Università della Calabria

DIPARTIMENTO DI FISICA
Corso di Dottorato di Ricerca in Fisica - XXIII ciclo

Aspects of Integrability in AdS/CFT duality

Settore scientifico disciplinare FIS/02

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Anno Accademico 2010-2011

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Introduction

AdS/CFT duality is actually one the most interesting field of research in theoretical Physics. String theory is the best model we have now for describing a complete Quantum Theory of Fundamental Interactions including Gravity. Non-renormalizability of Gravity reduces usual Quantum Field Theory only to an effective Theory and suggests that there should be a more fundamental level in description of Nature. String theory appears for the first time in Quantum Physics as a candidate for Strong Interactions. For many reasons, Strings cannot describe exactly Strong interactions and now we have QCD as a right description of that. However, some stringly behaviours remain in some approximations in QCD: hadron spectrum organizes into so called Regge-trajectories that represent an appropriate linear relationship, with universal slope, between the squared masses of hadronic resonances and their spins and this is what exactly a string theory on flat space predicts; then, another manifestation of stringy behaviour in gauge theory is the presence of flux tubes of the Chromodynamic field: they form, between two quarks when they are pulled apart. Using some approximation, they could be seen as one dimensionale objects with costant tension, i.e. strings. AdS/CFT duality goes even further: it propose an exact correspondence between a String Theory and a Gauge Theory. First proposed by Maldacena in 1998 (Maldacena conjecture, see first of [1]) and successively by others (see other works in [1]), this duality connects a String theory, living on a background composed by an Anti-de-Sitter space in five dimension times a five dimension maximally symmetric space ($AdS_5 \times S_5$), with a Conformal Field Theory, living in a four dimensional flat-spacetime, that is the boundary of the AdS_5 spacetime. Stringy observables in AdS_5 have correspondent gauge observables on the boundary and thier values are expected to match.

Importance of AdS/CFT duality is the possibility to study a complicated String Theory in extra dimensions in a more usual and comfortable Gauge Theory. Facilities could be given also by the strong/weak nature of the duality: weak coupling regime on gauge side is dual to strong coupling regime in strings and vice-versa. So, we could have the possibility to explore on both sides strong coupling regimes making easy perturbative calculations on the other side. It could be also a great advantage on the gauge side. Conformal Field Theory is a "cousin" of QCD, richer in symmetries. It is a very difficult task but not impossible in principle that we can, from CFT, obtain some insights in QCD. So, AdS/CFT duality, also in studying String through a simpler gauge theory, might permit to get some results at strong coupling in QCD. Unfortunately, AdS/CFT duality has not a formal proof yet! Without a formal proof, strong/weak duality becomes a great obstacle: tests of duality are difficult beacuse the two perturbative regimes do not overlap.

In this picture, a *miracolous* aspect emerges, that permits the possibility of a greatful test for the duality. In the framework of the duality, we can analyze the most symmetric setting choosing as partners type IIB String Theory on string side and $\mathcal{N} = 4$ Super Yang Mills Theory on gauge side. The latter is a supersymmetric and conformal field theory with the maximal number of supercharges ($\mathcal{N} = 4$). In the planar limit of this theory aspects of integrability emerge. Planar limit was first studied bu t'Hooft [2]: This is a limit for gauge theories with gauge symmetries SU(N_c), SO(N_c) or Sp(N_c), where N_c is the number of colors of the model, and consists in taking the rank of the group to infinity, i.e. $N_c \to \infty$, keeping the rescaled gauge coupling $\lambda = g_{YM} N_c$ finite. In this limit, dominating Feynman diagrams are those drawingable on a plane, whereas the others with crossing lines are suppressed. We can define local operators $O(x)$ as compositions of fundamental fields all at the same space-time point: two points correlation functions are determined by *scaling* dimension $\Delta(\lambda)$, where λ is the t'Hooft coupling:

$$\langle O_A O_B \rangle \propto \frac{\delta_{AB}}{|x-y|^{2\Delta(\lambda)}} \tag{1}$$

where

$$\Delta(\lambda) = L + \gamma(\lambda)\,. \tag{2}$$

Parameter L is the classical dimension, i.e. it is the number of constituent fields that generate local operator O, while $\gamma(\lambda)$ is the anomalous dimension, a quantum correction due to interaction among constituent fields. In the picture of AdS/CFT, anomalous dimensions are dual ti string energies. We can find anomalous dimensions as eigenvalues of some operators, called *Dilatation Operators*. Integrability emerges because it is possible to show that these Dilatation operators, in the planar limit defined above, could be written as some Hamiltonians of some integrable spin chains. An integrable system is a system with a number of conserved charges equal to the number of Hilbert spaces that constitutes the quantum space in which the system is defined. Integrability gives the possibility to compute anomalous dimensions, as energies of the spin chains, for all values of coupling λ. Through integrability, we can so test the duality, making the wanted overlaps between results on both sides.

First proof of integrability was given for the pure bosonic subsector of $\mathcal{N} = 4$ SYM, at one loop [3]. Dilatation operator of this subsector can be written as the Hamiltonian of the Heisenberg XXX spin $\frac{1}{2}$ chain. It is an integrable system [18] and it could be solved through a mathematical approach, the *Bethe Ansatz*. Bethe Ansatz permits to obtain, in an integrable system, all the conserved charges in terms of the so-called *Bethe roots*, solution of an algebraic trascendental set of equations, related to the system, that emerge naturally during the diagonalization of the conserved charges.

Subsequently, integrability was extended to all gauge sectors and at all loops in a weaker sense: any local operator has been seen as a state of a "spin chain" whose Hamiltonian is the dilatation operator related, although the latter does not have an explicit expression of the spin chain form, but for first few loops. Neverthless, large size (asymptotic) spectrum has turned out to be exactly described by certain Asymptotic Bethe Ansatz-like equations: the so called Beisert-Staudacher equa- tions [4, 5]. This approach, called *Asymptotic Bethe Ansatz* (ABA), suffers of a problem: its greatest limitation is the so-called *wrapping* problem [10]. When many loops contributions were computed in dilatation operator, related spin chains presents hamiltonian with interactions that have a growing complexity: one loop corresponds to nearest neighbour interaction, two loops corresponds to interaction involving three sites of the chain and so on. A spin chain of length L can describe in a right picture only dilatation operator until $L - 1$ loops. Over this point, phenomena of self-interaction makes losing of sense in this picture. Wrapping should be under control in some limits, as we will see immediatly.

Recently, great progresses have been made in application of other techniques of integrable models, as *Thermodynamic Bethe Ansatz* (TBA) [6], the equivalent Y-system [7] [8]. These techiques permit to solve exactly the models and to find anomalous dimensions, without the problem of wrapping corrections. Very recently [9] is has been showed that TBA equations could be reformulated as a finite set of integral equations. However, it has been showed that TBA equations present some difficulties when we have to disentagle excited state equations. In many cases, only numerical approach seems to be the possible way. So, it could be convenient to study what are the limit of validity of ABA approach. TBA computations are not argument of this work, but our interest is in study integrability through ABA, its limits and a way to better understand how wrapping corrections work.

Our main field of interest are anomalous dimensions related to dilatation operators belonging to $sl(2)$ gauge sector of $\mathcal{N}=4$ SYM. Local operators in this sector are defined as

$$O = \mathrm{Tr}\left(D^s Z^L + \cdots\right) \tag{3}$$

where we have s covariant derivative acting on L bosonic complex field in all possible combinations. Parameters s and L are called respectevely spin and twist. Importance of this gauge sector is due to their connection to twist operator in QCD [11, 12]. This gauge sector is usually studied in large spin limit $s \to \infty$. Integrability of $sl(2)$ sector is well studied in [4, 5] at all loops. Also, in [13] it is showed by perturbative calculations up to six loops that, for short operators ($L = 2, 3$) wrapping corrections start with terms of order $O\left(\frac{(\ln s)^2}{s^2}\right)$. In high spin limit, anomalous dimension for $sl(2)$ sector expands as:

$$\begin{aligned}\gamma(g, L, s) = f(g)\ln s + f_{sl}(g, L) + \sum_{n=1}^{\infty} \gamma^{(n)}(g, L)(\ln s)^{-n} + \\ + \frac{1}{s}\sum_{n=-1}^{\infty} \tilde{\gamma}^{(n)}(g, L)(\ln s)^{-n} + O\left(s^{-1}(\ln s)^{-\infty}\right) ,\end{aligned} \tag{4}$$

where g is linked to the t'Hooft coupling λ via the relation $\lambda = 8\pi^2 g^2$ and where the notation $O\left(s^{-1}(\ln s)^{-\infty}\right)$ denotes terms going to zero faster than $\frac{1}{s}$ times any inverse power of $\ln s$. Leading orders functions $f(g)$ and $f_{sl}(g, L)$ are well-known and studied (see introduction to Chapter 3). In this thesis, we present a scheme of calculation based on Asymptotic Bethe Ansatz and the equivalent Non-linear Integral Equation (NLIE) that permits to obtain generalised expression, at all loops, for subleading functions $\gamma^{(n)}(g, L)$ and $\tilde{\gamma}^{(n)}(g, L)$. Main purpose of this computations is to obtain suggestions in order to extend wrapping-free limit point $O\left(\frac{(\ln s)^2}{s^2}\right)$ to all loops and twist. Our computations, we will see, give very strong insights in confirming this idea. Shortly, this is due to the verification of some functional relation in $\mathcal{N}=4$ SYM. Anomalous dimension should satisfy this 'self-tuning' property [14]:

$$\gamma(g, L, s) = P\left(s + \frac{1}{2}\gamma(g, L, s)\right) \tag{5}$$

where the function P, when $s \to +\infty$ expands as follows,

$$P(s) = \sum_{n=0}^{\infty} \frac{a_n(\ln C(s))}{C(s)^{2n}} , \tag{6}$$

the quantity $C(s)$ being given by

$$C(s)^2 = \left(s + \frac{L}{2} - 1\right)\left(s + \frac{L}{2}\right) . \tag{7}$$

Second relation expresses the so-called 'reciprocity property' (in Mellin space). An analogous of these realtions exists also in QCD in the framework of Altarelli-Parisi equations, but it is valid only at one loop. Instead, it is supposed that in the framework of $sl(2)$ sector of $\mathcal{N}=4$ SYM it is valid at all-loops. We have verified that. Also, these properties produces some constraint relations among functions $\gamma^{(n)}(g,L)$ and $\check{\gamma}^{(n)}(g,L)$ that we can use to obtain strong insight about wrapping limits we have mentioned above.

Another limit is studied in this thesis: we perform the double limit $s\to\infty$, $L\to\infty$ with a relation among the parameters that reads as

$$j=\frac{L-2}{\ln s}\text{fixed}\,. \tag{8}$$

Main importance of this limit is that is completely wrapping free, because spin chains related have infinite lenghts. Anomalous dimension expansion is, in this case:

$$\gamma(g,L,s)=f(g,j)\ \ln s+f_{sl}(g,j)+\sum_{n=1}^{\infty}\gamma^{(n)}(g,j)\ (\ln s)^{-n}+O((\ln s)^{-\infty})\,. \tag{9}$$

Leading function $f(g,j)$ is well known and studied (see introduction in Chapter 4). We present in this work a systematic scheme for analyzing subleading term $f_{sl}(g,j)$. We show also explicit computations at weak coupling up three loops in the limit $j\ll 1$. Also, we give strong coupling limit.

Our computations are effectively made using an alternative, but equivalent, method to Bethe equations. Instead of solving a set on non-linear algebric equations, in the case of operators with high values of twist L it is more convenient to transform this set in a Non-Linear Integral Equation (NLIE). Problem of non-linearity still remains, but it is possible to show, under the limits we work in, that all non-linear terms in equations could be evaluated discarding terms of orders we do not need in results.

Outline of the Thesis

This work is divided into four chapters. First two chapters are introductory, last two contain the work.

In the first chapter we present integrable system techniques we use. After defining the Integrability of a system, we present Bethe Ansatz equations adopting the scheme presented in [18]. Subsequently, we show how to obtain the equivalent picture based on the Non-Linear Integral Equation, giving a general description of the method. All the chapter contains, for simplicity and for an example, continuos references to the practial case of the Heisenberg $\mathrm{XXX}_{1/2}$ spin chain. We will get, at the end, the energy of the chain of L sites in the limit of large L through NLIE approach. In particular, we will see how evaluate non-linear terms and the right corrections these terms give.

The second chapter regards the topic of AdS/CFT duality. It is showed a short overview of the subject, motivations behind it and all regards the various aspects mentioned also above, as advantages of a gauge/string duality, strong/weak duality, etc. We will introduce also the aspects of integrability, introducing it through the example of one-loop $su(2)$ gauge sector. We will show how one-loop dilatation operator belonging to this sector is exactly the Hamiltonian of the Heisenberg $\mathrm{XXX}_{1/2}$ spin chain.

Third chapter starts the investigation of anomalous dimensions of $sl(2)$ gauge sector, specifically here we will analyze high spin and fixed twist limit. We present a general method that permits to make a systematic study of anomalous dimension. In particular we apply this picture to compute generalized expression at all loops for functions $\gamma^{(n)}(g,L)$

and $\check{\gamma}^{(n)}(g, L)$. Functional relations belonging to reciprocity and self-tuning are verified at all loops, up to the order $(\ln s)^{-5}$. Mixing these two results, we will give some insights about wrapping problem: appearing of correction at order $O\left(\frac{(\ln s)^2}{s^2}\right)$ would seem a reasonable limit valid for all loops. We will not give a formal proof of it, but our results will give strong suggestions. Finally it is given a short discussion about excited states.

Fourth chapter, finally, presents the high spin and high twist double limit computations. We will focus on the first subleading function. We will give weak coupling results up to three loop showing how they are obtained, then it is given a general method that permits to make a systematic study. At the end, we show some strong coupling results.

Original contributions

Chapter 3 is based on results from [15] and from the non yet published work [16] (soon in archives). Original personal contributions are in the latter, in particular in computations of $\check{\gamma}^{(n)}(g, L)$ functions and in verifications of functional relation costraint belonging to reciprocity. The scheme of calculation presented has been obtained as a natural extension of that used in [15]. Also, a personal contribution is all regards excited states, these results has not been included, by now, in a work.

Chapter 4 is based on results from [17]. Original personal contributions reside in one-loop computation up to the order j^5.

Chapter 1

Integrability

1.1 Integrability of a system

1.1.1 Definition

A dynamical system with n degrees of freedom is called *integrable* if it has n quantities Q_j, $j = 1, \cdots, n$, including the Hamiltonian of the system, that commute among themselves:

$$[Q_i, Q_j] = 0 \quad , \quad i, j = 1, \cdots, n \, . \tag{1.1}$$

For a classical system, definition (1.1) has a clear meaning: a system is integrable if it has the same number of constants of motion as the number of degrees of freedom. For a quantum system instead, we have an integrable system if we have a number of conserved charges equal to the number of Hilbert spaces we need to describe the system. For example, a spin chain described by a lattice of N sites is integrable if there are N conserved charges associated to the system. We now give immediatly an example, showing the integrability of the **Heisenberg spin chain**.

1.1.2 The Heisenberg $\mathrm{XXX}_{1/2}$ spin chain

Now we show the integrability of a lattice made up by N sites where a vector of spin $s = \frac{1}{2}$ is associated to each site. The interaction is nearest neighbour like, so the Hamiltonian of the system is:

$$H = \sum_{n=1}^{N} \left[\frac{1}{4} - \vec{s}_n \cdot \vec{s}_{n+1} \right] = \frac{1}{4} \sum_{n=1}^{N} [1 - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}] \tag{1.2}$$

where $\vec{s}_n = \frac{1}{2}\vec{\sigma}_n$ are the spin matrices, with $\vec{\sigma}_n$ Pauli matrices, and periodic boundary conditions are imposed. This system is called *Heisenberg model.* It is also called XXX model because the interaction is the same for all the components of the spin. In general, a XYZ system has the Hamiltonian:

$$H = \sum_{n=1}^{N} \left[\frac{1}{4} - J^{\alpha} s_{\alpha,n} s_{\alpha,n+1} \right] \quad , \, \alpha = 1, 2, 3 \, . \tag{1.3}$$

The quantum space that describe the system is, naturally:

$$\mathcal{H} = \prod_{n=1}^{N} \otimes h_n \tag{1.4}$$

where h_n is the Hilbert space associated to each site, in this case $h = \mathbb{C}^2$. A spin matrix $\vec{s}_n$ acts nontrivially only on the Hilbert space associated as follows:

$$\vec{s}_n = \mathrm{I} \otimes \mathrm{I} \otimes \cdots \otimes \underbrace{\vec{s}}_{\text{n-th place}} \otimes \cdots \otimes \mathrm{I}\,. \tag{1.5}$$

Now we can give a short demonstration of the integrability of this system. What follows is based on [18], where one can find the complete proof of all the relations that will appear here. First, let us introduce a permutation operator:

$$P = \frac{1}{2}\left(\mathrm{I} \otimes \mathrm{I} + \vec{\sigma} \otimes \vec{\sigma}\right)\,. \tag{1.6}$$

We can rewrite, using it, the Hamiltonian as follows:

$$H = \frac{1}{2}\sum_{n=1}^{N}(1 - P_{n,n+1})\,. \tag{1.7}$$

Now define a local operator, called Lax operator, acting on the space $h_n \otimes V$, where V is an auxiliary space:

$$L_{n,a}(\lambda) = \lambda(\mathrm{I}_n \otimes \mathrm{I}_a + \vec{s}_n \otimes \vec{\sigma}_a) \tag{1.8}$$

where the label a indicates the auxiliary space. In our example, V coincides with $\mathbb{C}$ and it is the same of h_n but in general it does not have to be the same. The complex parameter λ is called spectral parameter. From (1.6) we have:

$$L_{n,a}(\lambda) = \left(\lambda - \frac{i}{2}\right)\mathrm{I}_{n,a} + P_{n,a}\,. \tag{1.9}$$

This relation permits to better understand the meaning of the Lax operator. Let ψ_n to be vector from $\mathcal{H} \otimes V$ associated to each site of the chain. Lax operator defines parallel transport of the chain, through the relation:

$$\psi_{n+1} = \frac{1}{\lambda}L_n\psi_n\,. \tag{1.10}$$

In other words, Lax operator is a connection on the lattice. Alternatively, we can put the Lax operator in the form of a 2×2 matrix, acting on V, with entries as operators in h_n:

$$L_{n,a}(\lambda) = \begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix} \tag{1.11}$$

where $s_n^{\pm} = s_n^1 \pm is_n^2$.

The next task is to define commutation relations among Lax operators. Since Lax operator is a 4×4 matrix, there are 16 matrix elements whose commutation relations we have to write. Compactly, we can write this set of relations as follows:

$$R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu)\,, \tag{1.12}$$

that is a relation in $\mathcal{H} \otimes V_1 \otimes V_2$ where the indices a_1 and a_2 and the variables λ and μ are associated respectevely to V_1 and V_2. In our case, the matrix R_{a_1,a_2} reads as:

$$R_{a_1,a_2}(\lambda) = \frac{1}{\lambda + i}(\lambda \mathrm{I}_{a_1,a_2} + iP_{a_1,a_2})\,. \tag{1.13}$$

Having introduced the Lax operator as a connection on the chain, we can define an operator that describes parallel transport once around the chain as follows:

$$T_a(\lambda) = L_{N,a}(\lambda)L_{N-1,a}\cdots L_{1,a}(\lambda)\,. \tag{1.14}$$

This operator is called *Monodromy Matrix* and, obviously, obeys the same commutation rule as Lax operator (1.12):

$$R_{a_1,a_2}(\lambda-\mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu)\,. \tag{1.15}$$

We can put the Monodromy Matrix in the form of a 2×2 matrix on V, analogously to what we have made with Lax operator (1.11), obtaining :

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{1.16}$$

where the matrix A, B, C, D act on the full quantum space $\mathcal{H}$. From the definition, it is clear that Monodromy Matrix is polynomial in λ of order N.

We now introduce the family of operators

$$F(\lambda) = \mathrm{Tr}_a(T_a(\lambda)) \tag{1.17}$$

that, after tracking on the auxiliary space, act on $\mathcal{H}$. Operators $F(\lambda)$ are also polynomial in λ and have a non-trivial expansion:

$$F(\lambda) = 2\lambda^N + \sum_{l=0}^{N-2} Q_l\lambda^l\,. \tag{1.18}$$

Using (1.15) and (1.13) it is possible to prove that

$$[F(\lambda), F(\mu)] = 0\,, \tag{1.19}$$

so, from (1.18), we have that all the $N-1$ operators Q_l commute among themselves. We are very close to the proof of integrability of the system. Next step is to show that the Hamiltonian is one of the Q_l. Precisely, Hamiltonian is the operator Q_2 and its expression is, from [18]:

$$H = \frac{N}{2} - \frac{i}{2}\frac{d}{d\lambda}\ln F(\lambda)\Big|_{\lambda=i/2}\,. \tag{1.20}$$

Now, adding the third component of the total spin S^3, we complete a family of N commuting operators including the Hamiltonian, i.e., the system is integrable.

1.2 The Bethe equations

Let us continue the analysis of integrable systems, in particular of the XXX Heisenberg model, finding the eigenstates and the eigenvalues for the Hamiltonian. Integrability says that eigenstates of the Hamiltonian are also eigenstates of the other $N-1$ commuting operators. We can simultaneously diagonalize all of them diagonalizing the operator $F(\lambda)$. We prooced as usual it is done in Quantum Mechanics: first we try to define a pseudo-vacuum state for $F(\lambda)$ and subsequently we shall build the other eigenstates using a kind of raising operator on the pseudo-vacuum state.

Let us start to build up the pseudo-vacuum state. For the $n-$th site of the chain, the vector that represents the spin up state is:

$$|+\rangle_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{1.21}$$

that, under the action of the raising spin operator s_n^+, obeys to the relation

$$s_n^+|+\rangle_n = 0 \tag{1.22}$$

and, under the action of Lax operator we have

$$L_{n,a}(\lambda)|+\rangle_n = \begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix} |+\rangle_n = \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} |+\rangle_n \, , \tag{1.23}$$

where the term $*$ is not relevant now. The last relation show that the Lax operator has a upper triangular form when applied on $|+\rangle_n$. We define as pseudo-vacuum state, or reference state, the state

$$\Omega = \prod_{n=1}^{N} \otimes |+\rangle_n \, . \tag{1.24}$$

From (1.23) we can immediatly see that the Monodromy Matrix is also upper triangular when applied to Ω:

$$T_a(\lambda) = \prod_{n=1}^{N} L_{n,a}(\lambda)\Omega = \begin{pmatrix} (\lambda + \frac{i}{2})^N & * \\ 0 & (\lambda - \frac{i}{2})^N \end{pmatrix} \Omega \tag{1.25}$$

and, finally, we get that Ω is an eigenstate of $F(\lambda) = A(\lambda) + D(\lambda)$:

$$F(\lambda)\Omega = [A(\lambda) + D(\lambda)]\Omega = \left[\left(\lambda + \frac{i}{2}\right)^N + \left(\lambda - \frac{i}{2}\right)^N \right] \Omega \, . \tag{1.26}$$

Now we pass to construct the other eigenstates of $F(\lambda)$. We follow a method presented by Bethe in [19], called *Bethe ansatz*, that is based on the crucial assumption to use $B(\lambda)$ as a raising operator, i.e., getting other eigenstates applying successively the operator $B(\lambda)$ to Ω:

$$\Phi(\{\lambda\}) = B(\lambda_1)B(\lambda_2) \cdots B(\lambda_l)\Omega \, . \tag{1.27}$$

We will see now that $\Phi(\{\lambda\})$ is an eigenstate of $F(\lambda)$ only for a specific set of values $\lambda_1, \cdots, \lambda_l$ of the spectral parameter. We need to analyze commutation rules between $B(\lambda)$ and $A(\lambda)$ and $D(\lambda)$: they follow from (1.15), (1.13) and (1.17):

$$[B(\lambda), B(\mu)] = 0 \tag{1.28}$$
$$A(\lambda)B(\mu) = \alpha(\lambda - \mu)B(\mu)A(\lambda) + \beta(\lambda - \mu)A(\mu)B(\lambda) \tag{1.29}$$
$$D(\lambda)B(\mu) = \gamma(\lambda - \mu)B(\mu)D(\lambda) + \delta(\lambda - \mu)D(\mu)B(\lambda) \tag{1.30}$$

where $\alpha(\lambda) = \dfrac{\lambda - i}{\lambda}$ and $\gamma(\lambda) = \dfrac{\lambda + i}{\lambda}$, while explicit form of functions $\beta(\lambda)$ and $\delta(\lambda)$ are now not relevant. If the second term on the right sides of (1.29) and (1.30) were absent, $\Phi(\{\lambda\})$ would have been an eigenstate of $F(\lambda)$ for all values of λ. The presence of these terms generates some uwanted terms but they could be erased for a specific set of values of λ, so, for them, Φ is at the end eigenstate of $F(\lambda)$. This set is determined by solving a trascendental equation. For this particular set, the eigenvalue equation reads as:

$$F(\lambda)\Phi(\{\lambda\}) = \left[\prod_{m=1}^{l}\left(\frac{\lambda_m-\lambda+i}{\lambda_m-\lambda}\right)\left(\lambda+\frac{i}{2}\right)^N + \prod_{m=1}^{N}\left(\frac{\lambda_m-\lambda-i}{\lambda_m-\lambda}\right)\left(\lambda-\frac{i}{2}\right)^N\right]\Phi(\{\lambda\}). \tag{1.31}$$

Right side of (1.31) has to be analytic as left side, so we have to cancel the poles $\lambda = \lambda_m$. Poles are removed if these relations are valid:

$$\left(\frac{\lambda_k+\frac{i}{2}}{\lambda_k-\frac{i}{2}}\right)^N = \prod_{m=1;m\neq k}^{l}\frac{\lambda_k-\lambda_m+i}{\lambda_k-\lambda_m-i} \quad , k=1,\cdots,l\,. \tag{1.32}$$

The set $\{\lambda_1,\cdots,\lambda_l\}$ is the same set mentioned above that we need for erasing unwanted terms in commutation relations (1.29) and (1.30). Thus, self-consistency of (1.31) provides the set of λ's for which it is valid. Equations (1.32) are called *Bethe equations* and the elements of the set $\{\lambda_1,\cdots,\lambda_l\}$ are called *Bethe roots*.

Bethe equations contain all the information about diagonalization of conserved charges of the system. It is possible to show that Bethe roots play a fundamental role in determining the eigenvalues of these physical quantities. Let see some examples (see [18] for completeness). Consider the shift operator:

$$U = e^{i\Pi} = i^{-N}F(\lambda)\Big|_{\lambda=i/2}. \tag{1.33}$$

When we apply U to Φ, since the second term of right side of (1.31) is zero for $\lambda = \frac{i}{2}$, we find a multiplicative spectrum for U and, subsequently, an additive spectrum for the momentum operator:

$$\Pi\Phi = \sum_{m=1}^{l} p(\lambda_m)\Phi \tag{1.34}$$

where

$$p(\lambda) = \frac{1}{i}\ln\left(\frac{\lambda+\frac{i}{2}}{\lambda-\frac{i}{2}}\right). \tag{1.35}$$

From (1.20) we can see that also the Hamiltonian has an additive spectrum:

$$H\Phi = \sum_{m=1}^{l} h(\lambda_m)\Phi \tag{1.36}$$

where

$$h(\lambda) = -\frac{1}{2}\frac{dp(\lambda)}{d\lambda} = \frac{1}{2}\frac{1}{\frac{1}{4}+\lambda^2}\,. \tag{1.37}$$

The parameter λ can be interpreted as the "rapidity". (Recall that the energy and the momentum of a relativistic particle can be parametrized in terms of the rapidity θ as $p = m\sinh\theta$, $E = m\cosh\theta$.) Eliminating the parameter λ we get the dispersion relation:

$$h(p) = \frac{1}{2}(1-\cos p)\,. \tag{1.38}$$

1.3 The Non-linear Integral equation

1.3.1 General idea

We present now a method completely equivalent to Bethe equations for solving integrable models. Before, we want to underline and summarize how Bethe ansatz works:

1. Starting from an integrable system, we find a set of Bethe-like equations as (1.32) for XXX Heisenberg model;

2. Solving Bethe-like equations, we find a set of M Bethe roots u_k, $k = 1, \cdots, M$;

3. Important Physical quantities have additive spectrum eigenvalues, i.e., they could be expressed in terms of sums of values of certain functions $O(u)$ computed in Bethe roots:
$$\sum_{k=1}^{M} O(u_k)\,. \tag{1.39}$$

Solving Bethe equations is not a difficult task for short chains. For long chains it could be difficult so it is better fo use an alternative method we now show. This method transforms the set of non-linear Bethe equations in only one non-linear integral equation. The starting idea is this: we want to evaluate a sum like (1.39) and, if we make the hypothesis that all Bethe roots u_k lie in a real interval $[A, B]$, we can compute “easily” the sum using Cauchy thorem:

$$2\pi i \sum_{k=1}^{M} O(u_k) = \oint_C dz\, O(z) f(z)\,, \tag{1.40}$$

where C is a contour of integration in the complex plane (see fig. (1.1)) and function $f(z)$ has to satisfy two conditions:

- $f(z)$ has poles in Bethe roots u_k;
- $\mathrm{Res}_{z=u_k}\,(f(z)) = 1$.

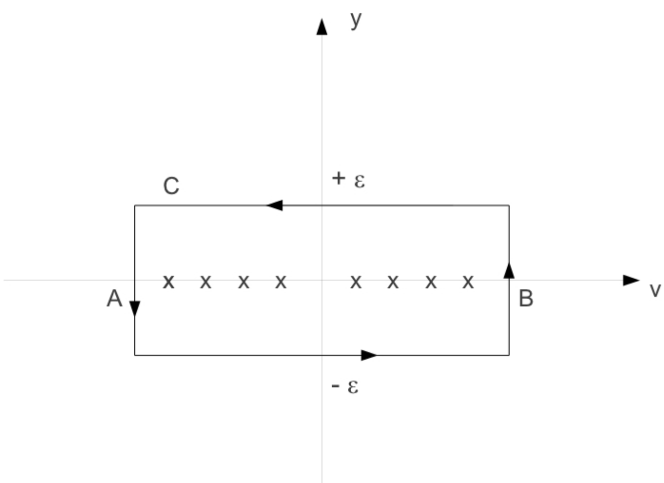

Figure 1.1: Contour of integration for Cauchy theorem

1.3.2 The Counting function

We have now to develop the idea showed above, introducing first a right function $f(z)$ satisfying the two properties listed above. We can choose

$$f(z) = \frac{i\delta Z'(z) e^{iZ(z)}}{1+\delta e^{iZ(z)}} \tag{1.41}$$

with $\delta = \pm 1$ and the property for $f(z)$ to have poles in Bethe roots with residues equal to one is satisfied if the function $Z(z)$ satisfies:

$$e^{iZ(u_k)} = -\delta \quad , \quad k = 1, \cdots, M \, , \tag{1.42}$$

with the interval $[A, B]$ defined through:

$$e^{iZ(A)} = e^{iZ(B)} = \delta \, . \tag{1.43}$$

Function $Z(z)$ plays the central role in this picture. It is called *counting function* (it will be clear later the motivation). Focus on property (1.42): it is an equation for Bethe roots and, more precisely, it replies Bethe equations. This property is the link between Bethe equations method and the picture we are developing now. In practical cases we will see that, starting from Bethe equations, we will define a counting function $Z(u)$ satisfying (1.43) and (1.42). It is not necessary, but we prefer choose counting function with another feature: we want $Z(u)$ monotone. This choice permits to fix a one-to-one correspondence between Bethe roots values on the axis and corresponding values $Z(u_k)$. We are completely free in choosing increasing or decreasing monotonicity. We now fix once for all our counting function as monotonously decreasing.

1.3.3 Developing the idea

Let us now enter in the main developing part of the method. First we explicit (1.40) using (1.41) and separating the countur C:

$$2\pi i \sum_{k=1}^{M} O(u_k) = \int_A^B dv O(v - i\epsilon) \frac{i\delta Z'(v-i\epsilon) e^{iZ(v-i\epsilon)}}{1+\delta e^{iZ(v-i\epsilon)}} + \int_B^A dv O(v + i\epsilon) \frac{i\delta Z'(v+i\epsilon) e^{iZ(v+i\epsilon)}}{1+\delta e^{iZ(v+i\epsilon)}} +$$
$$+ \int_{-\epsilon}^{\epsilon} idy O(B+iy) \frac{i\delta Z'(B+iy) e^{iZ(B+iy)}}{1+\delta e^{iZ(B+iy)}} + \int_{+\epsilon}^{-\epsilon} idy O(A+iy) \frac{i\delta Z'(A+iy) e^{iZ(A+iy)}}{1+\delta e^{iZ(A+iy)}} . \tag{1.44}$$

We are now in front of a crucial point: we note that function $f(z)$ could be written in a simple compact way:

$$f(z) = \frac{d}{dz} \ln \left[1 + \delta e^{iZ(z)} \right] \tag{1.45}$$

but we cannot make easily this substitution in the integrals due to the fact that we are integrating in the complex plane. We define the branch of the complex logarithm along the negative real semi-axis. So we must request that the argument of the logarithm in (1.45) does not be on the branch cut, i.e.

$$\mathrm{Re} \left[1 + \delta e^{iZ(z)} \right] = 1 + \left| e^{iZ(z)} \right| \cos\phi > 0 \Rightarrow \left| e^{iZ(z)} \right| \cos\phi > -1 \, . \tag{1.46}$$

We have two cases: if $\left| e^{iZ(z)} \right| \geq 1$ relation (1.46) could not be verified for $\cos\phi \in [-1, 0]$; if $\left| e^{iZ(z)} \right| < 1$ relation (1.46) is verified for all values of ϕ. So, with the goal of introducing complex logarithms, equation (1.44) is well defined only if the exponentials in it have the property $\left|e^{iZ(z)}\right| < 1$. If ϵ is suffiecient small (it is an arbitrary parameter) we have:

$$\left|e^{iZ(v+iy)}\right| \simeq e^{-yZ'(v)}\,. \tag{1.47}$$

Due to our choice of mononously decreasing counting function, we have that last relation is smaller than one only if $y < 0$, i.e. in the complex semi-plane with negative imaginary part. So, in equation (1.44), in all integrals in which variable of integration z lives in the upper complex semi-plane, we have to manipulate expression to have exponentials of the kind $e^{-iZ(v+iy)}$. In this case we have the correct behaviour:

$$\left|e^{-iZ(v+iy)}\right| \simeq e^{yZ'(v)} < 1 \quad \text{if} \quad y > 0\,. \tag{1.48}$$

With this prescription, (1.44) becomes:

$$\begin{aligned} 2\pi i \sum_{k=1}^{M} O(u_k) = & \int_A^B dv O(v-i\epsilon) \frac{i\delta Z'(v-i\epsilon)e^{iZ(v-i\epsilon)}}{1+\delta e^{iZ(v-i\epsilon)}} + \int_B^A dv O(v+i\epsilon) \frac{iZ'(v+i\epsilon)}{1+\delta e^{-iZ(v+i\epsilon)}} + \\ & + \int_{-\epsilon}^0 idy O(B+iy) \frac{i\delta Z'(B+iy)e^{iZ(B+iy)}}{1+\delta e^{iZ(B+iy)}} + \int_0^{\epsilon} idy O(B+iy) \frac{iZ'(B+iy)}{1+\delta e^{-iZ(B+iy)}} + \\ & + \int_0^{-\epsilon} idy O(A+iy) \frac{i\delta Z'(A+iy)e^{iZ(A+iy)}}{1+\delta e^{iZ(A+iy)}} + \int_{\epsilon}^0 idy O(A+iy) \frac{iZ'(A+iy)}{1+\delta e^{-iZ(A+iy)}}. \end{aligned} \tag{1.49}$$

Now we can further manipulate these expression in order to introduce the derivative of logarithms. We obtain, in few steps:

$$\begin{aligned} 2\pi i \sum_{k=1}^{M} O(u_k) = & -\int_A^B dv O(v+i\epsilon) iZ'(v+i\epsilon) + 2i \mathrm{Im} \int_A^B dv O(v-i\epsilon) \frac{d}{dv} \ln\left[1+\delta e^{iZ(v-i\epsilon)}\right] - \\ & - \int_0^{\epsilon} idy O(A+iy) iZ'(A+iy) + 2i \mathrm{Im} \int_0^{-\epsilon} dy O(A+iy) \frac{d}{dy} \ln\left[1+\delta e^{iZ(A+iy)}\right] - \\ & - \int_{\epsilon}^0 idy O(B+iy) iZ'(B+iy) + 2i \mathrm{Im} \int_{-\epsilon}^0 dy O(B+iy) \frac{d}{dy} \ln\left[1+\delta e^{iZ(B+iy)}\right]. \end{aligned} \tag{1.50}$$

Now, look to the first elements of every lines in (1.50). We can rewrite them in a compact and smart form in the following way: define a rectangular contour C' in the complex plane with the lower base coinciding with $[A,B]$; if there are no poles of $Z(v)$ and $O(v)$ in the area closed by C' we have:

$$\oint_{C'} O(v) iZ(v) = 0 \tag{1.51}$$

from which we have:

$$\begin{aligned} & \int_A^B dv O(v+i\epsilon) iZ'(v+i\epsilon) + \int_0^{\epsilon} idy O(A+iy) iZ'(A+iy) + \\ & + \int_{\epsilon}^0 idy O(B+iy) iZ'(B+iy) = -\int_B^A dv O(v) iZ'(v)\,. \end{aligned} \tag{1.52}$$

Putting (1.52) in (1.50) we finally obtain:

$$\begin{aligned} \sum_{k=1}^{M} O(u_k) = & -\int_A^B \frac{dv}{2\pi} O(v) Z'(v) + \mathrm{Im} \int_A^B \frac{dv}{\pi} O(v-i\epsilon) \frac{d}{dv} \ln\left[1+\delta e^{iZ(v-i\epsilon)}\right] + \\ & + \mathrm{Im} \int_0^{-\epsilon} \frac{dy}{\pi} O(A+iy) \frac{d}{dy} \ln\left[1+\delta e^{iZ(A+iy)}\right] + \mathrm{Im} \int_{-\epsilon}^0 \frac{dy}{\pi} O(B+iy) \frac{d}{dy} \ln\left[1+\delta e^{iZ(B+iy)}\right]. \end{aligned} \tag{1.53}$$

Equation (1.53) is what we call **Non-Linear Integral Equation** (NLIE). Note that discarding the non-linear terms, we have an usual and well-known equation, interpreting the function $-Z'(v)$ as a density (of Bethe roots in this case). So NLIE shows the exact structure of equations involving densities.

1.3.4 NLIE for the counting function

We are very close to complete our picture. What we still need is a way to evaluate counting function that is not completely known because, as we will see in practical case, it contains a part that depends on the Bethe roots themselves:

$$Z(u) = \Phi(u) - \sum_{m=1}^{k} \phi(u, u_k) \,. \tag{1.54}$$

We can evaluate $Z(u)$ applying (1.53) to the second term in the last equation:

$$Z(u) = \Phi(u) + \int_A^B \frac{dv}{2\pi}\phi(u,v)Z'(v) - \mathrm{Im}\int_A^B \frac{dv}{\pi}\phi(u,v-i\epsilon)\frac{d}{dv}\ln\left[1+\delta e^{iZ(v-i\epsilon)}\right] +$$
$$- \mathrm{Im}\int_0^{-\epsilon} \frac{dy}{\pi}\phi(u,A+iy)\frac{d}{dy}\ln\left[1+\delta e^{iZ(A+iy)}\right] - \mathrm{Im}\int_{-\epsilon}^{0} \frac{dy}{\pi}\phi(u,B+iy)\frac{d}{dy}\ln\left[1+\delta e^{iZ(B+iy)}\right]. \tag{1.55}$$

Our scheme is now complete: given an integrable system, we can derive Bethe-like equations or, alternatively, the counting function $Z(u)$ linked to them via (1.42), satisfying also (1.43). Then, we can work out the dipendence of $Z(u)$ on Bethe roots solving the NLIE (1.55) and, finally, we use the function so obtained for solving (1.53) an finding the eigenvalues of physical quantities such energy and momentum. Clearly, the main obstacle in this picture is the presence of non-linear terms. How to deal with non-linear terms depends on the specific system we are analyzing. Obviously, in this picture results should not depend on the kind of contour C choosen for the implementation of Cauchy theorem, i.e. on the thickness of the rectangle. So, making the limit $\epsilon \to 0$ in (1.53) and (1.55) we get:,

$$\sum_{k=1}^{M} O(u_k) = -\int_A^B \frac{dv}{2\pi}O(v)Z'(v) + \mathrm{Im}\int_A^B \frac{dv}{\pi}O(v)\frac{d}{dv}\ln\left[1+\delta e^{iZ(v-i0^+)}\right] \tag{1.56}$$

and

$$Z(u) = \Phi(u) + \int_A^B \frac{dv}{2\pi}\phi(u,v)Z'(v) - \mathrm{Im}\int_A^B \frac{dv}{\pi}\phi(u,v)\frac{d}{dv}\ln\left[1+\delta e^{iZ(v-i0^+)}\right]. \tag{1.57}$$

1.4 NLIE application: energy of the XXX Heisenberg chain

We now present an example of application of the NLIE picture studying the already seen Heisenberg model. Recall that a XXX chain state is described by Bethe equations:

$$\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^L = \prod_{k=1;k\neq j}^{M} \frac{u_j - u_k - i}{u_j - u_k + i} \quad , \, j = 1, \cdots, M \tag{1.58}$$

where L is the number of nodes of the lattice and M is the number of Bethe roots u_j that describe the state. We want to apply NLIE scheme in order to calculate the energy of the chain in the case of large L. We will able to compute the leading term and the first correction.

1.4.1 Counting function for the Heisenberg model

Let us start defining a counting function for the XXX chain. In order to do it, let start working on Bethe equations (3.48) applying the logarithm to both sides of the equation:

$$L \ln\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right) - \sum_{k=1;k\neq j}^{M} \ln\left(\frac{u_j - u_k - i}{u_j - u_k + i}\right) = 2\pi i n_j \, , \; j = 1, \cdots, M \tag{1.59}$$

where the integers n_j appear due to the polidromy of the complex logarithm and are related to each Bethe root position. Now, using the property

$$i \ln\left(\frac{x - ib}{x + ib}\right) - i \ln\left(\frac{ib - x}{ib + x}\right) = \pi \tag{1.60}$$

we can manipulate (1.59) and obtain:

$$iL \ln\left(\frac{\frac{i}{2} + u_j}{\frac{i}{2} - u_j}\right) - i \sum_{k=1}^{M} \ln\left(\frac{i + u_j - u_k}{i - u_j + u_k}\right) = \pi(2n_j + L - M + 1) \, , \; j = 1, \cdots, M. \tag{1.61}$$

Now we introduce a parameter that gives information about the parity of the chain:

$$\Delta = (L - M) \bmod 2 \; \Rightarrow \; L - M = \Delta + 2l \tag{1.62}$$

where $l = 0, 1$ and its specific value is related to the particular state of the chain, i.e. to the numbers of Bethe roots. We can so define a new parameter in the equations that takes account of distribution of roots for a specific state:

$$I_j = n_j + l + 1 \, . \tag{1.63}$$

The set $I_1, \cdots, I_M$ is a set of quantum numbers describing a specific state of the chain. With the introduction of Δ and I_j, we can finally put Bethe equations in the form:

$$iL \ln\left(\frac{\frac{i}{2} + u_j}{\frac{i}{2} - u_j}\right) - i \sum_{k=1}^{M} \ln\left(\frac{i + u_j - u_k}{i - u_j + u_k}\right) = \pi(2I_j + \Delta - 1) \, , \; j = 1, \cdots, M. \tag{1.64}$$

We are now able to define the right counting function for this model. Let start with identify the interval $[A, B]$ with all the real axis. Then, define a function $\phi(x, b)$ as follows:

$$\phi(x, b) = i \ln\left(\frac{ib + x}{ib - x}\right) \, , \; b > 0 \tag{1.65}$$

and the counting function as

$$Z(u) = L\phi\left(u, \frac{1}{2}\right) - \sum_{k=1}^{M} \phi(u - u_k, 1) \, . \tag{1.66}$$

With this definition we can write (1.64) in the compact form:

$$Z(u_j) = \pi(2I_j + \Delta - 1) \, , \; j = 1, \cdots, M \, . \tag{1.67}$$

This last way of writing Bethe equations permits to verify immediatly that the choosen counting function fulfills the property (1.42) necessary to define a correct counting function for NLIE picture:

$$e^{iZ(u_j)} = e^{i\pi(2I_j + \Delta - 1)} = e^{i\pi(\Delta - 1)} = (-1)^{\Delta - 1} = \pm 1 = -\delta \, . \tag{1.68}$$

Property (1.43) is also easily verified noting before that

$$\lim_{x\to\pm\infty}\phi(x,b)=\pm\pi \;\Rightarrow\; \lim_{u\to\pm\infty}Z(u)=\pm(L-M)\,. \tag{1.69}$$

So we have

$$e^{iZ(u\to\pm\infty)}=e^{\pm i\pi(L-M)}=(-1)^{L-M}=(-1)^{\Delta}=-(-1)^{\Delta-1}=\delta\,. \tag{1.70}$$

Note also that $Z(u)$ is monotone, but with increasing behaviour. It is not a problem beacuse we have said that the important feature we want to have is monotonicity.

1.4.2 Energy of the chain

Now we can apply NLIE scheme, using (1.56) with, for energy

$$O(u)=\frac{1}{u^2+\frac{1}{4}} \tag{1.71}$$

and (1.57) for the counting function (1.66). Note that some differences in signes are present due to the specific for the model increasing counting function, respect to decreasing one present in the general derivation. We have:

$$E=2\ln 2-\mathrm{Im}\int_{-\infty}^{+\infty}dv\frac{1}{\cosh\pi v}\frac{d}{dv}\ln\left[1+(-1)^{\Delta}e^{iZ(v+i0^+)}\right] \tag{1.72}$$

and

$$Z(u)=2L\arctan e^{\pi u}-\frac{\pi L}{2}+2\mathrm{Im}\int_{-\infty}^{\infty}dvG(u-v)\ln\left[1+(-1)^{\Delta}e^{iZ(v+i0^+)}\right] \tag{1.73}$$

where

$$G(x)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{ikx}\frac{1}{e^{|k|}+1}\,. \tag{1.74}$$

Evaluation of non-linear terms

We want now to put non-linear terms in (1.53) in a form that gives the possibility to better estimate them in some certain limits. Formulae we now present are not valid in general, but under certain conditions. It is possible to show, by direct calculations, that they are valid for non-linear terms we will compute in the example we are analyzing. We start with

$$\mathrm{NL}=\mathrm{Im}\int_A^B\frac{dv}{\pi}O(v-i\epsilon)\frac{d}{dv}\ln\left[1+\delta e^{iZ(v-i\epsilon)}\right]+\mathrm{Im}\int_0^{-\epsilon}\frac{dy}{\pi}O(A+iy)\frac{d}{dy}\ln\left[1+\delta e^{iZ(A+iy)}\right]+$$
$$+\mathrm{Im}\int_{-\epsilon}^{0}\frac{dy}{\pi}O(B+iy)\frac{d}{dy}\ln\left[1+\delta e^{iZ(B+iy)}\right]\,. \tag{1.75}$$

We can replace logarithmic terms in this way:

$$\ln\left[1+\delta e^{iZ(z)}\right]=\sum_{n=1}^{\infty}(-1)^{n+1}\delta^n\frac{e^{inZ(z)}}{n} \tag{1.76}$$

obtaining

$$\begin{aligned}\mathrm{NL} = \mathrm{Im}\int_A^B \frac{dv}{\pi} O(v-i\epsilon)\frac{d}{dv} + \sum_{n=1}^{\infty}(-1)^{n+1}\delta^n \frac{e^{inZ(v-i\epsilon)}}{n} + \\ + \mathrm{Im}\int_0^{-\epsilon} \frac{dy}{\pi} O(A+iy)\frac{d}{dy}\sum_{n=1}^{\infty}(-1)^{n+1}\delta^n \frac{e^{inZ(A+iy)}}{n} + \\ + \mathrm{Im}\int_{-\epsilon}^{0} \frac{dy}{\pi} O(B+iy)\frac{d}{dy}\sum_{n=1}^{\infty}(-1)^{n+1}\delta^n \frac{e^{inZ(B+iy)}}{n} .\end{aligned} \tag{1.77}$$

We can estimate such terms using these formulae:

$$\int^v dv O(v-i\epsilon)\frac{d}{dv}e^{inZ(v-i\epsilon)} = e^{inZ(v-i\epsilon)}\sum_{k=0}^{\infty}\left(\frac{i}{nZ'(v-i\epsilon)}\frac{d}{dv}\right)^k O(v) \tag{1.78}$$

$$\int^y dy O(c-iy)\frac{d}{dy}e^{inZ(c-iy)} = e^{inZ(c-iy)}\sum_{k=0}^{\infty}\left(\frac{i}{nZ'(c-iy)}\frac{d}{dy}\right)^k O(c-iy) \tag{1.79}$$

$$\tag{1.80}$$

which permit to obtain, exchanging series with integrals:

$$\begin{aligned}\mathrm{NL} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\delta^n}{\pi n}\sum_{k=0}^{\infty}\Bigg[\frac{i^{2k}}{n^{2k}}\sin(nZ(v))\left(\frac{1}{Z'(v)}\right)^{2k}\frac{d^{2k}}{dv^{2k}}O(v) + \\ + \frac{i^{2k+1}}{n^{2k+1}}\cos(nZ(v))\left(\frac{1}{Z'(v)}\right)^{2k+1}\frac{d^{2k+1}}{dv^{2k+1}}O(v)\Bigg]_{v=A}^{v=B} .\end{aligned} \tag{1.81}$$

Last step is valid if series in (1.78) and (1.79) are convergent or could be truncated. It could be verified, in practical cases, by direct calculations. Because the condition $\delta = e^{iZ(A)} = e^{iZ(B)}$ we have $\sin(nZ(A)) = \sin(nZ(B)) = 0$ and $\cos(nZ(A)) = \cos(nZ(B)) = \delta^n$. So, in (1.81), only the term containing cosine function survives when we evaluate functions in the extrema. Finally, summing over n we are left with:

$$\mathrm{NL} = -\sum_{k=0}^{\infty}\frac{(2\pi)^{2k+1}}{(2k+2)!}B_{2k+2}\left(\frac{1}{2}\right)\left[\left(\frac{1}{Z'(v)}\frac{d}{dv}\right)^{2k+1}O(v)\right]_{v=A}^{v=B} \tag{1.82}$$

where $B_k(x)$ is a Bernoulli polynomial. We can immediatly note from (1.82) that dependence on ϵ is canceled (as we want).

Recalling that we are in the limit of large L, let us now evaluate non-linear terms in (1.72) using (1.82), with the identification:

$$O(v) = \frac{\pi}{\cosh \pi v} . \tag{1.83}$$

From (1.73) we have

$$Z'(v) = L\frac{\pi}{\cosh \pi v} + \cdots \tag{1.84}$$

so we can note that the $k=0$ term in (1.82) gives a term of order $\frac{1}{L}$. We can so estimate the non-linear term of the energy as follows:

$$\mathrm{NL} = 2\pi B_2(1/2) \lim_{v\to\infty} \frac{O'(v)}{Z'(v)} + O\left(\frac{1}{L^2}\right) \tag{1.85}$$

where, with (1.83), we get

$$\lim_{v\to\infty} \frac{O'(v)}{Z'(v)} = \frac{-\pi}{L} + O\left(\frac{1}{L^2}\right) . \tag{1.86}$$

Since $B_2(1/2) = -\frac{1}{12}$ we finally have

$$\mathrm{NL} = \frac{\pi^2}{6L} + O\left(\frac{1}{L^2}\right) \tag{1.87}$$

and

$$E = 2L\ln 2 - \frac{\pi^2}{6L} + O\left(\frac{1}{L^2}\right) . \tag{1.88}$$

Chapter 2

The AdS/CFT duality

2.1 Maldacena conjecture

We now present a general and short overview about *AdS/CFT duality*. This correspondence, first introduced through the so-called *Maldacena conjecture* [1], matches two different physical models: on a side, we have a Quantum Field Theory with exact conformal spacetime symmetry, while on the other side there is a Superstring Theory with strings living on a background which contains an Anti-de-Sitter (AdS) space-time as a factor. We follow for this general description [20]. More precisely, the duality is formulated in these terms: the boundary of an AdS_{d+1} spacetime is a conformally flat $d-$dimensional spacetime on which CFT is formulated. The AdS/CFT duality relates the string partition function, with sources ϕ for string vertex operators fixed to value J at the boundary of AdS_{d+1}, to the CFT_d partition function with sources J for local operators:

$$Z_{str}[\phi|_{\partial AdS} = J] = Z_{CFT}[J]\,.$$

In other terms, for every string observable at the boundary of AdS_{d+1} there is a corresponding observable in the CFT_d (and vice versa) whose values are expected to match. AdS/CFT duality is suggested in [1] by some considerations about black holes geometries: they shows near horizon AdS backgrounds.

2.2 String/Gauge dualities

Main importance of AdS/CFT duality is due to the evidence, through it, of a relation between a string theory and a gauge theory. Insights about similarities among these two kind of models are as old as string theory itself. It is well known that hadron spectrum organizes into so called Regge-trajectories that represent an appropriate linear relationship, with universal slope, between the squared masses of hadronic resonances and their spins. This is what exactly a string theory on flat space predicts, so, for some time, String theory was considered a good candidate model for strong interactions. For many reasons this picture doesn't work and we have QCD as the right model describing strong interactions, but it is clear that, under some conditions, string theory could be a useful approximation to gauge theory phenomena. Another manifestation of stringy behaviour in gauge theory is the presence of flux tubes of the Chromodynamic field: they form, between two quarks when they are pulled apart. Using some approximation, they could be seen as one dimensionale objects with costant tension, i.e. strings.

AdS/CFT duality regards these aspects of relationship between Strings and Gauge theory in a deeper way. It proposes that in some cases a Gauge Theory is exactly dual to a String one. A very important aspect immediatly arises from the duality: today, String Theory is the best model able to describe a complete Quantum Theory of all interactions, including Gravity. If the duality is exact, we will be able to study Quantum Gravity using a more comfortable and usual Gauge Theory.

However, today it does not exist a formal proof of the duality. What we are able to do is testing duality predictions. A milestone in this direction has been given by t'Hooft in 1974 [2] with the discovery of the planar limit. This is a limit for gauge theories with gauge symmetries SU(N_c), SO(N_c) or Sp(N_c), where N_c is the number of colors of the model. It consists in taking the rank of the group to infinity, i.e. $N_c \to \infty$, keeping the rescaled gauge coupling $\lambda = g_{YM}^2 N_c$ finite. In this limit, dominating Feynman diagrams are those drawingable on a plane, whereas the others with crossing lines are suppressed. Now, main attention in testing AdS/CFT duality regards the most symmetric setting: on the gauge side we have the $\mathcal{N} = 4$ Super Yang-Mills theory, while on string side the partner is type IIB Superstring theory on $AdS_5 \times S_5$ background, where S_5 represents a maximally symmetric five-dimensional factor in a total space in which the other factor is an Anti de Sitter space in five dimensions. Suprisingly, in the planar limit of $\mathcal{N} = 4$ SYM there is the *miracolous* appearing of integrability. Before exploring how exactly integrability works and enters in $\mathcal{N} = 4$ SYM, we have to understand why appearing of integrability is so important in AdS/CFT duality, focusing on a specific feature of the duality, i.e. its *strong/weak* behaviour.

2.2.1 Strong/weak duality

The AdS/CFT duality is of the kind *strong/weak*. For understanding what it means, let discuss now about the mapping of the parameters of both theories. On the gauge side, two parameters are relevant: the t'Hooft coupling $\lambda = g_{YM}^2 N_c$ and the number of colors N_c that define the rank of the gauge group. On string side, we have the effective string tension $T = \dfrac{R^2}{2\pi\alpha'}$, where α' is the inverse string tension and R is the radius of AdS_5/S_5. Effective string tension also obeys to the relation of proportionality $T \propto \hbar^{-1}$. The AdS/CFT correspondence relates them as follows:

$$\lambda = 4\pi^2 T^2 \quad , \quad \frac{1}{N_c} = \frac{g_{str}}{4\pi^2 T^2} \, . \tag{2.1}$$

On the gauge side, the weak coupling regime is the region of parameter space where $\lambda \to 0$. In this regime, perturbative calculations on Feynman diagrams permit to find reliable results. Vice-versa, on the string side perturbative regime is in the parameter region around the point $g_{str} = 0$, $\lambda = \infty$. This double limit describes free and classical strings, this second feature is due to (2.1): $\lambda \to \infty \Rightarrow T \to \infty \Rightarrow \hbar \sim 0$. The main difficult is that this perturbative limit on the string side corresponds to the strong coupling limit on the gauge side. The two perturbative limits of the two theories do not overlap! This creates a strong obstacle in testing the duality, because we cannot compare directly results in calculations of matched observables into the duality. On the other hand, if we had a formal proof of the duality, we could have a great possibility from the strong/weak duality: we could get on a side exact results in strong coupling regime through perturbative results on the other side and vice-versa. This is where integrability comes to help. Let see it in details.

2.3 Integrability in $\mathcal{N}=4$ Super Yang-Mills

2.3.1 $\mathcal{N}=4$ Super Yang-Mills Lagrangian

In order to introduce integrability, we show briefly the field content of $\mathcal{N}=4$ SYM theory. The Action reads:

$$S=\frac{1}{g_{YM}^2}\int d^4x\mathrm{Tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+D_\mu\phi^2-\sum_{i<j}[\phi_i,\phi_j]^2+i\bar{\psi}\Gamma^\mu D_\mu\psi-\bar{\psi}\Gamma^i[\phi_i,\psi]\right\}. \tag{2.2}$$

The theory contains a gauge field A_μ, ($\mu=0,1,2,3$), six scalar fields Φ_i, ($i=1,\cdots,6$) and four Majorana fermions ψ_α^A ($\bar{\psi}_{\dot{\alpha}}^A$), ($A=1,\cdots,4$). Covariant derivative D_μ and the strenght field $F_{\mu\nu}$ are defined as usual:

$$D_\mu(*)=\partial_\mu(*)-i[A_\mu,(*)] \quad , \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu] \tag{2.3}$$

and, finally, $\Gamma^A=(\Gamma^\mu,\Gamma^i)$ are ten 16x16 Dirac matrices. This model has a much richer set of symmetries: first it has supersymmetry, i.e. bosonic degrees of freedom are in the same number as fermionic ones; then, it is present conformal symmetry, that implies that there are no massive particles whose spectrum we might wish to compute.

2.3.2 Spectrum of Scaling Dimensions

Let us take another step to the integrability. Let start with a comparison with QCD. An old (and never reached) target in QCD is to derive a quantitative description of the mass spectrum of hadronic particles and their excitations, expressing them (for example focus on masses of proton and neutron) as functions of the parameters of the theory:

$$m_p=f_1(\alpha_s,\alpha,\mu_{reg},\cdots) \quad , \quad m_n=f_2(\alpha_s,\alpha,\mu_{reg},\cdots).$$

In QCD, this result has to remain a dream: at low energies, coupling costant α_s is too large for meaningful approximation, so non-perturbative contributions dominate. Then, self-interactions of Chromodynamic field lead to a non-linear and highly complex problem. Clearly, confinement obscures the nature of fundamental particles in QCD at low energies.

However, an analogous to this scheme is present in $\mathcal{N}=4$ SYM. This model, roughly, is a cousin of QCD with a richer content of symmetries and fields. Conformal symmetry implies that non masses are present, but there is an analogue quantity that could play a similar role of mass, giving a spectrum depending on fundamental parameters: the *local operators*, defined as compositions of fundamental fields all residing at a common point in spacetime. Like in QCD, the color charges are balanced out making local operators gauge-invariant objects. Then, there is a characteristic quantity to replace the mass, the so-called *scaling dimension*, related to two points correlation function among local operators. Classically, scaling dimension equals the sum of constituents dimensions and, like the mass, it receives quantum corrections from interactions between constituents. This correction is called *anomalous dimension* and it is the eigenvalue of a quantum operator, the so-called *Dilatation Operator*:

$$D_O O=f(\lambda)O \tag{2.4}$$

where D_O is the dilatation operator related to the local operator O and $f(\lambda)$ is the anomalous dimension itself. Here, finally, we reach the point where integrability joins the picture: in planar limit of $N=4$ SYM equation (2.4) can be written as an eigenvalue equation for a Hamiltonian of an integrable system, i.e. dilatation operator assumes the form of a known integrable Hamiltonian. Integrability ensures that we can compute anomalous dimension $f(\lambda)$ using techinques belonging to integrable models at arbitrary values of λ. In AdS/CFT duality picture, anomalous dimensions are matched with string energies. So, integrability permits to obtain a wonderful test for AdS/CFT: the possibility to compute anomalous dimensions at arbitrary values of λ connects the regime of perturbative gauge theory with the perturbative regime of String theory.

The Thermodynamic Bethe Ansatz

Integration of the model means finding a solution for function $f(\lambda)$ through (2.4). In general, it is got as the solution of a set of integral equations: recently, great progresses has been showed: a set of *Thermodynamic Bethe Ansatz* (TBA) equations [6] or an equivalent $Y-$system of functional equations [7], together with certain additional information [8], provides a solid ground for exact predictions for anomalous dimensions of planar $\mathcal{N}=4$ SYM. Finally, a very recent work [9] shows the possibility of a reformulation of TBA equations as a finite set of integral equations.

TBA equations are not argument of this work. However, it is true that disentagling the TBA equations for excited states is a quite difficult task and at present a numerical approach seems to be the only viable strategy providing explicit results. Therefore, it is still convenient to study all the domains and all the limits in which it is possible to use the simpler Asymptotic Bethe Ansatz, showed in the previous chapter, in order to obtain correct anomaolous dimension. We will soon show a case as example and, subsequently, we give some insights about the limit of validity of ABA approach.

2.3.3 Integrability: the $su(2)$ sector of $\mathcal{N}=4$ SYM

Let us show now an example that gives evidence of integrability in $\mathcal{N}=4$ SYM. We are interesting in anomalous dimensions associated to the following local operators:

$$O = \text{Tr}\left(Z^{L-M}W^M + \text{permutations}\right) \tag{2.5}$$

where

$$Z = \phi_1 + i\phi_2 \quad , \quad W = \phi_3 + i\phi_4 \,. \tag{2.6}$$

We have local operators composed by L complex scalar fields, divided into $L-M$ of kind Z and M of kind W. The composition contains all possible permutations among these set of field operators. Operators defined in this way realize a subgroup SU(2) of symmetry in the global symmetry of the model.

Integrability of this model is prooved in [3]. We give now a short overview. Starting point is the two point correlator $\langle O_m(x)O_n(y)\rangle$: it has a UV divergence we can erase through renormalization. We renormalize, substantially, performing a linear superposition of all local operators:

$$O_A^{ren} = Z_A^m O_m^{bare} \tag{2.7}$$

which permits us to get:

$$\langle O_A^{ren} O_B^{ren}\rangle \propto \frac{\delta_{AB}}{|x-y|^{2\Delta(\lambda)}} \tag{2.8}$$

where

$$\Delta(\lambda) = L + \gamma(\lambda) \tag{2.9}$$

where L is the classical dimension, while $\gamma(\lambda)$ is the anomalous dimension we want to analyze. It is clear now what we have said previously: anomalous dimension is, from

(2.8), an effect due to the mixing of bare fields through renormalization. It is possible to find an operator for which anomalous dimension is an eigenvalue, i.e. what we have called dilatation operator. Of course, dilatation operator should be related to renormalization coefficients. In this case we have (see [3]):

$$\Gamma O = \gamma(\lambda) O \, , \, \Gamma = Z^{-1} \frac{dZ}{d \ln \Lambda} \tag{2.10}$$

where Λ is a UV cut-off.

We now see the explicit form of Γ for one-loop planar diagrams. Three cases are possible (see fig. 2.1): exchange of boson field, crossing and self-interaction. Computing

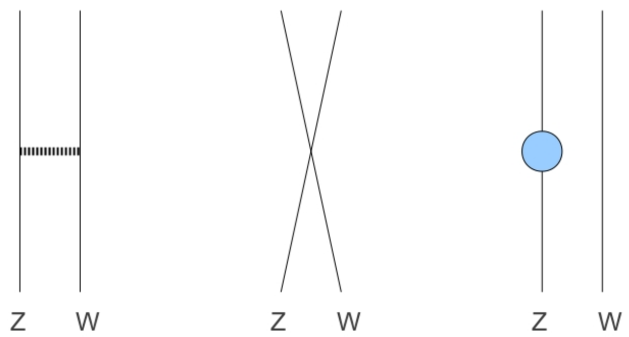

Figure 2.1: One loop planar diagrams in su(2) sector

all these contributions, it is possible to write dilatation operator in the following form:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (1 - P_{l,l+1}) \tag{2.11}$$

where $P_{l,l+1}$ are permutation matrices. But we have yet seen this operator. Recalling (1.7), it is exaclty the Hamiltonian of a $\text{XXX}_{1/2}$ Heisenberg chain times a factor $\dfrac{\lambda}{8\pi^2}$. We well know that this Hamiltonian is exaclty solvable due to integrability, using Bethe Ansatz equations or NLIE approach. Substantially, we can solve the problem making these identifications

$$\Gamma O = \gamma O \leftrightarrow H|\psi\rangle = E|\psi\rangle \, , \tag{2.12}$$

$$O = \text{Tr}(ZZZWWWZWZZ \cdots + \text{permutations}) \leftrightarrow |\psi\rangle = |\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow \cdots + \text{permutations}\rangle , \tag{2.13}$$

where H is (1.2) and E could be, for example in the case of large L, (1.88).

2.3.4 Limits of ABA integrability: the wrapping problem

We have seen in the previous subsection a natural way for appearing of integrability in $\mathcal{N} = 4$ SYM model. Substantially, dilatation operators could be written as hamiltonian operators of such integrable spin chains. Proof of integrability was extended to all gauge sectors of the theory and at all loops in a weaker sense: any local operator has been seen as a state of a "spin chain" whose Hamiltonian is the dilatation operator related, although the latter does not have an explicit expression of the spin chain form, but for first few loops. Neverthless, large size (asymptotic) spectrum has turned out to be exactly described by certain Asymptotic Bethe Ansatz-like equations: the so called Beisert-Staudacher equations [4], [5]. This picture suffers of a problem: focus now for simplicity on the $su(2)$

sector just seen. It could be interesting find two-loops contributions to anomalous dimension: drawing all possible planar diagrams and computing all contributions, one finds a spin chain Hamiltonian with an interaction that is no more of the kind nearest neighbour: the new interaction involves three fields. Continuing on this path, growing in number of loops corresponds to growing of number of internal fields involved in the interaction of the Hamiltonian related. Building these Hamiltonian is a very high diffcult task, as we have just said it is possible for few first loops but it is possible to derive, however, Bethe-like equations. The problem is that this approach ceases to be well defined when number of loops becomes equal to L, i.e. the length of the spin chain: in corresponding Hamiltonian interactions should now involve $L + 1$ fields. These phenomena of self-interactions of the chain give uncorrect results in ABA picture. ABA calculations are so limited: exact results seem to be only until $L - 1$ loops. This problem is called *wrapping*. We will return more precisely on this problem in the next chapter, when we will study $sl(2)$ sector of the theory. We can anticipate that it is possible to discard wrapping corrections under certain limits, in which ABA works correctly at all loops.

Chapter 3

High spin twist operators in $sl(2)$ sector of $\mathcal{N} = 4$ Super Yang-Mills Theory

3.1 Introduction

The set of composite operators

$$Tr(D^s Z^L) + \cdots , \tag{3.1}$$

where D is a covariant derivative acting in all possible ways on the complex scalar field Z, constitutes the so-called $sl(2)$ sector of $\mathcal{N} = 4$ SYM theory. The integer numbers s and L are called respectevely spin and twist. In the framework of AdS/CFT duality, this set of operators received particular attention, also because their connection to twist operators in QCD [11, 12].

Integrability in the $sl(2)$ sector is well known and studied [4, 5]. As it happens in $su(2)$ sector (showed in the previous chapter), every composite operator of the kind (3.1) can be thought as a state of a 'spin chain', whose hamiltonian is the dilatation operator itself. Although it is possible to find explicit expressions for these hamiltonians only for the first few loops, it is possible to describe exactly the spectrum of infinitely long operators up the desired loop order applying an approach based on Asymptotic Bethe Ansatz (ABA), obtaining Bethe-like equations, the Beisert-Staudacher equations [4, 5]. However, as we have mentioned in the previous chapter, anomalous dimensions of operators with finite size depend also by wrapping corrections [10], that start to play a role from L loops contributions. High spin limit of fixed twist operators seems to offer a partial solution. In the high spin (at fixed twist) limit the minimal anomalous dimension shows a leading high spin behaviour with the logaritmh of s. In this chapter we show that the high spin expansion goes on as follows:

$$\begin{aligned}\gamma(g, L, s) = f(g) \ln s + f_{sl}(g, L) + \sum_{n=1}^{\infty} \gamma^{(n)}(g, L)(\ln s)^{-n} + \\ + \frac{1}{s} \sum_{n=-1}^{\infty} \tilde{\gamma}^{(n)}(g, L)(\ln s)^{-n} + O\left(s^{-1}(\ln s)^{-\infty}\right) ,\end{aligned} \tag{3.2}$$

where g is linked to the t'Hooft coupling λ via the relation $\lambda = 8\pi g^2$. Peturbative calculations [13] (up to six loops) shows that for short (twist two and three) operators, wrapping contributions behave as $O\left(\frac{(\ln s)^2}{s^2}\right)$. It is natural to ask if this property extends to higher and possibly to all loops. If it is true, all functions appearing in (3.2) should be wrapping-free. Main purpose of this chapter is to show and provide strong evidence of this hypothesis.

Scaling functions $f(g)$ and $f_{sl}(g, L)$ are well-known and studied. Universal scaling function $f(g)$ [21, 22, 23] is twice the cusp anomalous dimension of Wilson loops [21]. It was obtained through the solution of a linear integral equation directly derived from the asymptotic Bethe ansatz via the root density approach [5]. Moreover, it was carefully studied and tested both in the weak [24, 5] and strong coupling limit [25, 26, 27, 28]. The subleading term $f_{sl}(g, L)$ has been derived as solution of a non-linear integral equation (NLIE) [29], while in [30] it has been obtained starting from a linear integral equation (LIE). Explicit weak and strong coupling expansions are showed in [31]. It is important to remark that strong coupling expansions of $f(g)$ and $f_{sl}(g, L)$, found in these works, agree with string theory computations [32]. This shows that there are good reasons to believe that these functions are wrapping-free and so anomalous dimension is wrapping-free at orders $\ln s$ and $(\ln s)^0$.

In this chapter we will focus on functions $\gamma^{(n)}(g, L)$ and $\tilde{\gamma}^{(n)}(g, L)$. Using NLIE method described in the first chapter, applyed to the ABA equations for the $sl(2)$ sector (the already mentioned Beisert-Staudacher equations), we are able to find all-loops general expressions for these functions. It is possible to show strong evidences of wrapping-free behaviour of these functions, using a very interesting costraining property acting on anomalous dimension [14]. Anomalous dimension should satisfy the following 'self-tuning' property:

$$\gamma(g, L, s) = P\left(s + \frac{1}{2}\gamma(g, L, s)\right) , \tag{3.3}$$

where the function P, when $s \to +\infty$, expands as follows,

$$P(s) = \sum_{n=0}^{\infty} \frac{a_n(\ln C(s))}{C(s)^{2n}} , \tag{3.4}$$

the quantity $C(s)$ being given by

$$C(s)^2 = \left(s + \frac{L}{2} - 1\right)\left(s + \frac{L}{2}\right) . \tag{3.5}$$

Relation (3.4) expresses the so-called 'reciprocity property' (in Mellin space). An analogous of these realtions exists also in QCD in the framework of Altarelli-Parisi equations, but it is valid only at one loop. Istead, it is supposed that in the framework of $sl(2)$ sector of $\mathcal{N} = 4$ SYM it is valid at all-loops. Relations (3.3), (3.4) provide exact and nontrivial information on the high spin limit of anomalous dimensions of twist operators. Firstly, we will show that our general expressions of the functions $\gamma^{(n)}(g, L)$ and $\tilde{\gamma}^{(n)}(g, L)$ fit exaclty these results from reciprocity. Subsequently, this exact fitting contains (finally) strong suggestions about the wrapping-free property of functions $\gamma^{(n)}(g, L)$ and $\tilde{\gamma}^{(n)}(g, L)$.

Results of this chapter are based on works [15] and [16].

3.2 Non-linear integral equation for $sl(2)$ sector of $\mathcal{N} = 4$ SYM

We start our path working to obtain the Non-linear integral equation for the $sl(2)$ sector, using the general method showed in the first chapter and already applied for the $su(2)$ sector - Heisenberg chain.

3.2.1 Algebraic Bethe ansatz equations for $sl(2)$ sector

Let start introducing the ABA equations for $sl(2)$ sector (Beisert-Staudacher equations) [4, 5]:

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L\left(\frac{1+\frac{g^2}{2x^-(u_k)^2}}{1+\frac{g^2}{2x^+(u_k)^2}}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{s}\frac{u_k-u_j-i}{u_k-u_j+i}\left(\frac{1-\frac{g^2}{2x^+(u_k)x^-(u_j)}}{1-\frac{g^2}{2x^-(u_k)x^+(u_j)}}\right)^2 e^{2i\theta(u_k,u_j)}\,, \quad (3.6)$$

where

$$x^{\pm}(u_k)=x(u_k\pm i/2)\,,\quad x(u)=\frac{u}{2}\left[1+\sqrt{1-\frac{2g^2}{u^2}}\right]\,,\quad \lambda=8\pi^2g^2\,, \qquad (3.7)$$

λ being the 't Hooft coupling. The so-called dressing factor [33, 5] $\theta(u,v)$ is given by

$$\theta(u,v)=\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)[q_r(u)q_{r+1+2\nu}(v)-q_r(v)q_{r+1+2\nu}(u)]\,, \qquad (3.8)$$

the functions $\beta_{r,r+1+2\nu}(g)=g^{2r+2\nu-2}2^{1-r-\nu}c_{r,r+1+2\nu}(g)$ being

$$c_{r,r+1+2\nu}(g)=2\sum_{\mu=\nu}^{\infty}\frac{g^{2r+2\nu+2\mu}}{2^{r+\mu+\nu}}(-1)^{r+\mu+1}\frac{(r-1)(r+2\nu)}{2\mu+1}\cdot$$
$$\cdot\begin{pmatrix}2\mu+1\\ \mu-r-\nu+1\end{pmatrix}\begin{pmatrix}2\mu+1\\ \mu-\nu\end{pmatrix}\zeta(2\mu+1) \qquad (3.9)$$

and $q_r(u)$,

$$q_r(u)=\frac{i}{r-1}\left[\left(\frac{1}{x^+(u)}\right)^{r-1}-\left(\frac{1}{x^-(u)}\right)^{r-1}\right]\,, \qquad (3.10)$$

being the expression of the r-th charge in terms of the rapidity u. Operators (3.1) of twist L correspond to zero momentum states of the $sl(2)$ spin chain described by an even number s of real Bethe roots u_k which satisfy (3.6). For a state described by the set of Bethe roots $\{u_k\}, k=1,\dots,s$, the eigenvalue of the r-th charge is

$$Q_r(g,L,s)=\sum_{k=1}^{s}q_r(u_k)\,. \qquad (3.11)$$

In particular, (asymptotic) anomalous dimension of (3.1) is

$$\gamma(g,L,s)=g^2Q_2(g,L,s)\,. \qquad (3.12)$$

3.2.2 Counting function

Recall now the formula for the NLIE introduced in the first chapter (1.56):

$$\sum_k O(v_k)=-\int_{-\infty}^{+\infty}\frac{dv}{2\pi}O(v)Z'(v)+\int_{-\infty}^{+\infty}\frac{dv}{\pi}O(v)\frac{d}{dv}\mathrm{Im}\ln[1+(-1)^L\,e^{iZ(v-i0^+)}]\,. \quad (3.13)$$

Equation (3.13) is solved for the counting function $Z(u)$ that, in the case (3.6), has the form:

$$Z(u) = \Phi(u) - \sum_{k=1}^{s} \phi(u, u_k) \tag{3.14}$$

where, distinguishing among one loop and higher loops contributions, we have:

$$\Phi(u) = \Phi_0(u) + \Phi_H(u) \,, \quad \phi(u,v) = \phi_0(u-v) + \phi_H(u,v) \,, \tag{3.15}$$

with

$$\Phi_0(u) = -2L \arctan 2u \,, \quad \Phi_H(u) = -iL \ln \left(\frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) \,, \tag{3.16}$$

$$\phi_0(u-v) = 2 \arctan(u-v) \tag{3.17}$$

$$\phi_H(u,v) = -2i \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u,v) \right] \,. \tag{3.18}$$

$Z(u)$ is choosen as monotonously decreasing function. In addition, in the limit $u \to \pm\infty$, since

$$\phi(u,v) + \phi(u,-v) \to \pm 2\pi - \frac{4}{u} + \frac{2ig^2}{u} \left(\frac{1}{x^-(v)} - \frac{1}{x^+(v)} \right) + O\left(\frac{1}{u^3} \right) \,, \tag{3.19}$$

the asymptotic behaviour is:

$$u \to \pm\infty \,, \quad Z(u) \to \mp(L+s)\pi + \frac{L + 2s + \gamma(g,L,s)}{u} + O\left(\frac{1}{u^3} \right) \,. \tag{3.20}$$

This means that there are $L+s$ real points v_k such that $e^{iZ(v_k)} = (-1)^{L+1}$. It is a simple consequence of the definition of $Z(u)$ that s of them coincide with the Bethe roots u_k. For Bethe equations (3.6) Bethe roots are all real and are contained in an interval $[-b, b]$ of the real axis. The remaining L points are called 'holes' [22, 29]. They are 'fake' solutions of Bethe equations, they are real and they will be denoted as x_h. Applying (3.13) we are evaluating a sum on the right side that contains both roots and holes. In order to evaluate correctly charges, that are given by sums on the roots, we have to specialize equation (3.13) subtracting to the left side holes contributions:

$$\begin{aligned} \sum_{k=1}^{s} O(u_k) = &- \int_{-\infty}^{+\infty} \frac{dv}{2\pi} O(v) Z'(v) - \sum_{h=1}^{L} O(x_h) + \\ &+ \int_{-\infty}^{+\infty} \frac{dv}{\pi} O(v) \frac{d}{dv} \mathrm{Im} \ln[1 + (-1)^L \, e^{iZ(v - i0^+)}] \,. \end{aligned} \tag{3.21}$$

We should distinguish between $L-2$ internal or 'small' holes $x_h, h = 1, ..., L-2$, which reside inside the interval $[-b, b]$, and two external or 'large' holes $x_{L-1} = -x_L$, with $x_L > b > 0$.

Application of (3.21) to (3.14) gives:

$$Z(u) = \Phi(u) + \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \phi(u,v) [Z'(v) - 2L'(v)] + \sum_{h=1}^{L} \phi(u, x_h) \,, \tag{3.22}$$

where

$$L(u) = \mathrm{Im} \ln[1 + (-1)^L \, e^{iZ(u - i0^+)}] \,. \tag{3.23}$$

Introducing the function $\sigma(u) = \frac{d}{du}Z(u)$ (also known as density function) we can derive immediatly the following NLIE:

$$\sigma(u) = \Phi'(u) + \int_{-\infty}^{+\infty} \frac{dv}{2\pi}\frac{d}{du}\phi(u,v)[Z'(v) - 2L'(v)] + \sum_{h=1}^{L}\frac{d}{du}\phi(u,x_h)\,. \tag{3.24}$$

3.2.3 NLIE for the auxiliary function $S(k)$

Now we use a general method for obtaining anomalous dimension working on a simpler NLIE written for an auxiliary function $S(k)$ linked to the Fourier transform $\hat{\sigma}(k)$ of $\sigma(u)$. So let us pass to Fourier space using

$$\hat{f}(k) = \int_{-\infty}^{+\infty} due^{-iku}f(u)\,,$$

we have:

$$\hat{\Phi}_0(k) = -\frac{2\pi Le^{-\frac{|k|}{2}}}{ik}\,, \tag{3.25}$$

$$\hat{\Phi}_H(k) = \frac{2\pi L}{ik}e^{-\frac{|k|}{2}}[1 - J_0(\sqrt{2}gk)]\,, \tag{3.26}$$

$$\hat{\phi}_0(k) = \frac{2\pi e^{-|k|}}{ik}\,, \tag{3.27}$$

$$\begin{aligned}\hat{\phi}_H(k,t) = {} & -8i\pi^2\frac{e^{-\frac{|t|+|k|}{2}}}{k|t|}\Big[\sum_{r=1}^{\infty}r(-1)^{r+1}J_r(\sqrt{2}gk)J_r(\sqrt{2}gt)\frac{1-\text{sgn}(kt)}{2}+\\ & + \text{sgn}(t)\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)(-1)^{r+\nu}\Big(J_{r-1}(\sqrt{2}gk)J_{r+2\nu}(\sqrt{2}gt)-\\ & - J_{r-1}(\sqrt{2}gt)J_{r+2\nu}(\sqrt{2}gk)\Big)\Big]\,.\end{aligned} \tag{3.28}$$

If we apply Fourier transform to (3.24) and use relations (3.25),(3.26),(3.27),(3.28), we obtain this following NLIE for $\hat{\sigma}(k)$:

$$\begin{aligned}\hat{\sigma}(k) = {} & -\frac{2\pi Le^{-\frac{|k|}{2}}}{1-e^{-|k|}} + \frac{2\pi Le^{-\frac{|k|}{2}}[1-J_0(\sqrt{2}gk)]}{1-e^{-|k|}} - \frac{2e^{-|k|}\hat{L}'(k)}{1-e^{-|k|}}+\\ & + \frac{2\pi e^{-|k|}}{1-e^{-|k|}}\sum_{h=1}^{L}e^{ikx_h} + \frac{ik}{1-e^{-|k|}}\sum_{h=1}^{L}\int_{-\infty}^{+\infty}\frac{dt}{2\pi}\;e^{itx_h}\;\hat{\phi}_H(k,t)+\\ & + \frac{ik}{1-e^{-|k|}}\int_{-\infty}^{+\infty}\frac{dt}{4\pi^2}\;\hat{\phi}_H(k,t)[\hat{\sigma}(t) - 2\hat{L}'(t)]\,.\end{aligned} \tag{3.29}$$

We can immediatly separate one-loop contributions:

$$\hat{\sigma}_0(k) = \frac{-2\pi Le^{-\frac{|k|}{2}} + 2\pi e^{-|k|}\sum_{h=1}^{L}e^{ik\bar{x}_h}\hat{\phi}_0(k)}{1-e^{-|k|}} - \frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'_0(k)\,, \tag{3.30}$$

where $\bar{x}_h$ are one loop contributions to holes positions and

$$\hat{L}'_0(k) = \int_{-\infty}^{+\infty} du\; e^{-iku}\frac{d}{du}\text{Im}\ln[1 + (-1)^L\, e^{iZ_0(u-i0^+)}]\,. \tag{3.31}$$

On the other hand, we can write the NLIE for higher loops density function $\hat{\sigma}_H(k) = \hat{\sigma}(k) - \hat{\sigma}_0(k)$:

$$\hat{\sigma}_H(k) + \frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'_H(k) = \frac{2\pi L e^{-\frac{|k|}{2}}[1 - J_0(\sqrt{2}gk)]}{1-e^{-|k|}} + \\ + \frac{2\pi e^{-|k|}}{1-e^{-|k|}} \sum_{h=1}^{L} \left(e^{ikx_h} - e^{ik\bar{x}_h}\right) + \frac{ik}{1-e^{-|k|}} \sum_{h=1}^{L} \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{itx_h} \hat{\phi}_H(k,t) + \\ + \frac{ik}{1-e^{-|k|}} \int_{-\infty}^{+\infty} \frac{dt}{4\pi^2}\, \hat{\phi}_H(k,t)[\hat{\sigma}(t) - 2\hat{L}'(t)]\,, \tag{3.32}$$

where $\hat{L}'_H(k) = \hat{L}'(k) - \hat{L}'_0(k)$.

We can now show that there is a direct connection among the right side of the (3.32) and the anomalous dimension. Let start with the NLIE for anomalous dimension:

$$\gamma(g,L,s) = g^2 \sum_{k=1}^{s} e(u_k) = -g^2 \sum_{h=1}^{L} e(x_h) - g^2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} e(v)[\sigma(v) - 2L'(v)]\,, \tag{3.33}$$

where

$$e(u) = q_2(u) = i\left[\frac{1}{x^+(u)} - \frac{1}{x^-(u)}\right]. \tag{3.34}$$

Then, passing to Fourier space, we have:

$$\gamma(g,L,s) = -g^2 \sum_{h=1}^{L} \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{itx_h} \hat{e}(t) - g^2 \int_{-\infty}^{+\infty} \hat{e}(t)[\hat{\sigma}(t) - 2\hat{L}'(t)] \tag{3.35}$$

where

$$\hat{e}(t) = \int_{-\infty}^{+\infty} du\, e^{-itu} e(u) = \frac{2\sqrt{2}\pi}{gt} e^{-\frac{|k|}{2}} J_1(\sqrt{2}gt). \tag{3.36}$$

Now we perform the limit for $k \to 0$ of the right side of(3.32). The first two terms of the left side are trivially equal to zero. Then, from the (3.28) we can compute the following limit:

$$\lim_{k\to 0} \frac{ik}{1-e^{-|k|}} \hat{\phi}_H(k,t) = -\pi g^2 \hat{e}(t) \tag{3.37}$$

and with this result we obtain:

$$\lim_{k\to 0}\left[\hat{\sigma}_H(k) + \frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'_H(k)\right] = \pi\gamma(g,L,s)\,. \tag{3.38}$$

Now we finally define the ausiliary function $S(k)$ as follows:

$$S(k) = \frac{\sinh\frac{|k|}{2}}{\pi|k|}\left\{\hat{\sigma}_H(k) + \frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'_H(k) - \frac{ik\hat{\phi}_0(k)}{1-e^{-|k|}} \sum_{h=1}^{L}\left[e^{ikx_h} - e^{ik\bar{x}_h}\right]\right\}. \tag{3.39}$$

It is evident, using (3.38) that the connection between $S(k)$ and the anomalous dimension is:

$$\lim_{k\to 0} 2S(k) = \gamma(g,L,s)\,. \tag{3.40}$$

Again, using (3.39) and the equation for one-loop density in Fourier space (3.30) into (3.32), we have a NLIE for $S(k)$ that reads as follows:

$$S(k) = \frac{L}{|k|}[1 - J_0(\sqrt{2}gk)] + \frac{ike^{\frac{|k|}{2}}}{2\pi|k|}\int_{-\infty}^{+\infty}\frac{dt}{4\pi^2}\hat{\phi}_H(k,t)\left[\frac{\pi|t|}{\sinh\frac{|t|}{2}}S(t) - \right.$$
$$\left. - \frac{2\hat{L}'(t)}{1-e^{-|t|}} - \frac{2\pi Le^{\frac{-|t|}{2}}}{1-e^{-|t|}} + \frac{2\pi}{1-e^{-|t|}}\sum_{h=1}^{L}e^{itx_h}\right]. \tag{3.41}$$

We can make another semplification for the previous equation introducing the *magic Kernel* [5] $\hat{K}(t,t')$:

$$\hat{K}(t,t') = \frac{2}{tt'}\left[\sum_{n=1}^{\infty}nJ_n(t)J_n(t') + 2\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}c_{2k+1,2l+2}(g)J_{2k}(t)J_{2l+1}(t')\right] \tag{3.42}$$

that has the property:

$$\int_{-\infty}^{+\infty}dt\,\hat{\phi}_H(k,t)\,f(t) = 8i\pi^2g^2\int_0^{+\infty}dt\,e^{-\frac{t+k}{2}}\,\hat{K}(\sqrt{2}gk,\sqrt{2}gt)f(t) \tag{3.43}$$

with $f(t)$ even function. Using (3.42) and (3.43) we can rewrite (3.41), restricting to $k \geq 0$, as follows:

$$S(k) = \frac{L}{k}[1 - J_0(\sqrt{2}gk)] - g^2\int_0^{+\infty}\frac{dt}{\pi^2}e^{\frac{-t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\left[\frac{\pi t}{\sinh\frac{t}{2}}S(t) - \right.$$
$$\left. - \frac{2\hat{L}'(t)}{1-e^{-t}} - \frac{2\pi Le^{\frac{-t}{2}}}{1-e^{-t}} + \frac{2\pi}{1-e^{-t}}\sum_{h=1}^{L}e^{itx_h}\right]. \tag{3.44}$$

3.3 High spin limit

The non-linear integral equation (3.44), together with relation (3.40), is the central point of our computations. Solving this NLIE we can obtain directly informations about anomalous dimension. Observing (3.44), we immediatly note that the solution depends on the positions of the holes x_h on the real axis. These positions are undeterminated and we need to evaluate them using the property that they are 'fake' solutions of Bethe equations, i.e. counting function in the holes positions satisfies (1.67), with a certain set of quantum numbers I_j which depend on the state we are analyzing. For simplicity, we will study now the minimal anomalous dimension state. For this state, positions of the internal holes satisfy the relation:

$$Z(x_h) = \pi(2h + 1 - L)\,,\; h = 1, \cdots, L-2 \tag{3.45}$$

while positions of external holes satisfy

$$Z(x_{L-1}) = -Z(x_L) = \pi(s + L - 1)\,. \tag{3.46}$$

A part of positions of holes, the other non-trivial term in the equation (3.44) is the term that contains the non linear function $\hat{L}'(h)$. If we work in the high spin limit, i.e. for $s \to \infty$, we are able to evaluate the non linear term of the equation and the position of internal holes exactly, discarding terms of order $O(s^{-2})$.

3.3.1 High spin limit of the non-linear term

Let start evaluating the high spin limit of the nonlinear terms appearing in our integral equations. In one loop equation (3.30) nonlinearity appears through the term

$$-\frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'_0(k)\,, \tag{3.47}$$

while, in 'all loops' equation (3.44), it appears in the following integral:

$$NL = g^2\int_0^{+\infty}\frac{dt}{\pi}e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{2it}{1-e^{-t}}\hat{L}(t)\;. \tag{3.48}$$

It is convenient to pass to the coordinates space and to define

$$\begin{aligned} I^\alpha(u) &= \int_{-\infty}^{+\infty}\frac{dt}{2\pi}e^{itu}\frac{-2e^{-\alpha|t|}}{1-e^{-|t|}}\hat{L}'(t) \\ &= \int_{-\infty}^{+\infty}\frac{dv}{i\pi}\left[\psi'(\alpha-iu+iv)-\psi'(\alpha+iu-iv)\right]L(v)\;. \end{aligned} \tag{3.49}$$

The case $\alpha = 1$ (specialised at one loop) is the Fourier antitransform of (3.47). The case $\alpha = \frac{1}{2}$ is connected to the all loops case (3.48). In all cases, following Appendix A of [29], we can restrict the integration to the region where no roots nor holes are present: what we miss are terms $O\left(\frac{1}{s^2}\right)$:

$$I^\alpha(u) = \int_{|v|>\frac{s}{\sqrt{2}}}\frac{dv}{i\pi}\left[\psi'(\alpha-iu+iv)-\psi'(\alpha+iu-iv)\right]L(v)+O\left(\frac{1}{s^2}\right)\;. \tag{3.50}$$

Developing the ψ functions for large v we get

$$I^\alpha(u) = \int_{|v|>\frac{s}{\sqrt{2}}}\frac{dv}{i\pi}\left[\psi'(iv)-\psi'(-iv)+O\left(\frac{1}{v^3}\right)\right]L(v)+O\left(\frac{1}{s^2}\right)\;. \tag{3.51}$$

From the definition (3.23), using (3.20), we have for large v that:

$$L(v) = \frac{L+2s+\gamma(g,L,s)}{2v}+\mathcal{O}\left(\frac{1}{v^3}\right)\,, \tag{3.52}$$

so we can also write

$$I^\alpha(u) = \int_{|v|>\frac{s}{\sqrt{2}}}\frac{dv}{i\pi}\left[\psi'(iv)-\psi'(-iv)\right]L(v)+O\left(\frac{1}{s^2}\right)\;. \tag{3.53}$$

The integral contained in such formula coincides with minus the derivative of the function $\mathcal{I}(u)$, evaluated at high spin in Appendix A of [29], the only difference being that in our case the counting function contains all loops contributions. However, in the computation of [29] only the asymptotic behaviour of $L(v)$, (3.52), is relevant. This means that in the high spin limit the only difference between the one loop and the all loops case is that the spin s should be replaced by $s+O\left(\frac{\ln s}{s}\right)$. This allows us to easily follow all the steps of [29], finding eventually

$$I^\alpha(u) = -2\ln 2+O\left(\frac{u^2}{s^2}\right)\;. \tag{3.54}$$

Therefore, in the region around the origin the non linear term is under control. The term $I^{\alpha=1}(u)$ appears in the expression (3.30) of the one loop density, which, consequently, is estimated using (3.54) only sufficiently close to the origin. $I^{\alpha=\frac{1}{2}}(u)$ appears through (3.48) in the equation (3.44) for $S(k)$: since in the coordinate space the kernel $K(u,v)$ is depressed for v large, we can safely plug (3.54) into (3.48) and find

$$NL = 2g^2 \ln 2\hat{K}(\sqrt{2}gk, 0) + O\left(\frac{1}{s^2}\right) . \tag{3.55}$$

3.3.2 Position of external holes

When the spin is large, Bethe roots near the two 'extrema' $\pm b$ scale with s. In the proximity of $\pm b$, it is therefore convenient to rescale the variable u of the counting function $Z(u)$: we will write $u = \bar{u}s$, where $\bar{u}$ will stay finite. From the definitions (3.14), (3.16), (3.18) of the counting function, we have

$$Z(\bar{u}s) = -2L \arctan 2\bar{u}s - 2\sum_{k=1}^{s} \arctan(\bar{u}s - \bar{u}_k s) + \frac{\gamma(g,L,s)}{\bar{u}s} + O\left(\frac{1}{s^2}\right) . \tag{3.56}$$

We observe that the only 'higher loops' effect is in the last term, proportional to the anomalous dimension.

For $\bar{u} = \bar{u}_l$, where $\bar{u}_l s$ is a Bethe root, we expand the various functions for large s and evaluate the sum over the Bethe roots contained in (3.56) as an integral term plus an 'anomaly' [34, 35]. We get

$$\begin{aligned} Z(\bar{u}_l s) = & -\pi L \epsilon(\bar{u}_l) + \frac{\gamma(g,L,s)+L}{\bar{u}_l s} + 2\int_{-\bar{b}}^{\bar{b}} d\bar{v}\rho(\bar{v}) P\frac{1}{\bar{u}_l - \bar{v}} + \\ & + \frac{\pi}{s}\rho'(\bar{u}_l)\coth \pi\rho(\bar{u}_l) - -\pi \sum_{\substack{k=1 \\ k\neq l}}^{s} \epsilon(\bar{u}_l - \bar{u}_k) - \\ & - \pi \sum_{h=1}^{L-2} \epsilon(\bar{u}_l - \bar{x}_h) + 2(L-2)\left[\frac{\pi}{2}\epsilon(\bar{u}_l) - \frac{1}{\bar{u}_l s}\right] + O\left(\frac{1}{s^2}\right) , \end{aligned} \tag{3.57}$$

where $\bar{x}_h = \frac{x_h}{s}$,

$$\rho(\bar{u}) = -\frac{1}{2\pi s}\frac{d}{d\bar{u}} Z(\bar{u}s) \tag{3.58}$$

and where we used the relation [34, 35]

$$\begin{aligned} & -2\sum_{k=1}^{s} \arctan(u_l - u_k) - 2\sum_{h=1}^{L-2} \arctan(u_l - x_h) + \pi \sum_{\substack{k=1 \\ k\neq l}}^{s} \epsilon(\bar{u}_l - \bar{u}_k) + \\ & + \pi \sum_{h=1}^{L-2} \epsilon(\bar{u}_l - \bar{x}_h) = \\ & = \frac{1}{i}\sum_{\substack{k=1 \\ k\neq l}}^{s} \ln \frac{u_l - u_k + i}{u_l - u_k - i} + \frac{1}{i}\sum_{h=1}^{L-2} \ln \frac{u_l - x_h + i}{u_l - x_h - i} = \\ & = 2\int_{-\bar{b}}^{\bar{b}} d\bar{v}\rho(\bar{v}) P\frac{1}{\bar{u}_l - \bar{v}} + \frac{\pi}{s}\rho'(\bar{u}_l)\coth \pi\rho(\bar{u}_l) + O\left(\frac{1}{s^2}\right) . \end{aligned} \tag{3.59}$$

We can make the identification

$$Z(\bar{u}_l s) = -\pi \sum_{\substack{k=1 \\ k\neq l}}^{s} \epsilon(\bar{u}_l - \bar{u}_k) - \pi \sum_{h=1}^{L-2} \epsilon(\bar{u}_l - \bar{x}_h) \tag{3.60}$$

which allows to simplify equation (3.57) as follows

$$\begin{aligned} 0 = & -2\pi\epsilon(\bar{u}_l) + \frac{4 - L + \gamma(g,L,s)}{\bar{u}_l s} + 2\int_{-\bar{b}}^{\bar{b}} d\bar{v}\rho(\bar{v}) P\frac{1}{\bar{u}_l - \bar{v}} + \\ & + \frac{\pi}{s}\rho'(\bar{u}_l)\coth\pi\rho(\bar{u}_l) + O\left(\frac{1}{s^2}\right) . \end{aligned} \tag{3.61}$$

At the leading order, $O(s^0)$, we know that the equation to be satisfied, for all $\bar{u}$, is:

$$0 = -2\pi\epsilon(\bar{u}) + 2\int_{-\bar{b}}^{\bar{b}} d\bar{v}\rho(\bar{v}) P\frac{1}{\bar{u} - \bar{v}} \tag{3.62}$$

whose solution is the well known Korchemsky [36] density

$$\rho(\bar{u}) = \frac{1}{\pi}\ln\left(\frac{\bar{b} + \sqrt{\bar{b}^2 - \bar{u}^2}}{\bar{u}}\right)^2 . \tag{3.63}$$

Using (3.63), we give an estimate of the last term in (3.61),

$$\frac{\pi}{s}\rho'(\bar{u}_l)\coth\pi\rho(\bar{u}_l) = \frac{1}{s}\left[\frac{1}{2\bar{b} + 2\bar{u}_l} - \frac{1}{2\bar{b} - 2\bar{u}_l} - \frac{2}{\bar{u}_l}\right] + O\left(\frac{1}{s^2}\right) , \tag{3.64}$$

which allows to find the function $\rho(\bar{u})$ which satisfies (3.61):

$$\begin{aligned} \rho(\bar{u}) = & \frac{1}{\pi}\ln\left(\frac{\bar{b} + \sqrt{\bar{b}^2 - \bar{u}^2}}{\bar{u}}\right)^2 - \frac{(2 + \gamma(g,L,s) - L)\delta(\bar{u})}{2s} - \\ & - \frac{\delta(\bar{u} + \bar{b}) + \delta(\bar{u} - \bar{b})}{2s} + O\left(\frac{1}{s^2}\right) . \end{aligned} \tag{3.65}$$

Using the form (3.65) of the solution, we can determine the position of the extremum $\bar{b}$ through the relation

$$\int_{-\bar{b}}^{\bar{b}} d\bar{u}\rho(\bar{u}) = 1 + \frac{L-2}{s} , \tag{3.66}$$

which gives

$$\bar{b} = \frac{1}{2}\left(1 + \frac{L - 1 + \gamma(g,L,s)}{2s}\right) + O\left(\frac{1}{s^2}\right) . \tag{3.67}$$

We now pass to determine the position $x_L = \bar{x}_L s$, $\bar{x}_L > \bar{b}$, of the positive external hole. We first compute equation (3.56) for $\bar{u} = \bar{x}_L$:

$$Z(\bar{x}_L s) = -2L\arctan 2\bar{x}_L s - 2\sum_{k=1}^{s}\arctan(\bar{x}_L s - \bar{u}_k s) + \frac{\gamma(g,L,s)}{\bar{x}_L s} + O\left(\frac{1}{s^2}\right) . \tag{3.68}$$

As before, we develop this equation for large s, use (3.59) and simplify the term containing coth using (3.64). We get the equation

$$
\begin{aligned}
Z(\bar{x}_L s) = & -(L+s)\pi + \frac{\gamma(g,L,s)-L+2}{\bar{x}_L s} + \frac{1}{2s}\frac{1}{\bar{b}+\bar{x}_L} - \frac{1}{2s}\frac{1}{\bar{b}-\bar{x}_L} + \\
& + 2\int_{-\bar{b}}^{\bar{b}} d\bar{v}\rho(\bar{v})\frac{1}{\bar{x}_L-\bar{v}} + O\left(\frac{1}{s^2}\right) .
\end{aligned} \tag{3.69}
$$

Inserting (3.65) into the last term of this equation and using the result

$$
\int_{-\bar{b}}^{\bar{b}} \frac{d\bar{v}}{\bar{x}_L-\bar{v}} \ln\left(\frac{\bar{b}+\sqrt{\bar{b}^2-\bar{v}^2}}{\bar{v}}\right)^2 = i\pi \ln\frac{\sqrt{\bar{b}^2-(\bar{x}_L)^2}-\bar{b}}{\sqrt{\bar{b}^2-(\bar{x}_L)^2}+\bar{b}} , \tag{3.70}
$$

we get

$$
Z(\bar{x}_L s) = -(L+s)\pi + 2i\ln\frac{\sqrt{\bar{b}^2-(\bar{x}_L)^2}-\bar{b}}{\sqrt{\bar{b}^2-(\bar{x}_L)^2}+\bar{b}} . \tag{3.71}
$$

Since $Z(\bar{x}_L s) = -(L+s)\pi + \pi$, we get

$$
\bar{x}_L = \sqrt{2}\bar{b} \Rightarrow \bar{x}_L = \frac{1}{\sqrt{2}}\left(1+\frac{L-1+\gamma(g,L,s)}{2s}\right) + O\left(\frac{1}{s^2}\right) \tag{3.72}
$$

and finally

$$
x_L = -x_{L-1} = \frac{s}{\sqrt{2}}\left(1+\frac{L-1+\gamma(g,L,s)}{2s} + O\left(\frac{1}{s^2}\right)\right) \tag{3.73}
$$

3.3.3 High spin NLIE for $S(k)$ and high spin expansions

Now, plugging (3.55) and (3.73) into (3.44), after some calculations, we obtain the high spin limit NLIE for $S(k)$ in the following form in which we are discarding terms of order $O\left(s^{-1}(\ln s)^{-\infty}\right)$:

$$
\begin{aligned}
S(k) = & \, 4g^2 \ln s \hat{K}(\sqrt{2}gk,0) + 4g^2\int_0^{+\infty}\frac{dt}{e^t-1}\hat{K}^*(\sqrt{2}gk,\sqrt{2}gt) + \frac{L}{k}[1-J_0(\sqrt{2}gk)] + \\
& + 4g^2\gamma_E\hat{K}(\sqrt{2}gk,0) + g^2(L-2)\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{1-e^{\frac{t}{2}}}{\sinh\frac{t}{2}} - \\
& - g^2\int_0^{+\infty} dt\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{\sum_{h=1}^{L-2}[\cos tx_h - 1]}{\sinh\frac{t}{2}} + \\
& + \frac{2g^2}{s}(L+\gamma(g,L,s)-1)\hat{K}(\sqrt{2}gk,0) - g^2\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{t}{\sinh\frac{t}{2}}S(t) + \\
& + O\left(s^{-1}(\ln s)^{-\infty}\right) ,
\end{aligned} \tag{3.74}
$$

where $\hat{K}^*(t,t') = \hat{K}(t,t') - \hat{K}(t,0)$.

Let analyze now equation (3.74): the first two lines contain driving terms of order $\ln s$ and $(\ln s)^0$. The term on the third line has the sum on internal holes. From the relation (3.45) we note that the leading order in s for internal holes should be $(\ln s)^{-1}$ so the sum starts to contribute with terms of order at least $(\ln s)^{-2}$. The first term on the fourth line has a recursive element due to the presence of $\gamma(g,L,s)$ that has the same s dependence of $S(k)$ due to the relation (3.40). So, these considerations suggest directly that $S(k)$ expands as follows:

$$S(k) = \sum_{n=-1}^{\infty} \frac{S^{(n)}(k)}{(\ln s)^n} + \sum_{n=-1}^{\infty} \frac{\check{S}^{(n)}(k)}{s(\ln s)^n} + O\left(s^{-1}(\ln s)^{-\infty}\right) \tag{3.75}$$

and subsequently

$$\gamma(g,L,s) = f(g)\ln s + f_{sl}(g,L) + \sum_{n=1}^{\infty} \gamma^{(n)}(g,L)(\ln s)^{-n} + \\ + \frac{1}{s}\sum_{n=-1}^{\infty} \check{\gamma}^{(n)}(g,L)(\ln s)^{-n} + O\left(s^{-1}(\ln s)^{-\infty}\right) . \tag{3.76}$$

We anticipated this result (eq. (3.2)) in the introduction of this chapter .

3.3.4 High spin first results: $\check{S}^{(-1)}(k)$ and $\check{S}^{(0)}(k)$

We have already spoken about the well known scaling functions $f(g)$ and $f_{sl}(g,L)$. We can quickly show now that, thanks to the equation (3.74), it is very simple to obtain coefficients $\check{S}^{(-1)}(k)$ and $\check{S}^{(0)}(k)$ of the expansion (3.75). Let us start writing the NLIE for the leading order term $S^{(-1)}(k)$:

$$S^{(-1)}(k) = 4g^2\hat{K}(\sqrt{2}gk,0) - g^2\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{t}{\sinh\frac{t}{2}}S^{(-1)}(t) \tag{3.77}$$

from which we have, using (3.40),

$$f(g) = 2S^{(-1)}(0) . \tag{3.78}$$

Next, let write NLIEs for the terms $\check{S}^{(-1)}(k)$ and $\check{S}^{(0)}(k)$:

$$\check{S}^{(-1)}(k) = 2g^2 f(g)\hat{K}(\sqrt{2}gk,0) - g^2\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{t}{\sinh\frac{t}{2}}\check{S}^{(-1)}(t) \tag{3.79}$$

$$\check{S}^{(0)}(k) = 2g^2\left(L-1+f_{sl}(g,L)\right)\hat{K}(\sqrt{2}gk,0) - g^2\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{t}{\sinh\frac{t}{2}}\check{S}^{(0)}(t) . \tag{3.80}$$

Comparing (3.79) and (3.80) with (3.77) we immediatly have:

$$\check{S}^{(-1)}(k) = \frac{f(g)}{2}S^{(-1)}(k) , \quad \check{S}^{(0)}(k) = \frac{f_{sl}(g,L)+L-1}{2}S^{(-1)}(k) , \tag{3.81}$$

which translate in terms of anomalous dimensions to the equalities [31, 32]

$$\check{\gamma}^{(-1)}(g,L) = \frac{1}{2}[f(g)]^2 , \quad \check{\gamma}^{(0)}(g,L) = \frac{1}{2}f(g)[L-1+f_{sl}(g,L)] . \tag{3.82}$$

3.4 High spin results: $\gamma^{(n)}(g,L)$ and $\check{\gamma}^{(n)}(g,L)$

Now we focus on the computations of coefficients of expansion of the anomalous dimension at orders $(\ln s)^{-n}$ and $s^{(-1)}(\ln s)^{-n}$, with $n \geq 1$. A central role in this computation is played by internal holes. Equation (3.74), together with condition (3.45), suggests this expansion for position of internal holes:

$$x_h = \sum_{n=1}^{\infty} \left(\alpha_{n,h} + \frac{\breve{\alpha}_{n,h}}{s} \right) (\ln s)^{(-n)} + O\left(s^{-1} (\ln s)^{-\infty}\right) . \tag{3.83}$$

The first step of the computation is determining coefficients $\alpha_{n,h}$ and $\breve{\alpha}_{n,h}$. It is possible to obtain them from the relation (3.45), using a mathematical trick known as *Faà di Bruno.*

Faà di Bruno

Suppose to have two functions f and g that expands in seres of powers of the same variable x:

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n}{n!} x^n \quad g(x) = \sum_{n=1}^{\infty} \frac{g_n}{n!} x^n . \tag{3.84}$$

Now, we want to evaluate the composite function $g\left(f(x)\right)$:

$$g\left(f(x)\right) = \sum_{n=1}^{\infty} \frac{g_n}{n!} \left(\sum_{m=1}^{\infty} \frac{f_m}{m!} x^m \right)^n . \tag{3.85}$$

It is natural to write the composite function also as a single power series in x:

$$g\left(f(x)\right) = \sum_{n=1}^{\infty} \frac{h_n}{n!} x^n . \tag{3.86}$$

Faà di Bruno permits to relate coefficients h_n of the (3.86) with the coefficients f_n and g_n of the power series (3.84). It is simple to verify that:

$$h_n = n! \sum_{k=1}^{n} g_k \sum_{\{j_1, \cdots, j_{n-k+1}\}} \prod_{m=1}^{n-k+1} \frac{(f_m)^{j_m}}{j_m! (m!)^{j_m}} \tag{3.87}$$

where the notation $\sum_{\{j_1,\cdots,j_{n-k+1}\}}$ means that we can do the product summing contributes for all set of integers $j_1, \cdots, j_m$ costrained by the conditions:

$$\sum_{m=1}^{n-k+1} j_m = k \quad , \quad \sum_{m=1}^{n-k+1} m j_m = n . \tag{3.88}$$

3.4.1 Determination of $\alpha_{n,h}$ and $\breve{\alpha}_{n,h}$

We can use Faà di Bruno for extracting expressions of coefficients $\alpha_{n,h}$ and $\breve{\alpha}_{n,h}$ from the relation (3.45)

$$Z(x_h) = \pi(2h + 1 - L) . \tag{3.89}$$

In order to apply Faà di Bruno, we have to expand in Taylor series the function $Z(u)$:

$$Z(u) = \sum_{n=1}^{\infty} \frac{d^n Z(u)}{du^n} \Big|_{u=0} \frac{u^n}{n!} = \sum_{n=1}^{\infty} \frac{d^{n-1} \sigma(u)}{du^{n-1}} \Big|_{u=0} \frac{u^n}{n!} \tag{3.90}$$

where the first term of the series, $Z(0)$, is zero because Z is an odd function. Trivially, function $\sigma(u)$ has the same high limit expansion of function $S(k)$, so we introduce the following notation:

$$\frac{d^r}{du^r} \sigma(u = 0) = \sum_{n=-1}^{\infty} \Sigma_r^{(n)} (\ln s)^{-n} + O\left(s^{-1} (\ln s)^{-\infty}\right) , \tag{3.91}$$

where

$$\Sigma_r^{(n)} = \sigma_r^{(n)} + \frac{\breve{\sigma}_r^{(n)}}{s}\,. \tag{3.92}$$

Write in a similar way the expansion of the positions of internal holes:

$$x_h = \sum_{n=1}^{\infty} A_{n,h} (\ln s)^{-n} + O\big(s^{-1} (\ln s)^{-\infty}\big)\,,\; A_{n,h} = \alpha_{n,h} + \frac{\breve{\alpha}_{n,h}}{s}\,. \tag{3.93}$$

Using (3.90), (3.91) and (3.93) we can immediatly write:

$$Z(x_h) = \sum_{n=1}^{\infty} \left(\sum_{l=-1}^{\infty} \Sigma_{n-1}^{(l)} (\ln s)^{-l} \right) \frac{1}{n!} \left(\sum_{m=1}^{\infty} A_{m,h} (\ln s)^{-m} \right)^n + O\big(s^{-1} (\ln s)^{-\infty}\big)\,. \tag{3.94}$$

Now we can apply Faà di Bruno to (3.94) with

$$g_n = \sum_{l=-1}^{\infty} \Sigma_{n-1}^{(l)} (\ln s)^{-l}\,,\; f_m = m! A_{m,h} \tag{3.95}$$

and obtain

$$Z(x_h) = \sum_{n=1}^{\infty} \sum_{l=-1}^{\infty} \sum_{k=1}^{n} (\ln s)^{-(n+l)} \Sigma_{k-1}^{(l)} \sum_{\{j_1,\cdots,j_n-k+1\}} \prod_{m=1}^{n-k+1} \frac{(A_{m,h})^{j_m}}{j_m!} \tag{3.96}$$

where the sets $j_1, \cdots, j_n$ are costrained by conditions (3.88). Now, comparing (3.89) and (3.96), working on the formula, we obtain the following equation for the $A_{n,h}$ coefficients:

$$\pi(2h+1-L) = \Sigma_0^{(-1)} A_{1,h} + \sum_{p=1}^{\infty} (\ln s)^{-p} \sum_{r=1}^{p+1} \Sigma_{r-1}^{(-1)} \sum_{\{j_1,\ldots,j_{p-r+2}\}} \prod_{m=1}^{p-r+2} \frac{(A_{m,h})^{j_m}}{j_m!} +$$

$$+ \sum_{p=1}^{\infty} (\ln s)^{-p} \sum_{l=0}^{p-1} \sum_{r=1}^{p-l} \Sigma_{r-1}^{(l)} \sum_{\{j_1,\ldots,j_{p-r-l+1}\}} \prod_{m=1}^{p-r-l+1} \frac{(A_{m,h})^{j_m}}{j_m!} \tag{3.97}$$

where the j_m contained in the second term of the r.h.s. are constrained by the conditions $\sum_{m=1}^{p-r+2} j_m = r$, $\sum_{m=1}^{p-r+2} m j_m = p+1$, the ones in the third term by $\sum_{m=1}^{p-r-l+1} j_m = r$, $\sum_{m=1}^{p-r-l+1} m j_m = p-l$. Equating l.h.s. and r.h.s. at all orders in $\ln s$ we obtain the following recursive equation

$$\begin{aligned} A_{p+1,h} &= -\sum_{r=1}^{p} \frac{\Sigma_r^{(-1)}}{\Sigma_0^{(-1)}} \sum_{\{j_1,\ldots,j_{p-r+1}\}} \prod_{m=1}^{p-r+1} \frac{(A_{m,h})^{j_m}}{j_m!} - \\ &\quad - \sum_{l=0}^{p-1} \sum_{r=1}^{p-l} \frac{\Sigma_{r-1}^{(l)}}{\Sigma_0^{(-1)}} \sum_{\{j_1,\ldots,j_{p-r-l+1}\}} \prod_{m=1}^{p-r-l+1} \frac{(A_{m,h})^{j_m}}{j_m!}, \quad p \geq 1 \\ A_{1,h} &= \frac{\pi(2h-1+L)}{\Sigma_0^{(-1)}}\,, \end{aligned} \tag{3.98}$$

where now the j_m contained in the first term of the r.h.s. are constrained by the conditions $\sum_{m=1}^{p-r+1} j_m = r+1$, $\sum_{m=1}^{p-r+1} m j_m = p+1$ and the ones in the second term by $\sum_{m=1}^{p-r-l+1} j_m = r$, $\sum_{m=1}^{p-r-l+1} m j_m = p-l$.

Now, we proceed restoring explicit expression for $\Sigma_r^{(n)}$ and $A_{m,h}$ (3.92,3.93). We can disentagle equations (3.98) developing for $s \to \infty$ and equating terms at the same order in powers of s (discarding terms of order $O(s^{-2})$. We obtain:

$$\begin{aligned}
\alpha_{p+1,h} &= -\sum_{r=1}^{p} \frac{\sigma_r^{(-1)}}{\sigma_0^{(-1)}} \sum_{\{j_1,\dots,j_{p-r+1}\}} \prod_{m=1}^{p-r+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!} - \\
&\quad - \sum_{l=0}^{p-1}\sum_{r=1}^{p-l} \frac{\sigma_{r-1}^{(l)}}{\sigma_0^{(-1)}} \sum_{\{j_1,\dots,j_{p-r-l+1}\}} \prod_{m=1}^{p-r-l+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!}, \quad p \geq 1 \\
\alpha_{1,h} &= \frac{\pi(2h-1+L)}{\sigma_0^{(-1)}}
\end{aligned} \tag{3.99}$$

and

$$\begin{aligned}
\check{\alpha}_{p+1,h} &= -\sum_{r=1}^{p} \left[\frac{\sigma_r^{(-1)}}{\sigma_0^{(-1)}} \sum_{m'=1}^{p-r+1} \frac{j_{m'}(\check{\alpha}_{m',h})}{(\alpha_{m',h})} + \frac{\check{\sigma}_r^{(-1)}}{\sigma_0^{(-1)}} - \frac{\sigma_r^{(-1)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^2} \right] \times \\
&\quad \times \sum_{\{j_1,\dots,j_{p-r+1}\}} \prod_{m=1}^{p-r+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!} - \\
&\quad - \sum_{l=0}^{p-1}\sum_{r=1}^{p-l} \left[\frac{\sigma_{r-1}^{(l)}}{\sigma_0^{(-1)}} \sum_{m'=1}^{p-r-l+1} \frac{j_{m'}(\check{\alpha}_{m',h})}{(\alpha_{m',h})} + \frac{\check{\sigma}_{r-1}^{(l)}}{\sigma_0^{(-1)}} - \frac{\sigma_{r-1}^{(l)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^2} \right] \times \\
&\quad \times \sum_{\{j_1,\dots,j_{p-r-l+1}\}} \prod_{m=1}^{p-r-l+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!} \quad p \geq 1 \\
\check{\alpha}_{1,h} &= \frac{-\pi\check{\sigma}_0^{(-1)}(2h+1-L)}{(\sigma_0^{(-1)})^2} .
\end{aligned} \tag{3.100}$$

In both sets of equations, coefficients j_m are costrained in the same way of those appearing in (3.98). These formulae are recursive, we show now for example the first coefficients of both sets:

$$\begin{aligned}
\alpha_{1,h} &= \frac{\pi(2h+1-L)}{\sigma_0^{(-1)}}, \quad \alpha_{2,h} = -\frac{\pi(2h+1-L)\sigma_0^{(0)}}{(\sigma_0^{(-1)})^2} \\
\alpha_{3,h} &= \frac{\pi(2h+1-L)}{\sigma_0^{(-1)}} \left(-\frac{\sigma_0^{(1)}}{\sigma_0^{(-1)}} + \left(\frac{\sigma_0^{(0)}}{\sigma_0^{(-1)}}\right)^2 - \frac{\pi^2}{6}(2h+1-L)^2 \frac{\sigma_2^{(-1)}}{(\sigma_0^{(-1)})^3} \right) \\
\alpha_{4,h} &= -\frac{\pi(2h+1-L)}{\left(\sigma_0^{(-1)}\right)^2} \left(\frac{\left(\sigma_0^{(0)}\right)^3}{\left(\sigma_0^{(-1)}\right)^2} + \sigma_0^{(2)} \right) + \\
&\quad + \frac{\pi^3(2h+1-L)^3}{\left(\sigma_0^{(-1)}\right)^4} \left(\frac{2\sigma_2^{(-1)}\sigma_0^{(0)}}{3\sigma_0^{(-1)}} - \frac{\sigma_2^{(0)}}{6} \right),
\end{aligned} \tag{3.101}$$

$$
\begin{aligned}
\check{\alpha}_{1,h} &= -\frac{\pi(2h+1-L)\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^2} \\
\check{\alpha}_{2,h} &= -\frac{\pi(2h+1-L)\check{\sigma}_0^{(0)}}{(\sigma_0^{(-1)})^2} + \frac{2\pi(2h+1-L)\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^3} \\
\check{\alpha}_{3,h} &= \pi(2h+1-L)\left[\frac{2\sigma_0^{(0)}\check{\sigma}_0^{(0)}}{(\sigma_0^{(-1)})^3} - \frac{3(\sigma_0^{(0)})^2\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^4}\right] - \\
&- \frac{\pi^3(2h+1-L)^3}{6}\left[\frac{\check{\sigma}_2^{(-1)}}{(\sigma_0^{(-1)})^4} - \frac{4\sigma_0^{(-1)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^5}\right] \qquad (3.102) \\
\check{\alpha}_{4,h} &= \pi(2h+1-L)\left[\frac{4(\sigma_0^{(0)})^3\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^5} - \frac{3(\sigma_0^{(0)})^2\check{\sigma}_0^{(0)}}{(\sigma_0^{(-1)})^4} + \frac{2\sigma_0^{(2)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^3} - \frac{\check{\sigma}_0^{(2)}}{(\sigma_0^{(-1)})^2}\right] + \\
&+ \frac{2\pi^3(2h+1-L)^3}{3}\left[-\frac{5\sigma_2^{(-1)}\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^6} + \frac{\sigma_0^{(0)}\check{\sigma}_2^{(-1)}}{(\sigma_0^{(-1)})^5} + \frac{\sigma_2^{(-1)}\check{\sigma}_0^{(0)}}{(\sigma_0^{(-1)})^5} + \right. \\
&+ \left.\frac{\sigma_2^{(0)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^5} - \frac{\check{\sigma}_2^{(0)}}{4(\sigma_0^{(-1)})^4}\right] .
\end{aligned}
$$

The expressions found are still quite involved. Substantially, we have 'inverted' relation (3.89), so now we express holes expansion coefficients in terms of expansions coefficients of the derivatives of the counting function. It is sufficient (we will see) for the main purpose of this work, but not in general: to obtain complete solutions, one should solve the NLIE for $Z(u)$. At weak coupling it is very easy. We will come back on this point later.

3.4.2 Sum on internal holes

In (3.74) we see that driving terms for orders $(\ln s)^{-n}$ and $s^{-1}(\ln s)^{-n}$, with $n \geq 1$, contain the sum on internal holes that reads as follows:

$$P(s,g,t) = \sum_{h=1}^{L-2}[\cos tx_h^{(i)} - 1] \, . \qquad (3.103)$$

Obviously, function $P(s,g,t)$ expands as:

$$P(s,g,t) = \sum_{n=1}^{\infty}\left(P^{(n)}(g,t) + \frac{\check{P}^{(n)}(g,t)}{s}\right)(\ln s)^{-n} + O\big(s^{-1}(\ln s)^{-\infty}\big) \, . \qquad (3.104)$$

It is possible to obtain coefficients $P^{(n)}(g,t)$ and $\check{P}^{(n)}(g,t)$ using again the Faà di Bruno. First, write (3.103) as follows:

$$
\begin{aligned}
P(s,g,t) &= \sum_{h=1}^{L-2}[\cos tx_h^{(i)} - 1] = \sum_{h=1}^{L-2}\sum_{n=1}^{\infty}\cos\Big(\frac{\pi n}{2}\Big)\frac{t^n}{n!}(x_h^{(i)})^n = \\
&= \sum_{h=1}^{L-2}\sum_{n=1}^{\infty}\cos\Big(\frac{\pi n}{2}\Big)\frac{t^n}{n!}\left[\sum_{m=1}^{\infty}\left(\alpha_{n,h} + \frac{\check{\alpha}_{n,h}}{s}\right)(\ln s)^{-m}\right]^n . \qquad (3.105)
\end{aligned}
$$

Comparing (3.104) and (3.105) we can use Faà di Bruno with

$$g_n = \cos\left(\frac{\pi n}{2}\right)t^n \quad , \quad f_m = m!\left(\alpha_{n,h} + \frac{\check{\alpha}_{n,h}}{s}\right) \qquad (3.106)$$

obtaining:

$$P^{(n)}(g,t) + \frac{\check{P}^{(n)}(g,t)}{s} = \sum_{r=1} t^r \cos\left(\frac{\pi r}{2}\right) \sum_{\{j_1,\cdots,j_{n-r+1}\}} \sum_{h=1}^{L-2} \prod_{m=1}^{n-r+1} \frac{\left(\alpha_{m,h} + \frac{\check{\alpha}_{m,h}}{s}\right)^{j_m}}{j_m!}, \tag{3.107}$$

where set $\{j_1, \cdots, j_{n-r+1}\}$ is costrained by (3.88). We can easily disentagle orders s^0 and s^{-1} remembering that we are in the high spin limit and developing on the right side. We have:

$$P^{(n)}(g,t) = \sum_{r=1} t^r \cos\left(\frac{\pi r}{2}\right) \sum_{\{j_1,\cdots,j_{n-r+1}\}} \sum_{h=1}^{L-2} \prod_{m=1}^{n-r+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!} \tag{3.108}$$

$$\check{P}^{(n)}(g,t) = \sum_{r=1} t^r \cos\left(\frac{\pi r}{2}\right) \sum_{\{j_1,\cdots,j_{n-r+1}\}} \sum_{h=1}^{L-2} \prod_{m=1}^{n-r+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!} \sum_{m'=1}^{n-r+1} j_{m'} \frac{\check{\alpha}_{m',h}}{\alpha_{m',h}}. \tag{3.109}$$

3.4.3 Solution of Linear Integral equation: order $(\ln s)^{-n}$

We are finally ready to conclude our computations. Let us write the order $(\ln s)^{-n}$ contributions of (3.74):

$$\begin{aligned} S^{(n)}(k) = &-g^2 \int_0^{+\infty} dt \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{P^{(n)}(g,t)}{\sinh\frac{t}{2}} - \\ &- g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh\frac{t}{2}} S^{(n)}(t) . \end{aligned} \tag{3.110}$$

Now we use a general procedure in the case of integral equation with separable kernel [37]. First step is the Newmann expansion for $S(k)$:

$$\begin{aligned} S(k) &= \sum_{p=1}^{\infty} \mathcal{S}_p(g,L,s) \frac{J_p(\sqrt{2}gk)}{k} \\ \mathcal{S}_p(g,L,s) &= \sum_{n=-1}^{\infty} \left[S_p^{(n)}(g,L) + \frac{1}{s}\check{S}_p^{(n)}(g,L) + O(1/s^2) \right] (\ln s)^{-n} \\ \gamma^{(n)}(g,L) &= \sqrt{2}g S_1^{(n)}(g,L) , \quad \check{\gamma}^{(n)}(g,L) = \sqrt{2}g \check{S}_1^{(n)}(g,L) . \end{aligned} \tag{3.111}$$

The Newmann expansion tranforms (3.110) into an infinite set of linear systems:

$$\begin{aligned} S_{2p-1}^{(n)}(g) &= -(2p-1) \int_0^{+\infty} \frac{dt}{t} \frac{P_n(g,t)\, J_{2p-1}(\sqrt{2}gt)}{\sinh\frac{t}{2}} - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{(n)}(g) , \\ S_{2p}^{(n)}(g) &= -2p \int_0^{+\infty} \frac{dt}{t} \frac{P_n(g,t)\, J_{2p}(\sqrt{2}gt)}{\sinh\frac{t}{2}} - 4p \sum_{m=1}^{\infty} Z_{2p,m}(g) (-1)^m S_m^{(n)}(g) , \end{aligned} \tag{3.112}$$

where

$$Z_{n,m}(g) = \int_0^{+\infty} \frac{dt}{t} \frac{J_n(\sqrt{2}gt) J_m(\sqrt{2}gt)}{e^t - 1} . \tag{3.113}$$

Plugging (3.108) into (3.112) we get:

$$S_p^{(n)}(g,L) = -2\pi \sum_{r=1}^{n} \tilde{S}_p^{(r/2)}(g) \, \cos\frac{\pi r}{2} \sum_{\{j_1,\dots,j_{n-r+1}\}} \frac{\sum\limits_{h=1}^{L-2} \prod\limits_{m=1}^{n-r+1} (\alpha_{m,h})^{j_m}}{\prod\limits_{m=1}^{n-r+1} j_m!} \tag{3.114}$$

$$\sum_{m=1}^{n-r+1} j_m = r \quad , \quad \sum_{m=1}^{n-r+1} m j_m = n \, ,$$

where the 'reduced coefficients' $\tilde{S}_p^{(r)}(g)$ satisfy the systems (see last of [29])

$$\begin{aligned} \tilde{S}_{2p-1}^{(r)}(g) &= \mathbb{I}_{2p-1}^{(r)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) \tilde{S}_m^{(r)}(g) \, , \\ \tilde{S}_{2p}^{(r)}(g) &= \mathbb{I}_{2p}^{(r)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,m}(g)(-1)^m \tilde{S}_m^{(r)}(g) \, , \end{aligned} \tag{3.115}$$

with

$$\mathbb{I}_p^{(r)}(g) = p \int_0^{+\infty} \frac{dh}{2\pi} h^{2r-1} \frac{J_p(\sqrt{2}gh)}{\sinh\frac{h}{2}} \, . \tag{3.116}$$

For $n = 1,2,3,4,5$ we have

$$S_p^{(1)}(g,L) = 0 \tag{3.117}$$

$$S_p^{(2)}(g,L) = 2\pi \tilde{S}_p^{(1)}(g) \sum_{h=1}^{L-2} \frac{(\alpha_{1,h})^2}{2!} \tag{3.118}$$

$$S_p^{(3)}(g,L) = 2\pi \tilde{S}_p^{(1)}(g) \sum_{h=1}^{L-2} (\alpha_{1,h})(\alpha_{2,h}) \tag{3.119}$$

$$\begin{aligned} S_p^{(4)}(g,L) &= 2\pi \tilde{S}_p^{(1)}(g) \sum_{h=1}^{L-2} \left[\frac{(\alpha_{2,h})^2}{2!} + (\alpha_{1,h})(\alpha_{3,h}) \right] - \\ &- 2\pi \tilde{S}_p^{(2)}(g) \sum_{h=1}^{L-2} \frac{(\alpha_{1,h})^4}{4!} \end{aligned} \tag{3.120}$$

$$\begin{aligned} S_p^{(5)}(g,L) &= 2\pi \tilde{S}_p^{(1)}(g) \sum_{h=1}^{L-2} \left[(\alpha_{1,h})(\alpha_{4,h}) + (\alpha_{2,h})(\alpha_{3,h}) \right] - \\ &- 2\pi \tilde{S}_p^{(2)}(g) \sum_{h=1}^{L-2} \frac{(\alpha_{1,h})^3}{3!} (\alpha_{2,h}) \end{aligned} \tag{3.121}$$

and, plugging explicit expressions (3.101) for the coefficients $\alpha_{n,h}$ we obtain:

$$S_p^{(1)}(g,L) = 0\,, \tag{3.122}$$

$$S_p^{(2)}(g,L) = \frac{\pi^3}{3(\sigma_0^{(-1)})^2}(L-3)(L-2)(L-1)\tilde{S}_p^{(1)}(g)\,, \tag{3.123}$$

$$S_p^{(3)}(g,L) = -2\frac{\pi^3\sigma_0^{(0)}}{3(\sigma_0^{(-1)})^3}(L-3)(L-2)(L-1)\tilde{S}_p^{(1)}(g)\,, \tag{3.124}$$

$$S_p^{(4)}(g,L) = 2\pi(L-3)(L-2)(L-1)\left\{\left[\frac{\pi^2\left(\sigma_0^{(0)}\right)^2}{2(\sigma_0^{(-1)})^4} - \right.\right. \tag{3.125}$$

$$\left.\left. - \frac{\pi^4\sigma_2^{(-1)}}{90(\sigma_0^{(-1)})^5}(5+3L(L-4))\right]\tilde{S}_p^{(1)}(g) - \frac{\pi^4}{360(\sigma_0^{(-1)})^4}(5+3L(L-4))\tilde{S}_p^{(2)}(g)\right\}$$

$$S_p^{(5)}(g,L) = (L-3)(L-2)(L-1)\left\{\left[\left(\frac{5\pi^5}{3}\frac{\sigma_2^{(-1)}\sigma_0^{(0)}}{\left(\sigma_0^{(-1)}\right)^6} - \frac{\pi^5}{3}\frac{\sigma_2^{(0)}}{\left(\sigma_0^{(-1)}\right)^5}\right)\times\right.\right.$$

$$\times\frac{(5+3L(L-4))}{15} + \left(-\frac{4\pi^3\left(\sigma_0^{(0)}\right)^3}{3\left(\sigma_0^{(-1)}\right)^5} - \frac{2\pi^3\sigma_0^{(2)}}{3\left(\sigma_0^{(-1)}\right)^3}\right)\Bigg]\tilde{S}_p^{(1)}(g) + \tag{3.126}$$

$$\left. + \frac{\pi^5\sigma_0^{(0)}}{45\left(\sigma_0^{(-1)}\right)^5}\tilde{S}_p^{(2)}(g)(5+3L(L-4))\right\}\,.$$

We finally obtain expressions for $\gamma^{(n)}(g,L)$ functions using (3.111):

$$\gamma^{(1)}(g,L)=0\,,$$
$$\gamma^{(2)}(g,L)=\sqrt{2}g\frac{\pi^3}{3(\sigma_0^{(-1)})^2}(L-3)(L-2)(L-1)\tilde{S}_1^{(1)}(g)\,,$$
$$\gamma^{(3)}(g,L)=-2\sqrt{2}g\frac{\pi^3\sigma_0^{(0)}}{3(\sigma_0^{(-1)})^3}(L-3)(L-2)(L-1)\tilde{S}_1^{(1)}(g)\,,$$
$$\gamma^{(4)}(g,L)=2\sqrt{2}g\pi(L-3)(L-2)(L-1)\left\{\left[\frac{\pi^2\left(\sigma_0^{(0)}\right)^2}{2(\sigma_0^{(-1)})^4}-\right.\right. \tag{3.127}$$
$$\left.\left.-\frac{\pi^4\sigma_2^{(-1)}}{90(\sigma_0^{(-1)})^5}(5+3L(L-4))\right]\tilde{S}_1^{(1)}(g)-\frac{\pi^4}{360(\sigma_0^{(-1)})^4}(5+3L(L-4))\tilde{S}_1^{(2)}(g)\right\}$$
$$\gamma^{(5)}(g,L)=\sqrt{2}g(L-3)(L-2)(L-1)\left\{\left[\left(\frac{5\pi^5}{3}\frac{\sigma_2^{(-1)}\sigma_0^{(0)}}{\left(\sigma_0^{(-1)}\right)^6}-\frac{\pi^5}{3}\frac{\sigma_2^{(0)}}{\left(\sigma_0^{(-1)}\right)^5}\right)\times\right.\right.$$
$$\times\frac{(5+3L(L-4))}{15}+\left.\left(-\frac{4\pi^3\left(\sigma_0^{(0)}\right)^3}{3\left(\sigma_0^{(-1)}\right)^5}-\frac{2\pi^3\sigma_0^{(2)}}{3\left(\sigma_0^{(-1)}\right)^3}\right)\right]\tilde{S}_1^{(1)}(g)+$$
$$\left.+\frac{\pi^5\sigma_0^{(0)}}{45\left(\sigma_0^{(-1)}\right)^5}\tilde{S}_2^{(2)}(g)(5+3L(L-4))\right\}\,.$$

3.4.4 Solution of Linear Integral equation: order $s^{-1}(\ln s)^{-n}$

We now follow the same procedure as the previous subsection to compute $\check{\gamma}^{(n)}$ functions. Start with LIE at order $s^{-1}(\ln s)^{-n}$:

$$\check{S}^{(n)}(k)=2g^2\gamma^{(n)}(g.L)\hat{K}(\sqrt{2}gk,0)-g^2\int_0^{+\infty}dt\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{\check{P}^{(n)}(g,t)}{\sinh\frac{t}{2}}-$$
$$-g^2\int_0^{+\infty}dte^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{t}{\sinh\frac{t}{2}}\check{S}^{(n)}(t)\,. \tag{3.128}$$

Newmann expansions (3.111) transforms (3.128) into the following system:

$$\check{S}_{2p-1}^{(n)}(g,L)=\sqrt{2}g\delta_{p,1}\gamma^{(n)}(g,L)-(2p-1)\int_0^{+\infty}\frac{dt}{t}\frac{\check{P}^{(n)}(g,t)\,J_{2p-1}(\sqrt{2}gt)}{\sinh\frac{t}{2}}-$$
$$-2(2p-1)\sum_{m=1}^{\infty}Z_{2p-1,m}(g)\check{S}_m^{(n)}(g,L)\,, \tag{3.129}$$
$$\check{S}_{2p}^{(n)}(g,L)=-2p\int_0^{+\infty}\frac{dt}{t}\frac{\check{P}^{(n)}(g,t)\,J_{2p}(\sqrt{2}gt)}{\sinh\frac{t}{2}}-4p\sum_{m=1}^{\infty}Z_{2p,m}(g)(-1)^m\check{S}_m^{(n)}(g,L)\,.$$

Plugging (3.109) into the system (3.129) we get:

$$\check{S}_p^{(n)}(g,L) = -2\pi \sum_{r=1}^{n} \tilde{S}_p^{(r/2)}(g)\ \cos\frac{\pi r}{2} \times$$
$$\times \sum_{\{j_1,\ldots,j_{n-r+1}\}} \frac{\sum_{h=1}^{L-2}\left(\prod_{m=1}^{n-r+1}(\alpha_{m,h})^{j_m}\right)\sum_{m'=1}^{n-r+1} j_{m'}\frac{\check{\alpha}_{m',h}}{\alpha_{m',h}}}{\prod_{m=1}^{n-r+1} j_m!} +$$
$$+\frac{S_p^{(-1)}(g)}{2}\gamma^{(n)}(g,L)\ , \qquad \sum_{m=1}^{n-r+1} j_m = r\,,\ \sum_{m=1}^{n-r+1} m j_m = n\,, \tag{3.130}$$

with the coefficients $\tilde{S}_p^{(r)}(g)$ satisfying (3.115). For $n = 1, 2, 3, 4, 5$ we get

$$\begin{aligned}
\check{S}_p^{(1)}(g,L) &= 0\\
\check{S}_p^{(2)}(g,L) &= 2\pi\tilde{S}_p^{(1)}(g)\sum_{h=1}^{L-2}\alpha_{1,h}\check{\alpha}_{1,h} + \frac{S_p^{(-1)}(g)}{2}\gamma^{(2)}(g,L)\\
\check{S}_p^{(3)}(g,L) &= 2\pi\tilde{S}_p^{(1)}(g)\sum_{h=1}^{L-2}(\alpha_{2,h}\check{\alpha}_{1,h}+\alpha_{1,h}\check{\alpha}_{2,h}) + \frac{S_p^{(-1)}(g)}{2}\gamma^{(3)}(g,L)\\
\check{S}_p^{(4)}(g,L) &= 2\pi\tilde{S}_p^{(1)}(g)\sum_{h=1}^{L-2}(\alpha_{1,h}\check{\alpha}_{3,h}+\alpha_{2,h}\check{\alpha}_{2,h}+\alpha_{3,h}\check{\alpha}_{1,h}) -\\
&\quad -\frac{\pi}{3}\tilde{S}_p^{(2)}(g)\sum_{h=1}^{L-2}(\alpha_{1,h})^3\check{\alpha}_{1,h} + \frac{S_p^{(-1)}(g)}{2}\gamma^{(4)}(g,L)\\
\check{S}_p^{(5)}(g,L) &= 2\pi\tilde{S}_p^{(1)}(g)\sum_{h=1}^{L-2}(\alpha_{1,h}\check{\alpha}_{4,h}+\alpha_{2,h}\check{\alpha}_{3,h}+\alpha_{3,h}\check{\alpha}_{2,h}+\alpha_{4,h}\check{\alpha}_{1,h})\\
&\quad -\frac{\pi}{3}\tilde{S}_p^{(2)}(g)\sum_{h=1}^{L-2}\left(3(\alpha_{1,h})^2\alpha_{2,h}\check{\alpha}_{1,h}+(\alpha_{1,h})^3\check{\alpha}_{2,h}\right) +\\
&\quad +\frac{S_p^{(-1)}(g)}{2}\gamma^{(5)}(g,L)\,.
\end{aligned} \tag{3.131}$$

and, inserting expressions (3.102) we obtain:

$$
\begin{aligned}
&\check{S}_p^{(1)}(g,L) = 0\\
&\check{S}_p^{(2)}(g,L) = -\frac{2\pi^3}{3}\tilde{S}_p^{(1)}(g)\frac{\check{\sigma}_0^{(-1)}}{\left(\sigma_0^{(-1)}\right)^3}(L-1)(L-2)(L-3)+\\
&+\frac{S_p^{(-1)}(g)}{2}\gamma^{(2)}(g,L)\\
&\check{S}_p^{(3)}(g,L) = \frac{2\pi^3}{3}\frac{\tilde{S}_p^{(1)}(g)}{\left(\sigma_0^{(-1)}\right)^3}\left[\frac{3\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} - \check{\sigma}_0^{(0)}\right](L-1)(L-2)(L-3)+\\
&+\frac{S_p^{(-1)}(g)}{2}\gamma^{(3)}(g,L)\\
&\check{S}_p^{(4)}(g,L) = \frac{2\pi^3}{3}\frac{1}{\left(\sigma_0^{(-1)}\right)^4}\left\{\frac{\pi^2}{6\sigma_0^{(-1)}}\left[\tilde{S}_p^{(1)}(g)\left(\frac{5\sigma_2^{(-1)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} - \check{\sigma}_2^{(-1)}\right)+\right.\right.\\
&\left.+\check{\sigma}_0^{(-1)}\tilde{S}_p^{(2)}(g)\,\right]\frac{(5+3L(L-4))}{5} + 3\sigma_0^{(0)}\tilde{S}_p^{(1)}(g)\left[\check{\sigma}_0^{(0)} - \frac{2\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}}\right]\Bigg\}\times\\
&\times(L-1)(L-2)(L-3) + \frac{S_p^{(-1)}(g)}{2}\gamma^{(4)}(g,L)+ \qquad (3.132)\\
&\check{S}_p^{(5)}(g,L) = \frac{2\pi^3}{3}\frac{1}{\left(\sigma_0^{(-1)}\right)^3}\left\{\frac{\pi^2}{2\left(\sigma_0^{(-1)}\right)^2}\left[\tilde{S}_p^{(1)}(g)\left(\frac{\sigma_2^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} + \frac{\sigma_0^{(0)}\check{\sigma}_2^{(-1)}}{\sigma_0^{(-1)}} + \frac{\sigma_2^{(-1)}\check{\sigma}_0^{(0)}}{\sigma_0^{(-1)}} -\right.\right.\right.\\
&\left.-\frac{\check{\sigma}_2^{(-1)}}{5} - \frac{6\sigma_2^{(-1)}\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\left(\sigma_0^{(-1)}\right)^2}\right) + \tilde{S}_p^{(2)}(g)\left(5\check{\sigma}_0^{(0)} - \frac{\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}}\right)\Bigg]\frac{(5+3L(L-4))}{3}+\\
&+\tilde{S}_p^{(1)}(g)\left[-\check{\sigma}_0^{(2)} + \frac{3\sigma_0^{(2)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} - \frac{6(\sigma_0^{(0)})^2\check{\sigma}_0^{(0)}}{(\sigma_0^{(-1)})^2} + \frac{10(\sigma_0^{(0)})^3\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^3}\right]\Bigg\}\times\\
&\times(L-1)(L-2)(L-3) + \frac{S_p^{(-1)}(g)}{2}\gamma^{(5)}(g,L)
\end{aligned}
$$

and for anomalous dimension, using (3.111):

$$\check{\gamma}^{(1)}(g,L) = 0$$

$$\check{\gamma}^{(2)}(g,L) = -\sqrt{2}g\frac{2\pi^3}{3}\tilde{S}_1^{(1)}(g)\frac{\check{\sigma}_0^{(-1)}}{\left(\sigma_0^{(-1)}\right)^3}(L-1)(L-2)(L-3)+ + \sqrt{2}g\frac{f(g)}{2}\gamma^{(2)}(g,L)$$

$$\check{\gamma}^{(3)}(g,L) = \sqrt{2}g\frac{2\pi^3}{3}\frac{\tilde{S}_1^{(1)}(g)}{\left(\sigma_0^{(-1)}\right)^3}\left[\frac{3\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} - \check{\sigma}_0^{(0)}\right](L-1)(L-2)(L-3)+ + \sqrt{2}g\frac{f(g)}{2}\gamma^{(3)}(g,L)$$

$$\check{\gamma}^{(4)}(g,L) = \sqrt{2}g\frac{2\pi^3}{3}\frac{1}{\left(\sigma_0^{(-1)}\right)^4}\left\{\frac{\pi^2}{6\sigma_0^{(-1)}}\left[\tilde{S}_1^{(1)}(g)\left(\frac{5\sigma_2^{(-1)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} - \check{\sigma}_2^{(-1)}\right)+ +\check{\sigma}_0^{(-1)}\tilde{S}_1^{(2)}(g)\right]\frac{(5+3L(L-4))}{5} + 3\sigma_0^{(0)}\tilde{S}_1^{(1)}(g)\left[\check{\sigma}_0^{(0)} - \frac{2\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}}\right]\right\}\times \times (L-1)(L-2)(L-3) + \sqrt{2}g\frac{f(g)}{2}\gamma^{(4)}(g,L)\,, \tag{3.133}$$

$$\check{\gamma}^{(5)}(g,L) = \sqrt{2}g\frac{2\pi^3}{3}\frac{1}{\left(\sigma_0^{(-1)}\right)^3}\left\{\frac{\pi^2}{2\left(\sigma_0^{(-1)}\right)^2}\left[\tilde{S}_1^{(1)}(g)\left(\frac{\sigma_2^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} + \frac{\sigma_0^{(0)}\check{\sigma}_2^{(-1)}}{\sigma_0^{(-1)}} + \frac{\sigma_2^{(-1)}\check{\sigma}_0^{(0)}}{\sigma_0^{(-1)}} - - \frac{\check{\sigma}_2^{(-1)}}{5} - \frac{6\sigma_2^{(-1)}\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\left(\sigma_0^{(-1)}\right)^2}\right) + \tilde{S}_1^{(2)}(g)\left(5\check{\sigma}_0^{(0)} - \frac{\sigma_0^{(0)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}}\right)\right]\frac{(5+3L(L-4))}{3} + \tilde{S}_1^{(1)}(g)\left[-\check{\sigma}_0^{(2)} + \frac{3\sigma_0^{(2)}\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} - \frac{6(\sigma_0^{(0)})^2\check{\sigma}_0^{(0)}}{(\sigma_0^{(-1)})^2} + \frac{10(\sigma_0^{(0)})^3\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^3}\right]\right\}\times \times (L-1)(L-2)(L-3) + \sqrt{2}g\frac{f(g)}{2}\gamma^{(5)}(g,L)\,.$$

3.5 Reciprocity and self-tuning: $\frac{1}{s}$ contributions from functional relations

3.5.1 Relations between $\gamma^{(n)}(g,L)$ and $\check{\gamma}^{(n)}(g,L)$

Relations (3.127) and (3.133) are the prediction of Asymptotic Bethe Ansatz at orders $(\ln s)^{-n}$ and $s^{-1}(\ln s)^{-n}$ with $n = 1, \cdots, 5$. These relations are not completely final results, because it is not explicited the g dependence, hidden in the presence of coefficients $\tilde{S}_p^{(n)}$ and of coefficients $\sigma_r^{(n)}$ and $\check{\sigma}_r^{(n)}$, belonging to the derivatives of the counting functions. However, these relations are valid at all-loops. Furthermore, in these relations are contained some important informations about wrapping. To obtain them, first we have to rewrite in a compact way expressions (3.132) and (3.133). Using the following relations:

$$f(g) = 2\frac{\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} = 2\frac{\check{\sigma}_2^{(-1)}}{\sigma_2^{(-1)}}\,, \tag{3.134}$$

$$f_{sl}(g,L) = 2\frac{\check{\sigma}_0^{(0)}}{\sigma_0^{(-1)}} - (L-1)\,, \tag{3.135}$$

$$\gamma^{(2)}(g,L) = 2\frac{\check{\sigma}_0^{(2)}}{\sigma_0^{(-1)}} + 4\frac{\sigma_0^{(2)}\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^2}\,. \tag{3.136}$$

that we prove at the end of this section and comparing (3.132) and (3.133) respectevely with (3.122) and (3.127), we end up with:

$$\check{S}_p^{(1)}(g,L) = 0 \tag{3.137}$$

$$\check{S}_p^{(2)}(g,L) = \frac{S_p^{(-1)}(g)}{2}\gamma^{(2)}(g,L) - f(g)S_p^{(2)}(g,L) \tag{3.138}$$

$$\begin{aligned}\check{S}_p^{(3)}(g,L) = {} & \frac{S_p^{(-1)}(g)}{2}\gamma^{(3)}(g,L) - (f_{sl}(g,L)+L-1)S_p^{(2)}(g,L) - \\ & -\frac{3}{2}S_p^{(3)}(g,L)f(g)\end{aligned} \tag{3.139}$$

$$\begin{aligned}\check{S}_p^{(4)}(g,L) = {} & \frac{S_p^{(-1)}(g)\gamma^{(4)}(g,L)}{2} - 2S_p^{(4)}(g,L)f(g) - \\ & -\frac{3}{2}S_p^{(3)}(g,L)(f_{sl}(g,L)+L-1)\end{aligned} \tag{3.140}$$

$$\begin{aligned}\check{S}_p^{(5)}(g,L) = {} & \frac{S_p^{(-1)}(g)}{2}\gamma^{(5)}(g,L) - \frac{5}{2}S_p^{(5)}(g,L)f(g) - \\ & -2S_p^{(4)}(g,L)(f_{sl}(g,L)+L-1) - S_p^{(2)}(g,L)\gamma^{(2)}(g,L)\,.\end{aligned} \tag{3.141}$$

For anomalous dimensions, such relations read

$$\check{\gamma}^{(1)}(g,L) = 0\,, \tag{3.142}$$

$$\check{\gamma}^{(2)}(g,L) = -\frac{f(g)}{2}\gamma^{(2)}(g,L)\,, \tag{3.143}$$

$$\check{\gamma}^{(3)}(g,L) = -f(g)\gamma^{(3)}(g,L) - (f_{sl}(g,L)+L-1)\gamma^{(2)}(g,L)\,, \tag{3.144}$$

$$\check{\gamma}^{(4)}(g,L) = -\frac{3}{2}f(g)\gamma^{(4)}(g,L) - \frac{3}{2}(f_{sl}(g,L)+L-1)\gamma^{(3)}(g,L)\,, \tag{3.145}$$

$$\check{\gamma}^{(5)}(g,L) = -2f(g)\gamma^{(5)}(g,L) - 2(f_{sl}(g,L)+L-1)\gamma^{(4)}(g,L) - \left(\gamma^{(2)}(g,L)\right)^2. \tag{3.146}$$

Relations (3.142-3.146) together with relations (3.82) show that the anomalous dimension at order $\frac{1}{s}$ completely depends on itself at order s^0. We want to stress that these conclusions holds for all values of the coupling g, i.e. it is a nonperturbative statement on the high spin expansion on anomalous dimension. In the following subsection, we will see that these results also belong from reciprocity and self-tuning properties.

3.5.2 $\frac{1}{s}$ contributions from functional relations

Self-tuning and reciprocity relations were summarised in formulæ (3.3, 3.4, 3.5). We remember expression (3.2) for the high spin expansion of the anomalous dimension:

$$\begin{aligned}\gamma(g,L,s) = {} & f(g)\ln s + f_{sl}(g,L) + \sum_{n=1}^{\infty}\frac{\gamma^{(n)}(g,L)}{(\ln s)^n} + \\ & + \sum_{n=-1}^{\infty}\frac{\check{\gamma}^{(n)}(g,L)}{s(\ln s)^n} + O\left(s^{-1}(\ln s)^{-\infty}\right)\,.\end{aligned} \tag{3.147}$$

Comparing (3.147) with (3.3, 3.4, 3.5), we get that the leading terms of $P(s)$ should read

$$P(s) = f(g)\ln C(s) + f_{sl}(g,L) + \sum_{n=1}^{\infty} \frac{\gamma^{(n)}(g,L)}{(\ln C(s))^n} + O(1/C^2)\,. \tag{3.148}$$

Developing $C(s)$ in the same regime

$$C(s)^2 = \left(s + \frac{L}{2} - 1\right)\left(s + \frac{L}{2}\right) \Rightarrow C(s) = s + \frac{L-1}{2} + O(1/s)\,, \tag{3.149}$$

and putting it together with (3.4, 3.148), we end up with the following prediction for the anomalous dimension

$$\begin{aligned}\gamma(g,L,s) &= f(g)\ln s + f_{sl}(g,L) + \sum_{n=1}^{\infty} \frac{\gamma^{(n)}(g,L)}{(\ln s)^n} + \\ &+ \frac{\ln s}{2s}[f(g)]^2 + \frac{1}{2s} f(g)(L-1+f_{sl}(g,L)) + \frac{f(g)}{2s}\sum_{n=1}^{\infty}\frac{\gamma^{(n)}(g,L)}{(\ln s)^n} - \\ &- \sum_{n=1}^{\infty} n \frac{\gamma^{(n)}(g,L)}{2s(\ln s)^{n+1}}\left[f(g)\ln s + f_{sl}(g,L) + L - 1 + \sum_{m=1}^{\infty}\frac{\gamma^{(m)}(g,L)}{(\ln s)^m}\right] + \\ &+ O\left(s^{-1}(\ln s)^{-\infty}\right)\,.\end{aligned} \tag{3.150}$$

Working out this formula for orders $\frac{1}{s(\ln s)^n}$, $n = -1, ..., 5$, we find formulæ which coincide with (3.82), for $n = -1, 0$ and with (3.142, ..., 3.146), for $n = 1, ..., 5$. Therefore, our findings in the previous subsection agree with self-tuning and reciprocity predictions. We have already underlined in the introduction if this chapter that the analogous of these properties in QCD holds only at one loop. We now have seen that, at the analyzed orders, reciprocity and self-tuning hold for all values of g. It is a strong hint in order to consider reciprocity and self-tuning as exact symmetries of the theory.

3.5.3 Useful relations

We want to prove now relations (3.134), (3.135) and (3.136). Let us start with equations (3.108) and (3.109). Comparing them, we find the following relation:

$$\check{P}^{(2)}(g,t) = -2\frac{\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} P^{(2)}(g,t). \tag{3.151}$$

Now, using the linear integral equation (3.74) we are able to find the following relations (we have already seen and used the first two of them, (3.82)):

$$\check{S}^{(-1)}(k) = \frac{f(g)}{2} S^{(-1)}(k)\,, \tag{3.152}$$

$$\check{S}^{(0)}(k) = \frac{f_{sl}(g,L) + L - 1}{2} S^{(-1)}(k)\,, \tag{3.153}$$

$$\check{S}^{(2)}(k) = \frac{\gamma_{as}^{(2)}(g,L)}{2} S^{(-1)}(k) - f(g) S^{(2)}(k)\,. \tag{3.154}$$

For what concerns $\hat{\sigma}(k)$ we have the exact expression:

$$\begin{aligned}\hat{\sigma}(k) &= -\frac{2\pi L e^{-\frac{|k|}{2}}}{1-e^{-|k|}} + \frac{2\pi L e^{-|k|}}{1-e^{-|k|}} + \hat{G}(k) - \\ &- \frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'(k) + \frac{2\pi e^{-|k|}}{1-e^{-|k|}}\sum_{h=1}^{L}(\cos kx_h - 1)\,,\end{aligned} \tag{3.155}$$

where

$$\hat{G}(k) = \frac{\pi|k|}{\sinh\frac{|k|}{2}}S(k)\,. \tag{3.156}$$

Then, applying inverse Fourier transform, we obtain:

$$\begin{aligned}\sigma(u) = {} & L\left[\psi\left(\frac{1}{2}-iu\right)+\psi\left(\frac{1}{2}+iu\right)\right]-(L-2)[\psi(1-iu)+\psi(1+iu)]- \\ & -\psi(1-ix_L-iu)-\psi(1+ix_L-iu)-\psi(1-ix_L+iu)-\psi(1+ix_L+iu)+ \\ & +\int_{-\infty}^{+\infty}dk e^{iku}\frac{e^{-|k|}}{1-e^{-|k|}}P(s,g,k)+G(u)-\left[2\ln 2+O\left(\frac{u^2}{s^2}\right)\right]\,.\end{aligned} \tag{3.157}$$

Using the position of the external holes (3.73) and computing $\sigma(u)$ at $u=0$, we get

$$\begin{aligned}\sigma(0) = {} & -4\ln s-4\gamma_E-4L\ln 2+G(0)-2f(g)\frac{\ln s}{s}-2\frac{f_{sl}(g,L)+L-1}{s}+ \\ & +\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}dk\frac{e^{-|k|}}{1-e^{-|k|}}\left(P^{(n)}(g,k)+\frac{\check{P}^{(n)}(g,k)}{s}\right)(\ln s)^{-n} \\ & -2\sum_{n=1}^{\infty}\frac{\gamma^{(n)}(g,L)}{s}(\ln s)^{-n}+O\left(s^{-1}(\ln s)^{-\infty}\right)\,.\end{aligned} \tag{3.158}$$

(3.159)

It is obvious that, expanding $G(0)$ in the same way of $S(k)$ (3.75), relations (3.152), (3.153) and (3.154) are also valid for the corresponding coefficients of $G(0)$. Using these relations and also (3.151) it is possible to find, from (3.158) and remembering notation (3.91), the following relations:

$$f(g) = 2\frac{\check{\sigma}_0^{(-1)}}{\sigma_0^{(-1)}} \quad , \quad f_{sl}(g,L) = 2\frac{\check{\sigma}_0^{(0)}}{\sigma_0^{(-1)}}-(L-1) \tag{3.160}$$

$$\gamma^{(2)}(g,L) = 2\frac{\check{\sigma}_0^{(2)}}{\sigma_0^{(-1)}}+4\frac{\sigma_0^{(2)}\check{\sigma}_0^{(-1)}}{\left(\sigma_0^{(-1)}\right)^2}\,. \tag{3.161}$$

Computing from (3.157) the second derivative of $\sigma(u)$ at $u=0$, it is also possible to show that:

$$f(g) = 2\frac{\check{\sigma}_2^{(-1)}}{\sigma_2^{(-1)}}\,. \tag{3.162}$$

3.6 Conclusions

We now summarize what we have seen in this chapter and try to give some important hints about wrapping problem. We have already commented that leading terms $f(g)$ and $f_{sl}(g,L)$ are wrapping free up to the order $O\left(\frac{(\ln s)^2}{s^2}\right)$: it is prooved until six loops calculations, but agreement at strong coupling with perturbative string calculations suggests that it holds at all loops. Then, if we suppose that self-tuning and reciprocity are exact symmetries, from (3.82, 3.150) it follows that also $\check{\gamma}^{(-1)}$ and $\check{\gamma}^{(0)}$ are wrapping-free. Then, one could expect that all the terms that are in between $f_{sl}(g,L)$ and $\check{\gamma}^{(-1)}(g,L)$ (the various $\gamma^{(n)}(g,L)$) are also not affected by wrapping. If we suppose this, use again self-tuning and reciprocity and compare (3.142-3.146) with (3.150), we are able to conclude that all the functions $\check{\gamma}^{(n)}(g, L)$ do not depend on wrapping diagrams.

Even if we are aware that our arguments do not provide a proof of the fact that at high spin wrapping diagrams start contributing at orders $\frac{(\ln s)^2}{s^2}$, we however think that our results provide some nonperturbative hints of this property.

Formulæ (3.81) and (3.137,...,3.141) seems to indicate that more generally functional relations similar to (3.3, 3.4) should hold for the high spin limit of the function $S(k)$. On the basis of our results, we are naturally led to make the following proposal for a self-tuning relation involving the coefficients of the Neumann expansion of the function $S(k)$,

$$\mathcal{S}_p(g, s, L) = P_p\left(s + \frac{1}{2}\gamma(g, s, L)\right) , \tag{3.163}$$

where $\mathcal{P}_p$ satisfies an high spin expansion analogous to (3.4):

$$P_p(s) = \sum_{n=0}^{\infty} \frac{a_p^{(n)}(\ln C(s))}{C(s)^{2n}} , \tag{3.164}$$

where $C(s)$ is given by (3.5). Formula (3.163) agrees with results in [38], where the following self-tuning relations involving also the higher conserved charges

$$Q_r(g, L, s) = P_r\left(s + \frac{1}{2}\gamma(g, s, L)\right) , \tag{3.165}$$

were proposed.

3.7 Weak coupling computations

Let come back to what we have written at the beginning of subsection (3.5.1). Relations (3.127) and (3.133) are valid for all values of g but are still implicit. It is very easy to explicit these relations at weak coupling.

Firstly, we need to solve system (3.115). At weak coupling, it is very simple to find a recursive solution that permits to find coefficients $\tilde{S}_p^{(n)}$ up to the desired order in g. Then, it is necessary to find coefficients $\sigma_r^{(n)}$ and $\check{\sigma}_r^{(n)}$, solving (3.24) up to the necessary order in function of s in high spin limit. In [15], using a program of symbolic manipulation, *Mathematica*®, it is showed the possibility to reach order g^{22}, eleven loops, for functions $\gamma^{(n)}(g, L)$. It is possible to reply all steps for functions $\check{\gamma}^{(n)}(g, L)$ but it is more convenient to obtain them using relations from reciprocity and self tuning (3.142-3.146). We report, for an example, some weak coupling results from [15]:

$$\begin{aligned}
\gamma^{(2)}(g, L) = [(L-2)^3 - (L-2)] \Bigg[& g^2 \frac{7\pi^2}{24}\zeta(3) + g^4\left(\frac{35\pi^4}{144}\zeta(3) - \frac{31\pi^2}{8}\zeta(5)\right) + \\
& + g^6\left(-\frac{73\pi^6}{4320}\zeta(3) - \frac{155\pi^4}{48}\zeta(5) + \frac{635\pi^2}{16}\zeta(7)\right) + \\
& + g^8\left(\frac{7\pi^8}{1728}\zeta(3) + \frac{91\pi^2}{24}\zeta^3(3) + \frac{7\pi^6}{60}\zeta(5) + \frac{3175\pi^4}{96}\zeta(7) - \frac{17885\pi^2}{48}\zeta(9)\right) + \cdots \Bigg] ,
\end{aligned} \tag{3.166}$$

$$\gamma^{(3)}(g,L) = -[(L-3)^3 - (L-2)]\Big[g^2\frac{7\pi^2}{12}(L\ln 2+\gamma_E)\zeta(3)+ \tag{3.167}$$
$$+g^4\frac{\pi^2}{72}\big(-147L\zeta^2(3) - 558L\ln 2\zeta(5) + 35\gamma_E\pi^2 zeta(3) - 558\gamma_E\zeta(5) + 7(11L-6)\pi^2\ln 2\zeta(3)\big)+$$
$$+g^6\frac{\pi^2}{2160}\big(135(651\zeta(3)\zeta(5)+1270\ln 2\zeta(7))L - 15\pi^2(2046\ln 2\zeta(5)L + 385\zeta^2(3)L-$$
$$-1116\ln 2\zeta(5)) + \gamma_E(-73\pi^4\zeta(3) - 13950\pi^2\zeta(5) + 171450\zeta(7)) + (767L-840)\pi^4\ln 2\zeta(3)\big)-$$
$$-g^8\frac{\pi^2}{4320}\big(\pi^6\ln 2\zeta(3)(307L-342) + 18\pi^4(1184\ln 2\zeta(5)L + 91\zeta^2(3)L - 1240\ln 2\zeta(5)+$$
$$+105\zeta^2(3)) - 15\pi^2(41910\ln 2\zeta(7)L + 15011\zeta(3)\zeta(5)L - 22860\ln 2\zeta(7) + 1302\zeta(3)\zeta(5))-$$
$$-90(L(756\ln 2\zeta^3(3) - 8649\zeta^2(5) - 17780\zeta(3)\zeta(7) - 35770\ln 2\zeta(9)) - 392\ln 2\zeta^3(3))-$$
$$-\gamma_E(35\pi^6\zeta(3) + 1008\pi^4\zeta(5) + 285750\pi^2\zeta(7) + 1260(26\zeta^3(3) - 2555\zeta(9))))\big) + \cdots\Big].$$

We want to stress that obtaining easily pertirbative results up to eleven loops is a great computational result compared with perturbative results in non-supesymmetric gauge theories without aspects of integrability.

3.8 About excited states

We have worked in this chapter, for simplicity, on minimal anomalous dimension. But it is not too much difficult working on excited states. The element that defines a state is the distribution of Bethe roots, and subsequently of holes, on real axis. The minimal state is described by $L-2$ holes near the origin, with no roots within. Internal holes are described by the relation (3.45):

$$Z(x_h) = \pi(2h+1-L)\,,\; h = 1,\cdots,L-2\,. \tag{3.168}$$

In general, for an excited state, we have:

$$Z(x_h) = \pi f(h)\,, \tag{3.169}$$

where $f(h)$ contains the right quantum numbers that describe the distribution of the holes for the excited state considered. If we watch the general method showed in this chapter, we note that the starting point in which the distribution of holes appears is the set of relations (3.99) and (3.100). So, in general, calculations of excited states are the same made for ground state, substituting to the already used $\alpha_{n,h}$ and $\check{\alpha}_{n,h}$, coefficients $\alpha^{(e)}_{n,h}$ and $\check{\alpha}^{(e)}_{n,h}$, through the substitution $(2h+1-L)\to f(h)$. Due to the recursive behaviour of (3.99) and (3.100) we just have to perform the substitution in the first ones of both sets:

$$\alpha^{(e)}_{1,h} = \frac{\pi f(h)}{\sigma_0^{(-1)}} \quad , \quad \check{\alpha}^{(e)}_{1,h} = -\frac{\pi f(h)\check{\sigma}_0^{(-1)}}{(\sigma_0^{(-1)})^2}\,. \tag{3.170}$$

Let focus now, only for shortness, to orders $(\ln s)^{-n}$ with $n\geq 1$. For excited states we have (analogously to (3.114)):

$$S_p^{(e)(n)}(g,L) = -2\pi\sum_{r=1}^{n}\tilde{S}_p^{(r/2)}(g)\,\cos\frac{\pi r}{2}\sum_{\{j_1,\ldots,j_{n-r+1}\}}\frac{\sum_{h=1}^{L-2}\prod_{m=1}^{n-r+1}(\alpha^{(e)}_{m,h})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!} \tag{3.171}$$

where j_m are costrained in the usual way (3.88). In order to get excited state is simpler to compute the difference $S_p^{(e)(n)}(g,L) - S_p^{(n)}(g,L)$:

$$\Delta S_p^{(e)(n)}(g,L) = -2\pi \sum_{r=1}^{n} \tilde{S}_p^{(r/2)}(g)\ \cos\frac{\pi r}{2} \sum_{\{j_1,\dots,j_{n-r+1}\}} \left\{ \frac{\sum_{h=1}^{L-2}\prod_{m=1}^{n-r+1}(\alpha_{m,h}^{(e)})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!} - \frac{\sum_{h=1}^{L-2}\prod_{m=1}^{n-r+1}(\alpha_{m,h})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!} \right\}. \tag{3.172}$$

Difference on the right side, when we develop the sum on internal holes, is made from differences from couples of terms linked to the same holes in the excited state and in the ground state. We now underline a specific property of function $f(h)$: an excited state configuration differs from the ground state configuration only for the position of those holes that change their position with respect to the ground state, i.e. for those holes that *migrate* from their ground state positions near the origin. So, function $f(h)$ differs from function $f_0(h) = 2h + 1 - L$ only in correspondece of migrated holes. Excited states very close to ground state differ for few migrated holes, while high excited states could contain configurations in which all holes are migrated. However, the difference (3.172) depends only on migrated holes:

$$\Delta S_p^{(e)(n)}(g,L) = -2\pi \sum_{r=1}^{n} \tilde{S}_p^{(r/2)}(g)\ \cos\frac{\pi r}{2} \sum_{\{k_1,\cdots,k_l\}} \sum_{\{j_1,\dots,j_{n-r+1}\}} \left\{ \frac{\prod_{m=1}^{n-r+1}(\alpha_{m,k}^{(e)})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!} - \frac{\prod_{m=1}^{n-r+1}(\alpha_{m,k})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!} \right\}, \tag{3.173}$$

where the set of indexes $k_1, \cdots, k_l$ represents the set of migrated holes.

Now we give an example, proposed and discussed also in [39]. On the ground state, two consecutive positions of holes differs of 2π. The first excited state has the two holes on edge positions (of the set of internal holes) that *jump* of 2π in the external directions. Function $f(h)$ is the following:

$$f(h) = \begin{cases} 1 - L\,, & h = 1 \\ 2h + 1 - L\,, & h = 2, \cdots, L-3 \\ L - 1\,, & h = L-2 \end{cases} \tag{3.174}$$

In this case, we have only two migrating holes, so it is very simple to evaluate (3.173). For example, we give now the first results (remembering (3.111)):

$$\Delta\gamma^{(2)}(g,L) = \sqrt{2}g\frac{8\pi^3}{(\sigma_0^{(-1)})^2}(L-2)\tilde{S}_1^{(1)}(g) \tag{3.175}$$

$$\Delta\gamma^{(3)}(g,L) = -16\sqrt{2}g\frac{\pi^3\sigma_0^{(0)}}{(\sigma_0^{(-1)})^3}(L-2)\tilde{S}_1^{(1)}(g)\,. \tag{3.176}$$

Chapter 4

High spin and high twist expansion in *sl(2)* sector of $\mathcal{N} = 4$ Super Yang-Mills Theory

4.1 Introduction

In this chapter we study the same sector $sl(2)$ of $\mathcal{N} = 4$ SYM in the following double scaling limit:

$$s \to +\infty \,, \quad L \to +\infty \,, \quad j = \frac{L-2}{\ln s} \text{ fixed} \,, \tag{4.1}$$

for which anomalous dimension expansion (3.2) reads as:

$$\gamma(g, L, s) = f(g, j) \, \ln s + f_{sl}(g, j) + \sum_{n=1}^{\infty} \gamma^{(n)}(g, j) \, (\ln s)^{-n} + O((\ln s)^{-\infty}) \,. \tag{4.2}$$

The leading logarithmic divergence has been well understood in a sequel of papers [22, 23, 29, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49], for both weak and strong coupling.

Importance of this limit is due mainly to the absence of wrapping effects, i.e., to the possibility to completely investigate (4.2) by using Asymptotic Bethe Ansatz. We can conjecture the absence of wrapping corrections because the spin chains associated to dilatation operators are infinite ($L \to \infty$).

At small j, the generalised scaling function $f(g, j)$

$$f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n \tag{4.3}$$

has been well studied: at weak coupling [29], at strong coupling [42, 43, 44, 46]; for $j \ll g$, $g \to +\infty$, was also shown [43] to coincide with the energy density [45] of the nonlinear $O(6)$ sigma model embedded into $AdS_5 \times S^5$, thus confirming the related previous proposal [40] by Alday and Maldacena. Results for $j \gg g$ are also available [50].

Object of our investigation is the subleading function $f_{sl}(g, L)$ for small j:

$$f_{sl}(g, j) = \sum_{n=0}^{\infty} f_{sl,n}(g) j^n \,. \tag{4.4}$$

We can use the same NLIE approach used in the previous chapter, but in this case we are interested only to the subleading term, so we can neglect all contributions of orders

$(\ln s)^{-n}$, $n \geq 1$. This brings to a quite simple linear integral equation that we will solve perturbatively at weak coupling up to the order j^5 at one loop and j^4 for two and three loops, as an example. Next, it is showed a generalised and systematic study of subleading terms and strong coupling limits. All results showed in this chapter are contained in [17].

4.2 Summing on holes

Working in the double scaling limit (4.1) has a great main difference respect to computations made in the previous chapter: summing on holes. We have seen that the main difficult in computations is to estimate position of holes on the real axis, in order to evaluate sums like those contained in (3.30) and (3.44). Now, we have an infinite numbers of holes! For the minimal anomalous dimension, we have still two external holes, outside the two bands containing Bethe roots, and infinite internal holes $(L-2)$ near the origin, simmetrycally distributed. First, we have to fix the interval $[-c, c]$ on the real axis in which internal holes condensate. Counting function counts on the real axis $L+s$ points, *jumping* of 2π between two consecutive points. Relation (3.45) is still valid, so, most external holes of the group of internal ones are in the positions defined by:

$$Z(x_1) = -\pi(L-3)\,,\; Z(x_{L-2}) = \pi(L-3)\,. \tag{4.5}$$

Jumping of 2π to the right, for example, the next position $Z(u_k) = -\pi(L-1)$ (remember that the counting function is monotonously decreasing) should be occupied by a root. So, the only condition on the interval $[-c, c]$ is:

$$-\pi(L-3) < Z(c) < -\pi(L-1)\,. \tag{4.6}$$

We can freely choose the exact relation and, for convenience that we will show later, we choose to fix c by the relation:

$$Z(c) = -\pi(L-2) + O\left(\frac{1}{\ln s}\right)\,, \tag{4.7}$$

where the introduction of the second term on the right is necessary because the first, for (4.1), is of order $\ln s$.

4.2.1 How to sum on internal holes

Let see now how the compute the sum on internal holes in the case of $L \to \infty$. Start with the NLIE:

$$\sum_{k=1}^{s} O(u_k) + \sum_{h=1}^{L} O(x_h) = -\int_{-\infty}^{+\infty} \frac{dv}{2\pi} O(v)\sigma(v) + \text{NL-terms}\,. \tag{4.8}$$

We can restrict the interval of integration to $[-c, c]$: in this case sum over roots is zero because no roots are in $[-c, c]$ and the sums on holes reduces only to the sum on internal ones. So we have an easy way to evaluate the sum on internal holes through a NLIE:

$$\sum_{h=1}^{L-2} O(x_h) = -\int_{-c}^{c} \frac{dv}{2\pi} O(v)\sigma(v) + \text{NL-terms}\,. \tag{4.9}$$

We have showed in the first chapter a way to evaluate non-linear terms in a NLIE (eq. (4.9)). This formula works in this case, for which we have:

$$\text{NL} = -\sum_{k=0}^{\infty} \frac{(2\pi)^{2k+1}}{(2k+2)!} \text{B}_{2k+2}\left(\frac{1}{2}\right) \left[\left(\frac{1}{Z'(v)}\frac{d}{dv}\right)^{2k+1} O(v)\right]_{v=-c}^{v=c}\,, \tag{4.10}$$

from which we can note, using (4.7) and (4.1), that

$$\text{NL} \sim O\big((\ln s)^{-(2k+1)}\big)\,, \tag{4.11}$$

so we are left with a linear integral equation, discarding terms of order $(\ln s)^{-n}$, $n \geq 1$:

$$\sum_{h=1}^{L-2} O(x_h) = -\int_{-c}^{c} \frac{dv}{2\pi} O(v)\sigma(v) + O\Big(\frac{1}{\ln s}\Big)\,. \tag{4.12}$$

In our calculations, we write for simplicity sum on internal holes in the following way (see for example (3.74), using symmetric distribution of holes around the origin:

$$\sum_{h=1}^{L-2} \big(\cos(tx_h) - 1\big) \tag{4.13}$$

so we have

$$\sum_{h=1}^{L-2} \big(\cos(tx_h) - 1\big) = -\int_{-c}^{c} \frac{dv}{2\pi} \big(\cos(tv) - 1\big)\sigma(v) + O\Big(\frac{1}{\ln s}\Big) \tag{4.14}$$

and, passing to Fourier space,

$$\sum_{h=1}^{L-2} \big(\cos(tx_h) - 1\big) = -\int_{-\infty}^{+\infty} \frac{dh}{2\pi^2} \hat{\sigma}(h) \left[\frac{\sin\big((t-h)c\big)}{t-h} - \frac{\sin(hc)}{h} \right] + O\Big(\frac{1}{\ln s}\Big)\,. \tag{4.15}$$

4.2.2 External holes

In the section (3.3.2) we have studied position of external holes, we have obtained equation (3.73):

$$x_L = -x_{L-1} = \frac{s}{\sqrt{2}} \left(1 + \frac{L - 1 + \gamma(g, L, s)}{2s} + O\left(\frac{1}{s^2}\right) \right)\,. \tag{4.16}$$

In our computations in the previous chapter, this formula enters specifically in the following way: seeing for example equation (3.74), leading terms of (4.16) gives the leading term of order $\ln s$, while subleading terms bring to terms of order s^{-1}. In this chapter we are only interested in analysis of subleading contributions $f_{sl}(g, j)$ so we can neglet subleading terms in (4.16) and consider, from now, positions of external holes as:

$$x_L = -x_{L-1} = \frac{s}{\sqrt{2}} \left(1 + +O\left(\frac{1}{s}\right) \right)\,. \tag{4.17}$$

4.3 Weak coupling results

We show in this section perturbative results ($g \ll 1$) for $f_{sl}(g, j)$ We separe one-loop contributions, i.e. order g^2, from many-loops ones.

4.3.1 One loop results

Our discussion starts from equation (3.30). It is simple to adapt this equation to our actual purposes. Keeping in mind that we are interested only in orders $\ln s$ and $(\ln s)^0$, we can discard in our computations all terms of orders $(\ln s)^{-n}$, $n \geq 1$ and $O(s^{-1})$. So, for external holes, we can use (4.17). Sum of internal holes could be easily written in the form (4.13), so we can evaluate it using (4.15), but taking the one-loop part of the interval $[-c, c]$, say $[-c_0, c_0]$.

For what concerns the non-linar term, we can refer to subsection (3.3.1) and we can make the substitution

$$-\frac{2e^{-|k|}}{1-e^{-|k|}}\hat{L}'_0(k) \to -4\pi\delta(k)\ln 2 \tag{4.18}$$

discarding terms of order $O(s^{-2})$. So we finally have:

$$\hat{\sigma}_0(k) = -2\pi\frac{\frac{j\ln s+2}{2}\left(1-e^{-\frac{|k|}{2}}\right)+e^{-\frac{|k|}{2}}\left(1-\cos\frac{ks}{\sqrt{2}}\right)}{\sinh\frac{|k|}{2}} - 4\pi\delta(k)\ln 2 -$$
$$-\frac{e^{-\frac{|k|}{2}}}{\sinh\frac{|k|}{2}}\int_{-\infty}^{+\infty}\frac{dh}{2\pi}\hat{\sigma}_0(h)\left[\frac{\sin(k-h)c_0}{k-h}-\frac{\sin hc_0}{h}\right]+O\left(\frac{1}{\ln s}\right). \tag{4.19}$$

We have to solve this equation together with the one-loop part of the condition (4.7), that reads:

$$Z(c_0) = \frac{1}{2}\int_{-c_0}^{c_0} du\,\sigma_0(u) = -\pi j\ln s + O\left(\frac{1}{\ln s}\right) \tag{4.20}$$

and, in Fourier space:

$$2\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\hat{\sigma}_0(k)\frac{\sin kc_0}{k} = -2\pi\, j\,\ln s + O\left(\frac{1}{\ln s}\right). \tag{4.21}$$

Equation (4.19) and (4.21) suggest that c_0 depends on j and $\ln s$ and expands, for $j \ll 1$, as

$$c_0 = \sum_{n=1}^{\infty} c_0^{(0,n)} j^n + [\sum_{n=1}^{\infty} c_0^{(1,n)} j^n]\frac{1}{\ln s} + O\left(\frac{1}{(\ln s)^2}\right). \tag{4.22}$$

Equations (4.19), (4.21) and (4.22) are the starting point of our computations. Using the notation

$$\frac{d^r\sigma(u)}{du^r}\Big|_{u=0} = \sigma_{0,r}(0) \tag{4.23}$$

we can expand condition (4.21) and get:

$$2\sigma_{0,0}(0)c_0 + \frac{1}{3}\sigma_{0,2}(0)c_0^3 + \ldots = -2\pi j\ln s\,. \tag{4.24}$$

For such quantities expansions similar to (4.22) hold, e.g.

$$\sigma_{0,0}(0) = [\sum_{n=0}^{\infty}\sigma_0^{(-1,n)}(0)j^n]\ln s + [\sum_{n=0}^{\infty}\sigma_0^{(0,n)}(0)j^n] + O\left(\frac{1}{\ln s}\right). \tag{4.25}$$

Looking to (4.19), the term on the second line is of leading order c_0^3, so it is a term of order $O(j^3)$. We can neglect the integral term and evaluate (4.19) up to order j^3. From it we can get:

$$\sigma_{0,0}(0) = [-4-4j\ln 2+O(j^3)]\ln s - [8\ln 2+4\gamma_E+O(j^3)] + O\left(\frac{1}{\ln s}\right) \tag{4.26}$$

and also that

$$\sigma_{0,2}(0) = [56\zeta(3)+O(j)] + O\left(\frac{1}{\ln s}\right). \tag{4.27}$$

Now, inserting the expansion (4.22) in the condition (4.24) and using also (4.26) we can start finding, by equating equal powers in j and $\ln s$, the various coefficients $c_0^{(0,n)}$, $c_0^{(1,n)}$.

We get, without much ado

$$c^{(0,1)} = \frac{\pi}{4}\,, \quad c^{(0,2)} = -\frac{\pi}{4}\ln 2\,, \quad c^{(0,3)} = \frac{\pi}{4}(\ln 2)^2\,, \quad c^{(1,1)} = -\frac{\pi}{4}(2\ln 2 + \gamma_E)\,,$$
$$c^{(1,2)} = \frac{\pi}{2}\ln 2(2\ln 2 + \gamma_E)\,, \quad c^{(1,3)} = -\frac{3}{4}\pi(\ln 2)^2(2\ln 2 + \gamma_E) + \frac{7}{192}\pi^3\zeta(3)\,,$$

in such a way that one can write

$$c_0 = \left[\frac{\pi}{4}j - \frac{\pi}{4}\ln 2\; j^2 + \frac{\pi}{4}(\ln 2)^2 j^3 + O(j^4)\right] + \left[-\frac{\pi}{4}(2\ln 2 + \gamma_E)j + \right. \tag{4.28}$$
$$\left. + \frac{\pi}{2}\ln 2(2\ln 2+\gamma_E)j^2 - \left(\frac{3}{4}\pi(\ln 2)^2(2\ln 2+\gamma_E) - \frac{7}{192}\pi^3\zeta(3)\right) j^3 + O(j^4)\right]\frac{1}{\ln s} +$$
$$+ O\left(\frac{1}{(\ln s)^2}\right)\,.$$

We can use these results to compute anomalous dimension. At one loop, equations (3.7), (3.10), (3.11) and (3.1) bring to:

$$\gamma_{g^2} = g^2 \sum_{k=1}^{s} e(u) = g^2 \sum_{k=1}^{s} \frac{1}{u_k^2 + \frac{1}{4}}\,. \tag{4.29}$$

We use the usual NLIE approach, so we have:

$$\gamma_{g^2} = -g^2 \int_{-\infty}^{+\infty} \frac{du}{2\pi} e(u)\sigma_0(u) + \mathrm{NL} + g^2 \sum_{h=1}^{L} e(x_h)\,. \tag{4.30}$$

We can easily evaluate Non-linear terms using (4.10) but on the interval $(-\infty,+\infty)$. Keeping in mind (4.1), we see from (3.20) that $Z(u \to \infty) \sim s$ so we get NL $\sim O(s^{-1})$. From (4.29) and (4.17) we immediatly see that external holes give contributions of order $O(s^{-2})$. Sum on internal holes can be evaluated using what we have said in (4.2.1), with its non-linear terms that are of order $O((\ln s)^{-1})$. We can discard all of these terms for our purposes so we are left with:

$$\gamma_{g^2} = -g^2 \int_{-\infty}^{+\infty} \frac{du}{2\pi} e(u)\sigma_0(u) + g^2 \int_{-c_0}^{c_0} \frac{du}{2\pi} e(u)\sigma_0(u) \tag{4.31}$$

and, in Fourier space, using (4.21) in computations:

$$\gamma_{g^2} = -\int_{-\infty}^{+\infty} \frac{dk}{4\pi^2}\hat{e}(k)\hat{\sigma}_0(k) + \int_{-\infty}^{+\infty} \frac{dk}{4\pi^2}\hat{e}(k)\times \tag{4.32}$$
$$\times \int_{-\infty}^{+\infty} \frac{dh}{2\pi}\hat{\sigma}_0(h)\left[2\frac{\sin(k-h)c_0}{k-h} - 2\frac{\sin hc_0}{h}\right] - \ln s j e(0)\,,$$

with

$$\hat{e}(k) = 2\pi e^{-\frac{|k|}{2}}\,. \tag{4.33}$$

Inserting all results collected, we get, expanding in powers of j:

$$\begin{aligned}
\gamma_{g^2} &= g^2 \ln s \Big[4 - 4\ln 2\, j + \frac{7\zeta(3)\pi^2}{24} j^3 - \frac{7\zeta(3)\pi^2 \ln 2}{12} j^4 + \\
&+ \left(\frac{7\pi^2 (\ln 2)^2 \zeta(3)}{8} - \frac{31\pi^4 \zeta(5)}{640} \right) j^5 + O(j^6) \Big] + \\
&+ g^2 \Big[4\gamma_E - \frac{7\zeta(3)\pi^2}{12} (2\ln 2 + \gamma_E)\, j^3 + \frac{7\zeta(3)\pi^2 \ln 2}{4} (2\ln 2 + \gamma_E) j^4 - \\
&- \left(\frac{7\pi^2 (\ln 2)^2 \zeta(3)}{2} - \frac{31\pi^4 \zeta(5)}{160} \right) (2\ln 2 + \gamma_E) j^5 + \\
&+ \frac{49\pi^4 \zeta^2(3)}{960} j^5 + O(j^6) \Big] + O\left(\frac{1}{\ln s} \right) .
\end{aligned} \tag{4.34}$$

4.3.2 Higher loops results

Let us now pass to compute weak coupling expansions in powers of j of anomalous dimension. In order to do that, we study the minimal anomalous dimension as a function of the coupling costant following the scheme showed in (3.2.3), introducing the auxiliary function $S(k)$. We can easily pass to the actual limit (4.1) we are working in, in the same way we have just used for one-loop computations. It is important to stress the validity of the relation (3.40):

$$\gamma(g, j) = 2S(0) . \tag{4.35}$$

For our purposes, we start to analyze the NLIE for $S(k)$ in the form (3.41):

$$\begin{aligned}
S(k) = \frac{L}{|k|}[1 - J_0(\sqrt{2}gk)] + \frac{ike^{\frac{|k|}{2}}}{2\pi|k|} \int_{-\infty}^{+\infty} \frac{dt}{4\pi^2} \hat{\phi}_H(k,t) \Bigg[\frac{\pi|t|}{\sinh \frac{|t|}{2}} S(t) - \\
- \frac{2\hat{L}'(t)}{1 - e^{-|t|}} - \frac{2\pi L e^{\frac{-|t|}{2}}}{1 - e^{-|t|}} + \frac{2\pi}{1 - e^{-|t|}} \sum_{h=1}^{L} e^{itx_h} \Bigg] .
\end{aligned} \tag{4.36}$$

Let us evaluate the various terms. Let begin from the non-linear term

$$-\frac{2\hat{L}'(t)}{1 - e^{-|t|}} . \tag{4.37}$$

Referring to the subsection (3.3.1) and using property (3.43), we can make the substitution:

$$-\frac{2\hat{L}'(t)}{1 - e^{-|t|}} \to -4\pi\delta(t) \ln 2 \tag{4.38}$$

discarding terms of order $O(s^{-2})$. Then, even distribution of holes around the origin permits to write the sum on holes as follows:

$$\sum_{h=1}^{L} e^{itx_h} = \sum_{h=1}^{L} [\cos(tx_h) - 1] + L \tag{4.39}$$

so we can evaluate sum on holes using (4.15) and (4.17). We get at the end:

$$
\begin{aligned}
S(k) = {} & \frac{j\ln s+2}{|k|}[1-J_0(\sqrt{2}gk)]+ \\
& + \frac{1}{\pi|k|}\int_{-\infty}^{+\infty}\frac{dt}{|t|}\Big[\sum_{r=1}^{\infty} r(-1)^{r+1}J_r(\sqrt{2}gk)J_r(\sqrt{2}gt)\frac{1-\mathrm{sgn}(kt)}{2}e^{-\frac{|t|}{2}}+ \\
& + \mathrm{sgn}(t)\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)(-1)^{r+\nu}e^{-\frac{|t|}{2}}\times \\
& \times\Big(J_{r-1}(\sqrt{2}gk)J_{r+2\nu}(\sqrt{2}gt)-J_{r-1}(\sqrt{2}gt)J_{r+2\nu}(\sqrt{2}gk)\Big)\Big]\cdot \\
& \cdot\Big\{\frac{\pi|t|}{\sinh\frac{|t|}{2}}S(t)-4\pi\ln 2\;\delta(t)-\pi j\ln s\frac{1-e^{\frac{|t|}{2}}}{\sinh\frac{|t|}{2}}-2\pi\frac{1-e^{-\frac{|t|}{2}}\cos\frac{ts}{\sqrt{2}}}{\sinh\frac{|t|}{2}}- \\
& - \frac{e^{\frac{|t|}{2}}}{\sinh\frac{|t|}{2}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\sigma}(p)\left[\frac{\sin(t-p)c}{t-p}-\frac{\sin pc}{p}\right]\Big\}+O\left(\frac{1}{\ln s}\right).
\end{aligned}
\tag{4.40}
$$

This relation has to be solved together with the condition (4.7), that reads in Fourier space as:

$$
2\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\hat{\sigma}(k)\frac{\sin kc}{k}=-2\pi j\ln s+O\left(\frac{1}{\ln s}\right). \tag{4.41}
$$

Two loops anomalous dimension

Let us expand the linear equation for $S(k)$ at order g^4. Before doing this, it is convenient to write the expression for $S(k)$ at the order g^2. From the general expression (4.40) we get, neglecting terms $O(1/\ln s)$,

$$
\begin{aligned}
S_{g^2}(k) = {} & \frac{g^2}{2}L|k|+\frac{1}{\pi|k|}\int_{-\infty}^{+\infty}\frac{dh}{|h|}\frac{\sqrt{2}gh}{2}\frac{\sqrt{2}gk}{2}\frac{1-\mathrm{sgn}(kh)}{2}e^{-\frac{|h|}{2}}\times \\
& \times\Big\{-4\pi\ln 2\;\delta(h)-\pi j\ln s\frac{1-e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}-2\pi\frac{1-e^{-\frac{|h|}{2}}\cos\frac{hs}{\sqrt{2}}}{\sinh\frac{|h|}{2}}- \\
& - \frac{e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\sigma}_0(p)\left[\frac{\sin(h-p)c_0}{h-p}-\frac{\sin pc_0}{p}\right]\Big\}.
\end{aligned}
\tag{4.42}
$$

Comparing (4.35) with (4.42) we get

$$
S_{g^2}(k)=\frac{g^2}{2}L|k|+\frac{1}{2}\gamma_{g^2}. \tag{4.43}
$$

Let us pass now to the order g^4. In this case the relevant equation reads, neglecting terms $O(1/\ln s)$,

$$
\begin{aligned}
S_{g^4}(k) = &-\frac{g^4}{16}L|k|^3 + \frac{1}{\pi|k|}\int_{-\infty}^{+\infty}\frac{dh}{|h|}\frac{\sqrt{2}gh}{2}\frac{\sqrt{2}gk}{2}\frac{1-\text{sgn}(kh)}{2}e^{-\frac{|h|}{2}}\times \\
&\times\left\{\frac{\pi|h|}{\sinh\frac{|h|}{2}}S_{g^2}(h) - \frac{e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\left[\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\sigma}(p)\left(\frac{\sin(h-p)c}{h-p}-\frac{\sin pc}{p}\right)\right]_{g^2}\right\}+ \\
&+\frac{1}{\pi|k|}\int_{-\infty}^{+\infty}\frac{dh}{|h|}\left[-\frac{1}{8}g^4k^2h^2 - \frac{(\sqrt{2}gh)^3}{16}\frac{\sqrt{2}gk}{2} - \frac{(\sqrt{2}gk)^3}{16}\frac{\sqrt{2}gh}{2}\right]\times \\
&\times\frac{1-\text{sgn}(kh)}{2}e^{-\frac{|h|}{2}}\left\{-4\pi\ln 2\;\delta(h) - \pi j\ln s\frac{1-e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}} - 2\pi\frac{1-e^{-\frac{|h|}{2}}\cos\frac{hs}{\sqrt{2}}}{\sinh\frac{|h|}{2}} - \right. \\
&\left.-\frac{e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\sigma}_0(p)\left[\frac{\sin(h-p)c_0}{h-p}-\frac{\sin pc_0}{p}\right]\right\}.
\end{aligned}
\tag{4.44}
$$

It follows that the two loops contribution to the anomalous dimension, γ_{g^4}, is given by[1]

$$
\begin{aligned}
\gamma_{g^4} = &-\frac{g^2}{2\pi}\int_{-\infty}^{+\infty}dh e^{-\frac{|h|}{2}}\left\{\frac{\pi|h|}{\sinh\frac{|h|}{2}}S_{g^2}(h) - \right. \\
&\left.-\frac{e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\left[\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\sigma}(p)\left(\frac{\sin(h-p)c}{h-p}-\frac{\sin pc}{p}\right)\right]_{g^2}\right\}+ \\
&+\frac{g^4}{8\pi}\int_{-\infty}^{+\infty}dh h^2 e^{-\frac{|h|}{2}}\left\{-4\pi\ln 2\;\delta(h) - \pi j\ln s\frac{1-e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}} - \right. \\
&\left.-2\pi\frac{1-e^{-\frac{|h|}{2}}\cos\frac{hs}{\sqrt{2}}}{\sinh\frac{|h|}{2}} - \frac{e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\sigma}_0(p)\left[\frac{\sin(h-p)c_0}{h-p}-\frac{\sin pc_0}{p}\right]\right\}+ \\
&+O(1/\ln s).
\end{aligned}
\tag{4.45}
$$

It is not difficult to compute, neglecting orders $1/\ln s$, the contributions to γ_{g^4} coming from the term in the first line, the terms in the third line and the first term in the fourth line. For these contributions we get

$$
\begin{aligned}
\gamma_{g^4,1} &= -2g^4\zeta(3)(j\ln s + 2) - \frac{g^2}{6}\pi^2\gamma_{g^2} + 6g^4\zeta(3)j\ln s - 2g^4\zeta(3) = \\
&= -6g^4\zeta(3) + 4g^4\zeta(3)j\ln s - \frac{g^2}{6}\pi^2\gamma_{g^2}.
\end{aligned}
\tag{4.46}
$$

The computation of the contribution, which we indicate with $\gamma_{g^4,2}$, coming from the last term in (4.45) relies on the one loop results. If one wants to restrict to terms containing powers of j not higher than 4, the relevant integral to compute is simple:

$$
\gamma_{g^4,2} = \frac{g^4}{24\pi}c_0^3\sigma_0(0)\int_0^{+\infty}dh\frac{h^4}{\sinh\frac{h}{2}} = \frac{g^4}{\pi}62\zeta(5)c_0^3\sigma_0(0).
\tag{4.47}
$$

[1]In this section and we will use the notation $[\ldots]_{g^2}$ (and analogous ones) to indicate the order g^2 of the quantity inside the square brackets.

Using one loop results we have, up to the desired order

$$
\begin{aligned}
c_0^3\sigma_0(0) &= \Big[-\frac{\pi^3}{16}j^3+\frac{\pi^3}{8}j^4\ \ln 2\Big]\ln s+ \\
&+ \Big[\frac{\pi^3}{8}(2\ln 2+\gamma_E)j^3-3\frac{\pi^3}{8}j^4\ \ln 2(2\ln 2+\gamma_E)\Big]+O\left(\frac{1}{\ln s}\right) ,
\end{aligned}
$$

in such a way that the corresponding contribution to the energy reads

$$
\begin{aligned}
\gamma_{g^4,2} &= \frac{g^4}{\pi}62\zeta(5)\Big[-\frac{\pi^3}{16}j^3+\frac{\pi^3}{8}j^4\ \ln 2\Big]\ln s+ \\
&+ \frac{g^4}{\pi}62\zeta(5)\Big[\frac{\pi^3}{8}(2\ln 2+\gamma_E)j^3-3\frac{\pi^3}{8}j^4\ \ln 2(2\ln 2+\gamma_E)\Big]+ \\
&+ O\left(\frac{1}{\ln s}\right) . \qquad (4.48)
\end{aligned}
$$

We are now left with the calculation of the contribution - $\gamma_{g^4,3}$ - to the anomalous dimension coming from the term in the second line of (4.45). Since we restrict to terms containing powers of j not higher that 4, at the relevant order such contribution equals

$$
\gamma_{g^4,3}=-\frac{g^2}{6\pi}[c^3\sigma(0)]_{g^2}\int_0^{+\infty}dh\frac{h^2}{\sinh\frac{h}{2}}=-g^2\frac{14}{3\pi}\zeta(3)[c^3\sigma(0)]_{g^2} . \qquad (4.49)
$$

We have now to compute $\sigma(0)$ and c up to the order g^2. For what concerns $\sigma(0)$ we have

$$
\sigma(0)=[-4-4j\ln 2+O(j^3)]\ln s-[8\ln 2+4\gamma_E+O(j^3)]+[\sigma_H(0)]_{g^2}+O\left(\frac{1}{\ln s}\right) , \qquad (4.50)
$$

where, from (3.39) (focusing only on leading terms of our actual interest), (4.43), one has, neglecting terms $O(j^3)$,

$$
[\sigma_H(0)]_{g^2}=\int_{-\infty}^{\infty}dh\frac{|h|}{2\sinh\frac{|h|}{2}}[S(h)]_{g^2}=14(j\ln s+2)g^2\zeta(3)+\frac{\pi^2}{2}\gamma_{g^2}+O\left(\frac{1}{\ln s}\right) . \qquad (4.51)
$$

On the other hand, the condition (4.41) for c at the desired order reads as

$$
2\sigma(0)c=-2\pi j\ln s . \qquad (4.52)
$$

Using such relation, we can get c at the order g^2 and up to the orders j^2 and $1/\ln s$:

$$
\begin{aligned}
[c]_{g^2} &= \left[\frac{\pi}{16\ln s}j(1-2\ln 2j)+\frac{\pi}{16(\ln s)^2}(\gamma_E+2\ln 2)(-2j+6\ln 2j^2)\right]\cdot \\
&\cdot \quad [28g^2\zeta(3)+\frac{\pi^2}{2}\gamma_{g^2}+14jg^2\zeta(3)\ln s]+O\left(\frac{1}{(\ln s)^2}\right) . \qquad (4.53)
\end{aligned}
$$

Coming back to the energy, we have

$$
\gamma_{g^4,3}=g^2\frac{14}{3\pi}\zeta(3)\pi j\ln s\ [c^2]_{g^2}+O(j^5,\ln s) , \qquad (4.54)
$$

which, after using the one loop expression for c_0 up to the order j^2,

$$
\begin{aligned}
c_0 &= \Big[\frac{\pi}{4}j-\frac{\pi}{4}\ln 2\ j^2+O(j^3)\Big]+\Big[-\frac{\pi}{4}(2\ln 2+\gamma_E)j+ \qquad (4.55)\\
&+ \frac{\pi}{2}\ln 2(2\ln 2+\gamma_E)\,j^2+O(j^3)\Big]\frac{1}{\ln s}+O\left(\frac{1}{(\ln s)^2}\right) ,
\end{aligned}
$$

gives

$$\gamma_{g^4,3} = \frac{7}{48}\zeta(3)g^2\pi^2 j^3\Big[(1-3j\ln 2) + \frac{2\ln 2+\gamma_E}{\ln s}(-3+12\ln 2\, j)\Big]\cdot$$
$$\cdot\,[28g^2\zeta(3) + \frac{\pi^2}{2}\gamma_{g^2} + 14jg^2\zeta(3)\ln s] + O\left(\frac{1}{\ln s}\right)\,. \tag{4.56}$$

Summing together (4.46, 4.48, 4.56) we get the energy at the order g^4 and up to the order j^4:

$$\begin{aligned}
\gamma_{g^4} &= -6g^4\zeta(3) + 4g^4\zeta(3)j\ln s - \frac{g^2}{6}\pi^2\gamma_{g^2} + \\
&+ \frac{g^4}{\pi}62\zeta(5)\Big[-\frac{\pi^3}{16}j^3 + \frac{\pi^3}{8}j^4\,\ln 2\Big]\ln s + \\
&+ \frac{g^4}{\pi}62\zeta(5)(2\ln 2+\gamma_E)\Big[\frac{\pi^3}{8}j^3 - 3\frac{\pi^3}{8}j^4\,\ln 2\Big] + \\
&+ \frac{7}{48}\zeta(3)g^2\pi^2 j^3\Big[(1-3j\ln 2) + \frac{2\ln 2+\gamma_E}{\ln s}(-3+12\,j\ln 2)\Big]\cdot \\
&\cdot\;\; [28g^2\zeta(3) + \frac{\pi^2}{2}\gamma_{g^2} + 14j\,g^2\zeta(3)\ln s] + O\left(\frac{1}{\ln s}\right)\,.
\end{aligned} \tag{4.57}$$

Alternatively, we can write

$$\begin{aligned}
\gamma_{g^4} = &-\frac{\pi^2}{6}g^2\gamma_{g^2} + g^4\ln s\Big[4\zeta(3)\,j - \frac{31}{8}\pi^2\zeta(5)\,j^3 + \frac{7}{24}\pi^4\zeta(3)\,j^3 + \\
&+ \frac{31}{4}\pi^2\ln 2\,\zeta(5)\,j^4 + \frac{49}{24}\pi^2\zeta(3)^2\,j^4 - \frac{7}{6}\pi^4\zeta(3)\ln 2\,j^4 + O(j^5)\Big] + \\
&+ g^4\Big[-6\zeta(3) + \frac{31}{4}\pi^2\zeta(5)(2\ln 2+\gamma_E)\,j^3 + \frac{49}{12}\pi^2\zeta(3)^2\,j^3 - \\
&- \frac{7}{12}\pi^4\zeta(3)(3\ln 2+\gamma_E)j^3 + \\
&+ \frac{7}{4}\pi^4\zeta(3)\ln 2(5\ln 2+2\gamma_E)\,j^4 - \frac{93}{4}\pi^2\zeta(5)\ln 2(2\ln 2+\gamma_E)\,j^4 - \\
&- \frac{49}{8}\zeta(3)^2\pi^2(4\ln 2+\gamma_E)\,j^4 + O(j^5)\Big] + O\left(\frac{1}{\ln s}\right)\,.
\end{aligned} \tag{4.58}$$

Three loops anomalous dimension

For the three loops anomalous dimension, the calculation follows the same route as in the two loops case. We simply report the final result. The three loops anomalous dimension γ_{g^6} expands as

$$\gamma_{g^6} = \ln s\sum_{n=0}^{\infty} f_{n,g^6}\,j^n + \sum_{n=0}^{\infty} f_{sl,n,g^6}\,j^n + O(1/\ln s)\,, \tag{4.59}$$

where the coefficients f_{n,g^6} were already computed [29] (at least, explicitly up to $n = 4$) and the new coefficients f_{sl,n,g^6}, still up to $n = 4$, read as

$$f_{sl,0,g^6} = 20\zeta(5) + \frac{2}{3}\pi^2\zeta(3) + \frac{11}{45}\pi^4\gamma_E\,, \tag{4.60}$$

$$f_{sl,1,g^6} = f_{sl,2,g^6} = 0\,, \tag{4.61}$$

$$\begin{aligned} f_{sl,3,g^6} = & -\frac{331}{1080}\pi^6\zeta(3)(2\ln 2 + \gamma_E) + \frac{1}{144}\pi^6\zeta(3)(42\ln 2 + 49\gamma_E) + \\ & + \frac{31}{3}\pi^4\zeta(5)(2\ln 2 + \gamma_E) - \frac{31}{8}\pi^4\gamma_E\zeta(5) + \frac{385}{72}\pi^4\zeta(3)^2 \\ & - \frac{635}{8}\pi^2\zeta(7)(2\ln 2 + \gamma_E) - \frac{651}{8}\pi^2\zeta(3)\zeta(5)\,, \end{aligned} \tag{4.62}$$

$$\begin{aligned} f_{sl,4,g^6} = & \frac{1905}{8}\pi^2\ln 2(2\ln 2 + \gamma_E)\zeta(7) + \frac{1953}{16}\pi^2(2\ln 2 + \gamma_E)\zeta(3)\zeta(5) + \\ & + \frac{1953}{8}\pi^2\ln 2\zeta(3)\zeta(5) - \frac{341}{8}\pi^4\ln 2(2\ln 2 + \gamma_E)\zeta(5) - \frac{93}{4}\pi^4(\ln 2)^2\zeta(5) - \\ & - \frac{385}{48}\pi^4(2\ln 2 + \gamma_E)\zeta(3)^2 - \frac{413}{12}\pi^4\ln 2\zeta(3)^2 + \frac{343}{8}\pi^2\zeta(3)^3 + \\ & + \frac{767}{720}\pi^6\ln 2(2\ln 2 + \gamma_E)\zeta(3) + \frac{35}{12}\pi^6(\ln 2)^2\zeta(3)\,. \end{aligned} \tag{4.63}$$

4.4 Systematics of the subleading term

We now want to put the linear integral equation (4.40) in a form which is more suitable for analysis of both the weak and the strong coupling limit. In brief, instead of working with one linear integral equation (4.40), we will be concerned with linear infinite (i.e. containing infinite equations) systems. As far as the dependence on $\ln s$ is concerned, equation (4.40) and its solution split in a part proportional to $\ln s$ and a part proportional to $(\ln s)^0$. Since the former has been extensively studied [29, 46], we concentrate on the latter, which we call $S^{(0)}(k)$. We restrict to the domain $k \geq 0$ and expand $S^{(0)}(k)$ in series of Bessel functions

$$S^{(0)}(k) = \sum_{r=1}^{\infty} S_r^{(0)}(g)\frac{J_r(\sqrt{2}gk)}{k}\,. \tag{4.64}$$

As a consequence of (4.40), the coefficients $S_r^{(0)}(g)$ satisfy the following system of equations,

$$\begin{aligned} S_{2p-1}^{(0)}(g) = & 2\sqrt{2}g\gamma_E\delta_{p,1} + 4(2p-1)\int_0^{+\infty}\frac{dh}{h}\frac{\tilde{J}_{2p-1}(\sqrt{2}gh)}{e^h - 1} + A_{2p-1}^{(0)}(g) - \\ & - 2(2p-1)\sum_{m=1}^{\infty} Z_{2p-1,m}(g)S_m^{(0)}(g)\,, \\ S_{2p}^{(0)}(g) = & 4 + 8p\int_0^{+\infty}\frac{dh}{h}\frac{J_{2p}(\sqrt{2}gh)}{e^h - 1} + A_{2p}^{(0)}(g) + 4p\sum_{m=1}^{\infty} Z_{2p,2m-1}(g)S_{2m-1}^{(0)}(g) - \\ & - 4p\sum_{m=1}^{\infty} Z_{2p,2m}(g)S_{2m}^{(0)}(g)\,, \end{aligned} \tag{4.65}$$

where, as usual, we introduced the notation

$$Z_{n,m}(g) = \int_0^{+\infty}\frac{dh}{h}\frac{J_n(\sqrt{2}gh)J_m(\sqrt{2}gh)}{e^h - 1}\,, \tag{4.66}$$

and where we introduced the function $\tilde{J}_{2p-1}(x)$ which coincides with the Bessel function $J_{2p-1}(x)$ for $p \geq 2$ and with $J_1(x) - \frac{x}{2}$ when $p = 1$. In (4.65) the terms $A_r^{(0)}(g)$ is the term proportional to $(\ln s)^0$ of the quantity:

$$A_r(g) = r \int_0^{+\infty} \frac{dh}{2\pi h} \frac{J_r(\sqrt{2}gh)}{\sinh \frac{h}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} 2 \left[\frac{\sin(h-p)c}{h-p} - \frac{\sin pc}{p} \right] \hat{\sigma}(p) \,. \tag{4.67}$$

Eventually, anomalous dimension at order $(\ln s)^0$, $\gamma^{(0)}$, is extracted from $S_1^{(0)}(g)$ by means of the formula

$$\gamma^{(0)} = \sqrt{2} g S_1^{(0)}(g) \,. \tag{4.68}$$

To be complete, we write down also the equations satisfied by the part of $S(k)$, which is linear in $\ln s$. Let us call such part $S^{(-1)}(k)$. Expanding in series of Bessel functions

$$S^{(-1)}(k) = \sum_{r=1}^{\infty} S_r^{(-1)}(g) \frac{J_r(\sqrt{2}gk)}{k} \,, \tag{4.69}$$

we find the following system of equations for the coefficients $S_r^{(-1)}(g)$,

$$\begin{aligned} S_{2p-1}^{(-1)}(g) &= 2\sqrt{2}g\delta_{p,1} - 2(2p-1)\, j \int_0^{+\infty} \frac{dh}{h} \frac{J_{2p-1}(\sqrt{2}gh)}{e^{\frac{h}{2}}+1} + A_{2p-1}^{(-1)}(g) - \\ &- 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{(-1)}(g) \,, \\ S_{2p}^{(-1)}(g) &= 2j - 4p\, j \int_0^{+\infty} \frac{dh}{h} \frac{J_{2p}(\sqrt{2}gh)}{e^{\frac{h}{2}}+1} + A_{2p}^{(-1)}(g) + \\ &+ 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(-1)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(-1)}(g) \,, \end{aligned} \tag{4.70}$$

where $A_r^{(-1)}(g)$ is the term proportional to $\ln s$ of the quantity (4.67).

This system has been used in the series of papers [42, 44, 46] in order to compute the (strong coupling limit of) the generalised scaling function $f(g, j)$. In the next subsections we will adapt the steps of [42, 44, 46] to system (4.65) in order to study the subleading function $f^{(0)}(g, j)$ as series in j, up to the order j^5.

4.4.1 Slicing in powers of j

In the limit (4.1) the function $S^{(0)}(k)$ admits an expansion in powers of j:

$$S^{(0)}(k) = \sum_{n=0}^{\infty} S^{(0,n)}(k) j^n \,. \tag{4.71}$$

And, this way of expanding extends to the coefficients $S_r^{(0)}(g)$ (4.64) as well:

$$S_r^{(0)}(g) = \sum_{n=0}^{\infty} S_r^{(0,n)}(g) j^n \,. \tag{4.72}$$

In expansion (4.72) the coefficients $S_r^{(0,0)}(g)$ - which give the part of the function $S^{(0)}(k)$ independent of j - satisfy the system of equations

$$S_{2p-1}^{(0,0)}(g) = 2\sqrt{2}g\gamma_E\delta_{p,1} + 4(2p-1)\int_0^{+\infty}\frac{dh}{h}\frac{\tilde{J}_{2p-1}(\sqrt{2}gh)}{e^h-1} - \\ -2(2p-1)\sum_{m=1}^{\infty} Z_{2p-1,m}(g)S_m^{(0,0)}(g)\,,$$

$$S_{2p}^{(0,0)}(g) = 4 + 8p\int_0^{+\infty}\frac{dh}{h}\frac{J_{2p}(\sqrt{2}gh)}{e^h-1} + 4p\sum_{m=1}^{\infty} Z_{2p,2m-1}(g)S_{2m-1}^{(0,0)}(g) - \\ - 4p\sum_{m=1}^{\infty} Z_{2p,2m}(g)S_{2m}^{(0,0)}(g)\,. \tag{4.73}$$

This system of equations is studied in the contemporaneous paper [51] - the coefficients $S_r^{(0,0)}(g)$ being there denoted as $S_r^{extra}(g)$. Therefore, we pass to study the function $S_r^{(0,n)}(g)$, when $n \geq 1$. For simplicity's sake we limit to the cases $n = 3, 4, 5$ (for $n = 1, 2$, $S_r^{(0,n)}(g) = 0$). The forcing terms $A_r^{(0,n)}(g)$ appearing in the system for $S_r^{(0,n)}(g)$, $n \geq 3$,

$$S_{2p-1}^{(0,n)}(g) = A_{2p-1}^{(0,n)}(g) - 2(2p-1)\sum_{m=1}^{\infty} Z_{2p-1,m}(g)S_m^{(0,n)}(g)\,,$$

$$S_{2p}^{(0,n)}(g) = A_{2p}^{(0,n)}(g) + 4p\sum_{m=1}^{\infty} Z_{2p,2m-1}(g)S_{2m-1}^{(0,n)}(g) - 4p\sum_{m=1}^{\infty} Z_{2p,2m}(g)S_{2m}^{(0,n)}(g)\,, \tag{4.74}$$

are obtained after developing (4.67) for small c up to c^5:

$$A_r^{(0,n)}(g) = r\int_0^{+\infty}\frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\left[-\frac{1}{3}h^2c^3\sigma(0) + \frac{1}{60}h^4c^5\sigma(0) - \frac{1}{10}h^2c^5\sigma_2(0)\right]_{(\ln s)^0, j^n} \tag{4.75}$$

with $3 \leq n \leq 5$ and where $\sigma_2(0)$ stands for the second derivative of $\sigma(u)$ at $u = 0$. Now, from the expansions in powers of $\ln s$:

$$\sigma(0) = \ln s\;\sigma^{(-1)}(0) + \sigma^{(0)}(0) + O\left(\frac{1}{\ln s}\right)\,,\; \sigma_2(0) = \ln s\;\sigma_2^{(-1)}(0) + \sigma_2^{(0)}(0) + O\left(\frac{1}{\ln s}\right)\,,$$
$$c = c^{(0)} + (\ln s)^{-1}\;c^{(1)} + O\left(\frac{1}{(\ln s)^2}\right)\,, \tag{4.76}$$

we get ($3 \leq n \leq 5$):

$$A_r^{(0,n)}(g) = r\int_0^{+\infty}\frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\Big[-\frac{1}{3}h^2\left({c^{(0)}}^3\sigma^{(0)}(0) + 3{c^{(0)}}^2c^{(1)}\sigma^{(-1)}(0)\right) + \tag{4.77}$$
$$+\frac{1}{60}h^4\left({c^{(0)}}^5\sigma^{(0)}(0) + 5{c^{(0)}}^4c^{(1)}\sigma^{(-1)}(0)\right) - \frac{1}{10}h^2\left({c^{(0)}}^5\sigma_2^{(0)}(0) + 5{c^{(0)}}^4c^{(1)}\sigma_2^{(-1)}(0)\right)\Big]_{j^n}\,.$$

Keeping in mind the expansions in powers of j:

$$c^{(k)} = \sum_{n=1}^{\infty} c^{(k,n)}j^n\,,\; \sigma^{(k-1)}(0) = \sum_{n=0}^{\infty}\sigma^{(k-1,n)}(0)j^n\,,\; \sigma_2^{(k-1)}(0) = \sum_{n=0}^{\infty}\sigma_2^{(k-1,n)}(0)j^n\,,\; k = 0, 1, \tag{4.78}$$

we can now specialise (4.77) to the cases $n = 3$,

$$A_r^{(0,3)}(g) = r\int_0^{+\infty} \frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\left[-\frac{1}{3}h^2\left({c^{(0,1)}}^3\sigma^{(0,0)}(0) + 3{c^{(0,1)}}^2 c^{(1,1)}\sigma^{(-1,0)}(0)\right)\right], \tag{4.79}$$

$n = 4$,

$$A_r^{(0,4)}(g) = r\int_0^{+\infty} \frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\Big[-\frac{1}{3}h^2\Big(3{c^{(0,1)}}^2 c^{(0,2)}\sigma^{(0,0)}(0) + 3{c^{(0,1)}}^2 c^{(1,1)}\sigma^{(-1,1)}(0) +$$
$$+ 3{c^{(0,1)}}^2 c^{(1,2)}\sigma^{(-1,0)}(0) + 6c^{(0,1)}c^{(0,2)}c^{(1,1)}\sigma^{(-1,0)}(0)\Big)\Big] \tag{4.80}$$

and $n = 5$:

$$A_r^{(0,5)}(g) = r\int_0^{+\infty} \frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\Big[-\frac{1}{3}h^2\Big(3{c^{(0,1)}}^2 c^{(0,3)}\sigma^{(0,0)}(0) + 3{c^{(0,2)}}^2 c^{(0,1)}\sigma^{(0,0)}(0) +$$
$$+ 3{c^{(0,1)}}^2 c^{(1,3)}\sigma^{(-1,0)}(0) + 6c^{(0,1)}c^{(0,2)}c^{(1,2)}\sigma^{(-1,0)}(0) + 6c^{(0,1)}c^{(0,3)}c^{(1,1)}\sigma^{(-1,0)}(0) +$$
$$+ 3{c^{(0,2)}}^2 c^{(1,1)}\sigma^{(-1,0)}(0) + 3{c^{(0,1)}}^2 c^{(1,2)}\sigma^{(-1,1)}(0) + 6c^{(0,1)}c^{(0,2)}c^{(1,1)}\sigma^{(-1,1)}(0)\Big) +$$
$$+ \frac{1}{60}h^4\left({c^{(0,1)}}^5\sigma^{(0,0)}(0) + 5{c^{(0,1)}}^4 c^{(1,1)}\sigma^{(-1,0)}(0)\right) -$$
$$- \frac{1}{10}h^2\left({c^{(0,1)}}^5\sigma_2^{(0,0)}(0) + 5{c^{(0,1)}}^4 c^{(1,1)}\sigma_2^{(-1,0)}(0)\right)\Big]. \tag{4.81}$$

On the other hand, the c can be expressed in terms of the $\sigma(0)$ by means of the condition (4.41). At the relevant order in c such condition reads

$$2\sigma(0)c + \frac{1}{3}\sigma_2(0)c^3 = -2\pi j \ln s. \tag{4.82}$$

The order $(\ln s)^0$ of such equation gives

$$2\sigma^{(-1)}(0)c^{(1)} + 2\sigma^{(0)}(0)c^{(0)} + \sigma_2^{(-1)}(0){c^{(0)}}^2 c^{(1)} + \frac{1}{3}\sigma_2^{(0)}(0){c^{(0)}}^3 = 0. \tag{4.83}$$

Specialising such equation at order j, we get the condition [2]

$$c^{(1,1)} = -\frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)}c^{(0,1)} = \pi\frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^2}. \tag{4.84}$$

At order j^2 we have

$$c^{(1,2)} = -\frac{\sigma^{(-1,1)}(0)}{\sigma^{(-1,0)}(0)}c^{(1,1)} - \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)}c^{(0,2)} = -2\pi\frac{\sigma^{(0,0)}(0)\sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^3}. \tag{4.85}$$

Going at order j^3 we get

$$c^{(1,3)} = -\frac{\sigma^{(-1,1)}(0)}{\sigma^{(-1,0)}(0)}c^{(1,2)} - \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)}c^{(0,3)} - \frac{1}{2}\frac{\sigma_2^{(-1,0)}(0)}{\sigma^{(-1,0)}(0)}{c^{(0,1)}}^2 c^{(1,1)} - \frac{1}{6}\frac{\sigma_2^{(0,0)}(0)}{\sigma^{(-1,0)}(0)}{c^{(0,1)}}^3 =$$
$$= 3\pi\frac{\sigma^{(0,0)}(0)[\sigma^{(-1,1)}(0)]^2}{[\sigma^{(-1,0)}(0)]^4} - \frac{2}{3}\pi^3\frac{\sigma^{(0,0)}(0)\sigma_2^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^5} + \frac{\pi^3}{6}\frac{\sigma_2^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^4}. \tag{4.86}$$

---

[2] We remember the formulæ (relations (5.18-5.20) of [46]):

$$c^{(0,1)} = -\frac{\pi}{\sigma^{(-1,0)}(0)}, \quad c^{(0,2)} = \pi\frac{\sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^2}, \quad c^{(0,3)} = \frac{\pi^3}{6}\frac{\sigma_2^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^4} - \pi\frac{[\sigma^{(-1,1)}(0)]^2}{[\sigma^{(-1,0)}(0)]^3}.$$

In such a way, we obtain

$$A_r^{(0,3)}(g) = r\int_0^{+\infty} \frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\left(-\frac{2}{3}\pi^3 h^2\right)\frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^3}\,, \tag{4.87}$$

and, also,

$$A_r^{(0,4)}(g) = r\int_0^{+\infty} \frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\,2\pi^3 h^2\frac{\sigma^{(0,0)}(0)\sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^4}\,. \tag{4.88}$$

For what concerns the term proportional to j^5, we get, after some calculation:

$$\begin{aligned} A_r^{(0,5)}(g) = r\int_0^{+\infty} \frac{dh}{2\pi h}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\Big[-\frac{1}{3}h^2\Big(\frac{\pi^5}{5}\frac{\sigma_2^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^5}+ \\ +12\pi^3\frac{\sigma^{(0,0)}(0)[\sigma^{(-1,1)}(0)]^2}{[\sigma^{(-1,0)}(0)]^5}-\pi^5\frac{\sigma^{(0,0)}(0)\sigma_2^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^6}\Big)+\frac{1}{15}h^4\pi^5\frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^5}\Big]\,. \end{aligned} \tag{4.89}$$

Now, in analogy to what done in [46], we introduce the 'reduced' coefficients $\tilde{S}_r^{(k)}(g)$ which satisfy the system (4.23) of [46], i.e. the 'usual' system with the BES kernel and with forcing terms

$$\mathbb{I}_r^{(k)}(g) = r\int_0^{+\infty}\frac{dh}{2\pi}h^{2k-1}\frac{J_r(\sqrt{2}gh)}{\sinh\frac{h}{2}}\,. \tag{4.90}$$

Using notations of [46], we can write the various f_n, $n=3,4,5$, in terms of $\tilde{S}_1^{(k)}(g)$ and of the density (of Bethe roots and holes) and its derivatives in zero

$$f_{sl,3}(g) = -\frac{2}{3}\pi^3\frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^3}\sqrt{2}g\tilde{S}_1^{(1)}(g)\,, \tag{4.91}$$

$$f_{sl,4}(g) = 2\pi^3\frac{\sigma^{(0,0)}(0)\sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^4}\sqrt{2}g\tilde{S}_1^{(1)}(g)\,, \tag{4.92}$$

$$\begin{aligned} f_{sl,5}(g) = \Big[-\frac{1}{3}\Big(\frac{\pi^5}{5}\frac{\sigma_2^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^5}+12\pi^3\frac{\sigma^{(0,0)}(0)[\sigma^{(-1,1)}(0)]^2}{[\sigma^{(-1,0)}(0)]^5}-\pi^5\frac{\sigma^{(0,0)}(0)\sigma_2^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^6}\Big)\times \\ \times\sqrt{2}g\tilde{S}_1^{(1)}(g)+\frac{\pi^5}{15}\frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^5}\sqrt{2}g\tilde{S}_1^{(2)}(g)\Big]\,. \end{aligned} \tag{4.93}$$

Expressions (4.91, 4.92, 4.93) are valid for all g, so they interpolate from weak to strong coupling. Their weak coupling expansion in powers of g^2 (at least for $f_3^{(0)}(g)$, $f_4^{(0)}(g)$) were given, up to g^6, in Section (4.3). In the next subsection, we will study and explicitly find their strong coupling limit.

Finally, for comparison, we write corresponding formulæ for the coefficients $f_{sl,n}(g)$ of the function $f(g,j)$, (5.30-32) of [46],

$$f_3(g) = \frac{\pi^3}{3}\frac{1}{[\sigma^{(-1,0)}(0)]^2}\sqrt{2}g\tilde{S}_1^{(1)}(g)\,, \tag{4.94}$$

$$f_4(g) = -\frac{2}{3}\pi^3\frac{\sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^3}\sqrt{2}g\tilde{S}_1^{(1)}(g)\,, \tag{4.95}$$

$$f_5(g) = \Big[\pi^3\frac{[\sigma^{(-1,1)}(0)]^2}{[\sigma^{(-1,0)}(0)]^4}-\frac{\pi^5}{15}\frac{\sigma_2^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^5}\Big]\sqrt{2}g\tilde{S}_1^{(1)}(g)-\frac{\pi^5}{60}\frac{1}{[\sigma^{(-1,0)}(0)]^4}\sqrt{2}g\tilde{S}_1^{(2)}(g)\,. \tag{4.96}$$

We notice the interesting proportionality between $f_{sl,n}(g)$ and $f_n(g)$ for $n = 3, 4$,

$$f_{sl,3}(g) = -2\frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)}f_3(g)\,, \quad f_{sl,4}(g) = -3\frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)}f_4(g)\,, \tag{4.97}$$

which, however, seems to be lost when $n = 5$. This could suggest a deep relation between $f^{(0)}(g, j)$ and $f(g, j)$, but at the moment our data are not conclusive about its presence and form.

4.4.2 Strong coupling

It is interesting to consider the strong coupling limit of the equations obtained in the last subsection. For that purpose, we need to know the values of the density and its derivatives in zero and the quantities $\tilde{S}_1^{(k)}(g)$ as well. Results concerning $\sigma^{(-1,n)}$, i.e. the part of the density proportional to $\ln s$, and $\tilde{S}_1^{(k)}(g)$ are reported in [46]. We remember [40, 43] that the contribution to the anomalous dimension proportional to $\ln s$, i.e. $f(g, j)$, in the limit (4.1) and when $j \ll g$, $g \to +\infty$, coincides with the energy density of the $O(6)$ sigma model. The nonperturbative (infrared) regime of the sigma model, $j \ll m(g)$, where (the dots stand for subleading corrections)

$$m(g) = kg^{\frac{1}{4}}e^{-\frac{\pi g}{\sqrt{2}}} + \ldots\,, \quad k = \frac{2^{\frac{5}{8}}\pi}{\Gamma\left(\frac{5}{4}\right)}\,, \tag{4.98}$$

makes contact with the double limit $j \ll 1$, $g \to \infty$, considered in this subsection (and also in [46]). In such regime the scale of the mass is given by $m(g)$ and expansion of $f(g, j)$ for small j and large g was successfully checked [44, 46] against analogous non perturbative expansions [45] in the sigma model.

Due to proportionality (4.97), one could ask if, as well as $f(g, j)$, also $f^{(0)}(g, j)$ shows connections with quantities of the $O(6)$ sigma model. This is a further motivation which leads to study the strong coupling limit of $f^{(0)}(g, j)$, which however is an important problem in itself, since it can be checked against string theory data. We concentrate on $f_n^{(0)}(g)$, for $n = 3, 4, 5$: as follows from (4.91, 4.92, 4.93), the calculation of their strong coupling limit can be finalised if we know the large g behaviour of the density and its second derivative in zero, $\sigma^{(0,0)}(0)$ and $\sigma_2^{(0,0)}(0)$. From results of [51], we know that at large g and at the leading order

$$S_r^{(0,0)}(g) = -\ln g\, S_r^{(-1,0)}(g) + \ldots\,, \tag{4.99}$$

where $S_r^{(0,0)}(g)$ is a solution of (4.73) and $S_r^{(-1,0)}(g)$ of (3.112) with $j = 0$ (i.e. the BES system). From this relation we can deduce that, at the leading order, the density in zero reads as

$$\sigma_H^{(0,0)}(0) = -\ln g\, \sigma_H^{(-1,0)}(0) + \ldots = -\ln g[4 - \pi m(g)] + \ldots \Rightarrow \sigma^{(0,0)}(0) = -4\ln g + \ldots \tag{4.100}$$

Analogously, for what concerns the second derivative, we get

$$\sigma_2^{(0,0)}(0) = 56\zeta(3) + \ldots \tag{4.101}$$

since the higher loops contributions, $\frac{\pi^3}{4}\ln g\, m(g)$, are exponentially depressed.

Using the results contained in [46], we get the strong coupling relations, which are valid only at the leading order, i.e. with $m(g)$ given by only its leading contribution (4.98),

$$
\begin{aligned}
f_{sl,3}(g) &= -\frac{8\ln g}{\pi m(g)} f_3(g) + ... = -\frac{\pi \ln g}{3m(g)^2} + ... \qquad (4.102)\\
f_{sl,4}(g) &= -\frac{12\ln g}{\pi m(g)} f_4(g) + ... = \frac{\ln g}{m(g)^3}\left(\ln 2 + \frac{\pi}{2}\right) + ... \qquad (4.103)\\
f_{sl,5}(g) &= -\frac{16\ln g}{\pi m(g)} f_5(g) + \frac{\pi^3 \ln g}{120 m(g)^4} + ... = \\
&= -\frac{2\ln g}{\pi m(g)^4}\left(\ln 2 + \frac{\pi}{2}\right)^2 + \frac{\pi^3 \ln g}{30 m(g)^4} + ... \qquad (4.104)
\end{aligned}
$$

The simple relation between $f_{sl,n}(g)$ and $f_n(g)$ at strong coupling could be the signal of a possible presence of the $O(6)$ sigma model also at the level of the subleading, order $(\ln s)^0$, scaling function $f_{sl}(g, j)$. We hope to address in a more general way this issue in the next future.

Conclusions

We summarize now all the results collected. First of all, this work contains a general mathematical method very useful to solve Non-linear integral equation in the high spin limit, for dilatation operators related to $sl(2)$ operators. We have seen how to evaluate non-linear terms, discarding terms of order $O\left(\frac{1}{s^2}\right)$. Great attention has been given to contribution of holes. We have seen that passing to NLIE approach *costs* the introduction of contributions due to these fake solutions of Bethe equations that we have to erase to obtain correct results. Evaluating position of holes has played a central role in our computations, but we have showed how to deal with them. In particular, in the case of fixed twist, it has been showed a recursive scheme that permits to express positions of internal holes in terms of derivatives of the counting function in zero. For external holes, we have computed exactly their position in high spin limit, discarding terms of order $O\left(\frac{1}{s}\right)$. In high spin and high twist limit, summing on internal holes has been treated using again a Non-linear integral equation approach and performing a specific evaluation for related non-linear terms that are of order $O\left(\frac{1}{\ln s}\right)$, discarded in this limit. Presence of holes and various evaluation of non-linear terms emerging in computations are the two greatest obstacle in NLIE picture. We have showed how to overcome them up to the desired order we are interesting in.

Let us now pass to specific results obtained. Main purpose in analyzing high spin and fixed twist limit was to understand the limit of validity of Asymptotic Bethe Ansatz approach without the introduction of wrapping corrections. We have obtained some general expressions, valid at all loops, for subleading terms of minimal anomalous dimension, up to orders $(\ln s)^{-n}$ and $(\ln s)^{-n}s^{-1}$, with $n = 1, \cdots, 5$. These expressions are not a complete solution, because they depend on derivatives of counting function in zero, that we should compute solving NLIE for the counting function. This has been made at weak coupling but we might, by a numerical approach, try to solve it at all values of coupling constant. This could be a natural following step for this work. However, expressions we have obtained have permitted to find some general relations among subleading terms: functions at order $(\ln s)^{-n}s^{-1}$ completely depend on functions at order $(\ln s)^{-n}$. This is a very important result: it gives strong hints for the validity of the appearing of wrapping contributions only at order $\frac{(\ln s)^2}{s^2}$ at all loops (for details, see Chapter 3). This limit for wrapping is formally prooved for short operators of twist two or three and up to six loops. What we have obtained is far away a formal proof, but it contains strong suggestions. Then, relations among subleading functions we have found fit exactly what is expected by reciprocity and self-tuning properties. We know that these properties are also valid in some aspects in QCD at one loop. We have found validity of them in $sl(2)$ subsector of $\mathcal{N} = 4$ SYM at all loops.

For what concerns high twist limit, we have studied it because in this limit wrapping does not appear, because it is an effect due to the eventual finite values of twist of operators. High spin and high twist limits are connected by the definition of the parameter $j = \frac{L-2}{\ln s}$ that we kept finite. We have focused our analysis on the subleading term of order $(\ln s)^0$, in the case $j \ll 1$. we have first obtained weak coupling results, giving explicit computations and results up to three loops. Subsequently, we have showed a more general systematic study of the subleading term and, at the end, we have obtained some results at strong coupling. Strong coupling results are very important in the framework of AdS/CFT duality, beacuse we can directly compare them with string theories. At strong coupling, our results show a possible presence of the $O(6)$ sigma model at the subleading $(\ln s)^0$ order. Very recently, relations between $O(6)$ sigma model and subleading correction has been explicitely showed in [52].

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