Psi}$; as a consequence, for conditional probabilities, equations

$$p(T|E) = \frac{\langle \mathit{\Psi}|TE\mathit{\Psi}\rangle}{\langle \mathit{\Psi}|E\mathit{\Psi}\rangle} = 1 = \frac{\langle \mathit{\Psi}|ET\mathit{\Psi}\rangle}{\langle \mathit{\Psi}|T\mathit{\Psi}\rangle} = p(E|T)$$

hold, so that outcome 1 (2) for T allows us to infer the passage through slit 1 (2). We have to notice that the Hamiltonian of the system would be dependent on T_{II}, because the emission of a photon causes a change in momentum; however, it has been shown [50] that such a dependence can be neglected; condition $[T,F(\mathit{\Delta})] = 0$ follows from the fact that the Hamiltonian acts only on Hilbert space $\mathcal{H}_I$ (see sect. 3.3); therefore, the measurement of T provides WS knowledge and moreover it can be performed together with the measurement of the final impact point: T can be called a *WS detector*.

By means of the operator V, another sorting of the particles is performed: if the photon is found in state $|+\rangle$ (resp. $|-\rangle$) then the particle is in state ψ_+ (resp. ψ_-). The projector operator V plays with respect to L the same role of T with respect to E. Equations $[V,L] = 0$ and $V\mathit{\Psi} = L\mathit{\Psi}$ hold, hence V (performed measurement) provides inferences about the outcome of L (not performed measurement); moreover $[V,F(\mathit{\Delta})] = 0$, so that it is possible to measure V together with the final impact point. So we may conclude that also

V is a detector, for the property represented by L.

A general definition of detector for a property of the kind R can be given [51]:

Definition 11. *A projection operator S of $\mathcal{H}$ is called a detector of property R with respect to a state Ψ if*

(i) $[S,F(\varDelta)] = \mathbf{0}$,

(ii) $[S,R] = \mathbf{0}$ *and* $S\Psi = R\Psi$.

Condition (i) ensures that S can be measured together with $F(\varDelta)$; condition (ii) allows us to infer the outcome of R (though not measured) from the outcome of S.

2.4.3 Interference in ESW experiments

In this section interference is investigated, in presence either of WS detection, by T, or of detection of L, by V.

If with respect to the state vector Ψ describing the particle, a which-slit detector is measured, then it is possible to ascertain which slit each particle passes through before hitting the final screen; in this case, indeed, the interference term in (2.4) disappears:

$$\begin{aligned} \langle \Psi_1 | F(\varDelta) \Psi_2 \rangle &= \langle E\Psi | F(\varDelta) E'\Psi \rangle = \langle T\Psi | F(\varDelta) T'\Psi \rangle = \\ &= \langle \Psi | T F(\varDelta) T'\Psi \rangle = \langle \Psi | F(\varDelta) T T'\Psi \rangle = 0. \end{aligned} \tag{2.5}$$

and assumption (C) holds.

Now we are interested in the distribution of only the particles detected by a detector of the kind $Z = \mathbf{1} \otimes Z_I$. Since $[Z,F(\varDelta)] = 0$, the probability

$p(Z \wedge F(\Delta))$ that a particle hits the final screen in a point within Δ, once detected by Z, is given by

$$\begin{aligned} p(Z \wedge F(\Delta)) &= \langle \Psi | Z F(\Delta) \Psi \rangle = \langle (E+E')\Psi | Z F(\Delta)(E+E')\Psi \rangle = \\ &= \langle \Psi_1 + \Psi_2 | Z F(\Delta)(\Psi_1 + \Psi_2) \rangle = \\ &= \langle \Psi_1 | Z F(\Delta) \Psi_1 \rangle + \langle \Psi_2 | Z F(\Delta) \Psi_2 \rangle + 2Re(\langle \Psi_1 | Z F(\Delta) \Psi_2 \rangle). \end{aligned}$$

where $\Psi_1 = E\Psi$ and $\Psi_2 = E'\Psi$. Interference pattern appears if and only if the last term is not vanishing.

If we select on the final screen only the particles detected by which-slit detector $T = \mathbf{1}_I \otimes |1\rangle\langle 1|$, then $Re(\langle \Psi_1 | Z F(\Delta) \Psi_2 \rangle)$ is 0; indeed, taking into account that $Z = T$, $F(\Delta) = F_I(\Delta) \otimes \mathbf{1}_{II}$ (see sect. 3.3),

$$\Psi_1 = E\Psi = \frac{1}{\sqrt{2}}(|\psi_1\rangle\langle\psi_1| \otimes \mathbf{1}_{II})(|\psi_1\rangle \otimes |1\rangle + |\psi_2\rangle \otimes |2\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle \otimes |1\rangle$$

and, similarly,

$$\Psi_2 = E'\Psi = \frac{1}{\sqrt{2}}|\psi_2\rangle \otimes |2\rangle,$$

we have

$$\begin{aligned} \langle \Psi_1 | T F(\Delta) \Psi_2 \rangle &= \tfrac{1}{2}\big\langle \Psi_1 \big| T F(\Delta)(|\psi_2\rangle \otimes |2\rangle)\big\rangle = \tfrac{1}{2}\big\langle \Psi_1 \big| T(F_I(\Delta)|\psi_2\rangle \otimes |2\rangle)\big\rangle = \\ &= \tfrac{1}{2}\big\langle \Psi_1 \big| F_I(\Delta)|\psi_2\rangle \otimes |1\rangle\langle 1|2\rangle\big\rangle = 0. \end{aligned}$$

On the contrary, the existence of a detector $V = \mathbf{1}_I \otimes |+\rangle\langle +|$ implies that

$$\begin{aligned} \langle \Psi_1 | T F(\Delta) \Psi_2 \rangle &= \tfrac{1}{2}\big\langle \Psi_1 \big| T F(\Delta)(|\psi_2\rangle \otimes |2\rangle)\big\rangle = \tfrac{1}{2}\big\langle \Psi_1 \big| T(F_I(\Delta)|\psi_2\rangle \otimes |2\rangle)\big\rangle = \\ &= \tfrac{1}{2}\big\langle \Psi_1 \big| F_I(\Delta)|\psi_2\rangle \otimes |+\rangle\langle +|2\rangle\big\rangle = \\ &= \tfrac{1}{4}\big\langle \langle\psi_1| \otimes \langle 1| \big| F_I(\Delta)|\psi_2\rangle \otimes (|1\rangle + |2\rangle)(\langle 1| + \langle 1|)|2\rangle)\big\rangle = \\ &= \tfrac{1}{4}\langle \psi_1 | F_I(\Delta)\psi_2\rangle. \end{aligned}$$

Hence, if we select on the final screen only those particles detected by V, we may conclude that they give rise to the interference pattern.

We have to notice, here, that the absence of the interference pattern for *all* the particles hitting the final screen does not implies the absence for those *selected* by the detector V. What happens is that particles *selected* by V give rise to an interference pattern (see the continuous line in fig. 2.7.b); particles *not selected* by V give rise to an “anti”-interference pattern (see the dashed line in fig. 2.7.b), so that, when we consider *all* particles, the sum of the two patterns does not exhibit interference (see fig. 2.7.a).

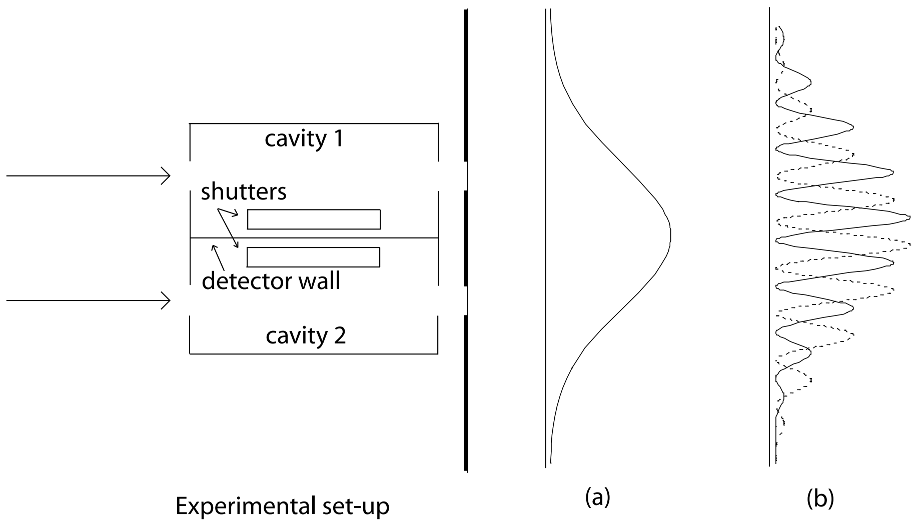

Figure 2.7. ESW experiment: (a) pattern for all particles, (b) patterns for selected (continuous line) and non-selected (dashed line) particles

Chapter 3

Detection of more incompatible properties

3.1 Introduction

In the previous chapter a typical double-slit experiment has been analyzed; according to the analysis in section 2.3, standard quantum theory denies the possibility of a simultaneous measurements of both WS property and the final impact point, since the corresponding projection operators do not commute. This notwithstanding, several devices have been conceived providing knowledge of WS property E by measuring a suitable property T, compatible with the measurement of the final impact point, and whose outcome *infers* the outcome of E.

In this chapter we face the question of ascertaining together more incompatible properties, in the framework of the double-slit experiment, by exploiting the notion of WS detector. Such a problem has been put and investigated in [7], on a theoretical ground, in the case of three incompatible properties (WS property, an incompatible one and the measurement of the final impact point); it is shown that properties, incompatible with WS property, can be *detected* together with the knowledge of which slit each particle passes through and together with the measurement of the point of impact on the final screen.

This kind of detection is made possible by the fact that besides the position of the centre-of-mass, our system possesses further degrees of freedom. As a consequence, the Hilbert space describing the entire system can be decomposed as $\mathcal{H}_I \otimes \mathcal{H}_{II}$, where $\mathcal{H}_I$ is the Hilbert space used to represent the observable position, and $\mathcal{H}_{II}$ is the Hilbert space used to represent the observable arising from the further degrees of freedom. The detection of WS property E is obtained by measuring an observable represented by a particular projection operator T acting on $\mathcal{H}_{II}$, which is correlated, in the particular quantum state of the system, with WS property, so that this last can be inferred from the outcome of T. This kind of detection was studied from a mathematical point of view in [5] and [6], where the interpretative questions were analyzed. The possibility of detecting an incompatible property G is provided by the existence of an observable represented by another projection operator Y acting on $\mathcal{H}_{II}$, but which can be measured together with T. A systematic investigation [7] establishes that the existence of such an observable (projection operator) depends on the dimension of space $\mathcal{H}_I$.

Section 2 shows an ideal experiment where a property incompatible with WS property is detected together with the measurement of the final impact point, without erasing WS knowledge provided by a WS detector, contrary to what happens in ESW experiment. However, in this (theoretically) similar experiment, the detections of E and L turn out to be completely correlated [51].

Section 3 is devoted to a systematic investigation of the problem of ascertaining together three incompatible properties: Which slit property, an incompatible one and the final impact point. An exhaustive treatment of the case $dim(\mathcal{H}_I) = 4$ is carried out: all solutions turn out to be correlated [7], as it happens in example presented in sect. 3.2. Therefore, non-correlated solutions can exist if and only if $dim(\mathcal{H}_I) > 4$. A wide family of non-correlated solutions is provided for $dim(\mathcal{H}_I) = 6$, and an ideal experiment which realizes the detection at issue is designed [8].

We notice that our approach to the problem of inferring the outcomes of more incompatible properties is not the only one. In [9] a procedure is described allowing to make inferences about the three cartesian components of a spin-$\frac{1}{2}$ particle. In section 4 we deal with the differences between our method and that proposed by Vaidman, Aharonov and Albert (VAA). Moreover, VAA claim that, according to their method, it is not possible to produce inferences (such that described in [9]) for more than three observables. Since our method runs in a quite different manner, maybe it shall admit solutions. Hence, in section 5 we treat the question whether two incompatible properties, G and L, can be detected together with the further incompatible property E (WS property) and together with measurement of the final impact point, in the same kind of ideal experiment. In particular, we show that such a question has an affirmative answer; as in the previous case, the existence of solutions depends on the dimension of space $\mathcal{H}_I$; we find a particular solution for $dim(\mathcal{H}_1) = 10$; nevertheless, in such a case the properties L and G turn out to be correlated [8]. In section 6 the detailed derivation of solutions is carried out.

3.2 Detection in ESW experiment

In section 2.4 ESW experiment has been analyzed: in that case the two properties E and L do not commute; though they cannot be measured together, detectors T and V exist providing *indirect* knowledge about E and L respectively. However, measurement of both T and V is not possible on the same system because $[T,V] \neq 0$. Therefore, a mutually exclusive choice must be done between either T or V.

Now we face the following question: *Are there situations in which the simultaneous knowledge of which slit passage and an incompatible property is possible for each particle hitting the final screen, together with the measurement of the final impact point?* We present an example showing that the answer is *yes*, where the two properties are directly correlated.

The system consists of a spin $\frac{1}{2}$-particle, so that we can take $\mathcal{H}_{II} = \mathbb{C}^2$, where $|1\rangle$ and $|-1\rangle$ are the eigenkets corresponding to the eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ of the spin along a fixed direction $\vec{\mathbf{n}}$. Let $dim(\mathcal{H}_I) = 4$ and M_1, M_2, M_3, M_4 be four mutually orthogonal subspaces of $\mathcal{H}_I$, with respective projections P_1, P_2, P_3, P_4, such that $E_I = P_1 + P_2$, $\mathbf{1} - E_I = P_3 + P_4$. Let the state of the particle be described by

$$\Psi = (1/2)\{(\psi_1 + \psi_2)|1\rangle + (\psi_3 + \psi_4)|2\rangle\},$$

where $\psi_k \in M_k$ and $\|\psi_k\| = 1$, $k = 1,2,3,4$. Therefore, the projection $T = \mathbf{1}_I \otimes |1\rangle\langle 1|$ measures whether the spin is up ($\frac{1}{2}$) or down ($-\frac{1}{2}$). Since equations $[T,E] = 0$ and $T\Psi = E\Psi$ continue to hold and, moreover $[T,F(\Delta)] = 0$, projection T above is a which-slit detector.

Now we consider the projection $L = L_I \otimes \mathbf{1}_{II}$, where $L_I = A + B + C + D$, with

$$
\begin{aligned}
A &= |\psi_1\rangle[(3/4)\langle\psi_1| + (1/4)\langle\psi_2| - (1/4)\langle\psi_3| + (1/4)\langle\psi_4|],\\
B &= |\psi_2\rangle[(1/4)\langle\psi_1| + (3/4)\langle\psi_2| + (1/4)\langle\psi_3| - (1/4)\langle\psi_4|],\\
C &= |\psi_3\rangle[-(1/4)\langle\psi_1| + (1/4)\langle\psi_2| + (1/4)\langle\psi_3| - (1/4)\langle\psi_4|],\\
D &= |\psi_4\rangle[(1/4)\langle\psi_1| - (1/4)\langle\psi_2| - (1/4)\langle\psi_3| + (1/4)\langle\psi_4|].
\end{aligned}
\tag{3.1}
$$

To grasp the physical meaning of L the choice of ψ_1,ψ_2,ψ_3,ψ_4 must be specified, of course. However, rule $[L,E] \neq \mathbf{0}$ holds for any choice; therefore projection L represents a property incompatible with E. Now, equality $T\Psi = L\Psi$ holds: T is a detector also for L, which is incompatible with E. Furthermore, since $E\Psi = T\Psi = L\Psi$, when the particles are prepared in the state Ψ, there is an entanglement between the two incompatible properties E and L.

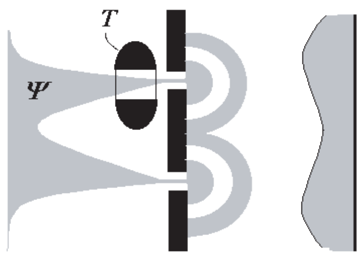

Figure 3.1. Ideal apparatus for detecting both E and L

Outcome 1 (0) for which slit detector T implies that both E and L have outcome 1 (0), E and L being incompatible with each other.

3.3 Detection of three incompatible properties

In this section we systematically face the question of ascertaining two mutually incompatible properties together with the measurement of a third property incompatible with both, in the framework of the double-slit experiment; in particular, we formalize the problem in precise mathematical terms and then a systematic investigation to find concrete solutions is performed.

The formalism introduced in section 2.3 is a valid base; let us suppose that, besides the position of the centre-of-mass, the system possesses further degrees of freedom, described in a different Hilbert space $\mathcal{H}_{II}$. As a consequence, the Hilbert space describing the entire system can be decomposed as $\mathcal{H}_I \otimes \mathcal{H}_{II}$. Let us suppose that the Hamiltonian operator H of the entire system is essentially independent of the degrees of freedom described by $\mathcal{H}_{II}$, so that we may assume the ideal case $H = H_I \otimes \mathbf{1}_{II}$. Let $E = E_I \otimes \mathbf{1}_{II}$ ($E' = \mathbf{1} - E$) be the projection operator corresponding to the event "the particle is localized in slit 1 (resp. 2) at the time t_1, i.e. when the particle passes through the slits"; for any interval Δ on the final screen, let $F(\Delta)$ be the projection operator corresponding to the event "the particle is localized in a point within Δ at the time t_2 of the impact on the final screen " and $F'(\Delta) = \mathbf{1} - F(\Delta)$ its negation. Quantum theory does not permit to measure the point of impact on the final screen together with WS property; indeed, in section 2.3, analyzing the time evolution of the system, we have seen that the corresponding operators do not commute, $[E, F(\Delta)] \neq 0$. Taking into account the notion of detector introduced in subsection 2.4.2, one can make inferences about WS property E by means of measurements of an observable represented by a projection operator T, acting on $\mathcal{H}_{II}$, whose outcome is correlated with the outcome of E. The independence of the Hamiltonian from the further degrees of freedom implies that equation $[T,F(\Delta)] = 0$ holds; indeed, the projection operator $F(\Delta,t_1)$ corresponding the localization in the interval Δ of the final screen at time t_1 must have the form $F(\Delta,t_1) = F_I(\Delta) \otimes \mathbf{1}_{II}$, since it is a localization operator at time t_1, like E; then

$$\begin{aligned}
F(\Delta) &= e^{iH(t_2-t_1)}F(\Delta,t_1)e^{-iH(t_2-t_1)} = \\
&= e^{i(H_I\otimes\mathbf{1}_{II})(t_2-t_1)}(F_I(\Delta)\otimes\mathbf{1}_{II})e^{-i(H_I\otimes\mathbf{1}_{II})(t_2-t_1)} = \\
&= \left(e^{iH_I(t_2-t_1)}\otimes\mathbf{1}_{II}\right)\left(F_I(\Delta)\otimes\mathbf{1}_{II}\right)\left(e^{-iH_I(t_2-t_1)}\otimes\mathbf{1}_{II}\right) = \\
&= \left(e^{iH_I(t_2-t_1)}F_I(\Delta)e^{-iH_I(t_2-t_1)}\right)\otimes\mathbf{1}_{II} = \\
&= F_I(\Delta,t_2)\otimes\mathbf{1}_{II}
\end{aligned}$$

so that $F(\Delta)$ commutes with any operator acting only on $\mathcal{H}_{II}$. Let G be another property, represented by a projection operator acting on $\mathcal{H}_I$, $G = G_I \otimes \mathbf{1}_{II}$, incompatible with WS property; let us suppose that G can be detected by means of an operator Y acting on $\mathcal{H}_{II}$; similarly, equation $[Y,F(\Delta)] = 0$ is satisfied.

The possibility of detecting G and E, together with $F(\Delta)$, would be ensured by detectors T of E and and Y of G, of the form $T = \mathbf{1}_I \otimes T_{II}$ and $Y = \mathbf{1}_I \otimes Y_{II}$, satisfying $[T,Y] = 0$ in such a way that T and Y can be measured together giving simultaneous information about E and G. Indeed, equations $[T,F(\Delta)] = 0$ and $[Y,F(\Delta)] = 0$ are automatically satisfied; moreover, if a detector T of E exists, then equations $[T,E] = 0$ and $T\Psi = E\Psi$ hold; equation $[T,E] = 0$ expresses the compatibility between the properties represented by T and E, so that simultaneous knowledge about them is allowed; furthermore, equation $T\Psi = E\Psi$ entails an entanglement between T and E, since it is mathematically equivalent to state that conditional probabilities satisfy

$$p(T \mid E) = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid E\Psi \rangle} = 1 = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid T\Psi \rangle} = p(E \mid T),$$

in other words, outcome 1 (0) for T reveals the passage of the particle through slit 1 (2). Similarly, the existence of a detector Y of G ensures that $[Y,G] = 0$ and $Y\Psi = G\Psi$ so that, the outcome of Y reveals the occurrence of the property G.

The problem at issue can be set out as follows:

Problem ($\mathcal{P}$). *Given the property $E = E_I \otimes \mathbf{1}_{II}$ we want to find a projection operator G_I of $\mathcal{H}_I$, two projection operators T_{II} and Y_{II} of $\mathcal{H}_{II}$, and a state vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$ such that the following conditions are satisfied:*

(C.1) $[E,G] \neq 0$ *i.e* $[E_I,G_I] \neq 0$

(C.2) $[T,E] = 0$ *and* $T\Psi = E\Psi$

(C.3) $[Y,G] = 0$ *and* $Y\Psi = G\Psi$

(C.4) $[T,Y] = 0$

(C.5) $\Psi \neq E\Psi \neq 0$ *and* $\Psi \neq G\Psi \neq 0$

(C.6) $[T,F(\Delta)] = [Y,F(\Delta)] = 0$

Condition (C.5) is added to exclude solutions corresponding to the uninteresting case that Ψ is an eigenvector of E or G.

3.3.1 General constraints

Conditions (C.1)-(C.6) can be expressed in a more useful form if the following matrix representation is adopted.

Let $(e_1,e_2,e_3,\ldots;r_1,r_2,r_3,\ldots)$ be an orthonormal basis of $\mathcal{H}_I$, formed by eigenvectors of E_I, such that $E_I e_k = e_k$ and $E_I r_j = 0$ for all k and j. Therefore, every vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$ can be uniquely decomposed as

$$\Psi = \sum_j \mathbf{e}_j \otimes \mathbf{x}_j + \sum_k \mathbf{r}_k \otimes \mathbf{y}_k,$$

where $\mathbf{x}_j,\mathbf{y}_k \in \mathcal{H}_{II}$. Now, condition (C.4) $[T,Y] = 0$ implies that four projection operators, A_i $(i = 1,\ldots,4)$, of $\mathcal{H}_{II}$ exist such that $\sum_1^4 A_i = \mathbf{1}$, $T_{II} = A_1 + A_2$ and $Y_{II} = A_1 + A_3$. Thereby, we choose to represent vectors $\mathbf{x}_j,\mathbf{y}_k \in \mathcal{H}_{II}$ as column vectors $\mathbf{x}_j = (\mathbf{a}_j,\mathbf{b}_j,\mathbf{c}_j,\mathbf{d}_j)^t$ and $\mathbf{y}_k = (\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k,\boldsymbol{\gamma}_k,\boldsymbol{\delta}_k)^t$, where $\mathbf{a}_j = A_1\mathbf{x}_j$, $\mathbf{b}_j = A_2\mathbf{x}_j$, $\mathbf{c}_j = A_3\mathbf{x}_j$, $\mathbf{d}_j = A_4\mathbf{x}_j$ and $\boldsymbol{\alpha}_j = A_1\mathbf{y}_j$, $\boldsymbol{\beta}_j = A_2\mathbf{y}_j$, $\boldsymbol{\gamma}_j = A_3\mathbf{y}_j$, $\boldsymbol{\delta}_j = A_4\mathbf{y}_j$. Then $\Psi \in \mathcal{H}$ shall be represented as a column vector

$$\Psi = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_j,\ldots;\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_k,\ldots)^t\,. \tag{3.2}$$

Once introduced suitable matrices $P = (p_{ij})$, $U = (u_{ij})$, $V = (v_{ij})$ and $Q = (q_{ij})$, we can write a linear operator G_I of $\mathcal{H}_I$ in the form $G_I = \begin{pmatrix} P & U \\ V & Q \end{pmatrix}$; let X_{II} be a linear operator of $\mathcal{H}_{II}$, then, according to such a representation, for any factorized linear operator $G_I \otimes X_{II}$ we have

$$(G_I \otimes X_{II})\Psi = \begin{pmatrix} \sum_j p_{1j} X_{II}\mathbf{x}_j + \sum_k u_{1k} X_{II}\mathbf{y}_k \\ \sum_j p_{2j} X_{II}\mathbf{x}_j + \sum_k u_{2k} X_{II}\mathbf{y}_k \\ \cdot \\ \sum_j v_{1j} X_{II}\mathbf{x}_j + \sum_k q_{1k} X_{II}\mathbf{y}_k \\ \sum_j v_{2j} X_{II}\mathbf{x}_j + \sum_k q_{2k} X_{II}\mathbf{y}_k \\ \cdot \end{pmatrix} \tag{3.3}$$

where $p_{ij} = \langle e_i \mid G_I \mid e_j\rangle$, $u_{ik} = \langle e_i \mid G_I \mid r_k\rangle$, $v_{kj} = \langle r_k \mid G_I \mid e_j\rangle$ and $q_{lk} = \langle r_l \mid G_I \mid r_k\rangle$. Then, in our representation $G_I \otimes X_{II}$ will be identified with the "four blocks" matrix

$$G_I \otimes X_{II} = \begin{pmatrix} p_{11}X_{II} & p_{12}X_{II} & \cdot & u_{11}X_{II} & u_{12}X_{II} & \cdot \\ p_{21}X_{II} & p_{22}X_{II} & \cdot & u_{21}X_{II} & u_{22}X_{II} & \cdot \\ . & . & . & . & . & . \\ v_{11}X_{II} & v_{12}X_{II} & \cdot & q_{11}X_{II} & q_{12}X_{II} & \cdot \\ v_{21}X_{II} & v_{22}X_{II} & \cdot & q_{21}X_{II} & q_{22}X_{II} & \cdot \\ . & . & . & . & . & . \end{pmatrix},$$

so that $(G_I \otimes X_{II})\Psi$ in (3.3) turns out to be the usual matrix product of this matrix with the column vector (3.2). In particular we have

$$E = \begin{pmatrix} \mathbf{1}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{1}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \end{pmatrix}, \quad G = \begin{pmatrix} p_{11}\mathbf{1}_{II} & p_{12}\mathbf{1}_{II} & . & u_{11}\mathbf{1}_{II} & u_{12}\mathbf{1}_{II} & . \\ p_{21}\mathbf{1}_{II} & p_{22}\mathbf{1}_{II} & . & u_{21}\mathbf{1}_{II} & u_{22}\mathbf{1}_{II} & . \\ . & . & . & . & . & . \\ v_{11}\mathbf{1}_{II} & v_{12}\mathbf{1}_{II} & . & q_{11}\mathbf{1}_{II} & q_{12}\mathbf{1}_{II} & . \\ v_{21}\mathbf{1}_{II} & v_{22}\mathbf{1}_{II} & . & q_{21}\mathbf{1}_{II} & q_{22}\mathbf{1}_{II} & . \\ . & . & . & . & . & . \end{pmatrix},$$

$$T = \begin{pmatrix} \mathbf{T}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{T}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{T}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{T}_{II} & . \\ . & . & . & . & . & . \end{pmatrix}, \quad Y = \begin{pmatrix} \mathbf{Y}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{Y}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{Y}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{Y}_{II} & . \\ . & . & . & . & . & . \end{pmatrix},$$

where $U = (u_{ij}) \neq 0$ is required by condition (C.1), and $G = G^* = G^2$. From condition (C.2) $T\Psi = E\Psi$, we obtain $\mathbf{x}_i =^t (\mathbf{a}_i,\mathbf{b}_i,\mathbf{0},\mathbf{0})$ and $\mathbf{y_j} =^t \big(\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_j,\boldsymbol{\delta}_j\big)$, i.e.

$$\Psi = (\underbrace{\mathbf{a}_1,\mathbf{b}_1,\mathbf{0},\mathbf{0}}_{\mathbf{x}_1^t},\underbrace{\mathbf{a}_2,\mathbf{b}_2,\mathbf{0},\mathbf{0}}_{\mathbf{x}_2^t},\cdot,\cdot;\ \underbrace{\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_1,\boldsymbol{\delta}_1}_{\mathbf{y}_1^t},\underbrace{\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_2,\boldsymbol{\delta}_2}_{\mathbf{y}_2^t},\cdot,\cdot)^t. \qquad (3.4)$$

Hence, condition (C.3) $Y\Psi = G\Psi$ is equivalent to

$$(i-A)\quad \begin{cases} \sum_i p_{ji}\mathbf{a}_i = \mathbf{a}_j \\ \sum_i p_{ji}\mathbf{b}_i = \mathbf{0} \end{cases} \qquad (ii-B)\quad \begin{cases} \sum_l u_{jl}\boldsymbol{\gamma}_l = \mathbf{0} \\ \sum_l u_{jl}\boldsymbol{\delta}_l = \mathbf{0} \end{cases} \tag{3.5}$$

$$(iii-C)\quad \begin{cases} \sum_i v_{ki}\mathbf{a}_i = \mathbf{0} \\ \sum_i v_{ki}\mathbf{b}_i = \mathbf{0} \end{cases} \qquad (iv-D)\quad \begin{cases} \sum_l q_{kl}\boldsymbol{\gamma}_l = \boldsymbol{\gamma}_k \\ \sum_l q_{kl}\boldsymbol{\delta}_l = \mathbf{0}. \end{cases}$$

Conditions (3.4) and (3.5) are general constraints to be satisfied in order that Ψ and G give rise to a solution of $(\mathcal{P})$. The following conditions (3.6) and (3.7) are straightforward consequences:

$$E\psi = T\Psi = (\mathbf{a}_1,\mathbf{b}_1,\mathbf{0},\mathbf{0},\mathbf{a}_2,\mathbf{b}_2,\mathbf{0},\mathbf{0},\cdots;\ \mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\cdots)^t; \tag{3.6}$$

$$G\Psi = Y\Psi = (\mathbf{a}_1,\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{a}_2,\mathbf{0},\mathbf{0},\mathbf{0},\cdots;\ \mathbf{0},\mathbf{0},\boldsymbol{\gamma}_1,\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_2,\mathbf{0},\cdots)^t. \tag{3.7}$$

So far we have established general constraints to be satisfied by every solution of the problem, independently of the ranks of E, G and A_i, with $i = 1,\ldots,4$, and therefore of dimensions of the spaces $\mathcal{H}_I$ and $\mathcal{H}_{II}$.

3.3.2 Correlated solutions of problem $(\mathcal{P})$

We shall restrict ourselves to the case that the two slits are symmetrical: this leads to exclude odd dimension of $\mathcal{H}_I$ and, moreover, to assume that $rank(E) = rank(\mathbf{1}) - E = dim(\mathcal{H}_I)/2$.

We begin our searching for solutions of $(\mathcal{P})$ by establishing that no solution exists if $dim(H_I) = 2$. Indeed, in this case $G = \begin{pmatrix} p\mathbf{1}_{II} & u\mathbf{1}_{II} \\ v\mathbf{1}_{II} & q\mathbf{1}_{II} \end{pmatrix}$, where p,u,v,q are complex numbers, with $u = \bar{v} \neq 0$ to satisfy $[G,E] \neq \mathbf{0}$, and $\Psi = (\mathbf{a},\mathbf{b},\mathbf{0},\mathbf{0};\mathbf{0},\mathbf{0},\boldsymbol{\gamma},\boldsymbol{\delta})^t$. Then (ii-B) and (iii-C) respectively imply $\boldsymbol{\gamma} = \boldsymbol{\delta} = \mathbf{0}$ and $\mathbf{a} = \mathbf{b} = \mathbf{0}$, i.e. $\Psi = \mathbf{0}$.

In the previous section 3.2 we have presented an example of solution of $(\mathcal{P})$ with $dim(\mathcal{H}_{II}) = 2$ and $dim(\mathcal{H}_I) = 4$. However, for this solution the outcomes of the two detections of E and G always coincide. Now we prove that this trivial character is shared by every solution of $(\mathcal{P})$, if $dim(\mathcal{H}_I) = 4$, independently of the dimension of $\mathcal{H}_{II}$. In this case, the representation of the projection operators E, G, T and Y in a solution of $(\mathcal{P})$ is made of 4×4 matrices, whereas Ψ in (3.4) is

$$\Psi = (\mathbf{a}_1,\mathbf{b}_1,\mathbf{0},\mathbf{0},\mathbf{a}_2,\mathbf{b}_2,\mathbf{0},\mathbf{0};\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_1,\boldsymbol{\delta}_1,\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_2,\boldsymbol{\delta}_2)^t.$$

As a consequence, conditions (3.5) become $(j = 1,2)$

$$(i-A) \quad \begin{cases} p_{j1}\mathbf{a}_1 + p_{j2}\mathbf{a}_2 = \mathbf{a}_j \\ p_{j1}\mathbf{b}_1 + p_{j2}\mathbf{b}_2 = \mathbf{0}, \end{cases} \qquad (ii-B) \quad \begin{cases} u_{j1}\boldsymbol{\gamma}_1 + u_{j2}\boldsymbol{\gamma}_2 = \mathbf{0} \\ u_{j1}\boldsymbol{\delta}_1 + u_{j2}\boldsymbol{\delta}_2 = \mathbf{0}; \end{cases}$$

$$(iii-C) \quad \begin{cases} v_{j1}\mathbf{a}_1 + v_{j2}\mathbf{a}_2 = \mathbf{0} \\ v_{j1}\mathbf{b}_1 + v_{j2}\mathbf{b}_2 = \mathbf{0}, \end{cases} \qquad (iv-D) \quad \begin{cases} q_{j1}\boldsymbol{\gamma}_1 + q_{j2}\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_j \\ q_{j1}\boldsymbol{\delta}_2 + q_{j2}\boldsymbol{\delta}_2 = \mathbf{0}. \end{cases} \tag{3.8}$$

In order that $[G,E] \neq \mathbf{0}$, at least one of the entries u_{ij} must be different from 0. This implies that the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ must be linearly dependent. Let us suppose that $\boldsymbol{\gamma}_2 = \lambda\boldsymbol{\gamma}_1$ and $\boldsymbol{\delta}_2 = \lambda\boldsymbol{\delta}_1$. By using these relations in equations (iv-D) of (3.8) we get

$$\begin{cases} q_{11}\boldsymbol{\gamma}_1 + q_{12}\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_1 = (q_{11} + \lambda q_{12})\boldsymbol{\gamma}_1 \\ q_{21}\boldsymbol{\gamma}_1 + q_{22}\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_2 = (q_{21} + \lambda q_{22})\boldsymbol{\gamma}_1 \\ q_{11}\boldsymbol{\delta}_1 + q_{12}\boldsymbol{\delta}_2 = 0 = (q_{11} + \lambda q_{12})\boldsymbol{\delta}_1 \\ q_{21}\boldsymbol{\delta}_1 + q_{22}\boldsymbol{\delta}_2 = 0 = (q_{21} + \lambda q_{22})\boldsymbol{\delta}_1\,. \end{cases}$$

If $\boldsymbol{\delta}_1 \neq \mathbf{0}$ then $(q_{11} + \lambda q_{12}) = (q_{21} + \lambda q_{22}) = 0$, which implies $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \mathbf{0}$. On the other hand, if $\boldsymbol{\delta}_1 = \mathbf{0}$ then $\boldsymbol{\delta}_2 = \lambda\boldsymbol{\delta}_1 = \mathbf{0}$, while $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$ can be non-vanishing with $\boldsymbol{\gamma}_2 = \lambda\boldsymbol{\gamma}_1$. Similarly, we can show that if $\mathbf{b}_1 \neq \mathbf{0}$ then $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$; if $\mathbf{b}_1 = \mathbf{0}$ then $\mathbf{b}_2 = \mathbf{0}$ and $\mathbf{a}_2 = \mu\mathbf{a}_1$. Hence, the following statements are implied by (3.8):

a) if $\mathbf{b}_1 = \mathbf{0}$ and $\boldsymbol{\delta}_1 = \mathbf{0}$, then

$\Psi = (\mathbf{a}_1,\mathbf{0},\mathbf{0},\mathbf{0},\mu\mathbf{a}_1,\mathbf{0},\mathbf{0},\mathbf{0};\ \mathbf{0},\mathbf{0},\boldsymbol{\gamma}_1,\mathbf{0},\mathbf{0},\mathbf{0},\lambda\boldsymbol{\gamma}_1,\mathbf{0})^t$,

b) if $\mathbf{b}_1 = 0$ and $\boldsymbol{\delta}_1 \neq 0$, then

$\Psi = (\mathbf{a}_1,\mathbf{0},\mathbf{0},\mathbf{0},\mu\mathbf{a}_1,\mathbf{0},\mathbf{0},\mathbf{0};\ \mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_1,\mathbf{0},\mathbf{0},\mathbf{0},\lambda\boldsymbol{\delta}_1)^t$,

c) if $\mathbf{b}_1 \neq 0$ and $\boldsymbol{\delta}_1 = 0$, then

$\Psi = (\mathbf{0},\mathbf{b}_1,\mathbf{0},\mathbf{0},\mathbf{0},\mu\mathbf{b}_1,\mathbf{0},\mathbf{0};\ \mathbf{0},\mathbf{0},\boldsymbol{\gamma}_1,\mathbf{0},\mathbf{0},\mathbf{0},\lambda\boldsymbol{\gamma}_1,\mathbf{0})^t$,

d) if $\mathbf{b}_1 \neq 0$ and $\boldsymbol{\delta}_1 \neq 0$, then

$\Psi = (\mathbf{0},\mathbf{b}_1,\mathbf{0},\mathbf{0},\mathbf{0},\mu\mathbf{b}_1,\mathbf{0},\mathbf{0};\ \mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_1,\mathbf{0},\mathbf{0},\mathbf{0},\lambda\boldsymbol{\delta}_1)^t$.

Cases (a) and (d) violate (C.5) because they respectively yield $G\Psi = \Psi$ and $G\Psi = 0$. Therefore, a state vector in a solution of $(\mathcal{P})$ must have one of the forms in (b), (c).

If case (b) (resp., case (c)) for Ψ is realized then, taking into account (3.6) and (3.7), we get $T\Psi = Y\Psi$ (resp., $T'\Psi = Y\Psi$). This is equivalent to state that conditional probability $P(T \mid Y)$ (resp., $P(T \mid Y')$) is equal to 1. As a consequence, we can conclude that *property G is detected by Y on a particle (i.e. the outcome for Y is 1) if and only if T detects the passage of that particle through slit 1 (resp., slit 2).* Thus we have perfect correlation.

Now we face the problem of singling out solutions G of (3.5) in the case of interest (b), where $\boldsymbol{\delta}_1 \neq \mathbf{0}$ and $\mathbf{b}_1 = 0$. At first, we find the solutions corresponding to $\mu = 0$ (or $\lambda = 0$), then solutions for $\mu \neq 0 \neq \lambda$ are singled out.

Case (c) can be treated in a similar way, giving rise to quite symmetrical results.

The case $\mu = 0$ or $\lambda = 0$

If $\mu = 0$ then $\Psi = (\mathbf{a}_1,\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0};\ \mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_1,\mathbf{0},\mathbf{0},\mathbf{0},\lambda\boldsymbol{\delta}_1)^t$. Since $\mathbf{a}_1 = \mathbf{0}$ implies $T\Psi = \mathbf{0}$, and $\boldsymbol{\delta}_1 = \mathbf{0}$ implies $T\Psi = \Psi$, which violate (C.5), we have to consider only the case that both $\mathbf{a}_1$ and $\boldsymbol{\delta}_1$ are non-vanishing. Therefore,

- (i-A) in (3.5) implies $p_{11} = 1$, $p_{21} = 0 = p_{12}$,
- (iii-C) in (3.5) implies $v_{11} = v_{21} = 0$, and $u_{11} = u_{12} = 0$ by self-adjointness,
- (ii-B) in (3.5) implies $u_{21} = -\lambda u_{22}$
- (iv-D) in (3.5) implies $q_{21} = -\lambda q_{22}$, $q_{11} = -\lambda q_{12} = -\lambda\overline{q_{21}} = |\lambda|^2 q_{22}$.

Then matrix G must have the form

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & -\lambda u & u \\ 0 & -\overline{\lambda}\overline{u} & |\lambda|^2 q & -\overline{\lambda} q \\ 0 & \overline{u} & -\lambda q & q \end{pmatrix}, \tag{3.9}$$

where $q \equiv q_{22}$ and $u \equiv u_{22}$. Now we find the solutions such that $rank(G) = 1 + p + |\lambda|^2 + q = 2$. Imposing idempotence to elements p and q we have that given any λ, matrix G in (3.9) is a solution if and only if $u = e^{i\theta}\sqrt{\frac{p-p^2}{1+|\lambda|^2}}$, $q = \frac{1}{1+|\lambda|^2}$ and $0 < p < 1$.

The case $\lambda = 0$ can be treated along the same logical lines, and lead to quite symmetrical results.

The case $\mu \neq 0 \neq \lambda$

Now we consider the remaining case $\mu \neq 0 \neq \lambda$. Self-adjointness together with (3.5) leads to

$$G = \begin{pmatrix} 1-|\mu|^2(1-p) & \bar{\mu}(1-p) & -\lambda u & u \\ \mu(1-p) & p & \frac{\lambda}{\bar{\mu}}u & -\frac{1}{\bar{\mu}}u \\ -\bar{\lambda}\bar{u} & \frac{\bar{\lambda}}{\mu}\bar{u} & |\lambda|^2 q & -\bar{\lambda}q \\ \bar{u} & -\frac{1}{\mu}\bar{u} & -\lambda q & q \end{pmatrix}. \tag{3.10}$$

Now, matrix G in (3.10) is a solution if numbers p, q, $u \neq 0$ can be chosen in such a way that G turns out to be idempotent.

The set of solutions is not empty. We show the solutions for the case $\lambda = \mu = 1$ and $rank(G) = 2$, so that (3.10) becomes

$$G = \begin{pmatrix} p & 1-p & -u & u \\ 1-p & p & u & -u \\ -\bar{u} & \bar{u} & q & -q \\ \bar{u} & -\bar{u} & -q & q \end{pmatrix}.$$

Idempotence imposes $0 < q < 1/2$, $p = 1-q$ and $u = e^{i\theta}\sqrt{(\frac{1}{2}-q)q}$, where θ is any real number. Therefore, for every $q \in (0{,}1/2)$ and every $\theta \in \mathbb{R}$ we have a solution

$$G = \begin{pmatrix} 1-q & q & -e^{i\theta}\sqrt{(\frac{1}{2}-q)q} & e^{i\theta}\sqrt{(\frac{1}{2}-q)q} \\ q & 1-q & e^{i\theta}\sqrt{(\frac{1}{2}-q)q} & -e^{i\theta}\sqrt{(\frac{1}{2}-q)q} \\ -e^{-i\theta}\sqrt{(\frac{1}{2}-q)q} & e^{-i\theta}\sqrt{(\frac{1}{2}-q)q} & q & -q \\ e^{-i\theta}\sqrt{(\frac{1}{2}-q)q} & -e^{-i\theta}\sqrt{(\frac{1}{2}-q)q} & -q & q \end{pmatrix}.$$

In [51] is presented the particular solution corresponding to $\theta = 0$ and $q = 1/4$.

3.3.3 Non-correlated solutions

Now we answer the question whether, by allowing the dimension of $\mathcal{H}_I$ to be at least 6, solutions of $(\mathcal{P})$ exist, without correlations between the detections of G and E, always present in the (even) cases $\dim(\mathcal{H}_I) < 6$. We assume that $rank(E_I) = rank(\mathbf{1}_I - E_I) = 3$, so that $\Psi = (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;\ \mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)^t$ and in (3.5) indexes i,j,k,l take values in $\{1,2,3\}$. No constraint is imposed to the ranks of A_i, with $i = 1,2,3,4$, and hence to the dimension of $\mathcal{H}_{II}$. If $dim(\mathcal{H}_I) = 6$, then $\Psi = (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)^t$, so that P, U, V and Q are 3×3 matrices. Now we show that in order to have non-correlated solutions everyone of the triples $\{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2,\boldsymbol{\delta}_3\}$, $\{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3\}$, $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$, must be generated by just one vector.

The detailed analysis of the derivation of such solutions is a rather technical matter; so, it is displaced in section 3.3.5; here we only sketch the idea, by investigating the linear dependencies in the triples $\{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2,\boldsymbol{\delta}_3\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$.

Since at least one of the entries u_{ij} must be non-zero to satisfy $[G,E] \neq \mathbf{0}$, general constraint $(ii - B)$ implies that one of the three vectors $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3$ - say $\mathbf{y}_1$ - is a linear combination of the remaining two:

$$\begin{cases} \boldsymbol{\delta}_1 = \lambda_2\boldsymbol{\delta}_2 + \lambda_3\boldsymbol{\delta}_3 \\ \boldsymbol{\gamma}_1 = \lambda_2\boldsymbol{\gamma}_2 + \lambda_3\boldsymbol{\gamma}_3. \end{cases}$$

Using these equations in (iv-D) we get

$$\begin{cases} (q_{j2} + \lambda_2 q_{j1})\boldsymbol{\delta}_2 + (q_{j3} + \lambda_3 q_{j1})\boldsymbol{\delta}_3 = \mathbf{0} \\ (q_{j2} + \lambda_2 q_{j1})\boldsymbol{\gamma}_2 + (q_{j3} + \lambda_3 q_{j1})\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_j, \end{cases} \quad j = 1,2,3. \tag{3.11}$$

If vectors $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ are linearly independent, then the first equation in (3.11) implies $(q_{j2} + \lambda_2 q_{j1}) = (q_{j3} + \lambda_3 q_{j1}) = 0$, so that second equation in (3.11) yields $\boldsymbol{\gamma}_j = 0$ for all j. Hence

$$(a.i) \qquad \boldsymbol{\delta}_2, \boldsymbol{\delta}_3 \text{ linearly independent} \quad \Rightarrow \quad \mathbf{y}_j = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{\delta}_j)^t, \ \forall j.$$

In a similar way we can prove that

$$(a.ii) \qquad \mathbf{b}_2, \mathbf{b}_3 \text{ linearly independent} \quad \Rightarrow \quad \mathbf{x}_j = (\mathbf{0}, \mathbf{b}_j, \mathbf{0}, \mathbf{0})^t, \ \forall j.$$

Now we derive the consequences of (a.i-ii) relative to our problem ($\mathcal{P}$). Given a state vector Ψ satisfying general constraint (3.4), a possibility is that $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ are linearly independent and also $\mathbf{b}_2$, $\mathbf{b}_3$ are linearly independent.
In this case $\mathbf{x}_j = (\mathbf{0}, \mathbf{b}_j, \mathbf{0}, \mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{\delta}_j)^t$. If a solution of ($\mathcal{P}$) exists, then $G\Psi = Y\Psi = \mathbf{0}$ would follow from (3.7), and condition (C.5) would be violated. If we consider the other cases, then we obtain the following implications.

b) $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ linearly independent and $\mathbf{b}_2$, $\mathbf{b}_3$ linearly dependent imply

   $\mathbf{x}_j = (\mathbf{a}_j, \mathbf{b}_j, \mathbf{0}, \mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{\delta}_j)^t$.

c) $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ linearly dependent and $\mathbf{b}_2$, $\mathbf{b}_3$ linearly independent imply

   $\mathbf{x}_j = (\mathbf{0}, \mathbf{b}_j, \mathbf{0}, \mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0}, \mathbf{0}, \boldsymbol{\gamma}_k, \boldsymbol{\delta}_k)^t$.

d) $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ linearly dependent and $\mathbf{b}_1$, $\mathbf{b}_2$ linearly dependent imply

   $\mathbf{x}_j = (\mathbf{a}_j, \mathbf{b}_j, \mathbf{0}, \mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0}, \mathbf{0}, \boldsymbol{\gamma}_j, \boldsymbol{\delta}_j)^t$.

Now we can see only case (d) leads to non correlated solutions. In case (b), if a solution of ($\mathcal{P}$) exists such that $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ are linearly independent and $\mathbf{b}_2$, $\mathbf{b}_3$ are linearly dependent, then (3.6), (3.7) imply that $YT\Psi = Y\Psi$ holds, which is equivalent to say that conditional probability

$$p(T \mid Y) = \frac{\langle \Psi \mid TY\Psi \rangle}{\langle \Psi \mid Y\Psi \rangle}$$

is equal to 1; this means that each time a particle is measured to have $T = 1$ then it certainly has $Y = 1$. In case (c) $TY\Psi = T\Psi$ holds, so that each time a particle is sorted by Y, then it is certainly sorted by T. Therefore, for all eventual solutions corresponding to cases (b) and (c), property G must be correlated with WS property E.

Hence, to concretely find non-correlated solutions, we have to take state vectors Ψ such that vectors $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are linearly dependent, say $\boldsymbol{\delta}_3 = \lambda\boldsymbol{\delta}_2$ and, similarly, vectors $\mathbf{b}_2$ and $\mathbf{b}_3$ are linearly dependent, say $\mathbf{b}_3 = \mu\mathbf{b}_2$, where λ and μ are complex numbers.

A similar reasoning for equations (i-A) and (iv-D) (see section 3.3.5), leads to conclude that also the triples $\{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3\}$ and $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ are generated by just one vector. As a consequence, we also attain the form Q and U have to do in order to satisfy (ii-B) and (iv-D); nevertheless, self-adjointness of Q yields to rather difficult calculation; hence we prefer making easier the search with the choice $\boldsymbol{\gamma}_2 = \mathbf{0}$.

Among general solutions of (3.5) with $\boldsymbol{\gamma}_2 = \mathbf{0}$, we select only those which satisfy ${G_I}^* = G_I$; we get

$$P = \begin{pmatrix} p & -\mu_2\left(p - \frac{|\mu_3|^2}{1+|\mu_3|^2}\right) & \mu_3(1-p) \\ -\overline{\mu}_2\left(p - \frac{|\mu_3|^2}{1+|\mu_3|^2}\right) & |\mu_2|^2\left(p - \frac{|\mu_3|^2}{1+|\mu_3|^2}\right) & \mu_3\overline{\mu}_2\left(p - \frac{|\mu_3|^2}{1+|\mu_3|^2}\right) \\ \overline{\mu}_3(1-p) & \overline{\mu}_3\mu_2\left(p - \frac{|\mu_3|^2}{1+|\mu_3|^2}\right) & 1-|\mu_3|^2(1-p) \end{pmatrix},$$

$$Q = \begin{pmatrix} q & -\lambda_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & \lambda_3(1-q) \\ -\overline{\lambda}_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & |\lambda_2|^2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & \lambda_3\overline{\lambda}_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) \\ \overline{\lambda}_3(1-q) & \overline{\lambda}_3\lambda_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & 1-|\lambda_3|^2(1-q) \end{pmatrix},$$

$$U = \begin{pmatrix} u & -\lambda_2 u & -\lambda_3 u \\ -\overline{\mu}_2 u & \lambda_2\overline{\mu}_2 u & \lambda_3\overline{\mu}_2 u \\ -\overline{\mu}_3 u & \lambda_2\overline{\mu}_3 u & \lambda_3\overline{\mu}_3 u \end{pmatrix},$$

$$\Psi = (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)^t,$$

where

$$\mathbf{x}_1 = (\mu_3\mathbf{a}_3,\tfrac{\mu_2}{|\mu_3|^2+1}\mathbf{b}_2,\mathbf{0},\mathbf{0})^t \qquad \mathbf{y}_1 = (\mathbf{0},\mathbf{0},\lambda_3\boldsymbol{\gamma}_3,\tfrac{\lambda_2}{|\lambda_3|^2+1}\boldsymbol{\delta_2},\mathbf{0})^t$$
$$\mathbf{x}_2 = (\mathbf{0},\mathbf{b}_2,\mathbf{0},\mathbf{0})^t \qquad \mathbf{y}_2 = (\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta_2})^t$$
$$\mathbf{x}_3 = \left(\mathbf{a}_3,-\tfrac{\mu_2\overline{\mu}_3}{|\mu_3|^2+1}\mathbf{b}_2,\mathbf{0},\mathbf{0}\right)^t \qquad \mathbf{y}_3 = \left(\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_3,-\tfrac{\lambda_2\overline{\lambda}_3}{|\lambda_3|^2+1}\boldsymbol{\delta_2}\right)^t$$

Self-adjointness of G_I implies that $V = \overline{U}^t$, moreover $\lambda = -\frac{\lambda_2\overline{\lambda}_3}{1+|\lambda_3|^2}$ and $\mu = -\frac{\mu_2\overline{\mu}_3}{1+|\mu_3|^2}$.

Such a solution completely solves the problem if numbers $p,u,q \neq 0$ can be chosen in such a manner that G turns out to be idempotent. It is easily shown that idempotence implies

1. $\frac{|\mu_3|^2}{1+|\mu_3|^2} < p < \frac{|\mu_3|^2}{1+|\mu_3|^2} + \frac{1}{1+|\mu_2|^2+|\mu_3|^2}$,
2. $u = e^{i\theta}\sqrt{\frac{\left(p-\frac{|\mu_3|^2}{1+|\mu_3|^2}\right)-\left(1+|\mu_2|^2+|\mu_3|^2\right)\left(p-\frac{|\mu_3|^2}{1+|\mu_3|^2}\right)^2}{1+|\lambda_2|^2+|\lambda_3|^2}}$, where θ is a real number,
3. $q = \frac{1-\left(1+|\mu_2|^2+|\mu_3|^2\right)\left(p-\frac{|\mu_3|^2}{1+|\mu_3|^2}\right)}{1+|\lambda_2|^2+|\lambda_3|^2}$.

Therefore for every real number $\frac{|\mu_3|^2}{1+|\mu_3|^2} < p < \frac{|\mu_3|^2}{1+|\mu_3|^2} + \frac{1}{1+|\mu_2|^2+|\mu_3|^2}$, every $\theta \in \mathbb{R}$ and every $\mu_2,\mu_3,\lambda_2,\lambda_3 \in \mathbb{C}$ we have a solution of $(\mathcal{P})$.

No constraint is imposed to $rank(G_I)$, i.e. to the trace of the projection operator G, hence these parameters are not all independent.

For every choice we attain a solution, ${G_I}^1$,${G_I}^2,\ldots$; then we can state there are several properties G^1,$G^2,\ldots$ incompatible with WS property E but detectable together with it. However, we notice that, taking into account (C.3), every G^i transforms Ψ in $Y\Psi$.

Our solutions form a rather wide family; however, it is not exhaustive, because of the choice $\boldsymbol{\gamma}_2 = \mathbf{0}$; if the case $\boldsymbol{\gamma}_2 \neq \mathbf{0}$ is taken into account, the problem would be completely solved. The family singled out in [7] is just a subfamily of the present one, corresponding to the particular choice $\mu_3 = \lambda_3 = 0$ and $\lambda_2 = \mu_2 = 1$.

In next sub-section an ideal experiment, not concretely performable, that realizes the detection at issue, is proposed.

3.3.4 Ideal experiment

Until now the treatment has been carried out on a theoretical ground only. Now we describe an ideal apparatus following the results of the previous sections.

The experimental set-up corresponds to the particular solution with parameters $\mu_2 = \lambda_2 = \sqrt{3}$ and $\mu_3 = \lambda_3 = 1$.

The system consists of an electrically neutral particle of spin $\frac{3}{2}$; the position of its centre-of-mass is described in space $\mathcal{H}_I$. The further degrees of freedom, described in $\mathcal{H}_{II}$, concerns the spin of the particle.

Let us suppose that, after crossing the screen with the slits, each particle passes through a non-uniform magnetic field, with gradient along the direction z (fig. 3.2). The beam splits into four beams and the deflection of each particle depends on the component of the spin in the direction of the magnetic field gradient. Hence the measurement of the amount of deflection of the particle indicates the value of its spin component.

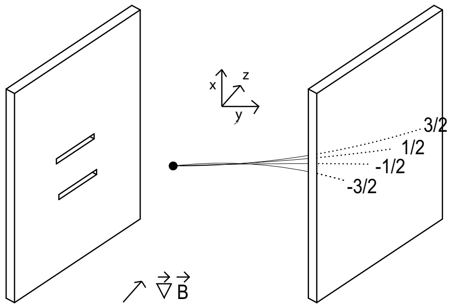

Figure 3.2. Experimental set-up for detecting E and G

Let A be the projection operator representing the event "the spin component in the z-direction is $\frac{3}{2}$". Similarly we define operators B, C and D associated to the spin components $\frac{1}{2}$, $-\frac{1}{2}$ and $-\frac{3}{2}$, respectively. We denote their respective eigenvectors relative to the eigenvalue 1 by $\mid \frac{3}{2}\rangle$, $\mid \frac{1}{2}\rangle$, $\mid -\frac{1}{2}\rangle$ and $\mid -\frac{3}{2}\rangle$. By ψ_i, $i = 1,\ldots,6$ we denote orthonormal eigenfunctions of $\mathcal{H}_I$ such that ψ_1, ψ_2, ψ_3 lie in $E_I\mathcal{H}_I$ and ψ_4, ψ_5, ψ_6 lie in $(I - E_I)\mathcal{H}_I$. Let the state vector of the entire system be

$$\psi = \frac{1}{3}\left\{(\psi_1 + \psi_2) \mid \frac{3}{2}\rangle + \left(\frac{\sqrt{3}}{2}\psi_1 + \psi_2 - \frac{\sqrt{3}}{2}\psi_3\right) \mid \frac{1}{2}\rangle\right\} + $$
$$+ \frac{1}{3}\left\{(\psi_4 + \psi_6) \mid -\frac{1}{2}\rangle + \left(\frac{\sqrt{3}}{2}\psi_4 + \psi_5 - \frac{\sqrt{3}}{2}\psi_6\right) \mid -\frac{3}{2}\rangle\right\},$$

which, within our representation, coincides with

$$\Psi = \frac{1}{3}\left(1,\frac{\sqrt{3}}{2},0,0,1,1,0,0,0,-\frac{\sqrt{3}}{2},0,0;0,0,1,\frac{\sqrt{3}}{2},0,0,0,1,0,0,1,-\frac{\sqrt{3}}{2}\right)^t.$$

According to the results of previous section, with respect to this state vector there exists a Which-Slit detector $T = I \otimes (A + B)$ and a detector $Y = I \otimes (A + C)$ of a property $G = G_I \otimes I$, incompatible with property E; G_I, with the particular choice, is the projection operator

$$G_I = \frac{1}{30}\begin{pmatrix} 20 & -5\sqrt{3} & 10 & \sqrt{5} & -\sqrt{15} & -\sqrt{5} \\ -5\sqrt{3} & 15 & 5\sqrt{3} & -\sqrt{15} & 3\sqrt{5} & \sqrt{15} \\ 10 & 5\sqrt{3} & 20 & -\sqrt{5} & \sqrt{15} & \sqrt{5} \\ \sqrt{5} & -\sqrt{15} & -\sqrt{5} & 16 & -\sqrt{3} & 14 \\ -\sqrt{15} & 3\sqrt{5} & \sqrt{15} & -\sqrt{3} & 3 & \sqrt{3} \\ -\sqrt{5} & \sqrt{15} & \sqrt{5} & 14 & \sqrt{3} & 16 \end{pmatrix}.$$

Therefore, the ideal experiment just described allows to make inferences about three non-commuting observables: the position of the final impact point is inferred from a direct measurement of $F(\Delta)$; WS property E is inferred from the outcome of detector T and property G is inferred from the outcome of detector Y.

We stress the ideal character of the experiment just described. In order to make it meaningful, we would be able to identify the observable represented by G by a physical point of view. Nevertheless, practical difficulties of creating the initial entangled state Ψ are the real obstacles in realizing the designed experiment. Hence, even if a real experiment for simultaneous detection of WS property, an incompatible one and the final impact point is not yet performed, a wide family of solutions is a contribution to increase the possibility of a concrete realization.

3.3.5 Derivation of a family of solutions

This section is devoted to find a detailed derivation of the family of solutions presented in previous section.

We are seeking for solutions such that the rank of L is 3, so that i, j, k, l take values in $\{1,2,3\}$. The detailed analysis at the beginning of section 3.3

shows that, in order to attain non-correlated solutions, the triples $\{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2,\boldsymbol{\delta}_3\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$ must be generated by just one vector; thus, we have also supposed that $\boldsymbol{\delta}_3 = \lambda\boldsymbol{\delta}_2$ and similarly $\mathbf{b}_3 = \mu\mathbf{b}_2$. Thereby, in (ii-B) we get

$$\begin{cases} (u_{j1}\lambda_2 + u_{j2})\boldsymbol{\gamma}_2 + (u_{j1}\lambda_3 + u_{j3})\boldsymbol{\gamma}_3 = \mathbf{0} \\ [(u_{j1}\lambda_2 + u_{j2}) + \lambda(u_{j1}\lambda_3 + u_{j3})]\boldsymbol{\delta}_2 = \mathbf{0}. \end{cases} \tag{3.12}$$

If $\boldsymbol{\delta}_2 = \mathbf{0}$ then second equation in (3.12) is satisfied and $\boldsymbol{\delta}_j = \mathbf{0}$ for all j, so that $\mathbf{y}_j =^t (\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_j,\mathbf{0})$. In a similar way, $\mathbf{b}_2 = \mathbf{0}$ implies $\mathbf{b}_j = \mathbf{0}$ for all j, so that $\mathbf{x}_j =^t (\mathbf{a}_j,\mathbf{0},\mathbf{0},\mathbf{0})$. Given a state Ψ, we obtain the following implications:

1. $\mathbf{b}_2 = \mathbf{0}$ and $\boldsymbol{\delta}_2 = \mathbf{0}$ imply $\mathbf{x}_j =^t (\mathbf{a}_j,\mathbf{0},\mathbf{0},\mathbf{0})$ and $\mathbf{y}_j =^t (\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_j,\mathbf{0})$; in this case if a solution exists, then $G\Psi = \Psi$. Therefore meaningful solutions cannot exist;

2. $\mathbf{b}_2 \neq \mathbf{0}$ and $\boldsymbol{\delta}_2 = \mathbf{0}$ imply $\mathbf{x}_j =^t (\mathbf{a}_j,\mathbf{b}_j,\mathbf{0},\mathbf{0})$ and $\mathbf{y}_j =^t (\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_j,\mathbf{0})$; if a solution exists, then $T'\Psi = T'Y'\Psi$, that is to say property G must be correlated with WS property E;

3. $\mathbf{b}_2 = \mathbf{0}$ and $\boldsymbol{\delta}_2 \neq \mathbf{0}$ imply $\mathbf{x}_j =^t (\mathbf{a}_j,0,0,0)$ and $\mathbf{y}_j =^t (\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_j,\boldsymbol{\delta}_j)$; if a solution exists, then $TY\Psi = T\Psi$. As in previous case property G must be correlated with WS property E;

4. $\mathbf{b}_2 \neq \mathbf{0}$ and $\boldsymbol{\delta}_2 \neq \mathbf{0}$ imply $\mathbf{x}_j =^t (\mathbf{a}_j,\mathbf{b}_j,\mathbf{0},\mathbf{0})$ and $\mathbf{y}_j =^t (\mathbf{0},\mathbf{0},\boldsymbol{\gamma}_j,\boldsymbol{\delta}_j)$; this is the only case that can lead to solution without correlation.

Hence, we are interested only in case (4).

Since $\boldsymbol{\delta}_2 \neq \mathbf{0}$, second equation in (3.12) is satisfied if and only if

$$(u_{j1}\lambda_2 + u_{j2}) = -\lambda(u_{j1}\lambda_3 + u_{j3}) \tag{3.13}$$

so that first equation in (3.12) becomes

$$(u_{j1}\lambda_3 + u_{j3})(\boldsymbol{\gamma}_3 - \lambda\boldsymbol{\gamma}_2) = \mathbf{0}. \tag{3.14}$$

If we suppose $\boldsymbol{\gamma}_3 = \lambda\boldsymbol{\gamma}_2$ then in (3.11) we get

$$\begin{cases} [(q_{j1}\lambda_2 + q_{j2}) + \lambda(q_{j1}\lambda_3 + q_{j3})]\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_j \\ [(q_{j1}\lambda_2 + q_{j2}) + \lambda(q_{j1}\lambda_3 + q_{j3})]\boldsymbol{\delta}_2 = \mathbf{0} \end{cases}$$

but $\boldsymbol{\delta}_2 \neq \mathbf{0}$ implies that $(q_{j1}\lambda_2 + q_{j2}) = -\lambda(q_{j1}\lambda_3 + q_{j3})$, hence $\boldsymbol{\gamma}_j = \mathbf{0}$ for all j. Therefore for all eventual solutions corresponding to both cases, property G must be correlated with WS property E. Since we are interested in non-correlated solutions, (3.14) yields

$$(u_{j1}\lambda_3 + u_{j3}) = 0. \tag{3.15}$$

If we consider all possibilities for $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, we get:

a. $\boldsymbol{\gamma}_3 = \lambda\boldsymbol{\gamma}_2$,

b. $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$ linearly independent,

c. $\boldsymbol{\gamma}_3 = \lambda_4\boldsymbol{\gamma}_2$,

d. $\boldsymbol{\gamma}_2 = \mathbf{0}$ and $\boldsymbol{\gamma}_3 \neq \mathbf{0}$.

Now we draw the consequences of (b), (c) and (d), since in case (a) eventual solutions lead to correlated properties.

Case (b).

If vectors $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$ are linear independent, then (3.11) becomes

$$\begin{cases} (q_{j1}\lambda_2 + q_{j2})\boldsymbol{\gamma}_2 + (q_{j1}\lambda_3 + q_{j3})\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_j \\ [(q_{j1}\lambda_2 + q_{j2}) + \lambda(q_{j1}\lambda_3 + q_{j3})]\boldsymbol{\delta}_2 = \mathbf{0}. \end{cases} \tag{3.16}$$

Since $\boldsymbol{\delta}_2 \neq \mathbf{0}$, then second equation in (3.16) implies $(q_{j1}\lambda_2+q_{j2}) = -\lambda(q_{j1}\lambda_3+q_{j3})$, so that first equation in (3.16) yields $(q_{j1}\lambda_3+q_{j3})(\boldsymbol{\gamma}_3-\lambda\boldsymbol{\gamma}_2) = \boldsymbol{\gamma}_j$. Using this relation we get

$$\begin{cases} (q_{11}\lambda_3+q_{13})(\boldsymbol{\gamma}_3-\lambda\boldsymbol{\gamma}_2) = \lambda_2\boldsymbol{\gamma}_2+\lambda_3\boldsymbol{\gamma}_3 \\ (q_{21}\lambda_3+q_{23})(\boldsymbol{\gamma}_3-\lambda\boldsymbol{\gamma}_2) = \boldsymbol{\gamma}_2 \\ (q_{31}\lambda_3+q_{33})(\boldsymbol{\gamma}_3-\lambda\boldsymbol{\gamma}_2) = \boldsymbol{\gamma}_3. \end{cases} \tag{3.17}$$

Second equation in (3.17) implies

$$\begin{cases} (q_{21}\lambda_3+q_{23}) = 0 \\ -\lambda(q_{21}\lambda_3+q_{23}) = 1. \end{cases} \tag{3.18}$$

Then system (3.18) is impossible.

Case (c).

If vectors $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$ are linearly independent no solution exists. Hence we may suppose that vectors $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$ are linearly dependent. Nevertheless, if $\boldsymbol{\gamma}_3 = \lambda\boldsymbol{\gamma}_2$ we proved that eventual solutions lead to correlated properties, so we may suppose the existence of a complex number $\lambda_4 \neq \lambda$, such that $\boldsymbol{\gamma}_3 = \lambda_4\boldsymbol{\gamma}_2$. As a consequence (iv-D) becomes

$$\begin{cases} [(q_{j1}\lambda_2+q_{j2})+\lambda_4(q_{j1}\lambda_3+q_{j3})]\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_j \\ [(q_{j1}\lambda_2+q_{j2})+\lambda(q_{j1}\lambda_3+q_{j3})]\boldsymbol{\delta}_2 = \mathbf{0}. \end{cases} \tag{3.19}$$

Since $\boldsymbol{\delta}_2 \neq \mathbf{0}$, second equation in (3.19) implies $(q_{j1}\lambda_2+q_{j2}) = -\lambda(q_{j1}\lambda_3+q_{j3})$, so that first equation in (3.19) yields $(q_{j1}\lambda_3+q_{j3})(\lambda_4-\lambda)\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_j$.

Straightforward calculations lead to a matrix Q of the form

$$Q = \begin{pmatrix} q_{11} & -\frac{\lambda(\lambda_2+\lambda_3\lambda_4)}{\lambda_4-\lambda} - q_{11}\lambda_2 & \frac{\lambda_2+\lambda_3\lambda_4}{\lambda_4-\lambda} - q_{11}\lambda_3 \\ q_{21} & -\frac{\lambda}{\lambda_4-\lambda} - q_{21}\lambda_2 & \frac{1}{\lambda_4-\lambda} - q_{21}\lambda_3 \\ q_{31} & -\frac{\lambda\lambda_4}{\lambda_4-\lambda} - q_{31}\lambda_2 & \frac{\lambda_4}{\lambda_4-\lambda} - q_{31}\lambda_3 \end{pmatrix}.$$

Nevertheless, self-adjointness of Q yields to rather difficult calculation.

Case (d).

Now we suppose $\boldsymbol{\gamma}_2 = \mathbf{0}$ and $\boldsymbol{\gamma}_3 \neq \mathbf{0}$ in order to make easier the search of solutions. Equations in (iv-D) become

$$\begin{cases} (q_{j1}\lambda_3 + q_{j3})\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_j \\ [(q_{j1}\lambda_2 + q_{j2}) + \lambda(q_{j1}\lambda_3 + q_{j3})]\boldsymbol{\delta}_2 = \mathbf{0}. \end{cases} \tag{3.20}$$

Thus, first equation of (3.20) gets

$$\begin{cases} (q_{11}\lambda_3 + q_{13})\boldsymbol{\gamma}_3 = \lambda_3\boldsymbol{\gamma}_3 \\ (q_{21}\lambda_3 + q_{23})\boldsymbol{\gamma}_3 = \mathbf{0} \\ (q_{31}\lambda_3 + q_{33})\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_3 \end{cases}$$

and, since $\boldsymbol{\gamma}_3 \neq 0$, this is equivalent to say

$$\begin{cases} q_{13} = \lambda_3(1 - q_{11}) \\ q_{23} = -\lambda_3 q_{21} \\ q_{33} = 1 - \lambda_3 q_{31} \end{cases}$$

Similarly, second equation in (3.20) implies

$$(q_{j1}\lambda_2 + q_{j2}) = -\lambda(q_{j1}\lambda_3 + q_{j3}),$$

then

$$\begin{cases} q_{12} = -\lambda\lambda_3 - \lambda_2 q_{11} \\ q_{22} = -\lambda_2 q_{21} \\ q_{32} = -\lambda - \lambda_2 q_{31} \end{cases}$$

By imposing self-adjointness, we find that $q_{11} = q$ is a real number, $\lambda =$

$-\frac{\lambda_3\overline{\lambda}_3}{|\lambda_3|^2+1}$ and

$$Q = \begin{pmatrix} q & -\lambda_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & \lambda_3\,(1-q) \\ & & \\ -\overline{\lambda}_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & |\lambda_2|^2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & \lambda_3\overline{\lambda}_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) \\ & & \\ \overline{\lambda}_3\,(1-q) & \overline{\lambda}_3\lambda_2\left(q - \frac{|\lambda_3|^2}{1+|\lambda_3|^2}\right) & 1-|\lambda_3|^2\,(1-q) \end{pmatrix}.$$

Taking into account (3.13) and (3.15), matrix U has the form

$$U = \begin{pmatrix} u_{11} & -u_{11}\lambda_2 & -u_{11}\lambda_3 \\ u_{21} & -u_{21}\lambda_2 & -u_{21}\lambda_3 \\ u_{11} & -u_{31}\lambda_2 & -u_{31}\lambda_3 \end{pmatrix}.$$

Analogous reasonings, for systems (i-A) and (iii-C), provides matrices P and V of a similar form; however, self-adjointness of G implies that $V = \overline{U}^t$, in such a manner that we attain the solution presented in section (3.3.3).

3.4 Comparison with the VAA method

In section 3.3 we have presented an approach to the problem of inferring simultaneously the outcomes of more incompatible properties; however it is not the only one. In [9] a procedure is described allowing to make inferences about the three cartesian spin-components of a spin-$\frac{1}{2}$ particle. The ideal experiment proposed by VAA works as follows: one of the three non-commuting observables, σ_x, σ_y or σ_z, is measured, on a system suitably prepared, by means of an apparatus which leaves the entire system in an eigenstate of the measured observable; after such a spin measurement, a suitable observable A is measured, having the property that the outcome of the measured spin can be inferred from the outcome of A, without knowing which spin component had been previously measured; this method yields inferences such as: if σ_x has been measured the outcome is $+\frac{1}{2}$, if σ_y has been measured the outcome is $-\frac{1}{2}$ and if σ_z has been measured the outcome is $\frac{1}{2}$. Hence, the inference of each run of the experiment actually concerns with the value of just one observable among the three non-commuting involved observables. For this reason, this kind of inferences are of a quite different nature with respect to the detections we are dealing with in the present work. According to VAA, from the outcome of A, the outcome of the performed spin-component measurement can be retrodicted; while the remaining inferences cannot be considered as detections in the sense of our definition; furthermore inferences can be drawn only under the hypothesis that the spin measurement actually performed leaves the system in an eigenstate of the measured observable. Our ideal experiment of the previous section allows to make inferences about all the three non-commuting observables: the position of the final impact point is inferred from a direct measurement of $F(\Delta)$; WS property E is inferred from the outcome of detector T and property G is inferred from the outcome of detector Y.

Moreover, VAA claim that, according to their method, it is not possible to produce inferences (like those described in [9]) for more than three observable. Since our method runs in a quite different matter, maybe it shall admit solutions. In next section we face this question; in particular, we show that such a problem has an affirmative answer.

3.5 Detection of four incompatible properties

In this section we deal with the question whether two incompatible properties, G and L, can be detected together with the further incompatible property E (WS property) and together with the measurement of the final impact point, in the same kind of ideal experiment.

More precisely, we seek for a concrete Hilbert space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$ for describing a double- slit experiment, where WS property is represented by a projection operator E acting on $\mathcal{H}_I$, such that a concrete state Ψ, and concrete projection operators G and L representing properties incompatible with each other and with E can be found in such a manner that:

- property E can be detected by means of a detector T, acting on $\mathcal{H}_{II}$;
- property G can be detected by means of a detector Y, acting on $\mathcal{H}_{II}$;
- property L can be detected by means of a detector W, acting on $\mathcal{H}_{II}$;
- the three detections can be carried out together, i.e. $[T,Y] = 0$, $[T,W] = 0$, $[Y,W] = 0$.

Again, we suppose that the Hamiltonian is independent of the further degrees of freedom, so that $H = H_I \otimes \mathbf{1}_{II}$.

In the rest of this section we formulate the question in formal way as problem $(\mathcal{P}')$; as before, we adopt a matrix representation and in this framework we establish the constraints to be satisfied in order that solutions exist. Then we present a concrete solution.

The systematic research of solution is presented in section 3.6. We set out such a research in order to answer the question whether non-correlated solutions exist or not. Then we fix the dimension of Hilbert space $\mathcal{H}_I$, $dim(\mathcal{H}_I) =$

10; a detailed analysis shows that solutions exist, however, with our assumptions, the properties L and G turn out to be always correlated.

3.5.1 Mathematical Formalism

Let $G = G_I \otimes \mathbf{1}_{II}$ and $L = L_I \otimes \mathbf{1}_{II}$ be properties incompatible with WS property E. Detection of both G, L and WS property E is possible if, with respect to the same state vector Ψ, there exist a which-slit detector $T = \mathbf{1}_I \otimes T_{II}$ of E, a detector $Y = \mathbf{1}_I \otimes Y_{II}$ of G and a detector $W = \mathbf{1}_I \otimes W_{II}$ of L. Let the detectors satisfy the condition $[T,Y] = [T,W] = [Y,W] = 0$ in such a manner that T, Y and W can be measured together; hence they provide simultaneous information about E, G and L. We notice that equations $[T,F(\Delta)] = [Y,F(\Delta)] = [W,F(\Delta)] = 0$ are automatically satisfied.

We formalize the problem into the following mathematical terms.

Problem ($\mathcal{P}'$). *Given the property $E = E_I \otimes \mathbf{1}_{II}$ we have to find*

- *two projection operators G_I and L_I of $\mathcal{H}_I$,*

- *three projection operators T_{II}, Y_{II} and W_{II} of $\mathcal{H}_{II}$,*

- *a state vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$,*

such that the following conditions are satisfied:

(C.1) $[E,G] \neq 0$ *i.e* $[E_I,G_I] \neq 0$,

(C.2) $[E,L] \neq 0$ *i.e* $[E_I,L_I] \neq 0$,

(C.3) $[L,G] \neq 0$ *i.e* $[L_I,G_I] \neq 0$,

(C.4) $[T,E] = 0$ *and* $T\Psi = E\Psi$,

(C.5) $[Y{,}G] = 0$ *and* $Y\Psi = G\Psi$,

(C.6) $[W{,}L] = 0$ *and* $W\Psi = L\Psi$,

(C.7) $[T{,}Y] = 0$,

(C.8) $[T{,}W] = 0$,

(C.9) $[Y{,}W] = 0$,

(C.10) $\Psi \neq E\Psi \neq 0$, $\Psi \neq G\Psi \neq 0$ *and* $\Psi \neq L\Psi \neq 0$.

Conditions (C.1)-(C.3) are equivalent to state that properties represented by projection operators E, G and L are mutually non-compatible. In the remaining items, the requirements that the commutators are zero ensures the compatibility (hence simultaneous measurability) of the properties represented by the involved projection operators. Moreover, for a given state Ψ, if equations $T\Psi = E\Psi$ in (C.4), $Y\Psi = G\Psi$ in (C.5) and $W\Psi = L\Psi$ in (C.6) hold, then it is also possible to detect which slit each particle passes through, to detect G and to detect L altogether, by means of a simultaneous measurement of T, Y and W; indeed, for instance, equation $T\Psi = E\Psi$ implies the following one

$$p(T \mid E) = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid E\Psi \rangle} = 1 = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid T\Psi \rangle} = p(E \mid T)$$

for the conditional probabilities, allowing us to infer the passage of particle 1 through slit 1 from the occurrence of outcome 1 for T: in this sense E and T are correlated properties; this argument can be repeated for (C.5) and (C.6). Condition (C.10) is added to exclude solutions corresponding to the uninteresting cases.

We introduce a suitable matrix representation, to make easier our task.

3.5.2 Matrix Representation

The matrix representation of $\mathcal{H}_I$ described in section 3.3 depends only upon WS property E, so that it is inherited here without modifications. The projection operators E_I,G_I and L_I in (C.1)-(C.10) have the following representations:

$$E_I = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad G_I = \begin{pmatrix} P & U \\ V & Q \end{pmatrix}, \quad L_I = \begin{pmatrix} M & Z \\ W & N \end{pmatrix}, \tag{3.21}$$

where $U \neq \mathbf{0}$ and $Z \neq \mathbf{0}$; $G_I = {G_I}^* = {G_I}^2$ and $L_I = {L_I}^* = {L_I}^2$. Constraints $U \neq \mathbf{0}$ and $Z \neq \mathbf{0}$ above are equivalent to $[E_I,G_I] \neq \mathbf{0}$ and $[E_I,L_I] \neq \mathbf{0}$ required by (C.1) and (C.2).

Following the same argument of the previous section for the representation of $\mathcal{H}_{II}$, this time eight projection operators A_i $(i = 1,\ldots,8)$ of $\mathcal{H}_{II}$ must exist, such that $\sum_1^8 A_i = \mathbf{1}$, $T_{II} = A_1 + A_2 + A_3 + A_5$, $Y_{II} = A_1 + A_2 + A_4 + A_6$, $W_{II} = A_1 + A_3 + A_4 + A_7$ (fig. 3.3). Then we choose to represent every

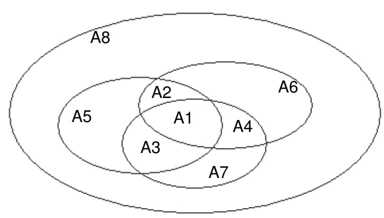

Figure 3.3. Representation for $\mathcal{H}_{II}$

vector $\mathbf{x} \in \mathcal{H}_{II}$ as a column vector $\mathbf{x} = (\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\mathbf{g},\mathbf{h})^t$ where $\mathbf{a} = A_1\mathbf{x}$,$\mathbf{b} = A_2\mathbf{x},\ldots,\mathbf{h} = A_8\mathbf{x}$. As a consequence, the projection operators T_{II}, Y_{II}, W_{II}

in (C.1)-(C.10) must satisfy the following constraints

$$T_{II} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad Y_{II} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$W_{II} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \qquad (3.22)$$

The representation of $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$ described in section 3.3 can be adopted here taking into account that every vector Ψ in the product space $\mathcal{H}_I \otimes \mathcal{H}_{II}$ shall be represented as a column vector

$$\Psi = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_i,\ldots;\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_j,\ldots)^t$$

where $\mathbf{x}_i = (\mathbf{a}_i,\mathbf{b}_i,\mathbf{c}_i,\mathbf{d}_i,\mathbf{e}_i,\mathbf{f}_i,\mathbf{g}_i,\mathbf{h}_i)^t$, $\mathbf{y}_j = \left(\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j,\boldsymbol{\gamma}_j,\boldsymbol{\delta}_j,\boldsymbol{\epsilon}_j,\boldsymbol{\zeta}_j,\boldsymbol{\eta}_j,\boldsymbol{\theta}_j\right)^t$ and $\mathbf{a}_i = A_1\mathbf{x}_i,\mathbf{b}_i = A_2\mathbf{x}_i,\ldots,\mathbf{h}_i = A_8\mathbf{x}_i,\boldsymbol{\alpha}_j = A_1\mathbf{y}_j,\boldsymbol{\beta}_j = A_2\mathbf{y}_j,\ldots,\boldsymbol{\theta}_j = A_8\mathbf{y}_j$. As a consequence, in our representation $E = E_I \otimes \mathbf{1}$ and $T = \mathbf{1}_I \otimes T_{II}$ will be identified with matrices

$$E = \begin{pmatrix} \mathbf{1}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{1}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \end{pmatrix}, \quad T = \begin{pmatrix} \mathbf{T}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{T}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{T}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{T}_{II} & . \\ . & . & . & . & . & . \end{pmatrix}.$$

From condition (C.4) $T\Psi = E\Psi$ we obtain that $\mathbf{d}_i = \mathbf{f}_i = \mathbf{g}_i = \mathbf{h}_i = \mathbf{0}$ and $\boldsymbol{\alpha}_j = \boldsymbol{\beta}_j = \boldsymbol{\gamma}_j = \boldsymbol{\epsilon}_j = \mathbf{0}$, so that $\mathbf{x}_i = (\mathbf{a}_i,\mathbf{b}_i,\mathbf{c}_i,\mathbf{0},\mathbf{e}_i,\mathbf{0},\mathbf{0},\mathbf{0})^t$ and $\mathbf{y}_j = \big(\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_j,\mathbf{0},\boldsymbol{\zeta}_j,\boldsymbol{\eta}_j,\boldsymbol{\theta}_j\big)$, i.e.

$$\Psi = (\cdot,\underbrace{\mathbf{a}_i,\mathbf{b}_i,\mathbf{c}_i,\mathbf{0},\mathbf{e}_i,\mathbf{0},\mathbf{0},\mathbf{0}}_{\mathbf{x}_i^t},\cdot,\cdot;\ \cdot\underbrace{\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_j,\mathbf{0},\boldsymbol{\zeta}_j,\boldsymbol{\eta}_j,\boldsymbol{\theta}_j}_{\mathbf{y}_j^t},\cdot,\cdot)^t. \tag{3.23}$$

Further constraints are imposed by condition (C.5) $Y\Psi = G\Psi$, where projection operators G and Y are represented as

$$G = G_I \otimes \mathbf{1}_{II} = \begin{pmatrix} p_{11}\mathbf{1}_{II} & p_{12}\mathbf{1}_{II} & . & u_{11}\mathbf{1}_{II} & u_{12}\mathbf{1}_{II} & . \\ p_{21}\mathbf{1}_{II} & p_{22}\mathbf{1}_{II} & . & u_{21}\mathbf{1}_{II} & u_{22}\mathbf{1}_{II} & . \\ . & . & . & . & . & . \\ v_{11}\mathbf{1}_{II} & v_{12}\mathbf{1}_{II} & . & q_{11}\mathbf{1}_{II} & q_{12}\mathbf{1}_{II} & . \\ v_{21}\mathbf{1}_{II} & v_{22}\mathbf{1}_{II} & . & q_{21}\mathbf{1}_{II} & q_{22}\mathbf{1}_{II} & . \\ . & . & . & . & . & . \end{pmatrix},$$

$$Y = \mathbf{1}_I \otimes Y_{II} = \begin{pmatrix} \mathbf{Y}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{Y}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{Y}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{Y}_{II} & . \\ . & . & . & . & . & . \end{pmatrix}, \tag{3.24}$$

so that

$$\begin{aligned} Y\Psi &= (\cdots, \mathbf{a}_i, \mathbf{b}_i, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}; \mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{\delta}_j, \mathbf{0}, \boldsymbol{\zeta}_j, \mathbf{0}, \mathbf{0}, \cdots)^t, \\ G\Psi &= (\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_i, \cdots; \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_j, \cdots)^t, \end{aligned} \tag{3.25}$$

where

$$\mathbf{z}_i = \begin{pmatrix} \sum_k p_{ik}\mathbf{a_k} \\ \sum_k p_{ik}\mathbf{b_k} \\ \sum_k p_{ik}\mathbf{c_k} \\ \sum_k u_{ik}\boldsymbol{\delta}_k \\ \sum_k p_{ik}\mathbf{e_k} \\ \sum_k u_{ik}\boldsymbol{\zeta}_k \\ \sum_k u_{ik}\boldsymbol{\eta}_k \\ \sum_k u_{ik}\boldsymbol{\theta}_k \end{pmatrix} \quad \text{and} \quad \mathbf{w}_j = \begin{pmatrix} \sum_k v_{jk}\mathbf{a}_k \\ \sum_k v_{jk}\mathbf{b}_k \\ \sum_k v_{jk}\mathbf{c}_k \\ \sum_k q_{jk}\boldsymbol{\delta}_k \\ \sum_k v_{jk}\mathbf{e}_k \\ \sum_k q_{jk}\boldsymbol{\zeta}_k \\ \sum_k q_{jk}\boldsymbol{\eta}_k \\ \sum_k q_{jk}\boldsymbol{\theta}_k \end{pmatrix}, \tag{3.26}$$

so that, taking into account (3.24) and (3.26), condition (C.5) $Y\Psi = G\Psi$ can be expressed as

$$\begin{aligned} &(i-A) \quad \begin{cases} \sum_k p_{ik}\mathbf{a}_k = \mathbf{a}_i \\ \sum_k p_{ik}\mathbf{b}_k = \mathbf{b}_i \\ \sum_k p_{ik}\mathbf{c}_k = \mathbf{0} \\ \sum_k p_{ik}\mathbf{e}_k = \mathbf{0} \end{cases} \qquad (ii-B) \quad \begin{cases} \sum_k u_{ik}\boldsymbol{\delta}_k = \mathbf{0} \\ \sum_k u_{ik}\boldsymbol{\zeta}_k = \mathbf{0} \\ \sum_k u_{ik}\boldsymbol{\eta}_k = \mathbf{0} \\ \sum_k u_{ik}\boldsymbol{\theta}_k = \mathbf{0} \end{cases} \\ &(iii-C) \quad \begin{cases} \sum_k v_{ik}\mathbf{a}_k = \mathbf{0} \\ \sum_k v_{ik}\mathbf{b}_k = \mathbf{0} \\ \sum_k v_{ik}\mathbf{c}_k = \mathbf{0} \\ \sum_k v_{ik}\mathbf{e}_k = \mathbf{0} \end{cases} \qquad (iv-D) \quad \begin{cases} \sum_k q_{ik}\boldsymbol{\delta}_k = \boldsymbol{\delta}_i \\ \sum_k q_{ik}\boldsymbol{\zeta}_k = \boldsymbol{\zeta}_i \\ \sum_k q_{ik}\boldsymbol{\eta}_k = \mathbf{0} \\ \sum_k q_{ik}\boldsymbol{\theta}_k = \mathbf{0}. \end{cases} \end{aligned} \tag{3.27}$$

Further constraints are imposed by condition (C.6) $W\Psi = L\Psi$. Projection operators L and W are represented as

$$L = L_I \otimes \mathbf{1}_{II} = \begin{pmatrix} m_{11}\mathbf{1}_{II} & m_{12}\mathbf{1}_{II} & . & z_{11}\mathbf{1}_{II} & z_{12}\mathbf{1}_{II} & . \\ m_{21}\mathbf{1}_{II} & m_{22}\mathbf{1}_{II} & . & z_{21}\mathbf{1}_{II} & z_{22}\mathbf{1}_{II} & . \\ . & . & . & . & . & . \\ w_{11}\mathbf{1}_{II} & w_{12}\mathbf{1}_{II} & . & n_{11}\mathbf{1}_{II} & n_{12}\mathbf{1}_{II} & . \\ w_{21}\mathbf{1}_{II} & w_{22}\mathbf{1}_{II} & . & n_{21}\mathbf{1}_{II} & n_{22}\mathbf{1}_{II} & . \\ . & . & . & . & . & . \end{pmatrix},$$

$$W = \mathbf{1}_I \otimes W_{II} = \begin{pmatrix} \mathbf{W}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{W}_{II} & . & \mathbf{0}_{II} & \mathbf{0}_{II} & . \\ . & . & . & . & . & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{W}_{II} & \mathbf{0}_{II} & . \\ \mathbf{0}_{II} & \mathbf{0}_{II} & . & \mathbf{0}_{II} & \mathbf{W}_{II} & . \\ . & . & . & . & . & . \end{pmatrix}, \tag{3.28}$$

so that

$$\begin{aligned} W\Psi &= (\cdots, \mathbf{a}_i, \mathbf{0}, \mathbf{c}_i, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}; \mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{\delta}_j, \mathbf{0}, \mathbf{0}, \boldsymbol{\eta}_j, \mathbf{0}, \cdots)^t, \\ L\Psi &= (\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_i, \cdots; \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_j, \cdots)^t, \end{aligned} \tag{3.29}$$

where

$$\mathbf{s}_i = \begin{pmatrix} \sum_k m_{ik}\mathbf{a}_k \\ \sum_k m_{ik}\mathbf{b}_k \\ \sum_k m_{ik}\mathbf{c}_k \\ \sum_k z_{ik}\boldsymbol{\delta}_k \\ \sum_k m_{ik}\mathbf{e}_k \\ \sum_k z_{ik}\boldsymbol{\zeta}_k \\ \sum_k z_{ik}\boldsymbol{\eta}_k \\ \sum_k z_{ik}\boldsymbol{\theta}_k \end{pmatrix} \quad \text{and} \quad \mathbf{t_j} = \begin{pmatrix} \sum_k w_{jk}\mathbf{a}_k \\ \sum_k w_{jk}\mathbf{b}_k \\ \sum_k w_{jk}\mathbf{c}_k \\ \sum_k n_{jk}\boldsymbol{\delta}_k \\ \sum_k w_{jk}\mathbf{e}_k \\ \sum_k n_{jk}\boldsymbol{\zeta}_k \\ \sum_k n_{jk}\boldsymbol{\eta}_k \\ \sum_k n_{jk}\boldsymbol{\theta}_k \end{pmatrix}. \tag{3.30}$$

Then condition (C.6) $W\Psi = L\Psi$, can be written as

$$(i - A') \quad \begin{cases} \sum_k m_{ik}\mathbf{a}_k = \mathbf{a}_i \\ \sum_k m_{ik}\mathbf{b}_k = \mathbf{0} \\ \sum_k m_{ik}\mathbf{c}_k = \mathbf{c}_i \\ \sum_k m_{ik}\mathbf{e}_k = \mathbf{0} \end{cases} \qquad (ii - B') \quad \begin{cases} \sum_k z_{ik}\boldsymbol{\delta}_k = \mathbf{0} \\ \sum_k z_{ik}\boldsymbol{\zeta}_k = \mathbf{0} \\ \sum_k z_{ik}\boldsymbol{\eta}_k = \mathbf{0} \\ \sum_k z_{ik}\boldsymbol{\theta}_k = \mathbf{0} \end{cases}$$

$$(iii - C') \quad \begin{cases} \sum_k w_{ik}\mathbf{a}_k = \mathbf{0} \\ \sum_k w_{ik}\mathbf{b}_k = \mathbf{0} \\ \sum_k w_{ik}\mathbf{c}_k = \mathbf{0} \\ \sum_k w_{ik}\mathbf{e}_k = \mathbf{0} \end{cases} \qquad (iv - D') \quad \begin{cases} \sum_k n_{ik}\boldsymbol{\delta}_k = \boldsymbol{\delta}_i \\ \sum_k n_{ik}\boldsymbol{\zeta}_k = \mathbf{0} \\ \sum_k n_{ik}\boldsymbol{\eta}_k = \boldsymbol{\eta}_j \\ \sum_k n_{ik}\boldsymbol{\theta}_k = \mathbf{0}. \end{cases} \tag{3.31}$$

3.5.3 A family of solutions

Until now we have established general constraints to be satisfied by any solution of the problem, independently of the ranks of matrices, and then of dimensions of the spaces $\mathcal{H}_I$ and $\mathcal{H}_{II}$. Here we present a concrete solution of the problem, whose rather technical derivation is displaced in the next section; our research is not at all exhaustive: we analyze a particular situation and, according to it, the detections of L and G turn out to be always correlated.

We notice that if in correspondence with a given state Ψ satisfying (3.23), matrices G and L satisfying (3.27) and (3.31) exist such that $[E,G]$, $[G,L]$ and $[L,E]$ are non-zero, $G = G^* = G^2$ and $L = L^* = L^2$, then, all (C-4)-(C-9) are automatically satisfied.

As in previous case, we restrict our research to the case that the two slits are symmetrical.

We shall proceed as follows: at the beginning, no constraint is imposed about the dimension of $\mathcal{H}_I$. We restrict our search by working with the equations in (ii-B), (iv-D) of (3.27) and (ii-B'), (iv-D') of (3.31), rather than with all of them. Since (i-A), (iii-C) in (3.27) are formally identical to (ii-B), (iv-D) of (3.27) and (i-A'), (iii-C') in (3.31) are formally identical to (ii-B'), (iv-D') of (3.31), we can extend to them the results found for (ii-B), (iv-D), (ii-B'), (iv-D').

We analyze the equations with respect to the linear dependence or independence of the involved vectors, according to a method described in the next section. Our task would be more meaningful if we attain non-correlated solutions of $(\mathcal{P}')$. However, the hypothesis introduced to restrict the kind of solutions, with the aim of simplifying the problem, lead us to correlated solutions. Our task is made easier by fixing the dimension of $\mathcal{H}_I$, $dim(\mathcal{H}_I) = 10$, and searching solutions corresponding to a particular state vector Ψ (see (3.47)). Taking into account self-adjointness of G_I and L_I, we get

$$Q = \begin{pmatrix} q & -\alpha_2\left(q-\frac{1}{\Gamma}\right) & -\alpha_3 q & -\beta_4\frac{\alpha_2}{\Gamma} & -\beta_5\frac{\alpha_2}{\Gamma} \\ -\overline{\alpha}_2\left(q-\frac{1}{\Gamma}\right) & \Lambda_3+\mid\alpha_2\mid^2\left(q-\frac{1}{\Gamma}\right) & \alpha_3\overline{\alpha}_2\left(q-\frac{1}{\Gamma}\right) & -\beta_4\Lambda_3 & -\beta_5\Lambda_3 \\ -\overline{\alpha}_3 q & -\overline{\alpha}_3\alpha_2\left(q-\frac{1}{\Gamma}\right) & \mid\alpha_3\mid^2 q & \overline{\alpha}_3\beta_4\frac{\alpha_2}{\Gamma} & \overline{\alpha}_3\beta_5\frac{\alpha_2}{\Gamma} \\ -\overline{\beta}_4\frac{\overline{\alpha}_2}{\Gamma} & -\overline{\beta}_4\Lambda_3 & \overline{\beta}_4\alpha_3\frac{\overline{\alpha}_2}{\Gamma} & \mid\beta_4\mid^2\Lambda & \overline{\beta}_4\beta_5\Lambda \\ -\overline{\beta}_5\frac{\overline{\alpha}_2}{\Gamma} & -\overline{\beta}_5\Lambda_3 & \overline{\beta}_5\alpha_3\frac{\overline{\alpha}_2}{\Gamma} & \overline{\beta}_5\beta_4\Lambda & \mid\beta_5\mid^2\Lambda \end{pmatrix},$$

where

- $\Gamma = \left(1+\mid\alpha_3\mid^2\right)+\left(\mid\beta_4\mid^2+\mid\beta_5\mid^2\right)\left(1+\mid\alpha_2\mid^2+\mid\alpha_3\mid^2\right)$,

- $\Lambda_2 = \frac{|\alpha_2|^2}{\Gamma}$, $\Lambda_3 = \frac{1+|\alpha_3|^2}{\Gamma}$ and $\Lambda = \Lambda_2 + \Lambda_3$;

$$N = \begin{pmatrix} n & -\alpha_2\left(n-\frac{1}{\mathit{\Gamma}}\right) & -\alpha_3\left(n-\frac{1}{\mathit{\Delta}}\right) & -\beta_4\frac{\alpha_2}{\mathit{\Gamma}}-\lambda_4\frac{\alpha_3}{\mathit{\Delta}} & -\beta_5\frac{\alpha_2}{\mathit{\Gamma}}-\lambda_5\frac{\alpha_3}{\mathit{\Delta}} \\ -\overline{\alpha}_2\left(n-\frac{1}{\mathit{\Gamma}}\right) & \mathit{\Delta}_3+\mid\alpha_2\mid^2\left(n-\frac{1}{\mathit{\Gamma}}\right) & \alpha_3\overline{\alpha}_2\left(n-\frac{1}{\mathit{\Gamma}}-\frac{1}{\mathit{\Delta}}\right) & -\beta_4\mathit{\Delta}_3+\lambda_4\overline{\alpha}_2\frac{\alpha_3}{\mathit{\Delta}} & -\beta_5\mathit{\Delta}_3+\lambda_5\overline{\alpha}_2\frac{\alpha_3}{\mathit{\Delta}} \\ -\overline{\alpha}_3\left(n-\frac{1}{\mathit{\Delta}}\right) & \alpha_2\overline{\alpha}_3\left(n-\frac{1}{\mathit{\Gamma}}-\frac{1}{\mathit{\Delta}}\right) & \mathit{\Sigma}_2+\mid\alpha_3\mid^2\left(n-\frac{1}{\mathit{\Delta}}\right) & \overline{\alpha}_3\beta_4\frac{\alpha_2}{\mathit{\Gamma}}-\lambda_4\mathit{\Sigma}_2 & \overline{\alpha}_3\beta_5\frac{\alpha_2}{\mathit{\Gamma}}-\lambda_5\mathit{\Sigma}_2 \\ -\overline{\beta}_4\frac{\overline{\alpha}_2}{\mathit{\Gamma}}-\overline{\lambda}_4\frac{\overline{\alpha}_3}{\mathit{\Delta}} & -\overline{\beta}_4\mathit{\Delta}_3+\alpha_2\overline{\lambda}_4\frac{\overline{\alpha}_3}{\mathit{\Delta}} & \overline{\beta}_4\alpha_3\frac{\overline{\alpha}_2}{\mathit{\Gamma}}-\overline{\lambda}_4\mathit{\Sigma}_2 & \mid\beta_4\mid^2\mathit{\Delta}+\mid\lambda_4\mid^2\mathit{\Sigma} & \overline{\beta}_4\beta_5\mathit{\Delta}+\overline{\lambda}_4\lambda_5\mathit{\Sigma} \\ -\overline{\beta}_5\frac{\overline{\alpha}_2}{\mathit{\Gamma}}-\overline{\beta}_4\frac{\overline{\alpha}_3}{\mathit{\Delta}} & -\overline{\beta}_5\mathit{\Delta}_3+\alpha_2\overline{\beta}_4\frac{\overline{\alpha}_3}{\mathit{\Delta}} & \overline{\beta}_5\alpha_3\frac{\overline{\alpha}_2}{\mathit{\Gamma}}-\overline{\lambda}_5\mathit{\Sigma}_2 & \overline{\beta}_5\beta_4\mathit{\Delta}+\overline{\lambda}_5\lambda_4\mathit{\Sigma} & \mid\beta_5\mid^2\mathit{\Delta}+\mid\lambda_5\mid^2\mathit{\Sigma} \end{pmatrix},$$

where

- $\lambda_4 = \frac{\alpha_2\overline{\alpha}_3}{\overline{\beta}_4\left(1+|\alpha_2|^2+|\alpha_3|^2\right)} - \lambda_5\frac{\overline{\beta}_5}{\overline{\beta}_4}$,
- $\mathit{\Delta} = \left(1+\mid\alpha_2\mid^2\right)+\left(\mid\lambda_4\mid^2+\mid\lambda_5\mid^2\right)\left(1+\mid\alpha_2\mid^2+\mid\alpha_3\mid^2\right)$,
- $\mathit{\Sigma}_2 = \frac{1+|\alpha_2|^2}{\mathit{\Delta}}$, $\mathit{\Sigma}_3 = \frac{|\alpha_3|^2}{\mathit{\Delta}}$ and $\mathit{\Sigma} = \mathit{\Sigma}_2 + \mathit{\Sigma}_3$;

and

$$U = \begin{pmatrix} u & -\alpha_2 u & -\alpha_3 u & 0 & 0 \\ -\overline{a}_2 u & \overline{a}_2\alpha_2 u & \overline{a}_2\alpha_3 u & 0 & 0 \\ -\overline{a}_3 u & \overline{a}_3\alpha_2 u & \overline{a}_3\alpha_3 u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Matrices P, M and Z have the same form of Q, N and U respectively, with p, m, z, a_i and b_j in place of q, n, u, α_i and β_j, where $i = 2{,}3$ and $j = 4{,}5$; A_i, B_i, C, D take the place of $\mathit{\Lambda}_i$, $\mathit{\Sigma}_i$, $\mathit{\Gamma}$, $\mathit{\Delta}$ and are defined in analogous manner; moreover, $V = \overline{U}^t$ and $W = \overline{Z}^t$. We notice that a_i,b_j,l_j,α_i,β_j,λ_j, with $i = 2{,}3$ and $j = 4{,}5$, are constant complex numbers arising from the linear dependence among the vector-components of $\mathbf{x_i}$ and $\mathbf{y_j}$, where $i,j = 1,\ldots,5$, as we shall see in the next section.

In order to solve the problem, such solution must be idempotent. By imposing idempotence we find that

1. $\frac{A_2}{1+|a_2|^2+|a_3|^2} < p < \frac{A_2+1}{1+|a_2|^2+|a_3|^2}$

2. $\frac{A_2+B_3}{1+|a_2|^2+|a_3|^2} < m < \frac{A_2+B_3+1}{1+|a_2|^2+|a_3|^2}$

3. $u = e^{i\theta_1}\sqrt{\frac{\left(p-\frac{1}{C}\right)(1-2A_3)-\left(p-\frac{1}{C}\right)^2\left(1+|a_2|^2+|a_3|^2\right)+A_3\frac{|\beta_4|^2+|\beta_5|^2}{C}}{1+|\alpha_2|^2+|\alpha_3|^2}}$

4. $z = e^{i\theta_2}\sqrt{\frac{\left(m-\frac{1}{C}\right)(1-2(A_3-B_3))-\left(m-\frac{1}{C}\right)^2\left(1+|a_2|^2+|a_3|^2\right)-\frac{(B_3-A_3)^2+(B_3-A_3)}{1+|a_2|^2+|a_3|^2}}{1+|\alpha_2|^2+|\alpha_3|^2}}$

5. $q = \frac{1+\Lambda_2+A_2-p\left(1+|a_2|^2+|a_3|^2\right)}{1+|\alpha_2|^2+|\alpha_3|^2}$

6. $n = \frac{1+A_2+B_3+\Lambda_2+\Sigma_3-m\left(1+|a_2|^2+|a_3|^2\right)}{1+|\alpha_2|^2+|\alpha_3|^2}$

where θ_1 and θ_2 are real numbers. Our family of solutions completely solves the problem if it satisfies $[G,L] \neq 0$.

Therefore, for every real number $\frac{A_2}{1+|a_2|^2+|a_3|^2} < p < \frac{A_2+1}{1+|a_2|^2+|a_3|^2}$, every real number $\frac{A_2+B_3}{1+|a_2|^2+|a_3|^2} < m < \frac{A_2+B_3+1}{1+|a_2|^2+|a_3|^2}$, every $a_2,a_3,b_4,b_5,l_5,\alpha_2,\alpha_3,\beta_4,\beta_5,\lambda_5 \in \mathbf{C}$, such that $[G,L] \neq 0$, we have a solution of $(\mathcal{P}')$. Since G_I and L_I are projection operators and no constraint is imposed to their traces, i.e. to $Rank(G_I)$ and $Rank(L_I)$, these parameters are not all independent.

For instance, the following solution of $(\mathcal{P}')$ is obtained in correspondence with the particular choice $a_2 = a_3 = \alpha_2 = \alpha_3 = 1$, $b_4 = b_5 = \beta_4 = \beta_5 = 1$,

$\lambda_5 = l_5 = 1$ $\theta_1 = \theta_2 = 0$, $p = \frac{11}{72}$ and $m = \frac{67}{456}$:

$$G_I = \frac{1}{72}\begin{pmatrix} 11 & -2 & -11 & -22 & -9 & 8\sqrt{2} & -8\sqrt{2} & -8\sqrt{2} & 0 & 0 \\ -2 & 20 & 2 & -18 & -18 & -8\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} & 0 & 0 \\ -11 & 2 & 11 & 9 & 9 & -8\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} & 0 & 0 \\ -9 & -18 & 9 & 27 & 27 & 0 & 0 & 0 & 0 & 0 \\ -9 & -18 & 9 & 27 & 27 & 0 & 0 & 0 & 0 & 0 \\ 8\sqrt{2} & -8\sqrt{2} & -8\sqrt{2} & 0 & 0 & 19 & -10 & -19 & -9 & -9 \\ -8\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} & 0 & 0 & -10 & 28 & 10 & -18 & -18 \\ -8\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} & 0 & 0 & -19 & 10 & 19 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & -9 & -18 & 9 & 27 & 27 \\ 0 & 0 & 0 & 0 & 0 & -9 & -18 & 9 & 27 & 27 \end{pmatrix},$$

$$L_I = \frac{1}{456}\begin{pmatrix} 67 & -10 & 5 & -9 & -129 & 32\sqrt{3} & -32\sqrt{3} & -32\sqrt{3} & 0 & 0 \\ -10 & 124 & -62 & -162 & -42 & -32\sqrt{3} & 32\sqrt{3} & 32\sqrt{3} & 0 & 0 \\ 5 & -62 & 139 & 153 & -87 & -32\sqrt{3} & 32\sqrt{3} & 32\sqrt{3} & 0 & 0 \\ -9 & -162 & 153 & 267 & 27 & 0 & 0 & 0 & 0 & 0 \\ -129 & -42 & -87 & 27 & 387 & 0 & 0 & 0 & 0 & 0 \\ 32\sqrt{3} & 32\sqrt{3} & -32\sqrt{3} & 0 & 0 & 171 & -114 & -99 & -9 & -129 \\ -32\sqrt{3} & 32\sqrt{3} & 32\sqrt{3} & 0 & 0 & -114 & 228 & 42 & -162 & -42 \\ -32\sqrt{3} & 32\sqrt{3} & 32\sqrt{3} & 0 & 0 & -99 & 42 & 243 & 153 & -87 \\ 0 & 0 & 0 & 0 & 0 & -9 & -162 & 153 & 267 & 27 \\ 0 & 0 & 0 & 0 & 0 & -129 & -42 & -87 & 27 & 387 \end{pmatrix},$$

$$\Psi = (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\mathbf{x}_5;\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{y}_4,\mathbf{y}_5)^t,$$

where

$$\mathbf{x}_1 = \left(-\tfrac{1}{3}\mathbf{a}_5,\mathbf{0}, -\tfrac{1}{3}\mathbf{c}_5,\mathbf{0},\tfrac{1}{3}\mathbf{e}_4 + 2\mathbf{e}_5,\mathbf{0},\mathbf{0},\mathbf{0},\right)^t$$

$$\mathbf{x}_2 = \left(-\tfrac{2}{3}\mathbf{a}_5,\mathbf{0},\tfrac{1}{3}\mathbf{c}_5,\mathbf{0},\mathbf{e}_4 + \mathbf{e}_5,\mathbf{0},\mathbf{0},\mathbf{0},\right)^t$$

$$\mathbf{x}_3 = \left(\tfrac{1}{3}\mathbf{a}_5,\mathbf{0}, -\tfrac{2}{3}\mathbf{c}_5,\mathbf{0}, -\tfrac{2}{3}\mathbf{e}_4 + \mathbf{e}_5,\mathbf{0},\mathbf{0},\mathbf{0},\right)^t$$

$$\mathbf{x}_4 = \left(\mathbf{a}_5,\mathbf{0}, -\tfrac{2}{3}\mathbf{c}_5,\mathbf{0},\mathbf{e}_4,\mathbf{0},\mathbf{0},\mathbf{0},\right)^t$$

$$\mathbf{x}_5 = (\mathbf{a}_5,\mathbf{0},\mathbf{c}_5,\mathbf{0},\mathbf{e}_5,\mathbf{0},\mathbf{0},\mathbf{0},)^t$$

$$\mathbf{y}_1 = \left(\mathbf{0},\mathbf{0},\mathbf{0}, -\tfrac{1}{3}\boldsymbol{\delta}_5,\mathbf{0},\mathbf{0}, -\tfrac{1}{3}\boldsymbol{\eta}_5,\tfrac{1}{3}\boldsymbol{\theta}_4 + 2\boldsymbol{\theta}_5\right)^t$$

$$\mathbf{y}_2 = \left(\mathbf{0},\mathbf{0},\mathbf{0}, -\tfrac{2}{3}\boldsymbol{\delta}_5,\mathbf{0},\mathbf{0},\tfrac{1}{3}\boldsymbol{\eta}_5,\boldsymbol{\theta}_4 + \boldsymbol{\theta}_5\right)^t$$

$$\mathbf{y}_3 = \left(\mathbf{0},\mathbf{0},\mathbf{0},\tfrac{1}{3}\boldsymbol{\delta}_5,\mathbf{0},\mathbf{0}, -\tfrac{2}{3}\boldsymbol{\eta}_5, -\tfrac{2}{3}\boldsymbol{\theta}_4 + \boldsymbol{\theta}_5\right)^t$$

$$\mathbf{y}_4 = \left(\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_5,\mathbf{0},\mathbf{0}, -\tfrac{2}{3}\boldsymbol{\eta}_5,\boldsymbol{\theta}_4\right)^t$$

$$\mathbf{y}_5 = (\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_5,\mathbf{0},\mathbf{0},\boldsymbol{\eta}_5,\boldsymbol{\theta}_5)^t.$$

We stress that our family of solutions is a particular one, obtained by making particular assumptions and corresponding to correlated detections. The question whether non-correlated solutions do exist or not, if $dim(\mathcal{H}_I) = 10$, remains open.

3.6 Derivation of a family of solutions

In this section we carry out the detailed derivation of the family of solutions of problem $(\mathcal{P}')$ presented in section 3.5.3. Our treatment is not at all exhaustive. Indeed, we shall consider solutions characterized by particular conditions of linear independence between some of their components. We prove that these particular solutions are correlated. The solutions concretely singled out correspond to the case $dim(\mathcal{H}_I) = 10$. Since we consider only particular classes of solutions, the question whether non-correlated solutions exist or not, for $dim(\mathcal{H}_I) \geq 10$, remains open.

In our derivation we analyze the equations involving vectors y_i, i.e. (ii-B), (iv-D) in (3.27) and (ii-B'), (iv-D') in (3.31); in order to solve them, we make some assumptions (e.g. vectors $\begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix}$ are supposed linearly independent, as well as $\boldsymbol{\theta}_4, \ldots, \boldsymbol{\theta}_n$; furthermore, $\boldsymbol{\zeta}_j = \mathbf{0}$ for all $j = 1, \ldots, n$,) which lead to particular forms for vectors y_i and for matrices Q, U, N, Z. Analogous results are token also for (i-A), (iii-C) in (3.27) and (i-A'), (iii-C') in (3.31), which have the same form of the previous ones. We fix the dimension of $\mathcal{H}_I$, $dim(\mathcal{H}_I) = 10$, then we obtain a particular family of solutions of problem $(\mathcal{P}')$, which show a correlation between L and G.

3.6.1 General constraints

Equations in (ii-B) imply $u_{j1}\mathbf{y}_1 + \cdots + u_{jn}\mathbf{y}_n = 0$. Therefore, since $U \neq \mathbf{0}$, vectors $\mathbf{y}_1, \ldots, \mathbf{y}_n$ must be linearly dependent. Let us suppose $\mathbf{y}_1 = \alpha_2\mathbf{y}_2 + \cdots + \alpha_n\mathbf{y}_n$. By using this relation in (iv-D) we get

$$\begin{cases} Q_{j2}\boldsymbol{\delta}_2 + \cdots + Q_{jn}\boldsymbol{\delta}_n = \boldsymbol{\delta}_j \\ Q_{j2}\boldsymbol{\zeta}_2 + \cdots + Q_{jn}\boldsymbol{\zeta}_n = \boldsymbol{\zeta}_j \\ Q_{j2}\begin{pmatrix} \boldsymbol{\eta}_2 \\ \boldsymbol{\theta}_2 \end{pmatrix} + \cdots + Q_{jn}\begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \end{cases} \tag{3.32}$$

where $Q_{jk} = q_{jk} + \alpha_k q_{j1}$, for all $k = 2, \ldots, n$.

In the second equation of (3.32), if we suppose that $\begin{pmatrix} \boldsymbol{\eta}_2 \\ \boldsymbol{\theta}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix}$ are linearly independent then $Q_{j2} = \ldots = Q_{j2} = 0$, which implies $\boldsymbol{\delta}_j = \boldsymbol{\zeta}_j = \mathbf{0}$, for all $j = 1,\ldots,n$, in the first equation of (3.32). A similar reasoning for (i-A) leads to $\mathbf{a}_j = \mathbf{b}_j = \mathbf{0}$, for all $j = 1,\ldots,n$; as a consequence $\mathbf{x}_j = (\mathbf{0},\mathbf{0},\mathbf{c}_j,\mathbf{0},\mathbf{e}_j,\mathbf{0},\mathbf{0},\mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\eta}_j,\boldsymbol{\theta}_j)^t$ so that, if a solution exists, then $Y\Psi = 0$, i.e. it would be correspond to the uninteresting case excluded by (C.10). If we consider all cases, also that of linear dependence of $\begin{pmatrix} \boldsymbol{\eta}_2 \\ \boldsymbol{\theta}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix}$, we obtain the following implications:

a.) $\mathbf{a}_j = \mathbf{b}_j = \mathbf{0}$ and $\boldsymbol{\delta}_j = \boldsymbol{\zeta}_j = \mathbf{0}$, for all $j = 1,\ldots,n$ imply that $\mathbf{x}_j = (\mathbf{0},\mathbf{0},\mathbf{c}_j,\mathbf{0},\mathbf{e}_j,\mathbf{0},\mathbf{0},\mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\eta}_j,\boldsymbol{\theta}_j)^t$ so that, if a solution exists, then $Y\Psi = 0$; hence, condition (C.10) is violated;

b.) $\mathbf{a}_j = \mathbf{b}_j = \mathbf{0}$ and $\boldsymbol{\delta}_j \neq \mathbf{0} \neq \boldsymbol{\zeta}_j$, for all $j = 1,\ldots,n$ imply that $\mathbf{x}_j = (\mathbf{0},\mathbf{0},\mathbf{c}_j,\mathbf{0},\mathbf{e}_j,\mathbf{0},\mathbf{0},\mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_j,\mathbf{0},\boldsymbol{\zeta}_j,\boldsymbol{\eta}_j,\boldsymbol{\theta}_j)^t$ so that, if a solution exists, then $TW\Psi = 0$, i.e. each time a particle is sorted by W than it is certainly not sorted by T; therefore, for all eventual solutions corresponding to this case, property L must be correlated with WS property E;

c.) $\mathbf{a}_j \neq \mathbf{0} \neq \mathbf{b}_j$ and $\boldsymbol{\delta}_j = \boldsymbol{\zeta}_j = \mathbf{0}$, for all $j = 1,\ldots,n$ imply that $\mathbf{x}_j = (\mathbf{a}_j,\mathbf{b}_j,\mathbf{c}_j,\mathbf{0},\mathbf{e}_j,\mathbf{0},\mathbf{0},\mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\eta}_j,\boldsymbol{\theta}_j)^t$ so that, if a solution exists, then $T'W'Y\Psi = 0$; this last equation expresses the impossibility that two probabilities are zero and the remaining is 1, for the occurrence of T', W' and Y;

d.) $\mathbf{a}_j \neq \mathbf{0} \neq \mathbf{b}_j$ and $\boldsymbol{\delta}_j \neq \mathbf{0} \neq \boldsymbol{\zeta}_j$, for all $j = 1,\ldots,n$ imply that $\mathbf{x}_j =$

$(\mathbf{a}_j,\mathbf{b}_j,\mathbf{c}_j,\mathbf{0},\mathbf{e}_j,\mathbf{0},\mathbf{0},\mathbf{0})^t$ and $\mathbf{y}_j = (\mathbf{0},\mathbf{0},\mathbf{0},\boldsymbol{\delta}_j,\mathbf{0},\boldsymbol{\zeta}_j,\boldsymbol{\eta}_j,\boldsymbol{\theta}_j)^t$ so that no correlated or meaningless solution immediately follows.

Now we investigate case (d.). Let $\beta_3,\ldots,\beta_n$ be complex numbers such that

$$\begin{pmatrix} \boldsymbol{\eta}_2 \\ \boldsymbol{\theta}_2 \end{pmatrix} = \beta_3 \begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix}. \tag{3.33}$$

Hence, (ii-B) and (iv-D) yield

$$\begin{cases} U_{j2} \begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} + \ldots + U_{jn} \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \\ (U_{j2}\beta_3 + U_{j3}) \begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix} + \ldots + (U_{j2}\beta_n + U_{jn}) \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{cases} \tag{3.34}$$

where $U_{jk} = u_{jk} + \alpha_k u_{j1}$, for all $k = 2,\ldots,n$, and

$$\begin{cases} Q_{j2} \begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{\delta}_j \\ \boldsymbol{\zeta}_j \end{pmatrix} \\ (Q_{j2}\beta_3 + Q_{j3}) \begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix} + \ldots + (Q_{j2}\beta_n + Q_{jn}) \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{cases}. \tag{3.35}$$

Let us suppose that the vectors in the second equation of (3.34) and (3.35) are linearly independent and, from now on, we draw the consequences of this hypothesis. From the second equation of (3.34) and (3.35), we get

$$U_{j2}\beta_3 + U_{j3} = \ldots = U_{j2}\beta_n + U_{jn} = 0, \tag{3.36}$$

$$Q_{j2}\beta_3 + Q_{j3} = \ldots = Q_{j2}\beta_n + Q_{jn} = 0; \tag{3.37}$$

respectively. Hence (3.34) and (3.35) become

$$\begin{cases} U_{j2}\left[\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} - \beta_3 \begin{pmatrix} \boldsymbol{\delta}_3 \\ \boldsymbol{\zeta}_3 \end{pmatrix} - \ldots - \beta_n \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}\right] = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \\ (U_{j2}\beta_3 + U_{j3}) \begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix} + \ldots + (U_{j2}\beta_n + U_{jn}) \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \end{cases} \tag{3.38}$$

$$\begin{cases} Q_{j2}\left[\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} - \beta_3 \begin{pmatrix} \boldsymbol{\delta}_3 \\ \boldsymbol{\zeta}_3 \end{pmatrix} - \ldots - \beta_n \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}\right] = \begin{pmatrix} \boldsymbol{\delta}_j \\ \boldsymbol{\zeta}_j \end{pmatrix} \\ (Q_{j2}\beta_3 + Q_{j3}) \begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix} + \ldots + (Q_{j2}\beta_n + Q_{jn}) \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \end{cases} \tag{3.39}$$

First equation in (3.38) implies either $U_{j2} = 0$ or

$$\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = \beta_3 \begin{pmatrix} \boldsymbol{\delta}_3 \\ \boldsymbol{\zeta}_3 \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}.$$

On the other hand, in the last hypothesis, (3.39) again implies $\boldsymbol{\delta}_j = \boldsymbol{\zeta}_j = \mathbf{0}$, for all $j = 1,\ldots,n$, which (together with $\mathbf{a}_j = \mathbf{b}_j = \mathbf{0}, \quad \forall j = 1,\ldots,n$) leads to meaningless or correlated solutions; hence we suppose $U_{j2} = 0$.

In the first equation of (3.39) we can suppose that vectors $\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix}, \ldots,$ $\begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}$ are either linearly independent or linearly dependent (with coefficients different from $b_3,\ldots,b_n$). In case they are independent, first equation in (3.39) yields

$$\begin{cases} Q_{n2} = 0 \\ \beta_3 Q_{n2} = 0 \\ \vdots \\ \beta_{n-1} Q_{n2} = 0 \\ \beta_n Q_{n2} = 1 \end{cases} \tag{3.40}$$

in correspondence with $j = n$; hence (3.40) has no solution. So we can suppose the existence of complex numbers, $c_3, \ldots, c_n$ such that

$$\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = c_3 \begin{pmatrix} \boldsymbol{\delta}_3 \\ \boldsymbol{\zeta}_3 \end{pmatrix} + \ldots + c_n \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}.$$

Again, with a reasoning similar to the previous one, carried out for $j = n$ and $j = n-1$, no solution is found for (3.35). A similar conclusion can be drawn if just two vectors among $\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}$ are linearly independent. Hence complex numbers $\gamma_2, \ldots, \gamma_{n-1}$ must exist such that

$$\begin{pmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = \gamma_2 \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{\delta}_{n-1} \\ \boldsymbol{\zeta}_{n-1} \end{pmatrix} = \gamma_{n-1} \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix}. \tag{3.41}$$

As a consequence of (3.41) and (3.33), in (iv-D') we get

$$\begin{cases} (N_{j2}\gamma_2 + \ldots + N_{jn-1}\gamma_{n-1} + N_{jn}) \begin{pmatrix} \boldsymbol{\delta}_n \\ \boldsymbol{\zeta}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{\delta}_j \\ \mathbf{0} \end{pmatrix} \\ (N_{j2}\beta_3 + N_{j3}) \begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix} + \ldots + (N_{j2}\beta_n + N_{jn}) \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta}_j \\ \mathbf{0} \end{pmatrix} \end{cases} \tag{3.42}$$

where $N_{jk} = n_{jk} + \alpha_k n_{j1}$. First equation in (3.42) implies that either $\boldsymbol{\zeta}_j = \mathbf{0}$, for all j, or $N_{ji+1}\gamma_{ji+1} + \ldots + N_{jn-1}\gamma_{n-1} + N_{jn} = 0$; hence $\boldsymbol{\delta}_j = \mathbf{0}$, for all j; in any case, solutions, if they exist, are correlated. Indeed, since we are taking symmetrical results for (i-A'), we obtain the following implications:

- if $\boldsymbol{\delta}_j = \mathbf{a}_j = \mathbf{0}$ for all $j = 1,\ldots,n$ then $WY\Psi = 0$, i.e. each time a particle is sorted by Y than it is certainly not sorted by W; therefore, for all eventual solutions corresponding to this case, property L must be correlated with the incompatible property G;

- if $\boldsymbol{\zeta}_j = \mathbf{b}_j = \mathbf{0}$ for all $j = 1,\ldots,n$ then $W'Y\Psi = 0$, i.e. each time a particle is sorted by Y than it is certainly not sorted by W'; therefore, for all eventual solutions corresponding to this case, property L must be correlated with the incompatible property G.

Let us suppose $\boldsymbol{\zeta_j} = \mathbf{0}$, for all j.

Equations in (3.42) can also be written as

$$\begin{cases} \left(N_{j2}\gamma_{j2} + \ldots + N_{jn-1}\gamma_{n-1} + N_{jn}\right)\boldsymbol{\delta}_n = \boldsymbol{\delta}_j \\ \mathcal{N}_{j3}\boldsymbol{\eta}_3 + \ldots + \mathcal{N}_{jn}\boldsymbol{\eta}_n = \boldsymbol{\eta}_j \\ \mathcal{N}_{j3}\boldsymbol{\theta}_3 + \ldots + \mathcal{N}_{jn}\boldsymbol{\theta}_n = \mathbf{0} \end{cases} \tag{3.43}$$

where $\mathcal{N}_{jk} = N_{j2}\beta_k + N_{jk}$, for all $k = 3,\ldots,n$. A reasoning similar to that carried out for (iv-D) (see equations (3.35)-(3.41)) implies the existence of coefficients $\lambda_4,\ldots,\lambda_n$ and $\mu_3,\ldots,\mu_{n-1}$ such that

$$\begin{aligned} &\boldsymbol{\theta}_3 = \lambda_4\boldsymbol{\theta}_4 + \ldots + \lambda_n\boldsymbol{\theta}_n \\ &\boldsymbol{\eta}_k = \mu_k\boldsymbol{\eta}_n \quad \forall k = 3,\ldots,n-1 \end{aligned} \tag{3.44}$$

where we have supposed $\boldsymbol{\theta}_4,\ldots,\boldsymbol{\theta}_n$ linearly independent. Such an independence implies

$$\mathcal{N}_{j3} = \ldots = \mathcal{N}_{jn} = 0, \tag{3.45}$$

so that last equation in (3.43) is always true. Hence, (iv-D) and (iv-D') can be written as

$$\begin{cases} Q_{j2}\left[\gamma_2 - \beta_3\gamma_3 - \ldots - \beta_{n-1}\gamma_{n-1} - \beta_n\right]\boldsymbol{\delta}_n = \boldsymbol{\delta}_j \\ \\ (N_{j2}\gamma_2 + \ldots + N_{jn-1}\gamma_{n-1} + N_{jn})\,\boldsymbol{\delta}_n = \boldsymbol{\delta}_j \\ \\ \mathcal{N}_{j3}\left[\mu_3 - \lambda_4\mu_4 - \ldots - \lambda_{n-1}\mu_{n-1} - \lambda_n\right]\boldsymbol{\eta}_n = \boldsymbol{\eta}_j \end{cases} \tag{3.46}$$

3.6.2 Concrete solutions

So far we have established some constraints in the hypothesis that vectors $\begin{pmatrix} \boldsymbol{\eta}_3 \\ \boldsymbol{\theta}_3 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{\eta}_n \\ \boldsymbol{\theta}_n \end{pmatrix}$ are linearly independent, as well as $\boldsymbol{\theta}_4, \ldots, \boldsymbol{\theta}_n$; furthermore, $\boldsymbol{\zeta}_j = \mathbf{0}$ for all $j = 1, \ldots, n$, independently of the ranks of matrices U, Q, N and A_i, with $i = 1 \ldots, 8$, and therefore of the dimensions of space $\mathcal{H}_I$ and $\mathcal{H}_{II}$.

Now we fix $dim(\mathcal{H}_I) = 10$, hence $n,j,k \in \{1, \ldots, 5\}$, and we shall see that concrete solutions exist. Our task is made easier if we search for solutions corresponding to a particular state vector Ψ such that

$$\begin{aligned} &\gamma_3 = \lambda_4\gamma_4 + \lambda_5, \qquad \alpha_4 = \alpha_5 = \beta_3 = 0 \\ &c_3 = l_4c_4 + l_5, \qquad a_4 = a_5 = b_3 = 0. \end{aligned} \tag{3.47}$$

where the coefficients that appear in the second line are those corresponding to the vectors $\mathbf{x}_i$ $(i = 1, \ldots, 10)$. Conditions (3.47) and (3.36), together with $U_{j2} = 0$ (arising from (3.38)), imply that U has the following form

$$U = \begin{pmatrix} u_{11} & -\alpha_2 u_{11} & -\alpha_3 u_{11} & 0 & 0 \\ . & . & . & . & . \\ u_{j1} & -\alpha_2 u_{j1} & -\alpha_3 u_{j1} & 0 & 0 \\ . & . & . & . & . \end{pmatrix}; \tag{3.48}$$

first equation in (3.46), together with (3.36) and the choice (3.47) imply that $Q = (q_{ij})_{5\times5}$ where $q_{j3} = -\alpha_3 q_{j1}$, $q_{j4} = -\beta_4(q_{j1}\alpha_2 + q_{j2})$ and $q_{j5} = -\beta_5(q_{j1}\alpha_2 + q_{j2})$. Similarly, independence of $\theta_4, \ldots, \theta_n$ in (3.43) and the second equation of (3.46), imply that $N = (n_{ij})_{5\times5}$ where $n_{j2} = q_{j2} + \alpha_2(q_{j1} - n_{j1})$, $n_{j4} = -\lambda_4\alpha_3 n_{j1} - \beta_4(q_{j1}\alpha_2 + q_{j2}) - \lambda_4 n_{j3}$ and $n_{j5} = -\lambda_5\alpha_3 n_{j1} - \beta_5(q_{j1}\alpha_2 + q_{j2}) - \lambda_5 n_{j3}$. (ii-B') has the same form of (ii-B), so that matrix Z can be obtained from U by means of the substitution $u_{ik} = z_{jk}$. Matrices P, V, N, W have similar forms. By imposing that G_I and L_I are self-adjoint matrices we find exactly matrices U, Q and N in section 3.5.3, and moreover $V = \overline{U}^t$.

Summary

Standard Quantum Theory [16] denies the possibility of measuring simultaneously (i.e. on the same individual specimen of the physical system) non-commuting observables; the best example describing this physical situation is the double-slit experiment: properties such as "the particle passes through the first (second) slit" and "the particle hits the final screen in a point within the interval Δ" cannot be measured together, since they are mathematically represented by non-commuting projection operators, E and $F(\Delta)$ respectively. This notwithstanding, several devices are conceived over the years providing *indirect* knowledge of WS property; among them, the experiment presented in [4] provides WS knowledge by exploiting recent advances in quantum optics: instead of measuring WS property, a different property T is measured, correlated with it and compatible with the measurement of the final impact point; this kind of detection is studied from a mathematical point of view in [5] and [6], where the interpretative questions are analyzed. In [7] the problem of detecting three incompatible properties in the framework of double-slit experiment is treated: in some circumstances, inferences about more incompatible properties can be done. This kind of detection is possible, for instance, if besides the position of the centre-of-mass, the system possesses further degrees of freedom; as a consequence, the Hilbert space describing the entire system can be decomposed

as $\mathcal{H}_I \otimes \mathcal{H}_{II}$, where $\mathcal{H}_I$ is the Hilbert space used to represent the position observable, and $\mathcal{H}_{II}$ is the Hilbert space used to represent the observable arising from the further degrees of freedom. The detection of Which Slit property E is obtained by measuring an observable represented by a particular projection operator T acting on $\mathcal{H}_{II}$, which is correlated, in the particular quantum state of the system, with which-slit property, so that this last can be inferred from the outcome of T. The possibility of detecting an incompatible property G is provided by the existence of an observable represented by another projection operator Y acting on $\mathcal{H}_{II}$, but which can be measured together with T. A systematic investigation establishes that the existence of such an observable (projection operator) depends on the dimension of space $\mathcal{H}_I$ [7]. In this work we are interested in the research of non-correlated solutions: a wide family of solutions is provided for $dim(\mathcal{H}_I) = 6$ and an ideal experiment realizing such kind of detection is designed. We have to notice that this approach to the problem of inferring the outcomes of more incompatible properties is not the only one. In [9], a procedure is described allowing to make inferences about the three cartesian components of a spin-$\frac{1}{2}$ particle. However, this kind of inferences are of a quite different nature; moreover, coherently with this second method, it is not possible to produces inferences about more than three observables. For this reason, the question whether two mutually incompatible properties, G and L, both incompatible with Which Slit property E, can be detected, together with the measurement of the final impact point (four incompatible properties), is investigated. In particular, we show that such a question has an affirmative answer; as in the previous case, the existence of solutions depends on dimension of space $\mathcal{H}_I$; we find a particular solution for $dim(\mathcal{H}_I) = 10$, nevertheless, in such a case the properties L and G turn out to be correlated.

Riassunto

La Teoria Quantistica standard [16] nega la possibilità di misurare simultaneamente (cioè sullo stesso campione individuale del sistema fisico) osservabili che non commutano; l'esempio che meglio descrive questa situazione fisica è quello della doppia fenditura: proprietà come "la particella passa attraverso la prima (seconda) fenditura" (proprietà WS) e "la particella impressiona lo schermo finale in un punto interno all'intervallo Δ" non possono essere misurate insieme, poichè sono rappresentate in termini matematici da operatori, E ed $F(\Delta)$ rispettivamente, che non commutano. Nonostante ciò, sono stati ideati vari dispositivi che forniscono informazione *indiretta* sulla proprietà WS; tra questi, l'esperimento presentato in [4] fornisce informazione sulla proprietà WS sfruttando recenti tecniche dell'ottica quantistica: piuttosto che misurare la proprietà WS, viene misurata un'altra proprietà T ad essa correlata e compatibile con la misurazione del punto d'impatto sullo schermo finale; questo tipo di rivelabilità è stato studiato da un punto di vista matematico in [5] a [6], in cui sono state analizzate quastioni interpretative. In [7] è stato trattato il problema di rivelare tre proprietà incompatibili nel contesto dell'esperimento della doppia fenditura: in alcuni casi, si possono fare inferenze su proprietà incompatibili. Questo tipo di rivelabilità è possibile, per esempio, se, oltre alla posizione del centro di massa, il sistema possiede ulteriori gradi di libertà; come conseguenza lo spazio di Hilbert che descrive l'intero sistema può essere descritto come $\mathcal{H}_I \otimes \mathcal{H}_{II}$, dove $\mathcal{H}_I$ è lo spazio di Hilbert usato per rappresentare l'osservabile posizione, e $\mathcal{H}_{II}$ è lo spazio di Hilbert usato per rappresentare l'osservabile che deriva dagli ulteriori gradi di libertà. La rivelabilità

della proprietà E può essere ottenuta misurando un'osservabile rappresentata da un particolare proiettore T che agisce su $\mathcal{H}_{II}$ e che è correlato, nel particolare stato quantistico del sistema, con la proprietà E, cosicché inferenze su quest'ultima possono essere fatte a partire dal risultato di T. La possibilità di rivelare una proprietà incompatibile G è data dall'esistenza di un altro proiettore ortogonale Y che agisce su $\mathcal{H}_{II}$, ma che può essere misurato insieme a T. Uno studio sistematico stabilisce che l'esistenza di una tale osservabile dipende dalla dimensione dello spazio $\mathcal{H}_I$ [7]. In questo lavoro siamo interessati alla ricerca di soluzioni non correlate: è data un'ampia famiglia di soluzioni per $dim(\mathcal{H}_1) = 6$, inoltre viene ideato un esperimento che realizza il tipo di rivelabilità in questione. Va notato che questo non è l'unico approccio al problema di fare inferenze su più proprietà che non commutano. In [9], è descritta una procedura che consente di fare inferenze sulle tre componenti cartesiane dello spin di una particella di spin 1/2. Tuttavia, questo tipo di inferenze sono di natura completamente diversa rispetto a quelle presentate in questo lavoro; inoltre, coerentemente con questo metodo, non è possibile fare inferenze su più di tre osservabili che non commutano. Per questo motivo, viene affrontato il problema di misurare due proprietà mutuamente incompatibili, L e G, entrambe incompatibili con la proprietà E, insieme alla misurazione del punto d'impatto sullo schermo finale (quattro proprietà incompatibili). In particolare, dimostriamo che questo problema ha risposta affermativa; come nel caso precedente, l'esistenza delle soluzioni dipende dalla dimensione dello spazio $\mathcal{H}_1$; troviamo una soluzione particolare per $dim(\mathcal{H}_1) = 10$, tuttavia, in questo caso le proprietà L e G risultano essere correlate.

Aknowledgements

I would like to express my heartfelt gratitude to my supervisor for his incessant help; I have learned a lot from his invaluable advice. Tanks for suggesting the problem, for discussing with me various aspect of the work and for kindly reading the entire typescript. Tanks also for his patience and for the encouragement in difficult moments.

Bibliography

[1] L. Rosenfeld, *Theory of electrons*, North- Holland Publishing Co., Amsterdam, 1951

[2] M. Born, *Atomic Physics*, Blakie, Glasgow, 1957

[3] M. Jammer, *The Conceptual Development of Quantum Mechanics*, McGraw-Hill Book Company, U.S.A., 1966

[4] M.O. Scully, B.-G. Englert, H. Walther, *Quantum optical test of complementarity*, Nature 351, 111 (1991).

[5] G. Nisticò, M.C. Romania, *Knowledge about noncommuting quantum observables by means of Einstein- Podolsky- Rosen correlations*, J. Math. Phys. **35** (9) (1994) 4534.

[6] G. Nisticò, *Empirical relations between noncommuting Observables*, Found. of Phys. **25** n. 12, 1757 (1995)

[7] G. Nisticò, *Detecting non compatible properties in double-slit experiment without erasure*, in I Fondamenti della Meccanica Quantistica, Analisi storica e Problemi aperti, C. Garola, A. Rossi, S. Sozzo (Eds.), World Scientific, Singapore (2006).

[8] G. Nisticò, A. Sestito, *Detection of Many Incompatible Quantum Properties in Double-Slit Experiment* in Foundations of Probability and Physics-4, G. Adenier, C. Fuchs, A.Y. Khrennikov, eds, AIP Conferences Proceeding **962**, Melville 2007

[9] L.Vaidman, Y.Aharonov, D.Z.Albert, *How to Ascertain the Values of* σ_x, σ_y, *and* σ_z *of a Spin-*$\frac{1}{2}$ *Particle*, Phys. Rev. Lett.58 (1987)1385.

[10] E. Schrödinger, Ann. Physik (4) **79** (1925) 361

[11] E. Schrödinger, Ann. Physik (4) **80** (1926) 437

[12] E. Schrödinger, Ann. Physik (4) **81** (1926) 109

[13] W Heisenberg, Z. Physik **33** (1925) 879

[14] L. de Broglie, Nature **112** (1923) 540

[15] P.A.M. Dirac, *The prindiples of Quantum Mechanics*, Clarendom Press, Oxford, 1930

[16] J. Von Neumann, *Mathematical Foundations of Quantum Mechanics* Princeton, Princeton University Press, 1955

[17] L.E Ballentine, *The Statistical Interpretation of Quantum Mechanics*, Reviews of Modern Physics **42**, n. 4, 1970

[18] C. Piron, *Foundations of Quantum Physisc*, W.A. Benjamin, Inc., New York, 1976

[19] L.E Ballentine, *Quantum Mechanics: a modern development* Singapore, World Scientific Publishing Co. Pte. Ltd. 1998

[20] J. Von Neumann, Annals of Math. **32** (1931)

[21] W Heisenberg, Z. Physik **43** (1927)

[22] K. Kraus, J. Schröter, Int. J.Thoer. Phys. **8** (1973) 431

[23] E.G Beltrametti, G. Cassinelli, *The Logic of Quantum Mechanics*, Addison-Wesley Publishing Company Inc., U.S.A, 1981

[24] N. Bohr, *Can Quantum-Mechanical Description of Physical reality be Considered Complete?* Physical Review **48** 696-702 (1935)

[25] N. Bohr, *The Quantum Postulate and the Recent Development of Atomic Theory*, Nature **121** 580-590 (1928), see also J.A. Wheeler, W.H. Zurek, *Quantum Theory and Measurement*, Princeton University Press, Princeton, New Jersey 1983

[26] W. Pauli, *General Principles in Quantum Mechanics* translated from the 1958 edition of *Die allgemeinen Prinzipen der Wellenmechanic* Berlin, Springer-Verlag 1933

[27] G. Ludwig, *Die Grundlagen der Quantenmechanik* Berlin, Springer-Verlag 1954

[28] G. Ludwig, *Versuch einer axiomatischen Grundlegung der Quantenmechanik und allgemeinerer physikalischer Theorien*, Zietschrift fur Physik **181**, 233-260, 1964

[29] G. Ludwig, *Foundations of Quantum Mechanics*, Springer-Verlag, Berlin, 1983

[30] P.J. Lahti, *Uncertainty and Complementarity in Axiomatic Quantum Mechanics*, Report Series No. D 2, Theoretical Physics, Dept. of Physical Sciences, University of Turku, Finland (1979). See also: *Report of Mathematical Physics, vol. 17 (1980)*

[31] P.J. Lahti, *Hilbertian Quantum Theory as the Theory of Complementarity*, International Journal of Theoretical Physics, **22**, n. 10, 911-929, 1983

[32] P. Busch, P.J. Lahti, *On various joint measurements of position and momentum observables in quantum theory*, Physical Review D **29**, n. 8, 1634-1646, 1984

[33] P.J. Lahti, *Characterization of Quantum Logic*, International Journal of

Theoretical Physics **19**, 905-923, 1980

[34] P. Busch, P.J. Lahti, *A note on Quantum Theory, Complementarity, and Uncertainty*, Philosophy of Science **52**, n. 1, 64-77, 1985

[35] S. Bugajski, P.J. Lahti, *Fundamental Principles of Quantum Theory*, International Journal of Theoretical Physics **29**, 499-514, 1980

[36] M.H. Stone, *Linear Transformation in Hilbert Spaces and Their Application to Analysis*, Amer. Math. Soc., New York 1932

[37] R. Paley, N. Wiener, *Fourier transforms in the complex domain*, Am. Math. Soc. Coll. Publ., vol. 19 (New York, 1934)

[38] P. Busch, P.J. Lahti, *To what extent do position and momentum commute?*, Physics Letters A **115**, n. 6, 259-264, 1986

[39] P. Busch, T.P. Schonbek, F.E. Schroeck, *Quantum observables: Compatibility versus commutativity and maximal information*, J. Math. Phys. **28** (12), 2866 (1987)

[40] F. Riesz, B.Sz.-Nagy, *Functional analysis*, New York, Dover Publications, Inc., 1990

[41] J. Dixmier, *Les Algèbres d'Opérateur dans l'Espace Hilbertien*, Gauthier-Villars, Paris, 1957

[42] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, Vol. I, Academic Press, New York, 1972

[43] M. Redhead, *Noncompleteness, nonlocality and realism*, Clarendon Press, Oxford, 1987

[44] R. Feynman, R. Leighton, M. Sand, *The Feynmann Lectures on Physics III*, Addison Wesley, Reading, 1965.

[45] N. Bohr, in *Albert Einstein: Philosopher-Scientist*, P.A. Schilpp, p.200, Library of Living Philosopher, Evanstone 1949.

[46] J.A. Wheeler, W.H. Zurek, *Quantum Theory and Measurement*, Princeton University Press, Princeton, 1983.

[47] N. Wiener *The Fourier Integral & Certain its Applications*, Cambridge University Press, Cambridge, 1933

[48] C.R. Putnam, *Commutation Properties of Hilbert Space operators*, Springer- Verlag, Berlin, 1967

[49] G. Cassinelli, N. Zanghì, Il Nuovo Cimento B, **73**, 1983

[50] B.-G. Englert, J. Schwinger, M.O. Scully, in *New frontiers in quantum electrodynamics and quantum optics*, p. 507, A.O. Barut. Ed., Plenum, New York 1990

[51] G.Nisticò, A. Sestito, Journal of Modern Optics, v. 51, n. 6-7, 2004 (1063)