Acknowledgments

At the end of this experience I would like to thank my supervisor Prof.ssa Annamaria Canino for the time she spent on my thesis and for allowing me to continue and complete my course of study.
I am also grateful to my supervisor Dott. Berardino Sciunzi, who was always willing to resolve my doubts and to support me during the writing of this work. His passion for research has been totally contagious.

To my Family and to Those who have made this thesis possible

Come in -*She said*- I'll give you shelter from the storm.

Contents

Chapter 1

Introduction

We consider the problem

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^{\beta}} + g(x,u) & \text{a.e. in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\beta > 0$ and g is a Carathéodory function fulfilling suitable assumptions that will be specified later. We will assume that the function $f(\cdot)$ is nonnegative and belongs to $L^{\infty}(\Omega)$ and bounded away from zero in some preliminary results. In our main applications we will only assume that $f(\cdot)$ is a nonnegative $L^1(\Omega)$-function if $\beta > 1$, while $f(\cdot)$ is a nonnegative $L^m(\Omega)$-function with $m = \frac{2N}{N+2+\beta(N-2)} = \left(\frac{2^*}{1-\beta}\right)'$ if $0 < \beta \leq 1$.
Singular problems like (1.1) where first considered in the seminal paper [11]. The literature on this topic is now wide and the interest on singular elliptic equations is growing up. Let us refer the interested readers to the papers [6, 7, 8, 16, 18, 19].
Note that the equation $-\Delta\, u = \frac{f}{u^{\beta}} + g(x,u)$ has to be understood in the weak distributional meaning as follows:

$$\int_{\Omega} (Du\,,\, D\varphi)\, dx \;=\; \int_{\Omega} \left(\frac{f}{u^{\beta}} + g(x,u)\right) \varphi\, dx \qquad \forall\, \varphi \in C^1_c(\Omega)\,,$$

so that each term make sense, being u positive in the interior of the domain.
In order to state our main results let us consider $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ the unique solution to the problem:

$$\begin{cases} -\Delta\, u_0 \;=\; \dfrac{f}{{u_0}^{\beta}} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We will prove the existence of u_0 exploiting two different approaches. The first one is based on the the scheme used in [4] which exploits a truncation argument. Actually, in a slightly more general setting, for $n \in \mathbb{N}$, we set

$$f_n(x) = \min\{f, n\}$$

and we consider the truncated regularized problem:

$$\begin{cases} -\mathrm{div}(M(x)\, Du_n) = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\beta}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Obtaining uniform bounds for u_n and passing to the limit the existence result follows.

The second approach we use is based on a sub-supersolution method as in [8]. We will describe this technique later, let us only mention that a very refined analysis is needed to take care about the singular nature of the problem near the boundary. In fact, the sub-supersolutions method that we use is based on minimization arguments and comparison principles that can not be proved in a standard way because of the lack of regularity of the solution up to the boundary and the singularity of the nonlinearity.

It is worthwhile to mention that it is needed to our purpose to present both the approaches. In fact, by the first one we can deduce important summability properties of the solution. On the other hand, the second approach allows us to prove that the solution is continuous up to the boundary under suitable assumptions. Both these information will be used in the applications.

A crucial point in all the thesis will be the fact that a solution u to (1.1) is generally not in $H_0^1(\Omega)$ but can be decomposed as

$$u = u_0 + w$$

where $w \in H_0^1(\Omega)$ and u_0 is defined in (1.2).
Note that by the Lazer-McKenna result [19], recalled in Chapter 2, we know that u_0 does not belong to $H_0^1(\Omega)$ when $\beta \geq 3$.
It also follows by [19] that a standard variational approach can not be carried out in our setting, since the energy functional might be identically infinity in $H_0^1(\Omega)$.

Under suitable assumptions, following the variational characterization in [8], we will prove that problem (1.1) can be reduced to find $u \in H^1_{loc}(\Omega)$ fulfilling the variational inequality:

$$\begin{cases} u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{loc}(\Omega), \\ \displaystyle\int_\Omega (Du, D(v-u))\, dx \geq \int_\Omega \frac{f}{u^\beta}\,(v-u)\, dx + \int_\Omega g(x,u)(v-u)\, dx \\ \qquad \forall v \in u + (H_0^1(\Omega) \cap L_c^\infty(\Omega)) \text{ with } v \geq 0 \text{ a.e. in } \Omega, \\ u \leq 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where, as customary, we say that $u \leq 0$ on $\partial\Omega$ if, for every $\varepsilon > 0$, it follows that $(u-\varepsilon)^+ \in H_0^1(\Omega)$.

Let now $G_0 : \Omega \times \mathbb{R} \to [0, +\infty]$ be defined as

$$G_0(x,s) = \phi(u_0(x) + s) - \phi(u_0(x)) + \frac{s}{u_0^\beta(x)}$$

where

$$\phi(s) = \begin{cases} -\displaystyle\int_1^s \frac{1}{t^\beta}\, dt & \text{if } s \geq 0, \\ +\infty & \text{if } s < 0\,. \end{cases}$$

We also set $g_1(x,s) = g(x, u_0(x) + s)$,

$$G_1(x,s) = \int_0^s g_1(x,t)\, dt$$

and $F : H_0^1(\Omega) \to (-\infty, +\infty]$ defined as

$$F(w) = \widetilde{F}_0(w) + \gamma(w),$$

where

$$\widetilde{F}_0(w) = \frac{1}{2}\int_\Omega |Dw|^2\, dx + \int_\Omega f\, G_0(x,w)\, dx$$

and

$$\gamma(w) = -\int_\Omega G_1(x,w)\, dx.$$

Then we have the following result:

Theorem 1.0.1. *Assume that f is a bounded positive function which is bounded away form zero. Then the following assertions are equivalent:*

(a) $u \in H^1_{loc}(\Omega) \cap L^{\frac{2N}{N-2}}(\Omega)$ *and*

$$\begin{cases} -\Delta u = \dfrac{f}{u^\beta} + g(x,u) \ \textit{in} \ D'(\Omega), \\ u > 0 \ \textit{a.e. in} \ \Omega, \ \textit{and} \ u^{-\beta} \in L^1_{loc}(\Omega), \\ u \le 0 \ \textit{on} \ \partial\Omega. \end{cases} \tag{1.5}$$

(b) $u \in u_0 + H^1_0(\Omega)$ *and* $w = u - u_0$ *is a critical point of* F.

The variational characterization provided by Theorem 1.0.1 allows to exploit variational methods in order to prove existence of solutions in $u_0 + H^1_0(\Omega)$. For example, in [6] a jumping problem in the spirit of [1] is considered in the setting of singular semilinear elliptic equations. In particular, the main ingredients in [6] are the variational characterization of [8] and the use of a generalized version (see [22]) of the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [2].

In this setting we consider the following jumping problem.
Let $\beta > 0$ and $t \in \mathbb{R}$ and denote by λ_1 the first eigenvalue of $-\Delta$ with zero homogeneous Dirichlet boundary condition and consider φ_1 an associated first eigenfunction chosen such that $\varphi_1 > 0$ in Ω. We study the existence of solutions to the problem

$$\begin{cases} -\Delta u = \dfrac{f}{u^\beta} + g(x,u) - t\,\varphi_1 & \text{a.e. in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

where Ω is a bounded smooth domain and we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the following:

(g_1) there exist a and b such that

$$|g(x,s)| \le a(x) + b(x)\,|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R},$$

with $a \in L^{\frac{2N}{N+2}}(\Omega)$ and $b \in L^{\frac{N}{2}}(\Omega)$ if $n \ge 3$; a, $b \in L^p(\Omega)$ for some $p > 1$ if $n = 2$; a, $b \in L^1(\Omega)$ if $n = 1$;

(g_2) there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{s \to +\infty} \frac{g(x,s)}{s} = \alpha \qquad \text{for a.e. } x \in \Omega.$$

As showed above, problem (1.6) is equivalent to find $u \in H^1_{loc}(\Omega)$ fulfilling the variational inequality:

$$\begin{cases} u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{loc}(\Omega), \\ \displaystyle\int_\Omega (Du, D(v-u))\,dx \ge \int_\Omega \frac{f}{u^\beta}\,(v-u)\,dx \\ \qquad\qquad\qquad\qquad \displaystyle + \int_\Omega (g(x,u) - t\,\varphi_1)\,(v-u)\,dx. \\ \qquad \forall v \in u + (H^1_0(\Omega) \cap L^\infty_c(\Omega)) \text{ with } v \ge 0 \text{ a.e. in } \Omega, \\ u \le 0 \text{ on } \partial\Omega. \end{cases} \tag{1.7}$$

Our main result, see also [9], is the following:

Theorem 1.0.2. *Let $\beta > 0$ and assume that $g(x,s)$ satisfies assumptions (g_1)-(g_2) with $\alpha > \lambda_1$ and $a(x), b(x) \in L^\infty(\Omega)$. Assume that f is a nonnegative $L^1(\Omega)$-function if $\beta > 1$, while f is a nonnegative $L^m(\Omega)$-function with $m = \frac{2N}{N+2+\beta(N-2)}$ if $0 < \beta \leq 1$. Moreover assume that, given any compact set $\mathcal{C} \subset \Omega$, there exists a constant $\theta = \theta(\mathcal{C}) > 0$ with*

$$f \geq \theta \qquad a.e.\ in \quad \mathcal{C}\,.$$

Then there exists $\bar{t} \in \mathbb{R}$ such that, for any $t \geq \bar{t}$ there exist $u_1\,,\, u_2 \in H^1_{loc}(\Omega)$ solutions to (1.6). Furthermore

$$u_i \,:=\, u_0 \,+\, w_i \qquad for \quad i = 1,2,$$

where $u_0 \in H^1_{loc}(\Omega)$ is defined in (1.2) and $w_i \in H^1_0(\Omega)$. Moreover u_i are strictly bounded away from zero in the interior of Ω.

The idea of the proof of Theorem 1.0.2 is the following.
We define the truncated function of f as follows

$$f_n \,:=\, \max\left\{\frac{1}{n}\,,\, \min\left\{n\,,\, f\right\}\right\}.$$

Following [6], we are able to prove that there exists $\bar{t} \in \mathbb{R}$, *not depending on* $n \in \mathbb{N}$, such that, for any $t \geq \bar{t}$, there exist two solutions of the truncated problem involving f_n, namely

$$\begin{cases} -\Delta\, u_n \,=\, \dfrac{f_n}{{u_n}^\beta} \,+\, g(x, u_n) - t\varphi_1 & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{for } x \in \partial\Omega\,, \end{cases} \tag{1.8}$$

where $\beta > 0$ and $u_n \in\, C(\overline{\Omega}) \cap C^2(\Omega)$.
As above,

$$u_n \,=\, u_{0n} \,+\, w_n \tag{1.9}$$

with $w_n \in H^1_0(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ and $u_{0n} \in H^1_{loc}(\Omega) \cap\, C(\overline{\Omega}) \cap C^2(\Omega)$ is the unique solution to the problem:

$$\begin{cases} -\Delta\, u_{0n} \,=\, \dfrac{f_n}{{u_{0n}}^\beta} & \text{in } \Omega, \\ u_{0n} > 0 & \text{in } \Omega, \\ u_{0n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

To deduce Theorem 1.0.2, we prove that w_n is uniformly bounded in $H^1_0(\Omega)$ and then pass to the limit. It is a crucial ingredient here to prove that the solutions u_n are strictly uniformly bounded away from zero in the interior of Ω, as proved in Lemma 6.1.2. This is needed in order to pass to the limit in the equation, since the nonlinearity is singular.

In the second part of the thesis, we prove a symmetry (and monotonicity) result, exploiting the moving plane method of James Serrin.
We consider the problem

$$\begin{cases} -\Delta\, u \,=\, \dfrac{1}{u^\beta} \,+\, g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $\beta > 0$, Ω is a bounded smooth domain and $u \in\, C(\overline{\Omega}) \cap C^2(\Omega)$.
Our main results will be proved under the following assumption

(H_p) $g(\cdot)$ is locally Lipschitz continuous, positive ($g(s) > 0$ for $s > 0$ and $g(0) \geq 0$) and non-decreasing.

As mentioned above, the proof of our symmetry result is based on the well known *Moving Plane Method* [20], that was used in a clever way in the celebrated paper [13] in the semilinear non-degenerate case. Actually our proof is more similar to the one of [3] and is based on the weak comparison principle in small domains.
Because of the singular nature of our problem, we have to take care of two difficulties, namely:

- generally u does not belong to $H^1_0(\Omega)$,
- $\frac{1}{s^\beta} + g(s)$ is not Lipschitz continuous at zero.

In fact, as follows by the variational characterization described here above, the solution u can be decomposed as $u = u_0 + w$ where $w \in H^1_0(\Omega)$ and $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ is the unique solution to problem (1.2) with $f = 1$, namely:

$$\begin{cases} -\Delta\, u_0 \;=\; \dfrac{1}{{u_0}^{\beta}} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

Let us state our symmetry result (see [10]):

Theorem 1.0.3. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to* (1.11) $(u = u_0 + w)$ *with $g(\cdot)$ satisfying (H_p). Assume that the domain Ω is strictly convex w.r.t. the $\nu-$direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T^ν_0, where*

$$T^\nu_0 = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

Then u is symmetric w.r.t. T^ν_0 and non-decreasing w.r.t. the $\nu-$direction in Ω^ν_0, where

$$\Omega^\nu_\lambda = \{x \in \Omega : x \cdot \nu < \lambda\}\,.$$

Moreover, if Ω is a ball, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.

For the reader's convenience, we describe here below the scheme of the proof of Theorem 1.0.3.

(i) We first remark that, by [4] (see Chapter 3), we get that u_0 is the limit of a sequence $\{u_n\}$ of solutions to a truncated problem. We exploit this to deduce symmetry and monotonicity properties of $\{u_n\}$ and, consequently, of u_0, since the moving plane procedure applies in a standard way to the truncated problem.

(ii) By (i), recalling the decomposition $u = u_0 + w$, we are reduced to prove symmetry and monotonicity properties of w. To do this we prove some comparison principles for w needed in the application of the moving plane procedure.

(iii) In Section 7.6, we give some details about the adaptation of the moving plane procedure to the study of the monotonicity and symmetry of w. It is worth emphasizing that the moving plane procedure is applied in our approach only to the $H^1_0(\Omega)$ part of u.
Note also that Theorem 1.0.3 is proved in Section 7.7, as a consequence of a more general result proved in Proposition 7.6.1.

The thesis is organized as follows. In Chapter 2 we present the Lazer-Mckenna result (see [19]), showing that the solution to problem (1.2) generally does not belong to $H^1_0(\Omega)$ if $\beta \geq 3$ and, therefore, a standard variational approach fails. In Chapter 3 we consider the equation $-\Delta\, u = f\, u^{-\beta}$: at first we present the results of existence and regularity due to Boccardo and Orsina (see [4]) while, in the second part, we extablish a result of existence and uniqueness for the problem, by improving an idea due to Canino and Degiovanni (see [8]). In Chapter 4 we present a variational approach to our problem, following the idea in [8]. In Chapter 5 we study a jumping problem for the equation $-\Delta\, u = f\, u^{-\beta} + g(x,u) - t\,\varphi_1$ with bounded f and in Chapter 6 we consider a more general setting, with nonnegative unbounded f in $L^1(\Omega)$. In Chapter 7 we prove a symmetry and monotonicity result, exploiting the the moving plane method of James Serrin for positive solutions to the equation $-\Delta\, u = u^{-\beta} + g(u)$.

1.1 Introduzione

Consideriamo il problema

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^{\beta}} + g(x,u) & \text{a.e. in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

dove Ω è un dominio limitato in $\mathbb{R}^N$, $\beta > 0$ e g è una funzione di Carathéodory che soddisfa alcune opportune condizioni che verranno specificate in seguito. Supporremo che la funzione $f(\cdot)$ sia non negativa, appartenente a $L^\infty(\Omega)$ e staccata dallo zero in alcuni risultati preliminari. Nel principale risultato, invece, assumeremo solo che $f(\cdot)$ sia una funzione non negativa in $L^1(\Omega)$ se $\beta > 1$, mentre che $f(\cdot)$ sia una funzione non negativa di $L^m(\Omega)$ con $m = \frac{2N}{N+2+\beta(N-2)} = \left(\frac{2^*}{1-\beta}\right)'$ se $0 < \beta \leq 1$.
Problemi singolari come (1.13) furono considerati per la prima volta nel lavoro [11]. La letteratura su quest'argomento è ampia e l'interesse verso le equazioni singolari ellittiche sta crescendo. Ai lettori interessati suggeriamo [6, 7, 8, 16, 18, 19].
L'equazione $-\Delta\, u = \frac{f}{u^\beta} + g(x,u)$ deve intendersi nel *weak distributional sense*, come segue:

$$\int_\Omega (Du\,,\, D\varphi)\, dx \;=\; \int_\Omega \left(\frac{f}{u^\beta} + g(x,u)\right) \varphi\, dx \qquad \forall\, \varphi \in C^1_c(\Omega)\,,$$

in modo che ogni termine abbia senso, poiché u è positiva nell'interno del dominio.
Per enunciare il nostro principale risultato, consideriamo $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ l'unica soluzione del problema:

$$\begin{cases} -\Delta\, u_0 \;=\; \dfrac{f}{{u_0}^{\beta}} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

L'esistenza di u_0 sarà dimostrata utilizzando due approcci differenti. Il primo è basato sullo schema usato in [4], dove è presente un argomento di troncamento. Più precisamente, in uno scenario leggermente diverso, per $n \in \mathbb{N}$, poniamo

$$f_n(x) = \min\{f, n\}$$

e consideriamo il problema troncato generalizzato:

$$\begin{cases} -\mathrm{div}(M(x)\, Du_n) = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\beta}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Il risultato di esistenza segue ottenendo limiti uniformi su u_n e passando al limite.

Il secondo approccio che utilizziamo è basato su un metodo di sotto-soprasoluzione come in [8]. Descriveremo questa tecnica nei dettagli dopo, per il momento mettiamo in evidenza solo come sia necessaria un'analisi molto raffinata a causa della natura singolare del problema vicino al bordo. Infatti, il metodo di sotto-soprasoluzione che utilizziamo, è basato su argomenti di minimizzazione e principi di confronto che non possono essere dimostrati in maniera standard, a causa della perdita di regolarità della soluzione vicino al bordo e poiché la non-linearità è singolare. Vale la pena sottolineare che entrambi gli approcci sono necessari alla nostra trattazione. Infatti, dal primo deduciamo importanti proprietà di sommabilità della soluzione. D'altra parte, il secondo metodo ci permette di dimostrare che, sotto opportune ipotesi, la soluzione è continua fino al bordo.

Un punto cruciale in tutta la tesi sarà il fatto che una soluzione u di (1.13) in genere non appartiene a $H^1_0(\Omega)$, ma può essere decomposta come

$$u = u_0 + w$$

dove $w \in H_0^1(\Omega)$ e u_0 è definita in (1.14).
Osserviamo che da un risultato di Lazer e McKenna [19], richiamato in 2, sappiamo che u_0 non appartiene a $H_0^1(\Omega)$ quando $\beta \geq 3$.
Inoltre, da [19] segue anche che non è possibile utilizzare un approccio variazionale standard nel nostro caso, poiché il funzionale energia può essere identicamente uguale a infinito in $H_0^1(\Omega)$.

Sotto opportune ipotesi, seguendo la caratterizzazione variazionale utilizzata in [8], dimostreremo che il problema (1.13) è equivalente a trovare $u \in H^1_{loc}(\Omega)$ che soddisfi la disuguaglianza variazionale:

$$\begin{cases} u > 0 \text{ a.e. in } \Omega \text{ e } u^{-\beta} \in L^1_{loc}(\Omega), \\ \displaystyle\int_\Omega (Du, D(v-u))\,dx \geq \int_\Omega \frac{f}{u^\beta}(v-u)\,dx + \int_\Omega g(x,u)(v-u)\,dx \\ \qquad \forall v \in u + (H_0^1(\Omega) \cap L_c^\infty(\Omega)) \text{ con } v \geq 0 \text{ a.e. in } \Omega, \\ u \leq 0 \text{ on } \partial\Omega, \end{cases} \tag{1.16}$$

dove, come usuale, diciamo che $u \leq 0$ su $\partial\Omega$ se, per ogni $\varepsilon > 0$, si ha che $(u-\varepsilon)^+ \in H_0^1(\Omega)$.

Sia ora $G_0 : \Omega \times \mathbb{R} \to [0, +\infty]$ definita come

$$G_0(x,s) = \phi(u_0(x)+s) - \phi(u_0(x)) + \frac{s}{u_0^\beta(x)}$$

dove

$$\phi(s) = \begin{cases} -\displaystyle\int_1^s \frac{1}{t^\beta}dt & \text{if } s \geq 0, \\ +\infty & \text{if } s < 0\,. \end{cases}$$

Poniamo inoltre $g_1(x,s) = g(x, u_0(x)+s)$,

$$G_1(x,s) = \int_0^s g_1(x,t)\,dt$$

e $F : H_0^1(\Omega) \to (-\infty, +\infty]$ definito come

$$F(w) = \widetilde{F}_0(w) + \gamma(w),$$

dove

$$\widetilde{F}_0(w) = \frac{1}{2}\int_\Omega |Dw|^2\,dx + \int_\Omega f\,G_0(x,w)\,dx,$$

$$\gamma(w) = -\int_\Omega G_1(x,w)\,dx.$$

Allora abbiamo il seguente risultato:

Theorem 1.1.1. *Supponiamo che f sia una funzione positiva limitata e staccata da zero. Allora le seguenti affermazioni sono equivalenti:*

(a) $u \in H^1_{loc}(\Omega) \cap L^{\frac{2N}{N-2}}(\Omega)$ *e*

$$\begin{cases} -\Delta u = \dfrac{f}{u^\beta} + g(x,u) \text{ in } D'(\Omega), \\ u > 0 \text{ a.e. in } \Omega, \text{ and } u^{-\beta} \in L^1_{loc}(\Omega), \\ u \leq 0 \text{ on } \partial\Omega. \end{cases} \tag{1.17}$$

(b) $u \in u_0 + H_0^1(\Omega)$ *e* $w = u - u_0$ *è un punto critico di* F.

La caratterizzazione variazionale dimostrata nel Theorem 1.1.1 ci permette di utilizzare metodi variazionali per dimostrare l'esistenza di soluzioni in $u_0 + H_0^1(\Omega)$. Per esempio, in [6] viene considerato un problema di jumping nello spirito di [1] nell'ambiente delle equazioni ellittiche semilineari singolari. In particolare gli ingredienti principali in [6] sono la caratterizzazione variazionale di [8] e l'uso di una versione generalizzata (vedi [22]) del famoso Mountain Pass Theorem di Ambrosetti e Rabinowitz [2].

Consideriamo il seguente problema di jumping.
Sia $\beta > 0$ e $t \in \mathbb{R}$ e indichiamo con λ_1 il primo autovalore di $-\Delta$ con condizione di Dirichlet sul bordo nulla e consideriamo φ_1 una prima autofunzione associata tale che $\varphi_1 > 0$ in Ω. Studiamo l'esistenza di soluzioni del problema

$$\begin{cases} -\Delta\, u = \dfrac{f}{u^\beta} + g(x,u) - t\,\varphi_1 & \text{a.e. in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

dove Ω è un dominio liscio limitato e assumiamo che $g : \Omega \times \mathbb{R} \to \mathbb{R}$ sia una funzione di Carathéodory che soddisfa le seguenti condizioni:

(g_1) esistono a e b tali che

$$|g(x,s)| \leq a(x) + b(x)\,|s| \qquad \text{per a.e. } x \in \Omega \text{ e ogni } s \in \mathbb{R}$$

con $a \in L^{\frac{2N}{N+2}}(\Omega)$ e $b \in L^{\frac{N}{2}}(\Omega)$ se $n \geq 3$; a, $b \in L^p(\Omega)$ per qualche $p > 1$ se $n = 2$; a, $b \in L^1(\Omega)$ se $n = 1$.

(g_2) esiste $\alpha \in \mathbb{R}$ tale che

$$\lim_{s \to +\infty} \frac{g(x,s)}{s} = \alpha \qquad \text{for a.e. } x \in \Omega.$$

Come detto prima, il problema (1.18) è equivalente a determinare $u \in H^1_{loc}(\Omega)$ che soddisfi la disuguaglianza variazionale:

$$\begin{cases} u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{loc}(\Omega), \\ \displaystyle\int_\Omega (Du, D(v-u))\,dx \geq \int_\Omega \frac{f}{u^\beta}\,(v-u)\,dx \\ \qquad\qquad\qquad + \displaystyle\int_\Omega (g(x,u) - t\,\varphi_1)\,(v-u)\,dx. \\ \qquad \forall v \in u + (H_0^1(\Omega) \cap L_c^\infty(\Omega)) \text{ with } v \geq 0 \text{ a.e. in } \Omega, \\ u \leq 0 \text{ on } \partial\Omega. \end{cases} \tag{1.19}$$

Il nostro principale risultato, vedi anche [9], è il seguente:

Theorem 1.1.2. *Sia $\beta > 0$ e supponiamo che $g(x\,,\,s)$ soddisfi le condizioni (g_1)-(g_2) con $\alpha > \lambda_1$ e $a(x), b(x) \in L^\infty(\Omega)$. Supponiamo che f sia una funzione non negativa in $L^1(\Omega)$ se $\beta > 1$, mentre f sia una funzione non negativa in $L^m(\Omega)$ con $m = \frac{2N}{N+2+\beta(N-2)}$ se $0 < \beta \leq 1$. Inoltre supponiamo che, dato un qualsiasi insieme compatto $\mathcal{C} \subset \Omega$, esista una costante $\theta = \theta(\mathcal{C}) > 0$ con*

$$f \geq \theta \qquad \text{a.e. in} \quad \mathcal{C}\,.$$

Allora esiste $\bar{t} \in \mathbb{R}$ tale che, per ogni $t \geq \bar{t}$ esistono u_1, $u_2 \in H^1_{loc}(\Omega)$ soluzioni di (1.18). Inoltre

$$u_i := u_0 + w_i \qquad \text{for} \quad i = 1, 2,$$

dove $u_0 \in H^1_{loc}(\Omega)$ è definita in (1.14) e $w_i \in H_0^1(\Omega)$. Inoltre u_i sono strettamente staccate dallo zero nell'interno di Ω.

L'idea della dimostrazione del Theorem 1.1.2 è la seguente.
Definiamo la funzione troncata di f come segue

$$f_n := \max\left\{\frac{1}{n}, \min\{n, f\}\right\}.$$

Seguendo l'idea in [6] è possibile dimostrare che esiste $\bar{t} \in \mathbb{R}$, *non dipendente da* $n \in \mathbb{N}$, tale che, per ogni $t \geq \bar{t}$, esistono due soluzioni del problema che riguarda f_n, cioè

$$\begin{cases} -\Delta u_n = \dfrac{f_n}{{u_n}^\beta} + g(x, u_n) - t\varphi_1 & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.20}$$

dove $\beta > 0$, $u_n \in C(\overline{\Omega}) \cap C^2(\Omega)$. Come prima,

$$u_n = u_{0n} + w_n \tag{1.21}$$

con $w_n \in H_0^1(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ e $u_{0n} \in H^1_{loc}(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ è l'unica soluzione del problema:

$$\begin{cases} -\Delta u_{0n} = \dfrac{f_n}{{u_{0n}}^\beta} & \text{in } \Omega, \\ u_{0n} > 0 & \text{in } \Omega, \\ u_{0n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

Per dimostrare il Theorem 1.1.2, mostriamo che w_n è uniformemente limitata in $H_0^1(\Omega)$ e quindi passiamo al limite. Un punto cruciale è mostrare che le soluzioni u_n sono uniformemente strettamente staccate da zero nell'interno di Ω, come mostrato in Lemma 6.1.2. Questo è necessario per passare al limite nell'equazione, poiché la non-linearità è singolare.

Nella seconda parte della tesi, dimostriamo un risultato di simmetria (e monotonia), utilizzando il metodo del moving plane di James Serrin.
Consideriamo il problema

$$\begin{cases} -\Delta u = \dfrac{1}{u^\beta} + g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

dove $\beta > 0$, Ω è un dominio liscio limitato e $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.
I nostri risultati principali saranno dimostrati sotto la seguente ipotesi:

(H_p) $g(\cdot)$ è locally Lipschitz continuous, positiva ($g(s) > 0$ for $s > 0$ and $g(0) \geq 0$) e non decrescente.

Come accennato prima, la dimostrazione del nostro risultato di simmetria è basata sul famoso metodo del moving plane, vedi [20], che fu usato nel celebre lavoro [13] nel caso semilineare non degenere. In realtà, la nostra dimostrazione è più simile a quella utilizzata in [3] (vedi Section 7.1) ed è basata sul principio del confronto debole in piccoli domini.
A causa della natura singolare del nostro problema, dobbiamo tener conto di due difficoltà, ovvero:

- generalmente u non appartiene a $H_0^1(\Omega)$,
- $\frac{1}{s^\beta} + g(s)$ non è Lipschitz continuous in zero.

Infatti, come segue dalla caratterizzazione variazionale descritta precedentemente, la soluzione u può essere decomposta come $u = u_0 + w$ dove $w \in H_0^1(\Omega)$ e $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ è l'unica soluzione del problema (1.14) con $f = 1$, cioè:

$$\begin{cases} -\Delta u_0 = \dfrac{1}{{u_0}^\beta} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

Enunciamo ora il nostro risultato di simmetria (vedi [10]):

Theorem 1.1.3. *Sia* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *una soluzione di* (1.23) $(u = u_0 + w)$ *con* $g(\cdot)$ *che soddisfa* (H_p). *Supponiamo che il dominio* Ω *sia strettamente convesso nella direzione* ν $(\nu \in S^{N-1})$ *e simmetrico rispetto a* T_0^ν, *dove*

$$T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

Allora u *è simmetrica rispetto a* T_0^ν *e non decrescente nella direzione* ν *in* Ω_0^ν, *dove*

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\}.$$

Inoltre, se Ω *è una palla, allora* u *è radiale con* $\frac{\partial u}{\partial r}(r) < 0$ *per* $r \neq 0$.

Per semplicità di lettura, descriviamo lo schema della dimostrazione del Theorem 1.1.3.

(i) Osserviamo innanzitutto che, da [4] (vedi Chapter 3), sappiamo che u_0 è il limite di una successione $\{u_n\}$ di soluzioni di un problema troncato. Utilizziamo ciò per dedurre proprietà di simmetria e monotonia di $\{u_n\}$ e, conseguentemente, di u_0, poiché la procedura di moving plane si applica in maniera standard al problema troncato.

(ii) Da (i), ricordando la decomposizione $u = u_0 + w$, segue che è necessario dimostrare solo la simmetria e la monotonia di w. Per fare ciò dimostriamo alcuni principi di confronto per w che sono necessari per applicare il moving plane.

(iii) In Section 7.6, diamo alcuni dettagli sull'adattamento del moving plane allo studio della simmetria e della monotonia di w. Vale la pena di evidenziare che il metodo del moving plane è applicato nel nostro caso solo alla parte $H_0^1(\Omega)$ di u.
Osserviamo anche che Theorem 1.1.3 è dimostrato in Section 7.7, come conseguenza di un risultato più generale dimostrato in 7.6.1.

La tesi è organizzata come segue. In Chapter 2 presentiamo il risultato di Lazer e Mckenna (vedi [19]), mostrando che la soluzione del problema (1.14) generalmente non appartiene a $H_0^1(\Omega)$ se $\beta \geq 3$ e, conseguentemente, un approccio variazionale standard fallisce. In Chapter 3 consideriamo l'equazione $-\Delta\, u = f\, u^{-\beta}$; all'inizio del capitolo presentiamo il risultato di esistenza e regolarità dovuto a Boccardo e Orsina (vedi [4]) mentre, nella seconda parte, dimostriamo un risultato di esistenza e unicità, sulla linea di un'idea di Canino e Degiovanni (vedi [8]). In Chapter 4 presentiamo un approccio variazionale al nostro problema, seguendo l'idea in [8]. In Chapter 5 studiamo un problema di jumping per l'equazione $-\Delta\, u = f\, u^{-\beta} + g(x, u) - t\, \varphi_1$ con f limitata e in Chapter 6 consideriamo una situazione più generale, con f in $L^1(\Omega)$ non negativa e non limitata. In Chapter 7 dimostriamo un risultato di simmetria e monotonia, utilizzando il metodo del moving plane di James Serrin per soluzioni positive dell'equazione $-\Delta\, u = u^{-\beta} + g(u)$

Chapter 2

The Lazer-McKenna result

In this chapter we recall the Lazer-McKenna result [19], which states that positive solutions to $-\Delta u = \frac{f}{u^\beta}$, under zero Dirichlet boundary condition, belongs to $W^{1,2}(\Omega)$ if and only if $\beta < 3$. This is important to deduce that a standard variational approach to the existence of solutions fails in this case. In fact the associated energy functional might be identically $+\infty$ in $H_0^1(\Omega)$.

2.1 Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 1$, f be a positive function defined in $\overline{\Omega}$, $\beta > 0$ a real number and let us consider the problem

$$\begin{cases} -\Delta u = \dfrac{f}{u^\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Problem (2.1) has been studied in [11] for general regions and in [12] under the assumption that Ω is the open unit ball of $\mathbb{R}^N$ and $f(x) = q(|x|)$, where q is a continuous function which is defined continuous and nonnegative on $[0, 1)$. In [11] it is shown that solutions exist under the assumption that Ω is C^3 and estimates are given for the behavior of the solutions as x approaches the boundary of Ω. In particular, if $\beta > 1$, it is shown that solutions fail to be in $C^1(\overline{\Omega})$.
Actually, in [11] the authors prove more general results for the existence of solutions, but $\beta > 1$ is the case where behavior near the boundary is studied. In [19] the authors show that if Ω has regular boundary, f is regular on $\overline{\Omega}$ and β is any positive number, it is possible to prove that there is a unique solution of (2.1) which is positive on Ω and in $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$. The proof does not involve in any sense the shape of Ω. In the paper a necessary and sufficient condition that this solution have a finite Dirichlet integral is also given.
The proof is based on the choice of suitable upper and lower solutions and, in particular, it is proved that problem (2.1) can have a classical solution but not a weak solution.
We have the following:

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}$ with $0 < \alpha < 1$). If $f \in C^\alpha(\overline{\Omega})$, $f(x) > 0$ for all $x \in \overline{\Omega}$ and $\beta > 0$, then there exists a unique $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ solution of problem (2.1). Moreover, denoting by λ_1 the first eigenvalue of $-\Delta$ with homogeneous Dirichlet condition and by φ_1 an associated eigenfunction with $\varphi_1 > 0$ in Ω and considering the case $\beta > 1$, then there exist positive constants b_1 and b_2 such that*

$$b_1\, \varphi_1^{\frac{2}{1+\beta}} \leq u \leq b_2\, \varphi_1^{\frac{2}{1+\beta}}$$

in $\overline{\Omega}$.

Theorem 2.1.2. *The solution u founded in Theorem (2.1.1) is in $W^{1,2}(\Omega)$ if and only if $\beta < 3$. If $\beta > 1$, then u is not in $C^1(\overline{\Omega})$.*

2.2 Proof of Theorem 2.1.1

Assume $\beta > 1$ and consider $\Psi(x) = b\varphi_1(x)^t$ where $b > 0$ is a constant and $t = \frac{2}{1+\beta}$. By direct calculation we have that

$$-\Delta\,\Psi = \frac{q(x,b)}{\Psi^\beta} \qquad \forall x \in \Omega \tag{2.2}$$

where $q(x,b) = b^{1+\beta}(t(1-t)|D\varphi_1|^2 + t\lambda_1\varphi_1^2)$. Since $0 < t < 1$, it is possible to choose b_1 and b_2 with $0 < b_1 < b_2$, such that

$$q(x,b_1) < u(x) < q(x,b_2) \qquad \forall x \in \overline{\Omega}. \tag{2.3}$$

For $k = 1,2$ let $u_k = b_k\varphi_1^t$. Since by (2.2) we have

$$-\Delta\,u_1 = \frac{q(x,b_1)}{u^\beta},$$

it follows that

$$-\Delta\,u_1 - \frac{f}{u_1^\beta} = \frac{q(x,b_1) - f}{u_1^\beta} \tag{2.4}$$

and by (2.3)

$$-\Delta\,u_1 - \frac{f}{u_1^\beta} < 0.$$

By similar considerations it follows that

$$-\Delta\,u_2 - \frac{f}{u_2^\beta} > 0.$$

If $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ and satisfies (2.1), then

$$u_1(x) \le u(x) \le u_2(x) \qquad \forall x \in \overline{\Omega}.$$

By contradiction, let us assume that there exists $x_0 \in \Omega$ such that $0 < u(x_0) < u_1(x_0)$ and that the minimum of the function $u - u_1$ in $\overline{\Omega}$ is assumed at x_0. Since u satisfies problem (2.1), by the hypothesis on x_0 and by (2.4) it follows that

$$\begin{aligned}
-\Delta\,(u-u_1)(x_0) &= -\Delta\,u(x_0) + \Delta\,u_1(x_0) = \frac{f(x_0)}{u^\beta} + \Delta\,u_1(x_0) \\
&> f(x_0)\left(\frac{1}{u^\beta(x_0)} - \frac{1}{u_1^\beta(x_0)}\right) > 0
\end{aligned}$$

which contradicts the fact that x_0 is the minimum of the function $u - u_1$. A similar argument shows that $u(x) \le u_2(x)$ for all $x \in \overline{\Omega}$.
For $\beta > 0$, let $u_* = \varepsilon\,\varphi_1$, where ε is a positive number sufficiently small. If $\delta > 0$, then

$$-\Delta\,u_* - \frac{f}{(u_* + \delta)^\beta} = -\frac{f}{(\varepsilon\,\varphi_1 + \delta)^\beta} + \varepsilon\,\lambda_1\,\varphi_1.$$

Therefore, since $\varphi_1 = 0$ on $\partial\Omega$, it is possible to choose $\varepsilon > 0$ and $\delta_0 > 0$ such that if $0 < \delta \le \delta_0$, then

$$-\Delta\,u_* - \frac{f}{(u_* + \delta)^\beta} < 0 \qquad \forall x \in \overline{\Omega}. \tag{2.5}$$

If $\beta > 1$, let $u^* = u_2 = b_2\,\varphi_1^t$. If δ_0 is as above, it follows that

$$-\Delta\,u^* - \frac{f}{(u^* + \delta)^\beta} > 0 \qquad \forall x \in \Omega. \tag{2.6}$$

If $\beta > 1$ we also suppose that $\varepsilon > 0$ is chosen so that $u_* = \varepsilon\,\varphi_1 < u_2 = u^*$.
If $0 < \beta \leq 1$, let s be chosen such that

$$\begin{cases} 0 < s < 1 \\ s(1+\beta) < 2. \end{cases} \tag{2.7}$$

Let $u^* = c\,\varphi_1^s$ where c is a suitable large constant. Then it turns out that

$$\begin{aligned} -\Delta\, u^* &= -c\,s(s-1)\,\varphi_1^{s-2}|D\,\varphi_1|^2 - c\,s\,\varphi_1^{s-1}\Delta\,\varphi_1 \\ &= -c\,s\,(s-1)\,\varphi_1^{s-2}|D\,\varphi_1|^2 + c\,s\,\lambda_1\varphi_1^s. \end{aligned}$$

The last equation can be rewritten as

$$\begin{aligned} &-\Delta\, u^* - \frac{f}{(u^*)^\beta} = \\ &\qquad -\varphi_1^{s-2}\left(c\,s\,(s-1)\,|D\,\varphi_1|^2 + \frac{f}{c^\beta\,\varphi_1^{2-s(1+\beta)}}\right) + c\,s\,\lambda_1\,\varphi_1^s. \end{aligned} \tag{2.8}$$

From (2.7) it follows that c can be chosen so large that $-\Delta\, u^* - \frac{f}{(u^*)^\beta} > 0$ for all $x \in \Omega$. Therefore if δ_0 is as above, then (2.6) holds for $0 < \delta \leq \delta_0$.
Since $0 < s < 1$ and $\varphi_1 = 0$ on $\partial\Omega$, we can assume c to be so large that

$$\varepsilon\,\varphi_1 < c\,\varepsilon\,\varphi_1^s.$$

It follows that both in the case $\beta > 1$ and in the case $0 < \beta \leq 1$

$$0 < u_*(x) < u^*(x) \qquad \forall x \in \Omega$$

holds true.
Let δ be a fixed number, with $0 < \delta < \delta_0$ and let $k > 0$ be so large that the function

$$h(x,\xi) = k\,\xi + \frac{f}{(\delta+\xi)^\beta}$$

is strictly increasing in ξ for $0 \leq \xi \leq M = \max\{u^*(x) : x \in \overline{\Omega}\}$ and $x \in \overline{\Omega}$. Let w a smooth function such that

$$\begin{cases} -\Delta\, w + k\,w = h(x,u^*) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

Since, according to (2.6), $-\Delta\, u^* + k\,u^* > h(x,u^*)$ in Ω, it follows that

$$-\Delta\,(u^* - w) + k\,(u^* - w) > 0 \qquad \text{in } \Omega.$$

Therefore, since $u^* - w = 0$ on $\partial\Omega$ and $u^* - w \in C(\overline{\Omega}) \cap C^2(\Omega)$, it follows that $u^* - w > 0$ for all x in Ω. Hence it follows from (2.9) and from the hypothesis on h that

$$\begin{cases} -\Delta\, w + k\,w > h(x,w) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

On the other hand, by (2.5) it follows that

$$\begin{cases} -\Delta\, u_* + k\,w < h(x,u_*) & \text{in } \Omega \\ u_* = 0 & \text{on } \partial\Omega. \end{cases}$$

By the same type of argument given above, it follows that if v is a smooth function such that

$$\begin{cases} -\Delta\, v + k\,v < h(x,v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.11}$$

then $u_* < v$ in Ω and therefore

$$\begin{cases} -\Delta\, v + k\, v < h(x,v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since

$$-\Delta\,(w - v) + k\,(w - v) = h(x,u^*) - h(x,u_*) > 0,$$

we have

$$v(x) < w(x) \qquad \forall x \in \Omega.$$

Since u and w are both smooth on $\overline{\Omega}$, it follows from the last inequality, (2.10), (2.11) and the basic result on the method of subsolutions and supersolutions (sse [14]) that there exists a smooth function z defined in $\overline{\Omega}$ which satisfies

$$\begin{cases} -\Delta\, z + k\, z = h(x,z) & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$u_*(x) < v(x) \le z(x) \le w(x) < u^*(x) \qquad \forall x \in \Omega,$$

namely

$$\begin{cases} -\Delta\, z + \frac{f}{(z+\delta)^\beta} = 0 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

For $n \ge 1$, let $\{\delta_n\}$ be a sequence of numbers such that $0 < \delta_{n+1} < \delta_n < \delta_0$ and $Z_n(x)$ be a smooth positive solution of (2.12) when $\delta = \delta_n$ such that $u_* < Z_n < u^*$ in Ω. From (2.12) and from the hypothesis on $\{\delta_n\}$ it follows that

$$-\Delta\, Z_n - \frac{f}{(\delta_{n+1} + Z_n)^\beta} < -\Delta\, Z_n - \frac{f}{(\delta_n + Z_n)^\beta} = 0.$$

Let us show now that $Z_{n+1}(x) > Z_n(x)$ for all $x \in \Omega$. By contradiction, assuming the contrary it follows, since $Z_n - Z_{n-1} = 0$ in $\partial\Omega$, that there exists a point $x_0 \in \Omega$ where $Z_n - Z_{n+1}$ assumes a nonnegative maximum. But from the above arguments

$$-\Delta(Z_n + Z_{n+1})(x_0) < f(x_0)\left(\frac{1}{(\delta_{n+1} + Z_n(x_0))^\beta} - \frac{1}{(\delta_{n+1} + Z_{n+1}(x_0))^\beta}\right) \le 0$$

which contradicts the fact that x_0 is a maximum for $Z_n - Z_{n+1}$.
Since $Z_n(x) < Z_{n+1}(x) < u^*(x)$ for all $x \in \overline{\Omega}$, it follows that

$$\lim_{n \longrightarrow +\infty} Z_n(x) = u^*(x)$$

exists for all $x \in \overline{\Omega}$ and

$$u_*(x) < u(x) < u_*(x) \qquad \forall x \in \overline{\Omega}. \tag{2.13}$$

The last point is to prove that u satisfies

$$-\Delta\, u - \frac{f}{u^\beta} = 0 \qquad \forall x \in \Omega \tag{2.14}$$

and $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$.
Let $x_0 \in \Omega$ and $r > 0$ to be chosen so that $\overline{B(x_0,r)} \subset \Omega$, where $B(x_0,r)$ is the open ball of radius r centered at x_0. Let Ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0,\frac{r}{2})}$ and equal to 0 off $B(x_0,r)$. Then

$$\Delta\,(\Psi\, Z_n) = 2D\,\Psi\, D\, Z_n + p_n$$

for $n \ge 1$, where $p_n = \Delta\Psi\, Z_n - \frac{f}{(Z_n+\delta_n)^\beta}$ is a term whose L^∞ norm does is bounded independently on n. Therefore

$$\Psi Z_n \Delta(\Psi Z_n) = \sum_{j=1}^{n} b_{n\,j}\,\frac{\partial\,(\Psi Z_n)}{\partial x_j} + q_n$$

where $b_{n\,j} = 2\Psi \frac{\partial Z_n}{\partial x_j}$ and $q_n = \sum_{j=1}^{n} 2\left(\frac{\partial}{\partial x_j}\Psi\right)^2 Z_n^2$ are terms bounded independently of n. Integrating the above equation, one gets

$$\int_{B(x_0,r)} |D(\Psi Z_n)|^2\,dx \le c_1 \left(\int_{B(x_0,r)} |D\Psi Z_n|^2\,dx\right)^{\frac{1}{2}} + c_2$$

where c_1, c_2 are two nonnegative constant not depending on n.
Therefore it follows that the $L^2(B(x_0,r))$-norm of $|D\Psi Z_n|$ is bounded independently of n and, consequently, the $L^2(B(x_0,\frac{r}{2}))$-norm of $|DZ_n|$ is bounded independently of n. Let now Ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_0,\frac{r}{4})}$ and equal to 0 off $B(x_0,\frac{r}{2})$. For $n \ge 1$ one has

$$\Delta(\Psi_1 Z_n) = 2D\Psi_1\,DZ_n + p_{1n},$$

where p_{1n} is a term whose $L^\infty(B\left(x_0,\frac{r}{2}\right))$-norm is bounded independently of n. From standard elliptic theory, the $W^{2,2}(B(x_0,\frac{r}{2}))$-norm of $\Psi_1 Z_n$ is bounded independently of n and, consequently, the $W^{2,2}(B(x_0,\frac{r}{4}))$-norm of Z_n is bounded independently of n. Since the $W^{1,2}(B(x_0,\frac{r}{4}))$-norms of the components of DZ_n are bounded independently of n, it follows from the Sobolev embedding Theorem that, if

$$\begin{cases} q = \frac{2N}{N-2} > 2 & \text{if } N > 2 \\ q > 2 \text{ arbitary} & \text{if } N \le 2, \end{cases}$$

then the $L^q(B(x_0,\frac{r}{4}))$-norm of DZ_n is bounded independently of n.
Let now Ψ_2 be a C^∞ function which is equal to 1 on $\overline{B(x_0,\frac{r}{8})}$. As above one gets

$$\Delta(\Psi_2 Z_n) = 2D\Psi_2\,DZ_n + p_{2n},$$

where p_{2n} is bounded independently of n in $L^\infty(B(x_0,\frac{r}{4}))$. Since the righthand side of the above equation is bounded in $L^q(B(x_0,\frac{r}{4}))$ independently of n, the $W^{2,q}(B(x_0,\frac{r}{8}))$-norm of $\Psi_2 Z_n$ is also bounded independently of n. Hence, the $W^{2,q}(B(x_0,\frac{r}{8}))$-norm of Z_n is bounded independently of n. Iterating this process, after a finite number of steps, we find $r_1 > 0$ and $q_1 > \frac{N}{1-\alpha}$ such that the $W^{2,q_1}(B(x_0,r_1))$-norm of Z_n is bounded independently of n. Hence, up to subsequence, $\{Z_n\}$ converges in $C^{1+\alpha}(\overline{B(x_0,r_1)})$. If θ is a C^∞ function which is equal to 1 on $\overline{B(x_0,\frac{r_1}{2})}$ and equal to 0 off $B(x_0,r_1)$, then

$$\Delta(\theta Z_n) = 2D\theta DZ_n + \widehat{p}_n$$

where $\widehat{p}_n = \theta\Delta Z_n + Z_n\Delta\theta$.
Since the righthand side of the above equation converges in $C^\alpha\overline{(B(x_0,r_1))}$, by Schauder theory, the sequence $\{\theta Z_n\}$ converges in $C^{2+\alpha}(\overline{B(x_0,r_1)})$ and hence $\{Z_n\}$ converges in $C^{2+\alpha}(\overline{B(x_0,\frac{r_1}{2})})$. From the arbitrariness of x_0 it follows that $u \in C^{2+\alpha(\Omega)}$. Clearly (2.14) holds.
Since $u_*(x) < u(x) < u^*(x)$ for $x \in \overline{\Omega}$ and both u_* and u^* vanish on the boundary of Ω, if $x_1 \in \partial\Omega$, then

$$\lim_{x \longrightarrow x_1} u(x) = 0 = u(x_1),$$

namely $u \in C(\overline{\Omega})$.
To prove the uniqueness of u, suppose that $\widehat{u}$ is also a function in $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ which is positive in Ω and which satisfies (2.14). If $u \ne \widehat{u}$, we may assume that $\widehat{u} - u$ assumes a positive value somewhere in Ω. This implies that $\widehat{u} - u$ attains a positive maximum at a point $x_0 \in \Omega$. But

$$-\Delta(\widehat{u} - u)(x_0) = -\frac{f}{u(x_0)^\beta - \widehat{u}(x_0)^\beta} < 0,$$

which contradicts the fact that x_0 is a maximum for the function.
Hence $\widehat{u} = u$ and this concludes the proof of Theorem 2.1.1.

2.3 Proof of Theorem 2.1.2

To prove Theorem 2.1.2, we need the following

Lemma 2.3.1. *The integral*

$$\int_\Omega \varphi_1^r \, dx < +\infty$$

if and only if $r > -1$.

Proof. Let $x_0 \in \partial\Omega$. By the smoothness of $\partial\Omega$, it is not restrictive to assume that $x_0 = 0$ and that there exists a neighborhood U of x_0 such that if $V = U \cap \Omega$, then

- V consists of points $x = (x_1, ..., x_N)$ such that $|x_j| < r$ for $1 \le j \le N-1$ and $0 < x^N < r$;
- $U \cap \partial\Omega$ the set of points x with $|x_j| < r$ for $1 \le j \le N-1$ and $x^N = 0$.

Since $\varphi_1 = 0$ on $\partial\Omega$ and $\frac{\partial \varphi_1}{\partial x_N}(x) > 0$ on $\partial\Omega$, we may assume that r is so small that there exist two positive constants c_1 and c_2 such that

$$c_1 x^N < \varphi_1 < c_2 x^N \qquad \text{in } V. \tag{2.15}$$

Since φ_1 is bounded from below by a positive constant on a compact subset of Ω, the thesis follows from (2.15) and from a partition of unity argument. □

In the following a proof of Theorem 2.1.2 is given.
First of all, we modify the definition of u_* as follows:

$$u_* = \begin{cases} \varepsilon\,\varphi_1 & \text{if } 0 < \beta \le 1 \\ u_1 = b_1 \varphi_1^t & \text{if } \beta > 1, \end{cases}$$

with $\varepsilon > 0$ and u_1 defined in the previous section. It follows from what was shown above that, if u is the unique solution of (2.1) positive on Ω, then (2.13) holds true for every $x \in \Omega$ even with the new definition of u_*.
Suppose that $1 < \beta < 3$, so $u_* = b_1 \varphi_1^t$ with $t = \frac{2}{1+\beta}$ and let $\{\delta_n\}$ and $\{Z_n\}$ be defined as above. Since $u_*(x) \le Z_n(x)$ for $x \in \Omega$, it follows that

$$f \frac{Z_n}{(Z_n + \delta_n)^\beta} \le \frac{f}{(Z_n + \delta_n)^{\beta-1}} \le \frac{f}{(u_* + \delta_n)^{\beta-1}} < M u_*^{\beta-1} \tag{2.16}$$

where M is the maximum of f in $\overline{\Omega}$. If $r = \frac{2(1-\beta)}{1+\beta}$, then $r > -1$ and therefore, by the Lemma 2.3.1,

$$\int_\Omega \frac{1}{u_*^{\beta-1}} \, dx < \infty. \tag{2.17}$$

Since for $n \ge 1$

$$\int_\Omega |DZ_n|^2 \, dx = \int_\Omega f \frac{Z_n}{(Z_n + \delta_n)^\beta} \, dx, \tag{2.18}$$

it follows from (2.16) and (2.17) that the $W^{1,2}$-norm of Z_n is bounded independently of n. Therefore, $\{Z_n\}$ converges weakly in $W^{1,2}(\Omega)$, up to subsequence, to a function $\widetilde{Z}$ in $W^{1,2}(\Omega)$. Since $\{Z_n\}$ converges pointwise to u in Ω, it follows that $Z = u$ and hence $u \in W^{1,2}(\Omega)$.
If $0 < \beta < 1$, then for every $x \in \Omega$

$$f(x) \frac{Z_n(x)}{(Z_n(x) + \delta_n)^\beta} \le \frac{f(x)}{(Z_n + \delta_n)^{\beta-1}} \le \frac{f(x)}{(u^*(x) + \delta_n)^{\beta-1}}$$

with $u^* = c\varphi_1^s$ and s is a positive number satisfying the inequalities (2.7). This, together with (2.18), gives that the sequence $\{Z_n\}$ is bounded in $W^{1,2}(\Omega)$, and it follows that $u \in W^{1,2}(\Omega)$.
Consider now the case $\beta \ge 3$. In this case $u^* = b_2 \varphi_1^t$, where $t = \frac{2}{1+\beta}$ so that $t(\beta+1) \ge 1$. Since

$u(x) \geq u^*(x)$ for every $x \in \Omega$ and $f(x) \geq m > 0$ for every $x \in \Omega$, it follows from Lemma 2.3.1 that

$$\int_\Omega \frac{f}{u^{\beta+1}}\, dx = +\infty. \tag{2.19}$$

Suppose, contrary to the assertion of the Theorem, that $u \in W^{1,2}(\Omega)$. Since $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, it follows that $u \in H_0^1(\Omega)$. Therefore, there exists a sequence $\{w_n\}$ in $C_c^\infty(\Omega)$ such that $w_n \underset{n\to+\infty}{\longrightarrow} u$ in $W^{1,2}(\Omega)$. If for each n we set $w_n^+ = \max\{w_n, 0\}$, then $w_n^+ \in H_0^1(\Omega)$, $Dw_n^+ = Dw_n$ where $w_n > 0$ and $Dw_n^+ = 0$ where $w_n < 0$. From this it follows that $\{w_n^+\}$ converges to u in $W^{1,2}(\Omega)$. Since $\frac{w_n^+(x)f(x)}{u(x)^\beta} \geq 0$ for every $x \in \Omega$ and, up to subsequence, $\{w_n^+\}$ converges to u almost everywhere in Ω, by (2.19) and Fatou's lemma it follows that

$$\lim_{n\to+\infty} \int_\Omega \frac{w_n^+ f}{u^\beta}\, dx = +\infty.$$

Since $-\Delta u = \frac{f}{u^\beta}$ in Ω and $w_n^+ \in H_0^1(\Omega)$, it follows that

$$\int_\Omega (Du, Dw_n^+)\, dx = -\int_\Omega w_n^+\, \Delta u\, dx = \int_\Omega \frac{w_n^+ f}{u^\beta}\, dx,$$

hence, passing to the limit as n tends to $+\infty$

$$\int_\Omega |Du|^2\, dx = \lim_{n\to+\infty} \int_\Omega (Du, Dw_n^+)\, dx = +\infty,$$

which contradicts the assumption that $u \in W^{1,2}(\Omega)$.
To prove the final statement, we note that if $x_0 \in \partial\Omega$ and $\overrightarrow{n}$ denotes the inner normal to $\partial\Omega$ at x_0, then $\varphi_1(x_0) = 0$ and

$$\lim_{s\to 0^+} \frac{\varphi_1(x_0 + s\overrightarrow{n})}{s} = \lim_{s\to 0^+} \frac{\varphi_1(x_0 + \overrightarrow{n}) - \varphi_1(x_0)}{s} = D\varphi_1(x_0) \cdot \overrightarrow{n}.$$

If $\beta > 1$, then $t = \frac{2}{1+\beta} < 1$ and, as showed before, $u(x) \geq b_1 \varphi_1(x)^t$ for every $x \in \Omega$, where $b_1 > 0$. Since $u(x_0) = 0$, it follows that, for $s > 0$,

$$\frac{u(x_0 + s\overrightarrow{n}) - u(x_0)}{s} \geq \frac{b_1}{\varphi_1(x_0 + s\overrightarrow{n})^{1-t}} \frac{\varphi_1(x_0 + \overrightarrow{n})}{s}.$$

Therefore

$$\lim_{s\to 0^+} \frac{u(x_0 + s\overrightarrow{n}) - u(x_0)}{s} = +\infty,$$

so that u does not belong to $C^1(\Omega)$.

Chapter 3

On the equation $-\Delta u = f\, u^{-\beta}$

Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 3$, $\beta \geq 0$, g a Carathéodory function and f a function such that there exist $\gamma > 0$ and $\Gamma < +\infty$ with $\gamma < f < \Gamma$. We are interested in the study of

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^\beta} \;+\; g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

In spite of the fact that (3.1) is formally the Euler equation of the functional

$$E(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx + \int_\Omega \phi(u)\, dx - \int_\Omega \int_0^{u(x)} g(x,s)\, ds\, dx \qquad u \in H_0^1(\Omega),$$

where

$$\phi(s) = \begin{cases} -\displaystyle\int_1^s \frac{1}{t^\beta} dt & \text{if } s \geq 0, \\ +\infty & \text{if } s < 0, \end{cases} \tag{3.2}$$

few existence and multiplicity results for (3.1) have been obtained through a direct variational approach, only in the case $\beta < 3$. This restriction is due to the fact that, according to Theorem 2.1.2 (see [19]), the functional E is identically $+\infty$ if $\beta \geq 3$. First of all, we provide a variational approach to the problem

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^\beta} + \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

with $\mu \in W^{-1,2}(\Omega)$. To do this, let us consider first the case $\mu = 0$. Our state is that problem

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

admits one and only one solution $u_0 \in C^\infty(\Omega)$.

3.1 Existence and regularity for a singular problem. The result of Boccardo and Orsina

In this chapter we recall the result of Boccardo and Orsina [4]. More precisely the problem of existence of positive solutions to the equation $-\mathrm{div}(M(x)\, Du) = \frac{f}{u^\beta}$, under zero Dirichlet boundary condition, is considered. The approach is based on a truncation argument.

3.1.1 Preliminaries

Let Ω be a bounded subset of $\mathbb{R}^N$, $N \geq 2$, $\beta > 0$ a real number and f a nonnegative function on Ω, belonging to some Lebesgue space specified later.
Let us consider the problem

$$\begin{cases} -\mathrm{div}(M(x)\,Du) = \dfrac{f}{u^\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where M is a bounded elliptic matrix, namely there exist $0 < a \leq b$ such that

$$a|\xi|^2 \leq M(x)\,\xi \times \xi, \qquad |M(x)| \leq b \tag{3.6}$$

for every $\xi \in \mathbb{R}^N$, for almost every $x \in \Omega$. A solution of (3.5) is a function $u \in W_0^{1,1}(\Omega)$ such that

$$\forall \Omega' \subset\subset \Omega \ \exists c_{\Omega'} \text{ such that } u \geq c_{\Omega'} > 0 \text{ in } \Omega', \tag{3.7}$$

and such that

$$\int_\Omega (M(x)\,Du, D\varphi)\,dx = \int_\Omega \frac{f\,\varphi}{u^\beta}\,dx \qquad \forall \varphi \in C_c^1(\Omega). \tag{3.8}$$

The righthand side in (3.8) is well defined, since φ has compact support.

In order to study problem (3.5), the authors work by approximation, truncating the singular term $\frac{1}{u^\beta}$ and studying the behavior of a sequence $\{u_n\}$ of solution of the approximated problems. Let f be a nonnegative measurable function (not identically zero), $n \in \mathbb{N}$,

$$f_n(x) = \min\{f, n\} \tag{3.9}$$

and consider the following problem:

$$\begin{cases} -\mathrm{div}(M(x)\,Du_n) = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^\beta} & \text{in } \Omega, \\ u_n = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.10}$$

Lemma 3.1.1. *Let M be a bounded elliptic matrix, $\beta > 0$ a real number, f_n defined in* (3.9). *Then problem* (3.10) *has a nonnegative solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let $n \in \mathbb{N}$ be fixed, $v \in L^2(\Omega)$ and define $w = S(v)$ the unique solution of

$$\begin{cases} -\mathrm{div}(M(x)\,Dw) = \dfrac{f_n}{\left(|v| + \frac{1}{n}\right)^\beta} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking w as test function and using (3.6), we get

$$\begin{aligned} a\int_\Omega |Dw|^2\,dx \leq \int_\Omega (M(x)\,Dw, Dw)\,dx &= \int_\Omega \frac{f_n\,w}{\left(|v| + \frac{1}{n}\right)^\beta}\,dx \\ &\leq n^{\beta+1}\int_\Omega |w|\,dx. \end{aligned}$$

By using Poincaré and Hölder inequalities we get

$$\int_\Omega |w|^2\,dx \leq C\,n^{\beta+1}\Big(\int_\Omega |w|^2\,dx\Big)^{\frac{1}{2}}$$

for some constant C independent on v. This means

$$\|w\|_{L^2(\Omega)} \le C\, n^{\beta+1},$$

so that the ball of $L^2(\Omega)$ of radius $C\, n^{\beta+1}$ is invariant for S. Using the Sobolev embedding, it is possible to prove that S is both continuous and compact on $L^2(\Omega)$, so that by Schauder's fixed point Theorem there exists u_n such that $u_n = S(u_n)$, namely u_n solves

$$\begin{cases} -\mathrm{div}(M(x)\, Du_n) = \dfrac{f_n}{\left(|u_n| + \frac{1}{n}\right)^{\beta}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\beta}} \ge 0$, the maximum principle implies that $u_n \ge 0$, so that u_n solves (3.10).
Since the righthand side of (3.10) belongs to $L^\infty(\Omega)$, by [21], Theorem 4.2, it follows that $u_n \in L^\infty(\Omega)$ (and its norm in $L^\infty(\Omega)$ may depend on n). □

Lemma 3.1.2. *The sequence* $\{u_n\}$ *defined in* (3.10) *is increasing with respect to* n, $u_n > 0$ *in* Ω *and for every* $\Omega' \subset\subset \Omega$ *there exists* $c_{\Omega'} > 0$ *(independent on* n*) such that*

$$u_n(x) \ge c_{\Omega'} > 0 \qquad \text{for every } x \in \Omega, \text{ for every } n \in \mathbb{N}. \tag{3.11}$$

Proof. Since $0 \le f_n \le f_{n+1}$ and $\beta > 0$, it follows that

$$-\mathrm{div}(M(x)\, Du_n) = \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\beta}} \le \frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^{\beta}}.$$

Combining this with

$$-\mathrm{div}(M(x)\, Du_{n+1}) = \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\beta}}$$

we get

$$\begin{aligned} -\mathrm{div}(M(x)\, D(u_n - u_{n+1}) &\le f_{n+1}\left(\frac{1}{\left(u_n + \frac{1}{n+1}\right)^{\beta}} - \frac{1}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\beta}}\right) \\ &= f_{n+1}\frac{\left(u_{n+1} + \frac{1}{n+1}\right)^{\beta} - \left(u_n + \frac{1}{n+1}\right)^{\beta}}{\left(u_n + \frac{1}{n+1}\right)^{\beta}\left(u_{n+1} + \frac{1}{n+1}\right)^{\beta}}. \end{aligned}$$

Choosing $(u_n - u_{n+1})^+$ as test function and observing that

$$\left(\left(u_{n+1} + \frac{1}{n+1}\right)^{\beta} - \left(u_n + \frac{1}{n+1}\right)^{\beta}\right)(u_n - u_{n+1})^+ \le 0,$$

we get, using (3.6),

$$0 \le a\int_\Omega |D(u_n - u_{n+1})^+|^2\, dx \le 0.$$

Hence $(u_n - u_{n+1})^+ = 0$ almost everywhere in Ω, namely $u_n \le u_{n+1}$.
By Lemma 3.1.1 it follows that $u_1 \in L^\infty(\Omega)$ and there exists a constant (only depending on Ω and N) such that

$$\|u_1\|_{L^\infty(\Omega)} \le C\|f_1\|_{L^\infty(\Omega)} \le C,$$

therefore

$$-\mathrm{div}(M(x)\, Du_1) = \frac{f_1}{(u_1+1)^{\beta}} \ge \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)}+1)^{\beta}} \ge \frac{f_1}{(C+1)^{\beta}}.$$

Since $\frac{f_1}{(C+1)^{\beta}}$ is not identically zero, the strong maximum principles implies that $u_1 > 0$ in Ω and that (3.11) holds for u_1, with $c_{\Omega'}$ depending on ω, N, f_1 and β. Since $u_n \ge u_1$ for every $n \in \mathbb{N}$, it follows that (3.11) holds for $\{u_n\}$, with the same constant $c_{\Omega'}$ which is independent on n. □

Remark 3.1.1. *If u_n and v_n are two solution of problem* (3.10)*, the first part of the proof of Lemma 3.1.2 shows that $u_n \le v_n$. By symmetry this implies that the solution of* (3.10) *is unique.*

Since $\{u_n\}$ is increasing in n, it is possible to define u as the pointwise limit of $\{u_n\}$, and since $u \ge u_n$, (3.11) holds for u, too. To get some a priori estimates on u_n, is useful to consider three different cases, depending on β.

3.1.2 The case $\beta = 1$

Lemma 3.1.3. *Let u_n be the solution of* (3.10) *with $\beta = 1$, and suppose that $f \in L^1(\Omega)$. Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. Choosing u_n as test function in (3.10), using (3.6) and the fact that $0 \le f_n \le f$, we get

$$a \int_\Omega |Du_n|^2\, dx \le \int_\Omega (M(x)\, Du_n, Du_n)\, dx = \int_\Omega \frac{f_n\, u_n}{u_n + \frac{1}{n}} \le \int_\Omega f_n\, dx$$
$$\le \int_\Omega f\, dx$$

as desired. □

Theorem 3.1.4. *Let $\beta = 1$ and let f be a nonnegative function in $L^1(\Omega)$ (not identically zero). Then there exists a solution $u \in H_0^1(\Omega)$ of* (3.5) *in the following sense:*

$$\int_\Omega (M(x)\, Du, D\varphi)\, dx = \int_\Omega \frac{f\, \varphi}{u}\, dx \qquad \forall \varphi \in C_c^1(\Omega).$$

Proof. Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$ by Lemma 3.1.2, hence, up to subsequences, weakly convergent, and since $\{u_n\}$ converges pointwise to u in Ω, then $\{u_n\}$ weakly converges to u in $H_0^1(\Omega)$, and therefore

$$\lim_{n\to+\infty} \int_\Omega (M(x)\, Du_n, D\varphi)\, dx = \int_\Omega (M(x)\, Du, D\varphi)\, dx, \tag{3.12}$$

for every $\varphi \in C_c^1(\Omega)$. Moreover, since $\{u_n\}$ satisfies (3.1.2), if we denote by Ω' the set $\{\varphi \ne 0\}$, it follows that

$$0 \le \left| \frac{f_n\, \varphi}{u_n + \frac{1}{n}} \right| \le \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_{\Omega'}}\, f,$$

for every $\varphi \in C_c^1(\Omega)$.
Whence, by Lebesgue Theorem it follows that

$$\lim_{n\to+\infty} \int_\Omega \frac{f_n\, \varphi}{u_n + \frac{1}{n}}\, dx = \int_\Omega \frac{f\, \varphi}{u}\, dx,$$

which, together with (3.12), yields the thesis. □

The summability of u depends on the summability of f, how the following Lemma shows.

Lemma 3.1.5. *Let $\beta = 1$ and let $f \in L^m(\Omega)$ with $m \ge 1$. Then the solution u of* (3.5)*, given by Theorem* 3.1.4 *satisfies the following:*

(*a*) *if $m > \frac{N}{2}$, then u belongs to $L^\infty(\Omega)$;*

(*b*) *if $1 \le m < \frac{N}{2}$, then u belongs to $L^s(\Omega)$, where $s = \frac{2Nm}{N-2m}$.*

Proof. (a) Let $k > 1$ and define $G_k(s) = (s-k)^+$. Taking $G_k(u_n)$ as test function in (3.10) and denoting by $\widehat{\Omega}$ the set $\{u_n - k \geq 0\}$, we get

$$a \int_\Omega |DG_k(u_n)|^2\,dx \leq \int_\Omega (M(x)\,DG_k(u_n), G_k(u_n))\,dx = \int_\Omega \frac{f_n\,G_k(u_n)}{u_n + \frac{1}{n}}\,dx$$
$$= \int_{\widehat{\Omega}} \frac{f_n\,G_k(u_n)}{u_n + \frac{1}{n}}\,dx \leq \int_{\widehat{\Omega}} f\,G_k(u_n)\,dx = \int_\Omega f\,G_k(u_n)\,dx,$$

because $u_n + \frac{1}{n} \geq k \geq 1$ in $\widehat{\Omega}$. Arguing as in [21], it follows that there exists a constant C, independent on n, such that

$$\|u_n\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^m(\Omega)}.$$

Since $\{u_n\}$ is bounded in $L^\infty(\Omega)$, u belongs to $L^\infty(\Omega)$ as well.
(b) If $m = 1$, then $s = \frac{2\,N}{N-2} = 2^*$, so that the thesis is true by Sobolev embedding.
If $1 < m < \frac{N}{2}$, let $\delta > 1$ and choose $u_n^{2\delta-1}$ as test function in (3.10). It is possible to choose a such test function since $u_n \in L^\infty(\Omega)$ for every n by Lemma 3.1.1, obtaining

$$a(2\delta-1)\int_\Omega |Du_n|^2 u_n^{2\delta-2}\,dx \leq \int_\Omega \frac{f_n\,u_n^{2\delta-1}}{u_n+\frac{1}{n}}\,dx \leq \int_\Omega f\,u_n^{2\delta-2}\,dx$$
$$\leq \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{(2\delta-2)m'}\,dx\right)^{\frac{1}{m'}}. \tag{3.13}$$

On the other hand, by Sobolev inequality it follows that

$$\int_\Omega |Du_n|^2\,u_n^{2\delta-2}\,dx = \int_\Omega |Du_n\,u_n^{\delta-1}|^2\,dx = \int_\Omega \frac{|Du_n^\delta|^2}{\delta^2}\,dx$$
$$\geq \frac{\mathcal{S}}{\delta^2}\left(\int_\Omega u_n^{2^*\delta}\,dx\right),$$

that, together with (3.1.10) yields

$$a\left(\int_\Omega u_n^{2^*\delta}\,dx\right)^{\frac{2}{2^*}} \leq \frac{\delta^2}{\mathcal{S}(2\delta-1)}\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(2\delta-2)\,m'}\right)^{\frac{1}{m'}}. \tag{3.14}$$

If δ is such that $2^*\delta = (2\delta-2)\,m'$, namely $\delta = \frac{m\,(N-2)}{N-2\,m}$, then $\delta > 1$ if and only if $m > 1$ and $2^*\delta = \frac{2\,N\,m}{N-2\,m} = s$, so that (3.14) becomes

$$\left(\int_\Omega u_n^s\,dx\right)^{\frac{2}{2^*}} \leq C(N,m)\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^s\,dx\right)^{\frac{1}{m'}}.$$

Since $\frac{2}{2^*} > \frac{1}{m'}$, we deduce from the previous inequality that $\{u_n\}$ is bounded in $L^s(\Omega)$ and therefore u belongs to $L^s(\Omega)$ as well. □

3.1.3 The case $\beta > 1$

The case $\beta > 1$ has many analogies with the case $\beta = 1$, even if in this case it is only possible to show that a positive power of u_n is bounded in $H_0^1(\Omega)$, while u_n is bounded only in $H_{loc}^1(\Omega)$.

Lemma 3.1.6. *Let u_n be the solution of* (3.10), *with $\beta > 1$, and $f \in L^1(\Omega)$.*
Then $\{u_n^{\frac{\beta+1}{2}}\}$ is bounded in $H_0^1(\Omega)$, while $\{u_n\}$ is bounded in $H_{loc}^1(\Omega)$ and in $L^s(\Omega)$ with $s = \frac{N(\beta+1)}{N-2}$.

Proof. Taking u_n^β as test function in (3.10), recalling (3.6), one gets

$$a\,\beta \int_\Omega |Du_n|^2\, u_n^{\beta-1} dx \le \int_\Omega \frac{f_n\, u_n^\beta}{\left(u_n + \frac{1}{n}\right)^\beta}\, dx \le \int_\Omega f_n\, dx \le \int_\Omega f\, dx$$

and, since

$$\int_\Omega |Du_n|^2\, u_n^{\beta-1}\, dx = \frac{4}{(\beta+1)^2} \int_\Omega |Du_n^{\frac{\beta+1}{2}}|^2\, dx,$$

it follows that $\{u_n^{\frac{\beta+1}{2}}\}$ is uniformly bounded in $H_0^1(\Omega)$.
Since $s = \frac{2^*(\beta+1)}{2}$, from Sobolev embedding, it follows that $\{u_n\}$ is bounded in $L^s(\Omega)$.
To prove that $\{u_n\}$ is bounded in $H^1_{loc}(\Omega)$, let $\Omega' \subset\subset \Omega$ and $\varphi \in C_c^1(\Omega)$ identically zero outside Ω'. Choosing $u_n\, \varphi^2$ as test function in (3.10), using (3.6) and (3.7), one obtains

$$\begin{aligned} a \int_\Omega |Du_n|^2 \varphi^2\, dx + 2 \int_\Omega (M(x)\, Du_n, u_n\, \varphi\, D\varphi)\, dx &\le \int_\Omega \frac{f_n\, u_n\, \varphi^2}{(u_n + \frac{1}{n})^\beta}\, dx \\ &\le \frac{1}{c_{\Omega'}^{\beta-1}} \int_\Omega f_n\, \varphi\, dx. \end{aligned}$$

Since, by Young inequality

$$2\, b \int_\Omega (Du_n, u_n\, \varphi\, D\varphi)\, dx \le \frac{a}{2} \int_\Omega |Du_n|^2\, \varphi^2\, dx + \frac{2\, b^2}{a} \int_\Omega |D\varphi|^2 u_n^2\, dx,$$

using the fact that $\{u_n\}$ is bounded in $L^s(\Omega)$, with $s \ge 2$, it follows that

$$\begin{aligned} \frac{a}{2} \int_\Omega |Du_n|^2 \varphi^2\, dx &\le \frac{1}{c_{\Omega'}^{\beta-1}} \int_\Omega f_n\, \varphi^2\, dx + \frac{2\, b^2}{a} \int_\Omega |D\varphi|^2\, u_n^2\, dx \\ &\le \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_{\Omega'}^{\beta-1}} \int_\Omega f\, dx + \frac{2\, b^2}{a} \|D\varphi\|^2_{L^\infty(\Omega)} \int_\Omega u_n^2\, dx \\ &\le C(f, \varphi), \end{aligned}$$

namely $\{u_n\}$ is bounded in $H^1_{loc}(\Omega)$. □

Once that Lemma 3.1.6 has been proved, passing to the limit, it is easy to deduce the following

Theorem 3.1.7. *Let $\beta > 1$ and f be a nonnegative function in $L^1(\Omega)$ (not identically zero). Then there exists a solution in $H^1_{loc}(\Omega)$ of (3.5). Furthermore, $u^{\frac{\beta+1}{2}}$ belongs to $H_0^1(\Omega)$.*

Lemma 3.1.8. *Let $\beta > 1$ and $f \in L^m(\Omega)$, with $m \ge 1$. Then the solution of (3.5) given by Theorem 3.1.7 is such that*

- (*a*) *if $m > \frac{N}{2}$, then $u \in L^\infty(\Omega)$;*
- (*b*) *if $1 \le m < \frac{N}{2}$, then $u \in L^s(\Omega)$, with $s = \frac{Nm(\beta+1)}{N-2m}$.*

Proof. (*a*) The proof of this part is the same as the corresponding one for Lemma 3.1.7.
(*b*) If $m = 1$, then $s = \frac{2\,N}{N-2} = 2^*$, so that the thesis is true by Sobolev embedding.
If $1 < m < \frac{N}{2}$, let $\delta > \frac{\beta+1}{2}$ and consider $u_n^{2\delta-1}$ as test function in 3.10. Arguing as in the proof of Lemma 3.1.7 and using the fact that $\delta > \frac{\beta+1}{2} > 1$, one gets

$$a \left(\int_\Omega u_n^{2^*\delta}\, dx \right)^{\frac{2}{2^*}} \le \frac{\delta^2}{\mathcal{S}(2\delta - 1)} \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{(2\delta-1-\beta)m'} \right)^{\frac{1}{m'}}.$$

Choosing δ in such a way that $2^*\delta = (2\delta - 1 - \beta)m'$, namely $\delta = \frac{m\,(N-2)\,(1+\beta)}{2\,(N-2m)}$, yields that $\delta > \frac{\beta+1}{2}$ if and only if $m > 1$ and that $2^*\delta = s$. Therefore, since from $m < \frac{N}{2}$ it follows that $\frac{2}{2^*} > \frac{1}{m'}$, the norm of u_n in $L^s(\Omega)$ is bounded with respect to n, and so $u \in L^s(\Omega)$. □

3.1.4 The case $\beta < 1$

If $\beta < 1$, an a priori estimate of u_n in $H_0^1(\Omega)$ will be obtained assuming that f is more regular than $L^1(\Omega)$, as the following theorem shows.

Theorem 3.1.9. *Let $\beta < 1$, u_n be the solution of (3.10) and $f \in L^m(\Omega)$, with $m = \frac{2N}{N+\beta(N-2)+2} = \left(\frac{2^*}{1-\beta}\right)'$. Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. Recalling (3.6) and using u_n as test function in (3.10), one gets

$$a\int_\Omega |Du_n|^2\,dx \le \int_\Omega \frac{f_n\,u_n}{\left(u_n+\frac{1}{n}\right)^\beta}\,dx \le \int_\Omega f\,u_n^{\beta-1}\,dx$$
$$\le \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(1-\beta)m'}\,dx\right)^{\frac{1}{m'}}.$$

Since $m = \frac{2N}{N+\beta(N-2)+2}$, it follow that $(\beta-1)m' = 2^*$, therefore, by Sobolev embedding

$$a\,\mathcal{S}\left(\int_\Omega u_n^{2^*}\,dx\right)^{\frac{2}{2^*}} \le a\int_\Omega |Du_n|^2\,dx \le \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{2^*}\,dx\right)^{\frac{1}{m'}}. \tag{3.15}$$

¿From $m < \frac{N}{2}$ it follows that $\frac{2}{2^*} > \frac{1}{m'}$, therefore $\{u_n\}$ is bounded in $L^{2^*}(\Omega)$; this, together with (3.15), yields the thesis. □

Passing to the limit it is now possible to obtain the following

Theorem 3.1.10. *Let $\beta < 1$, $f \in L^m(\Omega)$ not identically zero, with $m = \frac{2N}{N+\beta(N-2)+2} = \left(\frac{2^*}{1-\beta}\right)'$. Then there exists a solution $u \in H_0^1(\Omega)$.*

The summability of the solution given by Theorem 3.1.10 depends once again on the summability of f.

Theorem 3.1.11. *Let $\beta < 1$ and $f \in L^m(\Omega)$, with $m \ge \frac{2N}{N+\beta(N-2)+2}$. Then the solution given by Theorem 3.15 is such that*

(a) if $m > \frac{N}{2}$, then $u \in L^\infty(\Omega)$,

(b) if $\frac{2N}{N+\beta(N-2)+2} \le m < \frac{N}{2}$, then $u \in L^s(\Omega)$, with $s = \frac{Nm(\beta+1)}{N-2m}$.

Proof. (a) This part is identical to the corresponding in Lemma 3.1.5.
(b) The case $m = \frac{2N}{N+\beta(N-2)+2}$ is true by Sobolev embedding, since in this case $s = 2^*$.
If $\frac{2N}{N+\beta(N-2)+2} < m < \frac{N}{2}$, let $\delta > 1$ and use $u_n^{2\delta-1}$ as test function in (3.10). Arguing as in Lemma 3.1.5, one gets

$$a\left(\int_\Omega u_n^{2^*\delta}\,dx\right)^{\frac{2}{2^*}} \le \frac{\delta^2}{\mathcal{S}(2\delta-1)}\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(2\delta-\beta-1)\,m'}\right)^{\frac{1}{m'}}.$$

Choosing δ in such a way that $2^*\delta = (2\delta-\beta-1)\,m'$, namely $\delta = \frac{m\,(N-2)\,(1+\beta)}{2\,(N-2m)}$, yields that $\delta > 1$ if and only if $m > \frac{2N}{N+\beta(N-2)+2}$ and $2^*\delta = s$. Therefore, since $m < \frac{N}{2}$ implies $\frac{2}{2^*} > \frac{1}{m'}$, it follows that the norm of u_n is bounded in $L^s(\Omega)$ with respect to n, and so the limit $u \in L^s(\Omega)$. □

If $m < \left(\frac{2^*}{1-\beta}\right)'$, the solution does not belong to $H_0^1(\Omega)$, but to a larger Sobolev space, depending on m.

Theorem 3.1.12. *Let $\beta < 1$ and $f \in L^m(\Omega)$, with $1 \le m < \frac{2N}{N+\beta(N-2)+2}$. Then there exists a solution u of (3.5), with $u \in W_0^{1,q}(\Omega)$ and $q = \frac{Nm(\beta+1)}{N-m(1-\beta)}$.*

Proof. The thesis follows by proving an a priori estimate on $\{u_n\}$ in $W_0^{1,q}(\Omega)$, with q as in the statement, since the existence of a solution $u \in W^{1,q}(\Omega)$ of (3.5) can be proved by passing to the limit in (3.10) as in the proof of Theorem 3.1.4.
Since the gradient of the function $u_n^{2\delta-1}$, with $\frac{\beta+1}{2} \le \delta < 1$, is singular where $u_n = 0$, this function is not admissible as test function in (3.10). Therefore, for n fixed, let $\varepsilon < \frac{1}{n}$ and consider $(u_n+\varepsilon)^{2\delta-1} - \varepsilon^{2\delta-1}$. Using (3.6) one gets

$$\begin{aligned} a(2\delta-1)\int_\Omega |Du_n|^2 (u_n+\varepsilon)^{2\delta-2}\,dx &\le \int_\Omega \frac{f_n((u_n+\varepsilon)^{2\delta-1}-\varepsilon^{2\delta-1})}{\left(u_n+\frac{1}{n}\right)^\beta}\,dx \\ &\le \int_\Omega f\,(u_n+\varepsilon)^{2\delta-\beta-1}\,dx, \end{aligned} \tag{3.16}$$

where the last inequality follows since $f_n \le f$ and ε is such that $\varepsilon < \frac{1}{n}$.
On the other hand, by Sobolev inequality

$$\begin{aligned} \int_\Omega |Du_n|^2\,(u_n+\varepsilon)^{2\delta-2}\,dx &= \int_\Omega \frac{|D((u_n+\varepsilon)^\delta - \varepsilon^\delta)|^2}{\delta^2}\,dx \\ &\ge \frac{\mathcal{S}}{\delta^2}\left(\int_\Omega ((un_n+\varepsilon)^\delta - \varepsilon^\delta)^{2^*}\,dx\right)^{\frac{2}{2^*}} \end{aligned}$$

which, combining with (3.16), yields

$$a\left(\int_\Omega ((u_n+\varepsilon)^\delta - \varepsilon^\delta)^{2^*}\,dx\right)^{\frac{2}{2^*}} \le \frac{\delta^2}{\mathcal{S}\,(2\delta-1)}\int_\Omega f\,(u_n+\varepsilon)^{2\delta-\beta-1}\,dx.$$

Letting ε tend to zero, one obtains

$$a\left(\int_\Omega u_n^{2^*\delta}\,dx\right)^{\frac{2}{2^*}} \le \frac{\delta^2}{\mathcal{S}\,(2\delta-1)}\int_\Omega f\,u_n^{2\delta-\beta-1}\,dx. \tag{3.17}$$

If $m = 1$, the thesis follows by choosing $\delta = \frac{\beta+1}{2}$, in order to obtain by (3.17) that $\{u_n\}$ is bounded in $L^{\frac{N\,(\beta+1)}{N-2}}$, which is the value of s when $m = 1$.
If $m > 1$, from (3.17), arguing as in Lemma 3.1.8, it follows that $\{u_n\}$ is bounded in $L^s(\Omega)$, with $s = \frac{Nm\,(\beta+1)}{N-2m}$. Therefore, from the choice of δ, it follows that the righthand side of (3.16) is bounded with respect to n and ε. Since $\delta < 1$

$$\int_\Omega \frac{|Du_n|^2}{(u_n+\varepsilon)^{2-2\delta}}\,dx = \int_\Omega |Du_n|^2\,(u_n+\varepsilon)^{2\delta-2} \le C.$$

If $q = \frac{Nm(\beta+1)}{N-m(1-\beta)}$, since $q < 2$, by Hölder inequality it follows that

$$\int_\Omega |Du_n|^q\,dx = \int_\Omega \frac{|Du_n|^q}{(u_n+\varepsilon)^{(1-\delta)\,q}}\,dx \le C\left(\int_\Omega (u_n+\varepsilon)^{\frac{(2-2\delta)\,q}{2-q}}\,dx\right)^{1-\frac{q}{2}}.$$

A simple computation shows that the choices of δ and q are such that $s = \frac{(2-2\delta)q}{2-q}$, so that the righthand side of the previous inequality is bounded with respect to both n and ε, hence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$.

□

3.2 Existence and uniqueness for a singular problem. An improvement of the construction of u_0 by Canino-Degiovanni

In this chapter, following closely [8], we study the equation $-\Delta\,u = f\,u^{-\beta}$. Actually in [8] the case $f = 1$ was considered.
In all this section we will assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that

$$0 < \gamma \le f \le \Gamma.$$

3.2.1 Comparison principles

Let Ω be a bounded open subset of $\mathbb{R}^{\mathbb{N}}$ and let $\beta > 0$.

Definition 3.2.1. *Let $u \in H^1_{loc}(\Omega)$. We say that $u \le 0$ on $\partial\Omega$ if, for every $\varepsilon > 0$, the function $(u-\varepsilon)^+ \in H^1_0(\Omega)$.*

It is readly seen that, if $u \in H^1_0(\Omega)$, then $u \le 0$ on $\partial\Omega$.

Definition 3.2.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $\mu \in W^{-1,2}(\Omega)$ and $\varphi \in H^1_{loc}(\Omega)$. Let us consider the equation*

$$-\Delta u = g(x,u) + \mu. \tag{3.18}$$

We say that φ is a local subsolution of (3.18) if $g(x,\varphi) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega (D\varphi, Dv)\, dx \le \int_\Omega g(x,\varphi)\, v\, dx + \langle \mu, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega.$$

We say that φ is a local supersolution of (3.18) if $g(x,\varphi) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega (D\varphi, Dv)\, dx \ge \int_\Omega g(x,\varphi)\, v\, dx + \langle \mu, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega.$$

Definition 3.2.3. *Let $\mu \in W^{-1,2}(\Omega)$ and $\varphi \in H^1_{loc}(\Omega)$. Let us consider the variational inequality*

$$\int_\Omega (Du, D(v-u))\, dx \ge \int_\Omega \frac{f}{u^\beta}(v-u)\, dx + \langle \mu, v-u\rangle \qquad \forall v \ge 0. \tag{3.19}$$

We say that φ is a local subsolution of (3.19) if $\varphi > 0$ a.e. in Ω, $\varphi^{-\beta} \in L^1_{loc}(\Omega)$ and

$$\int_\Omega (D\varphi, Dv) \le \int_\Omega \frac{f}{\varphi^\beta} v\, dx + \langle \mu, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } 0 \le v \le \varphi \text{ a.e. in } \Omega.$$

We say that φ is a supersolution of (3.19) if $\varphi > 0$ a.e. in Ω, $\varphi^{-\beta} \in L^1_{loc}(\Omega)$ and

$$\int_\Omega (D\varphi, Dv)\, dx \le \int_\Omega \frac{f}{\varphi^\beta}\, dx + \langle \mu, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega.$$

Let us recall from [8] the following

Lemma 3.2.4. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$\forall S > 0 : \sup_{|s|\le S} |g(\cdot, s)| \in L^1_{loc}(\Omega),$$

let $\mu \in W^{-1,2}(\Omega)$ and let $\varphi, u, \psi \in H^1_{loc}(\Omega)$. Assume that φ is a subsolution of (3.18), ψ is a supersolution of (3.18), $\varphi \le u \le \psi$ a.e. in Ω, $g(x,u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega (Du, D(v-u))\, dx \ge \int_\Omega g(x,u)(v-u)\, dx + \langle \mu, v-u\rangle \qquad \forall v \in u + (H^1_0(\Omega) \cap L^\infty_c(\Omega)) \text{ with } \varphi \le v \le \psi \text{ a.e. in } \Omega.$$

Then $-\Delta u = g(x,u) + \mu$ in $D'(\Omega)$.

Let us denote by $\phi_k : \mathbb{R} \to \mathbb{R}$ the primitive of the function

$$\begin{cases} \max\{-s^{-\beta}, -k\} & \text{if } s > 0, \\ -k & \text{if } s \leq 0, \end{cases}$$

such that $\phi_k(1) = 0$.

Lemma 3.2.5. *Let $\mu \in W^{-1,2}(\Omega)$ and assume that $\varphi, \psi \in H^1_{loc}(\Omega)$ are, respectively, a subsolution and a supersolution of* (3.19)*, with $\varphi \leq 0$ on $\partial\Omega$.*
Then $\varphi \leq \psi$ a.e. in Ω.

Proof. Note that ψ is a supersolution also of the equation $-\Delta u = -\phi'_k(u) + \mu$. In fact, since ψ is supersolution of (3.19) we have

$$\int_\Omega (D\psi, Dv)\, dx \geq \int_\Omega \frac{f}{\psi^\beta}\, v\, dx + \langle \mu, v \rangle \geq \int_\Omega -f\, \varphi'_k(\psi)\, v\, dx + \langle \mu, v \rangle,$$

since $\psi^{-\beta} \geq -\phi'_k(\psi)$ by definition.
Let $\varepsilon > 0$, $k > \varepsilon^{-\beta}$ and ϕ_k and $f_{\mu,k}$ defined as above. Let u be the minimum of the functional $f_{\mu,k}$ in the set

$$\mathbb{K} = \{u \in H^1_0(\Omega) : 0 \leq u \leq \psi \text{ a.e. in } \Omega\}.$$

According to [17], we have

$$\int_\Omega (Du, D(v-u))\, dx \geq -\int_\Omega f\, \phi'_k(u)\, (v-u)\, dx + \langle \mu, v-u \rangle \qquad \forall v \in \mathbb{K}.$$

In particular, if $v \in C^\infty_c(\Omega)$ with $v \geq 0$ and $t > 0$, we can choose as test function $v_t = \min\{u + tv, \psi\}$ and, arguing as in Lemma 3.2.4, we find that

$$\int_\Omega (Du, Dv)\, dx \geq -\int_\Omega f\, \phi'_k(u)\, v\, dx + \langle \mu, v \rangle. \tag{3.20}$$

Actually, (3.20) holds for every $v \in H^1_0(\Omega)$ with $v \geq 0$ a.e. in Ω. In fact, let $v \in H^1_0(\Omega)$; then there exists $v_n \in C^\infty_c(\Omega)$ with $v_n \geq 0$ and $v_n \to v$ in $H^1_0(\Omega)$.
Then we have

$$\int_\Omega (Du, Dv_n)\, dx \geq -\int_\Omega f\, \phi'_k(u)\, v_n\, dx + \langle \mu, v_n \rangle.$$

Going to the limit as $n \to +\infty$ we get

$$\int_\Omega (Du, Dv)\, dx \geq -\int_\Omega f\, \phi'_k(u)\, v\, dx + \langle \mu, v \rangle.$$

In fact:

- $\displaystyle \Big| \int_\Omega (Du, Dv_n)\, dx - \int_\Omega (Du, Dv)\, dx \Big| \leq \int_\Omega |(Du, D(v_n - v))|\, dx \leq \|u\|_{H^1_0(\Omega)} \|v_n - v\|_{H^1_0(\Omega)} \underset{n\to+\infty}{\longrightarrow} 0;$
- $\displaystyle \Big| \int_\Omega f\, \phi'_k(u)\, (v_n - v)\, dx \Big| \leq \|f\|_\infty \|\phi'_k\|_\infty \|v_n - v\|_{L^2(\Omega)} \underset{n\to+\infty}{\longrightarrow} 0;$
- $\langle \mu, v_n \rangle \to \langle \mu, v \rangle$ by definition.

In particular, since $u \geq 0$, we have $(\varphi - u - \varepsilon)^+ \in H^1_0(\Omega)$, so that

$$\int_\Omega (Du, D(\varphi - u - \varepsilon)^+)\, dx \geq -\int_\Omega f\, \phi'_k(u)\, (\varphi - u - \varepsilon)^+\, dx + \langle \mu, (\varphi - u - \varepsilon)^+ \rangle. \tag{3.21}$$

Let now $v \in H_0^1(\Omega)$ such that $0 \le v \le \varphi$ a.e. in Ω and $D\varphi \in L^2(v > 0)$. Let $\{\widehat{v}_n\}$ a sequence in $C_c^\infty(\Omega)$ which converges to v in $H_0^1(\Omega)$ and let $v_n = \min\{\widehat{v}_n^+, v\}$. Then, since φ is subsolution,

$$\int_\Omega (Dv, Dv_n)\, dx \le \int_\Omega \frac{f}{\varphi^\beta}\, v_n\, dx + \langle \mu, v_n \rangle. \tag{3.22}$$

If $\varphi^{-\beta} v \in L^1(\Omega)$, going to the limit as $n \to +\infty$, we get

$$\int_\Omega (D\varphi, Dv)\, dx \le \int_\Omega \frac{f}{\varphi^\beta}\, v\, dx + \langle \mu, v \rangle. \tag{3.23}$$

In fact

- $\displaystyle \Big| \int_\Omega (D\varphi, D(v_n - v)\, dx \Big| \le \|D\varphi\|_{L^2(\Omega)} \|v_n - v\|_{H_0^1(\Omega)}\, dx \underset{n\to+\infty}{\longrightarrow} 0;$
- $\displaystyle \int_\Omega \frac{f}{\varphi^\beta}\, v_n\, dx \to \int_\Omega \frac{f}{\varphi^\beta}\, v\, dx$ by the Dominated Convergence Theorem.

On the other hand, if $\varphi^{-\beta} v \notin L^1(\Omega)$, formula (3.23) is obviously true.
In particular we have

$$\begin{aligned} \int_\Omega (D\varphi, D(\varphi - u - \varepsilon)^+)\, dx \le \int_\Omega & f\, \varphi^{-\beta} (\varphi - u - \varepsilon)^+\, dx \\ & + \langle \mu, (\varphi - u - \varepsilon)^+ \rangle. \end{aligned} \tag{3.24}$$

Since $\varepsilon < k$, from (3.21) and (3.24) we deduce that

$$\begin{aligned} \int_\Omega |D(\varphi - u - \varepsilon)^+|^2\, dx &= \int_\Omega (D(\varphi - u), D(\varphi - u - \varepsilon)^+)\, dx \\ &\le \int_\Omega f(\varphi^{-\beta} + \phi_k'(u))(\varphi - u - \varepsilon)^+\, dx \\ &= \int_\Omega f(-\phi_k'(\varphi) + \phi_k'(u))(\varphi - u - \varepsilon)^+\, dx \\ &\le 0, \end{aligned}$$

whence $(\varphi - u - \varepsilon)^+ = 0$, so that $\varphi \le u + \varepsilon \le \psi + \varepsilon$. The assertion follows from the arbitrariness of ε. □

3.2.2 Construction of the solution

We are now ready to prove the existence and uniqueness of the solution of (3.4).

Theorem 3.2.6. *Let $0 < \gamma \le f \le \Gamma$. There exists one and only one $u_0 \in C^2(\Omega)$ solution of*

$$\begin{cases} -\Delta u = \dfrac{f}{u^\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \le 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, if $u_1 \in H_0^1(\Omega) \cap C^\infty(\Omega)$ satisfies $-\Delta u_1 = 1$ in Ω, then

$$\gamma^{\beta+1} \|u_1\|_\infty u_1 \le u_0 \le (\Gamma(\beta+1)u_1)^{\frac{1}{\beta+1}}.$$

Proof. Let us set $\varphi = \gamma^{\beta+1}\|u_1\|_\infty u_1$ and $\psi = (\Gamma(\beta+1)u_1)^{\frac{1}{\beta+1}}$. Then it turns out that $\varphi \le \psi$ and φ and ψ are, respectively, a subsolution and a supersolution of $-\Delta u = -f\phi_k'(u)$, for any

$k \geq \|u_1\|_\infty^{\frac{-\beta}{\beta+1}}$. Let us show that φ is a subsolution.
Let $u_{0,k} \in H_0^1(\Omega)$ be the minimum of $f_{0,k}$, namely the weak solution of

$$\begin{cases} -\Delta u = -\phi_k'(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases} \tag{3.25}$$

Of course, $f_{0,k}$ admits one and only one minimum also on the convex set

$$\{u \in H_0^1(\Omega) : \varphi \leq u \leq \psi \text{ a.e in } \Omega\}$$

and such minimum is a solution of (3.25). It follows that $\varphi \leq u_{0,k} \leq \psi$ a.e. in Ω. Since $u_{0,k}$ is a subsolution of $-\Delta u = -f\,\phi_{k+1}'(u)$ ($-\phi_k'$ is decreasing w.r.t. the parameter k), it turns out that $u_{0,k} \leq u_{0,k+1}$. In fact, let $w = (u_{0,k} - u_{0,k+1})^+$. Then we have

$$\int_\Omega (Du_{0,k}, Dw)\,dx \leq -\int_\Omega f\,\phi_{k+1}'(u_{0,k}) w\,dx$$

and

$$\int_\Omega (Du_{0,k+1}, Dw)\,dx = -\int_\Omega f\phi_{k+1}'(u_{0,k+1}) w\,dx.$$

Subtracting we get

$$\begin{aligned} \int_\Omega |Dw|^2\,dx &\leq -\int_\Omega f(\phi_{k+1}'(u_{0,k}) - \phi_{k+1}'(u_{0,k+1}))\,w\,dx \\ &= \int_\Omega f(\phi_k'(u_{0,k}) - \phi_{k+1}'(u_{0,k+1}))(u_{0,k} - u_{0,k+1})^+\,dx \leq 0. \end{aligned}$$

So we have $w = (u_{0,k} - u_{0,k+1})^+ = 0$ and, consequently, $u_{0,k} \leq u_{0,k+1}$ a.e. in Ω. On the other hand, for every $\varepsilon > 0$ there exists $\bar{k} > \varepsilon^{-\beta}$ and for $k > \bar{k}$ we have

$$-\Delta(u_{0,\bar{k}}) = -\phi_{\bar{k}}'((u_{0,\bar{k}} + \varepsilon) - \varepsilon) \geq -\phi_k'(u_{0,\bar{k}} + \varepsilon),$$

namely $u_{0,\bar{k}}$ is a supersolution of $-\Delta u = -\phi_k'(u)$. Therefore $u_{0,k} < u_{0,\bar{k}} + \varepsilon$, namely $\{u_{0,k}\}$ is a Cauchy sequence in $L^\infty(\Omega)$.
Therefore $\{u_{0,k}\}$ is decreasing and convergent to some $u_0 \in L^\infty(\Omega)$. Moreover, since $\varphi \leq u_0 \leq \psi$ and $u_1 > 0$ in Ω, it follows that $u_0^{-\beta} \in L^\infty_{loc}(\Omega)$.
Given $\varepsilon > 0$, we have

$$\int_\Omega |D(u_{0,k} - \varepsilon)^+|^2\,dx = -\int_\Omega f\,\phi_k'(u_{0,k})\,(u_{0,k} - \varepsilon)^+\,dx \leq \varepsilon^{-\beta} \int_\Omega (u_{0,k} - \varepsilon)^+\,dx,$$

so that $(u_{0,k} - \varepsilon)^+$ is bounded in $H_0^1(\Omega)$ as $k \to +\infty$ and then

$$\forall \varepsilon > 0,\ (u_0 - \varepsilon)^+ \in H_0^1(\Omega),$$

namely $u \leq 0$ on $\partial\Omega$. Since $u_{0,k} \geq \varphi$, we deduce that $u_0 \in H^1_{loc}(\Omega)$. In fact, let K be a compact set of Ω. We have $u_0 \geq \varphi \geq c > 0$ on Ω. It is sufficiently to choose $\varepsilon < c$ to get $u_0 \geq c > \varepsilon$ in K and consequently $u_0 - \varepsilon > 0$ in K. Then it turns out that $(u_0 - \varepsilon)^+ \equiv u_0 - \varepsilon$ in K and since $(u_0 - \varepsilon)^+ \in H^1(K)$, $u_0 - \varepsilon \in H^1(K)$ too and, then, $u_0 \in H^1_{loc}(\Omega)$. On the other hand, it turns out that $\{Du_{0,k}\}$ is weakly convergent to Du_0 in $L^2_{loc}(\Omega)$. In fact, if K is a compact set in Ω, $D(u_{0,k} - \varepsilon)^+ \to D(u_0 - \varepsilon)^+$ and the the assertion follows from a suitable choice of ε.
¿From (3.25) it follows that $-\Delta u_0 = f u_0^{-\beta}$ in $D'(\Omega)$. In fact, let $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_\Omega (Du_{0,k}, D\varphi)\,dx \to \int_\Omega (Du, D\varphi)\,dx$$

because $Du_{0,k}$ is weakly convergent to Du_0.
On the other hand

$$
\begin{aligned}
&\Big|\int_\Omega f(-\phi_k'(u_{0,k}))\varphi\,dx - \int_\Omega \frac{f}{u_0^\beta}\,\varphi\,dx\Big| \\
&\le \int_\Omega |f|\,|u_0^{-\beta} + \phi_k'(u_{0,k})|\,|\varphi|\,dx \le \Gamma C \int_{\{\mathrm{supp}\varphi\}} |u_0^{-\beta} + \phi_k'(u_{0,k})|\,dx \\
&\le \Gamma C \int_{\{\mathrm{supp}\varphi\}} \Big|\frac{1}{u_{0,k}^\beta} - \frac{1}{u_0^\beta}\Big|\,dx \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

¿From interior regularity theory, it follows that $u_0 \in C^\infty(\Omega)$. The uniqueness of u_0 follows from Lemma 3.2.5. □

Corollary 3.2.7. *There exists one and only one $u_0 \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ solution of*

$$
\begin{cases}
-\Delta u = \dfrac{f}{u^\beta} & \text{in } \Omega,\\
u > 0 & \text{in } \Omega,\\
u = 0 & \text{on } \partial\Omega.
\end{cases}
$$

if and only if each $x \in \partial\Omega$ satisfies the Wiener criterion.

Proof. The proof is the same as in [8, Corollary 2.4]. □

Chapter 4

The variational carachterization and C^1 perturbations

In this chapter we provide a variational approach to our problem, expliting the arguments in [8].

4.1 A variational characterization

Let u_0 be the solution of (3.4), $\mu \in W^{-1,2}(\Omega)$, $\phi : \mathbb{R} \to (-\infty, +\infty]$ defined as in (3.2).
Let $G_0 : \Omega \times \mathbb{R} \to [0, +\infty]$ be defined as

$$G_0(x, s) = \phi(u_0(x) + s) - \phi(u_0(x)) + \frac{s}{u_0^\beta(x)}.$$

Then $G_0(x, 0) = 0$ and $G_0(x, \cdot)$ is convex and lower semicontinuous for any $x \in \Omega$. Moreover, $G_0(x, \cdot) \in C^1((-u_0(x), +\infty))$ with

$$D_s G_0(x, s) = \frac{1}{u_0^\beta(x)} - \frac{1}{(u_0(x) + s)^\beta}.$$

Define a functional $F_\mu : L^2(\Omega) \to (-\infty, +\infty]$ by

$$F_\mu(u) = \begin{cases} \frac{1}{2}\int_\Omega |D(u - u_0)|^2\,dx + \int_\Omega f G_0(x, u - u_0)\,dx \\ \qquad\qquad - \langle \mu, u - u_0 \rangle & \text{if } u \in u_0 + H_0^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then F_μ is strictly convex, lower semicontinuous and coercive, with $F_\mu(u_0) = 0$. Moreover, the effective domain of F_μ is

$$\{u \in H_0^1(\Omega) : G_0(x, u - u_0) \in L^1(\Omega)\} \subseteq H^1_{loc}(\Omega).$$

In the case $\mu = 0$, u_0 is just the minimum of F_0.

Theorem 4.1.1. *The following facts hold:*

(a) *for every* $\mu \in W^{-1,2}(\Omega)$ *and* $u \in H^1_{loc}(\Omega)$ *we have that* u *is the minimum of* F_μ *if and only if* u *satisfies*

$$\begin{cases} u > 0 \ a.e. \ in\ \Omega \ and\ \dfrac{1}{u^\beta} \in L^1_{loc}(\Omega), \\ \displaystyle\int_\Omega (Du, D(v-u))\, dx - \int_\Omega \frac{f}{u^\beta}(v-u)\, dx \geq \langle \mu, v-u \rangle \\ \qquad\qquad \forall v \in u + (H^1_0(\Omega) \cap L^\infty_c(\Omega)) \ with \ v \geq 0 \ a.e. \ in\ \Omega, \\ u \leq 0 \ on\ \partial\Omega; \end{cases} \tag{4.1}$$

in particular, for every $\mu \in W^{-1,2}(\Omega)$ *problem* (4.1) *admits one and only one solution* $u \in H^1_{loc}(\Omega)$;

(b) *if* $\mu_1, \mu_2 \in W^{-1,2}(\Omega)$ *and* $u_1, u_2 \in H^1_{loc}(\Omega)$ *are the corresponding solutions of* (4.1), *we have* $u_1 - u_2 \in H^1_0(\Omega)$ *and*

$$\|D(u_1 - u_2)\|_{L^2(\Omega)} \leq \|\mu_1 - \mu_2\|_{W^{-1,2}(\Omega)}.$$

Proof. (a) Let $\mu \in W^{-1,2}(\Omega)$. If $u \in u_0 + H^1_0(\Omega)$ is the only minimum of F_μ, it follows that $f\, G_0(x, u-u_0) \in L^1(\Omega)$ and then, since $\gamma \leq f \leq \Gamma$, we have that $g_0(x, u-u_0) \in L^1(\Omega)$. Hence, by the definition of $G_0(x,s)$, it turns out that $u \geq 0$ a.e. in Ω. Moreover

$$\begin{cases} \displaystyle\int_\Omega (D(u-u_0), D(v-u)\, dx + \int_\Omega f \Big(\frac{1}{u_0^\beta} - \frac{1}{u^\beta}\Big)(v-u)\, dx \geq \langle \mu, v-u \rangle, \\ \Big(\dfrac{1}{u_0^\beta} - \dfrac{1}{u^\beta}\Big)(v-u) \in L^1(\Omega), \end{cases} \tag{4.2}$$

for every $v \in u_0 + H^1_0(\Omega)$ with $G_0(x, v-u_0) \in L^1(\Omega)$.
In order to prove (4.2), let $v_\varepsilon = u + \varepsilon\,(v-u)$. Since u is the minimum of F_μ, we have $F_\mu(v_\varepsilon) - F_\mu(u) \geq 0$. Namely

$$\begin{aligned} F_\mu(v_\varepsilon) - F_\mu(u) &= \frac{1}{2}\int_\Omega |D(v_\varepsilon - u_0)|^2\, dx + \int_\Omega f\, G_0(x, v_\varepsilon - u_0)\, dx \\ &- \langle \mu, v_\varepsilon - u_0 \rangle - \frac{1}{2}\int_\Omega |D(u-u_0)|^2\, dx \\ &+ \int_\Omega f\, G_0(x, u-u_0)\, dx + \langle \mu, u - u_0 \rangle \\ &= \frac{1}{2}\varepsilon^2 \int_\Omega |D(v-u)|^2\, dx + \varepsilon \int_\Omega (D(u-u_0), D(v-u))\, dx \\ &+ \int_\Omega f\,(G_0(x, u-u_0+\varepsilon\,(v-u)) - G_0(x, u-u_0))\, dx \\ &- \varepsilon \langle \mu, v-u \rangle \geq 0, \end{aligned}$$

namely

$$\begin{aligned} \frac{1}{2}\varepsilon \int_\Omega |D(v-u)|^2\, dx &+ \int_\Omega (D(u-u_0), D(v-u))\, dx \\ &+ \int_\Omega \frac{G_0(x, u-u_0+\varepsilon(v-u)) - G_0(x, u-u_0)}{\varepsilon}\, dx \\ &\geq \langle \mu, v-u \rangle. \end{aligned}$$

Since $G_0(x, \cdot)$ is convex $\forall x \in \Omega$, passing to the limit as $\varepsilon \to 0^+$, we get

$$\int_\Omega (D(u-u_0), D(v-u))\, dx + \int_\Omega \left(\frac{1}{u_0^\beta} - \frac{1}{u^\beta}\right) dx \geq \langle \mu, v-u \rangle$$

which yields the fist part of (4.2). In order to prove the second part, we recall that F_μ is strictly convex so, if v_ε and u are such as above, we have

$$F_\mu(v_\varepsilon) = F_\mu(\varepsilon\, v + (1-\varepsilon)\, u) < \varepsilon\, F_\mu(v) + (1-\varepsilon)\, F_\mu(u)$$

namely

$$\frac{F_\mu(v_\varepsilon) - F_\mu(u)}{\varepsilon} < F_\mu(v) - F_\mu(u)$$

and the righthand term is finite, since u is the minimum of F_μ and $v \in u_0 + H_0^1(\Omega)$.
On the other hand, for any $v \in u_0 + H_0^1(\Omega)$, namely $v = u_0 + \varphi$ for some $\varphi \in H_0^1(\Omega)$ we have

$$\begin{aligned}
\Big| \int_\Omega (D(u-u_0), D(v-u))\, dx \Big| &= \Big| \int_\Omega (D(u-u_0), D(u_0+\varphi-u))\, dx \Big| \\
&\le \int_\Omega |D(u-u_0)|^2\, dx \\
&+ \int_\Omega |(D(u-u_0), D\varphi)|\, dx \\
&\le \int_\Omega |D(u-u_0)|^2\, dx + \frac{1}{2} \int_\Omega |D(u-u_0)|^2\, dx \\
&+ \frac{1}{2} \int_\Omega |D\varphi|^2\, dx < +\infty.
\end{aligned}$$

Where in the last inequality we use Young inequality.
Therefore

$$\begin{aligned}
0 \le\ & \frac{1}{2}\, \varepsilon \int_\Omega |D(v-u)|^2\, dx + \int_\Omega (D(u-u_0), D(v-u))\, dx \\
+ & \int_\Omega f\, \frac{(G_0(x, u-u_0+\varepsilon\,(v-u)) - G_0(x, u-u_0))}{\varepsilon}\, dx\ , \\
- & \langle \mu, v-u \rangle = \frac{F_\mu(v_\varepsilon) - F_\mu(u)}{\varepsilon} < F_\mu(v) - F_\mu(u)
\end{aligned}$$

which yields (4.2).
In particular from (4.2) we have

$$\Big(\frac{1}{u_0^\beta} - \frac{1}{u^\beta}\Big)\, v \in L^1(\Omega) \qquad \text{for every } v \in C_c^\infty(\Omega) \text{ with } v \ge 0. \tag{4.3}$$

In fact, let $v \in C_c^\infty(\Omega)$ with $v \ge 0$ and let us consider $v' = v+u$. Then $v' \ge 0$ and $v' \in u_0 + H_0^1(\Omega)$. Moreover

$$\begin{aligned}
0 \le G_0(x, v'-u_0) &= G_0(x, v+u-u_0) = \phi(v+u) - \phi(u_0) + (v+u-u_0)\, \frac{1}{u_0^\beta} \\
&\le \phi(u) - \phi(u_0) + (u-u_0)\, \frac{1}{u_0^\beta} + v\, \frac{1}{u_0^\beta} = G_0(x, u-u_0) + v\, \frac{1}{u_0^\beta}.
\end{aligned}$$

Both the last two terms belong to $L^1(\Omega)$, the first one because u is the minimum of F_μ and the second one because v has compact support and $u_0 > 0$ in Ω.
Whence $G_0(x, v'-u_0) \in L^1(\Omega)$ and we can use v' in (4.2). It follows that $u > 0$ a.e. in Ω and $u^{-\beta} \in L^1_{loc}(\Omega)$.
Let now $\varepsilon, \sigma > 0$ and let $v = \min\{u-u_0, \varepsilon - (u_0-\sigma)^+\}$. Clearly $v \in H_0^1(\Omega)$ and

$$\begin{cases} v = u - u_0 & \text{if } u - u_0 < \varepsilon - (u_0-\sigma)^+ \\ v = \varepsilon - (u_0-\sigma)^+ & \text{otherwise} \begin{cases} v = \varepsilon - u_0 + \sigma & \text{if } u_0 \ge \sigma \\ v = \varepsilon \le u - u_0 & \text{otherwise} \end{cases} \end{cases}$$

It follows that $G_0(x,v) \in L^1(\Omega)$, hence, by (4.2) with $v = v + u_0$,

$$\begin{cases} \displaystyle\int_\Omega (D(u-u_0), D(v+u_0-u))\,dx + \int_\Omega f\Big(\frac{1}{u_0^\beta} - \frac{1}{u^\beta}\Big)(v+u_0-u)\,dx \\ \qquad\qquad\qquad\qquad \geq \langle \mu, v+u_0-u\rangle, \\ \Big(\dfrac{1}{u_0^\beta} - \dfrac{1}{u^\beta}\Big)(v+u_0-u) \in L^1(\Omega). \end{cases} \tag{4.4}$$

In particular, both $u^{-\beta}(u-u_0-v)$ and $u_0^{-\beta}(u-u_0-v)$ belong to $L^1(\Omega)$.
In order to prove this, let us note that

$$((u_0-\sigma)+u-u_0-\varepsilon)^+ = u-u_0-v \in H_0^1(\Omega).$$

Let us consider the two possible cases:

- $((u_0-\sigma)^+ + u - u_0 - \varepsilon)^+ = 0$, this implies $(u_0-\sigma)^+ + u - u_0 - \varepsilon \leq 0$, whence $u - u_0 \leq \varepsilon - (u_0-\sigma)^+$ and then $v = u - u_0$;
- if $(u_0-\sigma)^+ + u - u_0 - \varepsilon \geq 0$, then $u - u_0 \geq \varepsilon - (u_0-\sigma)^+ = v$, whence $u - u_0 - v = u - u_0 - \varepsilon + (u_0-\sigma)^+$.

If $u-u_0-v=0$, of course both $u^{-\beta}(u-u_0-v)$ and $u_0^{-\beta}(u-u_0-v)$ belong to $L^1(\Omega)$.
When $u-u_0-v \neq 0$, since this implies $u > \varepsilon$, we have that $u^{-\beta}(u-u_0-v)$ and $u_0^{-\beta}(u-u_0-v)$ belong to $L^1(\Omega)$ also in this case.
On the other hand we have

$$\int_\Omega (D(u_0-\sigma), D\varphi)\,dx \leq \int_\Omega \frac{f}{u_0^\beta}\,\varphi\,dx \qquad \forall\varphi \in C_c^\infty(\Omega) \text{ with } \varphi \geq 0.$$

In fact, if we denote by Ω^+ the support of $(u_0-\sigma)^+$, we get

$$\int_{\Omega^+} (D(u_0-\sigma), D\varphi)\,dx - \int_{\Omega^+} \langle \varphi\, D(u_0-\sigma), \eta\rangle\,dx = \int_{\Omega^+} \frac{f}{u_0^\beta}\varphi\,dx$$

and then

$$\int_\Omega (D(u_0-\sigma)^+, D\varphi)\,dx \leq \int_{\Omega^+} \frac{f}{u_0^\beta}\,\varphi\,dx \leq \int_\Omega \frac{f}{u_0^\beta}\,\varphi\,dx \quad \forall\varphi \in C_c^\infty(\Omega)$$

with $\varphi \geq 0$.
Actually we can write

$$\int_\Omega (D(u_0-\sigma), D\varphi)\,dx \leq \int_\Omega \frac{f}{u_0^\beta}\,\varphi\,dx \qquad \forall\varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0 \text{ a.e. in } \Omega. \tag{4.5}$$

In fact let $\varphi_n \in C_c^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \to \varphi$ in $H_0^1(\Omega)$. We can choose $\{\varphi_n\}$ such that $\varphi_n < \varphi$. Then we have

$$\begin{cases} \displaystyle\Big|\int_\Omega (D(u_0-\sigma)^+, D\varphi_n)\,dx - \int_\Omega (D(u_0-\sigma)^+, D\varphi)\,dx\Big| \\ \qquad\qquad \displaystyle\leq \int_\Omega |((D(u_0-\sigma)^+, D(\varphi_n-\varphi))|\,dx \\ \qquad\qquad \leq \|(u_0-\sigma)^+\|_{H_0^1(\Omega)} \|\varphi_n - \varphi\|_{H_0^1(\Omega)} \underset{n\to\infty}{\longrightarrow} 0 \\ \displaystyle\int_\Omega \frac{f}{u_0^\beta}\,\varphi_n\,dx \underset{n\to\infty}{\longrightarrow} \int_\Omega \frac{f}{u_0^\beta}\,\varphi\,dx \text{ by the Dominated Convergence Theorem} \\ \qquad\qquad \text{if } \dfrac{f}{u_0^\beta} \in L^1(\Omega). \end{cases}$$

If $\frac{f}{u_0^\beta} \notin L^1(\Omega)$, formula (4.5) is obviously true.
In particular, chosen $\varphi = u - u_0 - v$, from (4.4) and (4.5) we get

$$\begin{aligned}\int_\Omega |D(u-u_0-v)|^2\,dx &= \int_\Omega (D((u-u_0)^+ + u - u_0 - \varepsilon), D(u-u_0-v))\,dx \\ &= \int_\Omega (D(u-u_0)^+, D(u-u_0-v))\,dx \\ &+ \int_\Omega (D(u-u_0-\varepsilon), D(u-u_0-v))\,dx \\ &\le \int_\Omega \frac{f}{u^\beta}\,(u-u_0-v)\,dx + \langle \mu, u-u_0-v\rangle \\ &\le \varepsilon^{-\beta} \int_\Omega f\,(u-u_0-v)\,dx + \langle \mu, u-u_0-v\rangle.\end{aligned}$$

Therefore, for any $\varepsilon > 0$, $((u_0-\sigma)^+ + u - u_0 - \varepsilon)^+$ is bounded in $H_0^1(\Omega)$ as $\sigma \to 0^+$. On the other hand, since $((u_0-\sigma)^+ + u - u_0 - \varepsilon)^+ \underset{\sigma\to 0^+}{\longrightarrow} (u-\varepsilon)^+$ a.e. and therefore $(u-\varepsilon)^+ \in H_0^1(\Omega)$, namely $u \le 0$ on $\partial\Omega$.
Let now $v \in u + (H_0^1(\Omega) \cap L_c^\infty(\Omega))$ with $v \ge 0$ a.e. in Ω and let $v_0 \in C_c^\infty(\Omega)$ with $v_0 \ge 0$ a.e. in Ω and $v_0 = 1$ when $v \ne u$. Then, for every $\varepsilon > 0$, $G_0(x, v + \varepsilon v_0 - u_0) \in L^1(\Omega)$. In fact

- ∘ near the boundary, $v_0 = 0$ and $v = u$, therefore

$$G_0(x, v + \varepsilon v_0 - u_0) = G_0(x, u - u_0) \in L^1(\Omega);$$

- ∘ in the interior of Ω $v = u + \varphi$, with $\varphi \in H_0^1(\Omega) \cap L_c^\infty(\Omega)$, therefore

$$\begin{aligned}G_0(x, v + \varepsilon v_0 - u_0) &= G_0(x, u + \varphi + \varepsilon - u_0) \\ &= \phi(u+\varphi+\varepsilon) - \phi(u_0) + (u+\varphi+\varepsilon-u_0)\frac{1}{u_0^\beta},\end{aligned}$$

  which is in $L^1(\Omega)$ since in the interior of Ω we have $u + \varphi \ge 0$ and $u_0 > 0$.

It follows by (4.2) that

$$\begin{aligned}\int_\Omega (D(u-u_0), D(v+\varepsilon v_0 - u))\,dx + \int_\Omega f\left(\frac{1}{u_0^\beta} - \frac{1}{u^\beta}\right)(v+\varepsilon v_0-u)\,dx \\ \ge \langle \mu, v + \varepsilon v_0 - u\rangle.\end{aligned}$$

¿From the arbitrariness of ε it follows

$$\int_\Omega (D(u-u_0), D(v-u))\,dx + \int_\Omega f\left(\frac{1}{u_0^\beta} - \frac{1}{u^\beta}\right)(v-u)\,dx \ge \langle \mu, v-u\rangle.$$

In fact, in the interior of Ω u_0 is bounded away from 0 and $u^{-\beta} \in L^1_{loc}(\Omega)$, and therefore

- ∘ $\left|\int_\Omega \frac{f}{u_0^\beta}\,(v - \varepsilon v_0 - u)\,dx - \int_\Omega \frac{f}{u_0^\beta}\,(v-u)\,dx\right| \underset{\varepsilon\to 0^+}{\longrightarrow} 0;$
- ∘ $\left|\int_\Omega \frac{f}{u^\beta}\,(v - \varepsilon v_0 - u)\,dx - \int_\Omega \frac{f}{u^\beta}\,(v-u)\,dx\right| \underset{\varepsilon\to 0^+}{\longrightarrow} 0.$

Since u_0 is solution of (3.4), it follows that

$$\int_\Omega (Du, D(v-u))\,dx - \int_\Omega \frac{f}{u^\beta}\,(v-u)\,dx \ge \langle \mu, v-u\rangle.$$

Conversely, let u be a solution of (4.1) and let $\widehat{u} \in H^1_{loc}(\Omega)$ be the minimum of F_μ. Then $\widehat{u}$ is a solution of (4.1) too. In particular, u and $\widehat{u}$ are both a subsolution and a supersolution of (3.19) and from Lemma 3.2.5 it follows that $u = \widehat{u}$, namely u is the minimum of F_μ.

(b) Let $\mu_1, \mu_2 \in W^{-1,2}(\Omega)$ and let $u_1, u_2 \in H^1_0(\Omega)$ be the corresponding minima of f_{μ_1} and f_{μ_2}. Then, from (4.1) it follows that

$$\int_\Omega (D(u_1 - u_0), D(v - u_1))\, dx + \int_\Omega f \left(\frac{1}{u_0^\beta} - \frac{1}{u_1^\beta} \right) (v - u_1) \geq \langle \mu_1, v - u_1 \rangle, \tag{4.6}$$

and

$$\int_\Omega (D(u_2 - u_0), D(v - u_2))\, dx + \int_\Omega f \left(\frac{1}{u_0^\beta} - \frac{1}{u_2^\beta} \right) (v - u_2) \geq \langle \mu_2, v - u_2 \rangle. \tag{4.7}$$

Since $u_1, u_2 \in u_0 + H^1_0(\Omega)$ and $G_0(x, u_1 - u_0), G_0(x, u_2 - u_0) \in L^1(\Omega)$, we can take u_1 and u_2 as v in (4.6) and (4.7) respectively and summing we get

$$\begin{aligned}\int_\Omega |D(u_1 - u_2)|^2\, dx &\leq \int_\Omega f \left(\frac{1}{u_1^\beta} - \frac{1}{u_2^\beta} \right) (u_1 - u_2)\, dx + \langle \mu_1 - \mu_2, u_1 - u_2 \rangle \\ &\leq \langle \mu_1 - \mu_2, u_1 - u_2 \rangle \\ &\leq \|\mu_1 - \mu_2\|_{W^{-1,2}(\Omega)} \|u_1 - u_2\|_{H^1_0(\Omega)},\end{aligned}$$

and therefore

$$\|D(u_1 - u_2)\|_{L^2(\Omega)} \leq \|\mu_1 - \mu_2\|_{W^{-1,2}(\Omega)}.$$

□

Theorem 4.1.2. *Let $\mu \in W^{-1,2}(\Omega)$ and $u \in W^{1,2}_{loc}(\Omega)$. If u satisfies*

$$\begin{cases} -\Delta\, u = \dfrac{f}{u^\beta} + g(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

then u is solution of (4.1).
If $\mu \in L^1_{loc}(\Omega) \cap W^{1,2}(\Omega)$ then (4.1) *and* (4.8) *are equivalent.*

Proof. Let $u \in W^{1,2}_{loc}(\Omega)$ be solution of (4.8). Our goal is to prove that

$$\int_\Omega (Du, D(v - u))\, dx - \int_\Omega \frac{f}{u^\beta} (v - u)\, dx \geq \langle \mu, v - u \rangle$$

$\forall v \in u + (H^1_0(\Omega) \cap L^\infty_c(\Omega))$ with $v \geq 0$ a.e. in Ω.
In order to prove this, let $\varphi \in H^1_0(\Omega) \cap L^\infty_c(\Omega)$ and let $\varphi_n \in C^\infty_c(\Omega)$ with $\varphi_n \underset{n \to +\infty}{\longrightarrow} \varphi$ in $H^1_0(\Omega)$. Then

$$\int_\Omega (Du, D\varphi_n)\, dx - \int_\Omega \frac{f}{u^\beta} \varphi_n\, dx \geq \langle \mu, \varphi_n \rangle.$$

Consider now Ω' such that $\Omega' \supseteq \{\operatorname{supp}(\varphi_n)\}$ we get

- $\displaystyle \Big| \int_\Omega (Du, D(\varphi_n - \varphi))\, dx \Big| \leq \int_\Omega |D(u)|\, |D(\varphi_n - \varphi)|\, dx \leq \|u\|_{H^1_0(\Omega')} \|\varphi_n - \varphi\|_{H^1_0(\Omega')} \underset{n \to +\infty}{\longrightarrow} 0;$

- $\displaystyle \int_\Omega \frac{f}{u^\beta} \varphi_n\, dx \to \int_\Omega \frac{f}{u^\beta} \varphi\, dx$ by the Dominated convergence Theorem.

- $\langle \mu, \varphi_n \rangle \to \langle \mu, \varphi \rangle$ by definition.

Then (4.1) holds.
Assume now $\mu \in W^{-1,2}(\Omega) \cap L^1_{loc}(\Omega)$ and that u is the solution of (4.1). Let us denote by $\bar\mu$ the L^1-function representing $\mu \in W^{-1,2}(\Omega)$. Then for every $v \in C_c^\infty(\Omega)$ with $v \geq 0$, we have that $v' = v + u$ satisfies (4.1) and then

$$\int_\Omega (Du, Dv)\, dx - \int_\Omega \frac{f}{u^\beta}\, v\, dx \geq \int_\Omega \bar\mu\, v\, dx \tag{4.9}$$

Let now $v \in C_c^\infty(\Omega)$ with $v \leq 0$, $t > 0$ and $v_t = (u + tv)^+$. Since $|v_t - u| \leq t|v|$, we get from 4.1

$$\int_\Omega (Du, D(v_t - u))\, dx - \int_\Omega \frac{f}{u^\beta}\, (v_t - u)\, dx \geq \int_\Omega \bar\mu\, (v_t - u)\, dx.$$

Dividing by t we get

$$\begin{aligned}\int_\Omega \frac{f}{u^\beta} \frac{v_t - u}{t}\, dx + \int_\Omega \bar\mu \frac{v_t - u}{t}\, dx &\leq \int_\Omega \Big(Du, D\frac{v_t - u}{t}\Big)\, dx \\ &= \int_{\{u+v_t \leq 0\}} \Big(Du, D\frac{v_t - u}{t}\Big)\, dx + \int_{\{u+v_t > 0\}} \Big(Du, D\frac{v_t - u}{t}\Big)\, dx \\ &= -\frac{1}{t}\int_\Omega |Du|^2\, dx + \int_\Omega (Du, Dv)\, dx \leq \int_{\{u+v_t > 0\}} \Big(Du, D\frac{v_t - u}{t}\Big)\, dx,\end{aligned}$$

namely

$$\int_{\{u+v_t > 0\}} \Big(Du, D\frac{v_t - u}{t}\Big)\, dx \geq \int_\Omega \frac{f}{u^\beta} \frac{v_t - u}{t}\, dx + \int_\Omega \bar\mu \frac{v_t - u}{t}\, dx.$$

Letting $t \to 0^+$ we get

- $\{u + tv > 0\} \to \Omega$ since $u > 0$ in Ω;
- $\frac{v_t - u}{t} \to v$ pointwise. Moreover, since $v \in C_c^\infty(\Omega)$ and $\left|\frac{v_t - u}{t}\right| \leq |v|$, by the Dominated convergence Theorem we get
$$\int_\Omega \frac{f}{u^\beta} \frac{v_t - u}{t}\, dx \to \int_\Omega \frac{f}{u^\beta} v\, dx;$$
- $\int_\Omega \bar\mu \frac{v_t - u}{t}\, dx \to \int_\Omega \bar\mu\, v\, dx$ by the Dominated Convergence Theorem;

therefore

$$\int_\Omega (Du, Dv)\, dx - \int_\Omega \frac{f}{u^\beta}\, v\, dx \geq \int_\Omega \bar\mu\, v\, dx \tag{4.10}$$

also in this case.
¿From (4.9) and (4.10) it follows that, $\forall v \in C_c^\infty(\Omega)$ with $v \geq 0$ or $v \leq 0$ we have

$$\int_\Omega (Du, Dv)\, dx - \int_\Omega \frac{f}{u^\beta}\, v\, dx = \int_\Omega \bar\mu\, v\, dx. \tag{4.11}$$

Let now $v \in C_c^\infty(\Omega)$. The thesis follows immediately from (4.11) by considering v^+ and v^-. □

Corollary 4.1.3. *Assume that each $x \in \partial\Omega$ satisfies the Wiener criterion (for istance, Ω has Lipschitz boundary) and that $\mu \in L^\infty(\Omega)$. Let $u \in W^{1,2}_{loc}(\Omega)$ be the solution of (4.8). Then $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ for any $p < \infty$ and satisfies*

$$\begin{cases} -\Delta\, u = \dfrac{f}{u^\beta} + \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Moreover, we have

$$t_\mu u_0 \leq u \leq T_\mu u_0 \qquad \text{in } \overline{\Omega}$$

for some $0 < t_\mu \leq T_\mu < +\infty$.

Proof. For t_μ and T_μ sufficiently small and big respectively, $t_\mu\, u_0$ and $T_\mu\, u_0$ are a subsolution and a supersolution of (3.19). Therefore, from Lemma 3.2.5 we deduce that $t_\mu\, u_0 \le u \le T_\mu\, u_0$ a.e. in Ω. Since, according to Corollary 3.2.7, $u_0 \in C(\overline{\Omega})$, also $u \in C(\overline{\Omega})$. Moreover, for any compact set $K \subset \Omega$, we have $u \ge t_\mu\, u_0 > c > 0$ and therefore $u^{-\beta} \in L^\infty_{loc}(\Omega)$ which yields $u \in W^{2,p}(K)$ for any $p < \infty$.

□

4.2 C^1 perturbations

Let $u_0 \in L^\infty(\Omega) \cap C^2(\Omega)$ be the solution of (3.4) and let $\widetilde{F}_0 : H^1_0(\Omega) \to (-\infty, +\infty]$ be the lower, semicountinuous, convex functional defined as

$$\widetilde{F}_0(v) = \frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v)\,dx \qquad \forall v \in H^1_0(\Omega). \tag{4.13}$$

Moreover, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and assume that

$$\begin{cases} \exists a \in L^{\frac{2N}{N+2}}(\Omega),\ b \in \mathbb{R} \text{ s.t.} \\ |g(x,s)| \le a(x) + b|s|^{\frac{N+2}{N-2}} \text{ a.e. in } \Omega \text{ and } \forall s \in \mathbb{R}. \end{cases} \tag{4.14}$$

Define a new Carathéodory function $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$g_1(x,s) = g(x, u_0(x) + s), \tag{4.15}$$

which also satisfies (4.14), since $u_0 \in L^\infty(\Omega)$.
Let $G_1(x,s) = \int_0^s g_1(x,t)\,dt$ and $F : H^1_0(\Omega) \to (-\infty,+\infty]$ be defined as $F(u) = \widetilde{F}_0(u) + \gamma(u)$, where γ is the C^1 functional defined as $\gamma(u) = -\int_\Omega G_1(x,u)\,dx$.

Definition 4.2.1. *We say that $u \in H^1_0(\Omega)$ is a critical point of the functional F if $\widetilde{F}_0(u) < +\infty$ and*

$$\forall v \in H^1_0(\Omega) \text{ we have } \langle \gamma'(u), v - u\rangle + \widetilde{F}_0(v) - \widetilde{F}_0(u) \ge 0.$$

Theorem 4.2.2. *For every u, the following assertions are equivalent:*

(a) $u \in H^1_{loc}(\Omega) \cap L^{\frac{2N}{N-2}}(\Omega)$ and

$$\begin{cases} -\Delta\, u = \dfrac{f}{u^\beta} + g(x,u) \text{ in } D'(\Omega), \\ u > 0 \text{ a.e. in } \Omega, \text{ and } u^{-\beta} \in L^1_{loc}(\Omega), \\ u \le 0 \text{ on } \partial\Omega. \end{cases} \tag{4.16}$$

(b) $u \in u_0 + H^1_0(\Omega)$ and $u - u_0$ is a critical point of F.

Proof. (a) $\Longrightarrow$ (b). Let $\mu = g(x,u) = g_1(x - u_0)$ and $\varphi \in H^1_0(\Omega)$.
By (4.14) we get

$$\int_\Omega |g(x,u)|^{\frac{2N}{N+2}}\,dx \le \alpha \int_\Omega |a(x)|^{\frac{2N}{N+2}}\,dx + \beta \int_\Omega |u(x)|^{\frac{2N}{N-2}}\,dx,$$

then $\mu \in W^{-1,2}(\Omega) \cap L^1_{loc}(\Omega)$ by the hypothesis on $a(x)$ and $u(x)$. By Theorems 4.1.1 and 4.1.2 it follows that u is solution of (4.1) and therefore $u \in u_0 + H^1_0(\Omega)$ and minimizes F_μ. This means that $u - u_0$ is a critical point of F.
In fact, $\forall u' \in u_0 + H^1_0(\Omega)$, we have $F_\mu(u') \ge F_\mu(u)$. This means that $\forall u' = u_0 + v$ with $v \in H^1_0(\Omega)$ we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v)\,dx - \langle \mu, v\rangle \ge{}& \frac{1}{2}\int_\Omega |D(u-u_0)|^2\,dx \\ &+ \int_\Omega f\,G_0(x,u-u_0)\,dx - \langle \mu, u-u_0\rangle,\end{aligned}$$

namely

$$\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v)\,dx \geq \frac{1}{2}\int_\Omega |D(u-u_0)|^2\,dx + \int_\Omega f\,G_0(x,u-u_0)\,dx - \int_\Omega g(x,u)\,(u-u_0-v)\,dx,$$

which means

$$\widetilde{F}_0(v) \geq \widetilde{F}_0(u-u_0) - \langle \gamma'(u), u_0+v-u\rangle.$$

(b) $\Longrightarrow$ (a). Conversely, assume that (b) holds. Then $u-u_0$ is a critical point of F, namely u is the minimum of F_μ. By Theorem 4.1.1 it follows that u satisfies (4.1) and by Theorem 4.1.2 it follows that u satisfies (4.8) and therefore satisfies (4.16), since from (4.1) we get $u \in L^{\frac{2N}{N-2}}(\Omega)$. □

Corollary 4.2.3. *Assume that each $x \in \partial\Omega$ satisfies the Wiener criterion, that*

$$\begin{cases} \exists b \in \mathbb{R} \text{ s.t.} \\ |g(x,s)| \leq b(1+|s|^{\frac{N+2}{N-2}}) \text{ for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}, \end{cases}$$

and that $u \in H^1_{loc}(\Omega) \cap L^{\frac{2N}{N-2}}(\Omega)$ is a solution of (4.16).
Then $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ for any $p < +\infty$ and satisfies

$$\begin{cases} -\Delta\,u = \dfrac{f}{u^\beta} + g(x,u) \text{ a.e. in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.17}$$

Proof. The thesis follows by Corollary 4.1.3 by proving that $u \in L^\infty(\Omega)$ and, consequently, $g(x,u) \in L^\infty(\Omega)$.
Let $z = (u-1)^+$. Then $z \in H^1_0(\Omega)$ and is a subsolution of the equation

$$-\Delta\,v = \widehat{g}(x,v) + \widehat{\mu}$$

where $\widehat{g}(x,s) = g(x,s+1)\chi_{\{u>1\}}$ and $\widehat{\mu} = \dfrac{f}{u^\beta}\chi_{\{u>1\}} \in L^\infty(\Omega)$.
In fact, if we set $\widehat{\Omega} = \{u>1\}$, $\forall\varphi \in H^1_0(\Omega) \cap L^\infty_c(\Omega)$ with $\varphi \geq 0$ a.e. in Ω, we get

$$\begin{aligned} &\int_\Omega (Dz, D\varphi)\,dx - \int_\Omega \widehat{g}(x,z)\,\varphi\,dx - \langle\widehat{\mu},\varphi\rangle \\ &\quad= \int_{\widehat{\Omega}} (Du, D\varphi)\,dx - \int_{\widehat{\Omega}} g(u,u)\,\varphi\,dx - \int_{\widehat{\Omega}} \frac{f}{u^\beta}\,\varphi\,dx \\ &\quad= \int_{\widehat{\Omega}} (-\Delta\,u)\,\varphi\,dx + \int_{\widehat{\Omega}} \varphi\,\frac{\partial u}{\partial\eta}\,dx - \int_{\widehat{\Omega}} g(x,u)\,\varphi\,dx \\ &\quad- \int_{\widehat{\Omega}} \frac{f}{u^\beta}\,\varphi\,dx = \int_{\widehat{\Omega}} \varphi\,\frac{\partial u}{\partial\eta}\,dx \leq 0. \end{aligned}$$

Therefore, by [5] that $z \in L^\infty(\Omega)$, whence $u \in L^\infty(\Omega)$ and $g(x,u) \in L^\infty(\Omega)$, which yields the thesis. □

Chapter 5

A jumping problem

In this chapter we study a jumping problem in the spirit of [1]. Actually we consider the problem

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^\beta} + g(x,u) - t\,\varphi_1 & \text{a.e. } x \in \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

assuming that there exist constants $\gamma\,,\,\Gamma \in \mathbb{R}$ such that

$$0 < \gamma \leq \, f \, \leq \Gamma\,.$$

We prove existence of two solutions for large values of the parameter t, following the approach in [6] and exploiting the variational characterization developed in the thesis.

5.1 The problem

Suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the following:

(g_1) there exist a and b such that

$$|g(x,s)| \leq a(x) + b(x)\,|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R};$$

with $a \in L^{\frac{2N}{N+2}}(\Omega)$ and $b \in L^{\frac{N}{2}}(\Omega)$ if $n \geq 3$; a, $b \in L^p(\Omega)$ for some $p > 1$ if $n = 2$; a, $b \in L^1(\Omega)$ if $n = 1$.

(g_2) there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{s\to+\infty} \frac{g(x,s)}{s} = \alpha \qquad \text{for a.e. } x \in \Omega.$$

Denote by λ_1 the first eigenvalue of $-\Delta$ with homogeneous Dirichlet condition and by φ_1 an associated eigenfunction with $\varphi_1 > 0$ in Ω and let us consider the problem, in dependence on $t \in \mathbb{R}$,

$$\begin{cases} -\Delta\, u \;=\; \dfrac{f}{u^\beta} + g(x,u) - t\,\varphi_1 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

5.2 A nonsmooth version of the Mountain Pass Theorem

In order to study problem (5.1), we recall from [22] an extension of the Mountain pass theorem of Ambrosetti and Rabinowitz [2].
Let X be a real Banach space and $F : X \to (-\infty, +\infty]$ a function. Assume that $F = \Psi + \Phi$, where $\Psi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous and $\Psi : X \to \mathbb{R}$ is of class C^1.

Definition 5.2.1. *We say that $u \in X$ is a critical point for F if*

$$\Psi(v) \geq \Psi(u) - \langle \Phi'(u), v - u\rangle \qquad \forall v \in X.$$

Definition 5.2.2. *We say that F satisfies the Palais-Smale (PS) condition if, for every sequence $\{u_n\}$ in X and $\{\mu_n\}$ in X^* such that $\sup_h |F(u_n)| < +\infty$, $\|\mu_n\|_{W^{-1,2}(\Omega)} \to 0$ and*

$$\Psi(v) \geq \Psi(u_n) - \langle \Phi'(u_n), v - u_n\rangle + \langle \mu_n, v - u_n\rangle \quad \forall v \in X$$

the sequence $\{u_n\}$ admits a convergent subsequence in X.

Theorem 5.2.3. *Assume that F satisfies (PS) and that there exist $r > 0$ and $\sigma > F(0)$ such that*

$$F(u) \geq \sigma \qquad \forall u \in X \text{ with } \|u\| = r,$$

$$F(u_1) \leq F(0) \qquad \text{for some } u_1 \in X \text{ with } \|u_1\| > r.$$

Then there exists a critical point u for F with $F(u) \geq \sigma$.

For the proof of Theorem 5.2.3 we refer the reader to [22].

5.3 Jumping for a class of singular variational inequalities

For every $t \in \mathbb{R}$, let $F_t : H^1_0(\Omega) \to (-\infty, +\infty]$ be the functional defined as

$$F_t(w) = \widetilde{F}_0(w) + \gamma_t(w), \tag{5.2}$$

where

$$\widetilde{F}_0(w) = \frac{1}{2}\int_\Omega |Dw|^2\,dx + \int_\Omega f\,G_0(x,w)\,dx,$$

$$\gamma_t(w) = -\int_\Omega G_1(x,w)\,dx + t\int_\Omega \varphi_1\,w\,dx - \langle \mu, w\rangle,$$

$\mu \in W^{-1,2}(\Omega)$ and $\widetilde{F}_0(w)$ defined in (4.13).

Theorem 5.3.1. *Let $w \in H^1_0(\Omega)$ such that*

$$\widetilde{F}_0(v) \geq \widetilde{F}_0(w) - \langle \gamma_t'(w)\,,\, v - w\rangle \qquad \forall v \in H^1_0(\Omega).$$

Then we have

$$\begin{cases} u_0 + w > 0 \text{ a.e. in } \Omega \text{ and } (u_0 + w)^{-\beta} \in L^1_{loc}(\Omega), \\ \displaystyle\int_\Omega (Dw, D(v-w))\,dx \geq \int_\Omega f\,(\,(u_0+w)^{-\beta} - u_0^{-\beta})\,(v-w)\,dx \\ \qquad\qquad \displaystyle + \int_\Omega (g(x, u_0+w) - t\,\varphi_1)\,(v-w)\,dx + \langle \mu, v - w\rangle. \\ \qquad \forall v \in w + (H^1_0(\Omega)\cap L^\infty_c(\Omega)) \text{ with } v \geq -u_0 \text{ a.e. in } \Omega, \\ u_0 + w \leq 0 \text{ on } \partial\Omega; \end{cases} \tag{5.3}$$

Proof. The thesis follows by Theorem 4.1.1, by observing that $g(x, u_0+u) - t\,\varphi_1 + \mu \in W^{-1,2}(\Omega)$. □

Lemma 5.3.2. *Let $\{w_n\}$ be a sequence in $H_0^1(\Omega)$ and $\{\eta_n\}$ a sequence in $W^{-1,2}(\Omega)$. If η_n is strongly convergent in $W^{-1,2}(\Omega)$ and*

$$\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v)\,dx \geq \frac{1}{2}\int_\Omega |Dw_n|^2\,dx + \int_\Omega f\,G_0(x,w_n)\,dx \\ + \langle \eta_n, v - w_n\rangle \tag{5.4}$$

for every $v \in H_0^1(\Omega)$.
Then $\{w_n\}$ is strongly convergent in $H_0^1(\Omega)$.

Proof. Setting $v = 0$ in (5.4), we get

$$\langle \eta_n, w_n\rangle \geq \frac{1}{2}\int_\Omega |Dw_n|^2\,dx + \int_\Omega f\,G_0(x,w_n)\,dx,$$

whence

$$0 \leq \frac{1}{2}\int_\Omega |Dw_n|^2\,dx \leq \frac{1}{2}\int_\Omega |Dw_n|^2\,dx + \int_\Omega f\,G_0(x,w_n)\,dx \\ \leq \langle \eta_n, w_n\rangle \leq C\|w_n\|,$$

which jelds that $\{w_n\}$ is bounded in $H_0^1(\Omega)$, hence weakly convergent up to a subsequence, to some $w \in H_0^1(\Omega)$. Also $G_0(x,w) \in L^1(\Omega)$. In fact, since $\int_\Omega f\,G_0(x,w_n) < \infty$ is bounded, by Fatou's Lemma also $f\,G_0(x,w) \in L^1(\Omega)$ and, since $\gamma < f < \Gamma$, $G_0(x,w) \in L^1(\Omega)$.
If we set $v = w$ in (5.4) we get

$$\frac{1}{2}\int_\Omega |Dw|^2\,dx + \int_\Omega f\,G_0(x,w)\,dx \geq \frac{1}{2}\int_\Omega |Dw_n|^2\,dx + \int_\Omega f\,G_0(x,w_n)\,dx \\ + \langle \eta_n, w - w_n\rangle.$$

Note now that

- $\limsup_{n\to+\infty} \int_\Omega f\,G_0(x,w_n)\,dx \leq \int_\Omega f\,G_0(v,w)\,dx$ by Fatou's Lemma,
- $\langle \eta_n, w - w_n\rangle = \langle \eta_n - \eta, w - w_n\rangle + \langle \eta, w - w_n\rangle$, where $\eta \in W^{-1,2}(\Omega)$ is the limit of $\{\eta_n\}$. Since $w_n \rightharpoonup w$ in $H_0^1(\Omega)$, the second term converges to zero, while for the first term we have:
$$\langle \eta_n - \eta, w - w_n\rangle \leq \|\eta_n - \eta\|_{W^{-1,2}(\Omega)}\|w - w_n\|_{H_0^1(\Omega)} \to 0.$$

Letting $n \to +\infty$ we therefore obtain

$$\limsup_{n\to+\infty} \int_\Omega |Dw_n|^2\,dx \leq \int_\Omega |Dw|^2\,dx. \tag{5.5}$$

Since w_n weakly converges to w, it follows now that $\liminf_{n\to+\infty} \|w_n\|_{H_0^1(\Omega)} \geq \|w\|_{H_0^1(\Omega)}$ which yields, together with (5.5), $w_n \to w$ in $H_0^1(\Omega)$ up to subsequence. Actually, all the sequence w_n converges to w in $H_0^1(\Omega)$. In fact passing to the limit in (5.4) we get

$$\lim_{n\to+\infty} \Big(\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v)\,dx\Big) \\ \geq \lim_{n\to+\infty} \Big(\frac{1}{2}\int_\Omega |Dw_n|^2\,dx + \int_\Omega f\,G_0(x,w_n)\,dx \\ + \langle \eta_n, v - w_n\rangle\Big) \quad \forall v \in H_0^1(\Omega),$$

namely

$$\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v)\,dx$$
$$\geq \frac{1}{2}\int_\Omega |Dw|^2\,dx + \int_\Omega f\,G_0(x,w)\,dx$$
$$+\langle \eta, v-w\rangle \quad \forall v \in H_0^1(\Omega),$$

which means that w is the minimum of the strictly convex functional $\widetilde{F}_0 - \eta$.
Therefore, the whole sequence $\{w_n\}$ converges to w in $H_0^1(\Omega)$. In fact, suppose that $\{w_{n_j}\}$ is a subsequence of w_n which converges to $\widehat{w} \neq w$. We can replay the entire proof with w_{n_j} instead of w_n, obtaining that $\widehat{w}$ is another minimum of $\widetilde{F}_0 - \eta$, which contradicts the uniqueness of the minimum of the functional. □

Theorem 5.3.3. *Assume that $\alpha > \lambda_1$ and assume that there exist $\gamma\,,\,\Gamma \in \mathbb{R}$ such that*

$$0 < \gamma \leq f \leq \Gamma\,.$$

Then, for every $t \in \mathbb{R}$, the functional $F_t = \widetilde{F}_0 + \gamma_t$ satisfies (PS).

Proof. Let $\{w_n\}$ be a sequence in $H_0^1(\Omega)$ and $\{\eta_n\}$ a sequence in $W^{-1,2}(\Omega)$ with $\sup_n |F_t(w_n)| < +\infty$, $\eta_n \to 0$ and

$$\begin{aligned}\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega f\,G_0(x,v) \geq \frac{1}{2}\int_\Omega |Dw_n|^2\,dx + \int_\Omega f\,G_0(x,w_n)\,dx \\ + \int_\Omega (g_1(x,w_n) - t\,\varphi_1)\,(v-w_n)\,dx \\ + \langle \mu + \eta_n, v - w_n\rangle\end{aligned} \tag{5.6}$$

for all $v \in H_0^1(\Omega)$. Hence, by Theorem 5.3.1, it follows that w_n satisfies (5.3). In particular a translation argument gives

$$\begin{aligned}\int_\Omega (Dw_n, Dv)\,dx \geq \int_\Omega f\,(\,(u_0+w_n)^{-\beta} - u_0^{-\beta})\,v\,dx \\ + \int_\Omega (g_1(x,w_n) - t\,\varphi_1)\,v\,dx + \langle \mu + \eta_n, v\rangle.\end{aligned} \tag{5.7}$$

for every $v \in H_0^1(\Omega) \cap L_c^\infty(\Omega)$ with $v \geq -u_0 - u_n$ a.e. in Ω.
We claim that $\{w_n\}$ is bounded in $H_0^1(\Omega)$. Assume by contradiction that $\|w_n\|_{H_0^1(\Omega)} \to +\infty$ and set $\sigma_n = \|w_n\|$ and $z_n = \dfrac{w_n}{\sigma_n}$. Since $\|z_n\|_{H_0^1(\Omega)} = 1$, up to subsequence $\{z_n\}$ is convergent to some $z \in H_0^1(\Omega)$, with $z \geq 0$ a.e. in Ω. To prove that $z \geq 0$ a.e. in Ω just use the fact that $w_n \geq -u_0$.
By an approximation argument we can choose $v = -w_n$ in (5.7), obtaining

$$\begin{aligned}-\int_\Omega |Dw_n|^2\,dx \geq \int_\Omega f\,((u_0+w_n)^{-\beta} - u_0^{-\beta})\,(-w_n)\,dx \\ + \int_\Omega (g_1(x,w_n) - t\,\varphi_1)\,(-w_n)\,dx + \langle \mu + \eta_n, -w_n\rangle,\end{aligned}$$

hence

$$\begin{aligned}\int_\Omega |Dw_n|^2\,dx \leq \int_\Omega f((u_0+w_n)^{-\beta} - u_0^{-\beta})\,w_n\,dx \\ + \int_\Omega (g_1(x,w_n) - t\,\varphi_1)\,w_n\,dx + \langle \mu + \eta_n, w_n\rangle.\end{aligned}$$

Dividing by σ_n^2 we get

$$1 = \int_\Omega |Dz_n|^2\,dx \leq \int_\Omega \frac{g_1(x,\sigma_n\,z_n)}{\sigma_n}\,z_n\,dx - \frac{t}{\sigma_n}\int_\Omega \varphi_1\,z_n\,dx + \frac{1}{\sigma_n}\langle \mu + \eta_n, z_n\rangle.$$

Letting $n \to +\infty$ we get

$$\begin{cases} \int_\Omega |Dz|^2\,dx \le \alpha \int_\Omega z^2\,dx \\ 1 \le \alpha \int_\Omega z^2\,dx\,, \end{cases} \tag{5.8}$$

so that in particular, $z \neq 0$. To prove this we use the fact that

$$\Big|\frac{t}{\sigma_n}\int_\Omega \varphi_1\, z_n\,dx\Big| \le \frac{|t|}{\sigma_n}\|\varphi_1\|_{L^2(\Omega)}\|z_n\|_{L^2(\Omega)} \le C\,\frac{|t|}{\sigma_n}\|\varphi_1\|_{L^2(\Omega)}\|z_n\|_{H^1_0(\Omega)} \underset{n\to+\infty}{\longrightarrow} 0,$$

and

$$\Big|\frac{1}{\sigma_n}\langle \mu+\eta_n, z_n\rangle\Big| \le \frac{1}{\sigma_n}\|w+\eta_n\|_{W^{-1,2}(\Omega)}\|z_n\|_{H^1_0(\Omega)} \le \frac{1}{\sigma_n}(\|\mu\|_{W^{-1,2}(\Omega)} + \|\eta_n\|_{W^{-1,2}(\Omega)}) \underset{n\to+\infty}{\longrightarrow} 0\,.$$

Furthermore

$$\int_\Omega \frac{g_1(x,\sigma_n\, z_n)}{\sigma_n}\, z_n\,dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega \alpha\, z^2\,dx\,,$$

since by [7] we know that

$$\lim_{k\to+\infty}\frac{g_1(x,\sigma_n\,z_n)}{\sigma_n} = \alpha\, z \qquad \text{strongly in } W^{-1,2}(\Omega)\,.$$

On the other hand, if we take $v \in C^\infty_c(\Omega)$ with $v \ge 0$ in (5.7), we get

$$\int_\Omega (Dz_n, Dv)\,dx \ge \frac{1}{\sigma_n}\int_\Omega f\Big(\frac{1}{(u_0+w_n)^\beta} - \frac{1}{u_0^\beta}\Big)\,v\,dx + \int_\Omega \frac{g_1(x,\sigma_n\,z_n)}{\sigma_n}\,v\,dx - \frac{t}{\sigma_n}\int_\Omega \varphi_1\,v\,dx + \frac{1}{\sigma_n}\langle \mu+\eta_n, v\rangle.$$

Hence, taking into account the fact that $(u_0+w_n)^{-\beta} - u_0^{-\beta} > 0$ where $w_n < 0$,

$$\int_\Omega (Dz_n, Dv)\,dx \ge \frac{1}{\sigma_n}\int_{\{w_n\ge 0\}} f\Big(\frac{1}{(u_0+w_n)^\beta} - \frac{1}{u_0^\beta}\Big)\,v\,dx + \int_\Omega \frac{g_1(x,\sigma_n\,z_n)}{\sigma_n}\,v\,dx - \frac{t}{\sigma_n}\int_\Omega \varphi_1\,v\,dx + \frac{1}{\sigma_n}\langle \mu+\eta_n, v\rangle.$$

As v has compact support and u_0 is positive in the interior of Ω, we can pass to the limit as $n \to +\infty$, obtaining

$$\int_\Omega (Dz, Dv)\,dx \ge \alpha\int_\Omega z\,v\,dx \qquad \forall v \in C^\infty_c(\Omega) \text{ with } v \ge 0.$$

Combining this last relation with (5.8) and arguing by density we get

$$\int_\Omega (Dz, D(v-z))\,dx \ge \alpha\int_\Omega z\,(v-z)\,dx \quad \forall v \in H^1_0(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega.$$

It follows by [8] that z is a positive nontrivial solution of $-\Delta\, z = \alpha\, z$, but this contradicts the assumption that $\alpha > \lambda_1$.
It follows therefore that $\{w_n\}$ is bounded in $H^1_0(\Omega)$ and therefore weakly convergent, up to subsequence, to some $w \in H^1_0(\Omega)$ and therefore, by (g_1), $\{g_1(x, w_n)\}$ is strongly convergent to $g_1(x, w)$ in $W^{-1,2}(\Omega)$.
The thesis follows now by Lemma 5.3.2 and (5.6). □

Theorem 5.3.4. *Let $\alpha > \lambda_1$ and assume that there exist γ, $\Gamma \in \mathbb{R}$ such that*

$$0 < \gamma \le f \le \Gamma\,.$$

Then the following hold:

(a) there exist $r, \bar{t}, \sigma > 0$ such that $F_t(w) \geq \sigma\, t^2$ for every $t > \bar{t}$ and every $w \in H_0^1(\Omega)$ with $\|w\| = t\, r$;

(b) there exists $v \in H_0^1(\Omega) \cap L_c^\infty(\Omega)$ such that $v \geq 0$ a.e. in Ω and

$$\lim_{s \to +\infty} F_t(s\, v) = -\infty \qquad \forall t \in \mathbb{R}.$$

Remarkably $r, \bar{t}, \sigma > 0$ do not depend on f.

Proof. (a) Let us set

$$F_t^f(w) := F_t(w)$$

defined in (5.2), where we emphasize the dependence on f of the functional for future use. Let also set

$$\widetilde{F}_t(w) := \widetilde{F}_t^f(w) = \frac{F_t^f(w)}{t^2} \qquad \text{for} \quad t > 0$$

and let

$$\widetilde{F}_\infty(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Dw|^2\, dx - \dfrac{\alpha}{2}\displaystyle\int_\Omega w^2\, dx + \displaystyle\int_\Omega \varphi_1\, w\, dx & \text{if } w \geq 0 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases}$$

By [15] we know that there exists $r > 0$ such that

$$\widetilde{F}_\infty(w) > 0 \qquad \forall w \in H_0^1(\Omega) \ \text{ such that } \ 0 < \|w\|_{H_0^1(\Omega)} \leq r. \tag{5.9}$$

To prove the result assume by contradiction that there exist:

- a sequence $\{w_n\}$ in $H_0^1(\Omega)$ with $\|w_n\|_{H_0^1(\Omega)} = r$,
- a sequence $\{t_n\} \in \mathbb{R}$ with $t_n \to +\infty$,
- a sequence of functions f_n such that $0 < \gamma_n \leq f_n \leq \Gamma_n$,

such that

$$\limsup_{k \to +\infty} \widetilde{F}_{t_n}^{f_n}(t_n\, w_n) \leq 0.$$

Therefore

$$\begin{aligned} 0 \geq \limsup_{n \to +\infty} \widetilde{F}_{t_n}(t_n\, w_n) &= \limsup_{n \to +\infty} \Big(\frac{1}{2}\int_\Omega |Dw_n|^2\, dx + \int_\Omega \frac{f_n(x) G_0(x, t_n\, w_n)}{t_n^2}\, dx \\ &\quad - \int_\Omega \frac{G_1(x, t_n\, w_n)}{t_n^2}\, dx + \int_\Omega \varphi_1\, w_n\, dx - \frac{1}{t_n}\langle \mu, w_n \rangle \Big) \\ &\geq \limsup_{n \to +\infty} \Big(\frac{1}{2}\int_\Omega |Dw_n|^2\, dx - \int_\Omega \frac{G_1(x, t_n\, w_n)}{t_n^2}\, dx \\ &\quad + \int_\Omega \varphi_1\, w_n\, dx - \frac{1}{t_n}\langle \mu, w_n \rangle \Big). \end{aligned}$$

Since $\|w_n\|_{H_0^1(\Omega)} = r$, the sequence $\{w_n\}$ weakly converges, up to subsequence, to some $w \in H_0^1(\Omega)$ with $\|w\|_{H_0^1(\Omega)} \leq \limsup_{n \to +\infty} \|w_n\|_{H_0^1(\Omega)} = r$. Also by [7] we know that

$$\lim_{n \to +\infty} \frac{G_1(x, t_n\, w_n)}{t_n^2} = \frac{\alpha}{2}\, w^2 \qquad \text{strongly in } L^1(\Omega),$$

and it follows that $w \neq 0$ and

$$\frac{1}{2}\int_\Omega |Dw|^2\, dx - \frac{\alpha}{2}\int_\Omega w^2\, dx + \int_\Omega \varphi_1\, w\, dx \leq 0. \tag{5.10}$$

To prove this just use the fact that:

- $\displaystyle \frac{1}{2}\int_\Omega |Dw|^2\,dx \le \frac{1}{2}\liminf_{n\to+\infty}\int_\Omega |Dw_n|^2\,dx \le \frac{1}{2}\limsup_{n\to+\infty}\int_\Omega |Dw_n|^2\,dx,$

- $\displaystyle \int_\Omega \varphi_1\, w_n\,dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega \varphi_1\, w\,dx$ by the definition of weak convergence,

- $\displaystyle \frac{1}{t_n}\langle \mu, w_n\rangle \le \frac{1}{t_n}\|\mu\|_{W^{-1,2}(\Omega)}\, r \underset{n\to+\infty}{\longrightarrow} 0.$

In order to exploit the results in [15] ending the proof, we need now to prove that $w \ge 0$ almost everywhere in Ω.
Actually, since $\widetilde{F}_{t_n}(t_n w_n) < +\infty$, from the definition of $G_0(x,s)$ it follows that $u_0 + t_n\, w_n \ge 0$ (and $u_n \ge -\frac{u_0}{t_n}$). Then, since $t_n \to \infty$, it follows that $u \ge 0$ almost everywhere in Ω.
This fact, with (5.10), contradicts (5.9) and therefore (a) holds.
(b) Let $v \in H_0^1(\Omega)\cap L_c^\infty(\Omega)$, with $v \ge 0$. Then

$$\begin{aligned} F_t(s\,v) &= \frac{1}{2}\int_\Omega |D(s\,v)|^2\,dx + \int_\Omega f\,G_0(x,s\,v)\,dx - \int_\Omega G_1(x,s\,v)\,dx \\ &+ t\int_\Omega \varphi_1\, s\,v\,dx - \langle \mu, s\,v\rangle = s^2\Big(\frac{1}{2}\int_\Omega |Dv|^2\,dx + \frac{1}{s^2}\int_\Omega f\,G_0(x,s\,v)\,dx \\ &- \frac{1}{s^2}\int_\Omega G_1(x,s\,v)\,dx + \frac{t}{s}\int_\Omega \varphi_1\,v\,dx - \frac{1}{s}\langle \mu, v\rangle\Big). \end{aligned}$$

Letting $s \to +\infty$ and taking into account the fact that both f and u_0 are bounded away from zero in the support of v, we get

- $\displaystyle \frac{1}{s^2}\int_\Omega f\,G_0(x,s\,v) \underset{s\to+\infty}{\longrightarrow} 0,$

- $\displaystyle \frac{1}{s^2}\,G_1(x,s\,v) \underset{s\to+\infty}{\longrightarrow} \frac{\alpha}{2}v^2$ strongly in $L^1(\Omega)$,

- $\displaystyle \frac{t}{s}\Big|\int_\Omega \varphi_1\,v\,dx\Big| \le \frac{t}{s}\|\varphi_1\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \underset{s\to+\infty}{\longrightarrow} 0$

- $\displaystyle \frac{1}{s}\langle \mu, v\rangle \le \frac{1}{s}\|\mu\|_{W^{-1,2}(\Omega)}\|v\|_{H_0^1(\Omega)} \underset{s\to+\infty}{\longrightarrow} 0$

Since $-\Delta\,\varphi_1 = \lambda_1\varphi_1$ with $\alpha > \lambda_1$, it follows that $\displaystyle\int_\Omega |D\varphi_1|^2\,dx < \alpha\int_\Omega \varphi_1^2\,dx$. An approximation argument gives therefore that we can take $v \in H_0^1(\Omega)\cap L_c^\infty(\Omega)$ such that

$$\int_\Omega |Dv|^2\,dx < \alpha\int_\Omega v^2\,dx. \tag{5.11}$$

For such a v (as in (5.11)) we get

$$\lim_{s\to+\infty} F_t(s\,v) \sim \lim_{s\to+\infty} s^2\Big(\frac{1}{2}\int_\Omega |Dv|^2\,dx - \frac{\alpha}{2}\int_\Omega v^2\,dx\Big) = -\infty\,,$$

that concludes the proof. □

Theorem 5.3.5. *Let $\alpha > \lambda_1$ and assume that there exist $\gamma\,,\,\Gamma \in \mathbb{R}$ such that*

$$0 < \gamma \le f \le \Gamma\,.$$

Then there exists $\bar t \in \mathbb{R}$ such that, for every $t > \bar t$, there exist $w_1 \in H_0^1(\Omega)$ and $w_2 \in H_0^1(\Omega)$ that fulfils the variational inequality (5.3).

Proof. Let $\sigma\,,\,\bar{t}\,,\,r>0$ be as in assertion (a) of Theorem 5.3.4 and take $t>\bar{t}$. Since $F_t(0)=0$, from Theorems 5.3.3 and 5.3.4, it follows that F_t satisfies the assumptions of Theorem 5.2.3, namely there exists $w_1\in H^1_0(\Omega)$ critical point of F_t, with $F_t(w_1)>\sigma$.
On the other hand, since F_t is weakly lower semicontinuous and bounded from below, it admits a minimum $w_2\in\{w\in H^1_0(\Omega):\|w\|\le t\,r\}$, with $F_t(w_2)\le 0$, which means that $w_1\ne w_2$. From $\|w_2\|_{H^1_0(\Omega)}<tr$ it follows that w_2 is a free local minimum of f_t, hence another critical point of F_t.
The conclusion follows now by Theorem 5.3.1. □

Theorem 5.3.6. *Let $\alpha>\lambda_1$ and assume that there exist $\gamma\,,\,\Gamma\in\mathbb{R}$ such that*

$$0<\gamma\le\ f\ \le\Gamma\,.$$

Consider $u_1\,,\,u_2\in H^1_{loc}(\Omega)$ given by

$$u_1\,:=\,u_0+w_1\qquad\textit{and}\qquad u_2\,:=\,u_0+w_2$$

with w_1 and w_2 given by Theorem 5.3.5 and assume that $g(x,s)$ satisfies (g_1)-(g_2) with $a(x)\,,\,b(x)\in L^\infty(\Omega)$.
Then $u_1\,,\,u_2\in L^\infty(\Omega)$ and

$$-\Delta\,u=\frac{f}{u^\beta}+g(x,u)-t\,\varphi_1\qquad\qquad \textit{in } D'(\Omega). \tag{5.12}$$

Moreover, if Ω is smooth with smooth boundary and (g_1) is fulfilled with $a(x)\,,\,b(x)\in L^\infty(\Omega)$, then

$$u_1\,,\,u_2\in\,C(\overline{\Omega})\cap\Big(\bigcap_{1\le p<\infty}W^{2,p}_{loc}(\Omega)\Big)\,, \tag{5.13}$$

and the equation is fulfilled a.e. in the classical sense.

Proof. The proof is the same as in [7, Theorem 3.7], that goes back to [8, Corollary 4.2]. □

Remark 5.3.7. *Note that by construction the solutions u_1 and u_2 given by Theorem 5.3.6 are positive a.e. in Ω. Actually the regularity of the solutions in (5.13) and the fact that f is nonnegative, allow us to use the Strong Maximum Principle and get that actually u is strictly positive in the interior of the domain. Consequently, standard regularity estimates apply and one can deduce more regularity of the solutions in the interior of the domain, depending on the regularity of f.*

Chapter 6

A jumping problem for more general equations

In this chapter we study a more general jumping problem considering

$$\begin{cases} -\Delta\, u \,=\, \dfrac{f}{u^{\beta}} + g(x,u) - t\,\varphi_1 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

assuming that f is a nonnegative $L^1(\Omega)$-function if $\beta > 1$, while f is a nonnegative $L^m(\Omega)$-function with $m = \frac{2N}{N+2+\beta(N-2)} = \left(\frac{2^*}{1-\beta}\right)'$ if $0 < \beta \leq 1$. The main idea is to consider a truncated problem and then to obtain uniform bounds in order to pass to the limit.

6.1 A-priori estimates for the truncated problem

Let us consider $f \geq 0$ a.e. in Ω such that

$$\begin{cases} f \in L^1(\Omega) & \text{if } \beta > 1; \\ f \in L^m(\Omega) \text{ with } m = \frac{2N}{N+2+\beta(N-2)} & \text{if } 0 < \beta \leq 1. \end{cases} \tag{6.2}$$

and define the truncated function of f as follows

$$f_n \,:=\, \max\left\{\frac{1}{n}\,,\, \min\{n\,,\, f\}\right\}.$$

By Theorem 5.3.6 we have that there exists $\bar{t} \in \mathbb{R}$ (given by Theorem 5.3.4) not depending on $n \in \mathbb{N}$ such that, for any $t \geq \bar{t}$, there exist two solutions of the truncated problem involving f_n. Let u_n one of these two solutions, so that

$$\begin{cases} -\Delta\, u_n \,=\, \dfrac{f_n}{{u_n}^{\beta}} \,+\, g(x,u_n) - t\varphi_1 & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{6.3}$$

where $\beta > 0$ and $u_n \in C(\overline{\Omega}) \cap C^2(\Omega)$. Let us recall that generally u_n does not belong to $H_0^1(\Omega)$, and can be decomposed as

$$u_n \,=\, u_{0n} \,+\, w_n \tag{6.4}$$

where $w_n \in H_0^1(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ and $u_{0n} \in H^1_{loc}(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ is the unique solution to the problem:

$$\begin{cases} -\Delta\, u_{0n} = \dfrac{f_n}{{u_{0n}}^{\beta}} & \text{in } \Omega, \\ u_{0n} > 0 & \text{in } \Omega, \\ u_{0n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

The equation has to be understood in the following sense:

$$\int_\Omega (Du_{0n}, D\varphi)\, dx = \int_\Omega \frac{f_n\, \varphi}{{u_{0n}}^{\beta}}\, dx \qquad \forall \varphi \in C^1_c(\Omega). \tag{6.6}$$

Actually the solution is fulfilled in the classical sense in the interior of Ω by standard regularity results, since u_{0n} is strictly positive in the interior of the domain.

Lemma 6.1.1. *The sequence w_n defined in* (7.5) *is uniformly bounded in $H_0^1(\Omega)$. More precisely there exists a constant C, not depending on n, such that*

$$\|w_n\|_{H_0^1(\Omega)} \le C\,.$$

Proof. Let us fix $n \in \mathbb{N}$ consider $\phi_j \in C_c^\infty(\Omega)$ such that

$$\phi_j \underset{j\to\infty}{\longrightarrow} w_n \qquad \text{in } H_0^1(\Omega)\,.$$

It is not restrictive to assume that ϕ_j is positive where w_n is positive, and ϕ_j is negative where w_n is negative. To do this just recall that $w_n \in H_0^1(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ and consider

$$w_n = w_n^+ - w_n^-\,.$$

Then let, for $t \ge 0$, $\mathcal{G}_\varepsilon(t) := t \cdot \chi_{\{t \ge 2\varepsilon\}} + (2t - 2\varepsilon) \cdot \chi_{\{\varepsilon \le t \le 2\varepsilon\}}$, and approximate in $H_0^1(\Omega)$ $\mathcal{G}_\varepsilon(w_n^+)$ and $\mathcal{G}_\varepsilon(w_n^-)$. The details are left to the reader.

Plugging ϕ_j as test function into (6.3), and exploiting the splitting in (7.5) and equation (7.7), we get

$$\begin{aligned} \int_\Omega (Dw_n\, D\phi_j)\, dx = &\int_\Omega f_n \Big(\frac{1}{(u_{0n} + w_n)^\beta} - \frac{1}{(u_{0n})^\beta} \Big) \phi_j\, dx \\ &+ \int_\Omega (g(x, u_n)\phi_j - t\varphi_1\phi_j)\, dx \le \int_\Omega (g(x, u_n)\phi_j - t\varphi_1\phi_j)\, dx\,. \end{aligned}$$

We now let $j \to +\infty$ and get

$$\int_\Omega |Dw_n|^2\, dx \le \int_\Omega (g(x, u_n)w_n - t\varphi_1 w_n)\, dx\,.$$

Assume now by contradiction that, up to subsequences, $\|w_n\|_{H_0^1(\Omega)}$ tends to infinity, and set

$$\sigma_n := \|w_n\|_{H_0^1(\Omega)} \qquad \text{and} \qquad z_n := \frac{w_n}{\sigma_n}\,.$$

Since $\|z_n\|_{H_0^1(\Omega)} = 1$. it follows that z_n weakly converges in $H_0^1(\Omega)$ to some $z \in H_0^1(\Omega)$. It is easy to see that $z \ge 0$ a.e. in Ω since $u_{0n} + w_n \ge 0$.
On the other hand we have

$$1 = \int_\Omega |Dz_n|^2\, dx \le \int_\Omega \frac{g_1(x, \sigma_n z_n)}{\sigma_n}\, z_n\, dx - \frac{t}{\sigma_n} \int_\Omega \varphi_1 z_n\, dx$$

g_1 defined in (4.15). Arguing as in Theorem 5.3.3 we get

$$1 = \int_\Omega |Dz|^2\, dx \le \alpha \int_\Omega z^2\, dx\,,$$

so that in particular $z \neq 0$.
Take now $v \in C_c^\infty(\Omega)$ with $v \ge 0$ in Ω, and put it as test function in (6.3), obtaining

$$\begin{aligned}\int_\Omega (Dz_n\, Dv)\, dx &= \frac{1}{\sigma_n}\int_\Omega f_n\Big(\frac{1}{(u_{0n}+w_n)^\beta} - \frac{1}{(u_{0n})^\beta}\Big) v\, dx \\ &+ \int_\Omega \Big(\frac{g_1(x,\sigma_n z_n)}{\sigma_n}\, v - \frac{t}{\sigma_n}\varphi_1 v\Big)\, dx \\ &\ge \frac{1}{\sigma_n}\int_{\{w_n\ge 0\}} f_n\Big(\frac{1}{(u_{0n}+w_n)^\beta} - \frac{1}{(u_{0n})^\beta}\Big) v\, dx \\ &+ \int_\Omega \left(\frac{g_1(x,\sigma_n z_n)}{\sigma_n}\, v - \frac{t}{\sigma_n}\varphi_1 v\right)\, dx\end{aligned}$$

Note now that, since u_{0n} is bounded away from zero in the interior of Ω, we get

$$\Big|\frac{1}{\sigma_n}\int_{\{w_n\ge 0\}} f_n\left(\frac{1}{(u_{0n}+w_n)^\beta} - \frac{1}{(u_{0n})^\beta}\right) v\, dx\Big| \le \frac{C}{\sigma_n}\|f\|_{L^1(\Omega)} \underset{n\to+\infty}{\longrightarrow} 0\,.$$

Therefore, arguing as above, we get

$$\int_\Omega (Dz\,,\, Dv)\, dx \ge \alpha \int_\Omega z\, v\, dx\,,$$

and consequently

$$\int_\Omega (Dz\,,\, D(v-z))\, dx \ge \alpha \int_\Omega z\,(v-z)\, dx\,.$$

It follows that z is a nonnegative solution to $-\Delta z = \alpha z$, which is not possible since we assumed $\alpha > \lambda_1$. Therefore the result is proved.

□

In what follows we will consider the further assumption:

Given any compact set $\mathcal{C} \subset \Omega$ there exists a constant $\theta = \theta(\mathcal{C}) > 0$ such that

$$f \ge \theta \qquad \text{a.e. in} \quad \mathcal{C}\,. \tag{6.7}$$

Lemma 6.1.2. *Let $u_n \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (6.3) and assume that f fulfills (6.7). Then, given any compact set $\mathcal{C} \subset \Omega$, there exists a constant $\mu = \mu(\mathcal{C}) > 0$ such that*

$$u_n \ge \mu \qquad \text{a.e. in} \quad \mathcal{C}\,,$$

uniformly in $n \in \mathbb{N}$.

Proof. Fix a compact set $\mathcal{C}' \subset \Omega$ and consider $v \in C^2(\mathcal{C}')$ positive solution to

$$\begin{cases} -\Delta\, v = 1 & \text{for } x \in \mathcal{C}', \\ v > 0 & \text{for } x \in \mathcal{C}', \\ v = 0 & \text{for } x \in \partial\mathcal{C}'\,, \end{cases}$$

and set

$$v_\lambda := \lambda\, v\,,$$

so that $-\Delta v_\lambda = \lambda$ in $\mathcal{C}'$.
We shall prove the thesis, showing that, for λ sufficiently small, we have that $u_n \ge v_\lambda$ in $\mathcal{C}'$. To prove this let us set

$$W_\lambda := (v_\lambda - u_n)^+$$

consider $\phi_j \in C_c^\infty(\Omega)$ such that

$$\phi_j \underset{j\to\infty}{\longrightarrow} W_\lambda \qquad \text{in } H_0^1(\Omega)\,.$$

It is not restrictive to assume that ϕ_j is positive and that supp $\phi_j \subseteq$ supp W_λ.

By (6.3) we have

$$\int_{\text{supp }\phi_j} (Du_n\,,\, D\phi_j)\,dx = \int_{\text{supp }\phi_j} \Big(\frac{f_n}{u_n^\beta} + g(x,u_n) - t\varphi_1\Big)\phi_j\,dx$$

Recall now that supp $\phi_j \subseteq$ supp $W_\lambda \subseteq \mathcal{C}'$, and that

$$u_n \le v_\lambda \le C\,\lambda \qquad \text{in the support of} \quad W_\lambda\,,$$

for some constant $C \in \mathbb{R}$.
Consequently, since $g(x,s)$ is bounded from above w.r.t. the variable s, and $\varphi_1 \in L^\infty(\Omega)$ and t is fixed, we can take λ small such that

$$\Big(\frac{f_n}{u_n^\beta} + g(x,u_n) - t\varphi_1\Big) \ge \lambda \qquad \text{in the support of} \quad W_\lambda\,,$$

and therefore we have

$$\int_{\text{supp }W_\lambda} (Du_n\,,\, D\phi_j)\,dx \ge \int_{\text{supp }W_\lambda} \lambda\phi_j\,dx = \int_{\text{supp }W_\lambda} (Dv_\lambda\,,\, D\phi_j)\,dx$$

that is

$$\int_{\text{supp }W_\lambda} (DW_\lambda\,,\, D\phi_j)\,dx \le 0\,.$$

Letting now $j \to \infty$, we get that $\int_{\text{supp }W_\lambda} |DW_\lambda|^2\,dx \le 0$ which shows that $W_\lambda = 0$ and therefore that

$$u_n \ge v_\lambda \qquad \text{in} \quad \mathcal{C}'\,.$$

The thesis follows now by the arbitrariness of the compact set $\mathcal{C}'$ (to get a bound on a compact set, just consider a bigger one).

□

6.2 Ending the proof of Theorem 1.0.2

We prove in this section Theorem 1.0.2. For the readers convenience we recall the statement.

Theorem 6.2.1. *Let $\beta > 0$ and assume that $g(x,s)$ satisfies assumptions (g_1)-(g_2) with $\alpha > \lambda_1$ and $a(x), b(x) \in L^\infty(\Omega)$. Let f satisfying* (6.2) *and* (6.7) [1]. *Then there exists $\bar{t} \in \mathbb{R}$ such that, for any $t \ge \bar{t}$ there exist $u_1\,,\, u_2 \in H^1_{loc}(\Omega)$ solutions to* (1.6). *Moreover*

$$u_i := u_0 + w_i \qquad \textit{for} \quad i = 1, 2,$$

where $u_0 \in H^1_{loc}(\Omega)$ is defined in (1.2) *and $w_i \in H_0^1(\Omega)$. Moreover u_i are strictly bounded away from zero in the interior of Ω.*

[1]Recall that f fulfills (6.7) if, given any compact set $\mathcal{C} \subset \Omega$ there exists a constant $\theta = \theta(\mathcal{C}) > 0$ such that $f \ge \theta$ a.e. in $\mathcal{C}$.

Proof. We showed in Lemma 6.1.1 that the sequence $\{w_n\} \in H_0^1(\Omega)$ defined in (7.5) is uniformly bounded in $H_0^1(\Omega)$, so that we can assume that

$$w_n \rightharpoonup w \qquad \text{in} \quad H_0^1(\Omega)$$

By [4], see also Section 3.1 in Chapter 3, taking into account (6.2), we also have that

- $\{u_{0n}\}$ is bounded in $H^1_{loc}(\Omega)$ for any $\beta > 0$,
- $\{u_{0n}\}$ is bounded in $H_0^1(\Omega)$ if $0 < \beta \le 1$,
- $\{u_{0n}^{\frac{\beta+1}{2}}\}$ is bounded in $H_0^1(\Omega)$.

Therefore we can assume that there exists u_0 such that

- u_{0n} weakly (and a.e.) converges to u_0 in $H^1_{loc}(\Omega)$,
- $u_0 \in H_0^1(\Omega)$ if $0 < \beta \le 1$, while $u_0^{\frac{\beta+1}{2}} \in H_0^1(\Omega)$ if $\beta > 1$,
- u_0 is strictly bounded away from zero in the interior of Ω.

Note now that, for any compact set $\mathcal{C} \subset \Omega$, we have

$$u_n \rightharpoonup u := u_0 + w \qquad \text{in} \quad H^1(\mathcal{C})\,,$$

and therefore $u_n \to u$ in $L^q(\mathcal{C})$ for $q < \frac{2N}{N-2}$ and a.e. in Ω.

Given $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega (Du_n\,,\, D\varphi)\,dx = \int_\Omega \frac{f_n}{u_n^\beta}\varphi\,dx + \int_\Omega (g(x,u_n)\varphi - t\varphi_1\varphi)\,dx\,.$$

By the definition of weak convergence, we have

$$\int_\Omega (Du_n\,,\, D\varphi)\,dx \longrightarrow \int_\Omega (Du\,,\, D\varphi)\,dx\,.$$

Note also that, on the support of φ, u_n is dominated by a function $h \in L^q$ (for $q < \frac{2N}{N-2}$). Therefore, recalling the assumption on $g(x\,,\,s)$, we can exploit the Dominated Convergence Theorem and get

$$\int_\Omega g(x,u_n)\varphi\,dx \to \int_\Omega g(x,u)\varphi\,dx\,.$$

Moreover

$$\int_\Omega \frac{f_n}{u_n^\beta}\varphi\,dx \to \int_\Omega \frac{f}{u^\beta}\varphi\,dx\,,$$

by the Dominated Convergence Theorem that applies thanks to Lemma 6.1.2. Therefore, passing to the limit for $n \to +\infty$, we get that $u \in H^1_{loc}(\Omega)$ is a weak solution to (5.1). Finally we note that u is strictly bounded away from zero on compact set of Ω, by Lemma 6.1.2.
To end the proof and get the existence of two distinct solutions, consider

$$u_n^1 = u_{0n} + w_n^1 \quad \text{and} \quad u_n^2 = u_{0n} + w_n^2,$$

the two different sequences of solutions given by Theorem 5.3.6. Arguing as above for the two sequences $\{u_n^1\}$ and $\{u_n^2\}$ we get that

$$u_n^1 \to u^1 \quad \text{and} \quad u_n^2 \to u^2,$$

where u^1 and u^2 are solutions to (6.1). In particular

$$w_n^1 \to w^1 \quad \text{and} \quad w_n^2 \to w^2.$$

It is now easy to deduce that actually $u^1 \neq u^2$, that is $w^1 \neq w^2$. To show this we argue by contradiction and assume that $w^1 = w^2$.
Recall that by Theorem 5.3.4 we have that

$$F_t(w_n^1) = \widetilde{f}_0(w_n^1) + \gamma_t(w_n^1) \geq \sigma > 0 \geq F_t(w_n^2) = \widetilde{f}_0(w_n^2) + \gamma_t(w_n^2)\,, \tag{6.8}$$

where σ does not depend on n. On the other hand, since w_n^1, w_n^2 are critical points (and are bounded sequences of $H_0^1(\Omega)$), we get

$$\left|\widetilde{f}_0(w_n^2) - \widetilde{f}_0(w_n^1)\right| \leq \left|\langle\gamma_t'(w_n^2)\,,\, w_n^1 - w_n^2\rangle\right| + \left|\langle\gamma_t'(w_n^1)\,,\, w_n^2 - w_n^1\rangle\right|,$$

and it is easy to see that

$$\left|\langle\gamma_t'(w_n^2)\,,\, w_n^1 - w_n^2\rangle\right| + \left|\langle\gamma_t'(w_n^1)\,,\, w_n^2 - w_n^1\rangle\right| \underset{n\to\infty}{\to} 0\,,$$

so that $\left|\widetilde{f}_0(w_n^2) - \widetilde{f}_0(w_n^1)\right| \underset{n\to\infty}{\to} 0$. It is also easy to check that, if we assume that $w^1 = w^2$, then

$$\left|\gamma_t(w_n^1) - \gamma_t(w_n^2)\right| \underset{n\to\infty}{\to} 0.$$

This is a contradiction because of (6.8) and therefore $w^1 \neq w^2$, so that $u^1 \neq u^2$. □

Chapter 7

Symmetry results via the moving plane method

In this chapter we consider the problem

$$\begin{cases} -\Delta\, u = \dfrac{1}{u^{\beta}} + g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\beta > 0$, Ω is a bounded smooth domain and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$. We prove a symmetry (and monotonicity) result exploiting the Moving Plane Method of James Serrin.
The adaptation of this procedure is not straightforward since the problem is singular and solutions are not in $H_0^1(\Omega)$. We start with a description of the moving plane procedure in the standard (not singular) case.

7.1 The moving plane method

The moving plane method goes back to James Serrin [20] in a paper regarding overdetermined boundary value problems. Here below we recall the result obtained in the celebrated paper [13], where the moving plane method is exploited to obtain symmetry and monotonicity results of the solutions to semilinear elliptic problems.

Theorem 7.1.1. *Let Ω be the open ball of radius R in $\mathbb{R}^N$, $u \in C^2(\overline{\Omega})$ be a solution to*

$$\begin{cases} -\Delta\, u = g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with g of class C^1. Then u is radially symmetric and the radial derivative satisfies

$$\frac{\partial u}{\partial r}(x) < 0, \qquad \text{for } 0 < r < R.$$

A more general result is the following (see [13]):

Theorem 7.1.2. *Let Ω be a domain of $\mathbb{R}^N$ of class C^2 which is convex in the x_1 direction and symmetric w.r.t. the hyperplane $x_1 = 0$. Let $u \in C^2(\Omega)$ be a solution to*

$$\begin{cases} -\Delta\, u = g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with g of class C^1.
Then u is symmetric w.r.t. x_1 *and*

$$\frac{\partial u}{\partial x_1}(x) < 0, \qquad \text{for } x_1 > 0.$$

We provide here the proof of an improved result from [3] where the authors present a new approach to the problem: it is important to remark that no assumptions on the smoothness of the domain are needed and the function only has to be locally Lipschitz continuous. In particular we have the following

Theorem 7.1.3. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *which is convex in the* x_1 *direction and symmetric w.r.t. the hyperplane* $x_1 = 0$. *Let* $u \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ *be a solution to*

$$\begin{cases} -\Delta\, u \;=\; g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where g is locally Lipschitz continuous.
Then u is symmetric w.r.t. x_1 *and*

$$\frac{\partial u}{\partial x_1}(x) < 0, \qquad \text{for } x_1 > 0.$$

This new approach enables to avoid careful study of the boundary, by using an improved version of the maximum principle in narrow domains.

7.1.1 Maximum principle in narrow domains

Consider a second order elliptic operator in a bounded domain Ω in $\mathbb{R}^N$:

$$L = M + c = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

with L^∞ coefficients and which is uniformly elliptic, namely:

$$c_0|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le C_0|\xi|^2, \qquad c_0, C_0 > 0, \forall \xi \in \mathbb{R}^N$$

and satisfying

$$\sqrt{\sum b_i^2} < \tilde{b}, \qquad |c| < \tilde{b}$$

for some $\tilde{b} \in \mathbb{R}$. The functions on which L will be applied will always be assumed to belong to $W^{2,n}_{loc}(\Omega)$.

Definition 7.1.4. *We say that the maximum principle holds for L in* Ω *if*

$$\begin{cases} Lz \ge 0 & \text{in } \Omega \\ \limsup_{x \to \partial\Omega} z(x) \le 0 \end{cases} \qquad \Longrightarrow z \le 0 \text{ in } \Omega. \tag{7.1}$$

It is well known in the literature (see [14], for example) that the following conditions are sufficient for the validity of the Maximum Principle:

- $c \le 0$;
- there exists a continuous positive function $h \in W^{2,\infty}(\Omega) \cap C(\overline{\Omega})$ satisfying $L\,h \le 0$. Therefore, z/h satisfies a new elliptic inequality with a new coefficient $c \le 0$;
- Ω lies in a narrow strip $\alpha < x < \alpha + \varepsilon$, with ε small.
  In this case, it is possible to construct a function $g(x_1)$ satisfying the conditions of the previous point.

In the general case we have the following proposition:

Proposition 7.1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\operatorname{diam}\Omega \le d$. Then there exists $\delta = \delta(N, d, c_0, \tilde{b}) > 0$ such that the maximum principle holds for L in Ω provided $\mathcal{L}(\Omega) < \delta$.*

Proof. In the proof only $c \le \tilde{b}$ is required, instead of $|c| \le \tilde{b}$.
The proof of this proposition is based on the theorem of Alexandroff, Bakelman and Pucci, which is used in the following form:
If $c \le 0$ and z satisfies $Lz \ge g$ and $\limsup_{x \to \partial\Omega} z(x) \le 0$, then

$$\sup_{\Omega} z \le \overline{C}\, \|g\|_{L^N(\Omega)}, \tag{7.2}$$

with $\overline{C}$ depending only on $N, c_0, \tilde{b}$ and d. For a proof of this result in a more general form, we refer to [14].
Let us consider z satisfying (7.1), that we can write as

$$(M - c^-)\, z \ge -c^+\, z^+$$

where $c = c^+ - c^-$. We can apply (7.2), which yields

$$\sup_{\Omega} z^+ \le \overline{C} \|c^+ \, z^+\|_{L^N(\Omega)} \le \overline{C} \left(\int_{\Omega} (b \sup_{\Omega} z)^N \, dx \right)^{\frac{1}{N}} \le \overline{C}\, \tilde{b} \sup_{\Omega} z^+ \, \left(\mathcal{L}(\Omega)\right)^{\frac{1}{N}}$$

and therefore $\sup_\Omega z^+ \le 0$ if $\overline{C}\, \tilde{b}\, \delta^{\frac{1}{N}} < 1$, namely the thesis holds. □

7.1.2 Proof of Theorem 7.1.3

Proposition 7.1.1 allow to prove Theorem 7.1.3.
Let us denote a generic point of $\mathbb{R}^N$ by (x_1, y) and use the short hand notation $u_{x_1} := \frac{\partial u}{\partial x_1}$. Our goal is to prove that u is symmetric w.r.t. x_1 and

$$u_{x_1}(x) < 0, \qquad \text{for } x_1 > 0.$$

If we denote by $u_\lambda(x_1, y) = u(2\lambda - x_1, y)$, since we may replace x_1 by $-x_1$ and use $u(-x_1, y)$, the symmetry follows by proving that

$$u(x_1, y) \le u_\lambda(x_1, y),$$

for every $\lambda < 0$. In what follows, let

$$a = \inf_{x \in \Omega} x_1$$

and, for $a < \lambda < 0$, let T_λ be the hyperplane $x_1 = 0$,

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\}$$

and

$$v(x, \lambda) = u_\lambda(x) - u(x).$$

Since g is Lipschitz, it follows that v satisfies the following problem

$$\begin{cases} -\Delta\, v \;=\; c(x,\lambda)\, v & \text{in } \Omega_\lambda, \\ u \ge 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

and v not identically zero on $\partial\Omega_\lambda$, for some bounded function $c(x, \lambda)$, with $|c| < \tilde{b}$ and $\tilde{b}$ not depending on $x \in \Omega$ and λ. For $\lambda - a > 0$ sufficiently small, the domain Ω_λ is narrow in the x_1 direction and it follows from Proposition 7.1.1 that $v(x, \lambda) \ge 0$. Actually

$$v(x, \lambda) > 0 \tag{7.3}$$

in Ω_λ by the Strong Maximum Principle (see [14]). Let (a, λ_0) be the largest open interval of values of λ such that (7.3) holds.
We want to prove that $\lambda_0 = 0$ and in order to do this we suppose that $\lambda_0 < 0$ and argue by contradiction. By continuity, $v(x, \lambda_0) \ge 0$ in Ω_{λ_0} and, taking into account the boundary condition on Ω_{λ_0}, it follows by the Strong Maximum Principle that $v(x, \lambda_0) > 0$ in Ω_{λ_0}.
Let now $\delta > 0$ and $\mathcal{K}$ be a compact set in Ω_{λ_0} such that $\mathcal{L}(\Omega_{\lambda_0} \setminus \mathcal{K}) < \frac{\delta}{2}$. By compactness, it is possible to find $\sigma > 0$ such that $v(x, \lambda_0) > 2\,\sigma > 0$ in $\mathcal{K}$.
Let now $\bar{\varepsilon} > 0$ sufficiently small so that for any $0 < \varepsilon \le \bar{\varepsilon}$ we have

- $\mathcal{L}(\Omega_{\lambda_0+\bar{\varepsilon}}) < \delta$;
- $v(x, \lambda_0 + \varepsilon) > \sigma > 0$ in $\mathcal{K}$.

In $\Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}$, if we set $v = v(x, \lambda_0 + \varepsilon)$ and $c = c(x, \lambda_0 + \varepsilon)$, it follows that v satisfies

$$\begin{cases} -\Delta\, v \,=\, c(x,\lambda)\, v & \text{in } \Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}, \\ v \geq 0 & \text{on } \partial(\Omega_{\lambda+\varepsilon} \setminus \mathcal{K}), \end{cases}$$

Therefore, by Proposition 7.1.1, it follows that $v \geq 0$ in $\Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}$. Actually, by the Strong Maximum Principle (see [14]), it follows that $v > 0$ in $\Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}$, namely $v > 0$ in $\Omega_{\lambda_0+\varepsilon}$, which contradicts the maximality of the interval (a, λ_0).
Finally, in order to show that $u_{x_1} > 0$ if $x_1 > 0$, let only observe that since $v(x, \lambda) > 0$ in Ω_λ and $v(x, \lambda) = 0$ on the hyperplane T_λ, it follows by the Hopf Lemma applied at the hyperplane $T_\lambda = 0$, that $v_{x_1} < 0$ on T_λ, namely $w_{x_1} = -2\, u_{x_1} < 0$, which concludes the proof.

7.2 The problem

We study symmetry and monotonicity properties of the solutions to the problem

$$\begin{cases} -\Delta\, u \,=\, \dfrac{1}{u^\beta} \,+\, g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.4}$$

where $\beta > 0$, Ω is a bounded smooth domain and $u \in\, C(\overline{\Omega}) \cap C^2(\Omega)$.
Our main results will be proved under the following assumption:

(H_p) $g(\cdot)$ is locally Lipschitz continuous, positive ($g(s) > 0$ for $s > 0$ and $g(0) \geq 0$) and non-decreasing.

The proof of our symmetry result is based on the well known *Moving Plane Method* ([13, 20]). Actually our proof is more similar to the one in [3], that we recalled in Section 7.1 and is based on the weak comparison principle in small domains that, we prove in our section in Section 7.5. Because of the singular nature of our problem, we have to take care of the fact that generally u does not belong to $H^1_0(\Omega)$, and $\frac{1}{s^\beta} \,+\, g(s)$ is not Locally Lipschitz continuous at zero.

In fact, by the variational characterization (see Chapter 4), the solution u can be decomposed as

$$u \,=\, u_0 \,+\, w \tag{7.5}$$

where $w \in H^1_0(\Omega)$ and $u_0 \in\, C(\overline{\Omega}) \cap C^2(\Omega)$ is the unique solution to the problem:

$$\begin{cases} -\Delta\, u_0 \,=\, \dfrac{1}{{u_0}^\beta} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.6}$$

The construction of the solution u_0 has been discussed in Chapter 3. Let us only recall that the equation $-\Delta\, u_0 \,=\, \frac{1}{{u_0}^\beta}$ has to be understood in the weak distributional sense with test functions with compact support in Ω. That is

$$\int_\Omega (Du_0, D\varphi)\, dx = \int_\Omega \frac{\varphi}{{u_0}^\beta}\, dx \qquad \forall \varphi \in C^1_c(\Omega). \tag{7.7}$$

Actually the solution is fulfilled in the classical sense in the interior of Ω by standard regularity results, since u_0 is strictly positive in the interior of the domain.

In any case, by the Lazer-McKenna result [19] (see Chapter 2), for $\beta \geq 3$ we have that u_0 does not belong to $H^1_0(\Omega)$, and consequently u does not belong to $H^1_0(\Omega)$, too.

Let us state our symmetry result:

Theorem 7.2.1. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (7.4) ($u = u_0 + w$) with $g(\cdot)$ satisfying (H_p). Assume that the domain Ω is strictly convex w.r.t. the $\nu-$direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν, where*

$$T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

Then u is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the $\nu-$direction in Ω_0^ν, where

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\}.$$

Moreover, if Ω is a ball, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.

For the reader's convenience, we describe here below the scheme of the proof.

(i) We first remark that, by [4] (see Chapter 3), we get that u_0 is the limit of a sequence $\{u_n\}$ of solutions to a truncated problem (7.10). We exploit this to deduce symmetry and monotonicity properties of u_n and, consequently, of u_0. At this step we use the fact that the moving plane procedure applies in a standard way to the truncated problem (7.10).

(ii) By (i), recalling the decomposition $u = u_0 + w$, we are reduced to prove symmetry and monotonicity properties of w. To do this, in Section 7.5, we prove some comparison principles for w needed in the application of the moving plane procedure.

(iii) In Section 7.6, we give some details about the adaptation of the moving plane procedure to the study of the monotonicity and symmetry of w. It is worth emphasizing that the moving plane procedure is applied in our approach only to the $H_0^1(\Omega)$ part of u.
Note also that Theorem 7.2.1 is proved in Section 7.7, exploiting the more general result Proposition 7.6.1.

7.3 Notations

To state the next results we need some notations. Let ν be a direction in $\mathbb{R}^N$ with $|\nu| = 1$. As customary, for a real number λ we set

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\}$$

and

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\}$$

and

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu,$$

which is the reflection trough the hyperplane T_λ^ν. Moreover

$$(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu)$$

recalling that in general $(\Omega_\lambda^\nu)'$ may be not contained in Ω. Also let

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu.$$

When $\lambda > a(\nu)$, since Ω_λ^ν is nonempty, we set

$$\Lambda_1(\nu) = \{\lambda : (\Omega_t^\nu)' \subset \Omega \text{ for any } a(\nu) < t \leq \lambda\},$$

and

$$\lambda_1(\nu) = \sup \Lambda_1(\nu).$$

Finally, for $a(\nu) < \lambda \leq \lambda_1(\nu)$, we set

$$u_\lambda^\nu(x) = u(x_\lambda^\nu).$$

7.4 Symmetry properties of u_0

Proposition 7.4.1. *Let $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ be the solution to (7.6). Then, for any*

$$a(\nu) < \lambda < \lambda_1(\nu)$$

we have

$$u_0(x) < {u_0}_\lambda^\nu(x), \qquad \forall x \in \Omega_\lambda^\nu \tag{7.8}$$

and

$$\frac{\partial u_0}{\partial \nu}(x) > 0, \qquad \forall x \in \Omega^\nu_{\lambda_1(\nu)}. \tag{7.9}$$

Proof. Let $u_n \in H_0^1(\Omega) \cap C(\overline{\Omega})$ be the unique solution to

$$\begin{cases} -\Delta\, u_n \;=\; \frac{1}{(u_n + \frac{1}{n})^\beta} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.10}$$

The existence of u_n was proved in [4] and the uniqueness follows by [8]. We refer the reader to Chapter 3.

Since the problem is no more singular, by standard elliptic estimates it follows that $u_n \in C^2(\overline{\Omega})$. Therefore we can use the moving plane technique exactly as in [3, 13, 20] (see, for example, Section 7.1), to deduce that the statement of our proposition holds true for each u_n. By [4] $\{u_n\}$ converges to u_0 a.e. as n tends to infinity and therefore (7.8) follows passing to the limit. Finally it follows in the same way

$$\frac{\partial u_0}{\partial \nu}(x) \geq 0, \qquad \forall x \in \Omega^\nu_{\lambda_1(\nu)}\,,$$

and therefore (7.9) follows via the strong maximum principle [14]. □

As a consequence of Proposition 7.4.1, we get

Proposition 7.4.2. *Let $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ be the solution of (7.6) and assume that the domain Ω is strictly convex w.r.t. the $\nu-$direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν, where*

$$T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

Then u_0 is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the $\nu-$direction in Ω_0^ν. Moreover, if Ω is a ball, then u_0 is radially symmetric with $\frac{\partial u_0}{\partial r}(r) < 0$ for $r \neq 0$.

7.5 Comparison principles

Let us prove the following

Lemma 7.5.1. *Let us consider the function*

$$\begin{aligned} r_\beta(x,y,z,h) \;:=\; & x^\beta(x+y)^\beta(z+h)^\beta \;+\; x^\beta z^\beta(z+h)^\beta \;-\; z^\beta(x+y)^\beta(z+h)^\beta \\ & -\; x^\beta z^\beta (x+y)^\beta \end{aligned}$$

and the domain $D \subset \mathbb{R}^4$ defined by

$$D \;:=\; \Big\{(x,y,z,h)\,|\, 0 \leq x \leq z\,;\, 0 \leq h \leq y\Big\}.$$

For any $\beta > 0$ it follows that $r_\beta \leq 0$ in D.

Proof. By direct calculation

$$\frac{\partial r_\gamma}{\partial y}(x,y,z,h) \;=\; \gamma x^\gamma (x+y)^{\gamma-1}(z+h)^\gamma \;-\; \gamma z^\gamma (x+y)^{\gamma-1}(z+h)^\gamma \;-\; \gamma x^\gamma z^\gamma (x+y)^{\gamma-1} \;\le 0$$

since $x \le z$. Therefore we are reduced to prove that $r_\beta \le 0$ in $D \cap \{h=y\}$, that is

$$r_\beta(x,y,z,y) = x^\gamma (x+y)^\gamma (z+y)^\gamma \;+\; x^\gamma z^\gamma (z+y)^\gamma \;-\; z^\gamma (x+y)^\gamma (z+y)^\gamma \;-\; x^\gamma z^\gamma (x+y)^\gamma \;\le 0\,.$$

For $x=0$ the thesis follows at once. For $x>0$ we note that

$$r_\beta(x,y,z,y) \;=\; -\Big(\frac{1}{x^\gamma} - \frac{1}{z^\gamma} \;+\; \frac{1}{(z+y)^\gamma} \;-\; \frac{1}{(x+y)^\gamma}\Big)(x^\gamma z^\gamma (z+y)^\gamma (x+y)^\gamma)$$

and the conclusion follows exploiting the fact that, for $0 < x \le z$ fixed, the function

$$r_\beta(t) \;:=\; x^{-\gamma} - z^{-\gamma} + (z+t)^{-\gamma} - (x+t)^{-\gamma}\,,$$

is increasing in $[0,+\infty)$ and $r_\beta(0)=0$. □

Lemma 7.5.2. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to problem* (7.4)*, with $\beta > 0$. Assume that Ω is a bounded smooth domain and that $g(\cdot)$ is positive ($g(s) > 0$ for $s > 0$ and $g(0) \ge 0$) and locally Lipschitz continuous. Consider the decomposition*

$$u = u_0 \,+\, w$$

where u_0 is the solution to (7.6) *and $w \in H^1_0(\Omega)$.*
Then it follows

$$w > 0 \qquad in \qquad \Omega\,.$$

Proof. Since $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$, then $w \in H^1_0(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$. Since $g(\cdot)$ is non-negative, it follows that u is a super-solution (see Definition 3.2.2) to the equation

$$-\Delta v = \frac{1}{v^\beta}\,.$$

Therefore, by Lemma 3.2.5 we get that

$$u \ge u_0 \qquad \text{and therefore} \qquad w \ge 0\,.$$

Let us now show that $w > 0$ in the interior of Ω. We will show this via the maximum principle exploited in regions where the problem is not singular. More precisely let us assume by contradiction that there exists a point $x_0 \in \Omega$ such that $w(x_0) = 0$ and let $r = r(x_0) > 0$ such that $B_r(x_0) \subset\subset \Omega$. We have, in the classical sense, in $B_r(x_0)$

$$-\Delta w \;=\; -\Delta u \,+\, \Delta u_0 \;=\; \frac{1}{(u_0+w)^\beta} \,+\, g(u) \,-\, \frac{1}{u_0^\beta} \ge \frac{1}{(u_0+w)^\beta} - \frac{1}{u_0^\beta}\,.$$

Since $u_0(x_0) > 0$ we may and do assume that u_0 is positive in $B_r(x_0)$. Recalling that w is nonnegative we therefore get that

$$\frac{1}{(u_0+w)^\gamma} - \frac{1}{u_0^\gamma} = c(x)\,(u_0 + w - u_0) = c(x)\,w$$

for some bounded coefficient $c(x)$. Therefore there exists $\Lambda > 0$ such that $\frac{1}{(u_0+w)^\gamma} - \frac{1}{u_0^\gamma} + \Lambda\, w \ge 0$ in $B_r(x_0)$, so that

$$-\Delta w \,+\, \Lambda\, w \ge 0 \qquad \text{in} \qquad B_r(x_0)\,.$$

By the strong maximum principle we would get $w \equiv 0$ in $B_r(x_0)$, and by a covering argument, that $w \equiv 0$ in Ω. Since the case $w \equiv 0$ in Ω is possible only if $f(\cdot) = 0$, we get a contradiction showing that such a point x_0 does not exist and that actually $w > 0$. □

Proposition 7.5.1 (A strong maximum principle)**.** *Let $\lambda < 0$ and Ω' a bounded sub-domain of Ω_λ^ν (say $\Omega' \subset\subset \Omega_\lambda^\nu$). Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to* (7.4) *with $g(\cdot)$ satisfying (H_p).*
Let w given by (7.5) *and assume that*

$$\frac{\partial w}{\partial \nu} \geq 0 \qquad in \qquad \Omega' .$$

Then it holds the alternative

$$\frac{\partial w}{\partial \nu} > 0 \quad in \quad \Omega' \qquad or \qquad \frac{\partial w}{\partial \nu} = 0 \quad in \quad \Omega'$$

Proof. Let us use the short hand notation $w_\nu := \frac{\partial w}{\partial \nu}$ (and $u_{0\nu} := \frac{\partial u_0}{\partial \nu}$), so that w_ν is nonnegative in Ω'. Differentiating the equation in (7.4) we get that w_ν solve

$$\begin{aligned} -\Delta w_\nu &= -\frac{\beta}{u^{\beta+1}} w_\nu \,+\, g'(u)(w_\nu + u_{0\nu}) \,+\, \beta \Big(\frac{1}{u_0^{\beta+1}} - \frac{1}{u^{\beta+1}} \Big) u_{0\nu} \\ &\geq -\frac{\beta}{u^{\beta+1}} w_\nu \,, \end{aligned}$$

where we used the fact that $g'(\cdot) \geq 0$ a.e.[1] by the assumption (H_p), the fact that $u_{0\nu} \geq 0$ in Ω' by Proposition 7.4.1, the fact that $u \geq u_0$ by Lemma 7.5.2 and finally the fact that $w_\nu \geq 0$ in Ω' by assumption.
We recall now that u is bounded away from zero in Ω', and therefore we find $\Lambda > 0$ such that

$$-\,\Delta w_\nu \geq -\frac{\beta}{u^{\beta+1}} w_\nu \geq -\Lambda \, w_\nu \,,$$

so that the conclusion follows by the standard strong maximum principle (see [14]). □

Proposition 7.5.2 (Weak Comparison Principle in small domains)**.** *Let $\lambda < 0$ and Ω' a bounded sub-domain of Ω_λ^ν. Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to* (7.4) *with $g(\cdot)$ satisfying (H_p).*
Assume that

$$w \leq w_\lambda^\nu \qquad on \qquad \partial\Omega' .$$

Then there exists a positive constant $\delta = \delta\,(u\,,\,g)$ such that, if we assume $\mathcal{L}(\Omega') \leq \delta$, then it holds

$$w \leq w_\lambda^\nu \quad in \ \Omega'.$$

Proof. We have

$$-\Delta(u_0 + w) \;=\; \frac{1}{(u_0 + w)^\beta} \,+\, g(u_0 + w) \qquad \text{in } \Omega, \tag{7.11}$$

$$-\Delta({u_0}_\lambda^\nu + w_\lambda^\nu) = \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\beta} \,+\, g({u_0}_\lambda^\nu + w_\lambda^\nu) \qquad \text{in } \Omega. \tag{7.12}$$

Since $(w - w_\lambda^\nu)^+ \in H_0^1(\Omega')$ we can consider a sequence of positive functions ψ_n such that

$$\psi_n \in C_c^\infty(\Omega') \qquad \text{such that} \qquad \psi_n \overset{H_0^1(\Omega')}{\longrightarrow} (w - w_\lambda^\nu)^+ \,.$$

Also we may and do assume that *supp* $\psi_n \subseteq$ *supp* $(w - w_\lambda^\nu)^+$. We plug ψ_n into the weak formulation of (7.11) and (7.12) and subtracting we get

[1]Note that, even if g' exist a.e., the term $g'(u)(w_\nu + u_{0\nu})$ make sense in the weak Sobolev meaning thanks to Stampacchia's Theorem.

$$\begin{aligned}
&\int_{\Omega'} (D(u_0 + w) - D({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda}) \,,\, D\psi_n)\, dx \\
&= \int_{\Omega'} \left(\frac{1}{(u_0 + w)^{\beta}} + g(u_0 + w) - \frac{1}{({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta}} - g({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda}) \right) \psi_n \, dx \,.
\end{aligned} \tag{7.13}$$

Since u_0 and ${u_0}^{\nu}_{\lambda}$ solve (7.6) we deduce

$$\begin{aligned}
&\int_{\Omega'} (D(w - w^{\nu}_{\lambda}) \,,\, D\psi_n)\, dx \\
&\quad = \int_{\Omega'} \left(\frac{1}{({u_0}^{\nu}_{\lambda})^{\beta}} - \frac{1}{(u_0)^{\beta}} + \frac{1}{(u_0 + w)^{\beta}} - \frac{1}{({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta}} \right) \psi_n \\
&\quad + \int_{\Omega'} \Big(g(u_0 + w) - g({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda}) \Big) \psi_n \, dx \\
&\quad = \int_{\Omega'} \Big(\frac{(u_0)^{\beta} (u_0 + w)^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta} + (u_0)^{\beta} ({u_0}^{\nu}_{\lambda})^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta}}{(u_0)^{\beta} ({u_0}^{\nu}_{\lambda})^{\beta} (u_0 + w)^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta}} \Big) \psi_n \, dx \\
&\quad - \int_{\Omega'} \Big(\frac{({u_0}^{\nu}_{\lambda})^{\beta} (u_0 + w)^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta} + (u_0)^{\beta} ({u_0}^{\nu}_{\lambda})^{\beta} (u_0 + w)^{\beta}}{(u_0)^{\beta} ({u_0}^{\nu}_{\lambda})^{\beta} (u_0 + w)^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta}} \Big) \psi_n \, dx \\
&\quad + \int_{\Omega'} \Big(g(u_0 + w) - g({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda}) \Big) \psi_n \, dx \,.
\end{aligned}$$

We now use the fact that $u_0 \leq {u_0}^{\nu}_{\lambda}$ in Ω^{ν}_{λ} and $w \geq w^{\nu}_{\lambda}$ on the support of ψ_n (since we assumed *supp* $\psi_n \subseteq$ *supp* $(w - w^{\nu}_{\lambda})^+$) to deduce from Lemma 7.5.1[2] that

$$\begin{aligned}
&(u_0)^{\beta} (u_0 + w)^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta} + (u_0)^{\beta} ({u_0}^{\nu}_{\lambda})^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta} \\
&- ({u_0}^{\nu}_{\lambda})^{\beta} (u_0 + w)^{\beta} ({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda})^{\beta} - (u_0)^{\beta} ({u_0}^{\nu}_{\lambda})^{\beta} (u_0 + w)^{\beta} \leq 0 \,,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{\Omega'} (D(w - w^{\nu}_{\lambda}) \,,\, D\psi_n)\, dx &\leq \int_{\Omega'} \Big(g(u_0 + w) - g({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda}) \Big) \psi_n \, dx \\
&\leq \int_{\Omega'} \Big(g({u_0}^{\nu}_{\lambda} + w) - g({u_0}^{\nu}_{\lambda} + w^{\nu}_{\lambda}) \Big) \psi_n \, dx \\
&\leq C \int_{\Omega'} (w - w^{\nu}_{\lambda}) \psi_n \, dx
\end{aligned}$$

for some constant $C > 0$ since $g(\cdot)$ is locally Lipschitz continuous and non-increasing by assumption. We now pass to the limit for $n \to \infty$ and get

$$\int_{\Omega'} |D(w - w^{\nu}_{\lambda})^+|^2 \, dx \leq C \int_{\Omega'} |(w - w^{\nu}_{\lambda})^+|^2 \, dx$$

and by Poincaré inequality

$$\int_{\Omega'} |D(w - w^{\nu}_{\lambda})^+|^2 \, dx \leq C\, C_p(\Omega') \int_{\Omega'} |D(w - w^{\nu}_{\lambda})^+|^2 \, dx \,.$$

For δ small it follows that $C\, C_p(\Omega') < 1$ which shows that actually

$$(w - w^{\nu}_{\lambda})^+ = 0$$

and the thesis follows.

□

[2] We exploit here Lemma 7.5.1 with $u_0 = x$, $w = y$, ${u_0}^{\nu}_{\lambda} = z$ and $w^{\nu}_{\lambda} = h$.

Lemma 7.5.3 (Strong Comparison Principle). *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to problem (7.4), with $g(\cdot)$ satisfying (H_p). Consider the decomposition $u = u_0 + w$, where u_0 is the solution to (7.6) and $w \in H_0^1(\Omega)$.*
Assume that, for some $a(\nu) < \lambda \le \lambda_1(\Omega)$, we have

$$w \le w_\lambda^\nu \qquad in \qquad \Omega_\lambda^\nu .$$

Then $w < w_\lambda^\nu$ in Ω_λ^ν unless $w \equiv w_\lambda^\nu$ in Ω_λ^ν.

Proof. Let us assume that there exists a point $x_0 \in \Omega_\lambda^\nu$ such that $w(x_0) = w_\lambda^\nu(x_0)$ and let $r = r(x_0) > 0$ such that $B_r(x_0) \subset\subset \Omega_\lambda^\nu$. We have, in the classical sense, in $B_r(x_0)$

$$\begin{aligned} -\Delta(w_\lambda^\nu - w) &= -\Delta(u_\lambda^\nu - {u_0}_\lambda^\nu) + \Delta(u - u_0) \\ &= \Big(\frac{1}{u_0^\beta} - \frac{1}{({u_0}_\lambda^\nu)^\beta} + \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\beta} - \frac{1}{(u_0 + w)^\beta} \Big) \\ &+ \Big(g({u_0}_\lambda^\nu + w_\lambda^\nu) - g(u_0 + w) \Big) \\ &= \Big(\frac{1}{u_0^\beta} - \frac{1}{({u_0}_\lambda^\nu)^\beta} + \frac{1}{({u_0}_\lambda^\nu + w)^\beta} - \frac{1}{(u_0 + w)^\beta} \Big) \\ &+ \Big(g({u_0}_\lambda^\nu + w_\lambda^\nu) - g(u_0 + w) \Big) \\ &+ \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\beta} - \frac{1}{({u_0}_\lambda^\nu + w)^\beta} . \end{aligned} \tag{7.14}$$

We use now the fact that $g(\cdot)$ is increasing by assumption, the fact that $u_0 \le {u_0}_\lambda^\nu$ in Ω_λ^ν (by Proposition 7.4.1) and the assumption $w \le w_\lambda^\nu$ in Ω_λ^ν to get that

$$g({u_0}_\lambda^\nu + w_\lambda^\nu) - g(u_0 + w) \ge 0.$$

In the same way, since for $0 < a \le b$ the function

$$h(t) := a^{-\beta} - b^{-\beta} + (b + t)^{-\beta} - (a + t)^{-\beta}$$

is increasing in $[0\,, +\infty)$, we also have

$$\left(\frac{1}{u_0^\beta} - \frac{1}{({u_0}_\lambda^\nu)^\beta} + \frac{1}{({u_0}_\lambda^\nu + w)^\beta} - \frac{1}{(u_0 + w)^\beta} \right) \ge 0 .$$

Consequently by (7.14)

$$-\Delta(w_\lambda^\nu - w) \ge \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\beta} - \frac{1}{({u_0}_\lambda^\nu + w)^\beta} .$$

Since ${u_0}_\lambda^\nu(x_0) > 0$, arguing as in Lemma 7.5.2, we find $\Lambda > 0$ such that, eventually reducing r, it results $\frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{({u_0}_\lambda^\nu + w)^\gamma} + \Lambda\,(w_\lambda^\nu - w) \ge 0$ in $B_r(x_0)$, so that

$$-\Delta(w_\lambda^\nu - w) + \Lambda\,(w_\lambda^\nu - w) \ge 0 \qquad \text{in} \qquad B_r(x_0) .$$

By the strong maximum principle [14] it follows now $(w_\lambda^\nu - w) \equiv 0$ in $B_r(x_0)$, and by a covering argument $(w_\lambda^\nu - w) \equiv 0$ in Ω_λ^ν, proving the result.

□

7.6 Symmetry

Proposition 7.6.1. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (7.4), decomposed as in (7.5) by*

$$u = u_0 + w\,.$$

Then, for any

$$a(\nu) < \lambda < \lambda_1(\nu)$$

we have

$$w(x) < w_\lambda^\nu(x), \qquad \forall x \in \Omega_\lambda^\nu. \tag{7.15}$$

Moreover

$$\frac{\partial w}{\partial \nu}(x) > 0, \qquad \forall x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{7.16}$$

Finally, (7.15) and (7.16) holds true replacing w by u.

Proof. Let $\lambda > a(\nu)$ and note that

$$w \leq w_\lambda^\nu \qquad \text{on} \qquad \partial\Omega_\lambda^\nu\,,$$

since $w > 0$ in Ω by Lemma 7.5.2. Therefore, assuming that $\mathcal{L}(\Omega_\lambda^\nu)$ is sufficiently small (say for $\lambda - a(\nu)$ sufficiently small) so that the Weak Comparison Principle Proposition 7.5.2 applies, we get

$$w \leq w_\lambda^\nu \qquad \text{in} \qquad \Omega_\lambda^\nu\,,$$

and actually $w < w_\lambda^\nu$ in Ω_λ^ν by the Strong Comparison Principle (Lemma 7.5.3).
Lets us now define

$$\Lambda_0 = \{\lambda > a(\nu) : w \leq w_t^\nu \text{ in } \Omega_t^\nu \text{ for all } t \in (a(\nu), \lambda]\}$$

and

$$\lambda_0 = \sup \Lambda_0.$$

To prove our result we have to show that actually $\lambda_0 = \lambda_1(\nu)$.
Assume otherwise that $\lambda_0 < \lambda_1(\nu)$ and note that, by continuity, we obtain $w \leq w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. Therefore, by the Strong Comparison Principle (Lemma (7.5.3)), it follows $w < w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$ since , taking into account the zero Dirichlet boundary condition and the fact that $w > 0$ in the interior of the domain, it follows that the case $w = w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$ is not possible if $\lambda_0 < \lambda_1(\nu)$.
We can now consider δ given by Proposition 7.5.2, so that the Weak Comparison Principle holds true in Ω' if $\mathcal{L}(\Omega') \leq \delta$. Fix a compact set $\mathcal{K} \subset\subset \Omega_{\lambda_0}^\nu$ so that $\mathcal{L}(\Omega_{\lambda_0}^\nu \setminus \mathcal{K}) \leq \frac{\delta}{2}$. Since we proved that $w < w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$, by compactness we find $\sigma > 0$ such that

$$w_{\lambda_0}^\nu - w \geq 2\sigma > 0 \quad \text{in} \quad \mathcal{K}\,.$$

Take now $\bar{\varepsilon} > 0$ sufficiently small so that $\lambda_0 + \bar{\varepsilon} < \lambda_1(\nu)$ and, for any $0 < \varepsilon \leq \bar{\varepsilon}$, we have

a) $w_{\lambda_0+\varepsilon}^\nu - w \geq \sigma > 0$ in $\mathcal{K}$,

b) $\mathcal{L}(\Omega_{\lambda_0+\varepsilon}^\nu \setminus \mathcal{K}) \leq \delta$.

Taking into account *a)* it is now easy to check that, for any $0 < \varepsilon \leq \bar{\varepsilon}$, we have that $w \leq w_{\lambda_0+\varepsilon}^\nu$ on the boundary of $\Omega_{\lambda_0+\varepsilon}^\nu \setminus \mathcal{K}$. Consequently, by *b)*, we can apply the Weak Comparison Principle (Proposition 7.5.2) and deduce that

$$w \leq w_{\lambda_0+\varepsilon}^\nu \qquad \text{in} \qquad \Omega_{\lambda_0+\varepsilon}^\nu \setminus \mathcal{K}\,,$$

and therefore $w \leq w_{\lambda_0+\varepsilon}^\nu$ in $\Omega_{\lambda_0+\varepsilon}^\nu$. Actually $w < w_{\lambda_0+\varepsilon}^\nu$ in $\Omega_{\lambda_0+\varepsilon}^\nu$ by the Strong Comparison Principle (Lemma (7.5.3)). We therefore get a contradiction with the definition of λ_0 and conclude that actually $\lambda_0 = \lambda_1(\nu)$ and (7.15).

It follows now directly from simple geometric considerations and by (7.15) that w is monotone non-decreasing in $\Omega^{\nu}_{\lambda_1(\nu)}$ in the $\nu-$direction. This gives

$$w_\nu := \frac{\partial w}{\partial \nu}(x) \geq 0 \quad \text{in} \quad \Omega^{\nu}_{\lambda_1(\nu)} \, .$$

Consequently it is standard to deduce (7.16) from Proposition 7.5.1.
To prove that (7.15) and (7.16) hold true replacing w with u, just recall that

$$u = u_0 + w \, ,$$

and exploit Proposition (7.4.1). □

7.7 Proof of Theorem 7.2.1

The proof of Theorem 7.2.1 is now a direct consequence of Proposition 7.6.1. Only note that

$$\lambda_1(\nu) = 0 \, ,$$

by assumption, and apply Proposition 7.6.1 in the $\nu-$direction to get

$$u(x) \leq u^{\nu}_{\lambda_1(\nu)}(x), \qquad \forall x \in \Omega^{\nu}_0.$$

and in the $(-\nu)-$direction to get

$$u(x) \geq u^{\nu}_{\lambda_1(\nu)}(x), \qquad \forall x \in \Omega^{\nu}_0.$$

and therefore $u(x) \equiv u^{\nu}_{\lambda_1(\nu)}(x)$ in Ω. The monotonicity of u follows by (7.16).

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