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Pairings and Symmetry Notions. A New Unifying Perspective in Mathematics and Computer Science

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Abstract

In this thesis we introduce two types of symmetry notions, local and global, that are applicable to many different mathematical contexts. To be more specific, let Ω be a fixed set. We call a triple $\mathfrak{P} = (U, F, \Lambda)$, where U and Λ are non-empty sets and $F : U \times \Omega \to \Lambda$ is a map, a *pairing* on Ω. Then, for any subset $A \subseteq \Omega$ we define a *local symmetry relation* on U given by $u \equiv_A u'$ if $F(u, a) = F(u', a)$ for any $a \in A$, and a *global symmetry relation* on the power set $\mathcal{P}(\Omega)$ given by $A \approx_{\mathfrak{P}} A'$ if $\equiv_A$ coincides with $\equiv_{A'}$.

We use the very general notion of pairing to investigate the above two types of symmetry notions for a broad spectrum of mathematical theories very far apart, such as simple undirected graphs, digraphs, vector spaces endowed of bilinear forms, information tables, metric spaces and group actions.

The relation $\approx_{\mathfrak{P}}$ induces a closure operator $M_{\mathfrak{P}}$ on Ω and we prove that any finite lattice is order isomorphic to the closure system induced by an appropriate operator $M_{\mathfrak{P}}$. On the other hand, the relation $\approx_{\mathfrak{P}}$ also induces a set operator $C_{\mathfrak{P}}$ on Ω whose fixed point set $MINP(\mathfrak{P})$ is an abstract simplicial complex, substantially dual to the closure system induced by $M_{\mathfrak{P}}$. In this way, we obtain a closure system and an abstract simplicial complex mutually interacting by means of three set systems having relevance in both theoretical computer science and discrete mathematics. As a matter of fact, any independent set family of a matroid on Ω can be represented as the set system of the minimal partitioners of some pairing $\mathfrak{P}$ on Ω. Hence, we are enabled to investigate $MINP(\mathfrak{P})$ in relation to classical operators derived from matroid theory.

Moreover, by means of the set operator $M_{\mathfrak{P}}$, it is possible to introduce a preorder $\geq_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ by setting $A \geq_{\mathfrak{P}} B$ if and only if $M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(B)$. This preorder (and the associated pair family $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$) satisfies the so-called *union additive property* according to which if $A'' \in \mathcal{P}(\Omega)$ and $A \geq_{\mathfrak{P}} A'$, $A \geq_{\mathfrak{P}} A''$ then $A \geq_{\mathfrak{P}} A' \cup A''$ and whose induced equivalence relation coincides exactly with $\approx_{\mathfrak{P}}$. In particular, there exists a bijection between closure operators on Ω and any preorder on $\mathcal{P}(\Omega)$ having also the above property.

Actually, the preorder $\geq_{\mathfrak{P}}$ represents a *local* version of a more general preorder on $\mathcal{P}(\mathcal{P}(\Omega)^2)$, denoted by $\rightsquigarrow_{\Omega}$, depending uniquely on the starting set Ω and mutually interrelating the preorders $\geq_{\mathfrak{P}}$, for each $\mathfrak{P} \in PAIR(\Omega)$.

In this context, the study of the preorder $\geq_{\mathfrak{P}}$ is relevant since it is possible to show that any finite lattice $\mathbb{L}$ is order-isomorphic to the symmetrization of a preorder induced by $\rightsquigarrow_{\Omega_{\mathbb{L}}}$ on the power set $\mathcal{P}(\Omega_{\mathbb{L}})$, for a suitable finite set $\Omega_{\mathbb{L}}$ and a pairing $\mathfrak{P}$ on it. On the other hand, we will see that the above preorder has an interpretation in terms of *symmetry transmission*. As a matter of fact, we define a set map $\Gamma_{\mathfrak{P}} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U)$ and, in the finite case, an associated numerical

quantity, thanks to which it is possible to evaluate the transmission of symmetry between two subsets $A, B \in \mathcal{P}(\Omega)$. In this way, the previous finite lattices representation theorem provides a refinement of order theory enabling us to connect order properties to topological, matroidal or set combinatorial properties. The epistemological consequence of all the aforementioned representation results is that we can consider closure system, finite lattice and matroid theories as sub-theories that are parts of the more general pairing paradigm.
In such a perspective, we relate the preorder $\geq_{\mathfrak{P}}$ with the closure system $MAXP(\mathfrak{P})$ from an operatorial standpoint by considering the map $DP : \mathcal{P}(\mathcal{P}(\Omega)^2) \to \mathcal{P}(\mathcal{P}(\Omega))$ and proving that $DP(\mathcal{D}^+) = DP(\mathcal{D})$ for any $\mathcal{D} \subseteq \mathcal{P}(\mathcal{P}(\Omega)^2)$ and that $DP(\mathcal{G}(\mathfrak{P})) = DP(\mathcal{Q}(\mathfrak{P})) = MAXP(\mathfrak{P})$, where $\mathcal{G}(\mathfrak{P}) := \{(A, B) \in \mathcal{P}(\Omega)^2 : A \geq_{\mathfrak{P}} B\}$ and $\mathcal{Q}(\mathfrak{P}) := \{(X, y) : X \in MINP(\mathfrak{P}), y \in M_{\mathfrak{P}}(X) \setminus X\}$. Thus, it follows that the study of specific subset pair families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ is strictly related to the set systems and to the set operators characterizing pairings. This explains the need to study the main properties of the specific families satisfying the same properties as the above two previous models.
Finally, the investigation of the set map $\Gamma_{\mathfrak{P}}$ leads us to the analysis of a particular structure, namely the *indistinguishability linear systems*. In particular, we focus our attention on the corresponding concept of *compatibility* from both a local and a global standpoint. The term *compatibility* is here used in analogy with the study of the compatibility of classical finite system of linear equations with coefficients in any field $\mathbb{K}$. An indistinguishability linear system $\mathcal{S}$ is a very general structure $\langle U_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}, F_{\mathcal{S}}, \Lambda_{\mathcal{S}}\rangle$ where $U_{\mathcal{S}}$ can be seen as the equation set, $C_{\mathcal{S}}$ as the variable set, $D_{\mathcal{S}}$ consists of at most an element $d_{\mathcal{S}}$ (*constant term*) and $F_{\mathcal{S}} : U_{\mathcal{S}} \times (C_{\mathcal{S}} \cup D_{\mathcal{S}}) \to \Lambda_{\mathcal{S}}$ is a map that plays the role of the coefficients matrix in a classic linear system. Then, from a local perspective, the study of the compatibility in $\mathcal{S}$ means to fix $W \subseteq U_{\mathcal{S}}$, $A \subseteq C_{\mathcal{S}}$ and to determine in what cases all equations $u \in W$ satisfy a specific condition related to local symmetry. On the other hand, from a global perspective, we try to determine the conditions to be satisfied by some operators formalizing the compatibility notion.

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Contents

Introduction

The study of the symmetry has always fascinated mathematicians. In his classical book [181], Hermann Weyl discusses the symmetry concept in geometry from several points of view. Nevertheless, currently, in literature there is no paradigm concerning symmetry that is sufficiently unifying and general enough to be applied to mathematical structures apparently very far apart. Therefore, it happens that in each specific research field, a characteristic notion of symmetry is given and there is no a common paradigm connecting them. The following contexts provide some of the most natural structures where specific symmetry questions can be investigated.

In graph theory, the symmetry notion is classically related to the properties of the automorphism group $Aut(G)$ of a graph G (see [83]) and it is also object of deep algebraic investigations (see [103]). However, even within graph theory itself, many notions of symmetry, different from the classical one proposed in [83] have been introduced and well studied. For example, in [115, 116, 117, 131, 155] several types of symmetry notions has been investigated for Cayley graphs, Moore graphs, attachment graphs, graph algebras and directed graphs. In fact, Cayley graphs of groups play an important role in symmetric graphs. In [36, 37, 38] it have been introduced some types of homotopy relations based on the symmetry of graphs, that are strictly related with database theory. Moreover, there is also a wide literature concerning the study of symmetry breaking [2, 90, 101, 102, 133].

In hypergraph theory some questions concerning the hyperedge symmetry have been investigated in [165, 173, 175].

In a more algebraic context, the symmetry is a basic notion when one studies the symmetric group [121, 154], the symmetric functions [130], the plane symmetry groups [158], the interrelations with Hopf bifurcations [91, 92, 93] and the association schemes [11, 12].

In order theory, the symmetry notion has been studied in relation to a specific types of symmetric functions and representations of the symmetric group [160, 167] and in relation to Ramsey theory [84]. In [21, 23, 24, 35] some classes of lattices with high symmetry degree have been investigated in terms of sequential and discrete dynamical systems.

In metric space theory, some types of symmetry transformations have been related in [136, 179] to the Wigner's theorem, that is a classical result in mathematical foundation of quantum mechanics [183]. From a more general perspective, if X is any set a map $d : X \times X \to [0, \infty[$ such that $d(u, v) = 0$ if and only if $u = v$ and such that $d(u, v) = d(v, u)$ has been studied in [100, 132] exactly for its symmetry properties.

In this work we introduce the unifying notion of *pairing* (see [20, 53, 55, 56, 58, 63, 64, 65]) or, equivalently, *function system* [61, 68]), that serves as paradigm to link each other, from an interdisciplinary outlook, several different mathematical structures, such as matroids, group actions, vector spaces endowed with bilinear forms, metric spaces, simple undirected graphs and digraphs. By means of this notion, we provide a new perspective to investigate a type of *generalized symmetry* in the above structures and in all other mathematical structures where it is natural to introduce an appropriate pairing.

Let us fix an arbitrary set Ω (finite of infinite). Our basic notion is given in the next definition.

Definition 0.0.1. A *pairing* on Ω is a triple $\mathfrak{P} = (U, F, \Lambda)$, where U, Λ are non-empty sets and $F : U \times \Omega \to \Lambda$ is a map having domain $U \times \Omega$ and codomain Λ. We denote by $PAIR(\Omega)$ the set of all pairings on Ω. When both the sets $U = \{u_1, \ldots, u_m\}$ and $\Omega = \{a_1, \ldots, a_n\}$ are finite, we say that $\mathfrak{P}$ is a *finite pairing.* For a finite pairing, we denote by $T[\mathfrak{P}]$ the $m \times n$ rectangular table having on the i-th row the element u_i, on the j-th column the element a_j and $F(u_i, a_j)$ in the place (i, j). In the finite case, we identify $\mathfrak{P}$ with $T[\mathfrak{P}]$ and call $T[\mathfrak{P}]$ the *functional table* of $\mathfrak{P}$.

By starting from the very general notion of pairing, we can develop new refined mathematical notions without additional hypotheses. In particular, through the comparison of specific partition symmetries induced on U by the subsets of Ω, we will construct a type of *symmetrization geometry* on $\mathcal{P}(\Omega)$.

In this thesis, we are going to present several new and original results obtained during our researches spread over the last three years, in relation to the aforementioned topic. In the next subsection we introduce the basic relations, operators and corresponding set systems which are naturally associated with any pairing $\mathfrak{P} \in PAIR(\Omega)$.

Local and Global Symmetry Relations on Pairings

Let $\mathfrak{P} = (U, F, \Lambda) \in PAIR(\Omega)$ and $A, A' \in \mathcal{P}(\Omega)$. For any $u, u' \in U$ we set

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a), \forall a \in A.$$

It easily follows that $\equiv_A$ is an equivalence relation on U that we call A-*symmetry relation* of $\mathfrak{P}$. For any $u \in U$ we denote by $[u]_A$ the equivalence class of u with respect to $\equiv_A$ and we call $[u]_A$ the A-*symmetry class* of u. Finally, we call the set partition $\pi_{\mathfrak{P}}(A) := \{[u]_A : u \in U\}$ induced by $\equiv_A$ on U the A-*symmetry partition* of $\mathfrak{P}$.

The terminology symmetry relation stems by the fact that in graph context (i.e. when we identify a graph with its adjacency matrix considered as a pairing), the previous equivalence relation takes actually account of a local type of symmetry (see [47, 49, 52, 54, 63, 71]). Consider for example the usual Petersen graph (briefly Pet) with vertex set $V(Pet) = \{1, \ldots, 10\}$ and let us take $A = \{1, 6\}$. Then, it is easy to verify that there are exactly the following three symmetry blocks of A in Pet: $\{1, 8, 9\}$, $\{2, 5, 6\}$ and $\{3, 4, 7, 10\}$. Hence $\pi_{Pet}(A) = 189|256|347(10)$. In fact, we can note that the vertices in each one of the previous blocks have all the same behavior with respect to the vertices 1 and 6. For example 1, 8 and 9 are all vertices of Pet adjacent only to 6; the vertices 2, 5 and 6 are all vertices adjacent only to 1 and the vertices 3, 4, 7 and 10 are all vertices non adjacent neither 1 nor 6. Finally, we note that there is no vertex

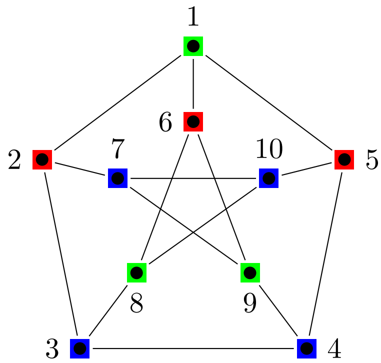


Figure 1:

adjacent to both 1 and 6. We can see better the symmetric position of the previous blocks with respect to A if we colour them with three different colours, as we did in Figure 1.

Therefore, from our perspective, symmetry with respect to $A \subseteq V(G)$ can be formally translated as the equivalence relation $\equiv_A$ and its equivalence classes can be seen as *symmetry blocks* relatively to A. In general, note that when we fix a vertex subset A, we implicitly assume A as an *observation point* (or, equivalently, a *reference system*), with respect to which observe the behavior of all other vertices in G. Then, if v, v' are two not adjacent vertices of G such that $v \equiv_A v'$, it results (see Corollary 2.1.4) that the induced subgraph by $A \cup \{v, v'\}$ is symmetric in the sense of Erdös-Rényi [83].

Now, let us note that $\pi_G(A)$ depends on both G and A, so we can not provide a general way to determine $\pi_G(A)$, except for some simple graph structures (for example the complete graph and the complete bipartite one). Therefore, given a finite undirected simple graph G, we must study (in some way to determine) the whole family of vertex partitions induced by the vertex subsets $A \in \mathcal{P}(\Omega)$. In this way we try to establish some type of parameters, depending only on the structure of G, which provide useful informations concerning the links among all the A-symmetry partitions.

We observe now that the above discussed A-symmetry relation can be considered a type of *local symmetry* on the set U, because it depends on the choice of the subset A. Then, by starting from the above A-symmetry relation, we introduce a type of *global* symmetry relation relatively to a given pairing. To this regard, we consider the following equivalence relation on $\mathcal{P}(\Omega)$:

$$A \approx_{\mathfrak{P}} A' :\Longleftrightarrow \pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A') \Longleftrightarrow (u \equiv_A u' \Longleftrightarrow u \equiv_{A'} u')$$

for all $u, u' \in U$. We call $\approx_{\mathfrak{P}}$ the *global symmetry relation* of $\mathfrak{P}$, and we denote by $[A]_{\approx_{\mathfrak{P}}}$ the equivalence class of A with respect to $\approx_{\mathfrak{P}}$, that we call *global symmetry class*. When the pairing is clear in the context, we omit the subscript $\mathfrak{P}$ when denoting the global symmetry relation.

Hence the global symmetry class $[A]_{\approx_{\mathfrak{P}}}$ contains all subsets of Ω which induce the same local symmetry on U. Moreover, a basic property of any such class is that it is a union-closed family

(Theorem 2.2.1, see [13, 76, 137, 152, 180, 187] for some studies on union-closed families arisen mainly from the famous Frankl's conjecture [85]), therefore it contains a maximum element (the union of all its members), that we denote by $M_{\mathfrak{P}}(A)$.

Then, $M_{\mathfrak{P}}(A)$ is the largest subset of Ω which induces on U the same local symmetry of A. Now, if we consider the set operator $M_{\mathfrak{P}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ such that $A \mapsto M_{\mathfrak{P}}(A)$ it results (see Theorem 2.2.1) that $M_{\mathfrak{P}}$ is a closure operator on Ω. As a direct consequence, the set system

$$MAXP(\mathfrak{P}) := \{M_{\mathfrak{P}}(A) : A \in \mathcal{P}(\Omega)\} = \{C \in \mathcal{P}(\Omega) : M_{\mathfrak{P}}(C) = C\}$$

is a closure system on Ω. On the other hand, we also consider the set partition family given by

$$PSYM(\mathfrak{P}) := \{\pi_{\mathfrak{P}}(A) : \ A \in \mathcal{P}(\Omega)\}.$$

Then, it results that the two posets $\mathbb{M}(\mathfrak{P}) := (MAXP(\mathfrak{P}), \subseteq^*)$ and $\mathbb{P}(\mathfrak{P}) := (PSYM(\mathfrak{P}), \preceq)$ (where $\preceq$ is the usual partial order on set partitions) are two order isomorphic complete lattice (see (ii) of Corollary 2.2.2). We call $\mathbb{M}(\mathfrak{P})$ the *maximum partitioner lattice* of $\mathfrak{P}$.

Again, we also associate with the pairing $\mathfrak{P}$ the family of all *minimal partitioners*, defined as follows:

$$MINP(\mathfrak{P}) := \bigcup \{\min([A]_{\approx_{\mathfrak{P}}}) : A \in MAXP(\mathfrak{P})\}.$$

Hence, it results that $MINP(\mathfrak{P})$ is an abstract simplicial complex [119] (see Theorem 2.6.1).

In this thesis, we are going to establish a representation result (Theorem 6.1.1), through which we prove that for any closure operator σ on an arbitrary finite set Ω it is possible to construct a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $\sigma = M_{\mathfrak{P}}$. By means of the aforementioned theorem and of a classical result of matroid theory (Theorem 1.6.5), we deduce that any independent set family of a matroid on Ω can be represented as the set system of the minimal partitioners of some pairing $\mathfrak{P}$ on Ω.

Therefore, by virtue of what we said before, we can study by means rectangular tables (in the finite case) or pairings (in the general case) both closure systems and matroids through the study of the local and global symmetry. In other terms, we can assert that we are going to develop a *symmetrization geometry on pairings* through the study of a closure system and an abstract simplicial complex, that are naturally associated with each pairing. In the remaining part of the introduction, we denote by $\mathfrak{P}$ a fixed pairing in $PAIR(\Omega)$.

The Global Symmetry Lattice

The deep link between maximum partitioners and minimal ones is provided by the following structure. Consider the poset $(GSYM(\mathfrak{P}), \sqsubseteq)$, where

$$GSYM(\mathfrak{P}) := \{[A]_{\approx_{\mathfrak{P}}} : A \in MAXP(\mathfrak{P})\}$$

is endowed with the partial order

$$[A]_{\approx_{\mathfrak{P}}} \sqsubseteq [B]_{\approx_{\mathfrak{P}}} :\iff A \subseteq^* B.$$

It can be shown (Theorem 2.2.8) that $\mathbb{G}(\mathfrak{P})$ is a complete lattice order isomorphic to $\mathbb{M}(\mathfrak{P})$, that we call *global symmetry lattice* of $\mathfrak{P}$.

	a_1	a_2	a_3	a_4
u_1	1	1	0	0
u_2	0	0	1	1
u_3	1	0	1	0
u_4	0	1	1	1
u_5	1	0	1	1

Figure 2: The functional table of Example 0.0.2

Let us note that each global symmetry class contains all the subsets of Ω inducing the same symmetry partition, and these subsets have a natural poset structure with respect to the usual set inclusion. Therefore these posets can be studied with the usual techniques derived from the order theory. Obviously, a relevant question is to understand *if*, and *how*, the inner order structure of each class $[A]_{\approx_{\mathfrak{P}}}$ *interacts* with the global order structure of the global symmetry lattice $\mathbb{G}(\mathfrak{P})$.

To this regard, we provide the following three type of results as a consequence of our study.

(1) results concerning the properties of the global symmetry classes.

(2) results concerning the global symmetry lattice.

(3) results that concern the possible interactions between the inner structure of any global symmetry class and the maximum partitioner lattice.

A basic property relative to (3) is the *global-local regularity*, which can be expressed as follows

(GLR) $M_{\mathfrak{P}}(A) \subsetneqq M_{\mathfrak{P}}(A') \implies Y \not\subseteq X$ for any $X \in [A]_{\approx_{\mathfrak{P}}}$ and $Y \in [A']_{\approx_{\mathfrak{P}}}$.

The global-local regularity property tells us that the inclusion between the maximum elements of any two global symmetry classes preserves the same type of inclusion between any two members of these classes.

Example 0.0.2. *Let us consider the pairing $\mathfrak{P}$ whose functional table is given in Figure 2. In Figure 3 we represent the maximum partitioner lattice of $\mathfrak{P}$. In Figure 4 we represent the global symmetry lattice of $\mathfrak{P}$.*

Symmetry Set Systems Related to the Global Symmetry Lattice

The subset $M_{\mathfrak{P}}(A)$ is the largest subset of Ω that induces on U the same local symmetry of A. On the other hand, given $A \in \mathcal{P}(\Omega)$, if one studies the A-symmetry in relation to the subsets of A there are two natural types of subsets that we can consider.
(A) The subsets C of A providing the same symmetry of A (i.e. $\pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$) and which are also $\subseteq$-minimals with respect to this property. Formally, we have the set system

$$BAS_{\mathfrak{P}}(A) := \{C \in \mathcal{P}(A) : \pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B) \,\text{and}\, \pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(B') \;\forall B' \subsetneqq B\},$$

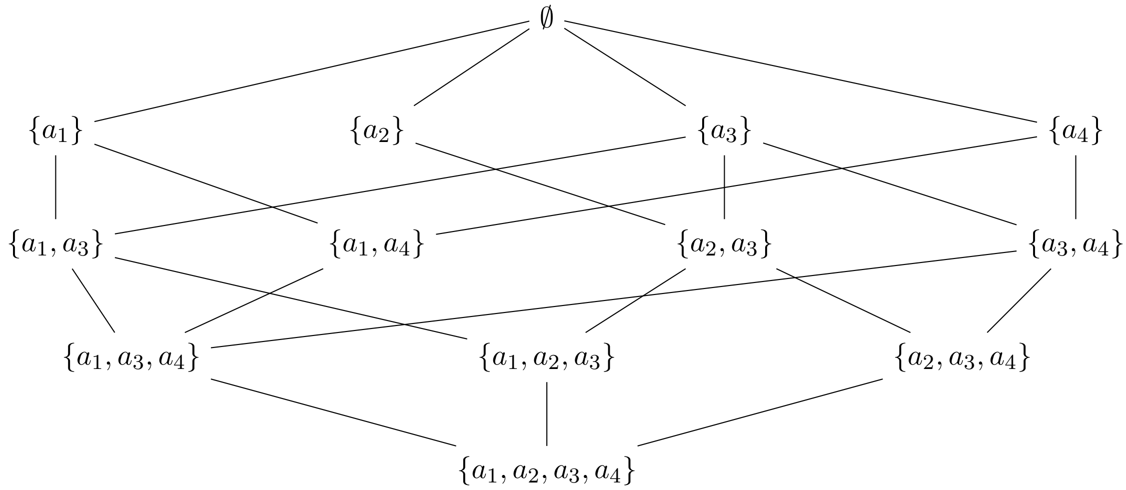


Figure 3: The Maximum Partitioner Lattice $\mathbb{M}(\mathfrak{P})$ of Example 0.0.2

and call any member of $BAS_{\mathfrak{P}}(A)$ an A-*symmetry base* of $\mathfrak{P}$.
(B) The subsets C of A which are "essential" to provide a maximum level of symmetry when we partition A (that is $\pi_{\mathfrak{P}}(A \setminus C) \neq \pi_{\mathfrak{P}}(A)$) and that are $\subseteq$-minimal with respect to this property. In such a case we have the set system

$$ESS_{\mathfrak{P}}(A) := \{C \in \mathcal{P}(A) : \pi_{\mathfrak{P}}(A \setminus C) \neq \pi_{\mathfrak{P}}(A) \,\text{and}\, \pi_{\mathfrak{P}}(A \setminus C') = \pi_{\mathfrak{P}}(A) \ \forall C' \subsetneqq C\},$$

and call any member of $ESS_{\mathfrak{P}}(A)$ an A-*symmetry essential* of $\mathfrak{P}$.

In [54], the basic properties of the previous set systems in relation to simple undirected graphs have been studied in relation to the notion of *local dissymmetry*. To this regard, we set

$$DIS(G) := \{\Delta_G(v_i, v_j) : v_i, v_j \in V(G)\},$$

where $\Delta_G(v_i, v_j) := N_G(v_i) \bigtriangleup N_G(v_j)$ is the set symmetric difference between the adjacency neighborhoods of v_i and v_j. We call the vertex subset family $DIS(G)$ and the subset $\Delta_G(v_i, v_j)$ respectively the *local dissymmetry set system* of G and the *dissymmetry neighborhood* of v_i and v_j. In fact, $\Delta_G(v_i, v_j)$ contains those vertices which *cut off* a potential 2-path between v_i and v_j (another kind of symmetric difference among neighborhoods of vertices has been introduced recently in [131] in order to study some types of symmetries in random graphs).

In this case, we can put within a table $\Delta[G]$ the members of the dissymmetry set system, i.e. if we denote by $N_G(v)$ the open neighborhood of all adjacent vertices to $v \in V(G)$, then in the place corresponding at the intersection between the v_i-row and the v_j-column there is the set symmetric difference $\Delta_G(v_i, v_j)$. Now, because G has no loops and multiple edges, it is immediate to see that G is uniquely characterized from this table. In other terms, if G and G' be two graphs on the same vertex set. Then

$$\Delta[G] = \Delta[G'] \iff E(G) = E(G'),$$

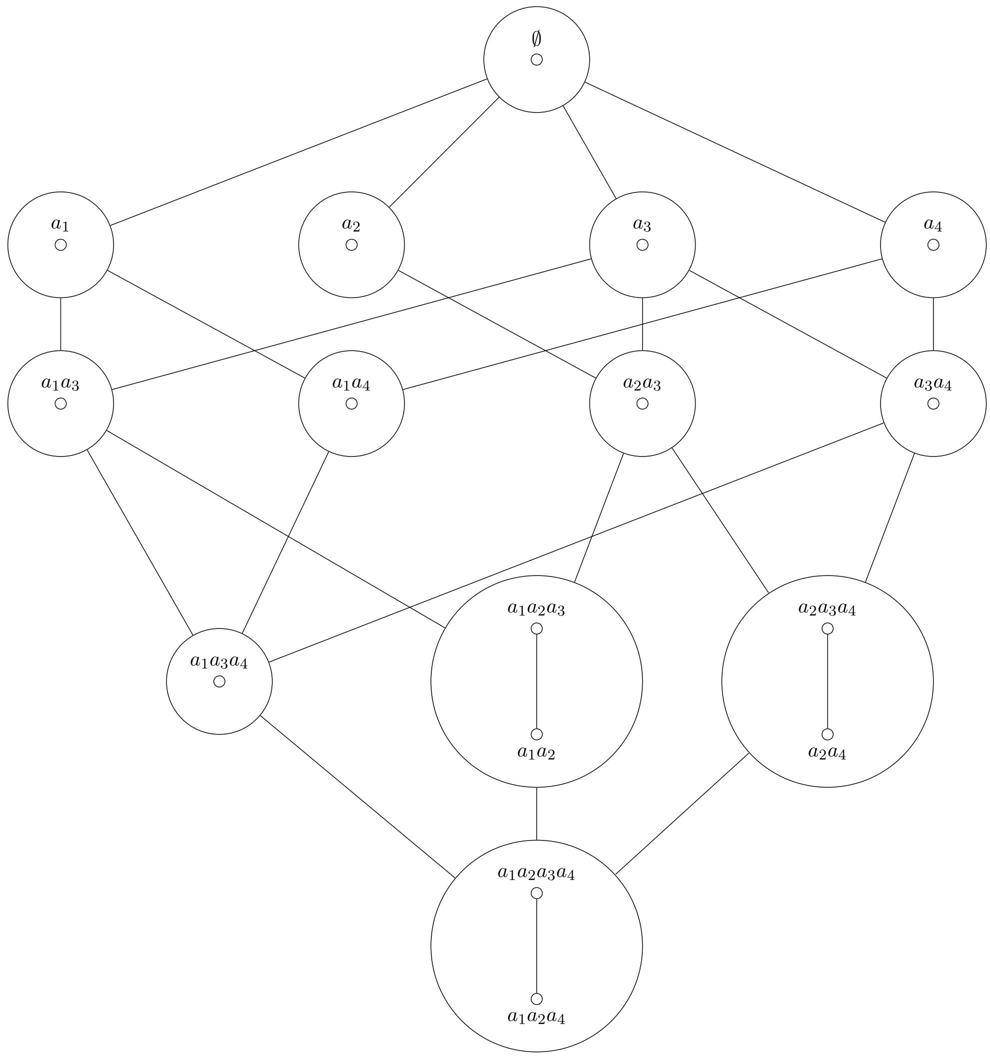
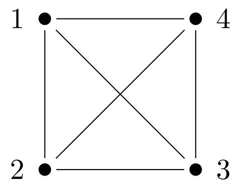


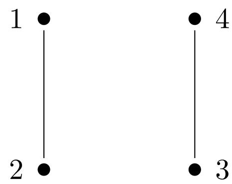
Figure 4: Global Symmetry Lattice $\mathbb{G}(\mathfrak{P})$ of Example 0.0.2.

where $E(G)$ denotes the edge set of G or, equivalently, two graphs G and G' coincide if and only if $\Delta_G(v_i, v_j) = \Delta_{G'}(v_i, v_j)$ for all $v_i, v_j \in V(G)$. An immediate and intuitive interpretation for the previous identities is that if $v_i \neq v_j$ then $v \in \Delta_G(v_i, v_j)$ if and only if v is adjacent to exactly one vertex between v_i and v_j, in other terms we can think any vertex in $\Delta_G(v_i, v_j)$ as a *local dissymmetry* vertex between v_i and v_j. Therefore, it is natural to think the entries of $\Delta[G]$ as *local dissymmetry neighborhoods* of G and to call G a *locally dissymmetric graph* when it happens that $\Delta_G(v_i, v_j) \neq \emptyset$ for all $v_i \neq v_j$. In particular, in [59] it has been provided an interpretation of local dissymmetry in graph context within the so-called *formal context analysis* (briefly FCA) [16, 27, 87, 88, 94, 184, 185, 186] (for the relation between graphs and FCA, see also [7, 8, 9, 10, 16]).

However, let us note that the the condition $DIS(G) = DIS(G')$ is weaker than $\Delta[G] = \Delta[G']$, in fact there are non isomorphic simple graphs G and G' having the same local dissymmetry set system, for example the following:



and



The above example contains in itself the following potential research development: *to classify the simple graphs by means of the local dissymmetry*, that is with respect to the condition $DIS(G) = DIS(G')$.

Moreover, in [71], it has been studied a generalization of the local dissymmetry set system, in fact it has been analyzed the behavior of the symmetric differences of vertex subsets. Again in [71], the local symmetry relation $\equiv_A$ has been studied in its interrelations with a specific type of binary operation $\circ$ defined on the power set $\mathcal{P}(V(G))$ and whose automorphism group is isomorphic to a subgroup of $Aut(G)$ (for other works on similar topics see also [25, 26, 103, 153]). In Section 4.2, we will investigate the main algebraic properties of the operation $\circ$ and see that it satisfies many group-like properties, even if it is not associative (see Proposition 4.2.3 and Theorems 4.2.18, 4.2.21 and 4.2.26).

Again in [54], it has been proved that $BAS(G)$ is the minimal transversal set system of $DIS(G)$, whereas $ESS(G)$ is the set system of all minimal (with respect to the usual inclusion) subsets of $DIS(G)$. The problem of compute the symmetry base set system is thus equivalent to that of finding the minimal transversal of a given hypergraph [80, 81, 82, 97, 99, 118] and it is a very difficult task that has many applications in mathematics and computer science [80, 81]. In Subsections 3.2.1 and 3.2.2, we treat the aforementioned problem for two graph models, namely the Petersen graph, for which we completely classified the subgraphs corresponding to the elements of $BAS(Pet)$, and the cycle on n vertices, for whose symmetry bases we provide an algebraic characterization (see Theorem 3.2.27).

We can generalize the local dissymmetry for graphs introducing the notion of dissymmetry in the more general pairing context as follows. For any $A \in \mathcal{P}(\Omega)$ we consider the map $\Delta_A^{\mathfrak{P}} : U \times U \to \mathcal{P}(\Omega)$ defined by

$$\Delta_A^{\mathfrak{P}}(u, u') := \{a \in A : F(u, a) \neq F(u', a)\},$$

for any $u, u' \in U$, and the following set system on Ω:

$$DIS_{\mathfrak{P}}(A) := \{\Delta_A^{\mathfrak{P}}(u, u') : u, u' \in U \text{ and } \Delta_A^{\mathfrak{P}}(u, u') \neq \emptyset\}.$$

We call $DIS_{\mathfrak{P}}(A)$ the A-*dissymmetry set system* of $\mathfrak{P}$. Then, the map $\Delta^{\mathfrak{P}} := \Delta_\Omega^{\mathfrak{P}}$ satisfies the following three basic properties:
(i) $\Delta^{\mathfrak{P}}(u, u) = \emptyset$ for each $u \in U$;
(ii) $\Delta^{\mathfrak{P}}(u, u') = \Delta^{\mathfrak{P}}(u', u)$ for all $u, u' \in U$;
(iii) $\Delta^{\mathfrak{P}}(u, u') = \bigcap_{u'' \in U}[\Delta^{\mathfrak{P}}(u, u'') \cup \Delta^{\mathfrak{P}}(u', u'')]$.
On the other hand, we prove (see Theorem 3.1.7) that if any map $D : U \times U \to \mathcal{P}(\Omega)$ satisfies the above properties, then there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $\Delta^{\mathfrak{P}} = D$. Therefore we can say that the properties (i), (ii) and (iii) can be considered as abstract axioms to define an abstract notion of *dissymmetry space* on U. In other terms, we call *dissymmetry space* (see [55]) on Ω a pair $\mathfrak{D} := \langle U, D \rangle$, where U is a non-empty set and $D : U \times U \to \mathcal{P}(\Omega)$ is a map which satisfies (i), (ii) and (iii).

Theorem 3.1.7 is the first of a series of representation theorems by pairings that are aimed to unify in a broader and general perspective several different theories, such as dissymetry spaces, finite lattices, closure systems and matroids.

Now, given a pairing $\mathfrak{P}$, for any $A \in \mathcal{P}(\Omega)$, based on the computation of the A-dissymmetry set system of $\mathfrak{P}$, it is possible also determine the A-symmetry bases and the A-symemtry essentials. In fact, as in the graph case [54], we can prove (Theorems 3.2.13 and 3.2.15) respectively that the A-symmetry essentials are the minimal elements of the set system $DIS_{\mathfrak{P}}(A)$ and that the A-symmetry bases are the minimal transversal of the same set system. Finally, in our treatment, a relevant role will be played by the subset of A given by

$$C_{\mathfrak{P}}(A) := \{a \in A : \pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})\},$$

that we call *core* of A. This subset can be identified with all singletons of the set system $ESS_{\mathfrak{P}}(A)$ and, it determines a set operator $C_{\mathfrak{P}}$ on Ω that we can substantially to consider a type of dual version of the set operator $M_{\mathfrak{P}}$ (see Proposition 3.2.8 and Theorem 3.2.9). Moreover, we also prove (see Theorem 3.2.9) that the set system $MINP(\mathfrak{P})$ is exactly the fixed point set of $C_{\mathfrak{P}}$.

Comparison Between A-Symmetry Relations

Relatively to the symmetry induced on U by the subsets of Ω, we want to specificate how these subsets are mutually related. In particular, we are going to compare the various symmetry partitions on U and, in the finite case, to determine a measure of transmission of symmetry. As a matter of fact, we consider the map $\Gamma_{\mathfrak{P}} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U)$ so defined:

$$\Gamma_{\mathfrak{P}}(A, B) := \{u \in U : [u]_A \subseteq [u]_B\}.$$

Furthermore, in the finite case, we can also define the *symmetry transmission measure* $\gamma_{\mathfrak{P}}$ as follows

$$\gamma_{\mathfrak{P}}(A, B) := \frac{|\Gamma_{\mathfrak{P}}(A, B)|}{|U|}.$$

Let us note (see [20, 53] for details) that

$$M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A') \iff \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A') \iff \Gamma_{\mathfrak{P}}(A, A') = U,$$

for all $A, A' \in \mathcal{P}(\Omega)$.

Then, in an operatorial perspective, which is based on the above three equivalences established, we introduce the binary relation $\geq_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ defined by

$$A \geq_{\mathfrak{P}} A' :\iff M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A'),$$

for any $A, A' \in \mathcal{P}(\Omega)$. We say that A is $\mathfrak{P}$-*symmetrically finer* than A' whenever $A \geq_{\mathfrak{P}} A'$. We can restate the notion of $\mathfrak{P}$-symmetrically finer by noting that $A \geq_{\mathfrak{P}} B \iff \Gamma_{\mathfrak{P}}(A, B) = U$. Since

$$A \approx_{\mathfrak{P}} A' \iff A \geq_{\mathfrak{P}} A' \text{ and } A' \geq_{\mathfrak{P}} A,$$

we can interpret the global symmetry relation $\approx_{\mathfrak{P}}$ as the equivalence relation on $\mathcal{P}(\Omega)$ induced by the preorder $\geq_{\mathfrak{P}}$.

The preorder $\geq_{\mathfrak{P}}$ has also the following property of union additivity: if $A'' \in \mathcal{P}(\Omega)$ and $A \geq_{\mathfrak{P}} A'$, $A \geq_{\mathfrak{P}} A''$ then $A \geq_{\mathfrak{P}} A' \cup A''$. Then, it is easy to see that any preorder on $\mathcal{P}(\Omega)$ having also the above union additive property (thus named *union additive relation* on Ω, and *union additive family* its corresponding version as subset family) is uniquely associated with a closure operator on Ω and vice versa. Hence the study of specific closure operators on Ω is equivalent to the study of corresponding types of union additive families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$. Then, we will show (Theorem 5.2.8) that there is a bijective correspondence between closure operators on Ω and union additive relations on $\mathcal{P}(\Omega)$. Moreover, with respect to this correspondence, the set operator $M_{\mathfrak{P}}$ can be identified with the union additive relation $\geq_{\mathfrak{P}}$. Let us note that the study of the preorder $\geq_{\mathfrak{P}}$ can also be seen as the abstract mathematical study of the *functional dependency relation* (for details see [161]) induced by any concrete model of data table associated with a pairing $\mathfrak{P}$ on the set Ω and that has recently been analyzed in relation to closure systems, abstract simplicial complexes [68] and to the corresponding convex geometries [111, 112, 113, 114].

Based on the previous considerations, given any family $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$, it is natural to ask how to characterize the smallest union additive family, denoted by $\mathcal{D}^+$, that contains $\mathcal{D}$ as a subfamily. To this regard, we provide three distinct recursive way to build $\mathcal{D}^+$ by starting from the elements of $\mathcal{D}$ (Theorem 5.3.1). Next, we address the links between order properties and subset structural properties from a perspective similar to that of model theory. In other terms, if $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$, we say that a pairing $\mathfrak{P} \in PAIR(\Omega)$ is a *model* of $\mathcal{D}$, denoted by $\mathfrak{P} \vdash \mathcal{D}$, if $X \geq_{\mathfrak{P}} Y$ for any ordered pair $(X, Y) \in \mathcal{D}$. At this point, we introduce a preorder on $\mathcal{P}(\mathcal{P}(\Omega)^2)$: if $\mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$ we say that the ordered pair $(\mathcal{D}, \mathcal{D}')$ is a *sequentiality* (relatively to the given set Ω, [66]), denoted by $\mathcal{D} \rightsquigarrow_{\Omega} \mathcal{D}'$, if for any pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $X \geq_{\mathfrak{P}} Y$ for all pairs $(X, Y) \in \mathcal{D}$ we also have that $X' \geq_{\mathfrak{P}} Y'$ for all pairs $(X', Y') \in \mathcal{D}'$. In this way we obtain a preorder

$\rightsquigarrow_\Omega$ between subset ordered pairs families of Ω that can be considered a global version of the preorder $\geq_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$. In fact, the preorder $\geq_{\mathfrak{P}}$ is related to the subsets of the starting set Ω, but it can be considered a type of *local* preorder because it depends on the choice of the particular model $\mathfrak{P}$. On the other hand, the preorder $\rightsquigarrow_\Omega$ depends only on the starting set Ω, therefore we can consider the study of $\rightsquigarrow_\Omega$ as an abstract global order study depending on the mutual interrelations between the preorders $\geq_{\mathfrak{P}}$, for each $\mathfrak{P} \in PAIR(\Omega)$.
In this way, we link order theory with specific subset properties (for example the above discussed union additivity). In such a perspective, we relate the preorder $\geq_{\mathfrak{P}}$ with the closure system $MAXP(\mathfrak{P})$ from an operatorial standpoint as follows. We consider the map $DP : \mathcal{P}(\mathcal{P}(\Omega)^2) \to \mathcal{P}(\mathcal{P}(\Omega))$ defined by

$$DP(\mathcal{D}) := \{Z \in \mathcal{P}(\Omega) : X \subseteq Z \implies Y \subseteq Z \ \forall (X,Y) \in \mathcal{D}\}$$

and we show that $DP(\mathcal{D}^+) = DP(\mathcal{D})$ for any $\mathcal{D} \subseteq \mathcal{P}(\mathcal{P}(\Omega)^2)$ (Theorem 5.5.1) and that $DP(\mathcal{G}(\mathfrak{P})) = MAXP(\mathfrak{P})$ (Theorem 5.5.2), where

$$\mathcal{G}(\mathfrak{P}) := \{(A,B) \in \mathcal{P}(\Omega)^2 : A \geq_{\mathfrak{P}} B\}.$$

Therefore it is natural to investigate the closure system $MAXP(\mathfrak{P})$ and its associated abstract complex $MINP(\mathfrak{P})$ from the perspective of the union additive relations.
Again by Theorem 5.5.2, it results that $DP(\mathcal{Q}(\mathfrak{P})) = MAXP(\mathfrak{P})$, where

$$\mathcal{Q}(\mathfrak{P}) := \{(X,y) : X \in MINP(\mathfrak{P}), y \in M_{\mathfrak{P}}(X) \setminus X\}.$$

In the pairing context, starting from the identities $DP(\mathcal{G}(\mathfrak{P})) = DP(\mathcal{Q}(\mathfrak{P})) = MAXP(\mathfrak{P})$, it can be seen that the study of specific subset pair families $\mathcal{D}$ is strictly related to the set systems and to the set operators characterizing pairings. Therefore, in this thesis, we feel the need to include as an additional part the investigation of the specific properties of such families, namely the *pointed pair system* on Ω [67], that is a family of ordered subset pairs whose second component is a singleton. In particular, in Subsections 5.6 and 5.7 we see that some classical notions that are usually investigated for several classes of set systems on Ω, such as, for instance, extensiveness and finitely inheritance, can be translated in the context of pointed pair systems and, furthermore, several links between specific classes of pointed pair systems, finitary abstract complexes (an example in pairing context has been given on the euclidean line, see Theorem 2.7.5) and weak types of MacLane-Steinitz set operators have been found.

Representation Theorems: Closure Systems, Finite Lattices and Matroids as Pairings

In this thesis we establish a representation result (Theorem 6.1.1), according to which it is possible to associate with any closure operator σ on an arbitrary finite set Ω a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $\sigma = M_{\mathfrak{P}}$. In other terms, for any closure operator σ on a finite set Ω we can find a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $\gamma_{\mathfrak{P}}$ uniquely characterize σ and vice versa. This equivalence implies that the symmetry transmission measure for pairings can provide to closure operator theory all possible concrete models of investigation. Then, as a consequence of such a result, we deduce some relevant facts concerning union additive relations, finite lattices and matroid theory. Firstly, we also show (Theorem 6.1.4) that *any* union additive relation on $\mathcal{P}(\Omega)$, when Ω is finite, is of the type $\geq_{\mathfrak{P}}$, for some pairing $\mathfrak{P} \in PAIR(\Omega)$. Hence, the study of a union additive relation becomes an abstract mathematical study of the functional dependency relation induced by any concrete model of data table associated with a pairing $\mathfrak{P}$ on Ω.

Secondly, in Theorem 6.1.5 and Theorem 6.1.7, we show that the partial order $\leq_{\mathbb{L}}$ of any finite lattice $\mathbb{L}$ can be described in terms of the symmetry transmission measure, for an appropriate finite set $\Omega_{\mathbb{L}}$ and a pairing $\mathfrak{P} \in PAIR(\Omega_{\mathbb{L}})$.
Moreover, the function $\gamma_{\mathfrak{P}}$ provides more information with respect to the partial order $\leq_{\mathbb{L}}$, because $\gamma_{\mathfrak{P}}$ can be computed on all pairs of $\mathcal{P}(\Omega_{\mathbb{L}}) \times \mathcal{P}(\Omega_{\mathbb{L}})$ that are in bijective correspondence with the pairs in $\mathbb{L} \times \mathbb{L}$, whereas the partial order $\leq_{\mathbb{L}}$ provides information only for the comparable pairs of $\mathbb{L} \times \mathbb{L}$. Hence the order relation of any finite lattice can be studied as a quotient relation of the preorder $\geq_{\mathfrak{P}}$ induced by some appropriate pairing. The theoretical consequence of this fact is that the whole finite lattice theory can be developed in a pairing perspective. To this regard, the real refinement that we can carry out in finite lattice theory by means of the study of the preorder $\geq_{\mathfrak{P}}$ is that we can connect order properties with set family properties. For example, the above described union additive property is explicitly a type of condition that naturally connects order theory with topological, matroidal or set combinatorial properties. More in general, it is natural to ask in which way a joint study of order properties and subset structural properties induced by pairings interact each other. We recall that there is a wide literature dealing with studies on the links between order theory, topological structures [95, 96, 138, 176], combinatorial and algebraic structures on subset families [1, 6, 78, 135, 177] and discrete dynamical systems [23, 24, 35].

Another consequence of Theorem 6.1.1 is that any independent set family of a matroid on Ω can be represented as the set system of the minimal partitioners of some pairing $\mathfrak{P}$ on Ω. Hence, we are enabled to investigate $MINP(\mathfrak{P})$ in relation to classical operators derived from matroid theory (see Theorems 7.0.14 and 7.0.16). To this regard, we relate $MINP(\mathfrak{P})$, the set operator $M_{\mathfrak{P}}$ and the classical rank-symmetry operator of a matroid. Finally (Theorem 7.0.18) we establish two equivalent conditions so that the rank-symmetry operator associated with $MINP(\mathfrak{P})$ coincides with $M_{\mathfrak{P}}$.
The epistemological consequence of the aforementioned results is that we can consider closure system, finite lattice and matroid theories as sub-theories that are parts of the more general pairing paradigm.

Applications in Computer Science

The notion of pairing generalizes the classical notion of Pawlak's *information table* [142, 144, 145, 146].

Definition 0.0.3. An *information table* is a table whose rows are labeled with the elements (called *objects*) of a finite set U and whose columns are labeled with the elements (called *attributes*) of a finite set Att and having at the entry (i, j) the value $F(u_i, a_j) \in Val$, where $F : U \times Att \to Val$ is the so-called *information map* and Val the set of *values*.

Remark 0.0.4. Our notion of pairing is actually a generalization of Pawlak's information table for two reasons. First of all, we can consider both U and Ω even infinite; on the other hand, Pawlak's notion of information table implicitily assumes a distinction between the nature of the objects and that of the attributes. In our case (see for example the pairing arising in graph context), we do not make this distinction.

Information tables arise from the necessity to study knowledge representation and data extraction. As a matter of fact, in database theory there is a very frequent need to study finite tables having a very large quantity of data, therefore many researches have been directed towards the purpose of reducing and simplifying the interpretation of these data. With such an aim, Pawlak [140, 141, 142, 144, 145, 146] developed the so called *rough set theory* (abbreviated RST). RST is an elegant and powerful methodology in extracting and minimizing rules from data tables. In 1979 the concept of *information granularity* was introduced by Zadeh [198] and it was related to the research on fuzzy sets. Next, the term *granular computing* (briefly GrC) was introduced again by Zadeh in 1997 (see [199]). Roughly speaking, information granules are collections of entities arranged together due to their similarity, functional or physical adjacency, indistinguishability, and so on. Since 1979, granular computing has become a very developed area of research in the scope of both applied and theoretical information science [148, 174]. GrC deals with representing and processing information in the form of some type of aggregates. These aggregates are generally called *information granules* or simply granules and they arise in the process of data abstraction and knowledge derivation from data. From a methodological perspective, GrC can be considered as an important attempt to investigate several research fields by means of the unifying granularity paradigm: rough set theory [17, 30, 74, 127, 128, 129, 144, 145, 146, 147, 148, 149, 151, 174, 190, 192, 194, 196] and its generalizations [28, 29, 31, 164, 166, 195], mathematical morphology [169], interval analysis [120], temporal dynamics [23, 24, 32, 33, 35, 73, 74], machine learning [197], preclusivity spaces [28], formal concept analysis [110, 172, 188], database theory [104, 105, 139], data mining [106, 125, 126, 191], fuzzy set theory [109, 150, 159, 199], interactive computing [162, 163], matroid theory [107, 108, 122, 123, 124, 200], hypergraph theory [34, 170, 171, 193], graph theory [39, 46, 47, 55, 156, 168], operative research [98], discrete dynamical systems [3, 4, 5, 21, 22, 40, 42, 43, 41, 44, 45, 143, 157].

The notion of A-symmetry represents the so-called *indiscernibility relation* [142]. The terminology indiscernibility relation derives by the classical Leibniz' indiscernibility principle, according to which two objects are indiscernible with respect to a given knowledge provided by a subset of attributes if and only if they share the same properties on these attributes. Formalizing, we obtain the following equivalence relation: let $A \in \mathcal{P}(Att)$ and $u, u' \in U$. Then

$$u \equiv_A u' :\Longleftrightarrow F(u,a) = F(u',a) \quad \forall a \in A.$$

Indistinguishability Linear Systems

In the last part of this thesis, we deal with *symmetry approximations.* Suppose we have a fixed set $Y \subseteq U$. We ask whether it is constituted of symmetry blocks with respect to a given

$A \in \mathcal{P}(\Omega)$ or it also contains part of a block. In the first case, we say that Y is A-*symmetry exact*, while in the second case, we can approximate it through two sets, namely the *lower symmetry approximation* and *upper symmetry approximation*. The lower symmetry approximation of a non-empty Y (briefly $\mathbf{l}_A(Y)$) represents the set of elements for which it can be said that they *surely* belong to Y in relation to the symmetry provided by A; the upper symmetry approximation of an element subset Y (briefly $\mathbf{u}_A(Y)$) for which it can be said that they *surely* or *possibly* belonging to Y, even in relation to the symmetry provided by A. Furthermore, it can be seen that Y is A-symmetry exact if $\mathbf{l}_A(Y) = \mathbf{u}_A(Y)$.
In particular, in Subsections 8.2.1 and 8.2.2 we compute A-lower and A-upper approximations for some basic graphs and digraphs families [47, 49, 54, 55, 52, 57, 60, 70, 72] and recall that the complete determination of A-lower and A-upper approximations represents a difficult task, even if the graphic structures at issue are very simple.
Some natural links of RST with both graph and hypergraph theory have recently been founded. In fact, in [47, 48, 49, 50, 51] the idea to study any simple undirected graph G as if it were a pairing was developed (in [34] this idea has been also extended to hypergraph theory). The basic tool to connect graphs and Boolean pairings is the adjacency matrix of G, which in [47, 48, 49, 50, 51] has been interpreted as the Boolean table of a particular information system. In particular, it can be shown that any Boolean pairing is induced from a bipartite graph [56].

Nevertheless, for finite sets it has been shown that the A-symmetry exactness of a subset $X \subseteq U$ is related to the value of the symmetry transmission measure $\gamma(A, \{d^X\})$, where d^X is a special element depending on A.

The choice of a special element allows us to introduce a new class of structures, namely the *indistinguishability linear systems* [62], that can be seen as a generalization of the so-called *decision tables* [69, 134, 144, 145, 146].

We call a structure $\mathcal{S} = \langle U_\mathcal{S}, C_\mathcal{S}, D_\mathcal{S}, F_\mathcal{S}, \Lambda_\mathcal{S}\rangle$, where $U_\mathcal{S}, C_\mathcal{S}, \Lambda_\mathcal{S}$ are non-empty sets, $D_\mathcal{S}$ is empty or consists of a single element $d_\mathcal{S}$ such that $d_\mathcal{S} \notin C_\mathcal{S}$. $F_\mathcal{S}$ is a map having domain $U_\mathcal{S} \times (C_\mathcal{S} \cup D_\mathcal{S})$ and codomain $\Lambda_\mathcal{S}$ an *indistinguishability linear system*.

In the last part of this work, we deal with the investigation of the main properties of indistinguishability linear systems. Indeed, we will study from an operatorial standpoint the notion of *compatibility* for indistinguishability linear systems.

Let us consider a system of linear equations $\mathcal{S}$ with coefficients in some field $\mathbb{K}_\mathcal{S}$. Let $U_\mathcal{S} := \{u_1, \ldots, u_m\}$ be the set of the m linear equations $u_1, \ldots, u_m$ and $C_\mathcal{S} = \{x_1, \ldots, x_n\}$ the set of the n unknowns $x_1, \ldots, x_n$ of $\mathcal{S}$. Let $Q_\mathcal{S} = (q_{ij})$ the coefficient matrix and $d_\mathcal{S} = (d_1, \ldots, d_m)^T$ the column of the constant terms of $\mathcal{S}$. If $W \subseteq U_\mathcal{S}$ and $A \subseteq \Omega_\mathcal{S}$, we denote by $\mathcal{S}_{W,A}$ the sub-system of $\mathcal{S}$ determined by the equations contained in W, with their corresponding constant terms, and by the unknowns contained in A. Then, it is well known that the compatibility of $\mathcal{S}$ do not imply the compatibility of $\mathcal{S}_{W,A}$. In order to know the compatibility of any single sub-system $\mathcal{S}_{W,A}$, it is sufficient to find the rank of the matrix and of the complete matrix associated with this sub-system, but what can we say about the *interrelations* between the compatibilities among all possible local sub-systems of $\mathcal{S}$? In Subsection 8.4, we reformulate the above problem in a combinatorial context. We say that $\mathcal{S}_{W,A}$ is *compatible* if there exists a map $f : \mathbb{K}^p \to \mathbb{K}$ such that $F_\mathcal{S}(u, d_\mathcal{S}) = f(F_\mathcal{S}(u, x_{i_1}), \ldots, F_\mathcal{S}(u, x_{i_p}))$ for any $u \in W$.

Hence, we propose a detailed investigation of the previous condition as a natural study of the notion of *local compatibility* for an indistinguishability linear system. We will see that the previous condition is logically equivalent to another condition relating an appropriate family of partitions of $U_{\mathcal{S}}$ induced by the subsets of $C_{\mathcal{S}}$ to the family of partitions induced by $d_{\mathcal{S}}$. Since set partitions are essential for compatibility, we can interpret it as a kind of *partition compatibility*. In fact, the aforementioned partitions are induced by the A-symmetry partition on the equation set $U_{\mathcal{S}}$. Then, it is easy to verify that the above condition is equivalent to the following inclusion:

$$[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$$

for any $u \in W$. Therefore the investigation of the compatibility condition for $\mathcal{S}_{W,A}$ ultimately becomes an analysis of the combinatorial properties of the partitions $\pi_{\mathcal{S}}(A)$ of $U_{\mathcal{S}}$ induced by $\equiv_A$.

The study of compatibility has been undertaken by defining some operators that we call *compatibility operators* (see Subsection 8.5). In particular, given $W \in \mathcal{P}(U_{\mathcal{S}})$ and $A \in \mathcal{P}(C_{\mathcal{S}})$, we set $\phi_A(X) = \Theta_{\mathcal{S}}(X, A)$, where

$$\Theta_{\mathcal{S}}(X, Y) := \{v \in X : [v]_Y \cap X \subseteq [v]_{D_{\mathcal{S}}} \cap X\}.$$

Then, if we fix a variable set A, it follows that $\phi_A(U_{\mathcal{S}}) = U_{\mathcal{S}}$, that is A gives rise to compatibility on $U_{\mathcal{S}}$ if and only if ϕ_A is a *kernel operator* [86]. In this way, we have that specific technical properties of the set operator at issue assumes a relevant interpretative role.

Chapter 1

Preliminaries

In this chapter we introduce notations we will use within the text and show some basic results.

1.1 Notations

In what follows, Ω will be an arbitrary (even infinite) set and $\mathcal{P}(\Omega)$ denotes its power set. We also set $SS(\Omega) := \mathcal{P}(\mathcal{P}(\Omega))$, $\mathbb{S}(\Omega) := (SS(\Omega), \subseteq)$. If we want to specify that Y is a finite subset of Ω, we will write $Y \subseteq_f \Omega$; moreover if n is a non-negative integer, we write $Y \subseteq_{\leq n} \Omega$ if $Y \subseteq \Omega$ and $|Y| \leq n$. If $|\Omega| = n$, we set $\Omega_n := \Omega$. If k is a positive integer, $Y \subseteq \Omega$ and $|Y| = k$, we say that Y is a *k-subset* of Ω. We write $\hat{n}$ to denote the set $\{1, \ldots, n\}$. We will denote by I_n the $n \times n$ identity matrix and by J_n the $n \times n$ matrix having 1 in all its entries. If X and Ω are two sets and we have two maps $\psi : X \to X$ and $f : X \to \mathcal{P}(\Omega)$ such that $f(\psi(x))) = \Omega \setminus f(x)$ for any $x \in X$, we say that f is a *ψ-complementary map.*

1.2 Posets and Lattices

A *partially ordered set* (abbreviated *poset*) is a pair $P = (\Omega, \leq)$, where Ω is a set and $\leq$ is a binary, reflexive, antisymmetric and transitive relation on Ω. If $P = (\Omega, \leq)$ is a partially ordered set and $x, y \in \Omega$, we also write $x < y$ if $x \leq y$ and $x \neq y$. If x, y are two distinct elements of Ω, we say that y *covers* x, denoted by $x \lessdot y$ if $x \leq y$ and there exists no element $z \in \Omega$ such that $x < z < y$. Moreover, we set $[x|P \uparrow] := \{z \in \Omega : x \lessdot z\}$ and call *covers* of x all its elements. Analogously, we set $[x|P \downarrow] := \{y \in \Omega : y \lessdot x\}$ and call *co-covers* of x its elements. We also set $\mathcal{L}_{\downarrow}(P) = \{x \in P : |[x|P \downarrow]| \leq 1\}$ and $\mathcal{L}_{\uparrow}(P) = \{x \in P : |[x|P \uparrow]| \leq 1\}$. Through the covering relation, it is possible to represent the so called *Hasse diagram* of P (see [18]): draw a small circle for any element of P and a segment connecting x to y whenever x covers y. An element $x \in \Omega$ is called *minimal* in P if $z \leq x$ implies $z = x$, and in a similar way one defines a *maximal* element in P. If there is an element $\hat{0}_\Omega \in \Omega$ such that $\hat{0}_\Omega \leq x$ then $\hat{0}_\Omega$ is unique and it is called the *minimum* of P. Analogously the *maximum* of P, usually denoted by $\hat{1}_\Omega$ is defined (if it exists). We call *upper bound* of a subset Ω of P an element $y \in P$ such that $x \leq y$ for any $x \in \Omega$. We call *least upper bound* the minimum of all the upper bounds. The notions of *lower bound* and *greatest lower bound* are dual. We call *lattice* a poset any two

of whose elements has both the least upper bound that the greatest lower bound. A lattice is *complete* when each of its subsets Ω has a least upper bound and greatest lower bound in the lattice (however a finite lattice is always complete). We say that a subset $I \subseteq \Omega$ is an ideal of the lattice $\mathbb{P} = (\Omega, \leq)$ if the following conditions holds:
(a) $I \neq \emptyset$;
(b) for any $x \in I$ and $y \leq x$, it follows that $y \in I$;
(c) for any $x, y \in I$ there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

A *chain* C of P is a subset $C \subset \Omega$ such that for all $x, y \in C$ we have $x \leq y$ or $y \leq x$. If a chain C has $n+1$ elements $x_0, \dots, x_n$ such that $x_0 < \cdots < x_n$, it is said that C is an $n+1$-*chain* and has *length* n; in this case often we write $C = \{x_0 < \cdots < x_n\}$. If Ω itself is a chain the poset P is said *linearly ordered.* If P is not linearly ordered, a *maximal chain* of P is a chain C of P which is not properly contained in any other chain of P.

A map $f : \mathbb{L} \to \mathbb{L}$ is said:

- *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$;
- *involution* if $f(f(x)) = x$.

Let $\mathbb{L}$ be a lattice endowed with an order-preserving involution $\psi : \mathbb{L} \to \mathbb{L}$ such that $\psi(\alpha) \wedge \alpha = \hat{0}_{\mathbb{L}}$ and $\psi(\alpha) \vee \alpha = \hat{1}_{\mathbb{L}}$. In this case we say that the pair $(\mathbb{L}, \psi)$ is a *complementary involution lattice.*

Two posets $P = (\Omega_1, \leq_1)$ and $P_2 = (\Omega_2, \leq_2)$ are said *isomorphic* if there exists a bijective map $\phi : \Omega_1 \to \Omega_2$ such that $x \leq_1 y \iff \phi(x) \leq_2 \phi(y)$, for all $x, y \in \Omega_1$. The *dual poset* of P is the poset $P^* := (\Omega, \leq^*)$, where $\leq^*$ is the partial order on Ω defined by $x \leq^* y :\iff y \leq x$, for all $x, y \in \Omega$. A poset P is called *self-dual* if there exists a bijection between P and P^*.

Let $\mathbb{P} := (\Omega, \leq)$ and $\mathbb{P}' := (\Omega', \leq')$ be two posets, $\alpha, \beta \in \Omega$ and $f : \Omega \to \Omega'$ a map. We say that f is $\{\alpha, \beta\}$-*preserving* if the restriction map $f_{|_{\{\alpha,\beta\}}} : \{\alpha, \beta\} \to \{f(\alpha), f(\beta)\}$ is an order isomorphism.

A poset $P = (\Omega, \leq)$ is said *graded of rank* l if all the maximal chains in P have length l, and in this case l is called *rank* of P.

Let $F_n = \{x_1, \dots, x_n\}$ with $x_1 > x_2 < x_3 > \cdots x_n$ (and no other comparabilities). A poset $P = (\Omega, \leq)$ which is isomorphic to F_n or to its dual F_n^* is called a *n-fence* (see [77]).

Let $C_n = \{x_1, y_1, \dots, x_n, y_n\}$ with $x_1 < y_1 > x_2 < y_2 > x_3 < \cdots < y_{n-1} > x_n < y_n > x_1$ (and no other comparabilities). A poset $P = (\Omega, \leq)$ which is isomorphic to C_n or C_n^* is called a *n-crown* (see [178]).

1.3 Set Partitions

A *set-partition* π on Ω is a collection of non-empty subsets $\{B_i : i \in I\}$ of Ω such that $B_i \cap B_j = \emptyset$ for all $i \neq j$ and such that $\bigcup_{i \in I} B_i = \Omega$. The subsets B_i are called *blocks* of π and, if I is finite, we write $\pi := B_1 | \dots | B_{|I|}$ to denote that π is a set partition having blocks $B_1, \dots, B_{|I|}$. If $u \in \Omega$, we denote by $\pi(u)$ the block of π which contains the element u. We use the standard notation $\pi := B_1 | \dots | B_{|\pi|}$, where $|\pi|$ is the number of distinct blocks of π. We denote by $\pi_{\mathfrak{P}}(\Omega)$

the set of all set-partitions of Ω. It is well known that on the set $\pi(\Omega)$ we can consider a partial order $\preceq$ defined as follows: if $\pi, \pi' \in \pi(\Omega)$, then

$$\pi \preceq \pi' :\Longleftrightarrow (\forall B \in \pi)\,(\exists B' \in \pi')\ :\ B \subseteq B' \iff (\forall u \in \Omega)\,(\pi(u) \subseteq \pi'(u)). \tag{1.1}$$

We will write $\pi \prec \pi'$ when $\pi \preceq \pi'$ and $\pi \neq \pi'$. The pair $(\pi_{\mathfrak{P}}(\Omega), \preceq)$ is a complete lattice which is called *partition lattice* of the set Ω.

Let $\pi_1 = A_1|\dots|A_m$ and $\pi_2 = B_1|\dots|B_n$ be two partitions on the same finite universe Ω, i.e., $\pi_1,\ \pi_2 \in \pi_{\mathfrak{P}}(\Omega)$, we firstly set

$$\mathcal{S}_{\pi_1, \pi_2} := \{C \subseteq \Omega :\ \text{ if } x \in C,\ \text{ then } \pi_1(x) \subseteq C \text{ and } \pi_2(x) \subseteq C\}$$

Then, the meet of π_1 and π_2, denoted by $\pi_1 \wedge \pi_2$, is the set partition of Ω whose blocks are given by

$$\pi_1 \wedge \pi_2 := \{A_i \cap B_j : i = 1, \dots m; j = 1, \dots, n\}. \tag{1.2}$$

On the other hand, the more simple way to describe the join of π_1 and π_2, denoted by $\pi_1 \vee \pi_2$, is the following:

$$\pi_1 \vee \pi_2 := C_1|\dots|C_k, \tag{1.3}$$

where $C_1, \dots, C_k$ are the minimal elements of the set family $\mathcal{S}_{\pi_1, \pi_2}$ with respect to the inclusion.

Example 1.3.1. *Let* $\Omega = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6\}$, $\pi_1 = \{x_1,\ x_2\}|\{x_3\}|\{x_4,\ x_5\}|\{x_6\}$ *and* $\pi_2 = \{x_1,\ x_3\}|\{x_2\}|\{x_4\}|\{x_5\}|\{x_6\}$ *be two set partitions of* Ω*. Then,* $\pi_1 \wedge \pi_2$ *is the partition*

$$\pi_1 \wedge \pi_2 = \{x_1\}|\{x_2\}|\{x_3\}|\{x_4\}|\{x_5\}|\{x_6\}.$$

The family $\mathcal{S}_{\pi_1, \pi_2}$ *is equal to:*

$$\mathcal{S}_{\pi_1, \pi_2} = \{\{x_1,\ x_2,\ x_3\},\ \{x_1,\ x_2,\ x_3,\ x_4,\ x_5\},\ \{x_1,\ x_2,\ x_3,\ x_6\},\ \{x_4,\ x_5\},\ \{x_4,\ x_5,\ x_6\},\ \{x_6\}\}.$$

Therefore, the meet of π_1 *and* π_2 *is the partition*

$$\pi_1 \vee \pi_2 = \{x_1,\ x_2,\ x_3\}|\{x_4,\ x_5\}|\{x_6\}.$$

1.4 Set Systems

We call an element $\mathcal{F} \in SS(\Omega)$ a *set system* on Ω. We call a set system family $\mathfrak{F} \subseteq SS(\Omega)$ a 2-*set system* on Ω. If $\mathfrak{F}$ is a 2-set system on Ω, we consider it as sub-posets of $\mathbb{S}(\Omega)$.

A hypergraph is a pair $H = (\Omega, \mathcal{F})$, where $\Omega = \{x_1, \dots, x_n\}$ is a finite set (called vertex set of H) and $\mathcal{F} = \{Y_1, \dots, Y_m\}$ is a non-empty family of subsets $Y_1, \dots, Y_m$ of Ω. The elements $x_1, \dots, x_n$ are called vertices of H and the subsets $Y_1, \dots, Y_m$ of Ω are called hyperedges of H.

If $\mathcal{F} \in SS(\Omega)$ we say that:

- $\mathcal{F}$ has *uniform cardinality* if each $X \in \mathcal{F}$ has the same cardinality. We denote by $||\mathcal{F}||$ the cardinality of each $X \in \mathcal{F}$.

- $\mathcal{F}$ is *k-uniform* if it is the family of all k-subsets of Ω. In this case, we denote it by $\binom{\Omega}{k}$.
- $\mathcal{F}$ is *union-closed* if whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cup\mathcal{F}' \in \mathcal{F}$. We denote by $UCL(\Omega)$ the collection of all union-closed set systems on Ω.
- $\mathcal{F}$ is *chain union-closed* if whenever $\mathcal{F}' \subseteq \mathcal{F}$ is a chain, then $\cup\mathcal{F}' \in \mathcal{F}$.
- $\mathcal{F}$ is a *complement-closed family* if $\Omega \in \mathcal{F}$ and $Z \in \mathcal{F}$ implies $Z^c \in \mathcal{F}$.
- $\mathcal{F}$ is an *abstract simplicial complex* (or simply an *abstract complex*) on Ω if $\emptyset \in \mathcal{F}$ and whenever $Y \in \mathcal{F}$ and $Z \subseteq Y$, then $Z \in \mathcal{F}$. We denote by $ASC(\Omega)$ the 2-set system on Ω whose elements are all abstract complexes on Ω.
- $\mathcal{F}$ is *finitary* if whenever $X \in \mathcal{P}(\Omega)$ and $F \in \mathcal{F}$ for any $F \subseteq_f X$, then $X \in \mathcal{F}$;
- a *finitary abstract complex* on Ω if $\mathcal{F}$ is a finitary set system and also an abstract complex on Ω. We denote by $FAC(\Omega)$ the 2-set system on Ω whose elements are all finitary abstract complexes on Ω.
- $\mathcal{F}$ is a *closure system* on Ω if $\Omega \in \mathcal{F}$ and whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cap\mathcal{F}' \in \mathcal{F}$. We denote by $CLSY(\Omega)$ the 2-set system on Ω whose elements are all closure systems on Ω.

We say that a subset $Y \subseteq \Omega$ is a *transversal* of a set system $\mathcal{F} \in SS(\Omega)$ if $Y \cap A \neq \emptyset$ for each non-empty $A \in \mathcal{F}$. Moreover, a transversal A of $\mathcal{F}$ is *minimal* if no proper subset of A is a transversal of $\mathcal{F}$. We denote by $Tr(\mathcal{F})$ the family of all minimal transversals of $\mathcal{F}$. We denote by $\max(\mathcal{F})$ the set system of all maximal members of $\mathcal{F}$ and by $\min(\mathcal{F})$ the set system of all minimal members of $\mathcal{F}$. If $A \in \mathcal{P}(\Omega)$, we set $\max(\mathcal{F}|A) := \max\{B \in \mathcal{P}(A) : B \in \mathcal{F}\}$ and $\min(\mathcal{F}|A) := \min\{B \in \mathcal{P}(A) : B \in \mathcal{F}\}$, so that $\max(\mathcal{F}|\Omega) = \max(\mathcal{F})$ and $\min(\mathcal{F}|\Omega) = \min(\mathcal{F})$.

We prove now that the 2-set system of all finitary abstract complexes on Ω is a complete lattice.

Theorem 1.4.1. *$FAC(\Omega)$ is a complete lattice such that if $\mathfrak{F} = \{\mathcal{F}_i\}_{i\in I} \subseteq FAC(\Omega)$ then $\bigwedge_{i\in I} \mathcal{F}_i = \bigcap_{i\in I} \mathcal{F}_i$. Moreover, if I is finite we also have $\bigvee_{i\in I} \mathcal{F}_i = \bigcup_{i\in I} \mathcal{F}_i$.*

Proof. We firstly show that given a collection $\{\mathcal{F}_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ of finitary abstract simplicial complexes on Ω, then $\bigcap_{i\in I} \mathcal{F}_i$ is a finitary abstract simplicial complex on Ω. For, let $X \in \mathcal{P}(\Omega)$. Suppose that $F \in \bigcap_{i\in I} \mathcal{F}_i$ for any $F \subseteq_f X$. Thus, $F \in \mathcal{F}_i$ for any $i \in I$ and, since each set system $\mathcal{F}_i$ is finitary, it results that $X \in \mathcal{F}_i$ for each index $i \in I$. We conclude that $X \in \bigcap_{i\in I} \mathcal{F}_i$, i.e. $\bigcap_{i\in I} \mathcal{F}_i$ is finitary.
On the other hand, we clearly have that $\emptyset \in \bigcap_{i\in I} \mathcal{F}_i$. Furthermore, let $Y \in \bigcap_{i\in I} \mathcal{F}_i$ and $Z \subseteq Y$. Since $Y \in \mathcal{F}_i$ for any $i \in I$, we conclude that $Z \in \mathcal{F}_i$ for each index $i \in I$, so $Z \in \bigcap_{i\in I} \mathcal{F}_i$. This shows that $\bigcap_{i\in I} \mathcal{F}_i$ is also an abstract simplicial complex, thus the claim has been shown.
Let $\mathcal{F}$ be such that $\mathcal{F} \subseteq \mathcal{F}_i$ for any $i \in I$. Then, $\mathcal{F} \subseteq \bigcap_{i\in I} \mathcal{F}_i$, so $\bigwedge_{i\in I} \mathcal{F}_i = \bigcap_{i\in I} \mathcal{F}_i$.

We now show the existence of the join for an arbitrary collection $\{\mathcal{F}_i : i \in I\} \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega) \in FAC(\Omega)$, the family of all upper bounds $\mathcal{U}$ of $\{\mathcal{F}_i : i \in I\}$ is non-empty. Let us take $\bigcap \mathcal{U}$. By the previous part, it is a finitary abstract simplicial complex on Ω; clearly, it belongs to $\mathcal{U}$ and is minimal in $\mathcal{U}$. This proves that $\bigvee\limits_{i \in I} \mathcal{F}_i = \bigcap \mathcal{U}$.
To conclude the proof, we note that the arbitrary union of abstract complexes is also an abstract complex.
On the other hand, given a finite collection $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ of finitary abstract simplicial complexes on Ω, we have that $\bigcup\limits_{i=1}^{n} \mathcal{F}_i$ is finitary. In fact, let $X \in \mathcal{P}(\Omega)$ such that $F \in \bigcup\limits_{i=1}^{n} \mathcal{F}_i$ for any $F \subseteq_f X$. Assume by contradiction that $X \notin \bigcup\limits_{i=1}^{n} \mathcal{F}_i$. Then, there exists $F_i \subseteq_f X$ such that $F_i \notin \mathcal{F}_i$ for any $i = 1, \ldots, n$. Let us take $F := \bigcup\limits_{i=1}^{n} F_i$. Thus, $F \subseteq_f X$, so there exists $j \in \{1, \ldots, n\}$ such that $F \in \mathcal{F}_j$ and, in particular, $F_j \in \mathcal{F}_j$, contradicting our choice of F_j. This shows our claim. □

Remark 1.4.2. When I is not a finite set, it turns out to be false in general that $\bigcup\limits_{i \in I} \mathcal{F}_i$ is a finitary abstract complex on Ω. As an example, just consider the 2-set system $\mathfrak{F} = \{\mathcal{F}_i : i \in I\}$, where $\mathcal{F}_i = \{F : F \subseteq_{\leq i} \Omega\} \cup \emptyset\}$.

1.5 Set Operators

A *set operator* on Ω is a map $\sigma : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$. We denote by $OP(\Omega)$ the set of all the set operators on Ω and we consider on $OP(\Omega)$ the partial order $\sqsubseteq$ defined by

$$\sigma \sqsubseteq \sigma' :\iff \sigma(X) \subseteq \sigma'(X) \quad \forall X \in \mathcal{P}(\Omega), \tag{1.4}$$

for any $\sigma, \sigma' \in OP(\Omega)$.

We will consider the maps $Fix : OP(\Omega) \to SS(\Omega)$ and $Int, Drk, Imx : SS(\Omega) \to OP(\Omega)$ defined respectively by

$$Fix(\sigma) := \{A \in \mathcal{P}(\Omega) : A \in \mathcal{P}(\Omega)\},$$
$$Int_{\mathcal{F}}(C) := \bigcap \{A \in \mathcal{F} : C \subseteq A\},$$
$$Drk_{\mathcal{F}}(C) := \{b \in \Omega : rk_{\mathcal{F}}(C \cup \{b\}) = rk_{\mathcal{F}}(C)\},$$
$$Imx_{\mathcal{F}}(C) := \bigcap \{A \in \max(\mathcal{F}|C)\},$$

for any $\sigma \in OP(\Omega)$, $\mathcal{F} \in SS(\Omega)$, $C \in \mathcal{P}(\Omega)$.

We call *fixed points* of σ the members of $Fix(\sigma)$, *intersecting operator* of $\mathcal{F}$ the set operator $Int_{\mathcal{F}}$, *rank-symmetry operator* of $\mathcal{F}$ the set operator $Drk_{\mathcal{F}}$ and *intersecting maximal operator* of $\mathcal{F}$ the set operator $Imx_{\mathcal{F}}$.

We consider now the operator $\Theta : OP(\Omega) \to SS(\Omega)$ defined by

$$\Theta(\sigma) := \{X \in \mathcal{P}(\Omega) : A \subsetneqq X \implies X \not\subseteq \sigma(A)\}, \tag{1.5}$$

for any $\sigma \in OP(\Omega)$. We call Θ the *minimality operator* on Ω. The following result easily follows.

Proposition 1.5.1. *(i) $(OP(\Omega), \sqsubseteq)$ is a complete lattice. In particular, given $\{\sigma_i :\in I\} \subseteq OP(\Omega)$, it results that $\bigvee_{i \in I} \sigma_i = \sigma^+$ and $\bigwedge_{i \in I} \sigma_i = \sigma^-$, where, respectively, $\sigma^+(X) := \bigcup_{i \in I} \sigma_i(X)$ and $\sigma^-(X) := \bigcap_{i \in I} \sigma_i(X)$ for any $X \in \mathcal{P}(\Omega)$.*
(ii) If $\sigma, \sigma' \in OP(\Omega)$ and $\sigma \sqsubseteq \sigma'$ then $\Theta(\sigma) \supseteq \Theta(\sigma')$.
(iii) Let $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{P}(\mathcal{P}(\Omega) \times \Omega)$ such that $\mathcal{D} \subseteq \mathcal{D}'$. Then, $\sigma_{\mathcal{D}} \sqsubseteq \sigma'_{\mathcal{D}}$.

Let $\sigma \in OP(\Omega)$. We say that:

- σ is *extensive* if $X \subseteq \sigma(X)$;
- σ is *intensive* if $\sigma(X) \subseteq X$;
- σ is *monotone* if $X \subseteq Y$ implies $\sigma(X) \subseteq \sigma(Y)$;
- σ is a *pre-closure operator* on Ω if it is extensive and monotone;
- σ is *idempotent* if $\sigma(\sigma(X)) = X$;
- σ is a *closure operator* on Ω if it an idempotent pre-closure operator. We denote by $CLOP(\Omega)$ the set of all closure operators on Ω;
- σ is a *kernel operator* [86] if it is an intensive, monotone and idempotent set operator;
- σ is *pseudo-monotone* if, whenever $A \subseteq B$, then $A \cap \sigma(B) \subseteq \sigma(A)$;
- σ is a *simplicial operator* if it is intensive and pseudo-monotone. We denote by $SOP(\Omega)$ the family of all simplicial operators on Ω;
- σ is *quasi-regular* if $b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))$ and $c \in \sigma(A \cup \{c\}) \setminus A$, then $c \in \sigma(A \cup \{b, c\})$;
- σ is a *core operator* if it is a simplicial quasi-regular operator. We denote by $COOP(\Omega)$ the family of all core operators on Ω;
- σ is *finitary*, if $\sigma(X) = \bigcup\{\sigma(Y) : Y \subseteq_f X\}$;
- σ is a *weak MacLane-Steinitz operator* (briefly σ is WMLS), if for any $A \in \Theta(\sigma)$ and any $x, y \in \Omega$ such that $x \in \sigma(A \cup \{y\}) \setminus \sigma(A)$, we have that $y \in \sigma(A \cup \{x\})$;
- σ is a *MacLane-Steinitz operator* (briefly σ is WMLS), if for any $A \in \mathcal{P}(\Omega)$ and any $x, y \in \Omega$ such that $x \in \sigma(A \cup \{y\}) \setminus \sigma(A)$, we have that $y \in \sigma(A \cup \{x\})$.

When the set operator σ is extensive, we have that $\Theta(\sigma)$ is an abstract simplicial complex.

Proposition 1.5.2. *If $\sigma \in OP(\Omega)$ is extensive, then $\Theta(\sigma) \in ASC(\Omega)$.*

Proof. Clearly, $\emptyset \in \Theta(\sigma)$. Let $X \in \Theta(\sigma)$ and $Y \subseteq X$. If were $Y \notin \Theta(\sigma)$, there would be some $y \in Y$ belonging to $\sigma(Y \setminus \{y\})$. In particular, by extensiveness, it must be $y \in \sigma(X \setminus \{y\})$, contradicting the fact that $X \in \Theta(\sigma)$. $\square$

In the following result we establish the basic links between abstract simplicial complexes and simplicial operators.

Theorem 1.5.3. *Let $\sigma \in SOP(\Omega)$ and $\mathcal{F} \in ASC(\Omega)$. Then, the following hold:*

(*i*) $Fix(\sigma) \in ASC(\Omega)$;

(*ii*) $Fix(Imx_{\mathcal{F}}) = \mathcal{F}$;

(*iii*) *if* $\mathcal{F} \in ASC(\Omega) \cap UCL(\Omega)$, *then* $Imx_{\mathcal{F}} \in SOP(\Omega)$ *and* $Imx_{\mathcal{F}}$ *is idempotent.*

Proof. (i) : Clearly, $\emptyset \in Fix(\sigma)$. On the other hand, let $B \in Fix(\sigma)$ and $A \subseteq B$. By intensivity, it follows that $\sigma(A) \subseteq A$. On the other hand, by pseudo-monotonicity it follows that $\sigma(A) \supseteq \sigma(B) \cap A = B \cap A = A$, so $A = \sigma(A)$, i.e. $A \in Fix(\sigma)$.
(ii) : Just observe that $Imx_{\mathcal{F}}(A) = A$ if and only if $A \in \mathcal{F}$.
(iii) : Clearly, $Imx_{\mathcal{F}}(A) \subseteq A$ for any $A \in \mathcal{P}(\Omega)$. Assume now that $A \subseteq B$ and let $x \in A \cap Imx_{\mathcal{F}}(B)$. Then, $x \in A \cap C$ for any $C \in \max(\mathcal{F}|B)$. Let us prove that there exists $D \in \max(\mathcal{F}|A)$ such that $x \in D$. For, consider the set system $\mathcal{G} := \{Z \in \mathcal{F} : A \cap C \subseteq D \subseteq A\}$. It is clearly non-empty, since $A \cap C \in \mathcal{G}$. Take a chain $\{Z_i : i \in I\} \subseteq \mathcal{G}$ e consider $Z := \bigcup_{i \in I} Z_i$. By the fact that $\mathcal{F}$ is union-closed, it follows that $Z \in \mathcal{F}$. Moreover, it is obvious that $A \cap C \subseteq Z \subseteq A$. Hence, we can apply Zorn's Lemma and prove that there exists an element $D \in \max \mathcal{G}$. Finally, it is also clear that $x \in D$ and it is straightforward to see that $D \in \max(\mathcal{F}|A)$.
We now show that $x \in E$ for any $E \in \max(\mathcal{F}|A)$. To this regard, assume by contradiction the existence of some $E' \in \max(\mathcal{F}|A)$ for which $x \notin E'$. Let us prove that $E' \subseteq C'$ for some $C' \in \max(\mathcal{F}|B)$. To this regard, set $\mathcal{G}' := \{Y \in \mathcal{F} : E' \subseteq Y \subseteq B\}$. It is clearly non-empty since $E' \in \mathcal{G}'$. So, take a chain $\{Y_i : i \in I\} \subseteq \mathcal{G}'$ and set $Y := \bigcup_{i \in I} Y_i$. It is immediate to see that $E' \subseteq Y \subseteq B$ and $Y \in \mathcal{F}$. By Zorn's Lemma, $\mathcal{G}'$ admits a maximal element C', that clearly belongs to $\max(\mathcal{F}|B)$.
Now, we have that $x \notin C'$, otherwise $C' \cap A \supsetneqq E'$ and $C' \cap A \in \max(\mathcal{F}|A)$, contradicting the maximality of E'. But x must belong to each subset of $\max(\mathcal{F}|B)$, so we reach to an absurd. This proves pseudo-monotonicity.
Finally, in order to prove that $Imx_{\mathcal{F}}$ is idempotent, just observe that $Imx_{\mathcal{F}}(A) \in \mathcal{F}$ so, by previous part (ii), we have $Imx_{\mathcal{F}}(Imx_{\mathcal{F}}(A)) = Imx_{\mathcal{F}}(A)$ □

Remark 1.5.4. It is easy to show that (iii) of Theorem 1.5.3 holds when Ω is a finite set, without assuming that $\mathcal{F} \in UCL(\Omega)$.

We now provide a characterization for $\Theta(\sigma)$ when $\sigma \in CLOP(\Omega)$.

Proposition 1.5.5. *Let $\sigma \in CLOP(\Omega)$. Then:*

$$\Theta(\sigma) = \{A \in \mathcal{P}(\Omega) : x \notin \sigma(A \setminus \{x\}) \ \forall x \in A\}. \tag{1.6}$$

Proof. Let $A \in \Theta(\sigma)$ and $x \in A$. Then, $A \setminus \{x\} \subseteq A$, therefore $\sigma(A \setminus \{x\}) \subseteq \sigma(A)$. If $x \in \sigma(A \setminus \{x\})$, we would have $A \subseteq \sigma(A \setminus \{x\})$, i.e. $\sigma(A) = \sigma(A \setminus \{x\})$, that is a contradiction. Vice versa, let $A \in \mathcal{P}(\Omega)$ such that $x \notin \sigma(A \setminus \{x\})$ for any $x \in A$. Let $B \subseteq A \setminus \{x\} \subsetneqq A$ for some $x \in A$. This obviously implies that $x \notin \sigma(B)$, i.e. $\sigma(A) \nsubseteq \sigma(B)$. □

In the next result, we characterize the condition for a closure operator to be WMLS.

Theorem 1.5.6. *Let $\sigma \in CLOP(\Omega)$. The following conditions are equivalent:*

(i) *σ is WMLS;*

(ii) *if $X \in \mathcal{P}(\Omega)$, $y \in \Omega$ and $x \in \sigma(X \cup \{y\}) \setminus \sigma(X)$ are such that $X \cup \{y\} \in \Theta(\sigma)$, then $y \in \sigma(X \cup \{x\})$;*

(iii) *for any $X \in \Theta(\sigma)$ and any $x \in \Omega \setminus \sigma(X)$, $X \cup \{x\} \in \Theta(\sigma)$;*

(iv) *if $X \in \mathcal{P}(\Omega)$ and $Y \in \max(\Theta(\sigma)|X)$, then $\sigma(Y) = \sigma(X)$.*

Proof. *(i)* $\implies$ *(ii)*: Let $X \in \mathcal{P}(\Omega)$, $y \in \Omega$ and $x \in \sigma(X \cup \{y\}) \setminus \sigma(X)$ be such that $X \cup \{y\} \in \Theta(\sigma)$. By Proposition 1.5.2, $X \in \Theta(\Omega)$, therefore by the fact that σ is WMLS, it follows that $y \in \sigma(X \cup \{x\})$.
(ii) $\implies$ *(iii)*: Let $X \in \Theta(\sigma)$, $x \in \Omega \setminus \sigma(X)$ and assume by contradiction that $X \cup \{x\} \notin \Theta(\sigma)$. So, there exists $z \in X$ such that $z \in \sigma(X \cup \{x\} \setminus \{z\}) \setminus \sigma(X \setminus \{z\})$ but, by our assumptions, this implies $x \in \sigma(X)$, that is a contradiction. Hence, $X \cup \{x\} \in \Theta(\sigma)$.
(iii) $\implies$ *(iv)* Let $X \in \mathcal{P}(\Omega)$ and $Y \in \max(\Theta(\sigma)|X)$. Since $\sigma \in CLOP(\Omega)$, we need only to prove that $\sigma(Y) \subseteq \sigma(X)$. To this regard, if $x \in \sigma(X) \setminus \sigma(Y)$, by our assumptions, it would be $Y \cup \{x\} \in \Theta(\sigma)$, contradicting maximality of Y.
(iv) $\implies$ *(i)*: Let $A \in \Theta(\sigma)$ and $x, y \in \Omega$ with $x \in \sigma(A \cup \{y\}) \setminus \sigma(A)$. It must necessarily be $A \cup \{x\} \in \Theta(\sigma)$ otherwise, $\sigma(A) = \sigma(A \cup \{x\})$, contradicting the choice of x. On the other hand, $A \cup \{x\} \cup \{y\} \notin \Theta(\sigma)$, since $x \in \sigma(A \cup \{y\})$. This entails that $\sigma(A \cup \{x\} \cup \{y\}) = \sigma(A \cup \{x\})$, i.e. $y \in A \cup \{x\}$. □

If $\mathcal{F} \in CLSY(\Omega)$, the map $\sigma_{\mathcal{F}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined by

$$C \in \mathcal{P}(\Omega) \mapsto \sigma_{\mathcal{F}}(C) := \bigcap \{A \in \mathcal{F} : C \subseteq A\} \in \mathcal{P}(\Omega). \tag{1.7}$$

is a closure operator on Ω. Moreover, if $\sigma \in CLOP(\Omega)$, the set system on Ω defined by

$$\mathcal{F}_{\sigma} := \{A \in \mathcal{P}(\Omega) : \sigma(A) = A\}. \tag{1.8}$$

is a closure system on Ω.

Theorem 1.5.7. *The maps $\mathcal{F} \in CLSY(\Omega) \mapsto \sigma_{\mathcal{F}} \in CLOP(\Omega)$ and $\sigma \in CLOP(\Omega) \mapsto \mathcal{F}_{\sigma} \in CLSY(\Omega)$ are inverses each other. In other terms, if $\mathcal{F} \in CLSY(\Omega)$ and $\sigma \in CLOP(\Omega)$ then $\mathcal{F}_{\sigma_{\mathcal{F}}} = \mathcal{F}$ and $\sigma_{\mathcal{F}_{\sigma}} = \sigma$. Moreover, if $\mathcal{F} \in CLSY(\Omega)$, then it is a complete lattice under set-inclusion in which meet means intersection.*

Proof. See [18]. □

By Theorem 1.5.7, we can consider equivalent the notions of closure system and closure operator on the same set Ω (clearly this equivalence is described by means of (1.7) and (1.8)). Therefore, for any closure system $\mathcal{F} \in CLSY(\Omega)$, we always associate with $\mathcal{F}$ the corresponding closure operator $\sigma_{\mathcal{F}} \in CLOP(\Omega)$, and vice versa.

Let $A, B, C, D \in \mathcal{P}(\Omega)$. We say that $A \in \mathcal{P}(\Omega)$ is *σ-complemented* if $\{A, \Omega \setminus A\} \subseteq \mathcal{F}_\sigma$. Moreover, if $\{A, B\} \subseteq \mathcal{F}_\sigma$ and $\{A, B\}$ is a covering on Ω, we say that $\{A, B\}$ is a *σ-covering* on Ω. If $C \cap D = \emptyset$ and $\{A, B\}$ is a covering on Ω, we say that $\{C, D\}$ *splits* $\{A, B\}$, denoted by $\{C, D\} \wr \{A, B\}$, if $C \cap B = \emptyset = D \cap A$. Furthermore, we say that $\{C, D\}$ *σ-splits* $\{A, B\}$, denoted by $\{C, D\} \wr_\sigma \{A, B\}$, if $\{C, D\} \wr \{A, B\}$ and $\{A, B\} \subseteq \mathcal{F}_\sigma$.

1.6 Matroids on Finite Sets

For general results on matroids, see [182]. In what follows, we provide the main results on matroid theory that we use in the thesis.

Definition 1.6.1. Let Ω be a finite set. We say that a set system $\mathcal{F} \in SS(\Omega)$ is a *matroid* on Ω if:

$(M1)$ $\mathcal{F} \in AC(\Omega)$;

$(M2)$ for any $U, V \in \mathcal{F}$ such that $|U| = |V|+1$, then there exists $x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{F}$.

In this case, we call *independent set* any element of $\mathcal{F}$. We denote by $MATR(\Omega)$ the set of all matroids on Ω.

In the next result, the condition for a set system $\mathcal{F}$ is a matroid will be characterized in terms of rank function.

Theorem 1.6.2. *A map $\rho : \mathcal{P}(\Omega) \to \mathbb{N}$ is the rank function of a matroid on Ω if and only if for any $A, B \in \mathcal{P}(\Omega)$, $a, b \in \Omega$, it results that:*

$(Rk1)$ $\rho(\emptyset) = 0$;

$(Rk2)$ *if $A \subseteq B$, then $\rho(A) \leq \rho(B)$;*

$(Rk3)$ *if $\rho(A \cup \{a\}) = \rho(A \cup \{b\}) = \rho(A)$, then $\rho(A \cup \{a\} \cup \{b\}) = \rho(A)$.*

In this case, the corresponding matroid is $\mathcal{F} = \{A \in \mathcal{P}(\Omega) : \rho(A) = |A|\}$.

If $\mathcal{F} \in MATR(\Omega)$ and $A \in \mathcal{P}(\Omega)$, an element $B \in \max(\mathcal{F}|A)$ is usually called an *A-basis* of $\mathcal{F}$. In particular, an element $B \in \max(\mathcal{F})$ is called a *basis* of $\mathcal{F}$.

In the next result, the condition for a subset family $\mathcal{F}$ is a matroid will be characterized in terms of basis family.

Theorem 1.6.3. *A non-empty set system $\mathcal{B} \in SS(\Omega)$ is the basis family of a matroid on Ω if and only if the following holds:*

$(B1)$ *if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.*

In this case, the corresponding matroid is $\mathcal{F} = \{A \in \mathcal{P}(\Omega) : A \in \bigcup_{B \in \mathcal{B}} \mathcal{P}(B)\}$.

Finally, we recall the following classical results concerning two further different characterizations of matroid.

Theorem 1.6.4. *A set system* $\mathcal{F} \in SS(\Omega)$ *is a matroid on* Ω *if and only if* $\max(\mathcal{F}|A)$ *has uniform cardinality for any* $A \in \mathcal{P}(\Omega)$.

Theorem 1.6.5. *A set operator* $\sigma \in OP(\Omega)$ *is the rank-symmetry operator of a matroid on* Ω *if and only if* σ *is a MLS closure operator on* Ω*. In this case, the corresponding matroid is* $\mathcal{F} = \{A \in \mathcal{P}(\Omega) : a \notin \sigma(A \setminus \{a\})\ \forall a \in A\}$.

1.7 Graphs

We refer to [79] for any general notion concerning graph theory. Let $G = (V(G), E(G))$ be a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. In this case, we also use the term *n-graph.* If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \nsim v'$ otherwise. We denote by $Adj(G)$ the usual adjacency matrix of G.

We call *neighborhood* of v in G the set $N_G(v) := \{w \in V(G) : v \sim w\}$. In particular, if $A \subseteq V(G)$ we call *neighborhood* of A in G the set

$$N_G(A) := \bigcup_{v \in A} N_G(v) \tag{1.9}$$

We say that $H = (V(H), E(H))$ is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $X \subseteq V(G)$, the *generated subgraph* by X in G, denoted by $G[X]$, is the graph having X as vertex set and such that if v and v' are two distinct vertices in X, then $\{v, v'\} \in E(G[X])$ if and only if $\{v, v'\} \in E(G)$. If G_1 and G_2 are two graphs such that with $V(G_1) \cap V(G_2) = \emptyset$, we call *disjoint union* of G_1 and G_2, denoted by $G_1 + G_2$, the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

If v and w are two vertices of G, we call a sequence of vertices $v_0 \ldots v_k$, where $v_0 = v$ and $v_1 = w$, a *path* between v and w. The number of edges of the path between v and w is called the *length* of the path. We denote by $d(v, w)$ the distance between v and w, i.e. the length of any shortest path between v and w. The *girth* of G is the minimum length of a cycle contained in G. A graph is said *connected* if there exists a path for any pair of vertices v and w. If v is a vertex, we call the maximal connected subgraph of G containing v the *connected component* of v in G. Each maximal connected subgraph of G is said *connected component* of G. An *automorphism* ϕ of G is a bijective map $\phi : V(G) \to V(G)$ such that for all $v, v' \in V(G)$ it results that

$$\{v, v'\} \in E(G) \Longleftrightarrow \{\phi(v), \phi(v')\} \in E(G).$$

The set of all automorphisms of G is a group with respect to the composition of maps and it is usually denoted by $Aut(G)$. The graph G is said *vertex-transitive* if $Aut(G)$ acts transitively on the set of vertices $V(G)$, i.e. for all v, $v' \in V(G)$ there exists $\phi \in Aut(G)$ such that $\phi(v) = v'$. The graph G is said *edge-transitive* if $Aut(G)$ acts transitively on the set of edges $E(G)$, i.e. for all $\{v, w\}$, $\{v', w'\} \in E(G)$ there exists $\phi \in Aut(G)$ such that $\phi(\{v, w\}) = \{v', w'\}$. The graph G is said *k-transitive* if for any two ordered $k-$tuples of vertices $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ such that $d(v_i, v_j) = d(w_i, w_j)$ for all $i, j \in \{1, \ldots, k\}$, there exists $\phi \in Aut(G)$ such that $\phi(v_i) = w_i$ for every $i \in \{1, \ldots, k\}$.

We give now a list of the graphs we use in the sequel:

- *n-null graph.* It is denoted by N_n and is the n-graph without edges;
- *Complete graph on n vertices.* It is denoted by K_n and is the graph such that $\{v_i, v_j\}$ is an edge, for each pair of indexes $i \neq j$;
- $(r_1, \dots, r_s)-$*complete multipartite graph on n vertices.* It is denoted by $K_{r_1,\dots,r_s}$, where $r_1 + \cdots + r_s = n$ and there exist s non-empty subsets $B_1, \dots, B_s$ of $V(G)$ such that

  (i) $|B_i| = r_i$;
  (ii) $B_i \cap B_j = \emptyset$ if $i \neq j$;
  (iii) $\bigcup_{i=1}^{s} B_i = V(G)$;
  (iv) $E(G) = \{\{x, y\} : x \in B_i,\, y \in B_j,\, i \neq j\}$.

  In this case, we also denote $K_{r_1,\dots,r_s}$ by the symbol $(B_1|\dots|B_s)$. Moreover, if $s = 2$, we say that K_{r_1,r_2} is a *complete bipartite graph*;
- *n-path.* It is denoted by P_n and has edge set $E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}$;
- *n-cycle.* It is denoted by C_n and has edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. In the sequel, all sums between vertex indexes of C_n are implicitly taken $\mathrm{mod}(n)$;
- *n-wheel* is denoted by W_{n+1} and has edge set $E(W_{n+1}) = E(C_n) \cup \{\{v_i, v_{n+1}\} : 1 \leq i \leq n\}$;
- *$\frac{n}{2}$-sticks.* It is denoted by F_n, where $n \geq 4$ is even and $E(F_n) = \{\{v_i, v_{i+1}\} : i = 1, 3, \dots, n-1\}$.

Let G and H be two graphs and let $V := V(G) \times V(H)$.

- The *Cartesian product* of G and H is the graph $G \square H$ on V whose edge set is

$$E(G \square H) := \{\{(u, u'), (v, v')\} : (u = v \wedge u' \sim v') \vee (u' = v' \wedge u \sim v)\}.$$

- The *tensor product* of G and H is the graph $G \otimes H$ whose vertex set is $V(G \otimes H) := V(G) \times V(H)$ and whose edge set is

$$E(G \otimes H) := \{\{(u, u'), (v, v')\} : u \sim v \wedge u' \sim v')\}.$$

Let us note that if $(u, u') \in V(G) \times V(H)$ then

$$N_{G \square H}(u, u') = (\{u\} \times N_H(u')) \cup (N_G(u) \times \{u'\})$$

and

$$N_{G \otimes H}(u, u') = N_G(u) \times N_H(u').$$

Some well studied Cartesian product graph families are the following:

- Let $m \geq 2$ and $n \geq 3$. The (m, n)-*prism graph* is the Cartesian product $P_m \square C_n$.
- Let $m,\, n \geq 2$. The (m, n)-*grid graph* is the Cartesian product $P_m \square P_n$.
- Let $m,\, n \geq 3$. The (m, n)-*rook's graph* is the Cartesian product $K_m \square K_n$.

1.8 Digraphs

A *digraph* (see [14]) is a pair $D = (V(D), Arc(D))$, where $V(D) = \{v_1, \dots, v_n\}$ and $Arc(D) = \{e_1, \dots, e_m\}$ are both finite sets. The elements of $V(D)$ are called *vertices* of D and the elements of $Arc(D)$ are called *arcs* of D. If the ordered pair (v, w) is an arc of D we also write $v \to w$. If $v \in V(D)$, we set

$$N_D^+(v) := \{z \in V(D) : (v, z) \in Arc(D)\}. \tag{1.10}$$

Usually $N_D^+(v)$ is called the *out-neighborhood* of v in D. We say that a vertex $v \in V(D)$ is an *initial node* of D if $N_D^-(v) = \emptyset$. We denote by $\mathbb{I}(D)$ the set of all initial nodes of D. A *path* P in D is an alternating sequence $v_1 e_1 \dots v_k e_k$, where $v_1, \dots, v_k \in V(D)$ are distinct vertices and $e_1, \dots, e_k \in A(D)$ are distinct arcs. If $v_1 = v_k$, we say that P is a *cycle*. A digraph is called *acyclic* if it has no cycles. It is easy to prove that if a digraph is acyclic, then $\mathbb{I}(D) \neq \emptyset$.

Particular digraphs we will study are:

- *Complete bipartite digraph.* If p and q be are two positive integers, we denote by $\vec{K}_{p,q}$ the digraph having vertex set $V(\vec{K}_{p,q}) = \{x_1, \dots, x_p, y_1, \dots, y_q\}$ and arc set $Arc(\vec{K}_{p,q}) = \{(x_i, y_j) : i = 1, \dots, p, j = 1, \dots, q\}$. We call $\vec{K}_{p,q}$ the (p, q)-*complete bipartite digraph*, or, simply *complete bipartite digraph* if p and q are clear from the context.

- *Direct path.* If n is a positive integer, we denote by $\vec{P}_n$ the direct path on n vertices , i.e. the digraph having arc set $Arc(\vec{P}_n) = \{(v_i, v_{i+1}) : i = 1, \dots, n-1\}$.

- *Direct cycle.* Let $n \geq 2$ we denote by $\vec{C}_n$ the direct cycle on n vertices , i.e. the digraph having arc set $Arc(\vec{C}_n) = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$.

- *Tournament.* A tournament is a digraph $\vec{T}$ obtained by assigning a direction to each edge in an undirected complete graph.

- *Transitive tournament.* A *n-transitive tournament* is a tournament $\vec{T}$ on n vertices such that, if (v_i, v_j) and (v_j, v_k) are in $Arc(\vec{T})$, then $(v_i, v_k) \in Arc(\vec{T})$.

Remark 1.8.1. It is easy to verify that all n-transitive tournaments are isomorphic to the tournament $\vec{T}_n$ such that

$$V(\vec{T}_n) = \{v_1, \dots, v_n\}, \qquad Arc(\vec{T}_n) = \{(v_i, v_j) : 1 \leq i \leq n-1,\ j = i+1, \dots, n\}.$$

Therefore in the sequel we refer to the above $\vec{T}_n$ when we use the term *n-transitive tournament.*

1.9 Formal Concept Analysis

We start recalling the general definition of formal contexts and their basic properties (for details see [87]).

Definition 1.9.1. A formal context is a triple $\mathbb{K} = (Z, M, \mathcal{R})$, where Z and M are sets and $\mathcal{R} \subseteq Z \times M$ is the binary relation involving them. The elements of Z and M are called objects and attributes (or *properties*) respectively. We write $g\mathcal{R}m$ instead of $(g,m) \in \mathcal{R}$. If $O \subseteq Z$ and $Q \subseteq M$, we set

$$O^{\uparrow} := \{m \in M : (\forall g \in O)\, g\mathcal{R}m\} \subseteq M$$

and

$$Q^{\downarrow} := \{g \in Z : (\forall m \in Q)\, g\mathcal{R}m\} \subseteq Z.$$

In this way the following two mappings are defined: ${}^{\uparrow} : \mathcal{P}(Z) \to \mathcal{P}(M)$, $O \mapsto O^{\uparrow}$ and ${}^{\downarrow} : \mathcal{P}(M) \to \mathcal{P}(Z)$, $Q \mapsto Q^{\downarrow}$.

By suitable compositions of the above two mappings we are able to construct the two new mappings ${}^{*} : \mathcal{P}(Z) \to \mathcal{P}(Z)$, $O \mapsto O^{*} := O^{\uparrow\downarrow}$, which is a transformation on $\mathcal{P}(Z)$, and ${}^{\diamond} : \mathcal{P}(M) \to \mathcal{P}(M)$, $Q \mapsto Q^{\diamond} := Q^{\downarrow\uparrow}$, which is a transformation on $\mathcal{P}(M)$.

Definition 1.9.2. A formal concept of the formal context $\mathbb{K} = (Z, M, \mathcal{R})$ is a pair (O, Q), where $O \subseteq Z$, $Q \subseteq M$, $O^{\uparrow} = Q$ and $Q^{\downarrow} = O$. If (O, Q) is a formal concept, O is called *extent* of (O, Q) and Q is called *intent* of (O, Q). We denote by $\mathfrak{B}(\mathbb{K})$ the set of all formal concepts of the formal context $\mathbb{K}$.

Remark 1.9.3. If $O \subseteq Z$, it is immediate to verify that :
(i) $O \subseteq O^{*}$.
(ii) The ordered pair $(O, O^{\uparrow})$ is a formal concept of $\mathbb{K}$ if and only if $O = O^{*}$.
(iii) The ordered pair $(O, O^{\uparrow})$ is a formal concept of $\mathbb{K}$ if and only if there exists an attribute subset $Q \subseteq M$ such that $O = Q^{\downarrow}$. In this case $O^{\uparrow}$ coincides with the largest attribute subset Q such that $O = Q^{\downarrow}$.
(iv) Obviously similar conditions to previous the (i), (ii) and (iii) also hold for the intent case.

Definition 1.9.4. Let $\mathbb{K} = (Z, M, \mathcal{R})$ be a Formal Context. We call *extent set system* of $\mathbb{K}$ the set system $\mathfrak{E}(\mathbb{K}) := (Z, EXT(\mathbb{K}))$, where $EXT(\mathbb{K})$ is the family of all extents of $\mathbb{K}$. Analogously, we call *intent set system* of $\mathbb{K}$ the set system $\mathfrak{I}(\mathbb{K}) := (M, INT(\mathbb{K}))$, where $INT(\mathbb{K})$ is the family of all intents of $\mathbb{K}$.

If (O_1, Q_1) and (O_2, Q_2) are two concepts in $\mathfrak{B}(\mathbb{K})$, it is usual to consider the relation $(O_1, Q_1) \sqsubseteq (O_2, Q_2)$ if and only if $O_1 \subseteq O_2$ (that is equivalent to $Q_1 \supseteq Q_2$). Then, $\sqsubseteq$ is a partial order on $\mathfrak{B}(\mathbb{K})$ and $(\mathfrak{B}(\mathbb{K}), \sqsubseteq)$ is a complete lattice, called *concept lattice* (or also *Galois lattice*) of the formal context $\mathbb{K}$, whose meet and join operations on an arbitrary family of formal concepts $\{(O_\alpha, Q_\alpha) : \alpha \in A\}$ are the following:

$$\begin{aligned} \bigwedge_{\alpha \in A} (O_\alpha, Q_\alpha) &= \Big(\bigcap_{\alpha \in A} O_\alpha, (\bigcup_{\alpha \in A} Q_\alpha)^{\diamond} \Big) \\ \bigvee_{\alpha \in A} (O_\alpha, Q_\alpha) &= \Big((\bigcup_{\alpha \in A} O_\alpha)^{*}, \bigcap_{\alpha \in A} Q_\alpha \Big) \end{aligned} \tag{1.11}$$

Proposition 1.9.5. *Both the set systems $\mathfrak{E}(\mathbb{K})$ and $\mathfrak{I}(\mathbb{K})$ are closure systems.*

Proof. It follows directly by the two identities (1.11). □

Chapter 2

Pairings and Symmetry Relations

In this chapter, we provide some basic examples in which pairings arise in a natural way. Furthermore, we study the main properties of the A-symmetry relation and of the indistinguishability relation. In particular, we will prove that each global symmetry class $[A]_{\approx_{\mathfrak{P}}}$ is union-closed, so it admits a maximum, that we call *maximum partitioner*. The family of the maximum partitioners is a closure system on Ω and the set operator $M_{\mathfrak{P}} : A \in \mathcal{P}(\Omega) \mapsto M_{\mathfrak{P}}(A) \in \mathcal{P}(\Omega)$ is a closure operator characterizing the pairing itself and satisfying many interesting topological properties. Finally, we also consider the family of all minimal elements of each global symmetry class and prove that it is an abstract simplicial complex on Ω.

2.1 Pairings

Let us consider an arbitrary set Ω (finite of infinite).

Definition 2.1.1. A *pairing* on Ω is a triple $\mathfrak{P} = (U, F, \Lambda)$, where U, Λ are non-empty sets and $F : U \times \Omega \to \Lambda$ is a map having domain $U \times \Omega$ and codomain Λ. We denote by $PAIR(\Omega)$ the set of all pairings on Ω. When both the sets $U = \{u_1, \dots, u_m\}$ and $\Omega = \{a_1, \dots, a_n\}$ are finite, we say that $\mathfrak{P}$ is a *finite pairing*. For a finite pairing, we denote by $T[\mathfrak{P}]$ the $m \times n$ rectangular table having on the i-th row the element u_i, on the j-th column the element a_j and $F(u_i, a_j)$ in the place (i, j). In the finite case, we identify $\mathfrak{P}$ with $T[\mathfrak{P}]$ and call $T[\mathfrak{P}]$ the *functional table* of $\mathfrak{P}$.

In the whole thesis, when Ω is finite, we always restrict our attention on finite pairings and use the notation $PAIR(\Omega)$ in order to denote the aforementioned family.
Let $\mathfrak{P} = (U, F, \Lambda) \in PAIR(\Omega)$ and $A, A' \in \mathcal{P}(\Omega)$. For any $u, u' \in U$ we set

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a), \ \forall a \in A. \tag{2.1}$$

It is immediate to show that $\equiv_A$ is an equivalence relation on U and we call it the *A-symmetry relation* of $\mathfrak{P}$. For any $u \in U$ we denote by $[u]_A$ the equivalence class of u with respect to $\equiv_A$ and we call $[u]_A$ the *A-symmetry class* of u. Finally, we call the set partition $\pi_{\mathfrak{P}}(A) := \{[u]_A : u \in U\}$ induced by $\equiv_A$ on U the *A-symmetry partition* of $\mathfrak{P}$.

We provide here some examples taken by various mathematical context within the notion of pairing arises in a natural way. In the examples we consider in the sequel, we omit the subscript $\mathfrak{P}$ in all the notations when the context is clear.

2.1.1 Pairings by Graphs

Let G be a n-graph. We now consider the pairing $\mathfrak{P}[G] := (V(G), F, \{0,1\}) \in PAIR(V(G))$, where

$$F(u,v) := \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}$$

Hereafter, we write G instead of $\mathfrak{P}[G]$. Let $A \subseteq V(G)$ a vertex subset. It is easy to prove that the A-symmetry relation $\equiv_A$ can be translated as follows:

$$v \equiv_A v' :\Longleftrightarrow N_G(v) \cap A = N_G(v') \cap A \tag{2.2}$$

We now prove some basic property of A-symmetry for graphs.

Proposition 2.1.2. *Let $v, v' \in G$ and $A \subseteq V(G)$. Then:*
(i) $v \equiv_A v'$ if and only if for all $z \in A$ it results that $v \sim z$ if and only if $v' \sim z$.
(ii) If $v \sim v'$ then $v \not\equiv_A v'$ or $\{v, v'\} \cap A = \emptyset$.
(iii) If $v \equiv_A v'$ and $\{v, v'\} \cap A \neq \emptyset$, then $v \nsim v'$.

Proof. (i): It follows immediately by (2.2).
(ii): We suppose that $v \sim v'$ and $v \equiv_A v'$. We must show that $\{v, v'\} \cap A = \emptyset$. We suppose by contradiction that $\{v, v'\} \cap A \neq \emptyset$. Without loss of generality, we can assume $v \in A$. Hence, we deduce that $F(v, v) = F(v', v)$, but $F(v, v) = 0$ since there are no loops in G while by our assumption $F(v', v) = 1$. So the equality $F(v, v) = F(v', v)$ does not hold, that is an absurd. The case $v' \in A$ is analogous.
On the other hand, a similar argument shows that if $v \sim v'$ and $\{v, v'\} \cap A \neq \emptyset$ then $v \not\equiv_A v'$. In fact we just observe that if $v \in A$, then $F(v, v) \neq F(v', v)$, hence we conclude by (2.2). This proves (ii).
(iii): It is the contra-nominal version of (ii). □

We now explain formally why the equivalence relation $\equiv_A$ is actually a symmetry relation with respect to the vertex subset A. Let $A \subseteq V(G)$ and let v, v' be two distinct vertices of G. We set $X := A \cup \{v, v'\}$ and we define a map $\phi^A_{v,v'} : X \to X$ as follows: if $x \in X$ then

$$\phi^A_{v,v'}(x) := \begin{cases} x & \text{if } x \in A \setminus \{v, v'\} \\ v' & \text{if } x = v \\ v & \text{if } x = v'. \end{cases} \tag{2.3}$$

Proposition 2.1.3. *With the above notations, if $v \nsim v'$ or $A \cap \{v, v'\} = \emptyset$, then*

$$v \equiv_A v' \Longleftrightarrow \phi^A_{v,v'} \in Aut(G[X]).$$

Proof. We set $\psi := \phi^A_{v,v'}$. Firstly we assume that $v \equiv_A v'$. In order to prove that $\psi \in Aut(G[X])$ we observe, by (2.3), that the map ψ is bijective. Therefore we only need to show that for all $x, y \in X$ it results that

$$x \sim y \Longleftrightarrow \psi(x) \sim \psi(y). \tag{2.4}$$

Let $x, y \in X$. Let us note that if $x, y \in A \setminus \{v, v'\}$ by (2.3) we have $\psi(x) = x$ and $\psi(y) = y$, therefore (2.4) is satisfied. We can assume then that $\{x, y\} \cap \{v, v'\} \neq \emptyset$. We distinguish two cases. Firstly, let us suppose that $x \in \{v, v'\}$ and $y \notin \{v, v'\}$ (the condition $x \notin \{v, v'\}$ and $y \in \{v, v'\}$ is similar. In this case we can assume without loss of generality that $x = v$. Since $y \in X$ and $y \notin \{v, v'\}$, it follows that $y \in A$. By (i) of Proposition 2.1.2 we obtain that

$$v \sim y \Longleftrightarrow v' \sim y. \tag{2.5}$$

On the other hand, since $y \neq v$ and $y \neq v'$, by (2.3) we deduce that $\psi(y) = y$, and this implies that (2.5) is equivalent to (2.4). The other case that we have to examine is $\{x, y\} = \{v, v'\}$. We can assume without loss of generality that $x = v$ and $y = v'$. By (2.3) it follows that $\psi(x) = y$ and $\psi(y) = x$, and due to the symmetry of the relation $\sim$ we obtain (2.4).
We suppose now that $\psi \in Aut(G[X])$ and we prove that for all $z \in A$ it results

$$z \sim v \Longleftrightarrow z \sim v'. \tag{2.6}$$

By (i) of Proposition 2.1.2, the equation (2.6) is equivalent to show that $v \equiv_A v'$. Let therefore $z \in A$. Firstly we assume that $z \notin \{v, v'\}$. By (2.3) we have then $\psi(z) = z$. From this last identity and since $\psi \in Aut(G[X])$ we deduce that

$$v \sim z \Longleftrightarrow \psi(v) \sim \psi(z) \Longleftrightarrow v' \sim z,$$

that is exactly (2.6). Now, if $A \cap \{v, v'\} = \emptyset$, since $z \in A$, it cannot be $z \in \{v, v'\}$. Hence, if $A \cap \{v, v'\} = \emptyset$ the proof is completed. We can suppose then that $v \nsim v'$ and $z \in \{v, v'\}$. Without loss of generality we can assume that $z = v$. Since G is a simple graph, $(z = v) \nsim v$, and by hypothesis $(z = v) \nsim v'$, hence (2.6) is satisfied. This complete the proof. □

Corollary 2.1.4. *In the same hypotheses of the previous theorem, if $v \equiv_A v'$ then $|Aut(G[X])| > 1$. Thus, the graph $G[X]$ is symmetric in the Erdös-Rényi terminology [83].*

The result described in Proposition 2.1.3 tells us that the relation $\equiv_A$ can be considered as a type of symmetry relation with respect to the fixed vertex subset A. In other terms, we can think the blocks of the set partition $\pi_G(A)$ as if they were symmetry blocks with respect to A.

2.1.2 Pairings by Digraphs

To any digraph D we can associate the pairing $\mathfrak{P}[D] := (V(D), F, \{0, 1\}) \in PAIR(\Omega)$ where $F(v_i, v_j) := 1$ if $v_i \to v_j$ and $F(v_i, v_j) := 0$ otherwise. Hereafter, we write D instead of $\mathfrak{P}[D]$.

We now provide a basic property of A-symmetry for digraphs.

Theorem 2.1.5. *Let $A \subseteq V(D)$ and $v, v' \in V(D)$. The following conditions are equivalent:*
(i) $v \equiv_A v'$.
(ii) For all $z \in A$ it results that $v \to z$ if and only if $v' \to z$.
(iii) $N^+_D(v) \cap A = N^+_D(v') \cap A$.

Proof. $(i) \implies (ii)$: Let $z \in A$ and $v \to z$, we show that $v' \to z$. By (i) we have that $F_D(v,a) = F_D(v',a)$ for all $a \in A$, therefore $F_D(v,z) = F_D(v',z)$. Since $v \to z$ it follows that $F_D(v,z) = 1$, and hence also $F_D(v',z) = 1$, that is $v' \to z$. By symmetry of the relation $\equiv_A$, if we assume that $v' \to z$, we obtain $v \to z$. This proves (ii)
$(ii) \implies (iii)$: By symmetry of the condition (ii), it is sufficient to prove that $N_D^+(v) \cap A \subseteq N_D^+(v') \cap A$. Let therefore $z \in N_D^+(v) \cap A$, then $v \to z$ and $z \in A$. By (ii) we have then that $v' \to z$, that is $z \in N_D^+(v')$. Hence $z \in N_D^+(v') \cap A$.
$(iii) \implies (i)$: We must show that $F_D(v,a) = F_D(v',a)$ for all $a \in A$. Let $a \in A$ such that $F_D(v,a) = 1$. Then $v \to a$, i.e. $a \in A \cap N_D^+(v)$. Since by hypothesis $N_D^+(v) \cap A = N_D^+(v') \cap A$, we have $v' \to a$, i.e. $F_D(v',a) = 1$. On the other hand, let $a \in A$ such that $F_D(v,a) = 0$. This means that $v \not\to a$, i.e. $a \notin A \cap N_D^+(v)$. Therefore, $a \notin N_D^+(v') \cap A$ and thus $v' \not\to a$, i.e. $F_D(v',a) = 0$. □

Thus, if $v \in V(D)$, it results by Theorem 2.1.5 that $v' \in [v]_A$ if and only if v and v' *"out-see"* all the vertices $z \in A$ in the same way, that is, $v \to z$ if and only if $v' \to z$ for all $z \in A$. So that an A-symmetry class for the pairing $\mathfrak{P}$ is exactly a vertex subset of $V(D)$ whose vertices have all the same out-adjacency relation with respect all vertices in A. Therefore, if we fix a vertex subset A of $V(D)$, we can interpret A as if it were a type of symmetry block against which to examine the out behavior of all vertices in D.

2.1.3 Pairings by Hypergraphs

Let us start by defining a pairing from a hypergraph.

Definition 2.1.6. Let H be a hypergraph on a set $\Omega = \{x_1, \dots, x_n\}$, with hyperedges $Y_1, \dots, Y_m$. We associate with H the pairing $\Gamma(H)$ on Ω defined as follows. The object set of $\Gamma(H)$ is $\{Y_1, \dots, Y_m\}$, $\Lambda = \{0,1\}$ and F is

$$F(Y_i, x_j) := \begin{cases} 1 & \textit{if } \ x_j \in Y_i \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 2.1.7. *Let H be a hypergraph on a set $\Omega = \{x_1, \dots, x_n\}$, with hyperedge family $\mathcal{F} = \{Y_1, \dots, Y_m\}$ and $\Gamma(H) = (\mathcal{F}, \{0,1\}, F)$. Let $A \subseteq \Omega$. Then, if $\equiv_A$ is the A-symmetry relation in $\Gamma(H)$, we have that*

$$Y_i \equiv_A Y_j \iff Y_i \cap A = Y_j \cap A$$

for all $i, j \in \{1, \dots, m\}$.

Proof. We assume that $Y_i \equiv_A Y_j$. If $a \in A \cap Y_i$, then $a \in Y_i$, whence $F(Y_i, a) = 1$. Therefore, we also have $F(Y_j, a) = 1$, i.e. $a \in Y_j$ and therefore $A \cap Y_i \subseteq A \cap Y_j$. The proof of other inclusion is similar.
On the other hand, suppose that $A \cap Y_i = A \cap Y_j$. Let $a \in A$. If $F(Y_i, a) = 1$, then $a \in A \cap Y_i = A \cap Y_j$ and hence $F(Y_j, a) = 1$. If $F(Y_i, a) = 0$, then $a \in A$ but $a \notin Y_i$, hence $a \notin A \cap Y_i = A \cap Y_j$, therefore $a \notin Y_j$, and this implies that $F(Y_j, a) = 0$. This proves that $Y_i \equiv_A Y_j$ because of the arbitrariness of a. □

Let us note that, by Proposition 2.1.7, if Y is a hyperedge of H then

$$[Y]_A = \{Y_i \in \mathcal{F} : Y_i \cap A = Y \cap A\}. \tag{2.7}$$

2.1.4 Pairings by Metric Spaces

Let (X, d_X) be a metric space and $\Omega = X$. We can consider the pairing $\mathfrak{P}[X, d_X] := (X, F, \{0, 1\}) \in PAIR(\Omega)$ where $F : X \times X \to \{0, 1\}$ defined by $F(u, u') := 1$ if $d_X(u, u') = 1$ and $F(u, u') := 0$ otherwise. It is immediate to see that the A-symmetry relation can be translated as follows: if $u, u' \in X$, then

$$u \equiv_A u' :\Longleftrightarrow N_X(u) \cap A = N_X(u') \cap A, \tag{2.8}$$

where $N_X(u) = \{u' \in X : d_X(u, u') = 1\}$. Therefore, also in this case, $\equiv_A$ can be naturally interpreted as a symmetry relation with respect the fixed subset A.

2.1.5 Pairings by Vector Spaces

If V is a vector space and $H \subseteq V$, we write $H \leqslant V$ when H is a vector subspace of V.

Let V be a vector space over a field $\mathbb{K}$ and $\varphi : V \times V \to \mathbb{K}$ a $\mathbb{K}$-bilinear form on V. Let $\Omega = V$. We associate with V the pairing $\mathfrak{P}[V, \varphi] = (V, \varphi, \mathbb{K}) \in PAIR(\Omega)$. We call it the $(V, \varphi, \mathbb{K})$-*vector bilinear pairing*, abbreviated $(V, \varphi, \mathbb{K})$-*VBP*. Hereafter, we write V instead of $\mathfrak{P}[V, \varphi]$.

Let A be an arbitrary fixed subset of V. We denote by $A^\perp$ the orthogonal subspace of A with respect to φ, i.e. $A^\perp := \{v \in V : \varphi(v, a) = 0 \ \forall a \in A\}$. In particular, the subspace $V^\perp$ is the classical *(left) radical* of V and the form is said *(left) non degenerate* if $V^\perp = \{0\}$. We denote by $V^* := Hom(V, \mathbb{K})$ the dual vector space of V. In particular, if V has finite dimension, then φ is non degenerate if and only if the map $d_\varphi : v \in V \to \varphi(v, \cdot) \in V^*$ is an isomorphism, that is if $Ker(d_\varphi) = \{v \in V : \varphi(v, w) = 0 \ \forall w \in V\} = V^\perp = \{0\}$.

Now, we examine the A-symmetry relation relative to the $(V, \varphi, \mathbb{K})$-VBP. For any $u, u' \in V$, we set

$$u \equiv_A u' \iff \varphi(u, a) = \varphi(u', a) \ \forall a \in A \iff \varphi(u - u', a) = 0 \ \forall a \in A. \tag{2.9}$$

In other terms,

$$u \equiv_A u' \iff u \equiv u' \mod A^\perp \iff u + A^\perp = u' + A^\perp.$$

Therefore $[u]_A = u + A^\perp$ for any $u \in V$ and

$$\pi_{\mathfrak{P}}(A) = \{u + A^\perp : u \in V\} = V/A^\perp.$$

So that, in this case, the set partition $\pi_{\mathfrak{P}}(A)$ has itself a vector space structure that coincides with that of the quotient vector space $V/A^\perp$.

2.1.6 Pairings by Group Actions

Let Ω be a generic set and G a group acting on it. Let us denote by $\psi : (g, x) \in G \times \Omega \mapsto \psi((g, x)) = gx \in \Omega$ this action. Then, we can associate with G the pairing $\mathfrak{P}[G, \psi] := (G, \psi, \Omega) \in PAIR(\Omega)$. Hereafter, we write (G, ψ) instead of $\mathfrak{P}[G, \psi]$.

Let $A \subseteq \Omega$. We recall that the stabilizer of A is the following subgroup of G:

$$\mathrm{Stab}_{G,\psi}(A) = \{g \in G : \, ga = a \ \forall a \in A\} = \bigcap_{a \in A} \mathrm{Stab}_{G,\psi}(a).$$

In this context, the A-symmetry relation becomes

$$g \equiv_A g' \iff ga = g'a \ \ \forall a \in A \iff (g')^{-1} g \in \ \mathrm{Stab}_{G,\psi}(a) \ \ \forall a \in A$$

so that

$$g \equiv_A g' \iff (g')^{-1} g \in \ \mathrm{Stab}_{G,\psi}(A).$$

This means that

$$[g]_A = g \ \mathrm{Stab}_{G,\psi}(A)$$

and

$$\pi_{G,\psi}(A) = \{g \ \mathrm{Stab}_{G,\psi}(A) : \, g \in G\}.$$

2.2 Global Symmetry and Maximum Partitioners

In this section we investigate the main properties of the global symmetry relation. To this regard, we first introduce the map $\pi_{\mathfrak{P}} : \mathcal{P}(\Omega) \to \pi_{\mathfrak{P}}(U)$ defined by

$$A \in \mathcal{P}(\Omega) \mapsto \pi_{\mathfrak{P}}(A), \tag{2.10}$$

and we denote by $PSYM(\mathfrak{P})$ the image of the operator $\pi_{\mathfrak{P}}$, i.e.

$$PSYM(\mathfrak{P}) := \{\pi_{\mathfrak{P}}(A) : \, A \in \mathcal{P}(\Omega)\}, \tag{2.11}$$

$$\mathbb{P}(\mathfrak{P}) := (PSYM(\mathfrak{P}), \preceq), \tag{2.12}$$

so that $\mathbb{P}(\mathfrak{P})$ is a sub-poset of the partition lattice $(\Pi(U), \preceq)$.

Let us note that

$$A \subseteq A' \implies \pi_{\mathfrak{P}}(A') \preceq \pi_{\mathfrak{P}}(A). \tag{2.13}$$

By starting from the A-symmetry relation, we introduce a type of global symmetry relation relatively to a given pairing. To this regard, let us consider the following equivalence relation on $\mathcal{P}(\Omega)$:

$$A \approx_{\mathfrak{P}} A' :\iff \pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A'),$$

that is equivalent to the condition

$$u \equiv_A u' \iff u \equiv_{A'} u', \tag{2.14}$$

for all $u, u' \in U$. We call $\approx_{\mathfrak{P}}$ the *global symmetry relation* of $\mathfrak{P}$, and we denote by $[A]_{\approx_{\mathfrak{P}}}$ the equivalence class of A with respect to $\approx_{\mathfrak{P}}$, that we call *global symmetry class*.

The basic properties of the global symmetry relation are established in the following result.

Theorem 2.2.1. *(i) If $A \approx_{\mathfrak{P}} A'$ and $D \subseteq \Omega$, then $A \cup D \approx_{\mathfrak{P}} A' \cup D$.*
(ii) The global symmetry class $[A]_{\approx_{\mathfrak{P}}}$ is a union-closed family.
(iii) The subset family $[A]_{\approx_{\mathfrak{P}}}$ has a maximum $M_{\mathfrak{P}}(A)$ that coincides with $\bigcup [A]_{\approx_{\mathfrak{P}}}$.
(iv) $M_{\mathfrak{P}}(A) = \{b \in \Omega : A \cup \{b\} \approx_{\mathfrak{P}} A\}$.
(v) $M_{\mathfrak{P}}(A) = \{a \in \Omega : (u, u' \in \Omega \wedge u \equiv_A u') \Longrightarrow F(u, a) = F(u', a)\}$.
(vi) $M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A') \iff \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A')$.
(vii) The set operator $M_{\mathfrak{P}} : B \in \mathcal{P}(\Omega) \mapsto M_{\mathfrak{P}}(B) \in \mathcal{P}(\Omega)$ is a closure operator on Ω.
(viii) The set family

$$MAXP(\mathfrak{P}) := \{M_{\mathfrak{P}}(B) : B \in \mathcal{P}(\Omega)\} = \{C \in \mathcal{P}(\Omega) : M_{\mathfrak{P}}(C) = C\} \tag{2.15}$$

is a closure system on Ω.
(ix) $A \approx_{\mathfrak{P}} A' \iff M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A')$.

Proof. (i): Let $u \equiv_{A \cup D} u'$, i.e. $F(u, a) = F(u', a)$ for any $a \in A \cup D$. This means that $u \equiv_A u'$ and, since $A \approx_{\mathfrak{P}} A'$, that $u \equiv_{A'} u'$. But, in particular, if follows that $F(u, d) = F(u', d)$ for any $d \in A' \cup D$, i.e. $u \equiv_{A' \cup D} u'$. The same argument holds when we start with $u \equiv_{A' \cup D} u'$.
(ii): Let $u, u' \in U$ such that $u \equiv_A u'$ and let $\{A_j : j \in J\} \subseteq [A]_{\approx_{\mathfrak{P}}}$. Hence, $u \equiv_{A_j} u'$ for all $j \in J$ by definition of the relation $\approx_{\mathfrak{P}}$. If $z \in \bigcup_{j \in J} A_j$ there exists some index $j \in J$ such that $z \in A_j$, so that $F(u, z) = F(u', z)$ because $u \equiv_{A_j} u'$. Hence $u \equiv_{\bigcup_{j \in J} A_j} u'$. This implies that $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(\bigcup_{j \in J} A_j)$. On the other hand, by (2.13) we have that $\pi_{\mathfrak{P}}(\bigcup_{j \in J} A_j) \preceq \pi_{\mathfrak{P}}(A_j) = \pi_{\mathfrak{P}}(A)$ because $A_j \approx_{\mathfrak{P}} A$. Since $\preceq$ is a partial order on $PSYM(\mathfrak{P})$ we deduce that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(\bigcup_{j \in J} A_j)$, that is equivalent to the condition $\bigcup_{j \in J} A_j \in [A]_{\approx_{\mathfrak{P}}}$.
(iii): Let $M_{\mathfrak{P}}(A) := \bigcup\{B : B \in [A]_{\approx_{\mathfrak{P}}}\}$. By part (ii), it follows that $M_{\mathfrak{P}}(A) \in [A]_{\approx_{\mathfrak{P}}}$, moreover we also have that $B \subseteq M_{\mathfrak{P}}(A)$ for all $B \in [A]_{\approx_{\mathfrak{P}}}$. Uniqueness is obvious.
(iv): Let $b \in M_{\mathfrak{P}}(A)$. Then, there exists $A' \in [A]_{\approx_{\mathfrak{P}}}$ such that $b \in A'$. Let now $u, u' \in U$ such that $u \equiv_{A \cup \{b\}} u'$, then $u \equiv_A u'$ because A is a subset of $A \cup \{b\}$. On the other hand, let assume that $u \equiv_A u'$. Since $A' \in [A]_{\approx_{\mathfrak{P}}}$, we have $u \equiv_{A'} u'$, and therefore $F(u, b) = F(u', b)$ because $b \in A'$. Hence $u \equiv_{A \cup \{b\}} u'$, so we deduce that $A \approx_{\mathfrak{P}} A \cup \{b\}$.
Let now $b \in \Omega$ such that $A \approx_{\mathfrak{P}} A \cup \{b\}$. Then, we have that $b \in A \cup \{b\} \in [A]_{\approx_{\mathfrak{P}}}$ and we deduce that $b \in M_{\mathfrak{P}}(A)$.
(v): Let $B := \{b \in \Omega : (u, u' \in U \wedge u \equiv_A u') \Longrightarrow F(u, b) = F(u', b)\}$. We show that $B \approx_{\mathfrak{P}} A$. In fact, let $u, u' \in U$ such that $u \equiv_A u'$ and let $b \in B$. By definition of B we have that $F(u, b) = F(u', b)$, so that $u \equiv_B u'$. Hence $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(B)$. Let now $C \in [A]_{\approx_{\mathfrak{P}}}$ and $c \in C$. Then, for all $u, u' \in U$ such that $u \equiv_A u'$, by definition of $\approx_{\mathfrak{P}}$ we have that $u \equiv_C u'$, so that $F(u, c) = F(u', c)$. It follows that $C \subseteq B$ for all $C \in [A]_{\approx_{\mathfrak{P}}}$, in particular $A \subseteq M_{\mathfrak{P}}(A) \subseteq B$. By (2.13) we obtain then $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(A)$. Therefore $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$, i.e. $B \in [A]_{\approx_{\mathfrak{P}}}$. By part (iii) we deduce then that $B \subseteq M_{\mathfrak{P}}(A)$. Hence $B = M_{\mathfrak{P}}(A)$.
(vi) : If A and A' are two subsets of Ω such that $M_{\mathfrak{P}}(A') \subseteq M_{\mathfrak{P}}(A)$ then $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \preceq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A'))$ by virtue of (2.13), which is equivalent to $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A')$ by virtue of part (ii). We assume now that $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \preceq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A'))$. Let $v \in M_{\mathfrak{P}}(A')$ and let us suppose by contradiction that $v \notin M_{\mathfrak{P}}(A)$, so that $M_{\mathfrak{P}}(A) \subsetneqq M_{\mathfrak{P}}(a) \cup \{v\}$. By (2.13) we obtain then $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \cup \{v\}) \preceq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$. On the other hand, by maximality of $M_{\mathfrak{P}}(A)$ proved in part (ii) we also deduce that $M_{\mathfrak{P}}(A) \not\approx_{\mathfrak{P}} M_{\mathfrak{P}}(A) \cup \{v\}$, i.e. $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \neq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \cup \{v\})$. Then, there exist $u, u' \in U$ such that $u \equiv_{M_{\mathfrak{P}}(A)} u'$ and $u \not\equiv_{M_{\mathfrak{P}}(A) \cup \{v\}} u'$, and this is possible

only if $F(u,v) \neq F(u',v)$. Now, since $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \preceq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A'))$ and $u \equiv_{M_{\mathfrak{P}}(A)} u'$, we obtain $u \equiv_{M_{\mathfrak{P}}(A')} u'$, therefore $F(u,v) = F(u',v)$ because $v \in M_{\mathfrak{P}}(A')$. This shows the contradiction and conclude the proof.
(vii): We now prove that the set operator $M_{\mathfrak{P}}$ is a closure operator. It is obvious that $A \subseteq M_{\mathfrak{P}}(A)$. Let now $A' \in \mathcal{P}(\Omega)$ such that $A \subseteq A'$. Let $b \in M_{\mathfrak{P}}(A)$. Thus $A \approx_{\mathfrak{P}} A \cup \{b\}$. Let $v, v' \in \Omega$ such that $v \equiv_{A'} v'$. Since A is a subset of A', we have that $v \equiv_A v'$, and this implies $F(v,b) = F(v',b)$ because $A \approx_{\mathfrak{P}} A \cup \{b\}$. This shows that $v \equiv_{A' \cup \{b\}} v'$, that is $b \in M_{\mathfrak{P}}(A')$. Hence $M_{\mathfrak{P}}(A) \subseteq M_{\mathfrak{P}}(A')$. Finally, we must prove that $M_{\mathfrak{P}}(M(A)) = M_{\mathfrak{P}}(A)$, for this it is sufficient to show that $M_{\mathfrak{P}}(M(A)) \subseteq M_{\mathfrak{P}}(A)$. Let then $b \in M_{\mathfrak{P}}(M(A))$ and $v, v' \in \Omega$ such that $v \equiv_A v'$. Since $A \approx_{\mathfrak{P}} M_{\mathfrak{P}}(A)$, it follows that $v \equiv_{M_{\mathfrak{P}}(A)} v'$, therefore $F(v,b) = F(v',b)$ by part (v), because b belongs to $M_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$. Hence $b \in M_{\mathfrak{P}}(A)$ again by part (v).
$(viii)$: $MAXP(\mathfrak{P})$ is the family of all closed sets for the closure operators $M_{\mathfrak{P}}$. Therefore the result follows by Theorem 1.5.7.
(ix) : If $A \approx_{\mathfrak{P}} A'$, by part (ii), it follows that $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A')$. On the other hand, let $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A')$. Then $A \approx_{\mathfrak{P}} M_{\mathfrak{P}}(A)$ and $A' \approx_{\mathfrak{P}} M_{\mathfrak{P}}(A')$ and we conclude that $A \approx_{\mathfrak{P}} A'$. □

We set now

$$\mathbb{M}(\mathfrak{P}) := (MAXP(\mathfrak{P}), \subseteq^*) \tag{2.16}$$

By Theorem 2.2.1 we deduce then the following result.

Corollary 2.2.2. (i) $\mathbb{M}(\mathfrak{P})$ *is a complete lattice such that if* $\{A_j : j \in J\} \subseteq MAXP(\mathfrak{P})$ *then:*

(a) $\bigvee_{j \in J} A_j = \bigcap_{j \in J} A_j$;

(b) $\bigwedge_{j \in J} A_j = M_{\mathfrak{P}}(\bigcup_{j \in J} A_j)$.

(ii) *The map* $\pi_{\mathfrak{P}}$ *induces an order isomorphism between the posets* $\mathbb{M}(\mathfrak{P})$ *and* $\mathbb{P}(\mathfrak{P})$. *Moreover, since* $\mathbb{M}(\mathfrak{P})$ *is a complete lattice, the same holds also for* $\mathbb{P}(\mathfrak{P})$.
(iii) *If* $A, A' \in MAXP(\mathfrak{P})$ *then*

$$A \supseteq A' \iff \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A') \tag{2.17}$$

Proof. (i): It follows by (vii) of Theorem 2.2.1 and by Theorem 2, p. 112 of [18].
(ii): It is an immediate consequence of (vi) of Theorem 2.2.1.
(iii): It is a special case of part (vi) of Theorem 2.2.1. □

Definition 2.2.3. We call the elements of $MAXP(\mathfrak{P})$ the *maximum partitioners* of $\mathfrak{P}$ and $\mathbb{M}(\mathfrak{P})$ the *symmetry partition lattice of* $\mathfrak{P}$. *We call the set operator* $M_{\mathfrak{P}}$ *the* $\mathfrak{P}$-maximum partitioner operator.

For the sake of completeness, we introduce a relativization of the notion of maximum partitioner. For any $A \in \mathcal{P}(\Omega)$, we set

$$MAXP_{\mathfrak{P}}(A) := \{B \cap A : B \in MAXP(\mathfrak{P})\}, \ \mathbb{M}_{\mathfrak{P}}(A) := (MAXP_{\mathfrak{P}}(A), \subseteq^*), \tag{2.18}$$

therefore we obtain a map $MAXP_{\mathfrak{P}} : \mathcal{P}(\Omega) \to \mathcal{P}(\mathcal{P}(\Omega))$ such that $MAXP_{\mathfrak{P}}(\Omega) = MAXP(\mathfrak{P})$ and

$$MAXP_{\mathfrak{P}}(A) \in CLSY(A), \tag{2.19}$$

for any $A \in \mathcal{P}(\Omega)$.

Definition 2.2.4. We call *A-relative maximum partitioner* of $\mathfrak{P}$ any element of $MAXP_{\mathfrak{P}}(A)$.

We now prove that the family $MAXP_{\mathfrak{P}}(A)$ coincides with the family of all maximum partitioners contained in A whenever $A \in MAXP(\mathfrak{P})$.

Proposition 2.2.5. *If $A \in MAXP(\mathfrak{P})$, then*

$$MAXP_{\mathfrak{P}}(A) = \{B \in MAXP(\mathfrak{P}) : B \subseteq A\}$$

Proof. Let $B \in MAXP_{\mathfrak{P}}(A)$. Then, there exists $C \in MAXP(\mathfrak{P})$ such that $B = A \cap C$. Therefore, $B \subseteq A$ and $B \in MAXP(\mathfrak{P})$ because $MAXP(\mathfrak{P})$ is a closure system. On the other hand, let $B \in MAXP(\mathfrak{P})$ and $B \subseteq A$. Then $B = A \cap B \in MAXP_{\mathfrak{P}}(A)$. □

2.2.1 Characterization of the Symmetry Partitions

In what follows, we want to understand when a given set partition π of U is an A-symmetry partition of U, for some $A \in \mathcal{P}(\Omega)$.

If $W \subseteq U$ we set

$$\Xi_{\mathfrak{P}}(W) := \{a \in \Omega \ : \ \forall\, u, u' \in W, F(u,a) = F(u',a)\}. \tag{2.20}$$

Let us observe that $\Xi_{\mathfrak{P}} : \mathcal{P}(U) \to \mathcal{P}(\Omega)$. Moreover, if $W' \in \mathcal{P}(U)$ it is clear that

$$W \subseteq W' \implies \Xi_{\mathfrak{P}}(W) \supseteq \Xi_{\mathfrak{P}}(W'). \tag{2.21}$$

We have the following basic result.

Proposition 2.2.6. *$\Xi_{\mathfrak{P}}(W) = A$ if and only if the following conditions hold:*

(i) $W \preccurlyeq \pi_{\mathfrak{P}}(A)$*;*

(ii) *if $W \preccurlyeq \pi_{\mathfrak{P}}(A')$, then $A' \subseteq A$.*

Proof. Straightforward. □

If $\pi = \{B_i : i \in I\} \in \pi_{\mathfrak{P}}(U)$ we set now

$$Max(\pi) := \bigcap_{i \in I} \Xi_{\mathfrak{P}}(B_i). \tag{2.22}$$

By means of the set operator $\Xi_{\mathfrak{P}}$ we can characterize now the symmetry partitions of $\mathfrak{P}$.

Theorem 2.2.7. *Let $\pi = \{B_i : i \in I\} \in \pi(U)$ and $A = Max(\pi)$. Then:*

(i) $A \in MAXP(\mathfrak{P})$*;*

(ii) $\pi \preceq \pi_{\mathfrak{P}}(A)$*;*

(iii) $\pi \in PSYM(\mathfrak{P}) \iff \pi = \pi_{\mathfrak{P}}(A)$*;*

(iv) Let $\pi(A') = \{C_l : l \in I'\}$. Then

$$M_{\mathfrak{P}}(A') = \bigcap_{l \in I'} \Xi_{\mathfrak{P}}(C_l).$$

Proof. (i): Since $MAXP(\mathfrak{P})$ is a closure system and $\Xi_{\mathfrak{P}}(B_i) \in MAXP(\mathfrak{P})$, the thesis holds.
(ii): Let $v, v' \in V(\mathfrak{P})$ belonging to a same block B_k of π. We must show that $v \equiv_A v'$. Therefore, let $a \in A$. Then one has $a \in \Xi_{\mathfrak{P}}(B_k)$, so $F(v, a) = F(v'a)$ by definition of $\Xi_{\mathfrak{P}}(B_k)$.
(iii): The implication $\Longleftarrow$ is obvious. We assume therefore that $\pi \in PSYM(\mathfrak{P})$. Let $C = Max(\pi)$. Then we have $\pi = \pi_{\mathfrak{P}}(C)$, and this implies that $F(v, c) = F(v', c)$ for all $v, v' \in B_i$, for each $i = 1, \ldots, s$ and each $c \in C$. By definition of $\Xi_{\mathfrak{P}}(B_i)$ we deduce then that $C \subseteq \Xi_{\mathfrak{P}}(B_i)$ for every $i = 1, \ldots, s$. Hence $C \subseteq A$.
Let now $a \in A$. By definition of A we have

$$F(v, a) = F(v', a), \ \ \forall v, v' \in B_i, \quad \text{for each } i \in I. \tag{2.23}$$

Let us suppose now by contradiction that $a \notin C$. Since $C \in MAXP(\mathfrak{P})$, by part (i) it follows that $\pi_{\mathfrak{P}}(C \cup \{a\}) \prec \pi_{\mathfrak{P}}(C) = \pi$. Therefore there exists at least an index $r \in \{1, \ldots, s\}$ and two distinct vertices $v, v' \in B_r$ such that $F(v, a) \neq F(v', a)$, and this in contrast with (2.23). This shows that $A \subseteq C$. Thus $A = C$ and $\pi = \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$. This concludes the proof of (iii).
(iv): We prove only the statement for A'. We set $\pi := \pi_{\mathfrak{P}}(A')$ and $A := \bigcap_{l=1}^{s'} \Xi_{\mathfrak{P}}(C_l)$, so that $\pi \in PSYM(\mathfrak{P})$. By part (iii), it follows that $\pi_{\mathfrak{P}}(A') = \pi = \pi_{\mathfrak{P}}(A)$, that is $A \approx_{\mathfrak{P}} A'$. Hence, by (vi) of Theorem 2.2.1 and again by part (iii), we obtain $M_{\mathfrak{P}}(A') = M_{\mathfrak{P}}(A) = A$. □

2.2.2 Global Symmetry Lattice

In this subsection we introduce the global symmetry lattice of a pairing $\mathfrak{P}$. Let $MAXP(\mathfrak{P}) = \{C_j : j \in J\}$. We set now

$$GSYM(\mathfrak{P}) = \{[C_j]_{\approx_{\mathfrak{P}}} : j \in J\}. \tag{2.24}$$

We introduce the following partial order $\sqsubseteq$ on $GSYM(\mathfrak{P})$. If $[C_l]_{\approx_{\mathfrak{P}}}, [C_j]_{\approx_{\mathfrak{P}}} \in GSYM(\mathfrak{P})$ we set

$$C_l \subseteq^* C_j \iff: [C_l]_{\approx_{\mathfrak{P}}} \sqsubseteq [C_j]_{\approx_{\mathfrak{P}}} \tag{2.25}$$

and

$$\mathbb{G}(\mathfrak{P}) := (GSYM(\mathfrak{P}), \sqsubseteq). \tag{2.26}$$

We have then the following immediate result

Theorem 2.2.8. *$\mathbb{G}(\mathfrak{P})$ is a lattice that is order isomorphic to the symmetry partition lattice $\mathbb{M}(\mathfrak{P})$.*

Proof. It follows immediately by (2.25). □

We call $\mathbb{G}(\mathfrak{P})$ the *global symmetry lattice* of $\mathfrak{P}$ and any $([C_j]_{\approx_{\mathfrak{P}}}, \subseteq)$ a *local symmetry poset* of $\mathfrak{P}$.

In the next result we show that the knowledge of $MAXP(\mathfrak{P})$ uniquely determine the knowledge of the whole global symmetry lattice $\mathbb{G}(\mathfrak{P})$.

Theorem 2.2.9. *Let $\mathfrak{P}, \mathfrak{P}' \in PAIR(\Omega)$. Then, the following are equivalent:*
(i) $\approx_{\mathfrak{P}}$ *coincides with* $\approx_{\mathfrak{P}'}$*;*
(ii) $\mathbb{G}(\mathfrak{P}) = \mathbb{G}(\mathfrak{P}')$*;*
(iii) $MAXP(\mathfrak{P}) = MAXP(\mathfrak{P}')$*.*

Proof. $(i) \implies (ii)$: Obvious.
$(ii) \implies (iii)$: Since $\mathbb{G}(\mathfrak{P}) = \mathbb{G}(\mathfrak{P}')$, for any $A \in \mathcal{P}(\Omega)$ we have that $[A]_{\approx_{\mathfrak{P}}} = [A]_{\approx_{\mathfrak{P}'}}$. Therefore, the maximum partitioners $M_{\mathfrak{P}}(A)$ and $M_{\mathfrak{P}'}(A)$ coincide for any $A \in \mathcal{P}(\Omega)$, so $MAXP(\mathfrak{P}) = MAXP(\mathfrak{P}')$.
$(iii) \implies (i)$: Since $M_{\mathfrak{P}}$ and $M_{\mathfrak{P}'}$ are the closure operators associate respectively to the closure systems $MAXP(\mathfrak{P})$ and $MAXP(\mathfrak{P}')$, by part $(viii)$ of Theorem 2.2.1 we obtain (i). □

2.3 Global Symmetry on Several Examples

In this section we provide some computations on the concrete pairings defined in Section 2.1.

2.3.1 Some Examples Taken From Graph Context

We now compute the maximum partitioner of some basic graph families.

Complete Multipartite Graphs

In what follows, we firstly determine the general form of the A-symmetry for $K_{r_1,\ldots,r_s}$ and then its symmetry partition lattice.

Proposition 2.3.1. *Let $G := K_{r_1,\ldots,r_s} = (B_1|\ldots|B_s)$. Then $A \in MAXP(G)$ if and only if $A = V(G)$ or $A = \bigcup_{q=1}^{t} B_{i_q}$, where $1 \le t \le s-2$.*

Proof. We need to compute the A-symmetry partition of $K_{r_1,\ldots,r_s}$. For, let $A \subseteq V(G)$, then we set
$$Q_G(A) := \{B_i : A \cap B_i \neq \emptyset\}.$$
We denote by r the quantity $|Q_G(A)|$. We claim that
$$\pi_G(A) = B_{i_1}|\ldots|B_{i_r}|Q_G(A)^c \tag{2.27}$$
For, just observe that two vertices u, u' belonging to the same subset B_i are not adjacent to the vertices of the same subset but they are adjacent to the other vertices. In particular, we deduce that $u \equiv_A u'$ if and only if $u, u' \in B_i$ for some $i = 1, \ldots, s$ or $u, u' \in Q_G(A)^c$. Therefore, we have $\pi_G(A) = B_{i_1}|\ldots|B_{i_r}|Q_G(A)^c$ and this proves (2.27). Let us also observe that if $r = n-1$, then $\pi_G(A) = B_1|\ldots|B_s = \pi_G(V(G))$. It is now straightforward to deduce that $A \in MAXP(G)$ if and only if $A = V(G)$ or $A = \bigcup_{q=1}^{t} B_{i_q}$, where $1 \le t \le s-2$. □

As a particular subcase, if G is a complete bipartite graph, then $MAXP(G)$ is a 2-chain. We now show that the only graph G for which $MAXP(G)$ is a 2-chain is the complete bipartite one.

Proposition 2.3.2. *$MAXP(G)$ is a 2-chain if and only if G is a complete bipartite graph.*

Proof. Let G be a graph such that $\mathbb{M}(G)$ is a 2-chain. If G is the n-null graph, then obviously $\mathbb{M}(G)$ has only the trivial partition and this contradicts our hypothesis. Therefore let v be a non isolated vertex of G. The partition induced by the singleton set $\{v\}$ has only two blocks: the neighborhood $N_G(v)$ of v in G and its complement in $V(G)$. Since $v \nsim v$ and $N_G(v) \neq \emptyset$, this partition is different from $\pi_G(\emptyset) = \hat{1}_G$. Moreover to say that $\mathbb{M}(G)$ is a 2-chain implies that $\pi_G(\{v\}) = \pi_G(V(G))$.

Let u, u' be vertices of G. If $u, u' \in N_G(v)$ or $u, u' \notin N_G(v)$, then $u \equiv_{\{v\}} u'$ and thus $u \equiv_{V(G)} u'$. By part (iii) of Proposition 2.1.2 we have therefore $u \nsim u'$. If $u \in N_G(v)$ and $u' \notin N_G(v)$, then $u \not\equiv_{\{v\}} u'$, which is equivalent to $u \not\equiv_{V(G)} u'$. In this case, since $v \in N_G(u)$, u is a non isolated vertex of G and so, as before, $\pi_G(\{u\}) = \pi_G(V(G))$. Thus $u \not\equiv_{\{u\}} u'$ and $u \nsim u$ implies $u \sim u'$. It follows that $G = (N_G(v), (V(G) \setminus N_G(v))$ is a complete bipartite graph.

Let $G = (V_1|V_2)$ be a complete bipartite graph. Let A be a non-empty subset of $V(G)$ and $u, u' \in V(G)$. If $u, u' \in V_1$ or $u, u' \in V_2$, then for each $v \in A$ we have $F(v, u) = F(v, u')$, so $u \equiv_A u'$. If $u \in V_1$ and $u' \in V_2$, then for each $v \in A$ we have $F(v, u) \neq F(v, u')$, so $u \not\equiv_A u'$. Thus $\pi_G(A) = V_1|V_2$. It follows that $PSYM(G)$ has only two elements: the trivial partition $\pi_G(\emptyset) = \hat{1}_G$ and the partition $\pi_G(B) = V_1|V_2$, induced by all non-empty subsets B of $V(G)$, with $\pi_G(B) \preceq \hat{1}_G$, so $\mathbb{M}(G)$ is a 2-chain. $\square$

The Complete Graph and The n-Cycle

In the next results we give the A-symmetry partition for the complete graph K_n and its symmetry partition lattice.

Proposition 2.3.3. *Let $n \geq 1$ and let $A = \{w_1, \ldots, w_k\}$ be a generic subset of $V(K_n) = \{v_1, \ldots, v_n\}$. Then*

$$\pi_{K_n}(A) = w_1|w_2|\ldots|w_k|A^c.$$

Proof. Let v, $v' \in V(K_n)$, with $v \neq v'$. By (iii) of Proposition 2.1.2, since $v \sim v'$, it holds that if $v \equiv_A v'$, then $v, v' \in A^c$. On the other hand, if $v, v' \in A^c$, then $\forall z \in A$, $F(z, v) = F(z, v') = 1$, namely $v \equiv_A v'$. The proposition is proved. $\square$

We now provide an explicit expression for $MAXP(K_n)$. To this aim, we first introduce the following notation: if $n \geq k$ we set

$$\mathcal{V}(n, k) := \{K \subseteq \hat{n} : |K| \leq n - k\} \cup \{\hat{n}\} \tag{2.28}$$

Proposition 2.3.4. *Let $n \geq 2$, $G = K_n$ and $V(G) = \hat{n}$. Then $MAXP(G) = \mathcal{V}(n, 2)$.*

Proof. By virtue of $(viii)$ of Theorem 2.2.1 the identity $\mathcal{V}(n, 2) = MAXP(G)$ is equivalent to the following:

$$\mathcal{V}(n, 2) = \{K \subseteq V(G) : K = M(K)\}. \tag{2.29}$$

Let $K \in \mathcal{V}(n,2)$. We must prove that $K = M(K)$. By definition of $M(K)$ we have that $K \subseteq M(K)$, therefore it remains to show that $M(K) \subseteq K$. By (v) of Theorem 2.2.1 we have that

$$M(K) = \{v \in V(G) : (u, u' \in V(G) \wedge u \equiv_K u') \Longrightarrow F(u,v) = F(u',v)\}. \tag{2.30}$$

If $K = \hat{n}$, then obviously $K = M(K)$. We can assume therefore $K \neq \hat{n}$. Let now $v \in \hat{n} \setminus K$. Since $|K| \leq n-2$, there exists a vertex let $z \in \hat{n} \setminus K$ such that $v \neq z$. Since $0 = F(v,v) \neq F(v,z) = 1$ and $v \equiv_K z$ by (2.30) we have that $v \notin M(K)$. It follows that $\hat{n} \setminus K \subseteq \hat{n} \setminus M(K)$, and thus $M(K) \subseteq K$. Hence if $K \in \mathcal{V}(n,2)$ then $K = M(K)$.
We assume now that $K \subseteq V(G)$ and $K = M(K)$. We must prove that $K = \hat{n}$ or $|K| \leq n-2$. Let $K \neq \hat{n}$ and suppose by contradiction that $|K| = n-1$. Then K^c is a singleton and therefore we have that $\pi_G(K) = v_1|\cdots|v_n = \pi_G(V(G))$, by Proposition 2.3.3. Thus $K \approx V(G)$, and this implies $K = M(K) = V(G)$ by virtue of part (ix) of Theorem 2.2.1, which gives a contradiction because $K \neq \hat{n} = V(G)$. This shows that $|K| \leq n-2$, i.e. $K \in \mathcal{V}(n,2)$. Hence the identity in (2.29) is proved. Since $\mathcal{V}(n,2) = MAXP(G)$, we have that $\mathbb{M}(G) = (\mathcal{V}(n,2), \subseteq^*)$, therefore the claim follows by (ii) of Corollary 2.2.2. $\square$

We provide below the A-symmetry partition of the cycle C_n on n vertices.

Proposition 2.3.5. (i) *Let $V = V(C_n) = \{v_1, \dots, v_n\}$. Fix a subset $A \subseteq V$, $A = \{v_{i_1}, \dots, v_{i_k}\}$, and let v_i, $v_j \in V$, with $i < j$. Then $v_i \equiv_A v_j$ if and only if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \emptyset$ or $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$.*
(ii) *Let us set $B_{C_n}(A) := (N_{C_n}(A))^c$, $C_{C_n}(A) := \{v_i \in A : v_{i-2} \notin A \wedge v_{i+2} \notin A\}$ and $K_{C_n}(A) := N_{C_n}(C_{C_n}(A))$. Then, if $v_{s_1}, \dots, v_{s_l}$ are the vertices in $S_{C_n}(A) := V(C_n) \setminus [B_{C_n}(A) \cup K_{C_n}(A)]$ and $v_{j_1}, \dots, v_{j_h}$ are the vertices in $C_{C_n}(A)$, we have*

$$\pi_{C_n}(A) = B_{C_n}(A)|v_{j_1-1}v_{j_1+1}|\cdots|v_{j_h-1}v_{j_h+1}|v_{s_1}|\cdots|v_{s_l}.$$

Proof. (i): Let us observe that for each $i \in \{1, \dots, n\}$, $N_{C_n}(v_i) = \{v_{i-1}, v_{i+1}\}$, where the index sums are all taken mod(n). The proof follows easily by observing that, since $v_i \neq v_j$, then $|N_{C_n}(v_i) \cap N_{C_n}(v_j)| \leq 1$ and the equality holds if and only if $j = i+2$. It follows that, if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A$, then $N_{C_n}(v_i) \cap A = (N_{C_n}(v_i) \cap N_{C_n}(v_j)) \cap A \subseteq N_{C_n}(v_i) \cap N_{C_n}(v_j)$. By (2.2), $v_i \equiv_A v_j$ if and only if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A$. Thus $|N_{C_n}(v_i) \cap A| = |N_{C_n}(v_j) \cap A| \leq 1$ and the equality holds if and only if $j = i+2$ and $v_{i+1} = v_{j-1} \in A$. This proves the thesis.
(ii): The proof follows directly by part (i). In fact, let v_i, $v_j \in V(C_n)$, with $i < j$ and $v_i \equiv_A v_j$. Then, either (a) $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \emptyset$ or (b) $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = v_{i+1} = v_{j-1}$. But (a) is equivalent to say that v_i, $v_j \in B_A$, (b) that $\{v_i, v_j\} = N_{C_n}(v)$, for some $v \in C_A$. The proposition is thus proved. $\square$

In the next result we provide a characterization for the maximum partitioners of C_n.

Theorem 2.3.6. *Let $A \subseteq V(C_n)$. Then $A \in MAXP(C_n)$ if and only if for any $v_i \notin A$ it holds one of the following conditions:*
(i) ($v_{i-2} \notin A$ or $v_{i+2} \notin A$) and ($B_{C_n}(A) \neq \{v_{i-1}, v_{i+1}\}$);
(ii) ($v_{i-2} \in A$ and $v_{i-4} \notin A$) or ($v_{i+2} \in A$ and $v_{i+4} \notin A$).

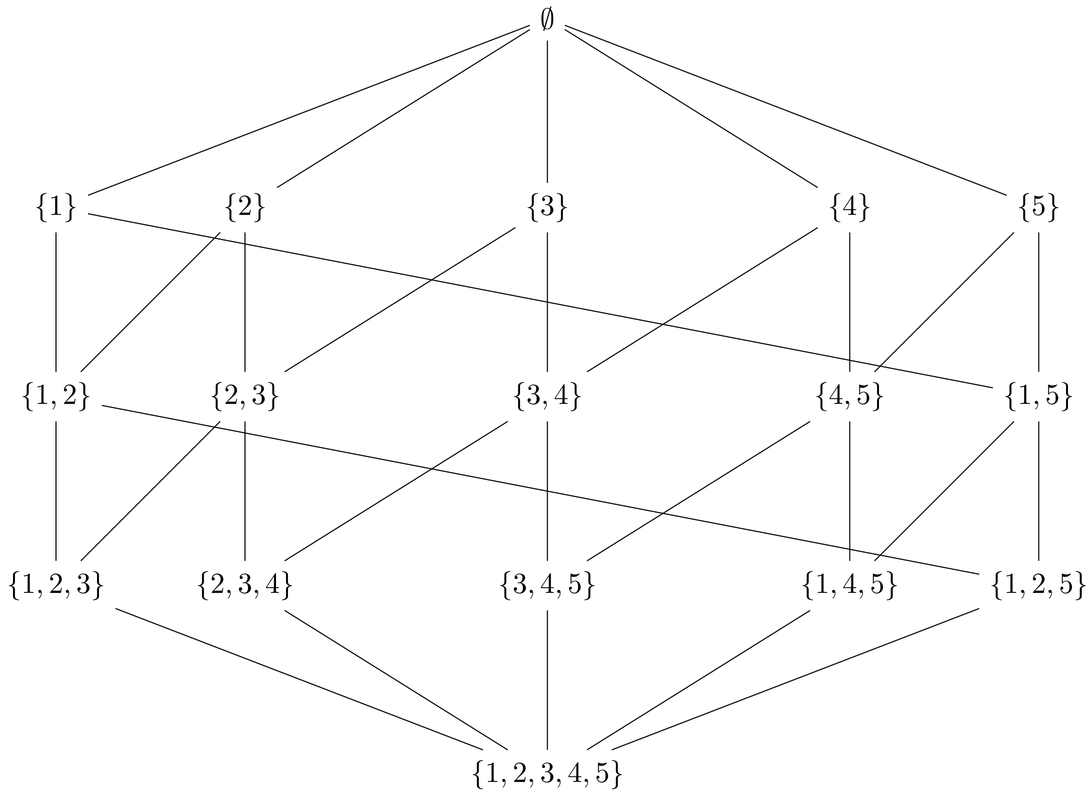


Figure 2.1: Symmetry Partition Lattice of C_5

Proof. If $A \subseteq V(C_n)$ is a maximum partitioner, then, for all $v_i \notin A$ there exist $v_j,\, v_k \in V(C_n)$ such that $v_j \equiv_A v_k$ and $F(v_i,\, v_j) \neq F(v_i,\, v_k)$. This holds if and only if it holds one of the two conditions: (1) $v_j,\, v_k$ are both in $B_{C_n}(A)$ and $|\{v_j,\, v_k\} \cap B_{C_n}(A \cup \{a\})| = 1$, or (2) $|k - j| = 2$, $v_{j+1} \in C_{C_n}(A)$ and either $v_i = v_{j-1}$ or $v_i = v_{k+1}$. Case (1) is equivalent to say that $v_{i-2} \notin A$ or $v_{i+2} \notin A$ and if both v_{i-2} and v_{i+2} are not in A, there exists another vertex $v_l \in B_{C_n}(A) \setminus \{v_{i-1},\, v_{i+1}\}$. This is clearly equivalent to condition (i) of the Theorem. Condition (2) holds if and only if $v_{j+1} \in C_{C_n}(A) \setminus C_{C_n}(A \cup \{a\})$. This is equivalent to say that $v_{j+1} \in A$, $v_{j-1} \notin A$, $v_{j+3} = v_{k+1} \notin A$ and $v_i \in \{v_{j-1},\, v_{k+1}\}$. This means that $v_{i+2} \in A$ and $v_{i+4} \notin A$ or $v_{i-2} \in A$ and $v_{i-4} \notin A$, that is condition (ii). The theorem is thus proved. $\square$

Example 2.3.7. *In Figure 2.1 we represent the symmetry partition lattice of C_5, while in Figure 2.2 that of C_6. Let us observe that $\mathbb{P}_{sym}(C_5)$ is self-dual, but that of C_6 is not. So far, there exists no geometric characterization for the symmetry partition lattice of C_n.*

2.3.2 Some Examples Taken From Digraph Context

In the next results we determine the form of any A-symmetry partition for the three digraph families, $\vec{P_n}$, $\vec{C_n}$ and $\vec{T_n}$. In order to simplify the exposure form of all the following statements we will use a convention.

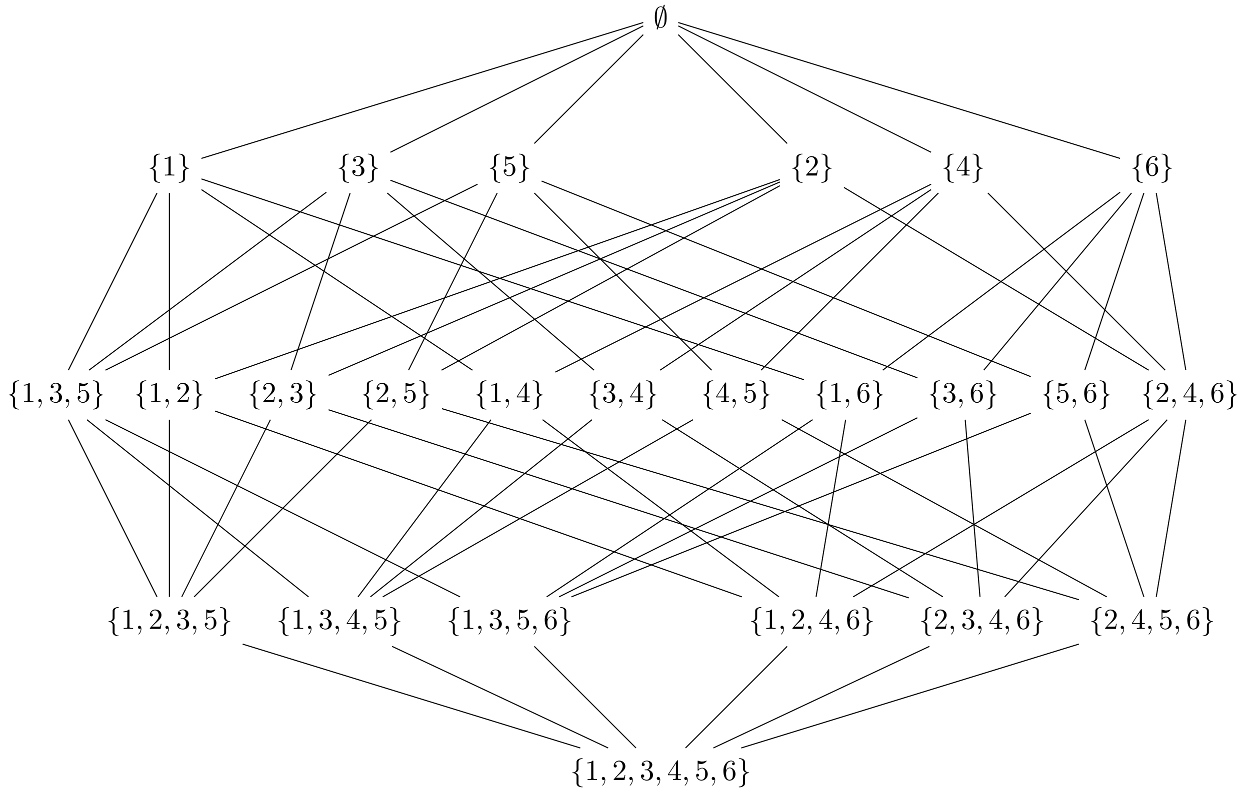


Figure 2.2: Symmetry Partition Lattice of C_6

Remark 2.3.8. Let D be one of the three digraphs $\vec{P_n}$, $\vec{C_n}$ or $\vec{T_n}$. If $A \subseteq V(D)$ is such that $|A| = k$, we implicitly assume that $1 \leq k < n$. Therefore, if $|A| = k$, we always have exactly one of the following two cases:
(i) $A = \{v_{i_1}, \ldots, v_{i_k}\}$,
(ii) $A = \{v_1, v_{i_2}, \ldots, v_{i_k}\}$,
where $1 < i_1 < i_2 < \cdots < i_k \leq n$.
Then, in the case (i) we will set $A_{-1} := \{v_{i_1-1}, \ldots, v_{i_k-1}\}$. In the case (ii) we will set $A_{-1} := \{v_{i_2-1}, \ldots, v_{i_k-1}\}$ if $D = \vec{P_n}$, whereas $A_{-1} := \{v_n, v_{i_2-1}, \ldots, v_{i_k-1}\}$ if $D = \vec{C_n}$.
Finally, only when $D = \vec{P_n}$ or $D = \vec{T_n}$, if we take a subset $A \subseteq V(D)$ such that $|A| = 1$ we implicitly assume that $A \neq \{v_1\}$.

Proposition 2.3.9. *We have that:*
(i)

$$\pi_{\vec{P_n}}(A) = \begin{cases} V & \text{if } A = \emptyset \text{ or } A \subseteq \{v_1\} \\ v_{i_1-1}|\ldots|v_{i_k-1}|A^c_{-1} & \text{if } A = \{v_{i_1}, \ldots, v_{i_k}\} \text{ and } 1 \leq |A| < n-1 \\ v_{i_2-1}|\ldots|v_{i_k-1}|A^c_{-1} & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \text{ and } 1 < |A| < n \\ v_1|\ldots|v_n & \text{if } A = V \text{ or } A = \{v_2, \ldots, v_n\} \end{cases}$$

(ii)

$$\pi_{\vec{C_n}}(A) = \begin{cases} V & \text{if } A = \emptyset \\ v_{i_1-1}|\ldots|v_{i_k-1}|A^c_{-1} & \text{if } A = \{v_{i_1}, \ldots, v_{i_k}\} \text{ and } 1 \leq |A| < n-1 \\ v_n|v_{i_2-1}|\ldots|v_{i_k-1}|A^c_{-1} & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \text{ and } 1 \leq |A| < n-1 \\ v_1|\ldots|v_n & \text{if } A = V \text{ or } |A| = n-1 \end{cases}$$

(iii)

$$\pi_{\vec{T_n}}(A) = \begin{cases} V & \text{if } A = \emptyset \text{ or } A = \{v_1\} \\ v_1 \ldots v_{i_1-1}|v_{i_1} \ldots v_{i_2-1}|\ldots|v_{i_k} \ldots v_n & \text{if } A = \{v_{i_1}, \ldots, v_{i_k}\} \text{ and } 1 \leq |A| < n-1 \\ v_1 \ldots v_{i_2-1}|\ldots|v_{i_k} \ldots v_n & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \text{ and } 2 \leq |A| < n \\ v_1|\ldots|v_n & \text{if } A = V \text{ or } A = \{v_2, \ldots, v_n\} \end{cases}$$

(iv) Let $\vec{K}_{p,q} = (B_1|B_2)$, where $B_1 = \{x_1, \ldots, x_p\}$ and $B_2 = \{y_1, \ldots, y_q\}$ and let $V = V(\vec{K}_{p,q}) = \{x_1, \ldots, x_p, y_1, \ldots, y_q\}$. Then

$$\pi^+_A(\vec{K}_{p,q}) = \begin{cases} V & \text{if } A = \emptyset \vee A \subseteq B_1 \\ B_1|B_2 & \text{otherwise} \end{cases}$$

Proof. (i): By the characterization of the symmetry relation given in Theorem 2.1.5, we have only to study the intersections $N^+_{\vec{P_n}}(v) \cap A$, when A and the vertex v vary. Therefore, if $A = \emptyset$, the result is trivial. Even if $A = V(\vec{P_n})$, computations are trivial. Let $A = \{v_1\}$, no neighborhood $N^+_{\vec{P_n}}(v)$ contains the element v_1, so every intersection $N^+_{\vec{P_n}}(v) \cap A$ is empty and $v_1, \ldots, v_n$ form a single block. Now, let A be a singleton different from $\{v_1\}$. The only neighborhood whose intersection with A is non-empty is $N^+_{\vec{P_n}}(v_{i-1})$. This means that, in this case, $\pi_{\vec{P_n}}(A) = v_{i-1}|\{v_{i-1}\}^c$. On the other hand, let $A = \{v_{i_1}, \ldots, v_{i_k}\}$ and $2 \leq k < n$.

it is clear that the neighborhoods intersecting A are $N^+_{\vec{P_n}}(v_{i_j-1})$, for $j = 1,\dots,k$. Since $N^+_{\vec{P_n}}(v_{i_j-1}) \neq N^+_{\vec{P_n}}(v_{i_l-1})$ if $j \neq l$, every vertex $v_{i_1-1},\dots,v_{i_k-1}$ form a single block and that the remaining vertices form the last block, hence $\pi_{\vec{P_n}}(A) = v_{i_1-1}|\dots|v_{i_k-1}|A^c_{-1}$. In particular, if $A = \{v_2,\dots,v_n\}$, the set A^c_{-1} consists of a single element, namely v_n, thus $\pi_{\vec{P_n}}(A) = v_1|\dots|v_n$. Furthermore, suppose $A = \{v_1, v_{i_2},\dots,v_{i_k}\}$, with $1 < |A| < n$. In this case, the neighborhoods intersecting A are $N^+_{\vec{P_n}}(v_{i_j-1})$, for $j = 2,\dots,k$, because v_1 is not contained in any neighborhood. So $\pi_{\vec{P_n}}(A) = v_{i_2-1}|\dots|v_{i_k-1}|A^c_{-1}$.
(ii): If $A = \emptyset$ or if $A = V(\vec{C_n})$ computations are obvious. Now, let $A = \{v_{i_1},\dots,v_{i_k}\}$ be a generic proper subset of $V(\vec{C_n})$ not containing v_1. Thus we have that the neighborhoods intersecting A are $N^+_{\vec{C_n}}(v_{i_j-1})$, with $j = 1,\dots,k$; furthermore, $N^+_{\vec{C_n}}(v_{i_j-1}) \neq N^+_{\vec{C_n}}(v_{i_l-1})$ if $j \neq l$, so every vertex $v_{i_1-1},\dots,v_{i_k-1}$ form a single block and the remaining vertices form the last block, since the intersections of their neighborhoods with A are empty. Let $A = \{v_1, v_{i_2}\dots,v_{i_k}\}$ be a proper subset of $V(\vec{C_n})$. As before, $N^+_{\vec{C_n}}(v_{i_j-1})$, with $j = 2,\dots,k$ and $N^+_{\vec{C_n}}(v_n)$ are the neighborhoods intersecting A. Since $N^+_{\vec{C_n}}(v_j) \neq N^+_{\vec{C_n}}(v_l)$ if $j,l \in \{i_2-1,\dots,i_k-1,n\}$ are two distinct indices, we deduce that the elements $v_{i_2-1},\dots,v_{i_k-1},v_n$ form a single block, while the others stay in another unique block. Finally, by the previous argument, if $|A| = n-1$, we observe that A^c_{-1} consists of a single element, therefore $\pi_{\vec{C_n}}(A) = v_1|\dots|v_n$.
(iii): It is clear that if $A = \emptyset$, then $\pi_{\vec{T_n}}(A)$ consists of a unique block whereas if $A = V(\vec{T_n})$, every vertex form a single block of $\pi_{\vec{T_n}}(A)$. Moreover, if $A = \{v_1\}$, we have that $N^+_{\vec{T_n}}(v_i)$ doesn't contain v_1 for all indices i, so $A \cap N^+_{\vec{T_n}}(v_i) = \emptyset$. Let $A = \{v_i\}$, where $i \neq 1$. Since $N^+_{\vec{T_n}}(v_j) = \{v_{j+1},\dots,v_n\}$, clearly $N^+_{\vec{T_n}}(v_j) \cap A = \{v_i\}$ if and only if $A \subseteq N^+_{\vec{T_n}}(v_j)$. This happens if and only if $j = 1,\dots,i-1$, otherwise, this intersection is empty. Now, we can provide the proof for the general case. Let $A = \{v_{i_1},\dots,v_{i_k}\}$ where $1 < i_1 < \dots < i_k \leq n$ and $2 \leq |A| < n-1$; we have to study the intersections $N^+_{\vec{T_n}}(v_j) \cap A$. Proceeding exactly as before, we deduce that the first block consists of $v_1,\dots,v_{i_1-1}$; the second block contains all the vertices such that $N^+_{\vec{T_n}}(v_j) \cap (A \setminus \{v_{i_1}\}) \neq \emptyset$, i.e. it contains exactly the $i_2 - i_1$ elements $v_{i_1},\dots,v_{i_2-1}$ and so on. In particular, if $A = \{v_2,\dots,v_n\}$, A^c_{-1} consists of a single element, therefore $\pi_{\vec{T_n}}(A) = v_1|\dots|v_n$. On the other hand, if $A = \{v_1, v_{i_2},\dots,v_{i_k}\}$ where $1 < i_2 < \dots < i_k \leq n$ and $2 \leq |A| < n$, the argument is the exactly the same as before, starting from the index i_2, since there is no vertex v_j whose neighborhood $N^+_{\vec{T_n}}(v_j)$ contains v_1. Thus $\pi_{\vec{T_n}}(A) = v_1 \dots v_{i_2-1}|\dots|v_{i_k}\dots v_n$ and the thesis follows.
(iv): We observe that

$$N^+_{\vec{K}_{p,q}}(v) = \begin{cases} B_2 & \text{if } v \in B_1 \\ \emptyset & \text{if } v \in B_2 \end{cases}$$

Therefore, if $A \subseteq B_1$, then $N^+_{\vec{K}_{p,q}}(v) \cap A = \emptyset$ for every vertex $v \in V$. Hence, $\pi^+_A(\vec{K}_{p,q}) = V$. In the other cases, it is easy to see that $N^+_{\vec{K}_{p,q}}(v) \cap A$ is the same subset of B_2 for every $v \in B_1$ and this intersection is empty for every $v \in B_2$. Hence, we conclude that $\pi_{\vec{K}_{p,q}}(A) = B_1|B_2$ in the other cases. □

In next result we provide the general form for the symmetry partition lattices of $\vec{T_n}$, $\vec{P_n}$

and $\vec{C_n}$.

Theorem 2.3.10. *(i) Let $D = \vec{T_n}$ or $D = \vec{P_n}$. Then*

$$MAXP(D) = \{\{v_1, v_{i_1}, \dots, v_{i_k}\} : 1 < i_1 < \dots < i_k \le n\}$$

and $(\mathbb{M}(D), \subseteq^*) \cong (\mathcal{P}([n-1]), \subseteq^*)$ by means of the isomorphism $\{v_1, v_{i_1}, \dots, v_{i_k}\} \mapsto \{i_1, \dots, i_k\}$.
(ii) If $V = V(\vec{C_n})$ we have

$$MAXP(\vec{C_n}) = \{A \subseteq V : |A| \le n-2\} \cup \{V\}$$

and $(\mathbb{M}(\vec{C_n}), \subseteq^*) \cong (\mathcal{V}(n,2), \subseteq^*)$ by means of the isomorphism $\{v_1, v_{i_1}, \dots, v_{i_k}\} \mapsto \{i_1, \dots, i_k\}$.

Proof. (i): Let $D = \vec{T_n}$ or $D = \vec{P_n}$, $A = \{v_{i_1}, \dots, v_{i_k}\} \subseteq V = V(D)$, where $1 < i_1 < \dots < i_k \le n$ and $B = \{v_1\} \cup A$. By Proposition 2.3.9, it is immediate that $A \approx_{\mathfrak{P}} B$. In particular, a set B is the maximum partitioner of $\pi_D(A)$ if and only if it is of the form $B = A \cup \{v_1\}$. In fact, let B be a maximum partitioner. Then, it must to contain the vertex v_1. On the other hand, let $B = \{v_1, v_{i_1}, \dots, v_{i_k}\}$, for $1 < i_1 < \dots < i_k \le n$. By definition of maximum partitioner, $B \subseteq M_{\mathfrak{P}}(B)$. Suppose by contradiction that $B \subsetneqq M_{\mathfrak{P}}(B)$ and let $C = M_{\mathfrak{P}}(B)$. This means that C contains at least another vertex distinct from those of B. So $\pi_D(C) \preceq \pi_D(B) = \pi_D(A)$ but $\pi_D(C) \neq \pi_D(B)$; therefore $C \not\approx_{\mathfrak{P}} B$, and this in contrast with $C = M_{\mathfrak{P}}(B)$. Therefore $MAXP(D) = \{\{v_1, v_{i_1}, \dots, v_{i_k}\}\}$, where $1 < i_1 < \dots < i_k \le n$. Moreover, it results that $A \subseteq^* B$ if and only if $B \subseteq A$. We define the bijective map $\phi : \mathbb{M}(D) \mapsto \mathcal{P}([n-1])$ such that $B = \{v_1, v_{i_1}, \dots, v_{i_k}\} \mapsto \phi(B) = \{i_1, \dots, i_k\}$. Let $A \subseteq^* B$, i.e. $B \subseteq A$; it's obvious that $\phi(B) \subseteq \phi(A)$, that's to say $\phi(A) \subseteq^* \phi(B)$. This means that $(\mathbb{M}(\vec{T_n}), \subseteq^*) \cong (\mathcal{P}([n-1]), \subseteq^*)$.
(ii): Let $A \subseteq V = V(\vec{C_n})$. By Proposition 2.3.9, it is then immediate that $[A]_{\approx_{\mathfrak{P}}} = \{A\}$ if $|A| < n-1$. Hence if $|A| > n-2$, then $\pi_{\vec{C_n}}(A) = \pi_{\vec{C_n}}(V(\vec{C_n}))$, otherwise $[A]_{\approx_{\mathfrak{P}}} = \{A\}$, i.e. the maximum partitioner of $[A]_{\approx_{\mathfrak{P}}}$ is A itself. This means that $MAXP(\vec{C_n}) = \{A \subseteq V : |A| \le n-2\} \cup \{V\}$. Define the bijective map $\phi : \mathbb{M}(\vec{C_n}) \mapsto \mathcal{V}(n,2)$ such that $A = \{v_{i_1}, \dots, v_{i_k}\} \mapsto \phi(A) = \{i_1, \dots, i_k\}$. Let $A \subseteq^* B$, i.e. $B \subseteq A$; hence $\phi(B) \subseteq \phi(A)$, i.e. $\phi(A) \subseteq^* \phi(B)$. This means that $(\mathbb{M}(\vec{C_n}), \subseteq^*) \cong (\mathcal{V}(n,2), \subseteq^*)$. □

2.3.3 Global Symmetry in Hypergraphs

By using Proposition 2.1.7 we obtain the following useful characterization of the A-symmetry classes in $\Gamma(\binom{\hat{n}}{k})$, where A is a subset of $\hat{n}$.

Proposition 2.3.11. *Let $\mathfrak{P}$ be the pairing $\Gamma(\binom{\hat{n}}{k})$ where $n \ge k$ are positive integer and let $A \subseteq \hat{n}$. Then, there exists a bijection between the set $\pi_{\mathfrak{P}}(A)$ of all A-symmetry classes in $\Gamma(H)$ and the set*

$$\Sigma_A := \{S \subseteq A : \max\{0, k+a-n\} \le |S| \le \min\{a, k\}\},$$

where $a := |A|$.

Proof. Let $S \in \Sigma_A$ and set $s := |S|$. By Proposition 2.1.7 all k-subsets of $\hat{n}$ whose intersection with A is equal to S, if non-empty, is an A-symmetry class in $\mathfrak{P}$. But if $\max\{0, k+a-n\} \le$

$s \leq \min\{a,k\}$ then there exists $\binom{n-a}{k-s}$ k-subsets of $\hat{n}$ such that their intersection with A is S. On the other hand if $x \in \binom{\hat{n}}{k}$, then the cardinality s of $S := x \cap A$ satisfies $\max\{0, k+a-n\} \leq s \leq \min\{a,k\}$. In fact $s \geq 0$, $s \leq a$ and $s \leq k$ by definition of S. Moreover it holds that $S = x \cap A = x \setminus (\hat{n} \setminus A)$ and so $s \geq k-(n-a) = k+a-n$. We proved that the function that associates with each $S \in \Sigma_A$ the A-symmetry class $C := \{x \in \binom{\hat{n}}{k} : x \cap A = S\}$ is bijective with inverse that associates with each $C \in \pi_{\mathfrak{P}}(A)$ the subset $S := A \cap \left(\bigcap_{x\in C} x\right)$. $\square$

Corollary 2.3.12. *Let $\mathfrak{P}$ be the Boolean information system $\Gamma(\binom{\hat{n}}{k})$ where $n \geq k$ are positive integers and let $A \subseteq \hat{n}$ and $x \in \binom{\hat{n}}{k}$. By setting $a := |A|$, $S := x \cap A$ and $s := |S|$ we have $\max\{0, k+a-n\} \leq s \leq \min\{a,k\}$ and $|[x]_A| = \binom{n-a}{k-s}$.*

Proof. As we have seen in the first part of the proof of Proposition 2.3.11 the A-symmetry class $[x]_A$ is the set of all k-subsets of $\hat{n}$ whose intersection with A is equal to S, $\max\{0, k+a-n\} \leq s \leq \min\{a,k\}$ and $|[x]_A| = \binom{n-a}{k-s}$. $\square$

Remark 2.3.13. Let $H = (\hat{n}, \mathcal{F})$ be a simple k-uniform hypergraph on n vertices (i.e. $\mathcal{F}$ is a sub-family of $\binom{\hat{n}}{k}$). Let $\mathfrak{P} := \Gamma(H)$ and A a subset of the attribute set $\hat{n}$.

In order to compute the number of distinct equivalence classes in the set $\pi_{\mathfrak{P}}(A)$ and the complete list of all the objects of any symmetry class $[x]_A \in \pi_{\mathfrak{P}}(A)$ it is convenient to see $\mathfrak{P}$ as a binary matrix $T(\mathfrak{P})$. With the subset $A \subseteq \Omega$, we can associate in a natural way a $\binom{n}{k} \times |A|$ submatrix $T_A(\mathfrak{P})$ of $T(\mathfrak{P})$ by choosing the columns corresponding to the elements of A. By using standard topological sort algorithms we can easily find the number of distinct rows of $T_A(\mathfrak{P})$ and partition the rows of $T_A(\mathfrak{P})$ with respect to the equality relation.

In the following result we determine the number of elements in any symmetry partition $\pi_{\mathfrak{P}}(A)$ when $\mathfrak{P} = \Gamma(\binom{\hat{n}}{k})$.

Proposition 2.3.14. *Let $\mathfrak{P} = \Gamma(\binom{\hat{n}}{k})$. If A is a subset of attributes of $\mathfrak{P}$ such that $|A| = l$, then:*
(i) if $l \leq k$ we have $|\pi_{\mathfrak{P}}(A)| = \sum_{i=0}^{\min\{l,n-k\}} \binom{l}{l-i}$;
(ii) if $l > k$ we have $|\pi_{\mathfrak{P}}(A)| = \sum_{i=0}^{\min\{k,n-l\}} \binom{l}{k-i}$.

Proof. Without loss of generality we can assume that $A = \{1, 2, \ldots, l\}$. By Proposition 2.1.7 we have that, given two k-subsets S_1, S_2 of $[n]$, $S_1 \equiv_A S_2$ if and only if $S_1 \cap A = S_2 \cap A$. Then we can associate with each class in $\pi_{\mathfrak{P}}(A)$ a subset of A. In particular for each subset B of A we can consider the k-subsets S of $[n]$ such that $S \cap A = B$. This set is either empty or it is an equivalence class in $\pi_{\mathfrak{P}}(A)$. Such a class has $\binom{n-l}{k-|B|}$ elements that can be obtained by choosing $k - |B|$ elements in $[n] \setminus A$.

Let us suppose now $l \leq k$. The k-subsets of $[n]$ containing A form a class in $\pi_{\mathfrak{P}}(A)$. More generally however we fix a subset B of A, the set of the k-subsets of $[n]$ containing B, if it is not empty, it matches an equivalence class of $\equiv_A$ with $\binom{n-l}{k-|B|}$ elements. Such a set is empty if and only if $k - |B| \leq n - l$ which is equivalent to $l - |B| \leq n - k$. Thus it follows that the number of classes in $\pi_{\mathfrak{P}}(A)$ is equal to

$$|\pi_{\mathfrak{P}}(A)| = \sum_{j=0}^{n-k} \binom{l}{j} = \sum_{i=0}^{n-k} \binom{l}{l-i}$$

and thus *(i)* holds.

If $l > k$ then obviously for each subset B of A such that $|B| > k$ there exists no k-subset S of $[n]$ such that $S \cap A = B$. When $|B| = k$ the unique k-subset S of $[n]$ containing B is B. So each k-subset in A uniquely identifies a class in $\pi_{\mathfrak{P}}(A)$. As before if $|B| < k$ and $k - |B| \leq n - l$ then the equivalence class of $\equiv_A$ of all the k-subsets S such that $S \cap A = B$ contains $\binom{n-l}{k-|B|}$ elements. Finally if $|B| < k$ and $k - |B| > n - l$ then there is not an element $S \in \binom{[n]}{k}$ such that $S \cap A = B$. Then when $l > k$ the number of classes in $\pi_{\mathfrak{P}}(A)$ is thus equal to

$$|\pi_{\mathfrak{P}}(A)| = \sum_{j=l-n+k}^{k} \binom{l}{j} = \sum_{i=0}^{n-l} \binom{l}{k-i}$$

and the proposition is proved. $\square$

By virtue of the result established in Proposition 2.3.14 it is convenient to set

$$c(n,l,k) := \begin{cases} \sum_{i=0}^{\min\{l,n-k\}} \binom{l}{l-i} & \textit{if } l \leq k \\ \\ \sum_{i=0}^{\min\{k,n-l\}} \binom{l}{k-i} & \textit{if } l > k \end{cases} \tag{2.31}$$

when n, l and k are three integers such that $0 \leq l, k \leq n$.

We determine now the symmetry partition lattice of $\binom{\hat{n}}{k}$ for all $n \geq k \geq 1$.

Theorem 2.3.15. *The lattice $(\mathbb{P}(\binom{\hat{n}}{k}), \preceq)$ is isomorphic to the dual lattice of $(\mathcal{V}(n, n-2), \subseteq)$.*

Proof. It is immediate to notice that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(\hat{n})$ when $A \subseteq \hat{n}$ has cardinality $|A| > n - 2$ and that $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(\hat{n})$ otherwise. Consider now the function $\phi : \mathbb{P}((\binom{\hat{n}}{k})) \longrightarrow \mathcal{V}(n, n-2)$ defined by:

$$\phi(\pi_{\mathfrak{P}}(A)) = \begin{cases} A & \text{if} \quad |A| \leq n-2, \\ \\ \hat{n} & \text{if} \quad |A| > n-2. \end{cases}$$

Let us prove now that ϕ is well defined. For this we have to prove that, if $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$ are two subsets of $\hat{n}$, then $\phi(\pi_{\mathfrak{P}}(A)) = \phi(\pi_{\mathfrak{P}}(B))$. Let A, $B \subseteq \hat{n}$, with $|A| \leq n - 2$, $B \leq n - 2$ and $\phi(\pi_{\mathfrak{P}}(A)) = A \neq B = \phi(\pi_{\mathfrak{P}}(B))$. Since $A \neq B$, we have $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A) \neq \emptyset$. Without loss of generality we can assume $A \setminus B \neq \emptyset$. Let $a \in A \setminus B$ and let K be a k-subset of $\hat{n}$ such that $a \in K$ and $|B \cap K| > |B| + k - n$. Note that such a K exists in this case. Set $S := B \cap K$ and let $K' \in \binom{n}{k}$ such that $B \cap K' = S$ and $a \notin K'$. By Proposition 2.1.7, $K \equiv_B K'$ but $K \not\equiv_A K'$ and thus $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(B)$.

The map ϕ is clearly both injective and surjective. Let us prove it is an isomorphism. For this let A, $B \subset \hat{n}$ such that $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(B)$. If $|A| \geq n - 1$, then $\phi(\pi_{\mathfrak{P}}(A)) = \hat{n} \supseteq \phi(\pi_{\mathfrak{P}}(B))$, for each B. Let us suppose $|A| < n-1$. In this case, since $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(B)$, we have $|B| < n-1$. Thus $\phi(\pi_{\mathfrak{P}}(A)) = A$ and $\phi(\pi_{\mathfrak{P}}(B)) = B$. Since $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(\hat{n})$, there exist K_1, $K_2 \in \binom{\hat{n}}{k}$ such that $K_1 \neq K_2$ By Proposition 2.3.11 it holds $k + a - n < |K_1 \cup A| = |K_2 \cup A| < k$, where $a = |A|$. Let $b \in B \setminus A$. Then there exist K_1', $K_2' \in \binom{\hat{n}}{k}$ such that $|K_1 \cup A| = |K_2 \cup A| = |K_1' \cup A| = |K_2' \cup A|$ and $b \in K_1' \setminus K_2'$. This implies $K_1' \equiv_A K_2'$ but $K_1' \not\equiv_B K_2'$, and this contradicts the condition $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(B)$. Thus $B \subseteq A$ and the theorem is proved. $\square$

2.3.4 Euclidean Real Line

Let us consider now the pairing $\mathcal{R} := \mathfrak{P}[\mathbb{R}, d_E]$, where d_E is the usual real euclidean distance.

Let $A \subseteq \mathbb{R}$ be fixed. We set

$$A^* := (A-1) \cup (A+1), \quad A_* := (A-1) \cap (A+1)^c \cap (A-3)^c \tag{2.32}$$

In the next result we provide the general form for an A-symmetry partition of $\mathcal{R}$.

Proposition 2.3.16. *We have that*

$$\pi_{\mathfrak{P}}(A) = \mathbb{R} \setminus A^* | \{x, x+2\}_{x \in A_*} | \{y\}_{y \in A^* \setminus (A_* \cup A_* + 2)},$$

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. It is easy to prove that $x \equiv_A y$ in $\mathbb{R}$ if and only if $x, y \in \mathbb{R} \setminus A^*$ or $x \in A_*$ and $y = x + 2$. Moreover, in this second case, since $A_* = \{x \in \mathbb{R} : x - 1 \notin A, x + 3 \notin A, x + 1 \in A\}$, there exist no other point equivalent to x beside y. □

We now characterize the maximum partitioners of $\mathbb{R}$.

Proposition 2.3.17. $A \in MAXP(\mathfrak{P})$ *if and only if for all* $a \notin A$

1′. *($a - 2 \notin A$ or $a + 2 \notin A$) and* $|\mathbb{R} \setminus A^*| \geq 2$*;*

2′. *($a + 2 \in A$ and $a + 4 \notin A$) or ($a - 2 \in A$ and $a - 4 \notin A$).*

Proof. A subset $A \subseteq \mathbb{R}$ is a maximum partitioner in $\mathbb{R}$ if and only if, for all $a \notin A$, there exist $x, y \in \mathbb{R}$ such that $x \equiv_A y$ but $x \not\equiv_{A \cup \{a\}} y$. By (2.3.16) it is equivalent to say that, for each choice of $a \notin A$, it holds at least one of the following conditions:

1. $\exists x, y \in \mathbb{R} \setminus A^*$ such that $x < y$ and $\{x, y\} \cap (A \cup \{a\})^* \neq \emptyset$;
2. $\exists x \in A_* \setminus (A \cup \{a\})_*$.

These conditions are equivalent respectively to 1′ and 2′. □

Proposition 2.3.18. *We have that*

$$M_{\mathfrak{P}}(A) = A \cup [(A-2) \cap (A+2) \cap (A-4) \cap (A+4)]. \tag{2.33}$$

Proof. It is an immediate consequence of Proposition 2.3.17. □

Corollary 2.3.19. $A \in MAXP(\mathcal{R})$ *if and only if* $(A-2) \cap (A+2) \cap (A-4) \cap (A+4) \subseteq A$.

As a consequence of Proposition 2.3.17 we obtain the following result.

Proposition 2.3.20. *If A is a bounded interval of* $\mathbb{R}$ *then* $A \in MAXP(\mathcal{R})$.

Proof. Let A be a bounded interval of the real line. Let us consider $A = [\alpha, \beta)$, the other cases are similar. Let $a \notin A$. If $a \in (-\infty, \alpha + 2)$ or $a \in [\beta - 2, +\infty)$, then $a - 2 \notin A$ or $a + 2 \notin A$, so condition 1′. is satisfied. If $a \in (-\infty, \alpha+2) \cup [\beta-2, +\infty)$, then $\beta - \alpha < 2$, $\{a-2, a+2\} \cap A \neq \emptyset$ and thus if $a - 2 \in A$, then $a - 4 \notin A$ and if $a + 2 \in A$, then $a + 4 \notin A$. It follows the thesis. □

2.3.5 Global Symmetry in $(V, \varphi, \mathbb{K})$-VBP

If $A, B \subseteq V$, then

$$A \approx_{\mathfrak{P}} B \iff V/A^{\perp} = V/B^{\perp} \iff A^{\perp} = B^{\perp}, \tag{2.34}$$

so that

$$M_{\mathfrak{P}}(A) = \{b \in V : (A \cup \{b\})^{\perp} = A^{\perp}\}. \tag{2.35}$$

We denote by $\mathrm{Span}(A)$ the vector subspace of V spanned by A. In next result, we study the relation between $M_{\mathfrak{P}}(A)$ and $\mathrm{Span}(A)$ and give a sufficient condition on φ to ensure that in finite dimensional vector spaces $\mathrm{Span}(A) = M_{\mathfrak{P}}(A)$.

Proposition 2.3.21. *(i) $M_{\mathfrak{P}}(A)$ is a vector subspace of V;*
(ii) $\mathrm{Span}(A) \subseteq M_{\mathfrak{P}}(A)$.
Moreover, let V be a finite dimensional vector space. Then:
(iii) if φ is (left) non degenerate, $\mathrm{Span}(A) = M_{\mathfrak{P}}(A)$;
(iv): If φ is (left) non degenerate, then $MAXP(V) = \{H : H \leqslant V\}$;
(v) if φ is (left) degenerate and $V^{\perp} \setminus \mathrm{Span}(A) \neq \emptyset$, we have $\mathrm{Span}(A) \subsetneqq M_{\mathfrak{P}}(A)$.

Proof. (i): Let $v, w \in M_{\mathfrak{P}}(A)$ and let us consider $\lambda v + \mu w$. It is obvious that $(A \cup \{\lambda v + \mu w\})^{\perp} \subseteq A^{\perp}$. On the other hand, let $x \in A^{\perp}$. Then $\varphi(x, a) = 0$ for any $a \in A$. Moreover, $\varphi(x, \lambda v + \mu w) = \lambda\varphi(x, v) + \mu\varphi(x, w) = 0$ since $(A \cup \{v\})^{\perp} = (A \cup \{w\})^{\perp} = A^{\perp}$, so $x \in (A \cup \{\lambda v + \mu w\})^{\perp}$ and $(A \cup \{v + w\})^{\perp} = A^{\perp}$. This means that $\lambda v + \mu w \in M_{\mathfrak{P}}(A)$ and $M_{\mathfrak{P}}(A)$ is a vector space.
(ii): Since $A \subseteq M_{\mathfrak{P}}(A)$ and $M_{\mathfrak{P}}(A)$ is a vector space, we have that $\mathrm{Span}(A) \subseteq M_{\mathfrak{P}}(A)$.
(iii): Since V is a finite dimensional vector space and φ is (left) non degenerate, for any $H \leqslant V$ it results that $H = (H^{\perp})^{\perp}$. Suppose now that $a \in M_{\mathfrak{P}}(A)$, i.e. $(A \cup \{a\})^{\perp} = A^{\perp}$. Let $W = \mathrm{Span}(\{a\})$, then $(\mathrm{Span}(A) + W)^{\perp} = \mathrm{Span}(A)^{\perp}$, so that $((\mathrm{Span}(A) + W)^{\perp})^{\perp} = (\mathrm{Span}(A)^{\perp})^{\perp}$. Hence $\mathrm{Span}(A) + W = \mathrm{Span}(A)$, i.e. $a \in \mathrm{Span}(A)$. Thus $M_{\mathfrak{P}}(A) \subseteq \mathrm{Span}(A)$ and $M_{\mathfrak{P}}(A) = \mathrm{Span}(A)$.
(iv): It follows immediately by part (iii).
(v): Let $b \in V^{\perp} \setminus \mathrm{Span}(A)$. In such a case, $\mathrm{Span}(A \cup \{b\})^{\perp} = A^{\perp}$, i.e. $\mathrm{Span}(A) \subsetneqq M_{\mathfrak{P}}(A)$. ☐

2.3.6 Global Symmetry by Group Actions

If $A, B \subseteq \Omega$ and $\mathfrak{P} = \mathfrak{P}[G, \psi]$, we have that

$$A \approx_{G,\psi} B \iff \mathrm{Stab}_{G,\psi}(A) = \mathrm{Stab}_{G,\psi}(B).$$

Let us consider the pairing $(S_n, \psi) := \mathfrak{P}[S_n, \psi]$ induced by the action of the symmetric group S_n on a set Ω having n elements. Then we have the following result:

Theorem 2.3.22. *$MAXP((S_n, \psi)) = \{A \subseteq \Omega : |A| \leq n - 2\} \cup \{\Omega\}$.*

Proof. Let $A \subseteq \Omega$ such that $|A| = k$. Hence, there exists exactly $(n - k)!$ permutations of S_n fixing all elements of A. Hence $|\mathrm{Stab}_{S_n,\psi}(A)| = (n - k)!$. Moreover, since the index of $\mathrm{Stab}_{S_n,\psi}(A) = \frac{n!}{(n-k)!}$, we deduce that there are $\frac{n!}{(n-k)!}$ classes of A-symmetry for any A. We conclude that if $0 \leq |A| \neq |B| \leq n - 2$, then $\pi_{S_n,\psi}(A) \neq \pi_{S_n,\psi}(B)$. Furthermore, if $1 \leq |A| = |B| \leq n - 2$, we clearly have $[Id]_A = \mathrm{Stab}_{S_n,\psi}(A) \neq \mathrm{Stab}_{S_n,\psi}(B) = [Id]_B$, so $\pi_{S_n,\psi}(A) \neq \pi_{S_n,\psi}(B)$. Finally, we have that $\pi_{S_n,\psi}(A) = \pi_{S_n,\psi}(\Omega)$ for any $A \subseteq \Omega$ such that $|A| = n - 1$. This gives $MAXP((S_n, \psi)) = \{A \subseteq \Omega : |A| \leq n - 2\} \cup \{\Omega\}$. ☐

2.4 Separators and Related Notions

In this section we assume a topological point of view for our investigation of the maximum partitioner family. To be more detailed, the maximum partitioners can be seen as a generalization of the closed sets of a topology on Ω. In this perspective, it is convenient to undertake an analysis of the family of all corresponding generalized open sets, i.e. the complement of each maximum partitioner. Therefore, if $A \subseteq \Omega$, we set

$$S_{\mathfrak{P}}(A) := \Omega \setminus M_{\mathfrak{P}}(A) \tag{2.36}$$

and

$$SEP(\mathfrak{P}) := \{S_{\mathfrak{P}}(A) : A \in \mathcal{P}(\Omega)\}.$$

We call *separator* of A any element $s \in S_{\mathfrak{P}}(A)$. The following characterization is immediate.

Proposition 2.4.1. *Let $s \in \Omega$. Then s is a separator of A if and only if there exist two point $u, u' \in U$ such that :*

(i) $F(u,a) = F(u',a)$ $\forall a \in A$;

(ii) $F(u,s) \neq F(u',s)$.

If $\mathcal{A} = \{A_i : i \in I\} \in SS(\Omega)$, we set

$$S_{\mathfrak{P}}(\mathcal{A}) := \bigcap_{i \in I} S_{\mathfrak{P}}(A_i). \tag{2.37}$$

Proposition 2.4.2. $S_{\mathfrak{P}}(\cup\mathcal{A}) \subseteq S_{\mathfrak{P}}(\mathcal{A})$.

Proof. Let $s \in S_{\mathfrak{P}}(\cup\mathcal{A})$, then there exist $u, u' \in U$ with the property that for any $a \in \bigcup_{i\in I} A_i$, $F(u,a) = F(u',a)$ and $F(u,s) \neq F(u',s)$. In other words, for any fixed $\tilde{i} \in I$ and for any choice of $a \in A_{\tilde{i}} \subseteq \bigcup_{i\in I} A_i$, it follows that $F(u,a) = F(u',a)$ and $F(u,s) \neq F(u',s)$, that is $s \in S_{\mathfrak{P}}(A_{\tilde{i}})$. It readily follows that $s \in S_{\mathfrak{P}}(A_i)$ for any $i \in I$ and hence $s \in S_{\mathfrak{P}}(\mathcal{A})$. $\square$

We say that a chain $\mathcal{A} \in SS(\Omega)$ is *union preserving separated* if $S_{\mathfrak{P}}(\mathcal{A}) \subseteq S_{\mathfrak{P}}(\cup\mathcal{A})$.

In what follows we define a new class of pairings, i.e. those for which any chain is union preserving.

Definition 2.4.3. We say that the pairing $\mathfrak{P}$ is *union chain preserving*, abbreviated *UCP*, if any chain $\mathcal{A} \in SS(\Omega)$ is union preserving separated.

Remark 2.4.4. Any finite pairing is UCP.

In the following result, we provide other alternative characterizations for UCP pairings.

Theorem 2.4.5. *The following three conditions are equivalent:*

(i) $\mathfrak{P}$ *is UCP;*

(ii) $MAXP(\mathfrak{P})$ *is a chain union-closed family;*

(iii) For any $A \in \mathcal{P}(\Omega)$ we have

$$M_{\mathfrak{P}}(A) = \bigcup\{M_{\mathfrak{P}}(F) : F \subseteq_f A\}. \tag{2.38}$$

Proof. $(i) \implies (ii)$: Let $\mathfrak{P}$ be a UCP pairing and let $\mathcal{B} = \{B_i : i \in I\}$ be a chain of maximum partitioners of Ω. Let $B := \cup\mathcal{B}$ and $s \in \Omega \setminus B$. Then $s \in \Omega \setminus B_i = \Omega \setminus M_{\mathfrak{P}}(B_i) = S(B_i)$ for all $i \in I$, because any B_i is a maximum partitioner and therefore $M_{\mathfrak{P}}(B_i) = B_i$. Hence $s \in \bigcap_{i \in I} S(B_i) := S(\mathcal{B})$, and $S(\mathcal{B}) \subseteq S(\cup\mathcal{B})$ by hypothesis, so that $s \in S(B)$. This shows that $\Omega \setminus B \subseteq S(B) = \Omega \setminus M_{\mathfrak{P}}(B)$, i.e. $M_{\mathfrak{P}}(B) \subseteq B$, and hence $B \in MAXP(\mathfrak{P})$. Therefore $MAXP(\mathfrak{P})$ is a chain union-closed family.
$(ii) \implies (i)$: Let $MAXP(\mathfrak{P})$ be a chain union-closed family and let $\mathcal{A} = \{A_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ be a chain. Since $A_i \subseteq A_j$ implies that $M_{\mathfrak{P}}(A_i) \subseteq M_{\mathfrak{P}}(A_j)$, it follows that $\mathcal{A}_M := \{M_{\mathfrak{P}}(A_i) : i \in I\}$ is a chain of maximum partitioners. Then by hypothesis we have that $K := \bigcup_{i \in I} M_{\mathfrak{P}}(A_i) \in MAXP(\mathfrak{P})$. Let us note now that

$$S(\mathcal{A}) := \bigcap_{i \in I} S(A_i) = \bigcap_{i \in I} \Omega \setminus M_{\mathfrak{P}}(A_i) = \Omega \setminus K. \tag{2.39}$$

Then since $K = M_{\mathfrak{P}}(K)$, by (2.39) we deduce that

$$S(\mathcal{A}) = \Omega \setminus M_{\mathfrak{P}}(K) = S(K). \tag{2.40}$$

Now, since $A_i \subseteq M_{\mathfrak{P}}(A_i)$ for all $i \in I$, we have $\cup\mathcal{A} := \bigcup_{i \in I} A_i \subseteq K$, therefore

$$S(K) \subseteq S(\cup\mathcal{A}). \tag{2.41}$$

Hence, by (2.40) and (2.41) we deduce that $S(\mathcal{A}) \subseteq S(\cup\mathcal{A})$. This shows that $\mathfrak{P}$ is an UCP.
$(ii) \implies (iii)$: If $\mathfrak{P}$ is a pairing on a finite set Ω, then the thesis is obvious. Let now $\mathfrak{P}$ be a pairing on an infinite set Ω and let $A \subseteq \Omega$. We endow A with a well-order $<$. Set

$$C(a) := \{b \in A : \, b < a\}$$

for any $a \in A$. Let us assume that the thesis holds for sets whose cardinality is less than $|A|$ and, furthermore, that for any $a \in A$, the set $C(a)$ has cardinality strictly less than $|A|$. In particular, the thesis holds for $C(a)$, for any $a \in A$. Let $\mathfrak{Q} := \{M_{\mathfrak{P}}(C(a)) : \, a \in A\}$. It is clearly a chain in $MAXP(\mathfrak{P})$. Since $MAXP(\mathfrak{P})$ is a chain union-closed family, we have that the union

$$D := \bigcup \mathfrak{Q}$$

belongs to $\mathfrak{Q}$. Let $K := \bigcup_{a \in A} C(a)$. Hence, we have that $M_{\mathfrak{P}}(C(a)) \subseteq M_{\mathfrak{P}}(K)$ for any $a \in A$, so $D \subseteq M_{\mathfrak{P}}(K)$. On the other hand, it is obvious that $M_{\mathfrak{P}}(K) \subseteq D$, thus $D = M_{\mathfrak{P}}(K)$. By transfinite inductive hypothesis, the thesis holds for $C(a)$, for any $a \in A$. We observe that for any $F \subseteq_f A$, there exists $a \in A$ such that $F \subseteq_f C(a)$. This shows the claim.
$(iii) \implies (ii)$: Let $\mathcal{C}$ be a non-empty chain of maximum partitioners in $MAXP(\mathfrak{P})$ and let $B := \bigcup \mathcal{C}$. We will show that $B \in MAXP(\mathfrak{P})$. To this regard, assume by contradiction that there exists $F \subseteq_f \bigcup \mathcal{C}$ such that $F \not\subseteq_f C$ for any $C \in \mathcal{C}$. Let us fix $a \in F$. Then, just take

$C_1, C_2 \in \mathcal{C}$ such that $x \in C_1$ and $F \setminus \{a\} \subseteq C_2$. Therefore, if $C_1 \subseteq C_2$, we would have $F \subseteq C_2$, that is an absurd and, analogously, if $C_2 \subseteq C_1$, then $F \subseteq C_1$, once again a contradiction. Thus, we conclude that for any $F \subseteq_f \bigcup \mathcal{C}$, there must be $C \in \mathcal{C}$ containing it. Furthermore, we must have $M_{\mathfrak{P}}(F) \subseteq C \subseteq B$ so, by our hypothesis, the claim has been showed. □

In the next result, we provide some sufficient condition in order to verify the existence of a maximal element for any non-empty subfamily $\mathcal{A}$ of $MAXP(\mathfrak{P})$.

Theorem 2.4.6. *If $MAXP(\mathfrak{P})$ is a chain union-closed family and for any $C \in MAXP(\mathfrak{P})$ there exists $F \subseteq_f C$ such that $F \approx_{\mathfrak{P}} C$, then any non-empty subfamily $\mathcal{A}$ of $MAXP(\mathfrak{P})$ admits a maximal element.*

Proof. Let $\mathcal{A}$ be a non-empty subfamily of $MAXP(\mathfrak{P})$. Since $MAXP(\mathfrak{P})$ is a chain union-closed family, then $\bigcup \mathcal{K} \in MAXP(\mathfrak{P})$ for any $\mathcal{K}$ chain in $MAXP(\mathfrak{P})$. Therefore, let us take a chain $\mathcal{K}$ in $\mathcal{A}$, so $C := \bigcup \mathcal{K} \in MAXP(\mathfrak{P})$. By our assumption, there exists $F \subseteq_f C$ such that $F \approx_{\mathfrak{P}} C$. Let us define the map $\phi : a \in F \mapsto A_a \in \mathcal{K}$, where A_a is a maximum partitioner in $\mathcal{K}$ containing a. Then, we obtain a finite chain $\{A_a : a \in F\}$ in $\mathcal{A}$ clearly having a maximal element, that we call A_b, for some $b \in F$. Thus, $A_b \in MAXP(\mathfrak{P})$. Since $F \subseteq A_b \subseteq C$, we deduce that $M_{\mathfrak{P}}(A_b) = A_b = C$. But since $\mathcal{K}$ is a maximal chain, we conclude that A_b is a maximal element in $\mathcal{A}$ and this completes the proof. □

As a consequence, we show that $\mathcal{R}$ is a UCP pairing.

Theorem 2.4.7. *$\mathcal{R}$ is a UCP pairing.*

Proof. Let us prove that $MAXP(\mathcal{R})$ is a chain union-closed family. Let $\mathcal{A} = \{A_i : i \in I\}$ be a chain of maximum partitioners of $\mathcal{R}$ and let $A := \bigcup_{i \in I} A_i$. Let $a \notin A$ and suppose $a-2$, $a+2$, $a+4 \in A$. Since $\mathcal{A}$ is a chain, there exists $i \in I$ such that $a-2$, $a+2$ and $a+4$ are all in A_i. It follows that $a-4 \notin A_i$ and thus for all $j \in I$ such that $A_j \subseteq A_i$, $a-4 \notin A_j$. Moreover if $j \in I$ is such that $A_i \subseteq A_j$, then $a-2$, $a+2$, $a+4 \in A_j$ thus, since $A_j \in MAXP(\mathcal{R})$, $a-4 \notin A_j$. We conclude that $a-4 \notin A$ and $A \in MAXP(\mathcal{R})$.
Let us suppose now $|\mathbb{R} \setminus A^*| \leq 1$ and that $a+2 \notin A$ and $a-2 \in A$. We have to prove that $a-4 \notin A$. If $a-4 \in A$ then for some $i \in I$, $|\mathbb{R} \setminus A_i^*| \leq 1$, $a+2 \notin A_i$, $a-2 \in A_i$ and $a-4 \in A_i$ and this contradicts the fact that A_i is a maximum partitioner. Moreover it can not hold simultaneusly that both $a+2$ and $a-2$ do not belong to A, because in this case, since $a \notin A$, it would hold that $|\mathbb{R} \setminus A^*| \geq 1$, contradicting our assumption.
Suppose $a+4 \in A$. If $a-2 \notin A$, then $a+2 \in A$ and there exists $i \in I$ such that $|\mathbb{R} \setminus A_i^*| \leq 1$, $a+2 \in A_i$, $a-2 \notin A_i$ and $a+4 \in A_i$, but this is impossible because $A_i \in MAXP(\mathcal{R})$. Finally suppose that both $a-4$ and $a+4$ are in A. Then, in this case for some $i \in I$, $|\mathbb{R} \setminus A_i^*| \leq 1$, $a-4 \in A_i$ and $a+4 \in A_i$, so A_i is not a maximum partitioner by Proposition 2.3.17, which is a contradiction. It follows that $A \in MAXP(\mathcal{R})$. □

2.4.1 Coverings and UCP Pairings

In this subsection, we analyze which conditions must be satisfied in order to find a *covering* on Ω constituted by maximum partitioners having well specific topological properties.

Let A, B be two non-empty subsets of Ω. We set

$$\mathbb{E}_{\mathfrak{P}}(A, B) := \{a \in \Omega : \ A \cap M_{\mathfrak{P}}(B \cup \{a\}) \neq \emptyset\}$$

and we call $\mathbb{E}_{\mathfrak{P}}(A, B)$ the *B-expansion of A*.

In the following proposition we establish some basic properties for the above expansion.

Proposition 2.4.8. *The following conditions hold:*
(i) $A \subseteq \mathbb{E}_{\mathfrak{P}}(A, B)$*;*
(ii) If $A \cap B \neq \emptyset$*, then* $M_{\mathfrak{P}}(A \cup B) \subseteq \mathbb{E}_{\mathfrak{P}}(A, B)$*;*
(iii) If $C \subseteq A$ *and* $D \subseteq B$*, then* $\mathbb{E}_{\mathfrak{P}}(C, D) \subseteq \mathbb{E}_{\mathfrak{P}}(A, B)$*;*
(iv) $\mathbb{E}_{\mathfrak{P}}(A, B) = \bigcup\limits_{a \in A} \mathbb{E}_{\mathfrak{P}}(\{a\}, B)$*.*

Proof. Straightforward. □

At this point we provide some sufficient conditions ensuring that $\mathbb{E}_{\mathfrak{P}}(A, B) \in MAXP(\mathfrak{P})$.

Theorem 2.4.9. *Let* $\mathfrak{P}$ *be a UCP pairing and* $A, B \in MAXP(\mathfrak{P})$ *be two non-empty maximum partitioners. Assume that* $\mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G)) \in MAXP(\mathfrak{P})$ *for any* $F \subseteq_f A$ *and* $G \subseteq_f B$*. Then* $\mathbb{E}_{\mathfrak{P}}(A, B) \in MAXP(\mathfrak{P})$*.*

Proof. Since $\mathfrak{P}$ is a UCP pairing, we must prove that the maximum partitioner of each finite subset of $\mathbb{E}_{\mathfrak{P}}(A, B)$ belongs to $\mathbb{E}_{\mathfrak{P}}(A, B)$. For this, let us consider $c_1, \ldots, c_n \in \mathbb{E}_{\mathfrak{P}}(A, B)$. By definition of B-expansion of A, there exist $a_1, \ldots, a_n$ such that $a_i \in A \cap M_{\mathfrak{P}}(B \cup \{c_i\})$. Since $\mathfrak{P}$ is a UCP pairing, for each index i there exists $G_i \subseteq_f B$ such that $a_i \in M_{\mathfrak{P}}(G_i \cup \{c_i\})$. Thus, we set $F := \{a_1, \ldots, a_n\}$ and $G := \bigcup\limits_{i=1}^{n} G_i$. It is obvious that $c_i \in \mathbb{E}_{\mathfrak{P}}(\{a_i\}, M_{\mathfrak{P}}(G))$. By (iv) of Proposition 2.4.8, it follows that $c_i \in \mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G))$ for any $i = 1, \ldots, n$ which is a maximum partitioner by our assumption. But by (iii) of Proposition 2.4.8, it follows that $\mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G)) \subseteq \mathbb{E}_{\mathfrak{P}}(A, B)$, so $M_{\mathfrak{P}}(\{c_1, \ldots, c_n\}) \subseteq \mathbb{E}_{\mathfrak{P}}(A, B)$ and, by the arbitrariness of $c_1, \ldots, c_n$, the claim has been proved. □

We now characterize the existence of a $M_{\mathfrak{P}}$-split for any pair of disjoint maximum partitioners of a UCP pairing in terms of the expansions of pairs of finite subsets whose maximum partitioners are disjoint from each other.

Theorem 2.4.10. *Let* $\mathfrak{P} \in PAIR(\Omega)$ *be a UCP pairing. Then the following conditions are equivalent:*
(i) For each $F, G \subseteq_f \Omega$ *such that* $M_{\mathfrak{P}}(F) \cap M_{\mathfrak{P}}(G) = \emptyset$*, there exists a covering* $\{A, B\}$ *on* Ω *such that* $\{M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G)\} \wr_{M_{\mathfrak{P}}} \{A, B\}$*.*
(ii) For any $F, G \subseteq_f \Omega$*, we have that* $\mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G)) \in MAXP(\mathfrak{P})$*.*
(iii) For any $A, B \in MAXP(\mathfrak{P})$ *such that* $A \cap B = \emptyset$*, there exists a* $M_{\mathfrak{P}}$*-complemented subset* $C \in \mathcal{P}(\Omega)$ *such that* $\{A, B\} \wr_{M_{\mathfrak{P}}} \{C, C^c\}$*.*

Proof. $(i) \implies (ii)$: Suppose by contradiction there exist $F, G \subseteq_f \Omega$ such that

$$\mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G)) \notin MAXP(\mathfrak{P})$$

Let $C := M_{\mathfrak{P}}(\mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G)))$ and let $a \in C \setminus \mathbb{E}_{\mathfrak{P}}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G))$. This means that $M_{\mathfrak{P}}(F) \cap M_{\mathfrak{P}}(M_{\mathfrak{P}}(G) \cup \{a\}) = \emptyset$. Let $A, B \in MAXP(\mathfrak{P})$ such that $\{M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G) \cup \{a\}\} \wr_{M_{\mathfrak{P}}} \{A, B\}$. Hence, $M_{\mathfrak{P}}(F) \cap A = \emptyset = (M_{\mathfrak{P}}(G) \cup \{a\}) \cap B$, i.e.

$$M_{\mathfrak{P}}(G) \cup \{a\} \subseteq A \setminus B \text{ and } M_{\mathfrak{P}}(F) \subseteq B \setminus A. \tag{2.42}$$

A fortiori, we have that $G \cup \{a\} \subseteq A \setminus B$. Nevertheless, it is straightforward that $C \subseteq B$ since if $b \in \mathbb{E}(M_{\mathfrak{P}}(F), M_{\mathfrak{P}}(G))$, then $M_{\mathfrak{P}}(F) \cap M_{\mathfrak{P}}(G) \cup \{b\} \neq \emptyset$, so $b \in B$. In particular, it follows that $a \in B$, contradicting (2.42).
$(ii) \implies (iii)$: Let $A, B \in MAXP(\mathfrak{P})$. Surely, we can always find two maximum partitioners A' and B' containing respectively A and B and maximal with respect to the property of being disjoint. Clearly, if one of them is empty, then Ω is the $M_{\mathfrak{P}}$-complemented subset we are searching for. Therefore, let us assume that A' and B' are both non-empty and let $a \in (B')^c$. By maximality, it follows that $M_{\mathfrak{P}}(B' \cup \{a\}) \cap A' \neq \emptyset$. This proves that $B' \subseteq \Omega \setminus \mathbb{E}_{\mathfrak{P}}(A', B')$, which belongs to $MAXP(\mathfrak{P})$ by Theorem 2.4.9. Thus, we have found a maximum partitioner $\mathbb{E}_{\mathfrak{P}}(A', B')$ such that $A' \subseteq \mathbb{E}_{\mathfrak{P}}(A', B')$ and $B' \subseteq \Omega \setminus \mathbb{E}_{\mathfrak{P}}(A', B')$. In particular, by maximality, it is immediate to prove that $B' = \Omega \setminus \mathbb{E}_{\mathfrak{P}}(A', B')$.
Finally, we must prove that $A' \cup B' = \emptyset$. Suppose by contradiction the claim is false. We firstly observe that $A' \subsetneqq \mathbb{E}_{\mathfrak{P}}(A', B')$. In fact, if $c \in \Omega \setminus (A' \cup B') = (A')^c \cap (B')^c$ then, since $B' = \Omega \setminus \mathbb{E}_{\mathfrak{P}}(A', B')$, it results that $A' \subsetneqq \mathbb{E}_{\mathfrak{P}}(A', B')$. Now, again by maximality, it follows that $\mathbb{E}_{\mathfrak{P}}(A', B') \cap B' \neq \emptyset$ or, equivalently, there exists $b \in B'$ such that $M_{\mathfrak{P}}(B' \cup \{b\}) \cap A' \neq \emptyset$, i.e. $A' \cap B' \neq \emptyset$, contradicting our assumptions. This proves the thesis.
$(iii) \implies (i)$: Obvious. □

2.5 Recursive Properties of $M_{\mathfrak{P}}$ in UCP Pairings

In this section, we investigate further properties of the set operator $M_{\mathfrak{P}}$. In particular, we find the conditions needed to express $M_{\mathfrak{P}}(A \cup \{b\})$ as the union of the maximum partitioners $M_{\mathfrak{P}}(\{a, b\})$ for any $a \in A$. In other terms, we provide some conditions whose main effect consists of a recursive-like action of the closure operator $M_{\mathfrak{P}}$ on subset $A \cup \{b\}$.

Definition 2.5.1. Let $A \in MAXP(\mathfrak{P})$ and $b \in \Omega$. We say that the pair (A, b) is $M_{\mathfrak{P}}$-*union decomposable* if

$$M_{\mathfrak{P}}(A \cup \{b\}) = \bigcup_{a \in A} M_{\mathfrak{P}}(\{a, b\}). \tag{2.43}$$

We say that the pairing $\mathfrak{P}$ is $M_{\mathfrak{P}}$-*union decomposable* if any pair $(A, b) \in MAXP(\mathfrak{P}) \times \Omega$ is $M_{\mathfrak{P}}$-union decomposable.

We now define a specific tipology of pairing that will be useful in the sequel.

Definition 2.5.2. We say that a pairing $\mathfrak{P} \in PAIR(\Omega)$ is *finitary* if $|M_{\mathfrak{P}}(F)| < \infty$ for any $F \subseteq_f \Omega$. We denote by $PAIR_{fin}(\Omega)$ the family of all finitary pairings.

We now show that a finitary pairing $\mathfrak{P}$ is $M_{\mathfrak{P}}$-union decomposable whenever the property of a maximum partitioner A to be covered by a finite number of maximum partitioner A_i is inherited by $A \cup \{b\}$, where $b \notin A$; in such a way all its covering maximum partitioners are of the form $M_{\mathfrak{P}}(A_i \cup \{b\})$.

Proposition 2.5.3. *Let $\mathfrak{P}$ be a finitary UCP pairing on Ω. The following conditions are equivalent:*

- (i) *$\mathfrak{P}$ is $M_{\mathfrak{P}}$-union decomposable;*
- (ii) *Let $F, F_1, \dots, F_n \subseteq_f \Omega$, $A := M_{\mathfrak{P}}(F)$, $A_i := M_{\mathfrak{P}}(F_i)$. If $A \subseteq \bigcup_{i=1}^{n} A_i$, for any $b \in \Omega \setminus A$ it results that*

$$M_{\mathfrak{P}}(A \cup \{b\}) \subseteq \bigcup_{i=1}^{n} M_{\mathfrak{P}}(A_i \cup \{b\}); \tag{2.44}$$

- (iii) *for any non-empty $F \subseteq_f \Omega$ and $b \notin M_{\mathfrak{P}}(F)$, it results that*

$$M_{\mathfrak{P}}(F \cup \{b\}) = \bigcup_{a \in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}). \tag{2.45}$$

Proof. $(i) \implies (ii)$: Let $\mathfrak{P}$ be a $M_{\mathfrak{P}}$-union decomposable pairing. Let $A, A_1, \dots, A_n \in MAXP(\mathfrak{P})$ such that $A \subseteq \bigcup_{i=1}^{n} A_i$ and $b \in \Omega \setminus A$. By hypothesis, for any $a \in A$ there exists A_i such that $a \in A_i$. Then $\{a, b\} \subseteq A_i \cup \{b\}$. Hence, it results that

$$\bigcup_{a \in A} M_{\mathfrak{P}}(\{a, b\}) \subseteq \bigcup_{i=1}^{n} M_{\mathfrak{P}}(A_i \cup \{b\}) \tag{2.46}$$

and, since $\mathfrak{P}$ is $M_{\mathfrak{P}}$-union decomposable, that

$$M_{\mathfrak{P}}(A \cup \{b\}) \subseteq \bigcup_{i=1}^{n} M_{\mathfrak{P}}(A_i \cup \{b\}). \tag{2.47}$$

$(ii) \implies (iii)$: Let $\emptyset \neq F \subseteq_f \Omega$. We clearly have that

$$\{a, b\} \subseteq M_{\mathfrak{P}}(F) \cup \{b\}$$

for any $a \in M_{\mathfrak{P}}(F)$. Hence

$$\bigcup_{a \in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}) \subseteq M_{\mathfrak{P}}(M_{\mathfrak{P}}(F) \cup \{b\})$$

or, equivalently, by (i) of Theorem 2.2.1,

$$\bigcup_{a \in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}) \subseteq M_{\mathfrak{P}}(F \cup \{b\}). \tag{2.48}$$

On the other hand, set $F_i = M_{\mathfrak{P}}(\{a_i\})$ for any $a_i \in M_{\mathfrak{P}}(F)$ and let $b \in \Omega \setminus F$. We observe that

$$M_{\mathfrak{P}}(F\cup\{b\}) = M_{\mathfrak{P}}(M_{\mathfrak{P}}(F)\cup\{b\}) \subseteq \bigcup_{i=1}^{n} M_{\mathfrak{P}}(F_i\cup\{b\}) = \bigcup_{i=1}^{n} M_{\mathfrak{P}}(\{a_i, b\}) = \bigcup_{a\in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}).$$

$(iii) \implies (i)$: Let $A \in MAXP(\mathfrak{P})$. If $b \in A$, then $M_{\mathfrak{P}}(A \cup \{b\}) = M_{\mathfrak{P}}(A) = A$ therefore it is straightforward to see that $A \subseteq \bigcup_{a\in A} M_{\mathfrak{P}}(\{a, b\})$.

Let now $b \in \Omega \setminus A$. Then

$$M_{\mathfrak{P}}(A \cup \{b\}) = \bigcup_{F\subseteq_f A} M_{\mathfrak{P}}(F \cup \{b\})$$

that, by our hypothesis, gives

$$\bigcup_{F\subseteq_f A} M_{\mathfrak{P}}(F \cup \{b\}) = \bigcup_{F\subseteq_f A} \bigcup_{a\in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}) = \bigcup_{a\in A} M_{\mathfrak{P}}(\{a, b\}).$$

This shows (i). □

Based on the results obtained in Proposition 2.5.3, we are also able to prove a more general recursive-like property of the operator $M_{\mathfrak{P}}$.

Theorem 2.5.4. *Let $\mathfrak{P}$ be a $M_{\mathfrak{P}}$-union decomposable finitary UCP pairing and $A_1, \ldots, A_n \in MAXP(\mathfrak{P})$ are non-empty, then*

$$M_{\mathfrak{P}}(\bigcup_{i=1}^{n} A_i) = \bigcup_{\substack{a_i\in A_i,\\ i=1,\ldots,n}} M_{\mathfrak{P}}(\{a_1, \ldots, a_n\}). \tag{2.49}$$

Proof. Let us observe that $M_{\mathfrak{P}}(A \cup B) = M_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \cup M_{\mathfrak{P}}(B))$ for any $A, B \subseteq \Omega$. We shall prove the theorem for two subsets $A, B \in MAXP(\mathfrak{P})$. Let $|A| = k$ and $|B| = l$. We proceed by induction on $k + l$. Fix k: if $l = 1$, the thesis follows immediately by (2.43). Let the thesis be true for $|B| \leq l - 1$ and suppose now that $|B| = l$. Let $a \in M_{\mathfrak{P}}(A \cup B)$ and fix $b \in B$. By (2.45), it results that

$$M_{\mathfrak{P}}((A \cup B \setminus \{b\}) \cup \{b\}) \subseteq \bigcup_{c\in M_{\mathfrak{P}}(A\cup B\setminus\{b\})} M_{\mathfrak{P}}(b, c),$$

i.e. there exists $c' \in M_{\mathfrak{P}}(A \cup B \setminus \{b\})$ such that $a \in M_{\mathfrak{P}}(\{b, c'\})$. By what we observed in the first part of the proof, we can use the inductive hypothesis on $A \cup (B \setminus \{b\})$, so there exist $a' \in M_{\mathfrak{P}}(A)$ and $b' \in M_{\mathfrak{P}}(B \setminus \{b\})$ such that $c' \in M_{\mathfrak{P}}(\{a', b'\})$. In other terms, it results that

$$a \in M_{\mathfrak{P}}(\{b, c'\}) = M_{\mathfrak{P}}(\{a', b, b'\}) = M_{\mathfrak{P}}(\{b, b'\} \cup \{a'\}).$$

Again by (2.45), there exists $b'' \in M_{\mathfrak{P}}(\{b, b'\})$ such that $a \in M_{\mathfrak{P}}(\{a', b''\})$. Since $\{b, b'\} \subseteq B$, it follows that $b'' \in B$ and, so, we have shown (2.49). We conclude by induction on the number of maximum partitioners. □

2.6 Minimal Partitioners

In this section, we analyze another basic structures arising from the global symmetry relation and, moreover, we focus our attention to the relations between this new structure and $MAXP(\mathfrak{P})$. To this regard, let us consider the subset family consisting of all the minimal members of each global symmetry class, that is $MINP(\mathfrak{P})$, defined as follows:

$$MINP(\mathfrak{P}) := \bigcup\{\min([A]_{\approx_{\mathfrak{P}}}) : A \in MAXP(\mathfrak{P})\}. \tag{2.50}$$

It is clear that the subset family $MINP(\mathfrak{P})$ can be considered a type of dual version of $MAXP(\mathfrak{P})$. We call *minimal partitioner* of $\mathfrak{P}$ any member of $MINP(\mathfrak{P})$.

In the next result, we show that $MINP(\mathfrak{P})$ is an abstract simplicial complex and, moreover, we provide the so-called *global-local regularity*, according to which the inclusion between the maximum elements of two any global symmetry classes preserves the same type of inclusion between any two members of these classes.

Theorem 2.6.1. *The following hold.*
(i) Let $C_i \subsetneqq C_j$ *be two subsets of* Ω*. Then, if* $X \in [C_i]_{\approx_{\mathfrak{P}}}$ *and* $Y \in [C_j]_{\approx_{\mathfrak{P}}}$ *we have* $Y \nsubseteq X$*.*
(ii) Let $A \in \mathcal{P}(\Omega)$ *and* $C \subseteq B \subseteq A$ *with* $B \in MINP(\mathfrak{P})$*. Then* $\pi_{\mathfrak{P}}(C \setminus \{x\}) \neq \pi_{\mathfrak{P}}(C)$ *for any* $x \in C$*.*
(iii) $MINP(\mathfrak{P}) \in ASC(\Omega)$*.*
(iv) $MINP(\mathfrak{P}) = \{A \in \mathcal{P}(\Omega) : a \notin M_{\mathfrak{P}}(A \setminus \{a\}) \, \forall a \in A\}$*.*

Proof. (i): Since C_i and C_j are two distinct maximum partitioners such that $C_i \subsetneqq C_j$, we have $\pi_{\mathfrak{P}}(C_j) \prec \pi_{\mathfrak{P}}(C_i)$. Now, by absurd, let $X \in [C_i]_{\approx_{\mathfrak{P}}}$ and $Y \in [C_j]_{\approx_{\mathfrak{P}}}$ such that $Y \subseteq X$. Then $\pi_{\mathfrak{P}}(C_i) = \pi_{\mathfrak{P}}(X) \preceq \pi_{\mathfrak{P}}(Y) = \pi_{\mathfrak{P}}(C_j)$, that is a contradiction.
(ii) : Let us suppose by contradiction that there exists $x \in C$ such that $\pi_{\mathfrak{P}}(C \setminus \{x\}) = \pi_{\mathfrak{P}}(C)$. Then, since $C \setminus \{x\} \subseteq B \setminus \{x\}$, we have

$$\pi_{\mathfrak{P}}(B \setminus \{x\}) \preceq \pi_{\mathfrak{P}}(C \setminus \{x\}) = \pi_{\mathfrak{P}}(C). \tag{2.51}$$

Therefore, if $v, v' \in \Omega$ by (2.51) it follows that

$$v \equiv_{B\setminus\{x\}} v' \implies v \equiv_C v' \implies F(v, x) = F(v', x) \implies v \equiv_B v',$$

therefore

$$\pi_{\mathfrak{P}}(B \setminus \{x\}) \preceq \pi_{\mathfrak{P}}(B). \tag{2.52}$$

On the other hand, since we also have $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(B \setminus \{x\})$, by (2.52) we deduce that $\pi_{\mathfrak{P}}(B \setminus \{x\}) = \pi_{\mathfrak{P}}(B)$, that is in contrast with the hypothesis that $B \in BAS_{\mathfrak{P}}(A)$. This concludes the proof.
(iii): Let $C \in MINP(\mathfrak{P})$. Then, there exists $B \in MAXP(\mathfrak{P})$ such that $C \in \min([B]_{\approx_{\mathfrak{P}}})$. Let $K \subsetneqq C$, then there exists $B' \in MAXP(\mathfrak{P})$ such that $K \in [B']_{\approx_{\mathfrak{P}}}$. Suppose by contradiction that $K \notin \min([B']_{\approx_{\mathfrak{P}}})$, then there exists $K' \subsetneqq K$ such that $K' \in \min([B']_{\approx_{\mathfrak{P}}})$. Hence, there is $x \in K \setminus K'$. We deduce that

$$K' \subseteq K \setminus \{x\} \subseteq K$$

i.e.

$$\pi_{\mathfrak{P}}(K) \preceq \pi_{\mathfrak{P}}(K \setminus \{x\}) \preceq \pi_{\mathfrak{P}}(K').$$

But since $\pi_{\mathfrak{P}}(K) = \pi_{\mathfrak{P}}(K') = \pi_{\mathfrak{P}}(B)$, we conclude that $\pi_{\mathfrak{P}}(K) = \pi_{\mathfrak{P}}(K \setminus \{x\})$, contradicting part (ii). Hence $K \in MINP(\mathfrak{P})$.
(iv): Let $A \in MINP(\mathfrak{P})$ and $a \in A$. Then $A \setminus \{a\} \notin [A]_{\approx_{\mathfrak{P}}}$, so $M_{\mathfrak{P}}(A \setminus \{a\}) \notin [A]_{\approx_{\mathfrak{P}}}$. Since $A \setminus \{a\} \subsetneqq A$, we have $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\})$. Moreover, suppose by contradiction that $a \in M_{\mathfrak{P}}(A \setminus \{a\})$. Then $A \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$ and $M_{\mathfrak{P}}(A) \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$. By (vi) of Theorem 2.2.1, we have $\pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(A)$, so $\pi_{\mathfrak{P}}(A \setminus \{a\}) = \pi_{\mathfrak{P}}(A)$ and $A \setminus \{a\} \in [A]_{\approx_{\mathfrak{P}}}$, that is an absurd. So $a \notin M_{\mathfrak{P}}(A \setminus \{a\})$ and $MINP(\mathfrak{P}) \subseteq \{A \in \mathcal{P}(\Omega) : a \notin M_{\mathfrak{P}}(A \setminus \{a\})\, \forall a \in A\}$. On the other hand, let $A \in \mathcal{P}(\Omega)$ such that $a \notin M_{\mathfrak{P}}(A \setminus \{a\})$ for any $a \in A$. Suppose by contradiction that $A \in MINP^c(\mathfrak{P})$, i.e. there exists $B \subsetneqq A$ such that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$. Then, there exists $a \in A$ such that $B \subseteq A \setminus \{a\} \subseteq A$. This implies that $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(B)$, i.e. $\pi_{\mathfrak{P}}(A \setminus \{a\}) = \pi_{\mathfrak{P}}(A)$. In other terms, we have $M_{\mathfrak{P}}(A \setminus \{a\}) = M_{\mathfrak{P}}(A) \supseteq A$, contradiction. □

By Theorem 2.6.1, it follows immediately that we obtain a map $MINP_{\mathfrak{P}} : \mathcal{P}(\Omega) \to SS(\Omega)$ such that $MINP_{\mathfrak{P}}(\Omega) = MINP(\mathfrak{P})$ and

$$MINP_{\mathfrak{P}}(A) \in ASC(A), \tag{2.53}$$

for any $A \in \mathcal{P}(\Omega)$.

2.7 Minimal Partitioners On Several Examples

In this section, we deal with the characterization of the minimal partitioners of some pairings, namely the simple undirected n-cycle, a vector space with a bilinear form and the euclidean real line.

2.7.1 Graphs: The n-Cycle Case

In the next result, we characterize the minimal partitioners of C_n.

Theorem 2.7.1. *Let $G = C_n$ and $A \subseteq V(C_n)$. Then $A \in MINP(G)$ if and only if for any $v_i \in A$ it holds one of the following conditions:*
(i) ($v_{i-2} \notin A$ or $v_{i+2} \notin A$) and ($B_G(A) \neq \emptyset$);
(ii) ($v_{i-2} \in A$ and $v_{i-4} \notin A$) or ($v_{i+2} \in A$ and $v_{i+4} \notin A$).

Proof. Let $A \subseteq V(G)$ and suppose there exists $v_i \in A$ such that both (i) and (ii) are false. If $B_G(A) = \emptyset$, then

$$B_G(A \setminus \{v_i\}) = \begin{cases} \{v_{i-1}\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \in A, \\ \{v_{i+1}\} & \text{if } v_{i-2} \in A \text{ and } v_{i+2} \notin A, \\ \{v_{i-1},\, v_{i+1}\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \notin A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover, since $v_{i-2} \in A \Longrightarrow v_{i-4} \in A$ and $v_{i+2} \in A \Longrightarrow v_{i+4} \in A$, then

$$C_G(A \setminus \{v_i\}) = \begin{cases} C_G(A) \setminus \{v_i\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \notin A, \\ C_G(A) & \text{otherwise.} \end{cases}$$

In all cases we obtain $A \approx_{\mathfrak{P}} A \setminus \{v_i\}$, so $A \notin MINP(G)$.
Let us suppose now that $A \notin MINP(G)$ and let $v_i \in A$ such that $A \approx_{\mathfrak{P}} A \setminus \{v_i\}$. Then, there exist $v_j,\, v_k \in V(G)$ such that $v_j \equiv_{A\setminus\{v_i\}} v_k$ and $v_j \not\equiv_A v_k$. This happens if and only if the following assertions are false:
(1) $v_j,\, v_k \in B_G(A \setminus \{v_i\})$, $\{v_j,\, v_k\} \nsubseteq B_G(A)$ and $B_G(A) \neq \emptyset$;
(2) $v_{i-2} \in C_G(A)$ or $v_{i+2} \in C_G(A)$.
These conditions are equivalent to conditions (i) and (ii) respectively. This proves the theorem. □

2.7.2 Vector Spaces

Let V be a vector space. If k is a non negative integer and $\dim((H) = k$, we write $H \leqslant_k V$. Let us assume that V is a vector space of dimension n over $\mathbb{K}$ and φ is (left) non degenerate. Let $\mathfrak{P}$ be the $(V, \mathbb{K}, \varphi)$-VBP. We have the following characterizations.

Theorem 2.7.2. *(i) $MAXP_{\mathfrak{P}}(A) = \{H \cap A : H \leqslant V\}$;*
(ii) $MINP_{\mathfrak{P}}(A) = \{C : C \subseteq_{li} A\} \cup \emptyset\}$;
(iii) if $A \in MAXP(\mathfrak{P})$, $u, u' \in V \setminus A$ and $M_{\mathfrak{P}}(A \cup \{u\}) = M_{\mathfrak{P}}(A \cup \{u'\})$, then $\mathrm{Span}(\{u\}) = \mathrm{Span}(\{u'\})$.

Proof. (i): By (iii) of Proposition 2.3.21, it is immediate to see that $MAXP(\mathfrak{P}) = \{H : H \leqslant V\}$. Hence, by the definition of $MAXP_{\mathfrak{P}}(A)$, it results that $MAXP_{\mathfrak{P}}(A) = \{H \cap A : H \leqslant V\}$.
(ii): Again by (iii) of Proposition 2.3.21, it is immediate to see that $MINP(\mathfrak{P}) = \{C : C \subseteq_{li} V\} \cup \emptyset\}$, that is the family consisting of all the possible bases of any vector subspace H of V and of the null space. Hence, let us consider $A \subseteq V$. Then $MINP_{\mathfrak{P}}(A) = \{C \in MINP(\mathfrak{P}) : C \subseteq A\}$, i.e. $MINP_{\mathfrak{P}}(A) = \{C : C \subseteq_{li} A\} \cup \emptyset\}$.
(iii): Let $x \in \mathrm{Span}(\{u\})$, then $x = \lambda u$ for some $\lambda \in \mathbb{K}$. So, $x \in \mathrm{Span}(A) + \mathrm{Span}(\{u\}) = \mathrm{Span}(A) + \mathrm{Span}(\{u'\})$ since φ is (left) nondegenerate, therefore $x \in \mathrm{Span}(A) + \mathrm{Span}(\{u'\})$, i.e. $x \in \mathrm{Span}(\{u'\})$. The converse is analogous. □

Remark 2.7.3. The property of the operator $M_{\mathfrak{P}}$ for a VBP established in (iii) of Theorem 3.3.1 can be considered a weak version of the classical exchange property for a closure operator.

2.7.3 The Euclidean Real Line

We now characterize the minimal partitioners of the real euclidean line and then show that $MINP(\mathfrak{R})$ is a finitary abstract complex on $\mathbb{R}$.

Proposition 2.7.4. *$A \in MINP(\mathfrak{R})$ if and only if for any $a \in A$ it results that $a - 4 \notin A$ or $a + 4 \notin A$.*

Proof. To say that $A \in MINP(\mathcal{R})$ is equivalent to require that $\pi_{\mathfrak{P}}(A \setminus \{a\}) \neq \pi_{\mathfrak{P}}(A)$. In other terms, for any $a \in A$ there exist $x, y \in \mathbb{R}$ such that $x \equiv_{A\setminus\{a\}} y$ and $x \not\equiv_A y$. It is straightforward to see that the only possibility is that $\{x, y\} = \{a-3, a-1\}$ or $\{x, y\} = \{a+1, a+3\}$. Therefore, one of the conditions $a - 4 \notin A$ or $a + 4 \notin A$ must hold. $\square$

Theorem 2.7.5. *$MINP(\mathcal{R})$ is a finitary abstract complex on $\mathbb{R}$.*

Proof. By *(iii)* of Proposition 2.6.1, $MINP(\mathcal{R})$ is an abstract simplicial complex on $\mathbb{R}$.
Let now $A \subseteq \mathbb{R}$. We prove that if $F \in MINP(\mathcal{R})$ for any $F \subseteq_f A$ then $A \in MINP(\mathcal{R})$. For, assume by contradiction that $A \notin MINP(\mathcal{R})$. Thus, there exists $a \in A$ such that $a-4, a+4 \in A$. Let us consider the finite set $F = \{a-4, a, a+4\}$. By Proposition 2.7.4, we conclude that $F \notin MINP(\mathcal{R})$, contradicting our assumption. $\square$

Chapter 3

Symmetry Set Systems Induced by a Pairing

In this chapter we will represent by pairings a particular structure on a finite set Ω, called *dissymmetry space*, that generalizes the symmetry relation since it takes account of the conditions to "discern in Ω" two elements x, x' of a given set X. Secondly, we introduce the main set systems arising from the symmetry relation, that's to say symmetry bases, core and symmetry essentials. We investigate their interrelations by means of an operatorial standpoint.

3.1 Dissymmetry Spaces

We now introduce the notion of *dissymmetry space* on Ω.

Definition 3.1.1. A *dissymmetry space* on Ω is a structure $\mathfrak{D} = \langle X, D\rangle$, where X is a non-empty set and $D : X \times X \to \mathcal{P}(\Omega)$ is a map, called *local dissymmetry map* of $\mathfrak{D}$ having the following properties:
(i) $D(x, x) = \emptyset$ for each $x \in X$.
(ii) $D(x, x') = D(x', x)$ for all $x, x' \in \Omega$.
(iii) $D(x, x') = \bigcap_{x'' \in X}[D(x, x'') \cup D(x', x'')]$.
We say that $\mathfrak{D}$ is a *finite dissymmetry space* if both X and Ω are finite.

Remark 3.1.2. The reader can easily verify that the property (iii) of Definition 3.1.1 is equivalent to the following:
(iii') $y \in D(x, x')$ if and only if, for all $x'' \in \Omega$, $y \notin D(x, x'') \Longrightarrow y \in D(x', x'')$.

In the above definition, the three properties (i)–(iii) are the counterpart of the reflexivity, symmetry and transitivity properties of an symmetry relation. Indeed, (i) says that an element x is not discernible from itself, (ii) says that what distinguishes x from x' is exactly the same of what distinguishes x' from x and (iii) expresses the idea that if x, x' differ on a property y, then there does not exist an element x'' which is equal to both x and x' on property y.

Given a pairing $\mathfrak{P} \in PAIR(\Omega)$ and fixed $A \in \mathcal{P}(\Omega)$, let us define the map $\Delta_A^{\mathfrak{P}} : U \times U \to \mathcal{P}(A)$ as follows

$$\Delta_A^{\mathfrak{P}}(u, u') := \{a \in A : F(u, a) \neq F(u', a)\}, \tag{3.1}$$

for any $u, u' \in U$.

In particular, we set $\Delta^{\mathfrak{P}} := \Delta_{\Omega}^{\mathfrak{P}}$, so that $\Delta_A^{\mathfrak{P}}(u,u') = \Delta^{\mathfrak{P}}(u,u') \cap A$. Let us observe that $\Delta_A^{\mathfrak{P}}(u,u')$ is the subset of all elements $a \in \Omega$ which discern u_i from u_j.

Therefore we call $\Delta_A^{\mathfrak{P}}(u,u')$ the A-*dissymmetry neighborhood* of u and u' and A-*dissymmetry set system* the subset family

$$DIS_{\mathfrak{P}}(A) := \{\Delta_A^{\mathfrak{P}}(u,u') : u, u' \in U \text{ and } \Delta_A^{\mathfrak{P}}(u,u') \neq \emptyset\}. \tag{3.2}$$

In particular, we set $DIS(\mathfrak{P}) := DIS_{\mathfrak{P}}(\Omega)$.

If $\mathfrak{P} \in PAIR(\Omega)$ is a finite pairing, we consider the A-*dissymmetry matrix* $\Delta[\mathfrak{P}]$ of $\mathfrak{P}$, that is the $m \times m$ matrix with entries (d_{ij}), where

$$d_{ij} := \Delta_A^{\mathfrak{P}}(u_i, u_j). \tag{3.3}$$

Let us note that $\Delta[\mathfrak{P}]$ is a symmetric matrix such that $\Delta^{\mathfrak{P}}(u_i,u_i) = \emptyset$ for all $i = 1, \dots, m$. Moreover, in the finite case, if $k \in \{1, \dots, n\}$, we set $DIS_k(\mathfrak{P}) := \{C \in DIS(\mathfrak{P}) : |C| = k\}$.

The following result relates the subsets $\Delta_A^{\mathfrak{P}}(u,u')$ to the local symmetry relations.

Proposition 3.1.3. *Let $D \subseteq A$ and $u, u' \in U$. Then:*
(i) $D = \Delta_A^{\mathfrak{P}}(u,u') \Longrightarrow u \equiv_{A \setminus D} u'$;
(ii) $u \equiv_{A \setminus D} u' \Longrightarrow \Delta_A^{\mathfrak{P}}(u,u') \subseteq D$;
(iii) $\Delta_A^{\mathfrak{P}}(u,u') \cap D = \emptyset \Longleftrightarrow u \equiv_D u'$.

Proof. (i) : Let $a \in A \setminus D$. Then, we have $F(v,a) = F(w,a)$, therefore $v \equiv_{A \setminus D} w$.
(ii) : By hypothesis, $F(v,a) = F(w,a)$ for all $a \in A \setminus D$. This means that the set of elements of A for which the values of $F(v,a)$ and $F(w,a)$ are different, i.e. $\Delta_A^{\mathfrak{P}}(v,w)$, stays in the complement of $A \setminus D$. Therefore $\Delta_A^{\mathfrak{P}}(v,w) \subseteq D$.
(iii) : Let $\Delta_A^{\mathfrak{P}}(v,w) \cap D = \emptyset$. This is equivalent to require that for all $c \in D$, $F(v,c) = F(w,c)$ or, equivalently, $v \equiv_D w$. $\square$

In what follows, we will prove that to each finite dissymmetry space corresponds a pairing. In fact, let us notice that the following result holds.

Proposition 3.1.4. *Let $\mathfrak{P} \in PAIR(\Omega)$ be a pairing. Then, the map $\Delta^{\mathfrak{P}} : U \times U \to \mathcal{P}(\Omega)$ satisfies the following properties:*
(i) $\Delta^{\mathfrak{P}}(u,u) = \emptyset$ for all $u \in U$;
(ii) $\Delta^{\mathfrak{P}}(u,u') = \Delta^{\mathfrak{P}}(u',u)$ for all $u, u' \in U$;
(iii) $\Delta^{\mathfrak{P}}(u,u') = \bigcap_{u'' \in U}[\Delta^{\mathfrak{P}}(u,u'') \cup \Delta^{\mathfrak{P}}(u',u'')]$.

Proof. Parts (i) and (ii) are obvious. Let us prove part (iii). Let $a \in \Delta^{\mathfrak{P}}(u,u')$ and let $u'' \in U$. If $a \notin \Delta^{\mathfrak{P}}(u,u'')$, then it means that $F(u,a) = F(u'',a)$. But since $a \in \Delta^{\mathfrak{P}}(u,u')$ we have $F(u',a) \neq F(u,a) = F(u'',a)$ and thus $a \in \Delta^{\mathfrak{P}}(u',u'')$. This proves the claim. $\square$

Remark 3.1.5. By Proposition 3.1.4, if $\mathfrak{P}$ is a pairing, the structure $\mathcal{D}[\mathfrak{P}] := \langle U, \Delta^{\mathfrak{P}} \rangle$ is a dissymmetry space, that we call *dissymmetry space of the pairing* $\mathfrak{P}$.

We describe now a method to associate a pairing $\mathfrak{P}_D$ to any finite dissymmetry space $\langle X, D\rangle$ on Ω in such a way that $\Delta^{\mathfrak{P}_D} = D$.

Let $X = \{x_1, \ldots, x_m\}$ be a finite set and let $D : X \times X \to \mathcal{P}(\Omega)$ be an arbitrary map. Let us construct a pairing $\mathfrak{P}_D = (U, \Lambda, F)$ in the following way. We set $U := X$ and define a map $F : U \times \Omega \to \mathbb{N}$ with the following recursive method:

1. For each $y_j \in \Omega$, set $F(x_1, y_j) := 1$.
2. Let $i \in \{2, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ and we assume to have inductively defined the values $F(x_s, y_j)$, for $s = 1, \ldots, i-1$.
3. We set $S_{ij} := \{s : s \in \{1, \ldots, i-1\} \wedge y_j \notin D(x_s, x_i)\}$ and $t_{ij} := \min S_{ij}$ if $S_{ij} \neq \emptyset$.
4. We set then
$$F(x_i, y_j) := \begin{cases} F(x_{t_{ij}}, y_j) & \text{if } S_{ij} \neq \emptyset \\ 1 + \max\limits_{1 \le s \le i-1} F(x_s, y_j) & \text{otherwise.} \end{cases}$$
5. Finally we set $\Lambda := Im(F)$.

Example 3.1.6. *Let $X = \{u_1, u_2, u_3, u_4\}$, $\Omega = \{a_1, a_2, a_3, a_4\}$ and $D : X \times X \to \mathcal{P}(\Omega)$ represented by the matrix below:*

	u_1	u_2	u_3	u_4
u_1	$\emptyset$	$\{a_1, a_3\}$	$\{a_1, a_2\}$	$\{a_4\}$
u_2	$*$	$\emptyset$	$\{a_1, a_2, a_4\}$	$\{a_3, a_4\}$
u_3	$*$	$*$	$\emptyset$	$\{a_2, a_4\}$
u_4	$*$	$*$	$*$	$\emptyset$

Firstly, we set $F(u_1, a_i) = 1$ for $i = 1, 2, 3, 4$. We represent below the table of $S_{i,j}$ for $i = 2, 3, 4$ and $j \in \{1, 2, 3, 4\}$.

$S_{2,1} = \emptyset$	$S_{2,2} = \{1\}$	$S_{2,3} = \emptyset$	$S_{2,4} = \{1\}$
$S_{3,1} = \emptyset$	$S_{3,2} = \emptyset$	$S_{3,3} = \{1, 2\}$	$S_{3,4} = \{1\}$
$S_{4,1} = \{1, 2, 3\}$	$S_{4,2} = \{1, 2\}$	$S_{4,3} = \{1, 3\}$	$S_{4,4} = \emptyset$

Therefore, we obtain the following pairing:

	a_1	a_2	a_3	a_4
u_1	*1*	*1*	*1*	*1*
u_2	*2*	*1*	*2*	*1*
u_3	*3*	*2*	*1*	*1*
u_4	*1*	*1*	*1*	*2*

and, in particular, $\Lambda = \{1, 2, 3\}$.

We have then the following result.

Theorem 3.1.7. *Let $\langle X, D\rangle$ be a dissymmetry space and let $\mathfrak{P}_D = (U, \Lambda, F)$ be the finite pairing built above. Then $\Delta^{\mathfrak{P}_D} = D$.*

Proof. Let $x_k,\ x_i \in U$. If $k = i$, then, by their definitions, both $\Delta^{\mathfrak{P}_D}(x_k, x_i)$ and $D(x_k, x_i)$ are equal to the empty set. Without loss in generality we can suppose in what follows $k < i$. Let us recall the definition of the map $\Delta^{\mathfrak{P}_D}$:

$$\Delta^{\mathfrak{P}_D}(x_k, x_i) := \{y_j \in \Omega :\ F(x_k, y_j) \neq F(x_i, y_j)\}.$$

In order to prove that $\Delta^{\mathfrak{P}_D}(x_k, x_i) \subseteq D(x_k, x_i)$, we claim that, $\forall h \in S_{ij}$, $S_{hj} = S_{ij} \cap \{1, \ldots, h-1\}$. In fact, if this holds then clearly, $\forall h \in S_{ij}$, $F(x_h, y_j) = F(x_i, y_j)$, so if $y_j \notin D(x_k, x_i)$, then $F(x_k, y_j) = F(x_i, y_j)$ and thus $y_j \notin \Delta^{\mathfrak{P}_D}(x_k, x_i)$. This is equivalent to say that $\Delta^{\mathfrak{P}_D}(x_k, x_i) \subseteq D(x_k, x_i)$.

Let us prove the claim. Let $h \in S_{ij}$. If $S_{hj} = \emptyset$, then obviously $S_{hj} \subseteq S_{ij} \cap \{1, \ldots, h-1\}$. Let us suppose $S_{hj} \neq \emptyset$ and let $s \in S_{hj}$. In this case it holds $y_j \notin D(x_s, x_h)$ and $y_j \notin D(x_h, x_i)$, so, by condition (iii') of Remark 3.1.2, $y_j \notin D(x_s, x_i)$ and thus $s \in S_{ij} \cap \{1, \ldots, h-1\}$. Conversely if $S_{ij} \cap \{1, \ldots, h-1\} = \emptyset$, then clearly $S_{ij} \cap \{1, \ldots, h-1\} \subseteq S_{hj}$. Le us suppose then that $S_{ij} \cap \{1, \ldots, h-1\} \neq \emptyset$ and let $s \in S_{ij} \cap \{1, \ldots, h-1\}$. As before this means that $y_j \notin D(x_s, x_i)$ and since $y_j \notin D(x_h, x_i)$, by (iii') of Remark 3.1.2, $y_j \notin D(x_s, x_h)$, so $s \in S_{hj}$ and the claim is proved.

Let $y_j \in D(x_k, x_i)$. If $S_{ij} = \emptyset$, by definition we have $F(x_i, y_j) = 1 + \max\limits_{1 \le s \le i-1} F(x_s, y_j) \neq F(x_k, y_j)$, so $y_j \in \Delta^{\mathfrak{P}_D}(x_k, x_i)$. Let $S_{ij} \neq \emptyset$. If $F(x_k, y_j) = F(x_i, y_j)$, by (iii') of Remark 3.1.2 it is straightforward to prove that there exists h such that $1 \le h \le k < i$, $y_j \notin D(x_h, x_k)$ and $y_j \notin D(x_h, x_i)$ or, equivalently, by (iii) of Definiton 3.1.1, that $y_j \notin D(x_k, x_i)$ and this is a contradiction. Then $F(x_k, y_j) \neq F(x_i, y_j)$ and thus $y_j \in \Delta^{\mathfrak{P}_D}(x_k, x_i)$. This proves the theorem. $\square$

3.2 Symmetry Set Systems Induced by a Pairing

In this section we introduce some new set systems induced by A-symmetry relation and study their main properties. To this regard, we introduce the notion of A-*symmetry base*.

Definition 3.2.1. Let $B \subseteq A$. We say that B is an A-*symmetry base* of $\mathfrak{P}$ if:

$(R1)$ $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$;

$(R2)$ $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(B')$ for all $B' \subsetneqq B$.

We denote by $BAS_{\mathfrak{P}}(A)$ the family of all A-symmetry bases of $\mathfrak{P}$ and we set $BAS(\mathfrak{P}) := BAS_{\mathfrak{P}}(\Omega)$. We call *symmetry bases* of $\mathfrak{P}$ the members of $BAS(\mathfrak{P})$. If $\mathfrak{P}$ is a finite pairing on Ω and if $k \in \{1, \ldots, n\}$, we set $BAS_k(\mathfrak{P}) := \{C \in BAS(\mathfrak{P}) : |C| = k\}$.

We now provide some basic properties of A-symmetry bases.

Theorem 3.2.2. (i) *$BAS_{\mathfrak{P}}(A) = \{B : B \in \min([A]_{\approx_{\mathfrak{P}}}),\ B \subseteq A\}$ for any $A \in \mathcal{P}(\Omega)$.*
(ii) *$BAS_{\mathfrak{P}}(A) \subseteq \max(MINP_{\mathfrak{P}}(A))$ for any $A \in \mathcal{P}(\Omega)$.*

	1	2	3	4	5
u_1	0	0	1	2	1
u_2	1	0	2	2	0
u_3	2	1	0	1	1
u_4	1	1	2	0	2
u_5	1	0	0	0	2
u_6	0	1	2	2	0

Figure 3.1: The Functional Table $\mathfrak{P}$

Proof. (i): Let $C \in BAS_{\mathfrak{P}}(A)$. Then, for any $C' \subsetneqq C$, we have that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(C) \neq \pi_{\mathfrak{P}}(C')$. This means that C is minimal in $[A]_{\approx_{\mathfrak{P}}}$.
Conversely, let $C \in \min([A]_{\approx_{\mathfrak{P}}})$. Then $\pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$. Moreover, let $C' \subsetneqq C$. By the minimality of C, it follows that $C' \notin [A]_{\approx_{\mathfrak{P}}}$, whence $\pi_{\mathfrak{P}}(C') \neq \pi_{\mathfrak{P}}(A)$. This shows that C satisfies both the conditions of Definition 3.2.1, i.e. $C \in BAS_{\mathfrak{P}}(A)$.
(ii): Let $C \in BAS_{\mathfrak{P}}(A)$, then $C \subseteq A$ and $C \in \min([A]_{\approx}) \subseteq MINP(\mathfrak{P})$ by part (i). This proves that $C \in MINP_{\mathfrak{P}}(A)$. Suppose by contradiction that $BAS_{\mathfrak{P}}(A) \nsubseteq \max(MINP_{\mathfrak{P}}(A))$, i.e. there exists $D \in MINP_{\mathfrak{P}}(A)$ such that $C \subsetneqq D$. Then, $\pi_{\mathfrak{P}}(D) \preceq \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$. On the other hand, by the fact that $D \in MINP_{\mathfrak{P}}(A)$, we deduce that $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(D)$. Therefore, we infer that $D \approx_{\mathfrak{P}} A$. It is now straightforward to see that it is in contrast with our choice of D. □

Let us note here that the inclusion established in (iii) of Theorem (3.2.2) cannot be reversed, as we see in next example.

Example 3.2.3. *Let $\mathfrak{P}$ be the functional table given in Figure 3.1 and $A = \{2, 3, 5\}$. Then, we have that*

$$BAS_{\mathfrak{P}}(A) = \{\{2, 5\}\} \subsetneqq \max(MINP_{\mathfrak{P}}(A)) = \{\{2, 3\}, \{2, 5\}, \{3, 5\}\}$$

We now provide the deep link between relative maximum partitioners and $BAS_{\mathfrak{P}}(A)$.

Proposition 3.2.4. *Let $B \in MAXP_{\mathfrak{P}}(A)$ such that $B \subsetneqq A$. Then $A \setminus B$ is a transversal of $BAS_{\mathfrak{P}}(A)$.*

Proof. Suppose by contradiction that $A \setminus B$ is not a transversal of $BAS_{\mathfrak{P}}(A)$, i.e. there exists $C \in BAS_{\mathfrak{P}}(A)$ such that $C \cap (A \setminus B) = \emptyset$. Thus, $C \subseteq B \subsetneqq A$. In particular, it results that

$$M_{\mathfrak{P}}(C) \subseteq M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(A),$$

but $M_{\mathfrak{P}}(C) = M_{\mathfrak{P}}(A)$, so $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(B)$. Since there exists $D \in MAXP(\mathfrak{P})$ such that $B = A \cap D$, we observe that $M_{\mathfrak{P}}(B) \subseteq D$, hence

$$A \subseteq M_{\mathfrak{P}}(A) \subseteq D$$

or, equivalently, $B = A \cap D = A$, contradiction. Therefore, $A \setminus B$ is a transversal of $BAS_{\mathfrak{P}}(A)$. □

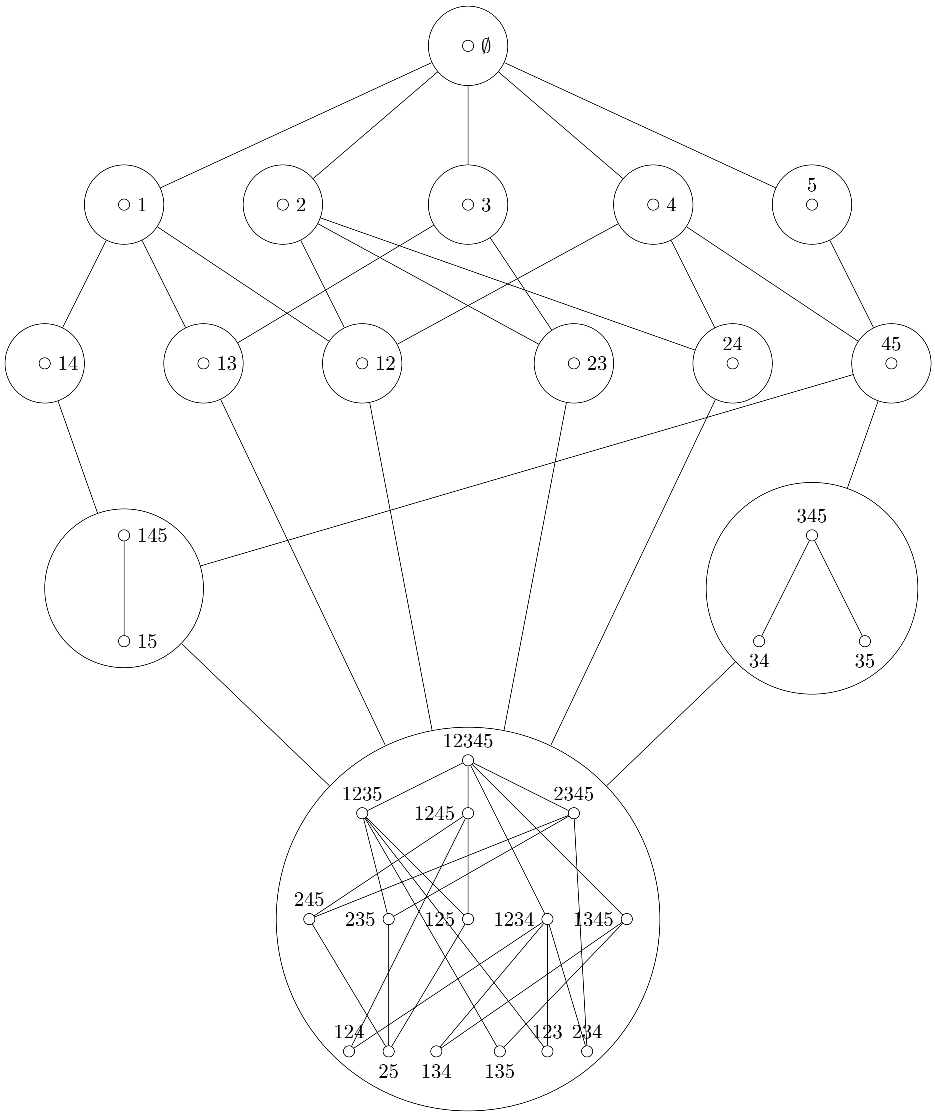


Figure 3.2: Diagram of $\mathbb{I}(\mathfrak{P})$.

We now provide the notion of *core*.

Definition 3.2.5. Let $A \in \mathcal{P}(\Omega)$. An element $a \in A$ is said *indispensable* if $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$. The subset of all indispensable elements of A is called the *core* of A and it is denoted by $C_{\mathfrak{P}}(A)$. In particular, we set $CORE(\mathfrak{P}) := C_{\mathfrak{P}}(\Omega)$.

Let us prove that the core of A is given by the intersection of all A-symmetry bases when Ω is finite.

Proposition 3.2.6. *We have that:*

$$C_{\mathfrak{P}}(A) := \bigcap\{C : C \in BAS_{\mathfrak{P}}(A)\}. \tag{3.4}$$

when Ω is finite.

Proof. Let $a \in \bigcap\{C : C \in BAS_{\mathfrak{P}}(A)\}$ and assume by contradiction that $a \notin C_{\mathfrak{P}}(A)$. This implies that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A \setminus \{a\})$ and, in particular, the existence of a subset $B \subseteq A \setminus \{a\}$ belonging to $BAS_{\mathfrak{P}}(A \setminus \{a\}$. But we also have $B \in BAS_{\mathfrak{P}}(A)$, contradicting the fact that $a \in \bigcap\{C : C \in BAS_{\mathfrak{P}}(A)\}$.
On the other hand, let $a \in C_{\mathfrak{P}}(A)$. Let $B \in BAS_{\mathfrak{P}}(A)$ and assume by contradiction that $a \notin B$. This implies $B \subseteq A \setminus \{a\}$, so $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$, but this would mean that $a \notin C_{\mathfrak{P}}(A)$, leading us to a contradiction. $\square$

Actually, the core can be interpreted from an operatorial perspective, therefore we use the following terminology.

Definition 3.2.7. We call the set operator $C_{\mathfrak{P}}$ the $\mathfrak{P}$- *core operator*.

We can now pass to examine the main properties of the core operator. In particular, the $\mathfrak{P}$-maximum partitioner operator and the $\mathfrak{P}$-core operator are related in the following way.

Proposition 3.2.8. *For any $A \in \mathcal{P}(\Omega)$ we have that:*

(*i*) $C_{\mathfrak{P}}(A) = \{a \in A : M_{\mathfrak{P}}(A \setminus \{a\}) \subsetneqq M_{\mathfrak{P}}(A)\}$;

(*ii*) $C_{\mathfrak{P}}(A) = \{a \in A : a \notin M_{\mathfrak{P}}(A \setminus \{a\})\}$;

(*iii*) $C_{\mathfrak{P}}(A) = \{a \in A : A \setminus \{a\} \in MAXP_{\mathfrak{P}}(A)\}$;

(*iv*) $M_{\mathfrak{P}}(A) = \{b \in \Omega : b \notin C_{\mathfrak{P}}(A \cup \{b\})\}$;

(*v*) $C_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \subseteq A$.

Proof. Part (*i*) follows directly by the definitions of $C_{\mathfrak{P}}(A)$ and $M_{\mathfrak{P}}(A)$.
For (*ii*), let us observe that $a \notin M_{\mathfrak{P}}(A \setminus \{a\})$ if and only if $M_{\mathfrak{P}}(A \setminus \{a\}) \neq M_{\mathfrak{P}}(A)$, i.e. if and only if $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$.
(*iii*): Let $a \in C_{\mathfrak{P}}(A)$. We must show that $A \setminus \{a\} \in MAXP_{\mathfrak{P}}(A)$, i.e. there exists $B \in MAXP(\mathfrak{P})$ such that $A \setminus \{a\} = A \cap B$. Let us assume by absurd that $A \setminus \{a\} \neq A \cap B$ for any $B \in MAXP(\mathfrak{P})$. Since $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$, then $A \setminus \{a\} \not\approx_{\mathfrak{P}} A$. Let B be the maximum

partitioner of $[A \setminus \{a\}]_{\approx_{\mathfrak{P}}}$. Clearly, $A \setminus \{a\} \subseteq B$ and we have $A \setminus \{a\} \subseteq A \cap B$ therefore, by our assumption, it follows that $A \setminus \{a\} \subsetneqq A \cap B$. Hence $A \cap B = A$, i.e. $A \subseteq B$. It follows that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\}) = \pi_{\mathfrak{P}}(B)$, i.e. $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$, that is equivalent to say that $A \approx_{\mathfrak{P}} B \approx_{\mathfrak{P}} A \setminus \{a\}$, that is an absurd. Thus $A \setminus \{a\} \in MAXP_{\mathfrak{P}}(A)$.
Conversely, let $a \in A$ such that $A \setminus \{a\} \in MAXP_{\mathfrak{P}}(A)$. Hence, there exists $B \in MAXP(\mathfrak{P})$ such that $A \setminus \{a\} = B \cap A$. If $B = A \setminus \{a\}$, then $A \setminus \{a\} \in MAXP(\mathfrak{P})$. In this case, by maximality of $A \setminus \{a\}$, then $A \notin [A \setminus \{a\}]_{\approx_{\mathfrak{P}}}$, so $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$, therefore $a \in MAXP_{\mathfrak{P}}(A) \subseteq C_{\mathfrak{P}}(A)$ and the claim is proved. Otherwise, let $A \setminus \{a\} \subsetneqq B$. Clearly, it results that $a \notin B$. Furthermore, it results $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\})$. Suppose by contradiction that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A \setminus \{a\})$. Then, we have $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(A)$, so

$$u \equiv_B u' \implies u \equiv_A u' \implies F(u, a) = F(u', a) \implies u \equiv_{B \cup \{a\}} u'.$$

In other terms, we have shown that $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(B \cup \{a\})$. Nevertheless, we also have $\pi_{\mathfrak{P}}(B \cup \{a\}) \preceq \pi_{\mathfrak{P}}(B)$, i.e. $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(B \cup \{a\})$. This contradicts the maximality of B, that is an absurd. Thus, $a \in C_{\mathfrak{P}}(A)$.
(iv): We simply observe that $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A \cup \{b\} \setminus \{b\})$ so $b \in M_{\mathfrak{P}}(A)$ if and only if $b \notin C_{\mathfrak{P}}(A \cup \{b\})$.
(v): Let $a \in C_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ and suppose by contradiction that $a \notin A$. Therefore we have $A \subseteq M_{\mathfrak{P}}(A) \setminus \{a\} \subseteq M_{\mathfrak{P}}(A)$ and, hence, that $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \preceq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(A)$. Since $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) = \pi_{\mathfrak{P}}(A)$, we conclude that $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus \{a\}) = \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$, contradicting the fact that $a \in C_{\mathfrak{P}}(A)$. ☐

In the next result we establish further new properties for the $\mathfrak{P}$-core operator.

Theorem 3.2.9. *$C_{\mathfrak{P}}$ is an idempotent core operator. Moreover, if Ω is finite and $A \in MAXP(\mathfrak{P})$, it results that*

$$a \in A \setminus C_{\mathfrak{P}}(A) \iff \text{there exists } B \in \mathcal{P}(\Omega) \text{ such that } a \notin B \text{ and } a \in M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(A). \tag{3.5}$$

Proof. We clearly have that $C_{\mathfrak{P}}(A) \subseteq A$. The set operator $C_{\mathfrak{P}}$ is pseudo-monotone. In fact, let $a \in C_{\mathfrak{P}}(B) \cap A$, then $a \in A$ and $\pi_{\mathfrak{P}}(B \setminus \{a\}) \neq \pi_{\mathfrak{P}}(A)$. Suppose that $a \in M_{\mathfrak{P}}(A \setminus \{a\})$, then $a \in M_{\mathfrak{P}}(B \setminus \{a\})$, contradicting the fact that $a \in C_{\mathfrak{P}}(B)$. Thus, $a \in C_{\mathfrak{P}}(A)$.
Let us prove that $C_{\mathfrak{P}}$ is quasi-regular. To this regard, let $A \in \mathcal{P}(\Omega)$ and $b, c \in \Omega \setminus A$. Assume that $b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))$. This means that $b \in M_{\mathfrak{P}}(A \cup \{b\} \setminus \{b\})$, i.e. $b \in M_{\mathfrak{P}}(A) \setminus A$. Moreover, assume that $c \in \sigma(A \cup \{c\})$. This implies that $c \in \Omega \setminus M_{\mathfrak{P}}(A \cup \{c\} \setminus \{c\}) = M_{\mathfrak{P}}(A)$. In particular, it results that $M_{\mathfrak{P}}(A \cup \{b, c\} \setminus \{c\}) = M_{\mathfrak{P}}(A \cup \{b\}) = M_{\mathfrak{P}}(A)$, so $c \in \Omega \setminus M_{\mathfrak{P}}(A)$ or, equivalently, $c \in M_{\mathfrak{P}}(A \cup \{b, c\})$.
Let us prove idempotency. By part (i), we have that $C_{\mathfrak{P}}(C_{\mathfrak{P}}(A)) \subseteq C_{\mathfrak{P}}(A)$. On the other hand, let $a \in C_{\mathfrak{P}}(A)$, then $\pi_{\mathfrak{P}}(A \setminus \{a\}) \neq \pi_{\mathfrak{P}}(A)$, i.e. $M_{\mathfrak{P}}(A \setminus \{a\}) \subsetneqq M_{\mathfrak{P}}(A)$. Now, since $C_{\mathfrak{P}}(A) \subseteq A$, we deduce that $a \in M_{\mathfrak{P}}(C_{\mathfrak{P}}(A) \setminus \{a\}) \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$ but this would imply that $A \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$ or, equivalently, $M_{\mathfrak{P}}(A \setminus \{a\}) = M_{\mathfrak{P}}(A)$, contradicting the fact that $a \in C_{\mathfrak{P}}(A)$.
Finally, we show that (3.5) holds. Let us assume Ω to be finite. Let $a \in A \setminus C_{\mathfrak{P}}(A)$. Then, just consider $B \in BAS_{\mathfrak{P}}(A)$ such that $a \notin B$. Nevertheless, we have $a \in M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A)$.

On the other hand, assume that $a \in A$ and let B not containing a and such that $a \in M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(A)$. Now, set $C := A \setminus \{a\}$. Clearly, $M_{\mathfrak{P}}(C) \subsetneq M_{\mathfrak{P}}(A) = A$. Moreover, we have that $B \subseteq C$ and $a \in M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(C)$ so, we conclude that $A \subseteq M_{\mathfrak{P}}(C)$ and, hence $A = M_{\mathfrak{P}}(C)$. In this way, we have shown the existence of a subset $C \in [A]_{\approx_{\mathfrak{P}}}$ not containing a. Now, let us observe that $a \notin D$ for any $D \in BAS_{\mathfrak{P}}(C)$. A fortiori, since $BAS_{\mathfrak{P}}(C) \subseteq BAS_{\mathfrak{P}}(A)$, it follows by (3.4) that $a \in A \setminus C_{\mathfrak{P}}(A)$. □

By means of both the operators $M_{\mathfrak{P}}$ and $C_{\mathfrak{P}}$ we can also provide the following new characterization for $BAS_{\mathfrak{P}}(A)$ and $MINP_{\mathfrak{P}}(A)$.

Theorem 3.2.10. *Let $A \in \mathcal{P}(\Omega)$. Then:*

(*i*) $MINP_{\mathfrak{P}}(A) = \{B \in \mathcal{P}(A) : C_{\mathfrak{P}}(B) = B\}$;

(*ii*) $MINP(\mathfrak{P}) = \{B \in \mathcal{P}(\Omega) : C_{\mathfrak{P}}(B) = B\}$;

(*iii*) $BAS_{\mathfrak{P}}(A) = \{B \in \mathcal{P}(A) : M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A) \text{ and } C_{\mathfrak{P}}(B) = B\}$;

(*iv*) $BAS(\mathfrak{P}) = \{B \in \mathcal{P}(\Omega) : M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(\Omega) \text{ and } C_{\mathfrak{P}}(B) = B\}$;

(*v*) $MINP(\mathfrak{P}) = \{B \in \mathcal{P}(\Omega) : BAS_{\mathfrak{P}}(B) = \{B\}\}$.

Proof. (*i*): Let $B \in MINP_{\mathfrak{P}}(A)$. Clearly, $C_{\mathfrak{P}}(B) \subseteq B$. Vice versa, let $b \in B$ and assume by contradiction that $\pi_{\mathfrak{P}}(B \setminus \{b\}) = \pi_{\mathfrak{P}}(B)$. This means that B is not a minimal partitioner of $\mathfrak{P}$, contradicting our assumption on B.
On the other hand, let $B \in \mathcal{P}(A)$ such that $C_{\mathfrak{P}}(B) = B$ and suppose by contradiction that $B \notin MINP_{\mathfrak{P}}(A)$. Then, there exists $B' \subsetneq B$ such that $B' \approx_{\mathfrak{P}} B$ and $B' \in MINP_{\mathfrak{P}}(A)$. In particular, $B' \subseteq B \setminus \{b\}$ for some $b \in B$. This entails that $\pi_{\mathfrak{P}}(B \setminus \{b\}) = \pi_{\mathfrak{P}}(B)$, i.e. $b \notin C_{\mathfrak{P}}(B)$ or, equivalently, $B \neq C_{\mathfrak{P}}(B)$, contradicting the choice of B. This proves that $MINP_{\mathfrak{P}}(A) = \{B \in \mathcal{P}(A) : C_{\mathfrak{P}}(B) = B\}$.
(*ii*): It follows by part (*i*) by replacing A with Ω.
(*iii*): Let $B \in BAS_{\mathfrak{P}}(A)$. This means that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ and $\pi_{\mathfrak{P}}(B') \neq \pi_{\mathfrak{P}}(A)$ for any $B' \subsetneq B$. In particular, we have that $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(B)$. Furthermore, if we take $B' = B \setminus \{b\}$, it follows that $\pi_{\mathfrak{P}}(B \setminus \{b\}) \neq \pi_{\mathfrak{P}}(A)$. By the arbitrariness of b, we deduce that $C_{\mathfrak{P}}(B) = B$.
On the contrary, let $B \in \mathcal{P}(A)$ such that $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(B)$ and $C_{\mathfrak{P}}(B) = B$. Then, $\pi_{\mathfrak{P}}(B \setminus \{b\}) \neq \pi_{\mathfrak{P}}(B)$ for any $b \in B$. In particular, if $B' \subsetneq B$, then there exists $b \in B$ for which $B' \subseteq B \setminus \{b\}$. So, $\pi_{\mathfrak{P}}(B') \neq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ and this proves that $B \in BAS_{\mathfrak{P}}(A)$.
(*iv*): It follows by part (*iii*) by replacing A with Ω.
(*v*): Let $A \in MINP(\mathfrak{P})$, then $A \in \min([A]_{\approx_{\mathfrak{P}}})$. Moreover, the unique subset satisfying both conditions of Definition 3.2.1 is exactly A, that is $BAS_{\mathfrak{P}}(A) = \{A\}$.
Vice versa, suppose that $BAS_{\mathfrak{P}}(A) = \{A\}$. Then, we have that $A \in \max(MINP_{\mathfrak{P}}(A))$ and, in particular, $A \in MINP(\mathfrak{P})$. □

We now show that any core operator on a finite set Ω is the $\mathfrak{P}$-core operator of some pairing $\mathfrak{P} \in PAIR(\Omega)$.

Theorem 3.2.11. *Let Ω be a finite set. If $\sigma \in COOP(\Omega)$, there exists $\mathfrak{P} \in PAIR(\Omega)$ such that $\sigma = C_{\mathfrak{P}}$.*

Proof. Let $\sigma \in COOP(\Omega)$. Let us consider the set operator $\varphi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined as follows:

$$\varphi_\sigma(A) := A \cup A^*, \tag{3.6}$$

where $A^* := \{b \in \Omega : b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))\}$. We will prove that φ is a closure operator. Clearly, φ is extensive. Let now $A \subseteq B$. It suffices to show that $A^* \subseteq \varphi_\sigma(B)$. Let $b \in A^*$. Assume that $b \in \Omega \setminus B$. Then, $A \cup \{b\} \subseteq B \cup \{b\}$, therefore, by pseudo-monotonicity, we have that $\sigma(A \cup \{b\}) \supseteq \sigma(B \cup \{b\}) \cap (A \cup \{b\})$. But since $b \in \Omega \setminus \sigma(A \cup \{b\})$, it follows that $b \in \Omega \setminus ((A \cup \{b\}) \cap \sigma(B \cup \{b\}))$. Clearly, it implies that $b \in \Omega \setminus \sigma(B \cup \{b\})$ and this proves that $b \in \Omega \setminus (B \cup \sigma(B \cup \{b\})) = B^*$, i.e. $b \in \varphi_\sigma(B)$. On the other hand, if $b \in B$, it is obvious that $b \in \varphi_\sigma(B)$. This proves monotonicity.
We now show that φ is idempotent. By extensiveness of φ_σ, it follows that $\varphi_\sigma(A) \subseteq \varphi_\sigma(\varphi_\sigma(A))$. Clearly, if $(\varphi_\sigma(A))^* = \emptyset$, then $\varphi_\sigma(\varphi_\sigma(A)) \subseteq \varphi_\sigma(A)$. Assume now that $(\varphi_\sigma(A))^* \neq \emptyset$ and let $b \in (\varphi_\sigma(A))^*$. Then, $b \in \Omega \setminus (\varphi_\sigma(A) \cup \sigma(\varphi_\sigma(A) \cup \{b\})) = \Omega \setminus ((A \cup A^*) \cup \sigma(A \cup A^* \cup \{b\}))$. Let us prove that the condition $b \in \Omega \setminus A \cup A^*$ implies $b \in \sigma(A \cup A^* \cup \{b\})$.
Let us observe that since $b \in \Omega \setminus \varphi_\sigma(A)$, then $b \in \sigma(A \cup \{b\})$. Therefore, let us fix an integer m and assume that $b \in \sigma(A \cup B \cup \{b\})$ for any $B \subseteq A^*$ such that $|B| = m - 1$. Let now $B \subseteq A^*$ such that $|B| = m$ and fix $c \in B$. We will prove that $c \in \Omega \setminus \sigma((A \cup (B \setminus \{c\})) \cup \{c\})$ and that $b \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$ so that it is possible to use quasi-regularity that, in this case, yields $c \in \sigma((A \cup B \setminus \{c\}) \cup \{b, c\}) = \sigma(A \cup B \cup \{b\})$, showing the claim for B. In particular, the claim holds for $B = A^*$. Then, $b \in \sigma(A \cup A^* \cup \{b\})$, contradicting our choice of b. Necessarily, it must be $(\varphi_\sigma(A))^* = \emptyset$.
Hence, let us firstly show that $c \in \Omega \setminus \sigma((A \cup (B \setminus \{c\})) \cup \{c\})$. As a matter of fact, we have that $c \in A^*$, so $c \in \Omega \setminus (A \cup \{c\})$. Furthermore, to say that $c \in B$ means that $A \cup \{c\} \subseteq A \cup B$ so, by pseudo-monotonicity, $\sigma(A \cup \{c\}) \supseteq \sigma(A \cup B) \cap (A \cup \{c\})$. In other terms, $c \in \Omega \setminus \sigma(A \cup B)$, i.e. $c \in \Omega \setminus \sigma((A \cup (B \setminus \{c\})) \cup \{c\})$.
On the other hand, we have that $b \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$, in fact we can use the inductive hypothesis on $B \setminus \{c\} \subseteq A^*$ since $|B \setminus \{c\}| = m - 1$. The two hypothesis of the definition of quasi-regular operator are satisfied and this proves that φ_σ is idempotent, so $\varphi_\sigma \in CLOP(\Omega)$. By part (i), there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $M_{\mathfrak{P}} = \varphi_\sigma$. To conclude, we have to show that $\sigma = C_{\mathfrak{P}}$.
Let $b \in C_{\mathfrak{P}}(A)$, then $b \in \Omega \setminus M_{\mathfrak{P}}(A \setminus \{b\}) = \varphi_\sigma(A \setminus \{b\})$. This entails that $b \in \Omega \setminus (A \setminus \{b\})^*$ or, equivalently, $b \in \sigma(A \setminus \{b\} \cup \{b\}) = \sigma(A)$.
On the other hand, let $b \in \sigma(A)$. By extensiveness, $b \in A$. Furthermore, assume by contradiction that $b \in M_{\mathfrak{P}}(A \setminus \{b\})$. Then, $b \in (A \setminus \{b\})^*$, i.e. $b \in \Omega \setminus \sigma(A \setminus \{b\} \cup \{b\}) = \Omega \setminus \sigma(A)$, contradicting our choice of b. This implies that $C_{\mathfrak{P}}(A) = \sigma(A)$ for any $A \in \mathcal{P}(\Omega)$ and conclude the proof. $\square$

We now provide the notion of A-symmetry essential.

Definition 3.2.12. Let $B \subseteq A$. We say that B is A-*symmetry essential* if:

$(E1)$ $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$;

$(E2)$ $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$ for all $B' \subsetneq B$.

We denote by $ESS_{\mathfrak{P}}(A)$ the family of all A-symmetry essential subsets of A and we set $ESS(\mathfrak{P}) := ESS_{\mathfrak{P}}(\Omega)$. We call *symmetry essentials* of $\mathfrak{P}$ the members of $ESS(\mathfrak{P})$. If $\mathfrak{P}$ is a finite pairing on Ω and $k \in \{1, \dots, n\}$, we set $ESS_k(\mathfrak{P}) := \{C \in ESS(\mathfrak{P}) : |C| = k\}$

The motivation to introduce the set system $ESS_{\mathfrak{P}}(A)$ is that it is a generalization of the core because $C_{\mathfrak{P}}(A) = \emptyset$ or, otherwise, $\{a\} \in ESS_{\mathfrak{P}}(A)$ for any $a \in C_{\mathfrak{P}}(A)$.

The next result tells us that the A-symmetry essentials are exactly the minimal elements of the A-dissymmetry set system.

Theorem 3.2.13. *We have that* $ESS_{\mathfrak{P}}(A) = \min(DIS_{\mathfrak{P}}(A))$.

Proof. Let $B \in ESS_{\mathfrak{P}}(A)$. By Definition 3.2.12 it results that $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$. Hence, there exist two distinct elements $v, w \in U$ such that $v \equiv_{A \setminus B} w$ and $v \not\equiv_B w$. Equivalently, we can express the previous condition by saying that $A \setminus B \subseteq A \setminus \Delta_A^{\mathfrak{P}}(v, w)$, i.e. $\Delta_A^{\mathfrak{P}}(v, w) \subseteq B$. This shows that any A-symmetry essential contains some subset of the A-dissymmetry set system. We now claim that $\Delta_A^{\mathfrak{P}}(v, w) = B$. Indeed, if $b \in B$ and $B' := B \setminus \{b\} \subsetneqq B$, we deduce that $v \not\equiv_{A \setminus B'} w$ by Definition 3.2.12. Therefore $b \in \Delta_A^{\mathfrak{P}}(v, w)$. By the arbitrariness of $b \in B$, it follows that $\Delta_A^{\mathfrak{P}}(v, w) = B$. This proves that $B \in DIS_{\mathfrak{P}}(A)$. Moreover, we proved that whenever two elements $v, w \in U$ satisfy the relation $\Delta_A^{\mathfrak{P}}(v, w) \subseteq B$, then $\Delta_A^{\mathfrak{P}}(v, w) = B$. This means that B is minimal in $DIS_{\mathfrak{P}}(A)$ with respect to set-theoretical inclusion.
Let now $B = \Delta_A^{\mathfrak{P}}(v,\, w) \neq \emptyset$ be minimal in the poset $(DIS_{\mathfrak{P}}(A), \subseteq)$, for some $v, w \in U$. Since B is non-empty, by (iii) of Proposition 3.1.3, it follows that $v \not\equiv_A w$. Moreover, by (i) of Proposition 3.1.3 we also obtain $v \equiv_{A \setminus B} w$. Then, we have $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$, and thus B satisfies (i) of Definition 3.2.12. Let now $B' \subsetneqq B$. B minimal in $DIS_{\mathfrak{P}}(A)$ implies that, for all $u,\, u' \in U$ such that $\Delta_A^{\mathfrak{P}}(u, u') \neq \emptyset$, $\Delta_A^{\mathfrak{P}}(u, u') \nsubseteq B'$. We claim that $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$. It is obvious that $u \equiv_A u'$ implies $u \equiv_{A \setminus B'} u'$; furthermore suppose that $u \equiv_{A \setminus B'} u'$ and assume by contradiction that $u \not\equiv_A u'$. Then, we have $\Delta_A^{\mathfrak{P}}(u, u') \subseteq B'$, that is an absurd. Hence $u \equiv_A u' \iff u \equiv_{A \setminus B'} u'$, so $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$. In this way, we have shown that B satisfies also condition (ii) of Definition 3.2.12. Then, $B \in ESS_{\mathfrak{P}}(A)$ and the theorem is proved. □

Remark 3.2.14. By Theorem 3.2.13, it is clear that $Tr(DIS_{\mathfrak{P}}(A)) = Tr(ESS_{\mathfrak{P}}(A))$.

The next result shows that the A-symmetry bases are exactly the minimal transversals of the family $ESS_{\mathfrak{P}}(A)$.

Theorem 3.2.15. *Let* $B \subseteq A$. *Then:*

(i) $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ *if and only if* B *is a transversal of* $\min(DIS_{\mathfrak{P}}(A))$.

(ii) $BAS_{\mathfrak{P}}(A) = Tr(DIS_{\mathfrak{P}}(A)) = Tr(ESS_{\mathfrak{P}}(A))$.

Proof. (i): Let $B \subseteq A$. Let us note that the equality $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ is obviously equivalent to the identity $\equiv_B = \equiv_A$. Therefore, we assume first that $\equiv_B = \equiv_A$; we must show that B is a transversal of $DIS_{\mathfrak{P}}(A)$. Let then $D \in DIS_{\mathfrak{P}}(A)$. By definition of $DIS_{\mathfrak{P}}(A)$ it results that D is non-empty and that there exist two distinct elements $u, u' \in U$ such that $D = \{a \in A : F(u, a) \neq F(u', a)\}$. Since D contains at least one element, we deduce that $u \not\equiv_A u'$, and this

also implies $u \not\equiv_B u'$ from the hypothesis $\equiv_B = \equiv_A$. Hence, by definition of $\equiv_B$, we find an element $b \in B$ such that $F(u,b) \neq F(u',b)$, i.e. $b \in B \cap D$. This shows that $B \cap D \neq \emptyset$, therefore B is a transversal of $DIS_{\mathfrak{P}}(A)$.
We suppose now that B is a transversal of $DIS_{\mathfrak{P}}(A)$ and let u, u' be two any distinct elements in U. If $u \equiv_A u'$ it is obvious that we also have $u \equiv_B u'$. We can assume therefore that $u \not\equiv_A u'$. By definition of $\equiv_A$ and by (3.1) it follows then that the $D := \Delta_A^{\mathfrak{P}}(u,u')$ is non-empty, so that $D \in DIS_{\mathfrak{P}}(A)$. Since B is a transversal of $DIS_{\mathfrak{P}}(A)$ we have that $B \cap D \neq \emptyset$. Let $b \in B \cap D$. Then, there exists $b \in B$ such that $F(u,b) \neq F(u',b)$, and this implies that $u \not\equiv_B u'$. Hence $\equiv_B = \equiv_A$. This proves that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ if and only if B is a transversal of $DIS_{\mathfrak{P}}(A)$. Clearly, it suffices to be transversal of $\min(DIS_{\mathfrak{P}}(A))$ in order to be also a transversal of $DIS_{\mathfrak{P}}(A)$. Hence, we showed that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$ if and only if B is a transversal of $\min(DIS_{\mathfrak{P}}(A))$.
(ii): Let $B \in BAS_{\mathfrak{P}}(A)$. By Definition 3.2.1 we have then $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$, and by part (i) this implies that B is a transversal of $DIS_{\mathfrak{P}}(A)$. Now, if $b \in B$, again by Definition 3.2.1 we have that $\pi_{\mathfrak{P}}(B \setminus \{b\}) \neq \pi_{\mathfrak{P}}(A)$, therefore by (i) it follows that $B \setminus \{b\}$ is not a transversal of $DIS_{\mathfrak{P}}(A)$. Hence B is a minimal transversal of $DIS_{\mathfrak{P}}(A)$. On the other hand, let B be a minimal transversal of $DIS_{\mathfrak{P}}(A)$, then by (i) it follows that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$. Now, if $b \in B$ the subset $B \setminus \{b\}$ is not a transversal of $DIS_{\mathfrak{P}}(A)$ by virtue of the minimality of B, therefore, again by (i) we obtain $\pi_{\mathfrak{P}}(B \setminus \{b\}) \neq \pi_{\mathfrak{P}}(A)$. Hence $B \in BAS_{\mathfrak{P}}(A)$ and the thesis follows. □

Based on the relevance of the set system $ESS_{\mathfrak{P}}(A)$, we now provide a new characterization for its structure in relation to the order structure of the poset $\mathbb{M}_{\mathfrak{P}}(A)$.

Theorem 3.2.16. $ESS_{\mathfrak{P}}(A) = \{B \in \mathcal{P}(A) : A \setminus B \in [A|\mathbb{M}_{\mathfrak{P}}(A) \uparrow]\}$.

Proof. Let B be a non-empty subset of A such that $A \setminus B \in [A|\mathbb{M}_{\mathfrak{P}}(A) \uparrow]$. Assume by contradiction that $\pi_{\mathfrak{P}}(A \setminus B) = \pi_{\mathfrak{P}}(A)$, i.e. $M_{\mathfrak{P}}(A \setminus B) = M_{\mathfrak{P}}(A)$. Since $A \setminus B = A \cap C$ for some $C \in MAXP(\mathfrak{P})$, we have that

$$M_{\mathfrak{P}}(A \cap C) = M_{\mathfrak{P}}(A \setminus B) = M_{\mathfrak{P}}(A) \subseteq M_{\mathfrak{P}}(A) \cap M_{\mathfrak{P}}(C) = M_{\mathfrak{P}}(A) \cap C,$$

i.e. $M_{\mathfrak{P}}(A) \subseteq C$. In other terms, we showed that $A = A \cap M_{\mathfrak{P}}(A) \subseteq A \cap C = A \setminus B$ or, equivalently $A = A \setminus B$, that is a contradiction. Hence $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus B)$. Furthermore, let $B' \subsetneqq B$ and assume that $\pi_{\mathfrak{P}}(A \setminus B') \neq \pi_{\mathfrak{P}}(A)$. Then, we have that $A \setminus B \subsetneqq A \setminus B' \subseteq M_{\mathfrak{P}}(A \setminus B') \cap A$. In particular, we also have that $M_{\mathfrak{P}}(A \setminus B') \cap A \subsetneqq A$, otherwise $A \subseteq M_{\mathfrak{P}}(A \setminus B')$, i.e. $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A \setminus B')$, contradicting our assumption. In this way, we showed that $A \setminus B$ is not a co-cover of A in $\mathbb{M}_{\mathfrak{P}}(A)$, that is a contradiction. This shows that $B \in ESS_{\mathfrak{P}}(A)$.
On the other hand, let $B \in ESS_{\mathfrak{P}}(A)$. Let us show that $A \setminus B = M_{\mathfrak{P}}(A \setminus B) \cap A$. Clearly, $A \setminus B \subseteq M_{\mathfrak{P}}(A \setminus B) \cap A$. Vice versa, let $a \in M_{\mathfrak{P}}(A \setminus B) \cap A$ and assume that $a \notin A \setminus B$. Hence $a \in B$. Let $u \equiv_{A \setminus B} u'$, then $F(u,c) = F(u',c)$ for any $c \in A \setminus B$ and, in particular, for any $c \in M_{\mathfrak{P}}(A \setminus B)$. Thus $F(u,c) = F(u',c)$ for any $c \in (A \setminus B) \cup \{a\}$, i.e. for any $c \in A \setminus (B \setminus \{a\})$. Set $B' := B \setminus \{a\}$. In other terms, we have shown that $u \equiv_{A \setminus B} u'$ implies $u \equiv_{A \setminus B'} u'$. But $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$, so $u \equiv_{A \setminus B} u'$ implies $u \equiv_A u'$ or, equivalently, $\pi_{\mathfrak{P}}(A \setminus B) = \pi_{\mathfrak{P}}(A)$, contradicting the fact that $B \in ESS_{\mathfrak{P}}(A)$. This proves that $A \setminus B \in \mathbb{M}_{\mathfrak{P}}(A)$. We must show that $A \setminus B \in [A|\mathbb{M}_{\mathfrak{P}}(A) \uparrow]$. Suppose by contradiction it were false. Then, there exists $A \setminus B \subsetneqq C \subsetneqq A$ such that $C \in [A|\mathbb{M}_{\mathfrak{P}}(A) \uparrow]$. But, this ensures that $C = A \setminus B'$ for some non-empty $B' \subsetneqq B$.

Let us prove that $\pi_{\mathfrak{P}}(C) \neq \pi_{\mathfrak{P}}(A)$. Assume by contradiction that $\pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$. Since $C = A \cap M$ for some $M \in MAXP(\mathfrak{P})$, so $C \subseteq M$ and $M_{\mathfrak{P}}(C) = M_{\mathfrak{P}}(A) \subseteq M$, i.e. $A \subseteq M$. But, this implies that $A \cap M = C \supsetneq A$ and this is impossible. So, $\pi_{\mathfrak{P}}(C) \neq \pi_{\mathfrak{P}}(A)$. But the existence of such a subset C contradicts the fact that $B \in ESS_{\mathfrak{P}}(A)$. □

3.2.1 The Petersen Graph

In this section we apply all previous general notions to study the Petersen graph, denoted by Pet. This graph has the following useful characterization: it is the graph whose vertices can be identified with the 2-subsets of $\hat{5} = \{1, 2, 3, 4, 5\}$, such that two vertices A and B are adjacent if and only if they are disjoint. Therefore, we write v_{ij} to denote the vertex identified with $\{i, j\}$.

We recall now, without giving any proofs, some important well known properties of the Petersen graph.

Proposition 3.2.17. *1. The valency of each vertex in Pet is equal to 3.*

2. The girth of Pet is equal to 5.

3. Pet is 3-transitive, thus both vertex-transitive and edge-transitive.

4. The automorphism group of the Petersen graph is isomorphic to the symmetric group S_5.

In particular an isomorphism ψ between the symmetric group S_5 and the automorphism group of Pet, $Aut(Pet)$, can be given in the following natural way. An element $\pi \in S_5$ permutes the element in $\hat{5}$ and thus induces a bijective map $\psi(\pi)$ of $V(Pet) = \binom{\hat{5}}{2}$ in itself. It is not hard to see that $\psi : \pi \longmapsto \psi(\pi)$ is an isomorphism.

The main aim of this section is to provide a complete description of $DIS(Pet)$, $ESS(Pet)$ and $BAS(Pet)$). In what follows the letters h, i, j, k, l will denote all elements in $\hat{5}$ in an arbitrary order. In this case, if we consider the vertex v_{ij}, it means an arbitrary vertex in $V(Pet)$. We begin to examine $DIS(Pet)$.

Proposition 3.2.18. *We have that $DIS(Pet) = DIS_4(Pet) \cup DIS_6(Pet)$, where*

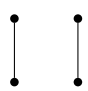
$$DIS_4(Pet) = \{A \subseteq V(Pet) : \ A = \Delta^{Pet}(v, w) \wedge v \nsim w\}$$

and

$$DIS_6(Pet) = \{B \subseteq V(Pet) : \ B = \Delta^{Pet}(v, w) \wedge v \sim w\}.$$

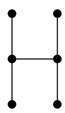
Moreover:

- *$A \in DIS_4(Pet)$ if and only if $Pet[A]$ is isomorphic to the following graph H_1:*



H_1

- $B \in DIS_6(Pet)$ *if and only if* $Pet[B]$ *is isomorphic to the following graph* H_2*:*

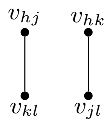


H_2

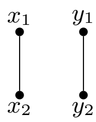
Proof. Let $v,\, w \in V(Pet)$ such that $v \nsim w$. In this case there exist distinct elements $i,\, j,\, k \in \hat{5}$ such that $v = v_{ij}$ and $w = v_{ik}$. If $h,\, l \in \hat{5}$ are such that $\{h,\, l\} = \hat{5} \setminus \{i,\, j,\, k\}$, then it holds:

$$\Delta_G(v, w) = \Delta_G(v_{ij}, v_{ik}) = \{v_{hj},\, v_{hk},\, v_{jl},\, v_{kl}\}.$$

If $A = \Delta_G(v, w)$, then $Pet[A]$ is the following graph:



Conversely, assume that $A = \{x_1,\, x_2,\, y_1,\, y_2\} \subseteq V(Pet)$ is such that the induced subgraph $Pet[A]$ is the following:

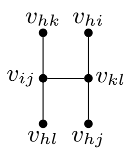


Since $x_1 \sim x_2$, there exist four distinct indexes $i,\, j,\, k,\, l \in \hat{5}$ such that $x_1 = v_{ij}$ and $x_2 = v_{kl}$. Now, y_1 and y_2 are adjacent to each other and both not adjacent to v_{ij} and v_{kl}. The only possibilities are that $\{y_1,\, y_2\} = \{v_{ik},\, v_{jl}\}$ or $\{y_1,\, y_2\} = \{v_{il},\, v_{jk}\}$. It is easy to see that $\{v_{ij},\, v_{kl},\, v_{ik},\, v_{jl}\} = \Delta^{Pet}(v_{hj},\, v_{hk})$ and $\{v_{ij},\, v_{kl},\, v_{il},\, v_{jk}\} = \Delta^{Pet}(v_{hj},\, v_{hl})$, so $A \in DIS_4(Pet)$.

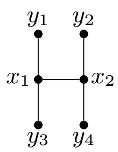
Suppose first that $v,\, w \in V(Pet)$ and $v \sim w$. Then, there exist four distinct indexes $i,\, j,\, k,\, l \in \hat{5}$ such that $v = v_{ij}$ and $w = v_{kl}$. If h is the only element in $\hat{5} \setminus \{i,\, j,\, k,\, l\}$ then

$$\Delta^{Pet}(v, w) = \Delta^{Pet}(v_{ij}, v_{kl}) = \{v_{ij},\, v_{kl},\, v_{hi},\, v_{hj},\, v_{hk},\, v_{hl}\}.$$

If $B = \Delta^{Pet}(v, w)$, the subgraph $Pet[B]$ is the following:



and thus it is isomorphic to H_2. Let now $B = \{x_1,\, x_2,\, y_1,\, y_2,\, y_3,\, y_4\} \subseteq V(Pet)$ such that $Pet[B]$ is the following graph:



Then, it is easy to see that $B = \Delta^{Pet}(x_1, x_2)$, and thus $B \in DIS_6(Pet)$. This concludes the proof. $\square$

Corollary 3.2.19. $\Delta[Pet] = 4(J_{10} - I_{10}) + 2Adj(Pet)$.

Proof. It follows by Proposition 3.2.18 and by definition of the adjacency matrix. $\square$

In order to give a complete description of the symmetry base for the Petersen graph, first we establish some useful results.

Proposition 3.2.20. *$A \in DIS_4(Pet)$ if and only if $V(Pet) \setminus A \in DIS_6(Pet)$.*

Proof. Let $A \in DIS_4(Pet)$. As in the proof of Proposition 3.2.18, there exist three distinct indexes $i, j, k \in \hat{5}$ such that

$$A = \Delta(v_{ij}, v_{ik}) = \{v_{hj}, v_{hk}, v_{jl}, v_{kl}\} = V(Pet) \backslash \{v_{hi}, v_{hl}, v_{ij}, v_{ik}, v_{il}, v_{jk}\} = V(Pet) \backslash \Delta(v_{hl}, v_{jk}),$$

where $\{h, l\} = \hat{5} \setminus \{i, j, k\}$. So $V(Pet) \setminus A \in DIS_6(Pet)$.

Similarly, if $B \in DIS_6(Pet)$, then there exist four distinct indexes $i, j, k, l \in \hat{5}$ such that $B = \Delta^{Pet}(v_{ij}, v_{kl}) = \{v_{ij}, v_{kl}, v_{hi}, v_{hj}, v_{hk}, v_{hl}\} = V(Pet) \setminus \{v_{ik}, v_{il}, v_{jk}, v_{jl}\} = V(Pet) \setminus \Delta(v_{hi}, v_{hj})$, where $\{h\} = \hat{5} \setminus \{i, j, k, l\}$. This concludes the proof. $\square$

Corollary 3.2.21. $|DIS_4(Pet)| = |DIS_6(Pet)| = 15$ *and* $|DIS(Pet)| = 30$.

Proof. It follows by Proposition 3.2.18, Proposition 3.2.20 and by using the fact that $|E(Pet)| = 15$. $\square$

Proposition 3.2.22. *We have that $DIS(Pet) = ESS(Pet)$.*

Proof. By Theorem 3.2.13 it is sufficient to prove that, if $A \in DIS_4(Pet)$, then there exists no local dissimmetry subset $B \in DIS_6(Pet)$ such that $A \subseteq B$. This follows by Proposition 3.2.18 and by using the fact that, if $A \subseteq B \in DIS_6(Pet)$ has cardinality $|A| = 4$, then $Pet[A] \cong H_2[A]$ and there exists no 4-subset of $V(H_2)$ such that the subgraph in H_2 generated by A is isomorphic to H_1. Thus the proposition is proved. $\square$

Proposition 3.2.23. *If $C \subseteq V(Pet)$ is a transversal of $DIS_4(Pet)$, then either $|C| \geq 5$ or $C \in DIS_4(Pet)$.*

Proof. Let $C \subseteq V(Pet)$ such that $C = \{v_{i_1 j_1}, v_{i_2 j_2}, v_{i_3 j_3}, v_{i_4 j_4}\} \notin DIS_4(Pet)$ and $|C| = 4$. In C we have thus 4 vertices associates to distinct pairs of elements in $\hat{5}$. We define the function $\eta_C : \hat{5} \longrightarrow \{0, 1, 2, 3, 4\}$ such that, if $i \in \hat{5}$, then $\eta_C(i)$ is the number of time that i occurrs in the sequences of number $i_1 j_1 i_2 j_2 i_3 j_3 i_4 j_4$. Thus $\sum_{i=1}^{5} \eta_C(i) = 2|C| = 8$. Since the pairs are mutually distinct, there exists at most one element $i \in \hat{5}$ such that $\eta_C(i) = 0$. Since $C \notin DIS_4(Pet)$, there exists $i \in \hat{5}$ such that $\eta(i) = 1$. In fact, if this does not hold, then there would exist $i, j, k, l \in \hat{5}$ such that $\eta_C(i) = \eta_C(j) = \eta_C(k) = \eta_C(l) = 2$. It is easy to see that in such a case $Pet[C] \cong H_1$ and thus $C \in DIS_4(Pet)$ that contradicts our assumptions. Let $h \in \hat{5}$ such that $\eta_C(h) = 1$. Without loss in generality we can suppose that $v_{hi} \in C$ and

v_{hj}, v_{hk}, $v_{hl} \notin C$. It is straightforward to prove that, for each choice of three vertices x, y and z in $V(Pet) \setminus \{v_{hi}, v_{hj}, v_{hk}, v_{hl}\}$, $C := \{v_{hi}, x, y, z\}$ is included in one of these subsets:

$$\begin{aligned} B_1 &= \{v_{hi}, v_{ij}, v_{jk}, v_{jl}, v_{kl}, v_{hj}\}, \\ B_2 &= \{v_{hi}, v_{ik}, v_{jk}, v_{jl}, v_{kl}, v_{hk}\}, \\ B_3 &= \{v_{hi}, v_{il}, v_{jk}, v_{jl}, v_{kl}, v_{hl}\}, \\ B_4 &= \{v_{hi}, v_{ij}, v_{ik}, v_{il}, v_{jk}, v_{hl}\}, \\ B_5 &= \{v_{hi}, v_{ij}, v_{ik}, v_{il}, v_{jl}, v_{hk}\}, \\ B_6 &= \{v_{hi}, v_{ij}, v_{ik}, v_{il}, v_{kl}, v_{hj}\}. \end{aligned}$$

Note that $Pet[B_k] \cong H_2$, for each $k \in \{1, 2, 3, 4, 5, 6\}$. By Proposition 3.2.20, they can not be transversals of $DIS_4(Pet)$ and the proposition is proved. □

Corollary 3.2.24. *Let $C \subseteq V(Pet)$. C is a transversal of the 2-local dissymmetry set system $DIS(Pet)$ if and only if C is a transversal of $DIS_4(Pet)$ and $C \notin DIS_4(Pet)$. In particular each symmetry base of Pet has cardinality greater than or equal to 5.*

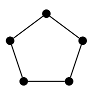
Proof. Let $C \subseteq V(Pet)$ be a transversal of $\mathfrak{D}_G$. Since $DIS_4(Pet) \subseteq DIS(Pet)$, the subset C is a transversal of $\mathfrak{D}_4(Pet)$. If $C \in DIS_4(Pet)$, then, by Corollary 3.2.21, $V(Pet)\setminus C \in DIS(Pet)$ and this contradicts the assumption on C.

Let now $C \subseteq V(Pet)$ be a transversal of $DIS_4(Pet)$ and we assume that $C \notin DIS_4(Pet)$. By Proposition 3.2.23, then $|C| \geq 5$. Let $D \in DIS(Pet)$. Then, by Proposition 3.2.18, $|D| = 4$ or $|D| = 6$. If $|D| = 4$, then $D \in DIS_4(Pet)$ and thus $D \cap C \neq \emptyset$. If $|D| = 6$, since $|C| \geq 5$ and $|V(Pet)| = 10$, we obtain $D \cap C \neq \emptyset$. So C is a transversal of $\mathfrak{D}_G$.

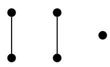
The last claim follows directly by definition of symmetry base. □

We prove now the main result of this section.

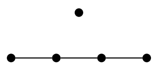
Theorem 3.2.25. *A subset $C \subseteq V(Pet)$ is a symmetry base of Pet if and only if $|C| = 5$ and $Pet[C]$ is isomorphic to one of the following graphs:*



C_5



$P_{2,2,1}$

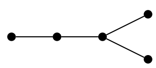


$P_{4,1}$

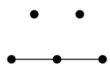
P_5

Proof. Note first that, if $C \subseteq V(Pet)$ is such that $Pet[C]$ is isomorphic to C_5, $P_{2,2,1}$, $P_{4,1}$ or P_5, then there is no element $B \in DIS_6(Pet)$ such that $C \subseteq B$. In fact, if $B \in DIS_6(Pet)$, then, by Proposition 3.2.18, $Pet[B]$ is isomorphic to H_2 but, however we choose a 5-subset $C \subseteq V(H_2)$, $H_2[C]$ is not isomorphic to C_5, $P_{2,2,1}$, $P_{4,1}$ or P_5. Thus if $C \subseteq V(Pet)$ is such that $Pet[C]$ is isomorphic to one of the graphs listed above, then C is a transversal of $DIS(Pet)$. By Corollary 3.2.24, it is minimal among the transversal of $DIS(Pet)$ with respect to the inclusion relation, so $C \in Tr(DIS(Pet)) = BAS(Pet)$ (see Theorem 3.2.15).

Consider the two graphs:



Y



$P_{3,1,1}$

We now prove that, if $C \subseteq V(Pet)$ has cardinality $|C| = 5$ and $C \subseteq B \in DIS_6(Pet)$, then $Pet[C]$ is isomorphic to Y or $P_{3,1,1}$. By Proposition 3.2.18, $Pet[B]$ is isomorphic to H_2 and if we remove from the graph H_2 one vertex (with all the edges incident to it), then we obtain a graph which is isomorphic to Y or $P_{3,1,1}$. On the other hand, if $C \subseteq V(Pet)$ is such that $Pet[C]$ is isomorphic to Y or $P_{3,1,1}$, then there exist exactly one element $B \in DIS_6(Pet)$ such that $C \subseteq B$. To prove that observe that, if $C \subseteq V(Pet)$ is such that $Pet[C] \cong Y$, then there exist $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{kl}, v_{jk}, v_{jl}, v_{hj}\}$ or $C = \{v_{hi}, v_{kl}, v_{jk}, v_{jl}, v_{ij}\}$. In both cases C is included in the unique $B \in DIS_6(Pet)$ given by $B := \{v_{hi}, v_{kl}, v_{jk}, v_{jl}, v_{hj}, v_{ij}\}$. Similarly, if $C \subseteq V(Pet)$ is such that $Pet[C] \cong P_{3,1,1}$, then there exist $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{hj}, v_{hk}, v_{hl}, v_{ij}\}$ and the unique $B \in DIS_6(Pet)$ containing C is $B = \{v_{hi}, v_{hj}, v_{hk}, v_{hl}, v_{ij}, v_{kl}\}$. Thus the number of 5-subsets of $V(Pet)$ that are not included in a local dissymmetry subset of Pet is equal to $b_5 = \binom{10}{5} - \binom{6}{5}|DIS_6(Pet)| = 252 - 90 = 162$. Each of these subsets is not included in any element in $DIS_6(Pet)$, so it is a symmetry base of Pet. We now characterize geometrically such subsets by proving that each of them is isomorphic to one of the graphs listed in the statement of the theorem. For this, we will count the number of 5-subsets of $V(Pet)$ whose corresponding induced subgraph is isomorphic to C_5, $P_{2,2,1}$, $P_{4,1}$ and P_5, respectively.

If $C \subset V(Pet)$, then the orbit of C among the 5-subsets of $V(Pet)$ is equal to

$$|O_{Aut(Pet)}(C)| = \frac{|Aut(Pet)|}{|Aut(Pet)_C|}, \tag{3.7}$$

where $Aut(Pet)_C$ is the stabilizer of C with respect to the action of $Aut(Pet)$ on $V(Pet)$. Moreover, since the Petersen graph is symmetric, the orbit of C is the set of all 5-subsets inducing the same graph (up to isomorphism), and so its cardinality is exactly the number of 5-subsets of $V(Pet)$ such that the induced subgraph is isomorphic to X.

The automorphism groups of the graphs C_5, $P_{2,2,1}$, $P_{4,1}$ and P_5 are isomorphic respectively to the dihedral group D_5, the dihedral group D_4, $\mathbb{Z}_2$ and $\mathbb{Z}_2$. We prove now that if $Pet[C]$ is isomorphic to X, where X is C_5, $P_{4,1}$ or P_5, then $Aut(Pet)_C \cong Aut(X)$, while if $X = P_{2,2,1}$, then $Aut(Pet)_C$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We recall that D_5 has order 10 and the following presentation:

$$D_5 = \langle \sigma, \rho | \sigma^5 = \rho^2 = (\sigma\rho)^2 = 1 \rangle$$

To prove that $Aut(Pet)_C \cong Aut(X)$ if $X = C_5$, it is sufficient to prove that each generator of the automorphism group of C_5 extends uniquely to an automorphism of Pet. If $C \subseteq V(Pet)$ is such that $Pet[C] \cong C_5$, then there exist different $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{jk}, v_{hl}, v_{ij}, v_{kl}\}$. So we can choose the generators $\sigma, \rho \in Aut(C_5)$ defined by:

$$\sigma(v_{hi}) = v_{jk},\ \sigma(v_{jk}) = v_{hl},\ \sigma(v_{hl}) = v_{ij},\ \sigma(v_{ij}) = v_{kl}, \sigma(v_{kl}) = v_{hi}$$

and

$$\rho(v_{hi}) = v_{hi},\ \rho(v_{jk}) = v_{kl},\ \rho(v_{hl}) = v_{ij},\ \rho(v_{ij}) = v_{hl}, \rho(v_{kl}) = v_{jk}.$$

Then, σ extends to the automorphism of Pet defined by the permutation in S_5 defined by:

$$\sigma'(h) = j,\ \sigma'(i) = k,\ \sigma'(j) = l,\ \sigma'(k) = h,\ \sigma'(l) = i,$$

while ρ extends to the automorphism of Pet defined by the permutation in S_5 defined by:

$$\rho'(h) = i,\ \rho'(i) = h,\ \rho'(j) = l,\ \rho'(k) = k,\ \rho'(l) = j.$$

Thus $Aut(Pet)_C \cong Aut(C_5)$ and the number of 5-subsets $C \subseteq V(Pet)$ such that $Pet[C] \cong C_5$ is equal to

$$\frac{|Aut(Pet)|}{|Aut(C_5)|} = \frac{120}{10} = 12.$$

Let τ be the generator of the automorphism groups of the graphs $P_{4,1}$ and P_5, that are both isomorphic to $\mathbb{Z}_2$.

A 5-subset C of $V(Pet)$ such that $Pet[C] \cong P_{4,1}$ ($Pet[C] \cong P_5$) is given by

$$C = \{v_{hi},\, v_{jk},\, v_{hl},\, v_{ij},\, v_{hj}\} \qquad (C = \{v_{hi},\, v_{jk},\, v_{hl},\, v_{ij},\, v_{hk}\})$$

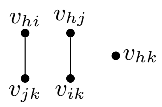
for a suitable choice of elements $h,\ i,\ j,\ k,\ l \in \hat{5}$. The automorphism τ of $P_{4,1}$ (P_5) corresponds to the permutation τ' of the elements in $\hat{5}$ given by:

$$\tau'(h) = j\,(\tau'(h) = h),\ \tau'(i) = i\,(\tau'(i) = k),\ \tau'(j) = h\,(\tau'(j) = j),$$

$$\tau'(k) = l\,(\tau'(k) = i),\ \tau'(l) = k\,(\tau'(l) = l).$$

Thus the orbit of C has cardinality $\frac{|Aut(Pet)|}{|\mathbb{Z}_2|} = \frac{120}{2} = 60$.

Now a 5-subset C of $V(Pet)$ satisfies $Pet[C] \cong P_{2,2,1}$ if and only if there exist $h,\ i,\ j,\ k,\ l \in \hat{5}$ such that $C = \{v_{hi},\, v_{jk},\, v_{hj},\, v_{ik},\, v_{hk}\}$. The graph $Pet[C]$ is the following graph:



We now find all automorphisms of Pet fixing $Pet[C]$. An automorphism ϕ of Pet that fixes $Pet[C]$ must also fix the vertex v_{hk}. It is straightforward to prove that the only permutations of the indices which give such an automorphism are id, α, β, $\alpha\beta$ where α is defined by:

$$\alpha(h) = h,\ \alpha(i) = j,\ \alpha(j) = i,\ \alpha(k) = k,\ \alpha(l) = l.$$

and β by

$$\beta(h) = k,\ \beta(i) = i,\ \beta(j) = j,\ \beta(k) = h,\ \beta(l) = l.$$

Thus the stabilizer of C is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the number of the 5-subsets of Pet that induce a subgraph isomorphic to $P_{2,2,1}$ is $\frac{120}{4} = 30$.

The sum of all the numbers found so far is equal to 162, so they are all the possible symmetry bases of Pet having cardinality 5.

Finally, observe that if $A \subseteq V(Pet)$ is such that $|A| > 5$ and $A \notin DIS_6(Pet)$, then $B \subsetneq A$, for some $B \in BAS_5(Pet)$ and therefore $A \notin BAS(Pet)$). This proves the theorem.

$\square$

3.2.2 The C_n Case

We now provide a general form for the symmetry essentials and the symmetry bases of C_n.

Proposition 3.2.26. *Let $n \geq 5$. Then:*
(i) $ESS_2(C_n) = DIS_2(C_n) = \{\{v_i, v_{i+4}\}\}$.
(ii)

$$ESS_4(C_n) = \begin{cases} \emptyset & \text{if } 5 \leq n \leq 7, \\ \{\{v_i, v_{i+2}, v_j, v_{j+2}\} : d(v_i, v_j) \notin \{0,\ 2,\ 4,\ 6,\ n-6\}\} & \text{otherwise.} \end{cases}$$

Proof. (i): Straightforward.
(ii): Let $G = C_n$ for $5 \leq n \leq 7$. By (iv) of Proposition 3.2.26, we deduce that the symmetry essential subsets of order two of G have the form $\{v_i, v_{i+n-4}\}$. Thus, since a $4-$subset of $V(G)$ must contain a pair of the form $\{v_i, v_{i+n-4}\}$, we deduce that $ESS_4(C_n) = \emptyset$ if $5 \leq n \leq 7$.
Let new assume that $n \geq 8$. Firstly, we note that $d(v_i, v_j) = 0$ if and only if $i = j$ and $d(v_i, v_j) = 2$ if and only if $j \equiv i + 2 \bmod(n)$ or $j \equiv i - 2 \bmod(n)$. In these cases we have $v_i = v_j$, $v_{i+2} = v_j$ or $v_{j+2} = v_i$, thus $|\{v_i, v_{i+2}, v_j, v_{j+2}\}| \neq 4$ and so $\{v_i, v_{i+2}, v_j, v_{j+2}\}$ can not be an element of $ESS_4(C_n)$. By part (i), we have that $ESS_2(C_n) = \{\{v_i, v_{i+4}\} : 1 \leq i \leq n\}$. It is straightforward to prove that a 4-subset of $V(C_n)$ of the form $\{v_i, v_{i+2}, v_j, v_{j+2}\}$ contains a symmetry essential subset of order two if and only if $d(v_i, v_j) \neq 4$, $d(v_i, v_j) \neq 6$ and $d(v_i, v_j) \neq n - 6$. This proves the claim. □

We now provide the following characterization for the symmetry bases of C_n.

Theorem 3.2.27. *Let $A \subseteq V(C_n)$. Then, A is a symmetry base of C_n if and only if $C_{C_n}(A) = \emptyset$ and for all $v_i \in A$ one of the following conditions holds:*

1. $|B_{C_n}(A)| = 1 \wedge (v_{i-2} \notin A \vee v_{i+2} \notin A)$;
2. $|B_{C_n}(A)| \leq 1 \wedge ((v_{i-4} \notin A \wedge v_{i-2} \in A) \vee (v_{i+2} \in A \wedge v_{i+4} \notin A))$.

Proof. Let $V = V(C_n)$. By Definition 3.2.1 it's clear that a vertex subset $A \subseteq V(C_n)$ is a symmetry base if and only if $|B_{C_n}(A)| \leq 1$ and $C_{C_n}(A) = \emptyset$. We claim that A is minimal with respect to (i) of Definition 3.2.1 if and only if (1) or (2) holds. Let A be a symmetry base of C_n and $v_i \in A$ such that $|B_{C_n}(A \setminus \{v_i\})| \geq 2$. If $|B_{C_n}(A)| = \emptyset$, we would have $\{v_{i-2}, v_{i+2}\} \cap A = \emptyset$ or, equivalently, $C_{C_n}(A) \neq \emptyset$, contradicting our assumptions. Thus $|B_{C_n}(A)| = 1$ and one of the two vertices v_{i-2} or v_{i+2} is not in A, which is condition (1). On the other hand, suppose that A is a symmetry base of C_n and $v_i \in A$ such that $C_{C_n}(A \setminus \{v_i\}) \neq \emptyset$. Hence $v_{i-2} \in C_{C_n}(A \setminus \{v_i\})$ or $v_{i+2} \in C_{C_n}(A \setminus \{v_i\})$. This is equivalent to condition (2). Conversely, suppose that $A \subseteq V$ is a vertex subset satisfying conditions (1) or (2) and such that $C_{C_n}(A) = \emptyset$. It's obvious that A satisfies (i) of Definition 3.2.1. Let $v_i \in A$ and let us consider the vertex subset $A \setminus \{v_i\}$. Suppose $B_{C_n}(A) = \{v_j\}$. Hence $j \neq i \pm 1$. If $v_{i-2} \notin A$, then $v_{i-1} \equiv_{A \setminus \{v_i\}} v_j$; similarly if $v_{i+2} \notin A$, then $v_{i+1} \equiv_{A \setminus \{v_i\}} v_j$. On the other hand, let $v_i \in A$ such that $v_{i-4} \notin A \wedge v_{i-2} \in A$. In this case $v_{i-3} \equiv_{A \setminus \{v_i\}} v_{i-1}$; similarly in the other case. In both cases $\pi_{\mathfrak{P}}(A \setminus \{v_i\}) \neq \pi_{\mathfrak{P}}(V)$. This completes the proof. □

Remark 3.2.28. So far, there exists no geometric characterization for the symmetry bases of C_n.

3.3 Symmetry Bases and Symmetry Essentials for $(V, \varphi, \mathbb{K})$-VBP

Let us consider a finite dimensional vector space V. If B, C are two non-empty subsets of V, we write $C \subseteq_{li} B$ when C is linearly independent and $C \subseteq B$. Let A be a fixed arbitrary subset of V. We denote by $\text{rank}(A)$ the *rank* of A, that is $\text{rank}(A) := \dim((\text{Span}(A))$. We have the following characterizations.

Theorem 3.3.1. *(i) $BAS_{\mathfrak{P}}(A) = \{C : C \subseteq_{li} A, |C| = \text{rank}(A)\}$;*
(ii) $DIS_{\mathfrak{P}}(A) = \{A \cap (V \setminus H) : H \leqslant_{n-1} V\}$;
(iii) $ESS_{\mathfrak{P}}(A) = \{C : C \subseteq A, A^{\perp} \subsetneqq (A \setminus C)^{\perp}, C' \subsetneqq A \implies A^{\perp} = (A \setminus C')^{\perp}\}$.

Proof. (i): Let $B \in BAS_{\mathfrak{P}}(A)$. Since $BAS_{\mathfrak{P}}(A) \subseteq MINP_{\mathfrak{P}}(A)$, it results that $B \subseteq_{li} A$. Let $C \subseteq_{li} A$ such that $|C| \neq \text{rank}(A)$, then $\text{rank}(C) \neq \text{rank}(A)$, hence $\text{Span}(C) \neq \text{Span}(A)$, so $\text{Span}(C)^{\perp} \neq \text{Span}(A)^{\perp}$. Thus $\pi_{\mathfrak{P}}(C) \neq \pi_{\mathfrak{P}}(A)$. Therefore, $C \in BAS_{\mathfrak{P}}(A)$ if and only if $|C| = \text{rank}(A)$, and this proves part (i).
(ii): Let us observe that $\Delta_A^{\mathfrak{P}}(u, u') = \{a \in A : \varphi(u, a) \neq \varphi(u', a)\}$, that is equivalent to $\{a \in A : \varphi(u - u', a) \neq 0\} = \{a \in A : a \notin (u - u')^{\perp}\}$. By varying u and u' over V, we obtain the family $\{u - u' : u, u' \in V\}$ of all 1-dimensional subspaces. Therefore, $DIS_{\mathfrak{P}}(A) = \{a \in A : a \in V \setminus (u - u')^{\perp}\}$ that is equivalent, since $\dim(((u - u')^{\perp}) = n - 1$, to say that $DIS_{\mathfrak{P}}(A) = \{A \cap (V \setminus H) : H \leqslant_{n-1} V\}$.
(iii): Let $B \in ESS_{\mathfrak{P}}(A)$. Then, $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus B)$ and, therefore, $A^{\perp} \neq (A \setminus B)^{\perp}$, i.e. $A^{\perp} \subsetneqq (A \setminus B)^{\perp}$. Moreover, let $B' \subsetneqq B$. One has $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A \setminus B')$, i.e. $A^{\perp} = (A \setminus B')^{\perp}$. Therefore $ESS_{\mathfrak{P}}(A) \subseteq \{C : C \subseteq A, A^{\perp} \subsetneqq (A \setminus C)^{\perp}, C' \subsetneqq A \implies A^{\perp} = (A \setminus C')^{\perp}\}$. The reverse inclusion is obvious. □

3.4 Structure of the Global Symmetry Classes

In this subsection, we study the inner structure of each global symmetry class and we relate this inner structure to the partial order of the poset $\mathbb{L}_{\mathfrak{P}}(A)$. The starting point of our analysis is the existence of a twofold link between $BAS_{\mathfrak{P}}(A)$ with the order structure of $\mathbb{L}_{\mathfrak{P}}(A)$ and with the deletion of elements from members of the class $[A]_{\approx_{\mathfrak{P}}}$, similarly to what happens to the indispensable elements whenever one defines the classical Pawlak's core. More in detail, we study a new class of elements preserving an global symmetry class. In this way, we formulate a concept similar to Pawlak's indispensability notion for the indistinguishability relation, though weaker. It allows us to investigate the effects of deletion of elements in each global symmetry class by means of the following terminology. In this section we assume that Ω is finite.

Let A be a fixed subset of Ω. We first recall the following notion (introduced in [20]) of *indistinguishability kernel* of A, that is the subset defined by

$$K_{\mathfrak{P}}(A) := \{a \in A : \exists\ B \in [A]_{\approx_{\mathfrak{P}}} \text{ such that } B \setminus \{a\} \notin [A]_{\approx_{\mathfrak{P}}}\} \tag{3.8}$$

Let us note that the indistinguishability kernel can be considered (by analogy) a version of the usual Pawlak's core relatively to the indistinguishability relation on $\mathfrak{P}$.

We recall then the following result.

Theorem 3.4.1. *The following conditions hold.*

(*i*) *Let $A \in MAXP(\mathfrak{P})$. Then, $a \in K_{\mathfrak{P}}(A)$ if and only if there exists $C \in MAXP_{\mathfrak{P}}(A)$ such that $a \notin C$ and $M_{\mathfrak{P}}(C \cup \{a\}) = A$.*

(*ii*) *Let $A \in MAXP(\mathfrak{P})$. Then, it results that $A \setminus K_{\mathfrak{P}}(A) = \bigcap\{B : B \in [A|\mathbb{M}(\mathfrak{P}) \uparrow]\}$.*

(*iii*) *$C_{\mathfrak{P}}(A) \subseteq K_{\mathfrak{P}}(A)$.*

(*iv*) *If $A \subseteq A'$ and $A \approx_{\mathfrak{P}} A'$, then $K_{\mathfrak{P}}(A) \supseteq K_{\mathfrak{P}}(A')$.*

(*v*) *If $A \in MINP(\mathfrak{P})$, then $K_{\mathfrak{P}}(A) = A$.*

(*vi*) *$A \approx_{\mathfrak{P}} K_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$.*

(*vii*) *$M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(K_{\mathfrak{P}}(M_{\mathfrak{P}}(A)))$.*

Proof. (*i*): Let $a \in K_{\mathfrak{P}}(A)$. By definition of $K_{\mathfrak{P}}(A)$ there exists $B \in [A]_{\approx_{\mathfrak{P}}}$ such that

$$B \setminus \{a\} \not\approx_{\mathfrak{P}} A. \tag{3.9}$$

Since $B \approx_{\mathfrak{P}} A$, by (3.9) we have that $a \in B$. Let $C := M_{\mathfrak{P}}(B \setminus \{a\})$. We first show that $a \notin C$. In fact, let us assume by absurd that $a \in C$. In this case $B \subseteq C$ because $B \setminus \{a\} \subseteq M_{\mathfrak{P}}(B \setminus \{a\}) = C$. Therefore $A = M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(C) = C$ because $A, C \in MAXP(\mathfrak{P})$ and $B \in [A]_{\approx_{\mathfrak{P}}}$, and this implies that

$$\pi_{\mathfrak{P}}(C) \preceq \pi_{\mathfrak{P}}(A). \tag{3.10}$$

On the other hand, we also have

$$\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(B \setminus \{a\}) = \pi_{\mathfrak{P}}(C). \tag{3.11}$$

Then, by (3.10) and (3.11) we deduce $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(C)$, that is $A \approx_{\mathfrak{P}} C$, and this implies (by definition of C) $A \approx_{\mathfrak{P}} B \setminus \{a\}$, that is in contrast with (3.9). Hence $a \notin C$. Let us observe now that

$$C = M_{\mathfrak{P}}(B \setminus \{a\}) \subseteq M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A) = A, \tag{3.12}$$

because $A \in MAXP(\mathfrak{P})$. By (3.12) we have then $C \in MAXP_{\mathfrak{P}}(A)$. Finally, since $B \setminus \{a\} \subseteq C$, we have $B \subseteq C \cup \{a\} \subseteq A$, so that

$$\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(C \cup \{a\}) \preceq \pi_{\mathfrak{P}}(B),$$

and this implies $\pi_{\mathfrak{P}}(C \cup \{a\}) = \pi_{\mathfrak{P}}(A)$ because $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$. Hence $M_{\mathfrak{P}}(C \cup \{a\}) = M_{\mathfrak{P}}(A) = A$ because $A \in MAXP(\mathfrak{P})$. This proves the first implication.
For the other implication, we assume that there exists $C \in MAXP_{\mathfrak{P}}(A)$ such that $a \notin C$ and $M_{\mathfrak{P}}(C \cup \{a\}) = A$. Then $a \in A$ and we set $B := C \cup \{a\}$. Therefore $B \in [A]_{\approx_{\mathfrak{P}}}$ and $B \setminus \{a\} = C$. Let us note that $C \notin [A]_{\approx_{\mathfrak{P}}}$. In fact, if $C \approx_{\mathfrak{P}} A$ then $C = A$ because $C \in MAXP_{\mathfrak{P}}(A)$ and $A \in MAXP(\mathfrak{P})$, that is in contrast with the conditions $a \in A$ and $a \notin C$. Hence we obtain an subset $B \in [A]_{\approx_{\mathfrak{P}}}$ such that $B \setminus \{a\} \notin [A]_{\approx_{\mathfrak{P}}}$ and $a \in A$, that is $a \in K_{\mathfrak{P}}(A)$.
(*ii*): Let $a \in \bigcap\{B : B \in [A|\mathbb{M}(\mathfrak{P}) \uparrow]\}$ and suppose by contradiction that $a \in K_{\mathfrak{P}}(A)$. By (*i*),

there exists $B \in MAXP_{\mathfrak{P}}(A)$ such that $a \notin B$ and $M_{\mathfrak{P}}(B \cup \{a\}) = A$. We clearly have that $B \notin [A|\mathbb{M}(\mathfrak{P}) \uparrow]$. Therefore, there exists $C \in MAXP_{\mathfrak{P}}(A)$ such that $B \subsetneq C \subsetneq A$. Hence

$$\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(C). \tag{3.13}$$

We claim that $a \notin C$. In fact, if $a \in C$, we would have $B \cup \{a\} \subseteq C$, hence

$$\pi_{\mathfrak{P}}(C) \preceq \pi_{\mathfrak{P}}(B \cup \{a\}) = \pi_{\mathfrak{P}}(A). \tag{3.14}$$

Thus, by (3.13) and (3.14), $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(C)$ and, so, $C = A$, contradicting our assumption on C. Proceeding in this way, we will find a subset $D \in [A|\mathbb{M}(\mathfrak{P}) \uparrow]$ not containing a, that is an absurd. Thus $a \in K^c(A)$.
Conversely, let $a \in K^c(A)$. Suppose by contradiction that there exists $B \in [A|\mathbb{M}(\mathfrak{P}) \uparrow]$ such that $a \notin B$. Hence $C := B \cup \{a\} \approx_{\mathfrak{P}} A$, so we have found an element $C \in [A]_{\approx_{\mathfrak{P}}}$ such that $C \setminus \{a\} \not\approx_{\mathfrak{P}} A$, i.e. $a \in K_{\mathfrak{P}}(A)$, that is an absurd.
(iii): Let $a \in CORE(A)$. Hence $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$, i.e. $a \in K_{\mathfrak{P}}(A)$.
(iv): Let $a \in K_{\mathfrak{P}}(A)$. Hence, for any $B \approx_{\mathfrak{P}} A$, it results that $B \setminus \{a\} \approx_{\mathfrak{P}} A$. Therefore, $a \in K_{\mathfrak{P}}(A')$.
(v): Since $A \in MINP(\mathfrak{P})$, we have that $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$ for any $a \in A$, hence $K_{\mathfrak{P}}(A) = A$.
(vi): Set $A' = K_{\mathfrak{P}}(M_{\mathfrak{P}}(A)))$. We have that $A' \subseteq M_{\mathfrak{P}}(A)$. Let $B \in \min([A]_{\approx_{\mathfrak{P}}})$, then $B \subseteq M_{\mathfrak{P}}(A)$, hence, by part (iv), we have that $K_{\mathfrak{P}}(B) \subseteq A'$. Since $B \in MINP(\mathfrak{P})$, by part (v) $K_{\mathfrak{P}}(B) = B$, so $B \subseteq A'$. We conclude that

$$B \subseteq A' \subseteq M_{\mathfrak{P}}(A).$$

Therefore, we have

$$\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) = \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A') \preceq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A),$$

so $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A')$ and $A \approx_{\mathfrak{P}} A' = K_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$.
(vii): It follows directly by part (vi). □

We establish now the following new results.

Theorem 3.4.2. *(i)* $\bigcup BAS_{\mathfrak{P}}(A) \subseteq K_{\mathfrak{P}}(A)$.
(ii) Let $A \in MAXP(\mathfrak{P})$. *Then* $BAS_{\mathfrak{P}}(A) \subseteq \mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(A))$.

Proof. (i): Let $a \in \bigcup BAS_{\mathfrak{P}}(A)$, then there exists $B \in BAS_{\mathfrak{P}}(A)$ containing a. In particular, $B \setminus \{a\} \notin [A]_{\approx_{\mathfrak{P}}}$, therefore $a \in K_{\mathfrak{P}}(A)$.
(ii): Just observe that $B \in BAS_{\mathfrak{P}}(A)$ if and only if $|[B|\mathbb{L}_{\mathfrak{P}}(A) \downarrow]| = 0$. □

Based on the results given in Theorems 3.4.1 and 3.4.2, we continue with the analysis of the structure of $[A]_{\approx_{\mathfrak{P}}}$.
Let us firstly note that the converse of (i) of Theorem 3.4.2 turns out to be false, as it can be seen in the next example.

Example 3.4.3. *Let* $\mathfrak{P}$ *be the pairing given in Example 3.2.3 and* $A = \{2, 4, 5\}$. *Then, we have that* $K_{\mathfrak{P}}(A) = A$ *but* $\bigcup BAS_{\mathfrak{P}}(A) = \{2, 5\}$.

$\mathfrak{P}$	RAM	$Memory$	$Color$	$BatteryLife$
u_1	Insufficient	Small	Black	Very Long
u_2	Sufficient	Middle	Blue	Short
u_3	Insufficient	Small	Black	Long
u_4	Sufficient	Big	Grey	Short
u_5	Excellent	Middle	Blue	Long

Table 3.1: Mobile Phone pairing

In the next result we show that the minimum union generator of the class $[A]_{\approx_{\mathfrak{P}}}$ coincides with the family of all elements of $[A]_{\approx_{\mathfrak{P}}}$ having at most one co-cover in the poset $\mathbb{L}_{\mathfrak{P}}(A)$.

Theorem 3.4.4. *$Mug([A]_{\approx_{\mathfrak{P}}}) = \mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(A))$.*

Proof. Let $B \approx_{\mathfrak{P}} A$ such that $|[B|\mathbb{L}_{\mathfrak{P}}(A) \downarrow]| \geq 2$. Firstly, we claim that if A_1, A_2, A_3 are three distinct non-empty subsets such that $A_1, A_2 \approx_{\mathfrak{P}} A_3$ and $A_1, A_2 \lessdot A_3$, then $A_1 \cup A_2 = A_3$. It clearly results that $A_1 \cup A_2 \subseteq A_3$, suppose therefore that $A_1 \cup A_2 \subsetneqq A_3$. Hence, $A_1 \subsetneqq A_1 \cup A_2 \subsetneqq A_3$, so A_3 does not cover A_1, contradicting our assumption. Thus, if $[B|\mathbb{L}_{\mathfrak{P}}(A) \downarrow] = \{D, E\}$, we have

$$B = D \cup E. \tag{3.15}$$

If D (or E) covers more than one subset, in (3.15) we can substitute D (or E) with the union of two of the subsets it covers; proceeding in this way, since B is finite, we will express B as the union of subsets of $\mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(A))$, so $\mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(A))$ spans $[A]_{\approx_{\mathfrak{P}}}$. In order to prove that $\mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(A))$ is minimum, just observe that each element B of $\mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(A))$ cannot be written as the union of elements of $[A]_{\approx_{\mathfrak{P}}}$ distinct from B. □

In the next example, we discuss the result provided in Theorem 3.4.4 in a concrete situation.

Example 3.4.5. *Let us consider the pairing $\mathfrak{P}$ represented in Table 3.1. We indicate by r,m,c and b respectively RAM, memory, color and battery life and, usually, we use string notation to enumerate the various set systems arising during our analysis.*

It is easy to see that

$$MAXP(\mathfrak{P}) = \{\emptyset, r, b, mc, rb, rmc, rmcb\}.$$

In particular, in Figure 3.3, we represent its maximum partitioner lattice $\mathbb{M}(\mathfrak{P})$, whereas in Figure 3.4 we represent the diagram $\mathbb{I}(\mathfrak{P})$.

By seeing Figure 3.4, we have that

$$MINP(\mathfrak{P}) = \{\emptyset, m, c, r, b, rm, rc, rb, cb, mb\}.$$

Let now $A = \Omega$. Then, it is easy to see that $\mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(\Omega)) = \{\{c, b\}, \{m, b\}, \{r, c, b\}, \{r, m, b\}\}$. As we see, $D := \bigcup_{E \in BAS(\mathfrak{P})} E \neq \Omega$, since $r \notin C$. In particular, $r \in K^c_{\mathfrak{P}}(\Omega)$. By Theorem 3.4.4, it follows that $Mug([\Omega]_{\approx_{\mathfrak{P}}}) = \mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(\Omega))$. As an example, we could consider the subfamily

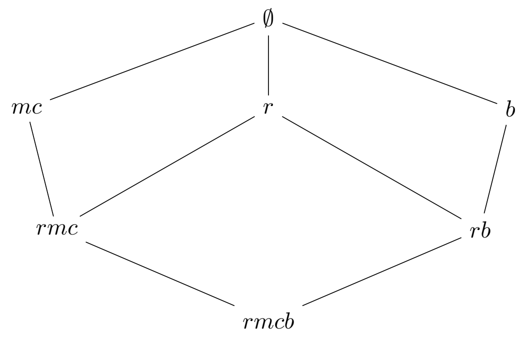


Figure 3.3: Diagram of $\mathbb{M}(\mathfrak{P})$.

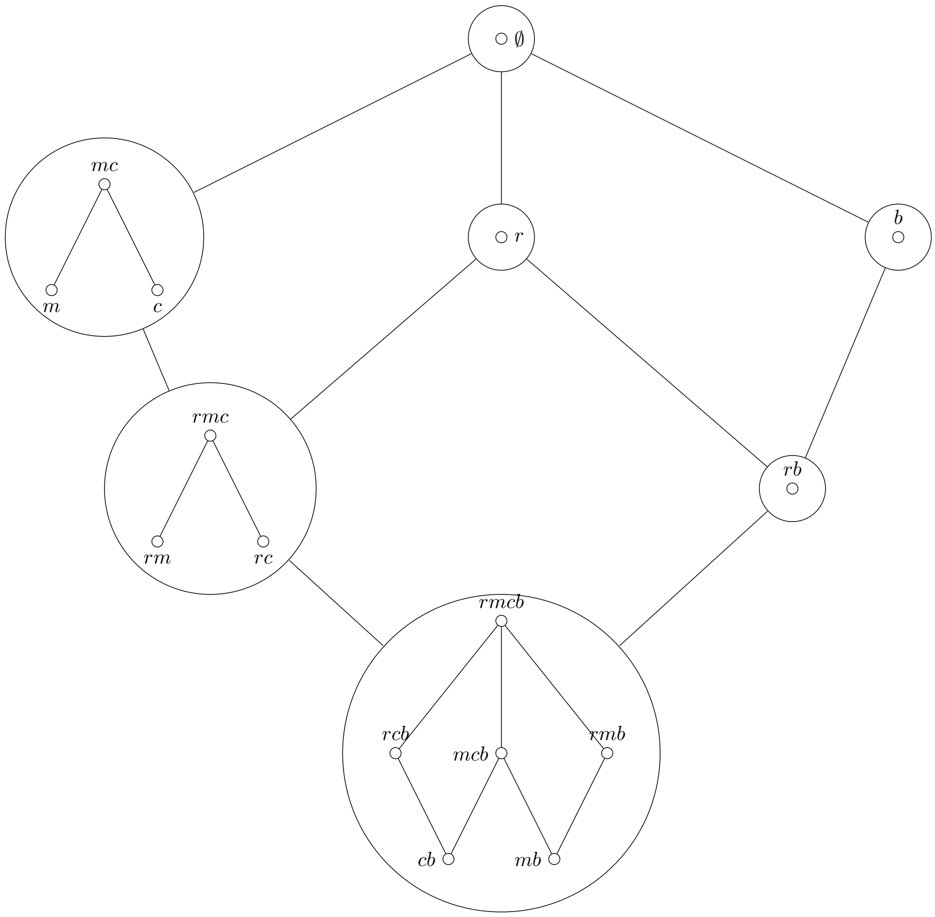


Figure 3.4: Diagram of $\mathbb{I}(\mathfrak{T})$.

$\{\{c,b\},\{m,b\},\{r,c,b\}\} \subsetneqq \mathcal{L}_{\downarrow}(\mathbb{L}_{\mathfrak{P}}(\Omega))$. *Then, it results that* $\pi_{\mathfrak{P}}(\{r,m,b\}) = \pi_{\mathfrak{P}}(\Omega)$, *but there is no way to obtain* $\{r,m,b\}$ *as union of the aforementioned family of subsets. Thus, their union does not span* $[\Omega]_{\approx_{\mathfrak{P}}}$.
Let us observe that $|[\Omega|\mathbb{L}_{\mathfrak{P}}(\Omega) \downarrow|] = 3$. *As we have seen in the proof of Theorem 3.4.4,* Ω *can be obtain as the union of two of its co-cover. In particular, the three co-covers can be obtained by deleting from* Ω *respectively* m *or* r *or* c. *It is obvious that the deletion of* b *is not allowed, otherwise there exists a symmetry base not containing* b, *that is an absurd since* $\{b\} = CORE(\mathfrak{P})$.

The preceding discussion enables us to show the following general result.

Proposition 3.4.6. *If* $|A| = k$, *then* $|[A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]| \leq k$.

Proof. Let $a_i, a_j \in A$ with $i \neq j$. Let us set $A_i = A \setminus \{a_i\}$ and $A_j = A \setminus \{a_j\}$. If $|[A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]| > k$, then there exist two subsets $B, C \subseteq A_i$ for some $i = 1, \ldots, k$. Thus, $B \subsetneqq B \cup C \subsetneqq A$, that is an absurd since $B, C \in [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]$. □

We conclude this subsection by finding find some conditions so that $BAS_{\mathfrak{P}}(A)$ coincides with the minimum union generator of $[A]_{\approx_{\mathfrak{P}}}$, when $A \in MAXP(\mathfrak{P})$.

Theorem 3.4.7. *Let* $A \in MAXP(\mathfrak{P})$ *and* $B, C \in [A]_{\approx_{\mathfrak{P}}}$ *such that*

(i) $B \cap C = \emptyset$,

(ii) $B, C \in [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]$.

Then $BAS_{\mathfrak{P}}(A) = Mug([A]_{\approx_{\mathfrak{P}}}) = [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow] = \{B, C\}$ *and* $[A]_{\approx} = \{A, B, C\}$.

Proof. By hypothesis, we have $\{B, C\} \subseteq [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]$. Suppose by contradiction that $\{B, C\} \subsetneqq [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]$ and let $D \in [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow] \setminus \{B, C\}$. By the proof of Theorem 3.4.4, we have that the union of any pair of elements of $[A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]$ is exactly A, hence we must have $B = A \setminus C \subseteq D$ and $C = A \setminus B \subseteq D$. This implies $B \cup C = A \subseteq D$, i.e. $D = A$, contradicting our assumption. Therefore $[A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow] = \{B, C\}$.
On the other hand, in order to prove that $BAS_{\mathfrak{P}}(A) = \{B, C\}$, it only suffices to prove that $\{B, C\} \subseteq BAS_{\mathfrak{P}}(A)$. Suppose that $B \notin BAS_{\mathfrak{P}}(A)$, thus there exists $D \subsetneqq B$ such that $D \in BAS_{\mathfrak{P}}(A)$. Hence, it results

$$C \subsetneqq C \cup D \subsetneqq A,$$

contradicting the fact that $C \in [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow]$. Similarly if $C \notin BAS_{\mathfrak{P}}(A)$. In this way, we have that $\{B, C\} \subseteq BAS_{\mathfrak{P}}(A)$. The reverse inclusion is now obvious. This also proves that $[A]_{\approx} = \{A, B, C\}$, therefore it follows that $BAS_{\mathfrak{P}}(A) = Mug([A]_{\approx_{\mathfrak{P}}}) = [A|\mathbb{L}_{\mathfrak{P}}(A)\downarrow] = \{B, C\}$ and the claim has been shown. □

To conclude this section, we investigate the geometric properties of the minimal transversal of each global symmetry class $[A]_{\approx_{\mathfrak{P}}}$.

Definition 3.4.8. We say that a subset B of Ω is A-*strongly asymmetric* if $B \subseteq M_{\mathfrak{P}}(A)$ and $C \not\approx_{\mathfrak{P}} M_{\mathfrak{P}}(A) \setminus B$, for any non-empty subset $C \subseteq B$. We denote by $\mathcal{A}_{\mathfrak{P}}(A)$ the family of all A-strongly asymmetric subsets of Ω.

An A-strongly asymmetric subset B of Ω is any subset no part of which provides the same degree of symmetry of its relative complement with respect to $M_{\mathfrak{P}}(A)$. In other terms, we cannot extend $M_{\mathfrak{P}}(A) \setminus B$ by means of any point or part of B without changing the symmetry with respect to $M_{\mathfrak{P}}(A) \setminus B$ itself.

In the next result we show that any non-empty subset B of $M_{\mathfrak{P}}(A)$ which is also a minimal transversal of the set system $[A]_{\approx_{\mathfrak{P}}}$ cannot induce the same symmetry partition of $M_{\mathfrak{P}}(A) \setminus B$.

Proposition 3.4.9. $Tr([A]_{\approx_{\mathfrak{P}}}) \cap \mathcal{P}(M_{\mathfrak{P}}(A)) \subseteq \mathcal{A}_{\mathfrak{P}}(A)$.

Proof. Let $B \in Tr([A]_{\approx_{\mathfrak{P}}}) \cap \mathcal{P}(M_{\mathfrak{P}}(A))$. We now prove that $B \in \mathcal{F}_{\mathfrak{P}}(A)$, where

$$\mathcal{F}_{\mathfrak{P}}(A) := \{B \subseteq M_{\mathfrak{P}}(A) : \ \forall C \in \mathcal{P}(B) \setminus \{\emptyset\}, \forall D \in \mathcal{P}(M_{\mathfrak{P}}(A)) \ \text{ with } M_{\mathfrak{P}}(D) \supseteq M_{\mathfrak{P}}(C), \ B \cap D \neq \emptyset\}.$$

For assume that $B \notin \mathcal{F}_{\mathfrak{P}}(A)$. Then, there exists a proper subset C of B and $D \in \mathcal{P}(M_{\mathfrak{P}}(A))$ such that $M_{\mathfrak{P}}(D) \supseteq M_{\mathfrak{P}}(C)$ and $B \cap D = \emptyset$.
Let us suppose, by contradiction, that there exists $E \in [A]_{\approx_{\mathfrak{P}}}$ such that $E \cap (B \setminus C) = \emptyset$ and set $E' := (E \setminus C) \cup D$. It is clearly disjoint from B. Because of our assumptions on D and E, it readily follows that $M_{\mathfrak{P}}(E') \subseteq M_{\mathfrak{P}}(E)$. On the other hand, let us prove that that $M_{\mathfrak{P}}(E) \subseteq M_{\mathfrak{P}}(E')$. By the condition $M_{\mathfrak{P}}(D) \supseteq M_{\mathfrak{P}}(C)$, we have that $C \subseteq M_{\mathfrak{P}}(D) \subseteq M_{\mathfrak{P}}(E')$ and $E \setminus C \subseteq M_{\mathfrak{P}}(E')$, then $E \subseteq M_{\mathfrak{P}}(E')$, i.e. $M_{\mathfrak{P}}(E) \subseteq M_{\mathfrak{P}}(E')$. In this way, we proved that $E' \approx_{\mathfrak{P}} E \approx_{\mathfrak{P}} A$. But the last claim contradicts the fact that $B \in Tr([A]_{\approx_{\mathfrak{P}}})$. This shows that $B \setminus C \in Tr([A]_{\approx_{\mathfrak{P}}}) \cap \mathcal{P}(M_{\mathfrak{P}}(A))$, contradicting the minimality of B. Hence $B \in \mathcal{F}_{\mathfrak{P}}(A)$.
We now prove that $\mathcal{F}_{\mathfrak{P}}(A) = \mathcal{H}_{\mathfrak{P}}(A)$, where

$$\mathcal{H}_{\mathfrak{P}}(A) := \{F \subseteq M_{\mathfrak{P}}(A) : \ M_{\mathfrak{P}}(F') \nsubseteq M_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus F), \ \forall F' \in \mathcal{P}(F) \setminus \{\emptyset\}\},$$

which is clearly contained in $\mathcal{A}_{\mathfrak{P}}(A)$. This will prove the thesis.
Let $F \in \mathcal{F}_{\mathfrak{P}}(A)$ and $F' \in \mathcal{P}(F) \setminus \{\emptyset\}$. Then, it is straightforward to see that $F' \nsubseteq M_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus F)$, otherwise, by the definition of $\mathcal{F}_{\mathfrak{P}}(A)$ we should have $F' \cap (M_{\mathfrak{P}}(A) \setminus F) \neq \emptyset$, that is an absurd. This implies that $F \in \mathcal{H}_{\mathfrak{P}}(A)$.
On the other hand, let $Z \in \mathcal{H}_{\mathfrak{P}}(A)$ and suppose there exist $Z' \in \mathcal{P}(Z \setminus \{\emptyset\})$ and $S \subseteq M_{\mathfrak{P}}(A)$ such that $Z' \subseteq M_{\mathfrak{P}}(S)$ but $S \cap Z = \emptyset$. Then $S \subseteq M_{\mathfrak{P}}(A) \setminus Z$, so

$$Z' \subseteq M_{\mathfrak{P}}(Z') \subseteq M_{\mathfrak{P}}(S) \subseteq M_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus Z),$$

contradicting our assumption on Z. This shows that $\mathcal{F}_{\mathfrak{P}}(A) = \mathcal{H}_{\mathfrak{P}}(A)$. The thesis is thus proved. $\square$

Chapter 4

A Dissymmetry Model in Graph Theory

For any graph G, it is immediate to verify that the dissymmetry matrix is linked to the neighboorhood of a graph as follows:

$$\forall v_i, v_j \in V(G) \quad \Delta_G(v_i, v_j) := \{a \in V(G) : F(v_i, a) \neq F(v_j, a)\} = N_G(v_i) \bigtriangleup N_G(v_j) \qquad (4.1)$$

where $\Delta_G(v_i, v_j)$ is the dissymmetry map of the pairing associated with G and Δ is the usual set symmetric difference: $A \bigtriangleup B := (A \cup B) \setminus (A \cap B)$.

Hence, the dissymmetry matrix of a graph represents a way to discern between two different vertices $v_i \neq v_j$ in the sense that $v \in \Delta_G(v_i, v_j)$ if and only if v is adjacent to exactly one vertex between v_i and v_j.

Example 4.0.10. *Let C_n be the cycle on n vertices. In this case, the indices are taken modulo n. We have*

$$N_{C_n}(v_i) \bigtriangleup N_{C_n}(v_j) = \begin{cases} \{v_{i-1}, v_{j+1}\} & \text{if } j = i+2 \\ \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\} & \text{if } i \leq j \leq n-1 \wedge j \neq i+2 \end{cases} \qquad (4.2)$$

where $1 \leq i < j \leq n$. Moreover, if we fix two vertices v_i and v_j such that $j \neq i+2$. It results by (4.1) that v_{i-1}, v_{i+1} are adjacent to v_i but not to v_j and, similarly, v_{j-1}, v_{j+1} are adjacent only to v_j. Therefore

$$\Delta_{C_n}(v_i, v_j) = \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\} = N_{C_n}(v_i) \bigtriangleup N_{C_n}(v_j).$$

If $j = i+2$, it is clear that v_{i+1} is adjacent to both v_i and v_j, while v_{i-1} is adjacent only to v_i and v_{j+1} only to v_j. Therefore, in this case,

$$\Delta_{C_n}(v_i, v_j) = \{v_{i-1}, v_{j+1}\} = N_{C_n}(v_i) \bigtriangleup N_{C_n}(v_j).$$

A first immediate result shows that the dissymmetry matrix $\Delta[G]$ uniquely characterizes its corresponding graph when G is a simple graph.

Proposition 4.0.11. *Let G_1 and G_2 be two simple graphs defined on the same set of vertices, i.e., $V(G_1) = V(G_2)$. Then:*

$$\Delta[G_1] = \Delta[G_2] \iff E(G_1) = E(G_2).$$

Proof. Suppose that $\Delta_{G_1}(v_i, v_j) = \Delta_{G_2}(v_i, v_j)$ for every v_i, v_j, then $F_{G_1}(v, w) = F_{G_2}(v, w)$ for every pair of vertices v and w, because in G_1 and G_2 there are no loops. Then, it follows immediately that $E(G_1) = E(G_2)$. $\square$

By Proposition 4.0.11, Δ uniquely determines the edge set of a graph.

In this subsection we deal with the cases of K_n and $K_{p,q}$.

Theorem 4.0.12. *Let G be a n-graph and $V = V(G)$. Then:*
(i) $DIS(G) = \{\{V\}\}$ if and only if $G = K_{p,q}$;
(ii) Let $DIS(G) = \binom{V}{2}$. Then $G = K_n$ if n is odd, while $G = K_n$ or $G = F_n$ if n is even.

Proof. (i): If $G = K_{p,q}$, then it is immediate to show that $DIS(G) = \{V\}$. Conversely, let $DIS(G) = \{V\}$ be the discernibility set system of G and let $v_i, v_j \in V(G)$ such that $\Delta_G(v_i, v_j) = V$. Then, in this case, by setting $B_1 := N_G(v_i)$ and $B_2 := N_G(v_j)$, then $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = \Delta_G(v_i, v_j) = V$. Now let v, w of B_1 (equivalently B_2). They cannot be adjacent, otherwise $\Delta_G(v, w) = V$, contradicting the fact that $v_i \in N_G(v) \cap N_G(w)$ (equivalently $v_j \in N_G(v) \cap N_G(w)$). If $v \in B_1$ and $w \in B_2$, then, since $N_G(v) \neq N_G(w)$, it must result that $\Delta_G(v, w) = V$, hence $v \sim w$. Thus, the claim is proved.
(ii): Firstly, we observe that any pair of vertices $\{v_i, v_j\}$, with $i < j$, occurs just once as $\Delta_G(v'_i, v'_j)$, with $v'_i, v'_j \in V(G)$ and $i' < j'$. If $n = 1$ or $n = 2$, there is nothing to prove. Let G be a n-graph such that $DIS(G) = \binom{V}{2}$. Without loss of generality, suppose moreover that $v_{n-1} \sim v_n$. Thus $\Delta_G(v_{n-1}, v_n) = \{v_{n-1}, v_n\}$. Therefore any other vertex must be adjacent to both v_{n-1} and v_n or non-adjacent to none of them. Let $v_i, v_j \in V(G)$ such that $v_i \sim v_{n-1}$, $v_i \sim v_n$, $v_j \nsim v_{n-1}$ and $v_j \nsim v_n$. In this case, $\Delta_G(v_i, v_j) = \{v_{n-1}, v_n\}$ but this contradicts our preliminary remark. Hence all vertices in $X := \{v_1, \ldots, v_{n-2}\}$ are all adjacent to v_{n-1}, v_n or are all non adjacent to them. Thus, by setting $H := G[X]$, we have that $|V(H)| = n - 2$ and $DIS(H) = \binom{V(H)}{2}$. So, by the inductive hypothesis, $H = K_{n-2}$ or, in the even case, $H = F_{n-2}$ or $H = K_{n-2}$. Let us suppose that $H = K_{n-2}$. If there exists $v_i \in V(G)$ such that $v_i \nsim v_{n-1}$ and $v_i \nsim v_n$, then $\Delta_G(v_i, v_{n-1}) = \{v_n\} \cup (X \setminus \{v_i\})$. Since $|\Delta_G(v_i, v_{n-1})| = 2$, the only possibility is that $n = 4$ and $G = F_n$. On the other hand, if there exists $v_i \in V(H)$ such that $v_i \sim v_{n-1}$ and $v_i \nsim v_n$, then clearly $G = K_n$. Let us suppose that n is even and $H = F_{n-2}$. Let $n \neq 4$ and $v_i, v_j \in V(H)$ such that $v_i \sim v_{n-1}$ and $v_i \nsim v_j$. Hence, $\Delta_G(v_i, v_{n-1})$ contains $\{v_i, v_j, v_{n-1}\}$, contradicting our assumption. Therefore, the only possibility in this case is $n = 4$ and $G = K_n$ or $G = F_n$. $\square$

4.1 Formal Context Analysis on Graphs

In this section we firstly recall some basic link notions between finite simple graphs and formal context as introduced in [48], next we discuss some new results concerning the formal contexts induced by simple graphs.

Definition 4.1.1. [48] We call *formal context of the graph* G the formal context $\mathbb{K}[G] := (V(G), V(G), \mathcal{R}_G)$, where $v\mathcal{R}_G v'$ if and only if $\{v, v'\} \in E(G)$ for all $v, v' \in V(G)$.

Hence the object subset and the attribute subset of the formal context $\mathbb{K}[G]$ are both equal to the vertex set $V(G)$, whereas the binary relation which defines this formal context is exactly the incidence relation between vertices of the graph G. Let us also note that, since the graph G is undirected, the relation $\mathcal{R}_G$ is symmetric.

Given the above considerations, in the formal context $\mathbb{K}[G]$ induced by a simple undirected graph G, the maps ${}^{\uparrow} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ and ${}^{\downarrow} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ are coincident. Therefore in the sequel we denote with the same symbol $'$ the map $' : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ such that $O \mapsto O' := O^{\uparrow} = O^{\downarrow}$, when O is any vertex subset of G. This implies obviously that also the two operators ${}^{*} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ and ${}^{\diamond} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ coincide. Therefore in the sequel we set $O \mapsto O'' := O^{*} = O^{\diamond}$, for all $O \subseteq V(G)$.

If O is a vertex subset of G, it is immediate to verify that

$$O' = \bigcap_{v \in O} N_G(v) = \{w \in V(G) : O \subseteq N_G(w)\}, \tag{4.3}$$

moreover, since G has not loops, we also have

$$O' \subseteq V(G) \setminus O. \tag{4.4}$$

Remark 4.1.2. The identity in (4.3) is also valid when $O = \emptyset$. In fact, in this case, we always have (by convention in FCA) $\emptyset^{\uparrow} = M$ and $\emptyset^{\downarrow} = Z$, that is $O' = O^{\uparrow} = O^{\downarrow} = V(G)$ in the formal context $\mathbb{K}[G]$. On the other hand, it is usual (in elementary set theory) to interpret the intersection $\bigcap_{v \in O} N_G(v)$ as coincident with the whole set $V(G)$ when O is the empty set.

Remark 4.1.3. If G is a finite simple undirected graph, a vertex subset $O \subseteq V(G)$ is the extent [intent] of some concept of the formal context $\mathbb{K}[G]$ if and only if $O'' = O$. In this case, both the pairs (O, O') and (O', O) are concepts of $\mathbb{K}[G]$. Moreover, since G has no loops, the cross table of the formal context $\mathbb{K}[G]$ (that is, the adjacency matrix of G) has zeroes in all its diagonal places, and this obviously implies that $V(G)' = \emptyset$. Hence both the pairs $(\emptyset, V(G))$ and $(V(G), \emptyset)$ are always concepts of the formal context $\mathbb{K}[G]$.

Proposition 4.1.4. *Let $A \subseteq V$. Then A is the extent [intent] of some concept of the formal context $\mathbb{K}[G]$ if and only if $A = O'$, for some $O \subseteq V$.*

By Proposition 4.1.4 and Equation (4.3) it follows that a vertex subset A is an extent [intent] of some concept in $\mathbb{K}[G]$ if and only if A is intersection of open neighborhoods of vertices of G.

We explicitly introduce now the notion of concept lattice for a finite simple undirected graph.

Definition 4.1.5. [48] We call concept lattice (or also *Galois lattice*) of the graph G the concept lattice of the formal context $\mathbb{K}[G]$ and we denote it simply by $(\mathfrak{B}(G), \sqsubseteq)$ instead of $(\mathfrak{B}(\mathbb{K}[G]), \sqsubseteq)$. We also call *Galois poset* of G the poset $(\mathfrak{B}^{\S}(G), \sqsubseteq)$, where $\mathfrak{B}^{\S}(G) := \mathfrak{B}(G) \setminus \{(\emptyset, V(G)), (V(G), \emptyset)\}$.

Remark 4.1.6. (i) Since the adjacency matrix of G is a symmetric matrix, we immediately deduce that the concept lattice $(\mathfrak{B}(G), \sqsubseteq)$ is self-dual.
(ii) From FCA general theory it is well known that is sufficient to represent any formal concept $B = (B_1, B_2)$ of $\mathbb{K}[G]$ simply with its extent B_1 (or, equivalently, with its intent B_2). As a consequence of this premise, in several cases we implicitly identify a formal concepts $B = (B_1, B_2)$ of $\mathbb{K}[G]$ with its extent B_1 (or, equivalently, with its intent B_2). Obviously, in these cases we can also identify the partial order $\sqsubseteq$ with the usual inclusion relation $\subseteq$ (or, respectively, with the dual inclusion $\subseteq^*$).

In the next two definitions we introduce two basic notions.

Definition 4.1.7. Let G, G' be two finite simple graphs.

- We say that G is *Galois equivalent* with G', denoted by $G \equiv_{gal} G'$, if the corresponding Galois lattices $\mathfrak{B}(G)$ and $\mathfrak{B}(G')$ are isomorphic.
- We call the *Galois class* of G the graph family $[G]_{gal} = \{G' : G \equiv_{gal} G'\}$.

Definition 4.1.8. Let s be a positive integer. We call s-*Galois class* of G the following graph family

$$[G, s]_{gal} := \{G' \in [G]_{gal} : |V(G')| = s\}.$$

Moreover, we say that the ordered pair (G, s) is:

- *Galois undetermined* if there exist at least two non-isomorphic locally dissymmetric graphs in $[G, s]_{gal}$;
- *Galois determined* if there is just one locally dissymmetric graph up to isomorphism in $[G, s]_{gal}$;
- *Galois inconsistent* if $[G, q]_{gal}$ is non-empty and there is not any locally dissymmetric graph in $[G, q]_{gal}$.

We also say that the graph G is:

- *Galois undetermined* if the pair $(G, |V(G)|)$ is Galois undetermined;
- *Galois determined* if the pair $(G, |V(G)|)$ is Galois determined;
- *Galois inconsistent* if the pair $(G, |V(G)|)$ is Galois inconsistent;.

Remark 4.1.9. If G is a locally dissymmetric n-graph then it is clear that G is Galois determined if and only if $[G, n]_{gal} = \{G\}$.

In the next propositions of this section we establish some useful properties of the Galois poset of a graph.

Proposition 4.1.10. *Let G, G' be simple graphs. Then:*
(i) $G \equiv_{gal} G'$ if and only if $\mathfrak{B}^{\S}(G) = \mathfrak{B}^{\S}(G')$.
(ii) Let $G_1, \ldots, G_k$ be the connected components of G. Then, the Galois poset of G is the direct sum of the Galois posets of its components, $\mathfrak{B}^{\S}(G) = \mathfrak{B}^{\S}(G_1) + \cdots + \mathfrak{B}^{\S}(G_k)$.
(iii) Let $v_{i_1}, \ldots, v_{i_k}$ be some isolated points in G and let $G'' := G[V(G) \setminus \{v_{i_1}, \ldots, v_{i_k}\}]$. Then $\mathfrak{B}^{\S}(G) = \mathfrak{B}^{\S}(G'')$.

Proof. (i) This is trivial and follows directly by the definition of the Galois poset.
(ii) Let $O \subseteq V(G)$ be a vertex subset. Then $O' \neq \emptyset$ if and only if $O' \subseteq V(G_i)$ for some $i \in \{1, \ldots, k\}$. More in general a vertex subset A of G, such that $A \neq V(G)$ and $A \neq \emptyset$, is an extent [intent] in G if and only if it is an extent [intent] in a connected component of G. The thesis thus holds.
(iii) This follows directly by part (ii). $\square$

Remark 4.1.11. (i) By Proposition 4.1.10, part (i), we can use the Galois poset of G instead of its Galois lattice in order to study the Galois class of G.
(ii) The notion of local dissymmetry is important in our context because each Galois class is uniquely determined by a locally dissymmetric graph. In fact, if G be a finite simple graph, then $G \equiv_{gal} G^{\simeq}$.
(iii) By (ii) of Proposition 4.1.10 we can restrict our analysis to graphs without isolated vertices.

Proposition 4.1.12. *Let G be a n-graph, $O \subseteq V(G)$ be an extent and $v_i \in O$. Then O contains all the vertices v_j such that $v_i \simeq v_j$.*

Proof. Let $v_j \in V(G)$ twin with v_i and $v_i \in O$ for some extent $O \subseteq V(G)$. By Proposition 4.1.4, there exists a vertex subset $A \subseteq V(G)$ such that $O = A' = \bigcap_{v \in A} N_G(v)$. Hence $v_i \in \bigcap_{v \in A} N_G(v)$, so $v_i \sim v$ for each $v \in A$. Since $N_G(v_i) = N_G(v_j)$, we deduce that $v_j \sim v$ for any $v \in A$, thus $v_j \in \bigcap_{v \in A} N_G(v)$, i.e. $v_j \in O$. $\square$

Proposition 4.1.13. *Let G be a graph. Then, if O is a vertex subset of G such that O' is minimal in $\mathfrak{B}^{\S}(G)$, then $|O'| = 1$. The maximal extents in $\mathfrak{B}^{\S}(G)$ are the neighborhoods of the elements which appear in the bottom part.*

Proof. Without loss in generality we can suppose $O = O''$. Let us suppose that $|O'| \geq 2$. Since there are not twin vertices in G, there exists a vertex $w \in O'$ such that $N_G(w)$ contains strictly O. Let $u \in N_G(w) \setminus O$ and set $U = O \cup \{u\}$. It follows that $w \in U'$, so U' is non-empty and contained strictly in O'. This contradicts the assumption that O' is minimal in $\mathfrak{B}^{\S}(G)$. Thus $|O'| = 1$. By duality, if $O' = \{v\}$, then $O'' = N_G(v)$ is maximal in $\mathfrak{B}^{\S}(G)$. This completes the proof. $\square$

In the last result of this scetion we show how the connectedness of the graph is related to the connectedness of the cover graph of the corresponding concept lattice.

Proposition 4.1.14. *Let G be a simple graph. If the cover graph of $\mathfrak{B}^{\S}(G)$ is connected, then G is a connected graph.*

Proof. Suppose that the graph G is not connected. Since G has no isolated vertices, every component has at least an edge. This means that we can find two different proper neighborhoods, which are extent in the concept lattice. Hence, $\mathfrak{B}^{\S}(G)$ can be expressed as the disjoint sum of lattices, i.e. its cover graph is not connected. $\square$

We will note in the next section that the reverse implication of that established in Proposition 4.1.14 is not true (see the next Remark 4.1.23).

4.1.1 Galois classes of K_n and $K_{p,q}$

In this section we describe the Galois classes of the complete graph and of the bipartite complete graph. The complete determination of the Galois class of a graph G is in general a difficult task, even in the case when the structure of G is relatively simple.

In the next result we provide the equivalent conditions that characterize a n-graph G that has the same Galois lattice of K_n.

Proposition 4.1.15. *Let G be a n-graph and let $V = V(G)$. Then, the following conditions are equivalent:*
(i) for any vertex subset $X \subseteq V$ we have $X' = V \setminus X$;
(ii) $G = K_n$;
(iii) $\mathfrak{B}(G) = \{(X, V \setminus X) : X \subseteq V\}$;
(iv) $(\mathfrak{B}(G), \sqsubseteq) \cong (\mathcal{P}(V), \subseteq)$.

Proof. $(i) \implies (ii)$: Let $v \in V$ and let $X = \{v\}$. By (i) we have that $N_G(v) = \{v\}' = V \setminus \{v\}$, therefore $G = K_n$.
$(ii) \implies (iii)$: Let $X \subseteq V$. Since $G = K_n$, by definition of X' we have that $X' = V \setminus X$, therefore $X'' = (X')' = V \setminus (V \setminus X) = X$. This implies that the ordered pair $(X, V \setminus X)$ is a formal concept of G. Hence (iii) is satisfied.
$(iii) \implies (iv)$: It follows directly by the definition of the partial order in the concept lattice.
$(iv) \implies (i)$: Let us suppose there exists $X \subseteq V$ such that $X' \neq V \setminus X$. Since X' is disjoint from X, we have that $X' \subsetneq X^c$ and thus that the cardinality of $X \cup X'$ is strictly less than n. By (iv) and by the duality property of the concept lattice associated with a graph, we have that each subset Y of V occurs exactly once as extent and once as intent in suitable formal concepts of G. Thus if we sum the cardinalities of all the extents and all the intents in $\mathfrak{B}(G)$, then we obtain:

$$\sum_{Y \in \mathcal{P}(V)} |Y \cup Y'| = 2 \sum_{Y \in \mathcal{P}(V)} |Y| = 2n2^{n-1} = n2^n.$$

Since the number of addends in the first sum is equal to 2^n and one of addends, $|X \cup X'| = |X| + |X'|$, is strictly less than n, then there exists $Y \subseteq V$ such that $|Y \cup Y'| > n$, but this contradicts the fact that Y and Y' are disjoint. This proves that $X' = V \setminus X$. $\square$

In the next result we provide the conditions under which a local dissymmetry graph G is Galois equivalent to K_n.

Proposition 4.1.16. *Let G be a locally dissymmetric graph. Then, $G \equiv_{gal} K_n$ if and only if there exists a vertex subset $U = \{u_1, u_2, \ldots, u_n\} \subseteq V(G)$ such that the following conditions hold:*
(i) $G[U] \cong K_n$;
(ii) for each $v \in V(G) \setminus U$, there exist $u_i \in U$ such that $N_G(v) \subseteq N_G(u_i)$;
(iii) if $v, w \in V(G) \setminus U$ are in $V(G)$, then $v \sim w$ if and only if $U \subseteq N_G(v) \cup N_G(w)$.

Proof. By Proposition 4.1.13, since $(\mathfrak{B}(G), \sqsubseteq) \cong (\mathcal{P}(V), \subseteq)$, there exist $u_1, \ldots, u_n \in V(G)$ such that the minimal elements in $\mathfrak{B}^{\S}(G)$ are the singletons $\{u_i\}$ and the maximal elements are $N_G(u_i)$, for $i = 1, 2, \ldots, n$. Then, for each $A \subseteq U := \{u_1, \ldots, u_n\}$, there exist just one extent B_A in $\mathfrak{B}(G)$ such that $A \subseteq B_A$. Let now $v \in V \setminus U$ and let us consider $N_G(v) = \{v\}'$. By maximality of the extents $N_G(u_i)$ in $\mathfrak{B}(G)$, there exists a vertex $u_i \in U$ such that $N_G(v) \subseteq N_G(u_i)$. So (ii) holds.
Let now $v, w \in V(G) \setminus U$ such that $U \subseteq N_G(v) \cup N_G(w)$. In this case $(U \cap N_G(v)) \cup (U \cap N_G(w)) = U$. By setting $A := N_G(v) \cap U$, then $B_A = N_G(v)$. On the other hand, $((U \cap N_G(w)) \setminus N_G(v))'$ is another extent such that its intersection with U is equal to A. This implies that $((U \cap N_G(w)) \setminus N_G(v))' = B_A = N_G(v)$ and thus $v \sim w$. Conversely let $v, w \in V(G) \setminus U$, with $v \sim w$. As an immediate consequence, we deduce that $\{w\}' = N_G(w)$ is an extent in $\mathfrak{B}(G)$ whose intersection with U is $A := (N_G(w) \cap U)$. Thus it coincides with $(U \setminus N_G(w))'$ and if $u \in U \setminus N_G(w)$, then, since $v \in \{w\}' = (U \setminus N_G(w))'$, we obtain $u \sim v$. It follows (iii).
Let us suppose now that $V(G)$ satisfies the condition (iii). Then, for each $A \subseteq U$, we set $B_A := (U \setminus A)'$. Let us prove now that the B_A are all the possible extents in $\mathfrak{B}(G)$. For this, let O be a vertex subset in G. Let us prove that $O' = (U \setminus O')'$. For this, let $v \in O'$ and $u \in U \setminus O'$. If $v \in O' \cap U$, then $u \neq v$ and so $u \sim v$ because they are different vertices in U and $G[U] \cong K_n$. Let $v \in O' \setminus U$. In this case, if $u \in O$, then $u \sim v$ because $v \in O'$. If $u \notin O$, since $u \notin O'$, then there exist $v' \in O \setminus U$ such that $u \nsim v'$. But since $v' \sim v$, by property (iii), we obtain that $u \sim v$, so $O' \subseteq (U \setminus O')'$. Let now $u \in (U \setminus O')'$. If $u \in U$, then $u \in O'$, so $u \sim v$, for all $v \in O$. Let us suppose thus $u \notin U$ and let $v \in O$. If $v \in U \cap O$, then $v \in U \setminus O'$, so $v \sim u$. If $u' \in U$ and $u' \nsim v$, then $u' \in U \setminus O'$, so $u \sim u'$. This proves that $U \subseteq N_G(v) \cup N_G(w)$ and thus, by property (iii), $u \sim v$. □

At this point we provide our results concerning the complete graph K_n in the following result.

Theorem 4.1.17. *(i) The complete graph K_n is Galois determined.*
(ii) $[K_n]_{gal}$ is the family of all the finite simple graph G such that $G^{\simeq}$ satisfies the conditions (i), (ii) and (iii) of Proposition 4.1.16.
(iii) If $s > n$ then the pair (K_n, s) is Galois undetermined or Galois inconsistent.

Proof. Part (i) follows by Proposition 4.1.15, whereas the parts (ii) and (iii) follow by Proposition 4.1.16. □

We devote now our attention to the complete bipartite graphs. We will prove that all these graphs are mutually equivalent to one another and, moreover, that a graph G is Galois equivalent to a complete bipartite graph if and only if it is a complete bipartite graph by itself.

Proposition 4.1.18. *The following conditions for a graph G with vertex set V are equivalent:*
(i) G is the complete bipartite graph having bipartition (B_1, B_2).
(ii) $\mathfrak{B}(G) = \{(\emptyset, V), (B_1, B_2), (B_2, B_1), (V, \emptyset)\}$, where $B_1|B_2$ is a set partition of V.
(iii) $(\mathfrak{B}(G), \sqsubseteq) \cong (\mathcal{P}(\hat{2}), \subseteq)$.

Proof. $(i) \Longrightarrow (ii)$: Let $O \subseteq V$ a non-empty vertex subset. Then, clearly:

$$O' = \begin{cases} B_2 & \text{if } O \subseteq B_1, \\ B_1 & \text{if } O \subseteq B_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

So the only contexts (extents) are $\emptyset$, B_1, B_2, V and the thesis follows.
$(ii) \Longrightarrow (iii)$: This is obvious.
$(iii) \Longrightarrow (i)$: Let (B_1, B_2) and (B_2, B_1) be the two concepts in $\mathfrak{B}(G)$ where both B_1 and B_2 are different from the empty set. Then, by definition of the operator $'$, all vertices in B_1 are adjacent to all vertices in B_2, so that $\{x, y\} \in E(G)$, for all $x \in B_1$ and $y \in B_2$. Let us suppose now there exist x_1, $x_2 \in B_1$ such that $\{x_1, x_2\} \in E(G)$. Set $O := N_G(x_i) \supseteq \{x_2\} \cup B_2$. In this case $x_1 \in O'$ and $x_2 \notin O'$, so $O' \neq B_1$, $O' \neq B_2$, $O' \neq \emptyset$ and $O' \neq V$. As observed in Remark 2.3, this contradicts the fact that the only extents in $\mathfrak{B}(G)$ are $\emptyset$, B_1, B_2 and V. The proposition is thus proved. □

For the complete bipartite graphs we obtain then the following result.

Corollary 4.1.19. *(i) The Galois class of any complete bipartite graph is is the family of all the complete bipartite graphs.*
(ii) If $s > 2$ then the pair $(K_{p,q}, s)$ is Galois inconsistent.

Proof. It follows directly by Proposition 4.1.18. □

4.1.2 Galois Posets of Cycles, Paths and Wheels

In this section we determine the Galois lattices of three well studied graph families: the n-cycles C_n, the n-paths P_n and the n-wheels W_{n+1}. The Galois posets of these graph families are graded of rank 1 (for C_n and P_n) or of rank 2 (for W_{n+1}).
In the next result we show that the only n-graph in the Galois class of C_n is just C_n.

Theorem 4.1.20. *Let G be a graph with $|V(G)| = n \geq 5$. Then, $G = C_n$ if and only if the Galois poset $(\mathfrak{B}^{\S}(G), \subseteq)$ is a n-crown when n is odd and is a disjoint sum of two $\frac{n}{2}$-crowns when n is even.*

Proof. Let $G = C_n$ and let $v_i \in V(G)$. Then, $N_G(v_i) = \{v_{i-1}, v_{i+1}\}$, where the index sums are taken mod(n). Now let i, $j \in \{1, \ldots, n\}$ such that $i < j$. Then,

$$\{v_i, v_j\}' = N_G(v_i) \cap N_G(v_j) = \begin{cases} \{v_1\} & \text{if } i = 2 \text{ and } j = n \\ \{v_n\} & \text{if } i = 1 \text{ and } j = n-1 \\ \{v_{i+1}\} & \text{if } j = i+2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Finally, note that if $O \subseteq V$ has cardinality greater than or equal to 3, then $O' = \emptyset$.
Therefore, we conclude that $\mathfrak{B}^{\S}(C_n) = \{\{v_i\}, \{v_{i-1}, v_{i+1}\} : i = 1, \dots, n\}$, where the index sums are taken $\mathrm{mod}(n)$. Now, let n be an odd integer. We set

$$x_k := \{v_{2k-1}\} \quad \text{for} \quad k = 1, \dots, n$$

and

$$y_k := \{v_{2k-1}, v_{2k+1}\} \quad \text{for} \quad k = 1, \dots, n$$

where all index sums are taken $\mathrm{mod}(n)$. It's immediate to see that, in this way, $\mathfrak{B}^{\S}(C_n)$ is an n-crown with respect to set theoretic inclusion.
Let n be an even integer. We set

$$x_k := \{v_{2k-1}\} \quad \text{for} \quad k = 1, \dots, \frac{n}{2}$$

$$y_k := \{v_{2k-1}, v_{2k+1}\} \quad \text{for} \quad k = 1, \dots, \frac{n}{2}$$

and

$$x_{k+\frac{n}{2}} := \{v_{2k}\} \quad \text{for} \quad k = 1, \dots, \frac{n}{2}$$

$$y_{k+\frac{n}{2}} := \{v_{2k}, v_{2k+2}\} \quad \text{for} \quad k = 1, \dots, \frac{n}{2}$$

where all index sums are taken $\mathrm{mod}(n)$. In this way, it's obvious that $\mathfrak{B}^{\S}(C_n)$ is the disjoint union of two $\frac{n}{2}$-crowns.
Vice versa, let $(\mathfrak{B}^{\S}(G), \sqsubseteq) = \{x_1, y_1, \dots, x_n, y_n\}$ with $x_1 < y_1 > x_2 < y_2 > x_3 < \cdots < y_{n-1} > x_n < y_n > x_1$. At first, we observe that, by minimality, the x_i are n pairwise disjoint subsets. Hence, we conclude necessarily that $x_i = \{v_i\}$. Then, it is also clear that $|y_i| = 2$ for every $i = 1, \dots, n$ and, by the maximality of y_i, we conclude that for each $i = 1, \dots, n$, there exists some $v_k \in V(G)$ such that $y_i = N_G(v_k)$. This means that the graph G is 2-regular and connected, i.e. $G = C_n$. In particular, n must be necessarily an odd integer.
In a similar way, let $(\mathfrak{B}^{\S}(G), \subseteq)$ be the disjoint sum of two $\frac{n}{2}$-crowns. It is obvious that the minimal elements of the first crown are disjoint from those of the second one and hence the maximal elements of the first crown are also disjoint from those of the second one. Arguing as before, it is easy to deduce that, even in this case, $G = C_n$. $\square$

Corollary 4.1.21. *(i) The n-cycle C_n is Galois determined, that is equivalent to the condition $[C_n, n]_{gal} = \{C_n\}$ because C_n is locally dissymmetric.*
(ii) Let G be a simple finite graph. Then, $G \equiv_{gal} C_n$ if and only if $G^{\simeq} = C_n$.

Proof. Both assertions follow directly by Theorem 4.1.20 and by (ii) of Remark 4.1.11. $\square$

Example 4.1.22. *In the figure we represent the concept lattice of C_5.*

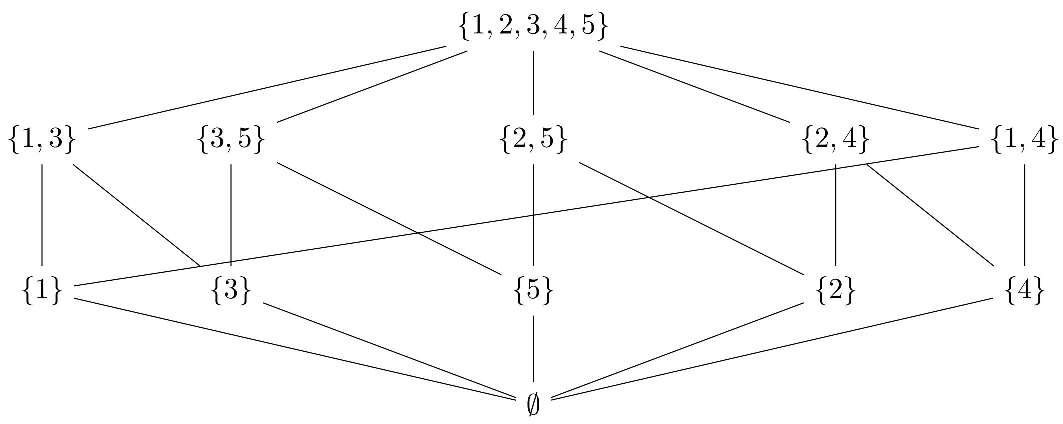


Figure 4.1: Concept Lattice of C_5

In the next figure we also represent the concept lattice of C_6.

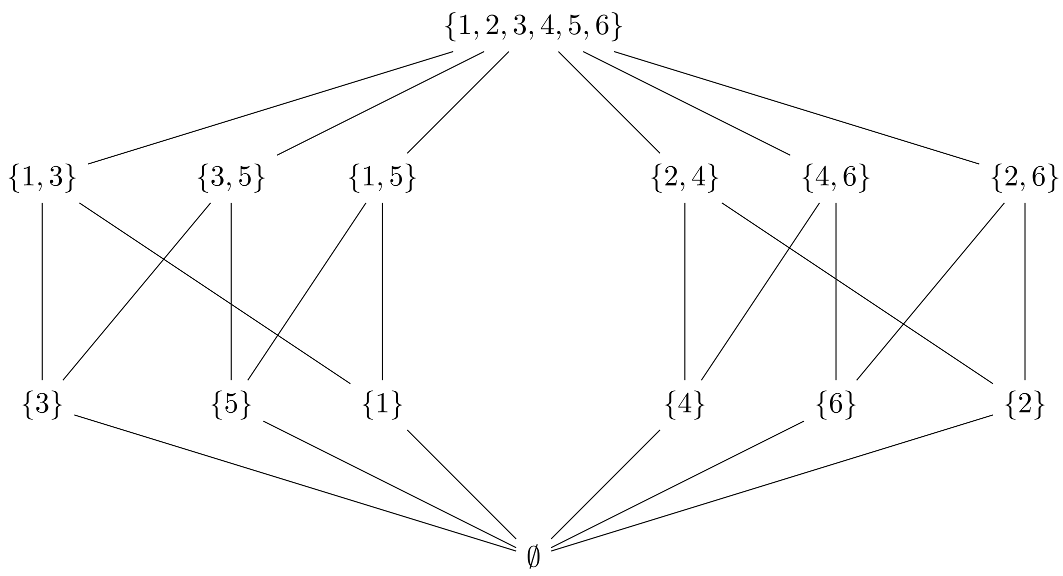


Figure 4.2: Concept Lattice of C_6

Remark 4.1.23. Let us note that the graph C_n, where n is even, is connected but the cover graph of its Galois poset is not connected.

We examine now the case $G = P_n$. For this graph we can establish the following result.

Theorem 4.1.24. *Let G be a graph with $|V(G)| = n \geq 5$. Then*

$$G = P_n \iff (\mathfrak{B}^{\S}(G), \subseteq) = F_{n-2} \cup^d F^*_{n-2}.$$

Proof. Let $G = P_n$ and let $v_i \in V(G)$. Then

$$N_G(v_i) = \{v_i\}' = \begin{cases} \{v_2\} & \text{if } i = 1 \\ \{v_{n-1}\} & \text{if } i = n \\ \{v_{i-1}, v_{i+1}\} & \text{otherwise.} \end{cases} \tag{4.5}$$

Now let $v_i, v_j \in V(G)$ such that $i < j$. Then

$$N_G(v_i) \cap N_G(v_j) = \{v_i,\, v_j\}' = \begin{cases} \{v_{i+1}\} & \text{if } j = i+2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover if $O \subseteq V(G)$ is such that $|O| \geq 3$, then $O' = \emptyset$. We conclude that $\mathfrak{B}^{\S}(P_n) = \{\{v_i\},\, \{v_j, v_{j+2}\} :\, i = 2, \ldots, n-1, j = 1, \ldots, n-2\}$.
Let n be an even integer. If we set, for $k = 1, \ldots, \frac{n-2}{2}$,

$$\{v_{2k}\} = x_{2k-1} \text{ and } \{v_{2k}, v_{2k+2}\} = x_{2k}$$

and

$$\{v_{2k-1}\} = x'_{2k-1} \text{ and } \{v_{2k-1}, v_{2k+1}\} = x'_{2k}$$

it's immediate to note that $\mathfrak{B}^{\S}(P_n)$ is the disjoint union of two $(n-2)$-fences. Similarly, let n be an odd integer. If we set

$$\{v_{2k}\} = x_{2k-1} \text{ for } k = 1, \ldots, \frac{n-1}{2}$$

$$\{v_{2k}, v_{2k+2}\} = x_{2k} \text{ for } k = 1, \ldots, \frac{n-3}{2}$$

and

$$\{v_{2k-1}, v_{2k+1}\} = x'_{2k-1} \text{ for } k = 1, \ldots, \frac{n-1}{2}$$

$$\{v_{2k+1}\} = x'_{2k} \text{ for } k = 1, \ldots, \frac{n-3}{2}$$

and we obtain an $(n-2)$-fence F_{n-2} and the dual $(n-2)$-fence F^*_{n-2}. Hence, the Galois poset $\mathfrak{B}^{\S}(P_n)$ is the disjoint union of two $(n-2)$-fences.
Vice versa, let $(\mathfrak{B}^{\S}(G), \subseteq) = F_{n-2} \cup^d F^*_{n-2}$. We observe that the minimal elements of a fence are clearly pairwise disjoint; moreover it is obvious the minimal elements of the first fence are disjoint from those of the second one. Hence the minimal elements are singletons and the maximal elements are pairs of vertices; in particular they are neighbourhoods of some vertex. If n is an even integer, then every fence has exactly $\frac{n-2}{2}$ minimal as many maximal elements, while if n is an odd integer, in the first fence there are $\frac{n-1}{2}$ minimal elements and $\frac{n-3}{2}$ maximal elements and in the second there are $\frac{n-3}{2}$ minimal elements and $\frac{n-1}{2}$ maximal elements. It is immediate to see that the maximal elements of a fence are the neighbourhoods of the minimal elements of the other fence; so, since G is a connected graph, we conclude that the $n-2$ vertices at issue form a $(n-2)$-path. Moreover, the remaining vertices can be linked only to the ends of the previous path and there are no edges between them, otherwise we obtain a n-cycle but this is impossible by Theorem 4.1.20. Thus, we deduce that $G = P_n$. $\square$

Corollary 4.1.25. *(i) The n-path P_n is Galois determined, that is equivalent to the condition $[P_n, n]_{gal} = \{P_n\}$ because P_n is locally dissymmetric.*
(ii) Let G be a simple finite graph. Then, $G \equiv_{gal} P_n$ if and only if $G^{\simeq} = P_n$.

Proof. Both assertions follow directly by Theorem 4.1.24 and by (ii) of Remark 4.1.11. $\square$

Example 4.1.26. *We represent the concept lattice of P_6 in the figure below.*

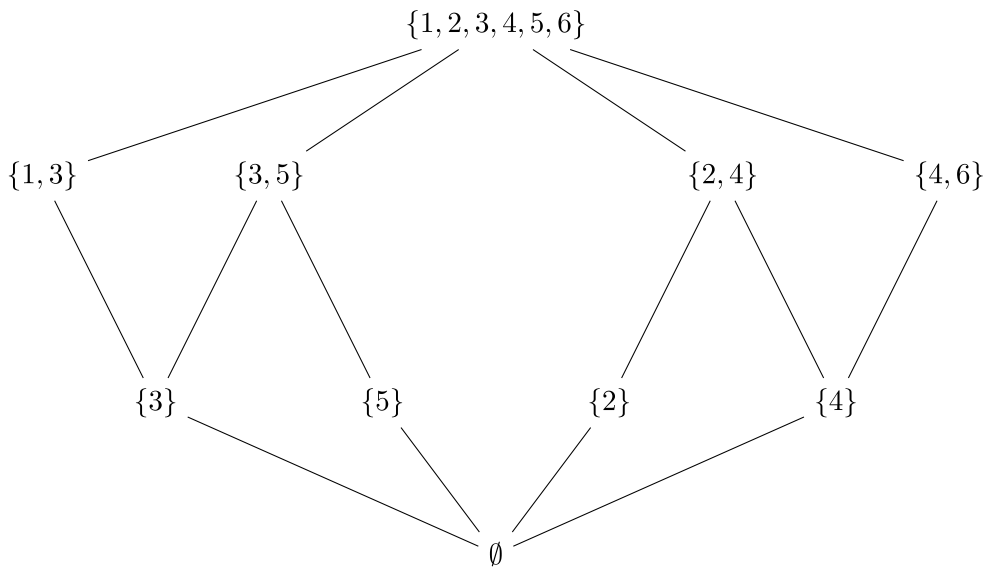


Figure 4.3: Concept Lattice of P_6

In the following theorem, we will prove that there not exists any graph whose concept lattice is isomorphic to a n-fence.

Theorem 4.1.27. *Let F_n be a n-fence. There exists no graph having F_n as Galois poset.*

Proof. Let us assume, by absurd, that there exists a graph G such that his Galois poset is isomorphic to F_n. Without loss in generality we can assume G locally dissymmetric. By Proposition 4.1.13 the minimal extents of G are singletons and the maximal elements in $\mathfrak{B}^{\S}(G)$ are the neighborhoods of the vertices that occur in the minimal part. Thus the number of minimal and maximal elements is the same.
Let F_n be the n-fence such that $x_1 < y_1 > x_2 < \cdots < x_n$ and let $v_1, v_2, \ldots, v_n \in V$ such that, for each i, x_i corresponds to the singleton $\{v_i\}$ and y_i corresponds to $N_G(v_{\sigma(i)})$, where σ is a permutation of the index set $\{1, 2, \ldots, n\}$. Since $v_i, v_{i+1} \in N_G(v_{\sigma(i)})$, where $i = 1, \ldots, n-1$, and $v_n \in N_G(v_{\sigma(n)})$, the induced subgraph on G by the vertex subset $\{v_1, v_2, \ldots, v_n\}$ has one vertex of degree 1 and all the others of degree 2. But there exists no such graph. This concludes the proof. $\square$

We close this section by determining the Galois lattice of an n-wheel.

Proposition 4.1.28. *Let $n \geq 5$ and W_{n+1} be the n-wheel. Then:*

$$(\mathfrak{B}^{\S}(W_{n+1}), \subseteq) = \{\{v_i\}, \{v_j, v_{j+2}\}, \{v_j, v_{n+1}\}, N_{W_{n+1}}(v_i)\},$$

for $i = 1, \ldots, n+1$, $j = 1, \ldots, n$ and the index sums $j+2$ are taken mod(n).

Proof. Let $G = W_{n+1}$ and $v_i \in V(G)$. Then

$$N_G(v_i) = \{v_i\}' = \begin{cases} \{v_2,\, v_n,\, v_{n+1}\} & \text{if } i = 1 \\ \{v_1,\, v_{n-1},\, v_{n+1}\} & \text{if } i = n \\ \{v_1,\, \ldots,\, v_n\} & \text{if } i = n+1 \\ \{v_{i-1}, v_{i+1}, v_{n+1}\} & \text{otherwise.} \end{cases}$$

It is easy to see that, if $O \subseteq V(G)$ have cardinality $|O| \geq 2$, then $|O'| \leq 2$ and $|O'| = 2$ if and only if $O' = \{v_j, v_{j+2}\}$ or $O' = \{v_j, v_{n+1}\}$ for some $j = 1, \ldots, n$. Moreover, if $i = 1, \ldots, n+1$, then:

$$\{v_i\} = \begin{cases} \{v_2,\, v_n,\, v_{n+1}\}' & \text{if } i = 1 \\ \{v_{i-1},\, v_{i+1},\, v_{n+1}\}' & \text{if } i \in \{2,\, \ldots,\, n-1\} \\ \{v_1,\, v_{n-1},\, v_{n+1}\}' & \text{if } i = n \\ \{v_1,\, v_2\}' & \text{if } i = n+1 \end{cases}$$

The Proposition is thus proved. □

Corollary 4.1.29. *The Galois poset of W_{n+1} is graded of rank 2.*

Example 4.1.30. *In the figure we represent the concept lattice of W_7.*

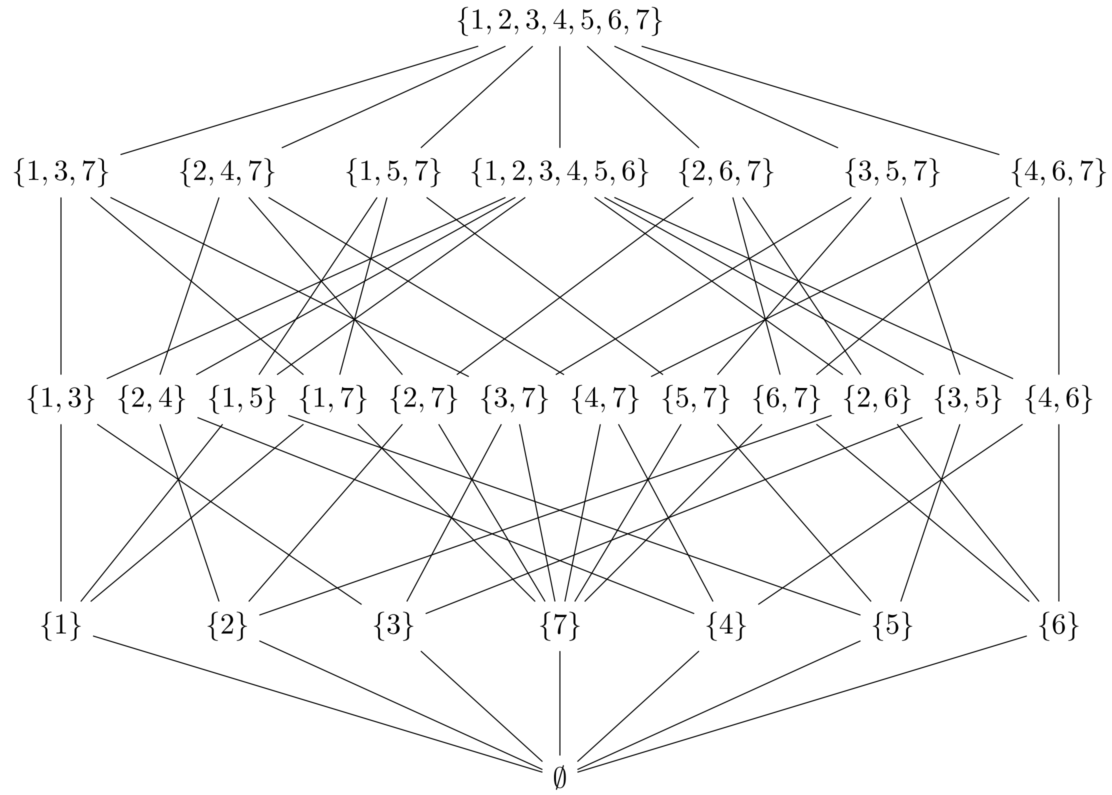


Figure 4.4: Concept Lattice of W_7

4.1.3 The Galois Lattice of Products of Cycles and Paths

By using the results of the previous section, we now investigate the the concept lattice of graphs obtained by composing cycles and paths by means of two classical graph operations: the Cartesian product and the tensor product.

We firstly establish a preliminary result.

Proposition 4.1.31. *Let G and H be two graphs. The Galois poset of the tensor product $G \otimes H$ is the direct product of the two Galois posets $\mathfrak{B}^{\S}(G)$ and $\mathfrak{B}^{\S}(H)$.*

Proof. Let $(u, u') \in V(G \otimes H)$. Then, $N_{G\otimes H}((u, u') = (u, u')' = N_G(u) \times N_H(u')$. By Equation (4.3) and Proposition 4.1.4 it follows easily that $A \subseteq V(G \otimes H)$ is an extent (intent) if and only if there exist $O \in V(G)$, $Q \in V(H)$ such that $A = O' \times Q'$. The thesis follows by definition of the product order in $\mathfrak{B}^{\S}(G \otimes H)$.

□

In the next result, we determine the extent family of the (m, n)-prism and we prove that its Galois poset is graded of rank 2.

Theorem 4.1.32. *Let $m \geq 2$, $n \geq 3$ and G be the (m,n)-prism $P_m \square C_n$. Then*

$$\mathfrak{B}^{\S}(G) = \{\{(i,j')\}, \{(k,r'),(k+1,r'-1)\}, \{(k,r'),(k+1,r'+1)\}, N_G((i,j'))\}, \tag{4.6}$$

where $i = 1,\ldots,m$, $j' = 1,\ldots,n$, $k = 1,\ldots,m-1$ and $r' = 1,\ldots,n$.
Moreover, the Galois poset $\mathfrak{B}^{\S}(G)$ is graded of rank 2.

Proof. All the index sums in this proof are taken mod(n). We have that $N_G((1,j')) = \{(1,j'-1),(1,j'+1),(2,j')\}$ for $j' = 1,\ldots,n$, $N_G((i,j')) = \{(i,j'-1),(i,j'+1),(i-1,j'),(i+1,j')\}$ for any $j' = 1,\ldots,n$ and $i = 2,\ldots,m-1$ and $N_G((m,j')) = \{(m,j'-1),(m,j'+1),(m-1,j')\}$ for $j' = 1,\ldots,n$. In particular, to distinct pairs correspond distinct neighborhoods. We obviously have $O'' = O$ if $O = \{(i,j')\}$ or $O = N_G((i,j'))$. Let $O \subseteq V(G)$ of the form $O = \{(i,r'),(i,s')\}$. Since these vertices belong to a C_n, we deduce that $O' = \emptyset$ unless $d(r',s') \equiv 2 \pmod n$. In this case, $O' = \{(i,r'-1)\}$ or $O' = \{(i,r'+1)\}$, depending on whether $s' = r'-2$ or $s' = r'+2$, but in both situations we have $O'' \neq O$. Let $O \subseteq V(G)$ of the form $O = \{(i,r'),(i+1,s')\}$ for some $r',s' \in \{1',\ldots,n'\}$ and $i \in \{1,\ldots,m-1\}$. It is easy to see geometrically the different cases occurring: if these two vertices belong to the same lateral face of a prism and are opposite, then O' consists of the other two vertices of the face; if these two vertices belong to the same lateral face of a prism and are adjacent, then $O' = \emptyset$ as well as if they belong to the different lateral faces of a prism. Hence $O'' = O$ if $O = \{(i,r'),(i+1,r'-1)\}$ or $O = \{(i,r'),(i+1,r'+1)\}$. If $O \subseteq V(G)$ is a vertex subset of cardinality 3, we have to consider two different cases; in the first, there exist a vertex (i,j') such that $O \subseteq N_G((i,j'))$, so $O'' = O$ if and only if $i = 1$ or $i = m$, i.e. if and only if $O = N_G((1,j'))$ or $O = N_G((m,j'))$, while in the second case, it is easy to show that $O' = \emptyset$. All other possible vertex subsets don't satisfy the condition $O'' = O$, since there are at least two adjacent vertices or two vertices belonging to different faces of a prism or there are at least three vertices which don't belong to the same neighborhood and this conclude the proof of (4.6).
Now, by (4.6) it is easy to deduce that any maximal chain in $B^{\S}(G)$ has one of the following forms:

- $\{(1,j')\} \subsetneqq \{(1,j'),(2,j'-1)\} \subsetneqq N_G((1,j'-1))$;
- $\{(1,j')\} \subsetneqq \{(1,j'),(2,j'-1)\} \subsetneqq N_G((2,j'))$;
- $\{(1,j')\} \subsetneqq \{(1,j'),(2,j'+1)\} \subsetneqq N_G((1,j'+1))$;
- $\{(1,j')\} \subsetneqq \{(1,j'),(2,j'+1)\} \subsetneqq N_G((2,j'))$;
- $\{(i,j')\} \subsetneqq \{(i,j'),(i-1,j'-1)\} \subsetneqq N_G((i-1,j'))$;
- $\{(i,j')\} \subsetneqq \{(i,j'),(i-1,j'-1)\} \subsetneqq N_G((i,j'+1))$;
- $\{(i,j')\} \subsetneqq \{(i,j'),(i-1,j'+1)\} \subsetneqq N_G((i-1,j'))$;
- $\{(i,j')\} \subsetneqq \{(i,j'),(i-1,j'+1)\} \subsetneqq N_G((i,j'+1))$.

where $i = 2,\ldots,m$ and $j' \in \{1',\ldots,n'\}$. Hence, any maximal chain has length 2, so the Galois poset $B^{\S}(G)$ is a graded poset of rank 2. $\square$

Example 4.1.33. *In the following figure we represent the concept lattice of the* $(2,3)$*-prism.*

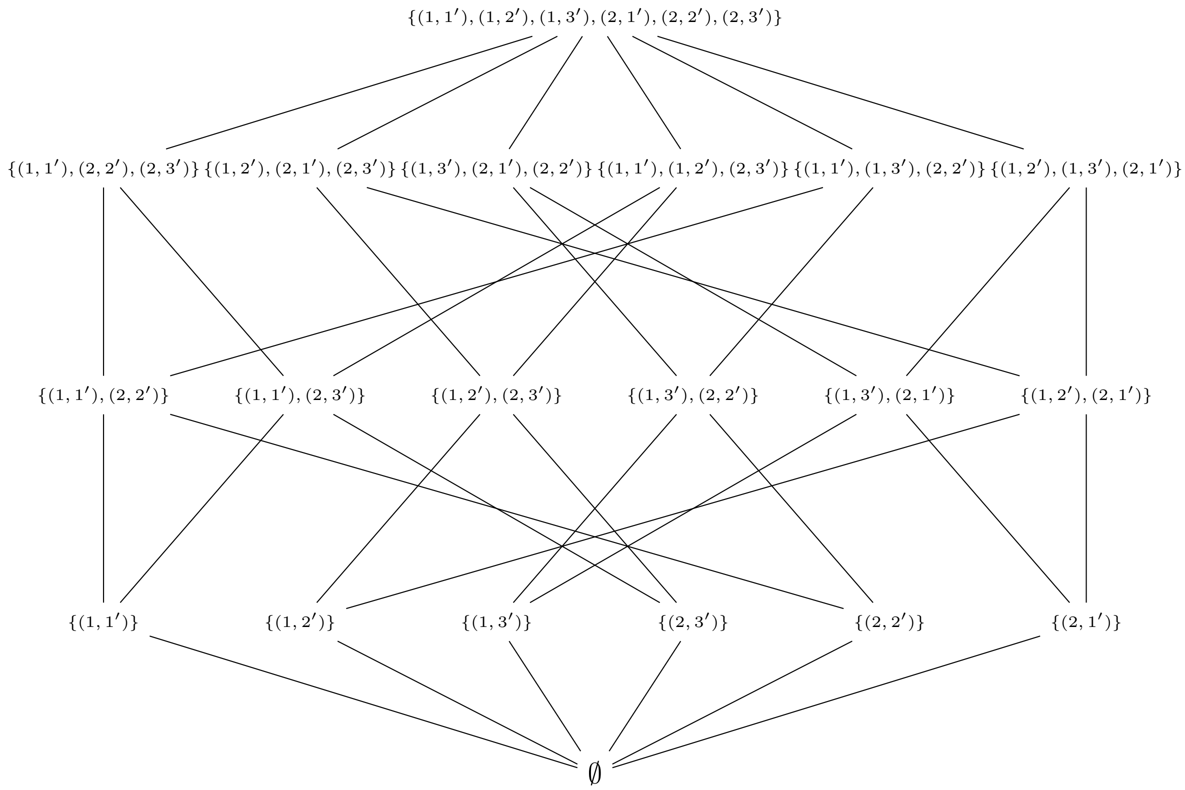


Figure 4.5: Concept Lattice of the $(2,3)$-prism.

In the next result we determine the extent family of the (m,n)-grid graph and we prove that its Galois poset is graded of rank 2.

Theorem 4.1.34. *Let* $m,n \geq 2$ *and let* $G = P_m \square P_n$ *be the* (m,n)*-grid graph. Then*

$$\mathfrak{B}^{\S}(G) = \{\{(i,j')\},\{(k,r'),(k+1,r'-1)\},\{(k,s'),(k+1,s'+1)\},N_G((i,j'))\} \tag{4.7}$$

where $i=1,\ldots,n$*,* $j'=1,\ldots,m$*,* $k=1,\ldots,m-1$*,* $r'=2,\ldots,n$ *and* $s'=1,\ldots,n-1$*. Moreover* $\mathfrak{B}^{\S}(G)$ *is a graded poset of rank 2.*

Proof. We have that $N_G((1,j')) = \{(1,j'-1),(1,j'+1),(2,j')\}$ for $j'=1,\ldots,m$, $N_G((i,j')) = \{(i,j'-1),(i,j+1),(i-1,j'),(i+1,j')\}$ for any $j'=1,\ldots,m$ and $i=2,\ldots,n-1$ and $N_G((n,j')) = \{(n,j'-1),(m,j'+1),(n-1,j')\}$ for $j'=1,\ldots,m$. At this point the remaining part of the proof of (4.1.34) is very similar to that provided in Theorem 4.1.32, therefore we omit it.
For the second part of the statement, by (4.7) we easily deduce that any maximal chain has one of the following form:

- $\{(1,1')\} \subsetneqq \{(1,1'),(2,2')\} \subsetneqq N_G((1,2'))$;

- $\{(1,1')\} \subsetneq \{(1,1'),(2,2')\} \subsetneq N_G((2,1'))$;
- $\{(1,n')\} \subsetneq \{(1,n'),(2,n'-1)\} \subsetneq N_G((1,2'))$;
- $\{(1,n')\} \subsetneq \{(1,n'),(2,n'-1)\} \subsetneq N_G((2,3'))$;
- $\{(m,1')\} \subsetneq \{(m,1'),(m-1,2')\} \subsetneq N_G((m,2'))$;
- $\{(m,1')\} \subsetneq \{(m,1'),(m-1,2')\} \subsetneq N_G((m-1,1'))$;
- $\{(m,n')\} \subsetneq \{(m,n'),(m-1,n'-1)\} \subsetneq N_G((m,n'-1))$;
- $\{(m,n')\} \subsetneq \{(m,n'),(m-1,n'-1)\} \subsetneq N_G((m-1,n'))$;
- $\{(1,j')\} \subsetneq \{(1,j'),(2,j'-1)\} \subsetneq N_G((2,j'))$;
- $\{(1,j')\} \subsetneq \{(1,j'),(2,j'+1)\} \subsetneq N_G((2,j'))$;
- $\{(m,j')\} \subsetneq \{(m,j'),(m-1,j'-1)\} \subsetneq N_G((m-1,j'))$;
- $\{(m,j')\} \subsetneq \{(m,j'),(m-1,j'+1)\} \subsetneq N_G((m-1,j'))$;
- $\{(i,1')\} \subsetneq \{(i,1'),(i-1,2')\} \subsetneq N_G((2,j'))$;
- $\{(i,1')\} \subsetneq \{(i,1'),(i+1,2')\} \subsetneq N_G((2,j'))$;
- $\{(i,n')\} \subsetneq \{(i,n'),(i-1,n'-1)\} \subsetneq N_G((m-1,j'))$;
- $\{(i,n')\} \subsetneq \{(i,n'),(i+1,n'-1)\} \subsetneq N_G((m-1,j'))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k-1,l'-1)\} \subsetneq N_G((k,l'-1))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k-1,l'-1)\} \subsetneq N_G((k-1,l'))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k-1,l'+1)\} \subsetneq N_G((k,l'+1))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k-1,l'+1)\} \subsetneq N_G((k-1,l'))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k+1,l'-1)\} \subsetneq N_G((k,l'-1))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k+1,l'-1)\} \subsetneq N_G((k+1,l'))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k+1,l'+1)\} \subsetneq N_G((k,l'+1))$;
- $\{(k,l')\} \subsetneq \{(k,l'),(k+1,l'+1)\} \subsetneq N_G((k+1,l'))$

where $i,k = 2,\ldots,m-1$ and $j',l' = 2',\ldots,n'-1$. Hence any maximal chain has length 2 and $P_m \Box P_n$ is a graded poset of rank 2. $\square$

Example 4.1.35. *In the following figure we represent the concept lattice of the* $(3,2)$*-grid graph.*

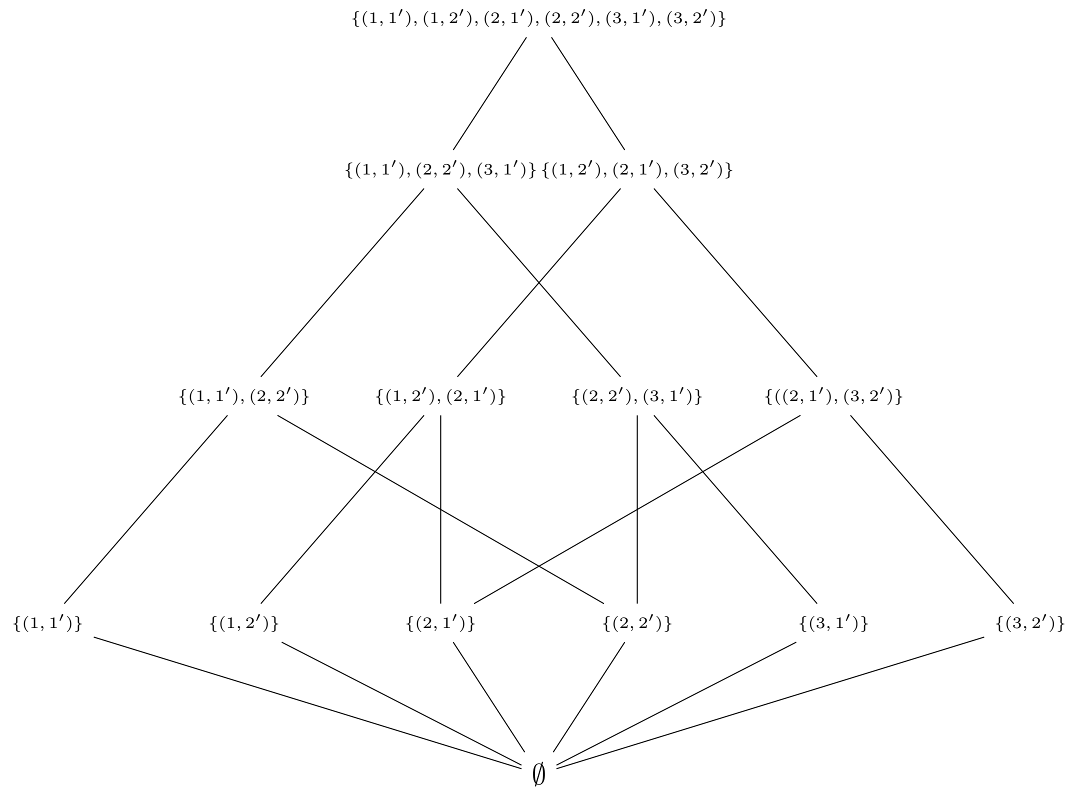


Figure 4.6: Concept Lattice of the $(3,2)$-grid graph.

For a rook's graph we obtain the following result.

Theorem 4.1.36. *Let* $m, n \geq 5$ *and* G *be the rook's graph* $K_m \square K_n$*. Then:*

$$\mathfrak{B}^{\S}(G) = \{\{(i,j')\}, \{(i,j'_1), \ldots, (i,j'_s)\}, \{(i_1,j'), \ldots, (i_r,j')\}, \{(i,j'), (k,l')\}, N_G((i,j'))\}$$

where $i, k = 1, \ldots, m$ *and* $j', l' = 1, \ldots, n$*,* $(i,j') \nsim (k,l')$*,* $i_1, \ldots, i_r \in \{1, \ldots, m\}$*,* $j'_1, \ldots, j'_s \in \{1, \ldots, n\}$ *and* $r = m-2$*,* $s = n-2$*.*

Proof. The rook's graph is $(m+n-2)$-regular. Let $O \subseteq V(G)$. It is immediate to see that

$$O' = \begin{cases} N_G((i,j')) & \text{if } O = \{(i,j')\} \\ \{(i,j')\} & \text{if } O = N_G((i,j')) \end{cases}$$

Now, let O be a vertex subset of cardinality 2. If the two elements (i,j') and (k,l') are not adjacent, we deduce that O' contains the pairs respectively (i,l') and (k,j'), so we have $O'' = O$. Moreover, if $O = \{(i,j'), (i,k')\}$, then their neighborhoods share $n-2$ points which are adjacent only to the previous two vertices, i.e. even in this case $O'' = O$. In particular, this is true for any vertex subset containing at most $n-2$ vertices of the same row. When we consider $n-1$ elements of the same row, their neighborhoods have in common one vertex, so $O'' \supsetneq O$, while if we consider all the vertices of the row, then $O' = \emptyset$ and $O'' = V$. The case in which O contains elements of the same column is similar. Let $O \subseteq V(G)$ containing 3 vertices, at least 2 of which non-adjacent. Then, $|O'| = 1$ when we consider two vertices of the same row (resp. column) and another vertex in one of the remaining columns (rows), so $O'' \supsetneq O$; we have $O' = \emptyset$ in the other cases, so $O'' = V$. In the other cases, we have three vertices as before, so $O'' \neq O$ and we conclude the proof. □

Remark 4.1.37. In general, the Galois poset of the rook's graph $G = K_m \square K_n$ is not graded. In fact, by Proposition 4.1.36 it is immediate to deduce that

- $\{(i,j')\} \subsetneq \{(i,j'),(k,l')\}$;
- $\{(i_1,1')\} \subsetneq \{(i_1,1'),(i_2,1')\} \subsetneq \cdots \subsetneq \{(i_1,1'),\ldots,(i_{m-2},1')\} \subsetneq N_G((1,1'))$;
- $\{(1,j_1')\} \subsetneq \{(1,j_1'),(1,j_2')\} \subsetneq \cdots \subsetneq \{(1,j_1'),\ldots,(1,j_{n-2}')\} \subsetneq N_G((1,1'))$,

where $(i,j') \nsim (k,l')$, are three maximal chains having length respectively 2, $m-1$ and $n-1$.

We conclude this section by determining the extent family of the tensor product of an m-path and an n-cycle. We also detremine the rank for the corresponding Galois poset.

Theorem 4.1.38. *Let $m,n \geq 2$ and $G = P_m \otimes C_n$ be the tensor product graph of the m-path and the n-cycle. Then*

$$(\mathfrak{B}^{\S}(G),\subseteq) = \{\{(i,j')\},\{(r,j'),(r+2,j')\},\{(i,j'),(i,j'+2)\},N_G((i,j'))\}, \tag{4.8}$$

where $r = 1,\ldots,m-2$, $i = 2,\ldots,m-1$ and $j' = 1,\ldots,n$.
Moreover, the Galois poset $\mathfrak{B}^{\S}(G)$ is graded of rank

$$\begin{cases} 1 & \text{if } m=3 \\ 2 & \text{if } m=4 \end{cases}$$

Proof. In this proof the index sums in the second component of the tensor product are taken mod(n). We firstly observe that $N_G((i,j')) = N_G(i) \times N_G(j')$. This means that

$$|N_G((i,j'))| = \begin{cases} 2 & \text{if } i = 1 \vee i = m \\ 4 & \text{otherwise} \end{cases}$$

It is immediate to see that all the singletons (i,j') such that $|N_G((i,j'))| = 4$ and their neighborhoods are extent of the formal context $\mathbb{K}[G]$. Let $O = \{(i,j')\}$ such that $|O'| = 2$. In order to fix ideas, let $O = \{(1,j')\}$. Then, $O' = \{(2,j'-1),(2,j'+1)\}$ and $O'' = \{(1,j'),(3,j')\}$, hence in this case O is not an extent. Let $O \subseteq V(G)$ of the form $O = \{(i,r'),(i,s')\}$, where $i = 2,\ldots,m-1$. Since these vertices belong to the same n-cycle, we deduce that $O' = \emptyset$ unless $d(r',s') \equiv 2 \pmod{n}$. In this case, $O' = \{(i-1,r'),(i+1,r')\}$. Moreover, let $O \subseteq V(G)$ of the form $O = \{(i,r'),(i+2,r')\}$, where $i = 1,\ldots,m-2$. Since these two vertices belong to the same m-path, it is easy to show that $O' = \{(i+1,r'-1),(i+1,r'+1)\}$, so we deduce that both the vertex subsets $O = \{(i, r'), (i, s')\}$, where $i = 2, \dots, m-1$, and $O = \{(i, r'), (i+2, r')\}$, where $i = 1, \dots, m-2$, are extent of $\mathbb{K}[G]$. Let $O \subseteq V(G)$ be a vertex subset of cardinality 2 different from the previous. If the two pairs are adjacent, then their first components different of one, but have their second components are the same, hence it is immediate to see that their neighborhoods do not intersect in any point. If the two pairs are non-adjacent, we have

$$|O'| = \begin{cases} 1 & \text{if } O = \{(i, j'), (i+2, j'+2)\} \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1, \dots, m-2$. In both cases, it's clear that $O'' \supsetneq O$, i.e. O is not an extent. Let $O \subseteq V(G)$ be any subset of cardinality greater than 2; then we have three cases: (a) there exist a vertex (i, j') such that $O \subsetneq N_G((i, j'))$, (b) there exist a vertex (i, j') such that $O = N_G((i, j'))$ or (c) O is not neighborhood of any vertex. In the first situation, it is clear that $O'' \neq O$, while in the third there are always two vertices whose neighborhoods don't have any point in common. Thus, O is an extent only in case (b) and we can conclude the proof of (4.8).
Let now $m = 3$. By (4.8) we deduce that $\{(2, j')\}$ and pairs $\{(1, j'-1), (1, j'+1)\}$, for $j' = 1, \dots, n$, are disjoint and they are minimal elements for the poset $B^{\S}(P_3 \otimes C_n)$. Moreover, singletons are contained only in the maximal elements $\{(2, j'-1), (2, j'+1)\}$, while pairs are contained only in the maximal elements $N_{P_3 \otimes C_n}((2, j'))$, for $j' = 1, \dots, n$. Thus, if $m = 3$, it results that $B^{\S}(P_3 \otimes C_n)$ is a graded lattice of rank 1.
Let $m \geq 4$. By 4.1.38 one can verify that the maximal chains of $\mathfrak{B}^{\S}(G)$ assume one of the following forms:

- $\{(2, j')\} \subsetneqq \{(2, j'), (2, j'-1)\} \subsetneqq N_G((3, j'+1))$;
- $\{(2, j')\} \subsetneqq \{(2, j'), (2, j'+1)\} \subsetneqq N_G((3, j'-1))$;
- $\{(2, j')\} \subsetneqq \{(2, j'), (4, j')\} \subsetneqq N_G((3, j'-1))$;
- $\{(2, j')\} \subsetneqq \{(2, j'), (4, j')\} \subsetneqq N_G((3, j'+1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i, j'-1)\} \subsetneqq N_G((i-1, j'+1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i, j'-1)\} \subsetneqq N_G((i+1, j'+1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i, j'+1)\} \subsetneqq N_G((i-1, j'-1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i, j'+1)\} \subsetneqq N_G((i+1, j'-1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i-2, j')\} \subsetneqq N_G((i-1, j'-1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i-2, j')\} \subsetneqq N_G((i-1, j'+1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i+2, j')\} \subsetneqq N_G((i+1, j'-1))$;
- $\{(i, j')\} \subsetneqq \{(i, j'), (i+2, j')\} \subsetneqq N_G((i+1, j'+1))$;
- $\{(m-1, j')\} \subsetneqq \{(m-1, j'), (m-1, j'-1)\} \subsetneqq N_G((m-2, j'+1))$;

- $\{(m-1,j')\} \subsetneqq \{(m-1,j'),(m-1,j'+1)\} \subsetneqq N_G((m-2,j'-1))$;
- $\{(m-1,j')\} \subsetneqq \{(m-1,j'),(m-3,j')\} \subsetneqq N_G((m-2,j'-1))$;
- $\{(m-1,j')\} \subsetneqq \{(m-1,j'),(m-3,j')\} \subsetneqq N_G((m-2,j'+1))$;

where $i = 3,\dots,m-2$ and $j' = 1,\dots,n$. Hence, when $m \geq 4$, the Galois lattice of $P_m \otimes C_n$ is graded of rank 2. $\square$

Example 4.1.39. *In the following figure we represent the concept lattice of the graph $P_3 \otimes C_3$.*

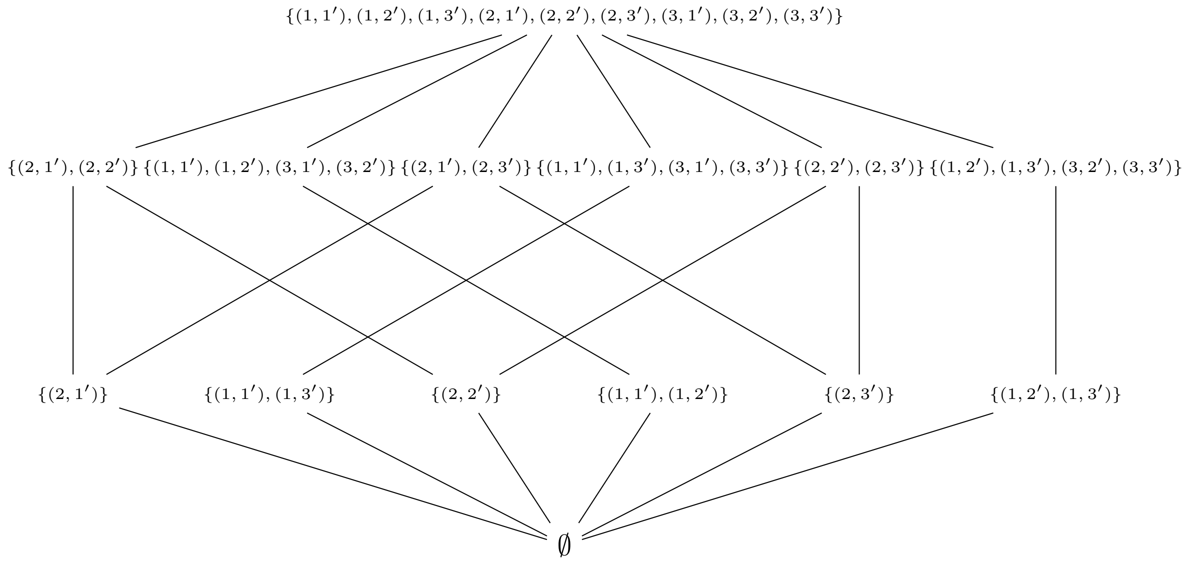


Figure 4.7: Concept Lattice of the graph $P_3 \otimes C_3$.

4.2 Algebraic Consequences of Dissymmetry and of its Generalizations On Graphs

In this section we study the properties of an algebraic operation arising whenever we consider the symmetric differences between non-empty vertex subsets of a graph G.

4.2.1 The Odd Extension of the Graph G

In this subsection we introduce the notion of *odd extension* $G^{\S}$ of a n-graph G, which stems from the odd number of order pairs between two any non-empty vertex subsets of G. We construct $G^{\S}$ by means of a binary operation having several interesting properties. We will show that $G^{\S}$ turns out connected and locally dissymmetric when G is connected and the determinant of its adjacency matrix is odd. In the remaining part of this section we assume $n \geq 3$.

Definition 4.2.1. Let G be a n-graph and let $A, B \in \mathcal{P}^*(V(G))$. We set

$$c_G(A, B) := |\{(v_i, v_j) \in A \times B : v_i \sim_G v_j\}|.$$

We denote by $\hat{G}$ the graph having vertex set $V(\hat{G}) := \mathcal{P}^*(V(G))$ and the following adjacency relation: if $A, B \in V(\hat{G})$ then

$$A \sim_{\hat{G}} B :\iff c_G(A, B) \quad \text{is} \quad \text{an} \quad \text{odd} \quad \text{integer}. \tag{4.9}$$

If $S \subseteq V(\hat{G})$, we denote by $\hat{G}_S$ the subgraph of $\hat{G}$ induced from the vertex set S.

We now introduce a binary operation on the vertex set of $\hat{G}$.

Definition 4.2.2. We set

$$A \circ B := \begin{cases} A & \text{if } A \nsim_{\hat{G}} B \\ A \bigtriangleup B & \text{if } A \sim_{\hat{G}} B, \end{cases} \tag{4.10}$$

for any $A, B \in V(\hat{G})$. It is clear that $\circ$ is a binary operation on $V(\hat{G})$, that we call *disparity operation.*

In the next result we establish some useful properties of the operation $\circ$.

Proposition 4.2.3. *The binary operation $\circ : V(\hat{G}) \times V(\hat{G}) \to V(\hat{G})$ has the following properties:*
(i): $A \circ A = A$;
(ii): $(A \circ B) \circ B = A$;
(iii): $(A \circ B) \circ C = (A \circ C) \circ (B \circ C)$;
(iv): if $A \circ B \neq A$, then $A \circ B = B \circ A$;
for any $A, B, C \in \mathcal{P}^*(V(G))$.

Proof. Straightforward. □

In what follows we will treat with other binary operations having the properties $(i) - (iv)$ established in Proposition 4.2.3, therefore we introduce the following general terminology.

Definition 4.2.4. (i) We call *delineated space* any pair $(X, \odot)$, where X is a non-empty set and $\odot$ is a binary operation on X that satisfies the properties $(i) - (iv)$ of Proposition 4.2.3.
(ii) If $(X_1, \odot_1)$ and $(X_2, \odot_2)$ are delineated spaces, we say that a map $f : X_1 \to X_2$ is a *homomorphism of delineated spaces* if $f(x \odot_1 x') = f(x) \odot_2 f(x')$ for all $x, x' \in X_1$. In particular, if f is a bijective homomorphism of delineated spaces, we say that f is an *isomorphism of delineated spaces.*
(iii) If f is an isomorphism of some delineated space $(X, \odot)$ into itself, we say that f is an *automorphism* of $(X, \odot)$. We denote by $Aut(X, \odot)$ the set of all the automorphisms of $(X, \odot)$. Obviously $Aut(X, \odot)$ is a group with respect to the usual composition of functions.

In this section we will investigate the class of the subsets of $V(\hat{G})$ that are closed with respect to the previous binary operation $\circ$. For such subsets we also consider a corresponding families of subgraphs of $\hat{G}$. We first introduce some terminology.

Definition 4.2.5. (i) If T is a subset of $V(\hat{G})$ that is closed with respect to the binary operation $\circ$ (therefore $(T, \circ)$ is a delineated space) we say that T is a *delineated subspace* of $\hat{G}$.
(ii) If $S \subseteq V(\hat{G})$, we call *delineated closure* of S in $V(\hat{G})$, denoted by S°, the smallest delineated subspace of $\hat{G}$ that contains S.
(iii) We denote by T_G the delineated closure of the subset $V(G) \subset V(\hat{G})$ in $V(\hat{G})$, and we denote by t_G the number of elements of T_G. We call *odd extension* of G the graph $G^{\S} := \hat{G}_{T_G}$. Hence T_G is the vertex subset of the graph $G^{\S}$ and t_G is its vertex number.
(iv) A *delineated subgraph* of $\hat{G}$ is a subgraph of the form $\hat{G}_T$, for some delineated subspace T of $\hat{G}$.

We now show that the algebraic automorphisms of any delineated subspace T of $\hat{G}$ coincide with the geometric automorphisms of the correspoding delineated subgraph $\hat{G}_T$.

Proposition 4.2.6. *Let T be a delineated subspace of $\hat{G}$, then $Aut(T, \circ) = Aut(\hat{G}_T)$. In particular, $Aut(T_G, \circ) = Aut(G^{\S})$.*

Proof. Let $\sigma \in Aut(T, \circ)$ and let A, $B \in T$ such that $A \sim_{\hat{G}_T} B$. Then, $\sigma(A) \neq \sigma(A \circ B) = \sigma(A) \circ \sigma(B)$, hence $\sigma(A) \sim_{\hat{G}_T} \sigma(B)$. On the other hand, suppose that $\sigma(A) \sim_{\hat{G}_T} \sigma(B)$, then $\sigma(A) \neq \sigma(A) \circ \sigma(B) = \sigma(A \circ B)$, so $A \neq A \circ B$ or, equivalently, $A \sim_{\hat{G}_T} B$. The case $A \nsim_{\hat{G}_T} B$ is similar. Hence $\sigma \in Aut(\hat{G}_T)$.
Conversely, let $\sigma \in Aut(\hat{G}_T)$ and suppose that $A \nsim_{\hat{G}_T} B$. Then,

$$\sigma(A \circ B) = \sigma(A) = \sigma(A) \circ \sigma(B).$$

On the other hand, if $A \sim_{\hat{G}_T} B$, we have that $A \circ B = A \bigtriangleup B$. Without loss of generality, let $x \in A \setminus B$. Hence $\sigma(x) \in \sigma(A) \setminus \sigma(B)$, therefore

$$\sigma(x) \in \sigma(A) \bigtriangleup \sigma(B)$$

or, equivalently,

$$\sigma(A \circ B) = \sigma(A) \circ \sigma(B).$$

Therefore, $\sigma \in Aut(T, \circ)$ and the claim follows. $\square$

Remark 4.2.7. (i) Let us note that G can be identified with a subgraph of $\hat{G}$ because the adjacency between two vertices of G coincides with the particular case of the adjacency between the singletons of $V(\hat{G})$.
(ii) By Definition of $\hat{G}$, since G is simple also $\hat{G}$ is a simple graph.
(iii) $\hat{G}$ has $2^{|G|} - 1 = 2^n - 1$ vertices.
(iv) Since we treat the elements of $\mathcal{P}^*(V(G))$ as vertices of the graph $\hat{G}$, often we also use the letters $u, v, w, \ldots$ instead of $A, B, \ldots$ in order to denote these elements.

Example 4.2.8. *Let $G = P_4$. It is easy to verify that the delineated closure of $V(G) = \{v_1, v_2, v_3, v_4\}$ in $V(\hat{G})$ is the subset*

$$T_G = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\}$$

In Figure 4.8, we give a representation for the odd extension of P_4:

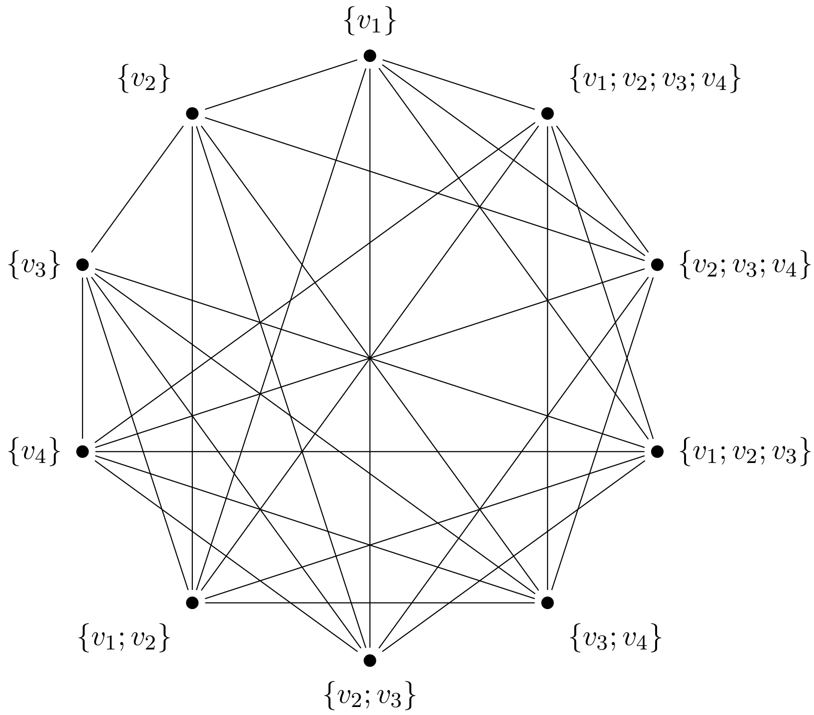


Figure 4.8: Odd Extension of P_4.

We denote by $Sym(V(\hat{G}))$ the permutation group of $V(\hat{G})$ and by Id the identity map on $V(\hat{G})$. We introduce now some specific elements of this group. If $u \in V(\hat{G})$ we set

$$\xi_u(v) := v \circ u, \tag{4.11}$$

for any $v \in V(\hat{G})$.

We have then the following result.

Proposition 4.2.9. *(i)* $\xi_u \in Sym(V(\hat{G}))$ *for any* $u \in V(\hat{G})$.
(ii) $\xi_u^2 = Id$ *for any* $u \in V(\hat{G})$.
(iii) If T *is a delineated subspace of* $\hat{G}$ *then* $\xi_{u|T} \in Aut(T, \circ)$ *for any* $u \in T$.
(iv) If $u, w \in T_G$ *and* $\sigma \in Aut(\hat{G})$, *it results*

$$\sigma^{-1}\xi_u\sigma = \xi_{\sigma^{-1}(u)}. \tag{4.12}$$

In particular, we have that

$$\xi_w^{-1}\xi_u\xi_w = \xi_{u \circ w}. \tag{4.13}$$

Proof. (i) : Since $V(\hat{G})$ is a finite set it is sufficient to prove that ξ_u is surjective. By (ii) of Proposition 4.2.3, if $v \in V(\hat{G})$, then $v = (v \circ u) \circ u = \xi_u(v \circ u)$.
(ii) : The thesis follows immediately by (ii) of Proposition 4.2.3.
(iii) : By (iii) of Proposition 4.2.3, we deduce immediately the thesis.
(iv) : Let us consider $\sigma^{-1}\xi_u\sigma(v)$. Then,

$$\sigma^{-1}\xi_u\sigma(v) = \sigma^{-1}(\sigma(v) \circ u) = v \circ \sigma^{-1}(u)$$

by Proposition 4.2.6. But the last term is exactly $\xi_{\sigma^{-1}(u)}(v)$ and the thesis has been shown. □

4.2.2 Regularity properties of $G^{\S}$

In what follows, we always assume that G is a connected graph. In the next result we relate the presence of isolated vertices in $\hat{G}$ to the parity of the determinant of the adjacency matrix of G.

Proposition 4.2.10. $\hat{G}$ *has at least an isolated vertex if and only if* $\det(Adj(G))$ *is even.*

Proof. Let us suppose $\hat{G}$ has an isolated vertex $A \in V(\hat{G}) = \mathcal{P}(V(G))$ and let $v_A = (x_1, \ldots, x_n)$ be the vector in $\mathbb{R}^n$ defined by

$$x_i := \begin{cases} 1 & \text{if } v_i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The vector $Adj(G)\, v_A = (y_1, \ldots, y_n)$ is the vector in $\mathbb{R}^n$ such that y_i is the number of vertices in A that are adjacent to v_i, for all $i = 1, \ldots, n$. Thus $y_i = c_G(\{v_i\}, A) \equiv 0 \pmod 2$, for all $i = 1, \ldots, n$ because $\{v_i\} \nsim_{\hat{G}} A$. By considering A as a square matrix of order n with entries in $\mathbb{Z}_2$, then $v_A \in \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ is in the kernel of the matrix A. So $\det(A) \equiv 0 \pmod 2$. For the reserve implication let us note that all of the previous arguments are reversible. In fact if $det(Adj(G)) \equiv 0 \pmod 2$, then, by considering $Adj(G)$ as a matrix in $\mathrm{Mat}_n(\mathbb{Z}_2)$, it has non-trivial kernel. So there exist $v \in \mathbb{Z}_2^n$ such that $Adj(G)\cdot v$ is the null vector in $\mathbb{Z}_2^n$. We can interpret v as a vertex subset of G. $v \in \ker(Adj(G))$ implies that, for all $i = 1, \ldots, n$, $c_G(\{v_i\}, v) \equiv 0 \pmod 2$. It follows that v is an isolated vertex of $\hat{G}$. This proves the proposition. □

In the next result we prove several regularity properties of the graph $G^{\S}$ when $det(Adj(G))$ is odd. In particular, under this hypothesis we will show that $G^{\S}$ is always a connected and locally dissymmetric graph.

Theorem 4.2.11. *Let $det(Adj(G))$ be odd. Then:*
(i) $G^{\S}$ is connected and locally dissymmetric.
(ii) There exists λ such that $\lambda = |N_{G^{\S}}(v) \cap N_{G^{\S}}(w)|$ for any two adjacent vertices v, w of $G^{\S}$.
(iii) $G^{\S}$ is not a complete graph.
(iv) $G^{\S}$ is a 2λ-regular graph.
(v) There exists $\mu \equiv 0 \pmod 4$ such that $\mu = |N_{G^{\S}}(v) \cap N_{G^{\S}}(w)|$ for any two non-adjacent vertices v, w of $G^{\S}$.

Proof. (i): Suppose by contradiction that $G^{\S}$ is not connected. Let S_G the connected component containing G and fix a vertex $v \notin S_G$. Then, $c_G(\{v_i\}, v) \equiv 0 \pmod 2$, for all $i = 1, \ldots, n$ because $\{v_i\} \nsim_{\hat{G}} v$. Thus, for all $w \in \mathcal{P}^*(V(G))$, it holds

$$c_G(w, v) = \sum_{v_i \in w} c_G(\{v_i\}, v) \equiv 0 \pmod 2,$$

so $w \nsim_{\hat{G}} v$. Then, w is an isolated vertex in $\hat{G}$, but this is impossible by Proposition 4.2.10.

Let us prove that $G^{\S}$ is locally dissymmetric. Suppose by contradiction that $G^{\S}$ is not locally dissymmetric. Hence there exist two distinct elements $v, w \in T_G$ such that $\Delta_{G^{\S}}(v, w) = \emptyset$. Thus, by setting $u := v \bigtriangleup w$, then, for all $z \in T_G$, $c_G(u, z) \equiv 0 \pmod 2$. This contradicts the previous part of the proof.

(ii): Let v and w be two adjacent vertices in $G^{\S}$. The application ξ_v defined in (4.11) induces a bijection between $N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$ and $N_{G^{\S}}(v) \setminus N_{G^{\S}}(w)$. In fact, if $u \in N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$, then $u \circ v = u \bigtriangleup v$ and thus

$$c_G(u \circ v, v) = c_G(u, v) + c_G(v, v) - 2c_G(u \cap v, v) \equiv 1 \pmod 2,$$

and

$$c_G(u \circ v, w) = c_G(u, w) + c_G(v, w) - 2c_G(u \cap v, w) \equiv 0 \pmod 2,$$

so $\xi_v(u) = u \circ v \in N_{G^{\S}}(v) \setminus N_{G^{\S}}(w)$. On the other side, if $u \in N_{G^{\S}}(v) \setminus N_{G^{\S}}(w)$, then in this case it results that

$$c_G(u \circ v, v) = c_G(u, v) + c_G(v, v) - 2c_G(u \cap v, v) \equiv 1 \pmod 2,$$

and

$$c_G(u \circ v, w) = c_G(u, w) + c_G(v, w) - 2c_G(u \cap v, w) \equiv 1 \pmod 2,$$

so $\xi_v(u) = u \circ v \in N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$. In the same way, ξ_w induces a bijection between $N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$ and $N_{G^{\S}}(w) \setminus N_{G^{\S}}(v)$, hence $|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)| = |N_{G^{\S}}(w) \setminus N_{G^{\S}}(v)|$. Therefore, the following relation holds for any two adjacent vertices $v, w \in S^{\circ}$

$$|N_{G^{\S}}(v)| = 2|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)| = |N_{G^{\S}}(w)|. \tag{4.14}$$

So, there exists λ such that $\lambda = |N_{G^{\S}}(v) \cap N_{G^{\S}}(w)|$ for any two adjacent vertices v, w of $G^{\S}$.
(iii): If G is not complete, then there are two non-adjacent vertices $v_i, v_j \in V(G)$, hence this holds also in $G^{\S}$. On the other hand, if G is the complete graph on n vertices, it is immediate to see that if u, v and w are three distinct vertices in $V(G)$, then u, $v \cup w$ are vertices in $G^{\S}$ such that $c_G(u, v \cup w) \equiv 0 \pmod 2$, hence $G^{\S}$ is not complete.
(iv): Since $G^{\S}$ is a connected graph, for any two non-adjacent vertices v and w we can take a path between them and iterate the equality (4.14) for any two adjacent vertices in the path. Hence, we conclude by part (ii).
(v): Let v, w be two non adjacent vertices of $G^{\S}$. Note that, if $u \in N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$, then $v \circ u$, $w \circ u$, $v \circ (w \circ u) \in N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$ and $|\{u,\ v \circ u,\ w \circ u,\ v \circ (w \circ u)\}| = 4$. Moreover, if $u' \in \{v \circ u,\ w \circ u,\ v \circ (w \circ u)\}$, then $\{u,\ v \circ u,\ w \circ u,\ v \circ (w \circ u)\} = \{u',\ v \circ u',\ w \circ u',\ v \circ (w \circ u')\}$. In fact, it is straightforward to prove that the vertices $v \circ u$ and $w \circ u$ are adjacent to both v and w. Let us prove now that $v \circ (w \circ u) \in N_{G^{\S}}(v) \cap N_{G^{\S}}(w)$ too. It holds

$$c_G(v \circ (w \circ u), v) = c_G(v, v) + c_G(w \circ u, v) - 2c_G(v \cap (w \circ u), v) \equiv 1 \pmod 2$$

and

$$c_G(v \circ (w \circ u), w) = c_G(v, w) + c_G(w \circ u, w) - 2c_G(v \cap (w \circ u), w) \equiv 1 \pmod 2,$$

so $v \circ (w \circ u) \sim_{G^{\S}} v$ and $v \circ (w \circ u) \sim_{G^{\S}} w$.

Let now $u' = v \circ u$. Then, we have

$$u' \circ v = (v \bigtriangleup u) \bigtriangleup v = v,$$

$$u' \circ w = (v \bigtriangleup u) \bigtriangleup w = v \bigtriangleup (w \bigtriangleup u) = v \circ (w \circ u)$$

and

$$u' \circ (v \circ (w \circ u)) = (v \bigtriangleup u) \bigtriangleup (v \circ (w \circ u)) = w,$$

because $(v \circ u) \sim_{G^{\S}} (v \circ (w \circ u))$. This prove that $|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)| \equiv 0 \pmod 4$.
We prove that $|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)|$ is constant for any two distinct non-adjacent vertices v, $w \in \hat{G}$. Let v', $w' \in \hat{G}$ two non-adjacent vertices distinct from v, w. First of all, we prove that $|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)| = |N_{G^{\S}}(v') \cap N_{G^{\S}}(w')|$ can be reduced to the case $|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)| = |N_{G^{\S}}(v) \cap N_{G^{\S}}(y)|$ for some $y \in T_G$ non-adjacent to v. By Proposition 4.2.9, if u, $v \in V(\hat{G})$, then

$$\xi_u(N_{\hat{G}}(v)) = N_{\hat{G}}(\xi_u(v)). \tag{4.15}$$

Let $P = s_0,\ s_1,\ \ldots,\ s_{l-1},\ s_l$ be a path between $v' = s_0$ and $v = s_n$. Since $s_i \sim s_{i-1}$, for each $i \in \{1, \ldots, l\}$, by applying sequentially $\xi_{s_i \circ s_{i-1}} =: \xi_{u_i}$ and by using Equation (4.15), it is easy to prove that

$$\xi_{u_n} \cdots \xi_{u_2} \xi_{u_1}(N_{G^{\S}}(v) \cap N_{G^{\S}}(w)) = N_{G^{\S}}(v) \cup N_{G^{\S}}(y),$$

where $y := (\ldots((w' \circ u_1) \circ u_2) \cdots \circ u_n)$. Moreover, by Proposition 4.2.9 again, $v \nsim_{G^{\S}} y$.
Let us note that $|N_{G^{\S}}(v) \cap N_{G^{\S}}(w)| \neq 0 \neq |N_{G^{\S}}(v) \cap N_{G^{\S}}(y)|$ since any path between two non-adjacent vertices can be shortened to a 2-path by replacing two adjacent vertices by their

symmetric difference. It is easy to prove that ξ_w induces a bijection between $N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y)$ and $(N_{G^\S}(v) \cap N_{G^\S}(w)) \setminus N_{G^\S}(y)$. Similarly ξ_y induces a bijection between $N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y)$ and $(N_{G^\S}(v) \cap N_{G^\S}(y)) \setminus N_{G^\S}(w)$. Thus $|(N_{G^\S}(v) \cap N_{G^\S}(w)) \setminus N_{G^\S}(y)| = |(N_{G^\S}(v) \cap N_{G^\S}(y)) \setminus N_{G^\S}(w)|$. By a simple computation, it is immediate to see that

$$|N_{G^\S}(v) \cap N_{G^\S}(w)| = |N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y)| + |N_{G^\S}(v) \cap N_{G^\S}(w) \setminus N_{G^\S}(y)| =$$

$$= |N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y)| + |N_{G^\S}(v) \cap N_{G^\S}(y) \setminus N_{G^\S}(w)| = |N_{G^\S}(v) \cap N_{G^\S}(y)|.$$

So the theorem holds in the first case. Suppose that $w \nsim y$. Then, there exists $x \in \Delta_{G^\S}(v, w)$ because $G^\S$ is a locally dissymmetric graph. Without loss in generality we can assume $y \sim x$. Let us observe that $(N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(x)) \setminus N_{G^\S}(y) \neq \emptyset$. In fact, we know that $|N_{G^\S}(v) \cap N_{G^\S}(w)| \neq 0$ so, there exists $z \in (N_{G^\S}(v) \cap N_{G^\S}(w)) \setminus N_{G^\S}(y)$. Furthermore, it is easy to show that

$$\xi_x : (N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(x)) \setminus N_{G^\S}(y) \to (N_{G^\S}(w) \cap N_{G^\S}(y) \cap N_{G^\S}(x)) \setminus N_{G^\S}(v)$$

$$\xi_x : (N_{G^\S}(w) \cap N_{G^\S}(x)) \setminus N_{G^\S}(v) \setminus N_{G^\S}(y) \to (N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y)) \setminus N_{G^\S}(x)$$
$$\xi_x : (N_{G^\S}(w) \cap N_{G^\S}(x)) \setminus N_{G^\S}(v) \setminus N_{G^\S}(y) \to N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y) \cap N_{G^\S}(x)$$
$$\xi_v : (N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(y)) \setminus N_{G^\S}(x) \to (N_{G^\S}(w) \cap N_{G^\S}(y) \cap N_{G^\S}(x)) \setminus N_{G^\S}(v)$$
$$\xi_v : (N_{G^\S}(v) \cap N_{G^\S}(w)) \setminus N_{G^\S}(y) \setminus N_{G^\S}(x) \to (N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(x)) \setminus N_{G^\S}(y)$$
$$\xi_y : (N_{G^\S}(w) \cap N_{G^\S}(y)) \setminus N_{G^\S}(v) \setminus N_{G^\S}(x) \to (N_{G^\S}(w) \cap N_{G^\S}(y) \cap N_{G^\S}(x)) \setminus N_{G^\S}(v)$$

are all bijections. Therefore, by some simple computations we deduce that

$$|N_{G^\S}(v) \cap N_{G^\S}(w)| = |N_{G^\S}(v) \cap N_{G^\S}(y)| = 4|(N_{G^\S}(v) \cap N_{G^\S}(w) \cap N_{G^\S}(x)) \setminus N_{G^\S}(y)|.$$

□

4.2.3 Two Algebraic Characterizations for the Odd Extension of G

In this section we associate with G two groups, denoted respectively by D_G and K_G. In terms of these groups we provide two algebraic characterizations related to the graph structure of $G^\S$.

Let T be a delineated subspace of $\hat{G}$. We set then

$$F_T := \{\xi_{u|T} : u \in T\},$$

and we will consider the following two subgroups of $Sym(T)$ (the permutation group on T):

$$K_T := \langle \xi_{u|T}\xi_{w|T} : u, w \in T \rangle \subseteq D_T := \langle F_T \rangle.$$

However, when T is specified and $u \in T$, we write simply ξ_u instead of $\xi_{u|T}$.

Proposition 4.2.12. *Let $w \in T$ be a fixed element. Then, $K_T = \langle \xi_u \xi_w : u \in T \rangle$.*

Proof. It is sufficient to observe that $\xi_u \xi_v = (\xi_u \xi_w)(\xi_v \xi_w)^{-1}$ for any $u, v \in T$. □

Definition 4.2.13. We call $D_G := D_{T_G}$ the *first normalization group* of G and $K_G := K_{T_G}$ the *second normalization group* of G.

Remark 4.2.14. (i): By Proposition 4.2.9 and by (4.12), it is immediate to see that D_G is a normal subgroup of $Aut(T_G)$.
(ii): Since D_G is a subgroup of $Sym(V(\hat{G}))$, it induces a natural group action on T_G.

In what follows, if H is a group acting on a set X and $h \in H$, we denote $hX := \{hx : x \in X\}$.

Definition 4.2.15. Let X be a finite set and H a group acting on X. We say that $S \subseteq X$ is a block for H if for any $h \in H$ we have $hS = S$ or $hS \cap S = \emptyset$. We say that H is primitive over X if its action on X is transitive and all its blocks are only singletons or the whole X. A group acting on a set X which is not primitive over X is called imprimitive over X.

Proposition 4.2.16. *Let T be a delineated subspace of $\hat{G}$ and let us consider the natural action of D_T over T. If $S \subseteq T$ is a block for D_T, then S is a delineated subspace of T.*

Proof. Let $S \subseteq T$ be a block for D_T and let u, $v \in S$. Then, since $v = v \circ v \in \xi_v S \cap S$, then $\xi_v S = S$, so $u \circ v \in \xi_v S = S$. □

Proposition 4.2.17. *D_G is primitive over T_G if and only if for any delineated subspace $S \subsetneqq T_G$ such that $|S| \geq 2$ it results*

$$\emptyset \subsetneqq \tau S \cap S \subsetneqq S, \tag{4.16}$$

for some $\tau \in D_G$.

Proof. If D_G is primitive over T_G, the claim is obvious. Vice versa, suppose that (4.16) holds and let $S \subsetneqq T_G$ be a block for D_G. By Proposition 4.2.16 S is a delineated subspace of T_G. Thus, by (4.16), S is a singleton.

Let us prove now that the action of D_G over T_G is transitive. Let us show that, for any $v \in T_G$, $D_G v := \{\tau(v) : \tau \in D_G\} = T_G$. Let $v \in V(G) \subseteq T_G$, then, since G is connected, there exists $w \in V(G)$ such that $v \sim w$. It follows $|D_G v| \geq 2$. By (4.16) it holds $D_G v = T_G$. If $v \in T_G$, then $v \in D_G w$ for some $w \in V(G)$. Thus $D_G w = D_G v = T_G$, so we proved that the action is transitive and the proposition holds. □

In the next result we provide an algebraic characterization for the condition that the odd extension of G is a graph connected and locally dissymmetric.

Theorem 4.2.18. *The following properties are equivalent:*
(a) The odd extension $G^{\S}$ is connected and locally dissymmetric.
(b) D_G is primitive over T_G.

Proof. $(a) \implies (b)$: Let us suppose that $G^{\S}$ is connected and locally dissymmetric and let v, w in T_G. Let us prove, by using induction on the distance between v and w, that there exists $\tau \in D_G$ such that $v = \tau(w)$. If $n = 1$, then $v \sim_{G^{\S}} w$, so $v \circ w \neq v$. By (iv) of Proposition 4.2.3, it holds $v \circ w = w \circ v$, so

$$v = (v \circ w) \circ w = (w \circ v) \circ v = \tau(w),$$

where $\tau := \xi_v\,\xi_w$. Let now $n > 1$. In this case $v \nsim_{G^\S} w$ and there exists $u \in T_G$ such that $v \sim_{G^\S} u$ and $d(u,w) = n-1$. By using the inductive hypothesis there exists τ_1, $\tau_2 \in D_G$ such that $\tau_1(w) = u$ and $\tau_2(u) = v$. Thus $v = \tau_2(u) = \tau_2(\tau_1(w))$. This proves transitivity. Let now S be a block for D_G. Then, by Proposition 4.2.16, S is a delineated subspace of T_G. Let us assume now that S is a proper delineated subspace of T_G such that $|S| \geq 2$ and let x, $y \in S$. Since S is a delineated subspace, $x \circ y \in S$. Let us consider the case $x \circ y \neq x$. If $v \in T_G$, then let us note that, if both $x \circ v$ and $y \circ v$ are not in S, then $(x \circ y) \circ v = x \circ y \in S$. By definition of block for the action of D_G, it holds $\xi_v S = S$. By transitivity of the action of D_G, it follows that $S = T_G$, so D_G is primitive over T_G. Consider now $x \circ y = x$. Since $G^\S$ is locally dissymmetric, then there exists $u \in T_G$ such that either $x \circ u = x$ and $y \circ u = y$ or $y \circ u \neq y$ and $x \circ u \neq x$. Still, by definition of block for D_G over T_G, in both cases we obtain $\xi_u S = S$. If $x \circ u \neq x$, then $u \in S$ and $u \neq x$, so the thesis follows as before. Similarly in the case $y \circ u \neq y$.
$(b) \implies (a)$: Let us assume now that D_G is primitive over T_G. Let v, $w \in T_G$ with $v \neq w$. Since D_G is primitive, its action over T_G is transitive. So there exist $\tau \in D_G$ such that $\tau(v) = w$. Suppose first that $\tau = \xi_u$, for some $u \in T_G$. In this case $w = v \circ u$, so $w \circ u = (v \circ u) \circ u = v$. Moreover, since $v \circ u \neq v$ and $w \circ u \neq w$, it holds $v \sim_{G^\S} u$ and $w \sim_{G^\S} u$. The connectedness of $G^\S$ follows by using induction over the minimum length of τ as reduced word in D_G. Let us prove now that $G^\S$ is locally dissymmetric. Let v, $w \in T_G$. If $N_{G^\S}(v) = N_{G^\S}(w)$, then $v \nsim_{G^\S} w$. Let us prove now that $S := \{v,\, w\}$ is a block for D_G. For this, let $u \in T_G$ such that $v \sim_{G^\S} u$ and $w \sim_{G^\S} u$. Then, in this case, $v \circ u \neq v$ and $w \circ u \neq w$. Moreover, since $v \circ u \sim_{G^\S} v$ and $w \circ u \sim_{G^\S} w$, then $v \circ u \neq w$ and $w \circ u \neq v$. Then, $\xi_u S \cap S = \emptyset$. Now let $u \in T_G$ such that $v \nsim_{G^\S} u$ and $w \nsim_{G^\S} u$. In this case $v \circ u = v$ and $w \circ u = w$, so $\xi_u S = S$. This proves that S is a block for D_G. Since D_G is primitive over T_G, then $|S| = 1$ and thus $v = w$.

□

Usually, K_G is not a commutative group. In the next result we provide a necessary and sufficient condition so that K_G is abelian.

Theorem 4.2.19. *Let $G^\S$ be a connected and locally dissymmetric graph. Then, K_G is an abelian group if and only if there exists $w \in T_G$ such that $K_G = \{\xi_u\xi_w : u \in T_G\}$.*

Proof. Suppose that $K_G = \{\xi_u\xi_w : u \in T_G\}$, where w is a fixed element of T_G. We will show that K_G is an abelian group. Let $\xi_u\xi_w \in K_G$. Then, it results that the map $\xi_u\xi_w \mapsto \xi_w^{-1}\xi_u\xi_w\xi_w = (\xi_u\xi_w)^{-1}$ is an automorphism of K_G mapping an element to its inverse. This ensures that K_G is an abelian group.
On the other hand, suppose that K_G is an abelian group. Let us consider the following map

$$F : T_G \to F_G \tag{4.17}$$
$$u \mapsto \xi_u.$$

It is obviously surjective. We show now that F is also injective. In fact, let u, $v \in T_G$ such that $\xi_u(w) = \xi_v(w)$, that is $w \circ u = w \circ v$, for any $w \in T_G$. Then, $w \notin N_{G^\S}(u)$ if and only if $w \notin N_{G^\S}(v)$, so $N_{G^\S}(u) = N_{G^\S}(v)$. This implies that $u = v$ because $G^\S$ is locally dissymmetric and F is thus injective.

Let now $v \in T_G$ be a fixed element. Then, for any $u \in T_G$ we define

$$u_v^k := \begin{cases} (\xi_u\xi_v)^i(v) & \text{if } \ k = 2i, \\ (\xi_u\xi_v)^i(u) & \text{if } \ k = 2i+1. \end{cases} \tag{4.18}$$

Since $\xi_u\xi_v$ has finite order in K_G, there exists a least positive integer n such that $u_v^n = v$. We also observe that

$$\xi_v(\xi_u\xi_v)^{-i} = (\xi_u\xi_v)^i\xi_v. \tag{4.19}$$

In fact, the subgroup of D_G spanned by the elements ξ_u, ξ_v is dihedral since $\langle \xi_u, \xi_v\rangle = \langle \xi_v, \xi_u\xi_v\rangle$ and it results

- $\xi_v^2 = Id$;
- There exists an integer j such that $(\xi_u\xi_v)^j = Id$;
- $\xi_v(\xi_u\xi_v)\xi_v = (\xi_u\xi_v)^{-1}$.

Hence, in particular, (4.19) holds.
By using (4.18), (4.19) and (4.12), it is easy to prove that

$$\xi_{u_v^k} = (\xi_u\xi_v)^k\xi_v \tag{4.20}$$

and, by means of the latter relation, we also conclude that, if $j < 2k$, then

$$\xi_{u_v^k}(u_v^j) = u_v^{2k-j}. \tag{4.21}$$

Now we prove that the smallest integer n such that $u_v^n = v$ is an odd integer for any $u \in T_G$. Suppose by contradiction that there exists an element $u \in T_G$ such that n is the least positive integer for which $u_v^n = v$, with $n = 2k$. Let l be the order of $\xi_u\xi_v$ in K_G, then by (4.20) $\xi_{u_v^l} = (\xi_u\xi_v)^l\xi_v$, hence $\xi_{u_v^l} = \xi_v$ so, by injectivity of F, $u_v^l = v$ and thus $n \leq l$. We also observe that, again by (4.20), we also have $(\xi_u\xi_v)^n = \xi_{u_v^n}\xi_v = \xi_v\xi_v = Id$, thus $l|n$ and this forces $l = n$. Therefore there exists $w \in T_G$ such that $(\xi_u\xi_v)^k(w) \neq w$.

Let us prove, by using that K_G is an abelian group, that $\xi_{\xi_u\xi_v(w)}\xi_w = (\xi_u\xi_v)^2$. In fact, by (4.12),

$$\xi_{\xi_u\xi_v(w)} = (\xi_u\xi_v)\xi_w(\xi_u\xi_v)^{-1} = (\xi_u\xi_v)\xi_w(\xi_v\xi_u),$$

so, by using the commutative property on K_G,

$$\xi_{\xi_u\xi_v(w)}\xi_w = \xi_u\xi_v(\xi_w\xi_v)(\xi_u\xi_w) = \xi_u\xi_v(\xi_u\xi_w)(\xi_w\xi_v) = (\xi_u\xi_v)^2.$$

It follows that k is the order of $\xi_{\xi_u\xi_v(w)}\xi_w$ in K_G. Moreover, it is easy to see that

$$(\xi_u\xi_v)^j(w) = \begin{cases} (\xi_{\xi_u\xi_v(w)}\xi_w)^i(w) & \text{if } \ j = 2i \\ (\xi_{\xi_u\xi_v(w)}\xi_w)^i(\xi_u\xi_v(w)) & \text{if } \ j = 2i+1. \end{cases}$$

Thus it follows that $(\xi_u\xi_v(w))_w^j = (\xi_u\xi_v)^j(w)$. By (4.20), it holds that

$$\xi_{(\xi_u\xi_v(w))_w^k} = (\xi_{\xi_u\xi_v(w)}\xi_w)^k\xi_w = \xi_w.$$

By the injectivity of F, we observe that $(\xi_u\xi_v)^k(w) = (\xi_u\xi_v(w))^k_w = w$, by contradicting our assumption. Hence n cannot be an even integer.
We finally show that $K_G = \{\xi_u\xi_w : u \in T_G\}$, where $w \in T_G$ is a fixed element. Let $u, v \in T_G$ two fixed elements such that $u^n_v = v$ and set $n = 2k+1$. Then, $(\xi_u\xi_v)^k(u) = v$, hence $(\xi_u\xi_v)^{k+1}(v) = u$. As direct consequence, we have $u^{2k+2}_v = u$. In order to show the claim, we must see that any composition and any inverse can be expressed as elements of $\{\xi_u\xi_w : u \in T_G\}$.
Let $w \in T_G$. Then, there exists $z \in T_G$ such that

$$\xi_w\xi_v\xi_u = \xi_z \tag{4.22}$$

In fact, $\xi_w\xi_v\xi_u = \xi_w\xi_v\xi_{u^{2k+2}}$ which, by means of (4.12) and (4.19), becomes $(\xi_u\xi_v)^{-(k+1)}\xi_w(\xi_u\xi_v)$, i.e., $\xi_{(\xi_u\xi_v)^{k+1}(w)}$. Therefore, we set $z = (\xi_u\xi_v)^{k+1}(w)$. In particular, $\xi_w\xi_v\xi_u\xi_v = \xi_z\xi_v$. Moreover, it results that

$$(\xi_u\xi_v)^{-1} = \xi_{u^{m-1}_v}\xi_v, \tag{4.23}$$

where m is the order of $\xi_u\xi_v$ in K_G. So $K_G = \{\xi_u\xi_w : u \in T_G\}$, where $w \in T_G$ is a fixed element. The theorem is thus proved. □

In reference to the formalism introduced in (4.18) we use the following terminology.

Definition 4.2.20. Let $u, v \in T_G$. We call *order* of u *with respect* to v the smallest positive integer n such that $u^n_v = v$.

In next result, we establish some structural properties of the group K_G when $G^\S$ is a connected locally dissymmetric graph.

Theorem 4.2.21. *Let $G^\S$ be connected and locally dissymmetric. Then:*
(a) K_G is the unique subgroup of D_G that satisfies the two following properties:

- *(i) $K_G \trianglelefteq D_G$;*
- *(ii) For any $\{Id_{T_G}\} \subsetneqq H \trianglelefteq D_G$ it results $K_G \subseteq H$.*

(b) K_G is either a simple group or it is the direct product of two its non-trivial normal conjugate subgroups.

Proof. (a): For the part (i), since $\xi_w^{-1}(\xi_u\xi_v)\xi_w = \xi_w^{-1}\xi_u\xi_w\xi_w^{-1}\xi_v\xi_w = \xi_{u\circ w}\xi_{v\circ w}$, we deduce that K_G is a normal group. We prove now part (ii). Let $H \neq \{Id_{T_G}\}$ be a normal subgroup of D_G. For any $v \in T_G$ we consider the H-orbit $Hv := \{h(v) : h \in H\}$ of v in T_G. Then, since $H \neq \{Id_{T_G}\}$ there exists a vertex $v_0 \in T_G$ such that $|Hv_0| > 1$. We show now that the H-orbit Hv_0 is a block for the group D_G.

To this aim, we assume that $\sigma \in D_G$ and $\sigma(Hv_0) \cap Hv_0 \neq \emptyset$, so that there is an element

$$w_0 = h_1(v_0) = \sigma h_2(v_0) \in Hv_0 \cap \sigma(Hv_0),$$

for some $h_1, h_2 \in H$. We must prove then that $\sigma(Hv_0) = Hv_0$.
Let first $v = h_3(v_0) \in Hv_0$, for some $h_3 \in H$. Then,

$$v = h_3h_1^{-1}(w_0) = h_3h_1^{-1}\sigma h_2(v_0) = \sigma(\sigma^{-1}h_3\sigma)(\sigma^{-1}h_1^{-1}\sigma)h_2(v_0) \in \sigma(Hv_0)$$

because H is normal in D_G. Hence $Hv_0 \subseteq \sigma(Hv_0)$.
On the other hand, let now $v = \sigma h_4(v_0) \in \sigma(Hv_0)$, for some $h_4 \in H$. Then

$$v = \sigma h_4 h_2^{-1} \sigma^{-1}(w_0) = (\sigma(h_4 h_2^{-1})\sigma^{-1}) h_1(v_0) \in Hv_0$$

because H is normal in D_G. Hence $\sigma(Hv_0) \subseteq Hv_0$. This shows that Hv_0 is a block for D_G. By Theorem 4.2.18, D_G is primitive over T_G, so any block for D_G is a singleton or the whole T_G. In our case, it must results that $Hv_0 = T_G$. In particular, we observe that $Hv_0 = Hv$ for any $v \in T_G$. Thus H is transitive over T_G. In other terms, if $u, v \in T_G$, there exists $\rho \in H$ such that $\rho(v) = u$. By (4.12), we have:

$$\xi_u \xi_v = \xi_u^{-1} \rho^{-1} \xi_u \rho \in H. \tag{4.24}$$

So $K_G \subseteq H$, i.e. K_G is the minimum non-trivial normal subgroup of D_G. Clearly, K_G is the unique subgroup of D_G satisfying both (i) and (ii).
(b): Suppose that K_G is not a simple group, we want to show that it can be expressed as the direct product of two simple normal subgroups $N, M \neq \{Id_{T_G}\}$ conjugate in D_G, i.e. $K_G \cong M \times N$. Since K_G is not simple, there exists $\{Id_{T_G}\} \subsetneqq N \triangleleft K_G$. For any $u, v \in T_G$, we have that $\xi_u, \xi_v \in D_G$ and $\xi_u \xi_v \in K_G$. Since N is normal in K_G we have $(\xi_v \xi_u)^{-1} N (\xi_v \xi_u) = N$, therefore we deduce that

$$\xi_u N \xi_u = \xi_v N \xi_v, \tag{4.25}$$

since $\xi_u^2 = \xi_v^2 = Id_{T_G} = Id_{T_G}$. Let $M := \xi_z N \xi_z$ for some, and thus for all, $z \in T_G$. We have to verify that $M \cap N = \{Id_{T_G}\}$ and that MN is a normal subgroup of D_G contained in K_G. In order to prove that $M \cap N = \{Id_{T_G}\}$, we firstly show that $M \cap N$ is a normal subgroup of D_G. Let then $\phi \in M \cap N$ and $\xi_v \in D_G$. Therefore, there exists $\phi' \in N$ such that $\phi = \xi_z \phi' \xi_z$. Then, $\xi_v^{-1} \phi \xi_v = (\xi_z \xi_v)^{-1} \phi' (\xi_z \xi_v) \in N$ because N is normal in K_G. On the other hand, by taking $u = z$ in (4.25), it follows that $\xi_v^{-1} \phi \xi_v = \xi_v \phi \xi_v = \xi_z \phi'' \xi_z \in M$, where $\phi'' \in N$. Thus $M \cap N$ is normal in D_G. Now, if $M \cap N \neq \{Id_{T_G}\}$, by (ii) of the previous part (a), we would have $K_G \subseteq M \cap N$, that is, $K_G = M \cap N$, that is in contrast with the hypothesis $N \subsetneqq K_G$. So, it must be necessarily $M \cap N = \{Id_{T_G}\}$. We observe that $M \subseteq K_G$: in fact K_G consists of all the automorphisms of D_G given by the product of an even number of elements of F_G, and any element of M is also given by products of an even number of elements of F_G. So, we conclude that $M \subseteq K_G$. We now prove that $M \trianglelefteq K_G$. In fact, $\xi_u \xi_v M \xi_v \xi_u = \xi_u \xi_v \xi_v N \xi_v \xi_v \xi_u = \xi_u N \xi_u = M$. We clearly deduce that MN is a subgroup contained in K_G. We claim that $MN \triangleleft D_G$. In fact, $\xi_v MN \xi_v = \xi_v \xi_v N \xi_v N \xi_v = N \xi_v N \xi_v = NM$. Hence, by (ii) of previous part (a), we have that $K_G \subseteq MN$, but since $MN \subseteq K_G$, we deduce that $MN = K_G$. So, the thesis follows. ☐

4.2.4 Solvability of K_G

In this subsection, we provide a sufficient condition on T_G in order to ensure that K_G is a solvable group when the odd extension $G^{\S}$ is connected and locally dissymmetric.

Definition 4.2.22. We say that a delineated subspace T of $\hat{G}$ is *transitively spanned* if for any $u, v \in T$ there exists $\xi_w \in F_T$, for some $w \in T$, such that $\xi_w(u) = v$.

Any subset of a transitively spanned is also transitively spanned.

Lemma 4.2.23. *Let $T_1 \subseteq T_2 \subseteq V(\hat{G})$ be two delineated subspaces of $\hat{G}$. If T_2 is transitively spanned, then also T_1 is.*

Proof. Let u, $v \in T_1$. If $u = v$, then by taking $w = u$, it holds $\xi_w(u) = v$ and clearly $w \in T_1$. Let $u \neq v$. Since u, $v \in T_2$, there exists $x \in T_2$ such that $\xi_x(u) = v$. Thus, $x \sim_{\hat{G}} u$, so $u \circ x = x \circ u$. If we set $w = v \circ u \in T_1$, then we obtain

$$u \circ w = u \circ (v \circ u) = u \circ ((x \circ u) \circ u) = u \circ x = v.$$

□

Now, we want to find an algebraic condition on the group K_G ensuring that T_G is transitively spanned. In order to do this, we need some preliminary classical notions and results:

Definition 4.2.24. [89] Let H be a group. We denote by M_H the core of H, i.e. the maximal normal subgroup of odd order in H. Furthermore, we denote by $Z^*(H)$ the subgroup of H containing M_H such that $Z^*(H)/M_H = Z(H/M_H)$, where $Z(H/M_H)$ denotes the center of H/M_H.

Theorem 4.2.25. *[89] Let τ be an involution of a group H. Then, $\tau \in Z^*(H)$ if and only if for any $g \in H$ the element $g^{-1}\tau^{-1}g\tau$ has odd order.*

In next result we provide a necessary and sufficient on K_G condition so that T_G is transitively spanned.

Proposition 4.2.26. *Let $G^{\S}$ be connected and locally dissymmetric. Then, following conditions are equivalent:*

- (i) *For any $u, v \in T_G$, u has odd order with respect to v;*
- (ii) *T_G is transitively spanned;*
- (iii) *K_G has odd order.*

Proof. $(i) \iff (ii)$: Suppose that for any $u, v \in T_G$, the order of u with respect to v is odd. Fix two vertices $u, v \in T_G$ such that the order of $u \in T_G$ with respect to v is $n = 2k+1$. Hence, if we take u_v^{k+1}, by (4.21), we have $\xi_{u_v^{k+1}}(u) = v$. Hence u is mapped to v. By the arbitrariness of u and v, we conclude that T_G is transitively spanned.
On the other hand, assume that T_G is transitively spanned and suppose by contradiction that there are two elements $u, v \in T_G$ such that the order of u with respect to v is even, i.e. $n = 2k$. Let us note that the delineated subspace T consisting of the distinct elements of $\{v, u, \ldots, u_v^{n-1}\}$ is transitively spanned by Lemma 4.2.23. By (4.21), we have $\xi_{u_v^k}(u_v^j) = u_v^{n-j}$, for any $0 \leq j \leq n-1$. Now, since the order n coincides with the order of $\xi_u\xi_v$ in K_G, it is immediate to see that $\xi_{u_v^k}(u_v^j) = u_v^{n-j} = u_v^{-j} = \xi_v(u_v^j)$. Since the restriction to T of the map F defined in (4.17) is also bijective, we deduce that $\xi_{u_v^k} = \xi_v$ and $u_v^k = v$ and this contradicts the minimality of n.
$(ii) \iff (iii)$: Suppose that K_G has odd order and let $u, v \in T_G$. In the proof of Theorem 4.2.19 we showed that the order m of $\xi_u\xi_v$ in K_G coincides with the order of u with respect to

v. Let $m = 2k+1$. Then, by (4.21), it follows that $\xi_{u_v^{k+1}}(u) = v$. By the arbitrariness of u and v, we conclude that T_G is transitively spanned.
On the other hand, let T_G be transitively spanned, hence, for any $u, v \in T_G$, there exists $w \in T_G$ such that $\xi_w(u) = v$. For any $\xi_u \in F_G$ we have $F_G \cap S_{D_G}(\xi_u) = \{\xi_u\}$, where $S_{D_G}(\xi_u)$ denotes the stabilizer of ξ_u in D_G. Now, since the group spanned by the single element ξ_u fixes only $u \in T_G$, we deduce that $|F_G| = |T_G| = 2k+1$, for some integer k. Let $H := \langle \xi_u, \xi_v \rangle$ be the group spanned by two fixed elements for some $u, v \in T_G$. Let us consider the set $\xi_u^H := \{\rho^{-1}\xi_u\rho,\ \rho \in H\}$. The group H acts by conjugation on ξ_u^H and the only element in ξ_u^H fixed by ξ_u is ξ_u, then its cardinality must be odd. In particular, also ξ_v fixes only v, hence we obtain

$$\xi_v = (\xi_u\xi_v)^{-j}\xi_u(\xi_u\xi_v)^j,$$

for some integer j. But, since H is dihedral, we deduce that

$$\xi_v = (\xi_u\xi_v)^{-j}\xi_u(\xi_u\xi_v)^j = \xi_u(\xi_u\xi_v)^{2j}, \tag{4.26}$$

i.e. $(\xi_u\xi_v)^{2j-1} = Id$. Thus $\xi_u\xi_v$ has odd order. It is immediate to see that $[D_G : K_G] = 2$. Let D'_G be the derived subgroup of D_G. We claim that $K_G = D'_G$. Since T_G is transitively spanned, for any u, $v \in T_G$ there exists $\xi_w \in F_G$ such that $\xi_v = \xi_{u \circ w} = \xi_w\xi_u\xi_w$, so $\xi_u\xi_v$ is a commutator in D_G and $K_G \subseteq D'_G$. On the other hand, it is easy to prove that $D'_G \subseteq K_G$ since any commutator in D_G is an element of K_G. In particular, for any $\sigma \in D_G$, it results that the element $\sigma^{-1}\xi_u^{-1}\sigma\xi_u$ is in K_G, then it has odd order and, by Theorem 4.2.25, we deduce that $\xi_u \in Z^*(D_G)$ for any $u \in T_G$, i.e. $D_G = Z^*(D_G)$. Therefore, by Definition 4.2.24, D_G/M_G is an abelian group and since $D'_G = K_G$ is the smallest normal subgroup such that D_G/K_G is abelian, we deduce that M_G contains K_G. Thus, we have shown that K_G has odd order. □

Theorem 4.2.27. *Let $G^\S$ be a connected and locally dissymmetric graph such that T_G is transitively spanned. Then, K_G is a solvable group.*

Proof. It is an immediate consequence of Proposition 4.2.26 and of the classical theorem of Feit-Thompson. □

Chapter 5

The Pre-Order Induced by Symmetry Relation

By means of the aforementioned $\mathfrak{P}$-maximum partitioner operator, in this chapter we investigate the set map $\Gamma_{\mathfrak{P}}$ as the preorder $\geq_{\mathfrak{P}}$ (reflexive and transitive binary relation) on $\mathcal{P}(\Omega)$ which is exactly the symmetrization of the above indistinguishability relation $\approx_{\mathfrak{P}}$.

5.1 Symmetry Transmission on Pairings

In this section, we focus our attention to the map $\Gamma_{\mathfrak{P}} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U)$ so defined:

$$\Gamma_{\mathfrak{P}}(A, B) := \{u \in U : [u]_A \subseteq [u]_B\}. \tag{5.1}$$

Furthermore, in the finite case, we can also define the *symmetry transmission measure* $\gamma_{\mathfrak{P}}$ as follows

$$\gamma_{\mathfrak{P}}(A, B) := \frac{|\Gamma_{\mathfrak{P}}(A, B)|}{|U|}. \tag{5.2}$$

Since we have that

$$M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A') \iff \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A') \iff \Gamma_{\mathfrak{P}}(A, A') = U, \tag{5.3}$$

for all $A, A' \in \mathcal{P}(\Omega)$, we introduce the binary relation $\geq_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ defined by

$$A \geq_{\mathfrak{P}} A' :\iff M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A'), \tag{5.4}$$

for any $A, A' \in \mathcal{P}(\Omega)$. We say that A is $\mathfrak{P}$-*symmetrically finer* than A' whenever $A \geq_{\mathfrak{P}} A'$. Let us observe that

$$A \geq_{\mathfrak{P}} B \iff \Gamma_{\mathfrak{P}}(A, B) = U. \tag{5.5}$$

Some basic properties of the map $\Gamma_{\mathfrak{P}}$ are described in the next proposition.

Proposition 5.1.1. *Let A, A', B, $B' \in \mathcal{P}(\Omega)$. Then, the following hold:*
(i) If $A \subseteq A'$ and $B \subseteq B'$, then

$$\Gamma_{\mathfrak{P}}(A, B) \subseteq \Gamma_{\mathfrak{P}}(A', B) \text{ and } \Gamma_{\mathfrak{P}}(A, B) \supseteq \Gamma_{\mathfrak{P}}(A, B').$$

(ii) If $\mathfrak{P}$ is a finite pairing, then $A \geq_{\mathfrak{P}} A' \iff \gamma_{\mathfrak{P}}(A, A') = 1$.

Proof. (i): Let $A \subseteq A'$, then $\pi_{\mathfrak{P}}(A') \preceq \pi_{\mathfrak{P}}(A)$. This proves that $\Gamma_{\mathfrak{P}}(A,B) \subseteq \Gamma_{\mathfrak{P}}(A',B)$. Let $u \in \Gamma_{\mathfrak{P}}(A,B)$, then $[u]_A \subseteq [u]_B$ but since $[u]_{A'} \subseteq [u]_A$, we deduce that $[u]_{A'} \subseteq [u]_B$, i.e. $u \in \Gamma_{\mathfrak{P}}(A',B)$.
On the other hand, let $B \subseteq B'$ and $u \in \Gamma_{\mathfrak{P}}(A,B')$. Then, $[u]_A \subseteq [u]_{B'} \subseteq [u]_B$, hence $u \in \Gamma_{\mathfrak{P}}(A,B)$.
(ii): It follows immediately by (5.2) and by (5.3). □

We can interpret the global symmetry relation $\approx_{\mathfrak{P}}$ as the equivalence relation induced by the preorder $\geq_{\mathfrak{P}}$. Indeed, by $(viii)$ of Theorem 2.2.1 and (5.4) it follows that

$$A \approx_{\mathfrak{P}} A' \iff A \geq_{\mathfrak{P}} A' \text{ and } A' \geq_{\mathfrak{P}} A,$$

consequently, if we consider the preordered set $\mathbb{H}(\mathfrak{P}) := (\mathcal{P}(\Omega), \geq_{\mathfrak{P}})$, we have that

$$\mathbb{H}(\mathfrak{P})/\approx_{\mathfrak{P}} = \mathbb{G}(\mathfrak{P}) \cong \mathbb{M}(\mathfrak{P}). \tag{5.6}$$

We will show that, in addition to the reflexivity and transitivity, there is a property that uniquely characterize the above binary relation $\geq_{\mathfrak{P}}$. We call *union additivity* this property, which can be expressed as follows:

$$A \geq_{\mathfrak{P}} A' \text{ and } A \geq_{\mathfrak{P}} A'' \implies A \geq_{\mathfrak{P}} A' \cup A'' \tag{5.7}$$

In order to investigate in a deeper way the link between the closure operator $M_{\mathfrak{P}}$ and the preorder $\geq_{\mathfrak{P}}$ we introduce the following ordered subset pair family

$$\mathcal{G}(\mathfrak{P}) := \{(A,B) \in \mathcal{P}(\Omega)^2 : \Gamma_{\mathfrak{P}}(A,B) = U\} \tag{5.8}$$

and we call any pair $(A,B) \in \mathcal{G}(\mathfrak{P})$ a *semi-symmetry* of $\mathfrak{P}$.

In the next result we establish eight basic properties that are satisfied by the family $\mathcal{G}(\mathfrak{P})$. Next, in order to show a deep link between closure operators and specific families of ordered pairs of elements, we use seven of this properties as potential properties for generic families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$.

Theorem 5.1.2. *Let $X, Y, Z, W, T \in \mathcal{P}(\Omega)$. The family $\mathcal{G}(\mathfrak{P})$ satisfies the following properties:*

$(I1)$ *if $X \supseteq Y$ then $(X,Y) \in \mathcal{G}(\mathfrak{P})$;*

$(I2)$ *if $(X,Y) \in \mathcal{G}(\mathfrak{P})$ then $(X \cup Z, Y) \in \mathcal{G}(\mathfrak{P})$;*

$(I3)$ *if $(X,Y) \in \mathcal{G}(\mathfrak{P})$ and $(X,Z) \in \mathcal{G}(\mathfrak{P})$ then $(X, Y \cup Z) \in \mathcal{G}(\mathfrak{P})$;*

$(I4)$ *if $(X, Y \cup Z) \in \mathcal{G}(\mathfrak{P})$ then $(X,Y) \in \mathcal{G}(\mathfrak{P})$;*

$(I5)$ *if $(X,Y) \in \mathcal{G}(\mathfrak{P})$ and $(Y,Z) \in \mathcal{G}(\mathfrak{P})$ then $(X,Z) \in \mathcal{G}(\mathfrak{P})$;*

$(I6)$ *if $(X,Y) \in \mathcal{G}(\mathfrak{P})$ and $(Y \cup Z, W) \in \mathcal{G}(\mathfrak{P})$ then $(X \cup Z, W) \in \mathcal{G}(\mathfrak{P})$;*

$(I7)$ *if $(X, Y \cup Z) \in \mathcal{G}(\mathfrak{P})$ and $(Z, T \cup W) \in \mathcal{G}(\mathfrak{P})$ then $(X, Y \cup Z \cup T) \in \mathcal{G}(\mathfrak{P})$.*

Proof. $(I1)$: Since $X \supseteq Y$, for any $\mathfrak{P} \in PAIR(\Omega)$ we have that $\Gamma_{\mathfrak{P}}(X,Y) = U$.
$(I2)$: Let $\mathfrak{P} \in PAIR(\Omega)$ be such that $\Gamma_{\mathfrak{P}}(X,Y) = U$. Since $X \subseteq X \cup Z$, we deduce that $[u]_{X\cup Z} \subseteq [u]_X \subseteq [u]_Y$ for any $u \in U$, therefore $\Gamma_{\mathfrak{P}}(X \cup Z, Y) = U$.
$(I3)$: Let $\mathfrak{P} \in PAIR(\Omega)$ be such that $\Gamma_{\mathfrak{P}}(X,Y) = \Gamma_{\mathfrak{P}}(X,Z) = U$. Hence, $[u]_X \subseteq [u]_Y$ and $[u]_X \subseteq [u]_Z$ for any $u \in U$. Therefore, $[u]_X \subseteq [u]_Y \cap [u]_Z$ for any $u \in U$ or, equivalently, $\pi_{\mathfrak{P}}(X) \preceq \pi_{\mathfrak{P}}(Y) \wedge \pi_{\mathfrak{P}}(Z)$ and the latter coincides with $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(Y \cup Z)) = \pi_{\mathfrak{P}}(Y \cup Z)$. Therefore, $[u]_X \subseteq [u]_{Y\cup Z}$ for any $u \in U$, i.e. $\Gamma_{\mathfrak{P}}(X, Y \cup Z) = U$.
$(I4)$: Let $\mathfrak{P} \in PAIR(\Omega)$ be such that $\Gamma_{\mathfrak{P}}(X, Y \cup Z) = U$. Then, $[u]_X \subseteq [u]_{Y\cup Z} \subseteq [u]_Y$ for any $u \in U$. Therefore $\Gamma_{\mathfrak{P}}(X,Y) = U$.
$(I5)$: Let $\mathfrak{P} \in PAIR(\Omega)$ be such that $\Gamma_{\mathfrak{P}}(X,Y) = \Gamma_{\mathfrak{P}}(Y,Z) = U$. Then, $[u]_X \subseteq [u]_Y \subseteq [u]_Z$ for any $u \in U$.
$(I6)$: Let $\mathfrak{P} \in PAIR(\Omega)$ be such that $\Gamma_{\mathfrak{P}}(X,Y) = \Gamma_{\mathfrak{P}}(Y \cup Z, W) = U$. Then, $[u]_X \subseteq [u]_Y$ and $[u]_{Y\cup Z} \subseteq [u]_W$ for any $u \in U$. Thus, we have that $[u]_{X\cup Z} = [u]_X \cap [u]_Z$ and $[u]_{Y\cup Z} = [u]_Y \cap [u]_Z$ for any $u \in U$. Since $[u]_X \cap [u]_Z \subseteq [u]_Y \cap [u]_Z = [u]_{Y\cup Z} \subseteq [u]_W$, so $\Gamma_{\mathfrak{P}}(X \cup Z, W) = U$.
$(I7)$: Let $\mathfrak{P} \in PAIR(\Omega)$ be such that $\Gamma_{\mathfrak{P}}(X, Y \cup Z) = \Gamma_{\mathfrak{P}}(Z, T \cup W) = U$. This implies that, for any $u \in U$, the following holds:

$$[u]_X \subseteq [u]_{Y\cup Z} \subseteq [u]_Z \subseteq [u]_{T\cup W} \subseteq [u]_T.$$

In other terms, if $u \equiv_X u'$ then $u \equiv_{Y\cup Z} u'$ and $u \equiv_T u'$ and this is equivalent to require that $F(u,a) = F(u,a')$ for any $a \in Y \cup Z \cup T$, i.e. $u \equiv_{Y\cup Z\cup T} u'$. By the arbitrariness of $u \in U$, we conclude that $\Gamma_{\mathfrak{P}}(X, Y \cup Z \cup T) = U$. □

5.2 Union Additive Relations

We consider now the conditions established in Proposition 5.1.2 as specific properties that can be satisfied by suitable families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$. To this regard, let $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ be a given family on Ω.

Some Conditions for a Generic Family $\mathcal{D}$ on $\mathcal{P}(\Omega)^2$. *Let $X, Y, Z, W, T \in \mathcal{P}(\Omega)$. We give the following possible axioms for $\mathcal{D}$:*

$(D1)$ if $X \supseteq Y$ then $(X,Y) \in \mathcal{D}$;

$(D2)$ if $(X,Y) \in \mathcal{D}$ then $(X \cup Z, Y) \in \mathcal{D}$;

$(D3)$ if $(X,Y) \in \mathcal{D}$ and $(X,Z) \in \mathcal{D}$ then $(X, Y \cup Z) \in \mathcal{D}$;

$(D4)$ if $(X, Y \cup Z) \in \mathcal{D}$ then $(X,Y) \in \mathcal{D}$;

$(D5)$ if $(X,Y) \in \mathcal{D}$ and $(Y,Z) \in \mathcal{D}$ then $(X,Z) \in \mathcal{D}$;

$(D6)$ if $(X,Y) \in \mathcal{D}$ and $(Y \cup Z, W) \in \mathcal{D}$ then $(X \cup Z, W) \in \mathcal{D}$;

$(D7)$ if $(X, Y \cup Z) \in \mathcal{D}$ and $(Z, T \cup W) \in \mathcal{D}$ then $(X, Y \cup Z \cup T) \in \mathcal{D}$;

$(D8)$ $(X,Y) \in \mathcal{D}$ if and only if $(M_{\mathfrak{P}}(X), M_{\mathfrak{P}}(Y)) \in \mathcal{D}$.

We have then the following result.

Proposition 5.2.1. *The following conditions are equivalent:*

(i) $\mathcal{D}$ *satisfies the properties* $(D1)$*,* $(D3)$ *and* $(D5)$*;*

(ii) $\mathcal{D}$ *satisfies the properties* $(D1)$*,* $(D2)$ *and* $(D6)$*;*

(iii) $\mathcal{D}$ *satisfies the properties* $(D1)$*,* $(D4)$ *and* $(D7)$*;*

(iv) $\mathcal{D}$ *satisfies all properties* $(D1)$*–*$(D7)$*.*

Proof. $(i) \implies (ii)$: We have to derive the properties $(D1)$, $(D2)$ and $(D6)$ starting from $(D1)$, $(D3)$ and $(D5)$. For, let us consider $(X, Y) \in \mathcal{D}$. By $(D1)$, also $(X \cup Z, X) \in \mathcal{D}$ and, by $(D5)$, it follows that $(X \cup Z, X), (X, Y) \in \mathcal{D}$ implies $(X \cup Z, Y) \in \mathcal{D}$.
On the other hand, let us prove that $(D6)$ derives by $(D1), (D2), (D3), (D5)$. To this regard, assume that $(X, Y), (Y \cup Z, W) \in \mathcal{D}$. By $(D2)$, it results that $(X \cup Z, Y) \in \mathcal{D}$ and, by $(D1)$, this turns out to be true also for the pairs (Z, Z) and $(X \cup Z, Z)$. Thus, by $(D3)$ applied to the pairs $(X \cup Z, Y), (X \cup Z, Z)$, we have that $(X \cup Z, Y \cup Z) \in \mathcal{D}$ and, by $(D5)$ applied to the pairs $(X \cup Z, Y \cup Z), (Y \cup Z, W)$ we have that $(X \cup Z, W) \in \mathcal{D}$. So, the claim has been shown.
$(ii) \implies (iii)$: Let us derive the properties $(D1)$, $(D4)$ and $(D7)$ by using $(D1)$, $(D2)$ and $(D6)$. To this regard, let $(X, Y \cup Z) \in \mathcal{D}$. By $(D1)$, $(Y \cup Z, Y) \in \mathcal{D}$. Now, by $(D6)$ applied to the pairs $(X, Y \cup Z), ((Y \cup Z) \cup \emptyset, Y)$ we have that $(X, Y)\mathcal{D}$. This shows that $(D4)$ can be derived by $(D1)$, $(D2)$ and $(D6)$.
We now prove the claim for $(D7)$. Let now assume that $(X, Y \cup Z), (Z, T \cup W) \in \mathcal{D}$. By $(D2)$, $(Y \cup Z, T \cup W) \in \mathcal{D}$. Moreover, applying $(D6)$ to the pairs $(Y \cup Z, T \cup W), (T \cup W, T)$, we have that $(Y \cup Z, T) \in \mathcal{D}$. Again, we apply $(D6)$ to the pair $(X, Y \cup Z)(Y \cup Z, T)$, so $(X, T) \in \mathcal{D}$ and, by $(D6)$ applied to the pairs $(X, Y \cup Z, Y \cup Z \cup T, Y \cup Z \cup T)$, we deduce $(X \cup T, Y \cup Z \cup T) \in \mathcal{D}$. Finally, starting from the pairs $(X, T), (X \cup T, Y \cup Z \cup T)$, by $(D6)$ we obtain that $(X, Y \cup Z \cup T)$.
$(iii) \implies (iv)$: Let us derive the properties $(D2)$, $(D3)$,$(D5)$ and $(D6)$ by using $(D1)$, $(D4)$ and $(D7)$. For, let $(X, Y) \in \mathcal{D}$. We will prove $(D2)$. By $(D1)$, $(X \cup Z, X \cup Z) \in \mathcal{D}$, so, by applying $(D7)$ to the pairs $(X \cup Z, X \cup Z), (X, Y)$ we have that $(X \cup Z, X \cup Y \cup Z) \in \mathcal{D}$. Finally, by applying $(D4)$ to the latter pair, we deduce that $(X \cup Z, Y) \in \mathcal{D}$.
We now show that $(D6)$ can be derived by $(D1)$, $(D4)$ and $(D7)$. Let $(X, Y), (Y \cup Z, W) \in \mathcal{D}$. By $(D1)$, $(X \cup Z, X \cup Z) \in \mathcal{D}$ so, by $(D7)$ applied to the pairs $(X \cup Z, X \cup Z), (X, Y)$ we deduce that also $(X \cup Z, X \cup Y \cup Z) \in \mathcal{D}$. Again by $(D7)$, this time applied to $(X \cup Z, X \cup Y \cup Z)$ and $(Y \cup Z, W)$, we have that $(X \cup Z, X \cup Y \cup Z \cup W) \in \mathcal{D}$. Then, by $(D4)$, $(X \cup Z, W) \in \mathcal{D}$. Furthermore, $(D5)$ is nothing but $(D6)$ with $Z = \emptyset$.
We prove the claim for $(D3)$. We will use $(D1)$, $(D2)$ and $(D6)$ that have been already derived by $(D1), (D4)$ and $(D7)$. Assume $(X, Y), (X, Z) \in \mathcal{D}$. By $(D1)$, we also have $(Y \cup Z, Y \cup Z)$ and, by $(D6)$ applied to the pairs $(X, Y), (Y \cup Z, Y \cup Z)$, we deduce that $(X \cup Z, Y \cup Z) \in \mathcal{D}$. By applying again $(D6)$ to the pairs $(X, Z), (X \cup Z, Y \cup Z)$, it results that $(X, Y \cup Z) \in \mathcal{D}$. This shows the thesis.
$(iv) \implies (i)$: Obvious. $\square$

Definition 5.2.2. If $\mathcal{D}$ satisfies one of the equivalent conditions established in Proposition 5.2.1 we say that $\mathcal{D}$ is a *union additive family* on Ω. We denote by $UAF(\Omega)$ the set of all union additive families on Ω.

Since any $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ can be uniquely identified with the binary relation that it induces on $\mathcal{P}(\Omega)$, the conditions established in Proposition 5.1.2 can also be expressed in terms of conditions on binary relations on $\mathcal{P}(\Omega)$ as follows.

Some Conditions for a Generic Binary Relation $\geq$ on $\mathcal{P}(\Omega)$. *Let $\geq$ a binary relation on $\mathcal{P}(\Omega)$ and $X, Y, Z, W, T \in \mathcal{P}(\Omega)$. We give the following possible axioms for $\geq$:*

$(R1)$ if $X \supseteq Y$ then $X \geq Y$;

$(R2)$ if $X \geq Y$ then $X \cup Z \geq Y$;

$(R3)$ if $X \geq Y$ and $X \geq Z$ then $X \geq Y \cup Z$;

$(R4)$ if $X \geq Y \cup Z$ then $X \geq Y$;

$(R5)$ if $X \geq Y$ and $Y \geq Z$ then $X \geq Z$;

$(R6)$ if $X \geq Y$ and $Y \cup Z \geq W$ then $X \cup Z \geq W$;

$(R7)$ if $X \geq Y \cup Z$ and $Z \geq T \cup W$ then $X \geq Y \cup Z \cup T$;

$(R8)$ $X \geq Y$ if and only if $M_{\mathfrak{P}}(X) \geq M_{\mathfrak{P}}(Y)$.

Then, Proposition 5.2.1 can be reformulated in terms of equivalences between conditions on binary relations on Ω as follows.

Proposition 5.2.3. *The following conditions are equivalent for any binary relation $\geq$ on Ω.*

(i) $\geq$ *satisfies the properties* $(R1)$, $(R3)$ *and* $(R5)$*;*

(ii) $\geq$ *satisfies the properties* $(R1)$, $(R2)$ *and* $(R6)$*;*

(iii) $\geq$ *satisfies the properties* $(R1)$, $(R4)$ *and* $(R7)$*;*

(iv) $\geq$ *satisfies all properties* $(R1)$– $(R7)$*.*

Definition 5.2.4. We call *union additive relation* on Ω a binary relation $\geq$ on $\mathcal{P}(\Omega)$ that satisfies one of the equivalent conditions established in Proposition 5.2.3. We denote by $UAR(\Omega)$ the set of all union additive relations on Ω.

It is clear that the notions of union additive family and union additive relation are equivalent. In fact, if $\geq$ a union additive relation on Ω then $\mathcal{D}_{\geq} := \{(X, Y) \in \mathcal{P}(\Omega)^2 : X \geq Y\}$ is a union additive family on Ω. On the other hand, if $\mathcal{D}$ is a union additive family on Ω, the binary relation $X \geq_{\mathcal{D}} Y :\iff (X, Y) \in \mathcal{D}$ is a union additive relation on Ω. Moreover, the maps $\mathcal{D} \in UAF(\Omega) \mapsto \geq_{\mathcal{D}} \in UAR(\Omega)$ and $\geq \in UAR(\Omega) \mapsto \mathcal{D}_{\geq} \in UAF(\Omega)$ are inverses each other. So that, in what follows we use the terms union additive relation and union additive

family in equivalent way.
Thus, by Proposition 5.1.2 it follows that $\geq_{\mathfrak{P}}$ is a union additive relation on Ω, or, equivalently, $\mathcal{G}(\mathfrak{P})$ is a union additive family on Ω.

In the following result we establish some first basic properties of the union additive relations.

Proposition 5.2.5. *(i) If $\{\mathcal{D}_i : i \in I\} \subseteq UAF(\Omega)$, then $\bigcap_{i \in I} \mathcal{D}_i \in UAF(\Omega)$.*
(ii) Let $\geq \in UAR(\Omega)$ and $A, B, C, D \in \mathcal{P}(\Omega)$. Then, the following facts hold:

(a) If $A \geq B$, $C \supseteq A$ and $B \supseteq D$, then $C \geq D$;

(b) If $A \geq B$ and $C \geq D$, then $A \cup C \geq B \cup D$.

Proof. (i): Let $Y \subseteq X \in \mathcal{P}(\Omega)$. Then, $(X, Y) \in \mathcal{D}$ for any $\mathcal{D} \in UAF(\Omega)$, so $(X, Y) \in \bigcap_{i \in I} \mathcal{D}_i$. Let $(X, Y), (Y, Z) \in \bigcap_{i \in I} \mathcal{D}_i$. Then, $(X, Y), (Y, Z) \in \mathcal{D}_i$ for any $i \in I$. Therefore $(X, Z) \in \mathcal{D}_i$ for any $i \in I$, i.e. $(X, Z) \in \bigcap_{i \in I} \mathcal{D}_i$. Finally, let $(X, Y), (X, Z) \in \bigcap_{i \in I} \mathcal{D}_i$, hence $(X, Y), (X, Z) \in \mathcal{D}_i$ for any $i \in I$. So, $(X, Y \cup Z) \in \mathcal{D}_i$ for any $i \in I$, i.e. $(X, Y \cup Z) \in \bigcap_{i \in I} \mathcal{D}_i$. This shows that $\bigcap_{i \in I} \mathcal{D}_i \in UAF(\Omega)$.
(ii): (a): By $(R1)$, it follows that $C \geq A$ and $B \geq D$. Moreover, by applying $(R5)$ on the three conditions $C \geq A$, $A \geq B$ and $B \geq D$, we conclude that $C \geq D$.
(b): Let us observe that $A \cup C \geq A \geq B$ and $A \cup C \geq C \geq D$. Moreover, we have that $B \cup D \geq B$ by $(R1)$. So, by applying $(R3)$ and then $(R5)$, we conclude that $A \cup C \geq B \cup D$. $\square$

It is natural to define the smallest union additive family on Ω containing $\mathcal{D}$ by setting

$$\mathcal{D}^+ := \bigcap \{\mathcal{E} \in UAF(\Omega) : \mathcal{D} \subseteq \mathcal{E}\} \tag{5.9}$$

Then, the following result holds.

Proposition 5.2.6. *$\mathcal{D}^+$ is the smallest union additive family on Ω containing $\mathcal{D}$ as a subfamily and the set operator $^+ : \mathcal{D} \in \mathcal{P}(\mathcal{P}(\Omega)^2) \mapsto \mathcal{D}^+ \in \mathcal{P}(\mathcal{P}(\Omega)^2)$ is a closure operator on $\mathcal{P}(\Omega)^2$.*

Proof. Let us observe that the set $\{\mathcal{E} \in UAF(\Omega) : \mathcal{D} \subseteq \mathcal{E}\}$ is non-empty since $\mathcal{P}(\Omega)^2 \in UAF(\Omega)$. Thus, by (i) of Proposition 5.2.5, it follows that $\mathcal{D}^+ \in UAF(\Omega)$. Moreover, it is obvious that $\mathcal{D}^+$ is the smallest union additive family on Ω containing $\mathcal{D}$.
We now prove that the set operator $^+ : \mathcal{D} \in \mathcal{P}(\mathcal{P}(\Omega)^2) \mapsto \mathcal{D}^+ \in \mathcal{P}(\mathcal{P}(\Omega)^2)$ is a closure operator on $\mathcal{P}(\Omega)^2$. Let $\mathcal{D} \in \mathcal{P}(\Omega)^2$. Then, $\mathcal{D} \subseteq \mathcal{D}^+$. On the other hand, if $\mathcal{D} \subseteq \mathcal{F}$, it is immediate to see that $\mathcal{D}^+ \subseteq \mathcal{F}^+$ since the union additive families containing $\mathcal{F}$ also contain $\mathcal{D}$. Finally, since $\mathcal{D}^+$ is a union additive family, we have $(\mathcal{D}^+)^+ = \mathcal{D}^+$ and the claim has been proved. $\square$

We call $\mathcal{D}^+$ the *union additive envelope* of $\mathcal{D}$. If $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$, we say that $\mathcal{D}$ and $\mathcal{D}'$ are *union additive equivalent* (denoted by $\mathcal{D} \simeq \mathcal{D}'$) if $\mathcal{D}^+ = (\mathcal{D}')^+$.

5.2.1 Union Additive Families and Closure Operators

In this subsection, given a fixed finite set Ω, we show (Theorem 5.2.8) that the notions of union additive family and closure operator are equivalent. To this aim we consider the set operator $c_{\mathcal{D}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined by

$$c_{\mathcal{D}}(X) := \bigcup \{Y \in \mathcal{P}(\Omega) : (X, Y) \in \mathcal{D}\}. \tag{5.10}$$

So that, for any $z \in \Omega$ we have that

$$z \in c_{\mathcal{D}}(X) \iff \exists Z \in \mathcal{P}(\Omega) : z \in Z \text{ and } (X, Z) \in \mathcal{D} \tag{5.11}$$

When $\mathcal{D}$ satisfies the conditions $(D1)$ and $(D3)$, another equivalent way to characterize the subset $c_{\mathcal{D}}(X)$ is the following.

Proposition 5.2.7. *If $\mathcal{D}$ satisfies the conditions $(D1)$ and $(D3)$, then $c_{\mathcal{D}}(X)$ is the largest subset W of Ω such that $(X, W) \in \mathcal{D}$.*

Proof. By $(D1)$, the subset family $\mathcal{F} := \{Y \in \mathcal{P}(\Omega) : (X, Y) \in \mathcal{D}\}$ is non-empty because it contains at least X. On the other hand, by means of repeated applications of $(D3)$ we deduce that $W := \bigcup \mathcal{F} \in \mathcal{F}$. Since $c_{\mathcal{D}}(X) = W$, the thesis follows. □

On the other hand, for any set operator $\sigma \in OP(\Omega)$ we also consider the subset family

$$\mathcal{C}_\sigma := \{(X, Y) \in \mathcal{P}(\Omega)^2 : \sigma(X) \supseteq Y\}. \tag{5.12}$$

Then, we obtain the following result.

Theorem 5.2.8. *(i) If $\mathcal{D} \in UAF(\Omega)$ then $c_{\mathcal{D}} \in CLOP(\Omega)$.*
(ii) If $\sigma \in CLOP(\Omega)$ then $\mathcal{C}_\sigma \in UAF(\Omega)$.
(iii) The map $\mathcal{D} \in UAF(\Omega) \mapsto c_{\mathcal{D}} \in CLOP(\Omega)$ is bijective and the map $\sigma \in CLOP(\Omega) \mapsto \mathcal{C}_\sigma$ is its inverse. That is, if $\mathcal{D} \in UAF(\Omega)$ and $\sigma \in CLOP(\Omega)$ we have that $\mathcal{C}_{c_{\mathcal{D}}} = \mathcal{D}$ and $c_{\mathcal{C}_\sigma} = \sigma$.

Proof. (i): Let $\mathcal{D} \in UAF(\Omega)$ and $X \in \mathcal{P}(\Omega)$. By $(D1)$ we have $(X, X) \in \mathcal{D}$, so that $X \subseteq c_{\mathcal{D}}(X)$.

Let now $Y \in \mathcal{P}(\Omega)$ and $X \subseteq Y$. Again by $(D1)$ we have $(Y, X) \in \mathcal{D}$. Let now $a \in c_{\mathcal{D}}(X)$. By (5.11) there exists $Z \in \mathcal{P}(\Omega)$ such that $(X, Z) \in \mathcal{D}$ and $a \in Z$. Then, by $(D5)$ we deduce that $(Y, Z) \in \mathcal{D}$, therefore by (5.11) it follows that $a \in c_{\mathcal{D}}(Y)$. Hence $c_{\mathcal{D}}(X) \subseteq c_{\mathcal{D}}(Y)$.

In order to show that $c_{\mathcal{D}}(c_{\mathcal{D}}(X)) = c_{\mathcal{D}}(X)$, let $W := c_{\mathcal{D}}(X)$ and $a \in c_{\mathcal{D}}(W)$. By (5.11) there exists $Z \in \mathcal{P}(\Omega)$ such that $(W, Z) \in \mathcal{D}$ and $a \in Z$. By Proposition 5.2.7 we have that $(X, W) \in \mathcal{D}$, therefore, by $(D5)$, it follows that $(X, Z) \in \mathcal{D}$. Then, by (5.11) we have that $a \in c_{\mathcal{D}}(X)$. This shows that $c_{\mathcal{D}}(W) \subseteq W$. Finally, by the first part of this proof, we also have $W \subseteq c_{\mathcal{D}}(W)$. Hence $c_{\mathcal{D}}(W) = W$.

This proves that $c_{\mathcal{D}}$ is a closure operator on Ω.

(ii): If $Y \subseteq X \in \mathcal{P}(\Omega)$, then $Y \subseteq X \subseteq \sigma(X)$, so $(X, Y) \in \mathcal{C}_\sigma$. Let now $(X, Y), (Y, Z) \in \mathcal{C}_\sigma$, then $Z \subseteq \sigma(Y)$ and $Y \subseteq \sigma(X)$. Moreover, we have $Z \subseteq \sigma(Y) \subseteq \sigma(\sigma(X)) = \sigma(X)$. Finally, if $(X, Y), (X, Z) \in \mathcal{C}_\sigma$, it results that $Y \subseteq \sigma(X)$ and $Z \subseteq \sigma(X)$, so $Y \cup Z \subseteq \sigma(X)$, i.e. $(X, Y \cup Z) \in \mathcal{C}_\sigma$. Thus we showed that $\mathcal{C}_\sigma \in UAF(\Omega)$.

(iii): Let us prove that $\mathcal{C}_{c_\mathcal{D}} = \mathcal{D}$. Let $(X, Y) \in \mathcal{D}$, then clearly $Y \subseteq c_\mathcal{D}(X)$ by definition of $c_\mathcal{D}(X)$, i.e. $(X, Y) \in \mathcal{C}_{c_\mathcal{D}}$. On the other hand, if $(X, Y) \in \mathcal{C}_{c_\mathcal{D}}$, it is obvious, again by definition of $c_\mathcal{D}(X)$, that $(X, Y) \in \mathcal{D}$. Vice versa, let us prove that $c_{\mathcal{C}_\sigma} = \sigma$. Clearly, if $X \in \mathcal{P}(\Omega)$ is a fixed subset, we have

$$c_{\mathcal{C}_\sigma}(X) = \bigcup\{Y \in \mathcal{P}(\Omega) : Y \subseteq \sigma(X)\} = \sigma(X),$$

and the claim follows. □

If $\sigma \in CLOP(\Omega)$ and $\mathcal{D} \in UAF(\Omega)$ are such that $\mathcal{D} = \mathcal{C}_\sigma$ (and therefore equivalently $\sigma = c_\mathcal{D}$, by Theorem 5.2.8), we say that $\mathcal{C}_\sigma$ is the *union additive family induced* by σ and $c_\mathcal{D}$ is the *closure operator induced* by $\mathcal{D}$.

As a direct consequence, we prove that $\mathcal{G}(\mathfrak{P})$ coincides exactly with the union additive family on Ω induced by the closure operator $M_\mathfrak{P}$.

Proposition 5.2.9. $\mathcal{C}_{M_\mathfrak{P}} = \mathcal{G}(\mathfrak{P})$ *and* $c_{\mathcal{G}(\mathfrak{P})} = M_\mathfrak{P}$. *That is,* $M_\mathfrak{P}$ *is the closure operator on* Ω *induced by* $\mathcal{G}(\mathfrak{P})$ *and* $\mathcal{G}(\mathfrak{P})$ *is the union additive family on* Ω *induced by* $M_\mathfrak{P}$.

Proof. Let $(X, Y) \in \mathcal{C}_{M_\mathfrak{P}}$. Then, $Y \subseteq M_\mathfrak{P}(X)$, so we have $\pi_\mathfrak{P}(X) = \pi_\mathfrak{P}(M_\mathfrak{P}(X)) \preceq \pi_\mathfrak{P}(Y)$. Therefore, $[u]_X \subseteq [u]_Y$ for any $u \in U$ or, equivalently, $\Gamma_\mathfrak{P}(X, Y) = U$. This implies that $\mathcal{C}_{M\mathfrak{P}} \subseteq \mathcal{G}(\mathfrak{P})$. Vice versa, let us suppose that $(X, Y) \in \mathcal{G}(\mathfrak{P})$, so $\Gamma_\mathfrak{P}(X, Y) = U$. Therefore, $[u]_X = [u]_{M_\mathfrak{P}(X)} \subseteq [u]_Y = [u]_{M_\mathfrak{P}(Y)}$, i.e. $\pi_\mathfrak{P}(M_\mathfrak{P}(X)) \preceq \pi_\mathfrak{P}(M_\mathfrak{P}(Y))$ and this is equivalent to require that $M_\mathfrak{P}(Y) \subseteq M_\mathfrak{P}(X)$ hence, a fortiori, $Y \subseteq M_\mathfrak{P}(X)$. This shows that $\mathcal{C}_{M_\mathfrak{P}} = \mathcal{G}(\mathfrak{P})$. On the other hand, by (5.10), it is immediate to verify that $c_{\mathcal{G}(\mathfrak{P})} = M_\mathfrak{P}$. □

5.3 Union Additive Families from an Abstract Mathematical Perspective

Starting now from $\mathcal{D}$, we inductively define three sequences of different subset ordered pair families and next we show that all their unions coincide with the above additive envelope $\mathcal{D}^+$. In this way, we provide a new interpretation for $(D1) - (D7)$. First of all, let us set $\hat{\mathcal{D}} := \mathcal{D} \cup \{(X, Y) \in \mathcal{P}(\Omega)^2 : Y \subseteq X\}$.

I) Set $\mathcal{D}_0^* := \hat{\mathcal{D}}$. Moreover, if $k \geq 1$, we inductively define $\mathcal{D}_k^*$ as follows. A subset ordered pair (X', Y') belongs to $\mathcal{D}_k^*$ if one of the following holds:

$(A1)$ there exist $Y, Z \in \mathcal{P}(\Omega)$ such that (X', Y'), $(X', Z) \in \mathcal{D}_{k-1}^*$ and $Y' = Y \cup Z$;

$(A2)$ there exists $Y \in \mathcal{P}(\Omega)$ such that (X', Y), $(Y, Y') \in \mathcal{D}_{k-1}^*$.

II) Set $\mathcal{D}_0^\S := \hat{\mathcal{D}}$. Moreover, if $k \geq 1$, we inductively define $\mathcal{D}_k^\S$ as follows. A subset ordered pair (X', Y') belongs to $\mathcal{D}_k^\S$ if one of the following holds:

$(B1)$ there exists $X \subseteq X'$ such that $(X, Y') \in \mathcal{D}_{k-1}^\S$;

(B2) there exist $X, Y, Z \in \mathcal{P}(\Omega)$ such that $(X, Y) \in \mathcal{D}^{\S}_{k-1}$, $(Y \cup Z, Y') \in \mathcal{D}^{\S}_{k-1}$ and $X' = X \cup Z$.

III) Set $\mathcal{D}^{\flat}_0 := \hat{\mathcal{D}}$. Moreover, if $k \geq 1$, we inductively define $\mathcal{D}^{\flat}_k$ as follows. A subset ordered pair (X', Y') belongs to $\mathcal{D}^{\flat}_k$ if one of the following holds:

(C1) there exists $Z \in \mathcal{P}(\Omega)$ such that $(X', Y' \cup Z) \in \mathcal{D}^{\flat}_{k-1}$;

(C2) there exist $Y, Z, W, T \in \mathcal{P}(\Omega)$ such that $(X', Y \cup Z)$, $(Z, T \cup W) \in \mathcal{D}^{\flat}_{k-1}$ and $Y' = Y \cup Z \cup T$.

We have then the following result.

Theorem 5.3.1. *We have that*

$$\bigcup_{k\geq 0} \mathcal{D}^*_k = \bigcup_{k\geq 0} \mathcal{D}^{\S}_k = \bigcup_{k\geq 0} \mathcal{D}^{\flat}_k = \mathcal{D}^+$$

Proof. Let us prove that $\bigcup_{k\geq 0} \mathcal{D}^*_k \in UAF(\Omega)$. Let $X \supseteq Y$. Then, $(X, Y) \in \mathcal{D}^*_0$ and so $(X, Y) \in \bigcup_{k\geq 0} \mathcal{D}^*_k$. By $(A1)$, it can be readily proved that $\mathcal{D}^*_k \subseteq \mathcal{D}^*_{k+1}$ for any $k \in \mathbb{N}$. Hence, if $(X, Y), (X, Z) \in \bigcup_{k\geq 0} \mathcal{D}^*_k$, then there always exists an index j such that (X, Y), $(X, Z) \in \mathcal{D}^*_j$. Thus $(X, Y \cup Z) \in \mathcal{D}^*_{j+1}$ hence $(X, Y \cup Z) \in \bigcup_{k\geq 0} \mathcal{D}^*_k$. Finally, if $(X, Y), (Y, Z) \in \bigcup_{k\geq 0} \mathcal{D}^*_k$, there exists an index i such that (X, Y), $(Y, Z) \in \mathcal{D}^*_i$. Thus, by $(A2)$, $(X, Z) \in \bigcup_{k\geq 0} \mathcal{D}^*_k$. This shows that $\bigcup_{k\geq 0} \mathcal{D}^*_k \in UAF(\Omega)$. Moreover, $\mathcal{D} \subseteq \bigcup_{k\geq 0} \mathcal{D}^*_k$ and it is immediate to see that any $(X, Y) \in \bigcup_{k\geq 0} \mathcal{D}^*_k$ also belongs to any $\mathcal{E} \in UAF(\Omega)$ containing $\mathcal{D}$. So $\mathcal{D}^+ = \bigcup_{k\geq 0} \mathcal{D}^*_k$.
Let us prove now that

$$\bigcup_{k\geq 0} \mathcal{D}^*_k = \bigcup_{k\geq 0} \mathcal{D}^{\S}_k.$$

Obviously, $\mathcal{D}^*_0 \subseteq \bigcup_{k\geq 0} \mathcal{D}^{\S}_k$. Fix an integer n and assume that $\mathcal{D}^*_l \subseteq \bigcup_{k\geq 0} \mathcal{D}^{\S}_k$ for any $l \leq n-1$. We will prove that $\mathcal{D}^*_n \subseteq \bigcup_{k\geq 0} \mathcal{D}^{\S}_k$. For, let $(X', Y') \in \mathcal{D}^*_n$. If $(A1)$ holds, there exist $Y, Z \in \mathcal{P}(\Omega)$ such that $(X', Y), (X', Z) \in \mathcal{D}^*_{n-1}$ and $Y' = Y \cup Z$. By the inductive hypothesis, $(X', Y) \in \mathcal{D}^{\S}_i$ and $(X', Z) \in \mathcal{D}^{\S}_j$, for two integers i and i'. Since $\mathcal{D}^{\S}_k \subseteq \mathcal{D}^{\S}_{k+1}$ for any $k \in \mathbb{N}$, then (X', Y), $(X', Z) \in \mathcal{D}^{\S}_j$ for some $\mathcal{D}^{\S}_j$. Moreover, since $(Y \cup Z, Y \cup Z) \in \mathcal{D}^{\S}_0$, it also belongs to $\mathcal{D}^{\S}_j$. By $(B2)$, the pair $(X' \cup Z, Y \cup Z) \in \mathcal{D}^{\S}_{j+1}$ can be obtained starting from the pairs (X', Y) and $(Y \cup Z, Y \cup Z)$ and, again by $(B2)$, we obtain the pair $(X', Y \cup Z) = (X', Y') \in \mathcal{D}^{\S}_{j+2}$ from to the pairs (X', Z) and $(X' \cup Z, Y \cup Z)$. Thus, $(X', Y') \in \bigcup_{k\geq 0} \mathcal{D}^{\S}_k$. If $(A2)$ holds, then there exists $Y \in \mathcal{P}(\Omega)$ such that $(X', Y), (Y, Y') \in \mathcal{D}^*_{n-1}$. Hence, by inductive hypothesis and by the previous remark, it results that $(X', Y), (Y, Y') \in \mathcal{D}^{\S}_j$ for some integer j. By $(B2)$, we obtain the pair $(X', Y') \in \mathcal{D}^{\S}_{j+1}$ from the pairs $(X', Y), (Y, Y')$, so $(X', Y') \in \bigcup_{k\geq 0} \mathcal{D}^{\S}_k$. This shows that $\bigcup_{k\geq 0} \mathcal{D}^*_k \subseteq \bigcup_{k\geq 0} \mathcal{D}^{\S}_k$.

On the other hand, $\mathcal{D}^{\S}_0 \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^*_k$. Fix an integer n and assume that $\mathcal{D}^{\S}_l \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^*_k$ for any $l \leq n-1$. We will show that $\mathcal{D}^{\S}_n \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^*_k$. For, let $(X', Y') \in \mathcal{D}^{\S}_n$. If $(B1)$ holds, then there exists $X \subseteq X'$ such that $(X, Y') \in \mathcal{D}^{\S}_{n-1}$ and, by the inductive hypothesis, $(X, Y') \in \mathcal{D}^*_i$ for some integer i. Since $(X', X) \in \mathcal{D}^*_0$, it also belongs to $\mathcal{D}^*_i$. By $(A2)$, the pairs (X', X) and (X, Y') yield $(X', Y') \in \mathcal{D}^*_{i+1}$, so $(X', Y') \in \bigcup\limits_{k\geq 0} \mathcal{D}^*_k$. If $(B2)$ holds, then there exist $X, Y, Z \in \mathcal{P}(\Omega)$ such that $(X, Y), (Y \cup Z, Y') \in \mathcal{D}^{\S}_{k-1}$ and $X' = X$. By the inductive hypothesis, $(X, Y) \in \mathcal{D}^*_i$ and $(Y \cup Z, Y') \in \mathcal{D}^*_{i'}$ for some integers i and i'. In particular, there exists j such that $(X, Y) \in \mathcal{D}^*_j$. Moreover, since $(X \cup Z, X), (X \cup Z, Z) \in \mathcal{D}^*_0$, they belong to $\mathcal{D}^*_j$ for any integer j. Now, by $(A2)$, the pair $(X \cup Z, Y) \in \mathcal{D}^*_{j+1}$ can be obtained from the pairs $(X \cup Z, X)$ and (X, Y) and, by $(A1)$, we obtain the pair $(X \cup Z, Y \cup Z) \in \mathcal{D}^*_{j+2}$ from the pairs $(X \cup Z, Y)$ and $(X \cup Z, Z)$. Finally, again by $(A2)$, the pairs $(X \cup Z, Y \cup Z)$ and $(Y \cup Z, Y')$ yield $(X \cup Z, Y') = (X', Y') \in \mathcal{D}^*_{j+3}$, thus $(X', Y') \in \bigcup\limits_{k\geq 0} \mathcal{D}^{\S}_k$. This implies that $\bigcup\limits_{k\geq 0} \mathcal{D}^*_k = \bigcup\limits_{k\geq 0} \mathcal{D}^{\S}_k$. Let us prove that

$$\bigcup_{k\geq 0} \mathcal{D}^{\S}_k = \bigcup_{k\geq 0} \mathcal{D}^{\flat}_k.$$

Obviously, $\mathcal{D}^{\S}_0 \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$. Fix an integer n and assume that $\mathcal{D}^{\S}_l \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$ for any $l \leq n-1$. We will prove that $\mathcal{D}^{\S}_n \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$. For, let $(X', Y') \in \mathcal{D}^{\S}_n$. If $(B1)$ holds, then there exists $X \subseteq X'$ such that $(X, Y') \in \mathcal{D}^{\S}_{n-1}$. By the inductive hypothesis, $(X, Y') \in \mathcal{D}^{\flat}_i$ for some integer i. Since $(X', X') \in \mathcal{D}^{\flat}_0$, then $(X', X') \in \mathcal{D}^{\flat}_i$. Let $Z = X' \setminus X$. By $(C2)$, by starting from the pairs $(X', X \cup Z), (X, Y')$ we obtain that $(X', X \cup Z \cup Y') \in \mathcal{D}^{\flat}_{i+1}$. Moreover, by $(C1)$, it follows that $(X', Y') \in \mathcal{D}^{\flat}_{i+2}$, so $(X', Y') \in \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$. If $(B2)$ holds, then there exist $X, Y, Z \in \mathcal{P}(\Omega)$ such that $(X, Y), (Y \cup Z, Y') \in \mathcal{D}^{\S}_{k-1}$ and $X' = X \cup Z$. By the inductive hypothesis $(X, Y) \in \mathcal{D}^{\flat}_i$ and $(Y \cup Z, W) \in \mathcal{D}^{\flat}_{i'}$ for some integers i, i'. Since $\mathcal{D}^{\flat}_k \subseteq \mathcal{D}^{\flat}_{k+1}$ for any $k \in \mathbb{N}$, we have that $(X, Y) \in \mathcal{D}^{\flat}_j$ for some $j \in \mathbb{N}$. Moreover, since $(X \cup Z, X \cup Z) \in \mathcal{D}^{\flat}_0$, then $(X \cup Z, X \cup Z) \in \mathcal{D}^{\flat}_j$. Then, by $(C2)$, the pairs $(X \cup Z, X \cup Z)$ and (X, Y) yield $(X \cup Z, X \cup Y \cup Z) \in \mathcal{D}^{\flat}_{j+1}$. Moreover, again by $(C2)$, we obtain that $(X \cup Z, X \cup Y \cup Z \cup W) \in \mathcal{D}^{\flat}_{j+2}$ from the pairs $(X \cup Z, X \cup Y \cup Z)$ and $(Y \cup Z, W)$. Finally, by $(C1)$, by starting from $(Y \cup Z, W)$, we obtain that $(X \cup Z, W) = (X', Y') \in \mathcal{D}^{\flat}_{j+3}$, so $(X', Y') \in \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$. This shows that $\bigcup\limits_{k\geq 0} \mathcal{D}^{\S}_k \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$.

On the other hand, $\mathcal{D}^{\flat}_0 \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^{\S}_k$. Fix an integer n and suppose that $\mathcal{D}^{\flat}_l \subseteq \bigcup\limits_{k\geq 0} \mathcal{D}^{\S}_k$ for any $l \leq n-1$. Let $(X', Y') \in \mathcal{D}^{\flat}_n$. If $(C1)$ holds, then there exists $Z \in \mathcal{P}(\Omega)$ such that $(X', Y' \cup Z) \in \mathcal{D}^{\flat}_{n-1}$ and, by the inductive hypothesis, $(X', Y' \cup Z) \in \mathcal{D}^{\S}_i$ for some integer i. Since $(Y' \cup Z, Y') \in \mathcal{D}^{\S}_0$, it is contained in $\mathcal{D}^{\S}_i$, therefore, from the pairs $(X', Y' \cup Z)$ and $(Y' \cup Z, Y')$ and by using $(B2)$, we obtain that $(X', Y') \in \mathcal{D}^{\S}_{i+1}$, thus $(X', Y') \in \bigcup\limits_{k\geq 0} \mathcal{D}^{\flat}_k$. If $(C2)$ holds, then there exist $Y, Z, W, T \in \mathcal{P}(\Omega)$ such that $(X', Y \cup Z)$, $(Z, T \cup W) \in \mathcal{D}^{\flat}_{n-1}$

and $Y' = Y \cup Z \cup T$. By the inductive hypothesis we have that $(X', Y \cup Z) \in \mathcal{D}_i^{\S}$ and $(Y \cup Z, T \cup W) \in \mathcal{D}_{i'}^{\S}$ for some integers i and i'. Since $\mathcal{D}_i^{\S} \subseteq \mathcal{D}_j^{\S}$ whenever $i < j$, we can assume that $(X', Y \cup Z) \in \mathcal{D}_j^{\S}$. By applying $(B1)$, we obtain that $(Y \cup Z, T \cup W) \in \mathcal{D}_{j+1}^{\S}$ from the pair $(Z, T \cup W)$. Moreover, the pair $(T \cup W, T) \in \mathcal{D}_0^{\S}$ also belongs to $\mathcal{D}_{j+1}^{\S}$. Hence, by $(B2)$ we obtain that $(Y \cup Z, T) \in \mathcal{D}_{j+2}^{\S}$ from the pairs $(Y \cup Z, T \cup W), (T \cup W, T)$. Again by $(B2)$, we also obtain that $(X, T) \in \mathcal{D}_{j+3}^{\S}$ from the pairs $(X, Y \cup Z), (Y \cup Z, T)$. Now, $(Y \cup Z \cup T, Y \cup Z \cup T) \in \mathcal{D}_0^{\S}$ also belongs to $\mathcal{D}_{j+3}^{\S}$. Therefore, by $(B2)$ we obtain $(X \cup T, Y \cup Z \cup T) \in \mathcal{D}_{j+4}^{\S}$ from $(X, Y \cup Z)$ and $(Y \cup Z \cup T, Y \cup Z \cup T)$. Finally, again by $(B2)$, we have the pair $(X, Y \cup Z \cup T) \in \mathcal{D}_{j+5}^{\S}$ from (X, T) and $(X \cup T, Y \cup Z \cup T)$. This proves that $\bigcup_{k \geq 0} \mathcal{D}_k^{\S} = \bigcup_{k \geq 0} \mathcal{D}_k^{\flat}$ and the thesis holds. $\square$

At this point, we introduce the notion of *model* for $\mathcal{D}$.

Definition 5.3.2. We say that a pairing $\mathfrak{P} \in PAIR(\Omega)$ is a *model* for $\mathcal{D}$, denoted by $\mathfrak{P} \vdash \mathcal{D}$, if for any pair $(X, Y) \in \mathcal{D}$ we have that $X \geq_{\mathfrak{P}} Y$, i.e. if $\mathcal{D}^+ \subseteq \mathcal{G}(\mathfrak{P})$.

In particular, if $\mathcal{D} = \{(X_1, Y_1), \dots, (X_k, Y_k)\}$, we write simply $\mathfrak{P} \vdash (X_1, Y_1), \dots, (X_k, Y_k)$ instead of $\mathfrak{P} \vdash \{(X_1, Y_1), \dots, (X_k, Y_k)\}$, that is therefore equivalent to say that $X_1 \geq_{\mathfrak{P}} Y_1, \dots, X_k \geq_{\mathfrak{P}} Y_k$.

Let us introduce now a preorder between families of $\mathcal{P}(\Omega)^2$.

Definition 5.3.3. Let $\mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$. We say that the ordered pair $(\mathcal{D}, \mathcal{D}')$ is a *sequentiality* on Ω, in symbols $\mathcal{D} \rightsquigarrow_{\Omega} \mathcal{D}'$, if, for any $\mathfrak{P} \in PAIR(\Omega)$ such that $X \geq_{\mathfrak{P}} Y$ for all $(X, Y) \in \mathcal{D}$, we also have $X' \geq_{\mathfrak{P}} Y'$ for all $(X', Y') \in \mathcal{D}'$. In particular, if $\mathcal{D} = \emptyset$, then $\emptyset \rightsquigarrow_{\Omega} \mathcal{D}'$ if and only if $X' \geq_{\mathfrak{P}} Y'$ for any $\mathfrak{P} \in PAIR(\Omega)$ and for any $(X', Y') \in \mathcal{D}'$.

Moreover, if $\mathcal{D} = \{(X_1, Y_1), \dots, (X_k, Y_k)\}$ and $\mathcal{D}' = \{(X_1', Y_1'), \dots, (X_l', Y_l')\}$, we will write

$$(X_1, Y_1), \dots, (X_k, Y_k) \rightsquigarrow_{\Omega} (X_1', Y_1'), \dots, (X_l', Y_l')$$

instead of $\{(X_1, Y_1), \dots, (X_k, Y_k)\} \rightsquigarrow_{\Omega} \{(X_1', Y_1'), \dots, (X_l', Y_l')\}$.
Let us note that

$$\mathcal{D} \rightsquigarrow_{\Omega} \mathcal{D}' \iff \mathcal{D} \rightsquigarrow_{\Omega} (X', Y'), \;\; \forall (X', Y') \in \mathcal{D}' \tag{5.13}$$

and that $\rightsquigarrow_{\Omega}$ is a preorder between ordered subset pairs families. In the next result, we establish some basic properties concerning this preorder.

Theorem 5.3.4. *Let $X, Y, Z, W \in \mathcal{P}(\Omega)$. Then, the following sequentiality on Ω hold:*

$(I1)$ *if $X \supseteq Y$, then $\emptyset \rightsquigarrow_{\Omega} (X, Y)$ and, in particular, $\emptyset \rightsquigarrow_{\Omega} (X, X)$;*

$(I2)$ $(X, Y) \rightsquigarrow_{\Omega} (X \cup Z, Y)$;

$(I3)$ $(X, Y), (X, Z) \rightsquigarrow_{\Omega} (X, Y \cup Z)$;

$(I4)$ $(X, Y \cup Z) \rightsquigarrow_{\Omega} (X, Y)$;

(*I*5) $(X,Y),(Y,Z) \rightsquigarrow_\Omega (X,Z)$;

(*I*6) $(X,Y),(Y\cup Z,W) \rightsquigarrow_\Omega (X\cup Z,W)$;

(*I*7) $(X,Y\cup Z),(Z,T\cup W) \rightsquigarrow_\Omega (X,Y\cup Z\cup T)$.

Proof. It is the same of that of Theorem 5.1.2. □

In the next section we use some properties established in Theorem 5.3.4 to connect our investigation to particular types of acyclic digraphs.

5.4 Tail Decomposition Digraphs

In this section we link the study of the preorder $\rightsquigarrow_\Omega$ to the existence of some particular kinds of acyclic digraphs having as vertex sets suitable subsets of $\mathcal{P}(\Omega)$.

Let $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$.

Definition 5.4.1. We call $\mathcal{D}$-*tail decomposition digraph* an acyclic digraph $D = (V(D), Arc(D))$ such that $V(D) \subseteq \mathcal{P}(\Omega)$ and for any $(W,Z) \in Arc(D)$, there exist $X_1, X_2, \ldots, X_k, Z' \in \mathcal{P}(\Omega)$ such that:

(*i*) $X_1 = W$ and $Z \subseteq Z'$;

(*ii*) $(X_1 \cup X_2 \cup \cdots \cup X_k, Z') \in \mathcal{D}$;

(*iii*) $(X_i, Z) \in Arc(D)$ for all $i = 1,\ldots,k$.

Let D be a $\mathcal{D}$-tail decomposition digraph. We call a pair $(A,B) \in \mathcal{D}$ that satisfies the above conditions a $\mathcal{D}$-*extension* of the arc (W,Z) and we denote by $Ext_{\mathcal{D}}(W,Z)$ the set of all $\mathcal{D}$-extensions of (W,Z). Moreover, we also set

$$\mathcal{U}_{\mathcal{D}}(D) := \bigcup\{Ext_{\mathcal{D}}(W,Z) : (W,Z) \in Arc(D)\}.$$

If $(X,Y) \in \mathcal{P}(\Omega)^2$, we say that D is a $\mathcal{D}|(X,Y)$-*tail decomposition digraph* if $X = \bigcup\{W : W \in \mathbb{I}(D)\}$ and $Y = \bigcup\{Z : Z \in \mathcal{F}\}$, for some $\mathcal{F} \subseteq V(D)$.

Example 5.4.2. *Let* $\mathcal{D} = \{(X_1\cup X_2, Z_1\cup Z_2), (Z_2,Z_3), (Z_1\cup Z_2\cup Z_3, Y)\} \subseteq \mathcal{P}(\Omega)^2$. *In Figure 6.2, we illustrate an example of* $\mathcal{D}|(X_1\cup X_2, Y)$*-tail decomposition digraph. All the edges have been introduced by using Definition 5.4.1.*

In the next Theorem 5.4.4, we show that the condition $\mathcal{D} \rightsquigarrow_\Omega (X,Y)'$ is equivalent to require both that $(X,Y) \in \mathcal{D}^+$ and also that there exists a $\mathcal{D}|(X,Y)$-tail decomposition digraph. In order to prove this equivalence, we firstly introduce the following notion.

Definition 5.4.3. We call $\mathcal{D}|(X,Y)$-*tail resolution* a sequence of pairs

$$\mathcal{S} = ((X_0,Y_0),(X_1,Y_1)\ldots,(X_r,Y_r))$$

such that:

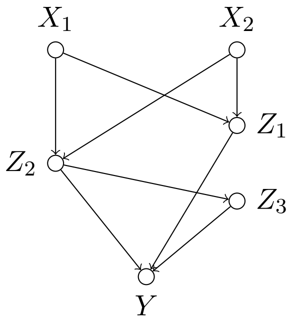


Figure 5.1: $\mathcal{D}|(X_1 \cup X_2|Y)$-tail decomposition digraph

(i) $(X_0, Y_0) = (X, X)$;

(ii) $(X_1, Y_1) \in \hat{\mathcal{D}}$;

(iii) if $2 \leq k \leq r-1$, (X_k, Y_k) satisfies one of the following conditions:

(a) $(X_k, Y_k) \in \hat{\mathcal{D}}$;

(b) $X_k = X_{k-2} = X$ and there exist $T, W, Z \in \mathcal{P}(\Omega)$ such that $Y_{k-2} = T \cup X_{k-1}$, $Y_{k-1} = Z \cup W$ and $Y_k = Y_{k-2} \cup Z$.

(iv) $(X_r, Y_r) = (X, Y)$ and satisfies one of the following conditions:

(a) there exist $T, W, Z \in \mathcal{P}(\Omega)$ such that $Y_{r-2} = T \cup X_{r-1}$, $Y_{r-1} = Z \cup W$, $X_{r-2} = X$ and $Y = Y_{r-2} \cup Z$;

(b) there exists $Z \in \mathcal{P}(\Omega)$ such that $Y_{r-1} = Y \cup Z$.

(v) No other pair (X_k, Y_k), for all $k = 0, \ldots, r-1$, is (X, Y).

We denote by $\mathcal{R}^{\mathcal{S}}_{\mathcal{D}}(X, Y)$ the set of all pairs in $\mathcal{D}$ occurring in the $\mathcal{D}$-tail resolution $\mathcal{S}$ of (X, Y).

We have then the following result.

Theorem 5.4.4. *Let $(X, Y) \in \mathcal{P}(\Omega)^2$. The following conditions are equivalent:*
(i) $\mathcal{D} \rightsquigarrow_\Omega (X, Y)$;
(ii) $(X, Y) \in \mathcal{D}^+$;
(iii) there exists a $\mathcal{D}|(X, Y)$-tail decomposition digraph.

Proof. $(i) \implies (ii)$: Let us suppose that $\mathcal{D} \rightsquigarrow_\Omega (X, Y)$ and assume by contradiction that $(X, Y) \in \mathcal{P}(\Omega)^2 \setminus \mathcal{D}^+$. We will prove that there exists $\mathfrak{P} \in PAIR(\Omega)$ such that $\Gamma_{\mathfrak{P}}(W, Z) = U$ for any $(W, Z) \in \mathcal{D}^+$ but $\Gamma_{\mathfrak{P}}(X, Y) \neq U$. Since $(X, X) \in \mathcal{D}^+ = \bigcup\limits_{k \geq 0} \mathcal{D}^*_k$, we can consider

$\overline{X} := \max\{Y \in \mathcal{P}(\Omega) : (X,Y) \in \mathcal{D}^+\}$. Let us consider the pairing $\mathfrak{P} \in PAIR(\Omega)$ with $U := \{u, u'\}$, $\Lambda := \{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n\}$ and F so defined:

$$F(v, a_i) = \begin{cases} \lambda_i & \text{if } v = u \vee (v = u' \wedge a_i \in \overline{X}), \\ \mu_i & \text{otherwise.} \end{cases} \tag{5.14}$$

Firstly, let us show that $\Gamma_{\mathfrak{P}}(W,Z) = U$ for any $(W,Z) \in \mathcal{D}^+$. For, we distinguish two cases, namely $W \not\subseteq \overline{X}$ and $W \subseteq \overline{X}$.
Let $W \not\subseteq \overline{X}$. Then, $u \not\equiv_W u'$, i.e. $\pi_{\mathfrak{P}}(W) = u|u'$, therefore $\Gamma_{\mathfrak{P}}(W,Z) = U$ for any $Z \in \mathcal{P}(\Omega)$ and, in particular $\Gamma_{\mathfrak{P}}(W,Z) = U$ for any pair $(W,Z) \in \mathcal{D}^+$.
Suppose now that $W \subseteq \overline{X}$. Hence, $u \equiv_W u'$. Since $(W,Z) \in \mathcal{D}^+$, there exists an index i such that $(W,Z) \in \mathcal{D}^*_i$. Moreover, we have that $(\overline{X}, W) \in \mathcal{D}^*_0$. Thus $(\overline{X}, W) \in \mathcal{D}^*_k$, for any integer k. By applying $(A2)$ on the pairs $(\overline{X}, W)$ and (W,Z), it follows that $(\overline{X}, Z) \in \mathcal{D}^*_{i+1}$. On the other hand, by $(A1)$, we obtain $(\overline{X}, \overline{X} \cup Z) \in \mathcal{D}^*_{i+2}$ from the pairs $(\overline{X}, \overline{X})$ and $(\overline{X}, Z)$. Furthermore, by $(A2)$ we construct the pair $(\overline{X}, W)$ starting from the pairs $(\overline{X}, \overline{X} \cup Z)$ and $(\overline{X} \cup Z, W)$. By the definition of $\overline{X}$, it results that $(X, \overline{X}) \in \mathcal{D}^+$, hence we can find a $\mathcal{D}^*_j$, for some $j > i$, such that $(X, \overline{X}), (\overline{X}, W) \in \mathcal{D}^*_j$. In particular, by applying $(A2)$ on the latter pairs, we obtain the pair $(X, W) \in \mathcal{D}^*_{j+1}$. Finally, since $(W,Z) \in \mathcal{D}^*_i \subseteq \mathcal{D}^*_{j+1}$, by $(A2)$ we obtain $(X,Z) \in \mathcal{D}^*_{j+2}$ starting from the pairs (X,W) and (W,Z). This means that $(X,Z) \in \mathcal{D}^+$. In other terms, we showed that $Z \subseteq \overline{X}$, thus $u \equiv_Z u'$, i.e. $\Gamma_{\mathfrak{P}}(W,Z) = U$.
At this point, we must prove that $\Gamma_{\mathfrak{P}}(X,Y) = \emptyset$. By (5.14), we can deduce that $u \equiv_X u'$. Suppose by contradiction that $u \equiv_Y u'$; it must necessarily result that $Y \subseteq \overline{X}$. Since $(X, \overline{X}) \in \mathcal{D}^+$, then $(X, \overline{X}) \in \mathcal{D}^*_i$ for some $i \in \mathbb{N}$. Therefore, we can obtain the pair $(X,Y) \in \mathcal{D}^*_{i+1}$ starting from the pairs $(X, \overline{X})$ and $(\overline{X}, Y)$. This means that $(X,Y) \in \mathcal{D}^+$, contradicting our choice of (X,Y). Therefore, $u \not\equiv_Y u'$, so $\Gamma_{\mathfrak{P}}(X,Y) = \emptyset$. This shows the claim.

$(ii) \implies (i)$: Let us suppose that $(X,Y) \subseteq \mathcal{D}^+ = \bigcup_{k \geq 0} \mathcal{D}^\flat_k$ and let $\mathfrak{P}$ be a pairing such that $\Gamma_{\mathfrak{P}}(W,Z) = U$, for any $(W,Z) \in \mathcal{D}$. We will prove that $\Gamma_{\mathfrak{P}}(X,Y) = U$. If $(X,Y) \in \hat{\mathcal{D}}$, then obviously $\Gamma_{\mathfrak{P}}(X,Y) = U$. Let n be a fixed integer and suppose that for any $k \leq n-1$ we have that $\Gamma_{\mathfrak{P}}(W', Z') = U$ for any $(W', Z') \in \mathcal{D}^\flat_k$. We now show that for any pair $(X', Y') \in \mathcal{D}^\flat_n$, it results that $\Gamma_{\mathfrak{P}}(X', Y') = U$. Let $(X', Y') \in \mathcal{D}^\flat_n$. If $(C1)$ holds, there exists $Z \in \mathcal{P}(\Omega)$ such that $(X', Y' \cup Z) \in \mathcal{D}^\flat_{n-1}$. By the inductive hypothesis, $\Gamma_{\mathfrak{P}}(X', Y' \cup Z) = U$, then, by $(I4)$, $(X', Y' \cup Z) \rightsquigarrow_\Omega (X', Y')$, so $\Gamma_{\mathfrak{P}}(X', Y') = U$.
On the other hand, if $(C2)$ holds, there exist $Y, Z, W, T \in \mathcal{P}(\Omega)$ such that $(X', Y \cup Z), (Z, T \cup W) \in \mathcal{D}^\flat_{n-1}$ and $Y' = Y \cup Z \cup T$. By the inductive hypothesis, $\Gamma_{\mathfrak{P}}(X', Y \cup Z) = \Gamma_{\mathfrak{P}}(Z, T \cup W) = U$ hence by $(I7)$, it follows that $(X', Y \cup Z), (Z, T \cup W) \rightsquigarrow_\Omega (X', Y')$, i.e. $\Gamma_{\mathfrak{P}}(X', Y') = U$ and the claim has been showed.
$(ii) \implies (iii)$: Let us suppose that $(X,Y) \in \mathcal{D}^+$. We divide the proof of (iii) in two steps. Firstly, we show that there exists a $\mathcal{D}|(X,Y)$-tail resolution $\mathcal{S} = ((X_0, Y_0), (X_1, Y_1) \ldots, (X_r, Y_r))$.

By virtue of the identity $\mathcal{D}^+ = \bigcup_{k \geq 0} \mathcal{D}^\flat_k$ and by Theorem 5.3.1, there exists a sequence of ordered subsets pairs $(X_0, Y_0), \ldots, (X_s, Y_s)$ such that $(X_0, Y_0) \in \hat{\mathcal{D}}$ and $(X_s, Y_s) = (X, Y)$. In particular, since the pair $(X, X) \in \hat{\mathcal{D}}$, we can always assume that $(X_0, Y_0) = (X, X)$.

Suppose that a pair $(W,Z) \neq (X,Y)$ in the sequence has been obtained from some pair $(W, Z \cup Z')$ by $(C1)$. If (W,Z) is not used elsewhere to obtain some other pair of sequence, we can remove it from the sequence. On the contrary, suppose that (W,Z) is used to yield another pair. If this second pair has been obtained through $(C1)$, we can suppose directly that it has been obtained by the pair $(W, Z \cup Z')$, hence (W,Z) can be removed from the sequence. Nevertheless, the second pair can be obtained from (W,Z) through $(C2)$. In this case, two possibilities are allowed:

(a) there exist S, T, $U \in \mathcal{P}(\Omega)$, with $S \subseteq Z$, such that the pairs $(W,Z), (S, T \cup U)$ yield the pair $(W, T \cup Z)$;

(b) there exist U, $V \in \mathcal{P}(\Omega)$, with $V \supseteq W$, such that the pairs $(U,V), (W,Z)$ yield the pair $(U, W \cup V)$.

In case (a) we can use the ordered pair $(W, Z \cup Z')$ to obtain $(W, T \cup Z \cup Z')$ instead of $(W, T \cup Z)$. This implies that we can remove (W,Z) in the above sequence replacing all the pairs derived from $(W, T \cup Z)$ with the ones obtained by starting from $(W, T \cup Z \cup Z')$. In case (b), we can directly replace (W,Z) with $(W, Z \cup Z')$ without changing the resulting pair $(U, W \cup V)$. In other terms, in both cases we replaced a pair (W,Z) with a pair $(W, Z \cup Z')$. Let us observe that with these replacements, it can happen that we derive the pair $(X, Y \cup Y')$ instead of (X,Y). The latter can be yielded by applying $(C1)$ to the pair $(X, Y \cup Y')$.
However, by iterating the previous procedure, we find a sequence $(X_0, Y_0), \ldots, (X_m, Y_m)$, with $(X_0, Y_0) = (X,X)$ and $(X_m, Y_m) = (X,Y)$ and where each intermediate pair can be derived only through $(C2)$.
Let us prove now that, if (X,Z) can be obtained by $(C2)$ from some pairs of form (X, V') and $(S, T \cup U)$ of the above sequence, then we can obtain a pair of the form (X,W), with $Z \subseteq W$, starting from a pair of the form (X,V) and belonging to the previous sequence and a pair $(S,R) \in \mathcal{D}$.
For, let us assume that the pair (X,Z) has been yielded by $(X,V), (S, T \cup U)$ by applying $(C2)$. Firstly, let us note that by the previous argument we can suppose that $(S, T \cup U)$ cannot be obtained by using some application of $(C1)$. We distinguish two cases: if $S \supseteq T \cup U$, then we can add the pair $(S, T \cup U)$ in the sequence; otherwise, $(S, T \cup U)$ has been obtained by means of $(C2)$ from two pairs $(S,R), (R', T' \cup U')$, both belonging to the above sequence. In this case, we have that $R \cup T' = T \cup U$ and, thus, another application of $(C2)$ to the pairs $(X, V \cup T'), (S,R)$ yields $(X, Z \cup U)$. The only problem that may arise is that $(S,R) \notin \hat{\mathcal{D}}$ but, if so, just repeat the same argument for it. Hence, we can remove $(S, T \cup U)$ and all the pairs in our sequence not belonging in $\mathcal{D}$ and not having the form (X,Z) for some $Z \in \mathcal{P}(\Omega)$. This proves the existence of a $\mathcal{D}|(X,Y)$-tail resolution.
The second step of our proof consists of constructing a $\mathcal{D}|(X,Y)$-tail decomposition digraph starting from the above $\mathcal{D}|(X,Y)$-tail resolution.
Let $\mathcal{S}$ be a $\mathcal{D}|(X,Y)$-tail resolution and let (X, Z_0), (X, Z_1), $\ldots$, (X, Z_k) be the ordered subset pairs occurring in $\mathcal{S}$, with $Z_0 = X$ and $Z_k = Y$. We will construct a collection D_i of $\mathcal{D}|(X, Z_i)$-tail decomposition digraphs, for $i = 0, \ldots, k$. When $i = 0$, the graph with just a node X is clearly a $\mathcal{D}|(X, Z_0)$-tail decomposition digraph, where $(X, Z_0) = (X,X)$. Let $i \leq k$ be a fixed integer and suppose inductively that D_j is a $\mathcal{D}|(X, Z_j)$-tail decomposition digraphs for

$j = 0, \ldots, i-1$; we now construct a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph obtained from D_{i-1} by adding suitable vertices and edges. For, let us consider (X, Z_i) in $\mathcal{S}$. If $(X, Z_i) \in \mathcal{D}$, then we can add to D_{i-1} a vertex Z_i and an edge (X, Z_i). Thus we obtain a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph D_i.
On the other hand, suppose that $(X, Z_i) \in \mathcal{D}^+ \setminus \hat{\mathcal{D}}$. This implies that one of the conditions $(C1)$, $(C2)$ holds. Suppose that there exists $T \in \mathcal{P}(\Omega)$ such that the pair (X, Z_i) have been obtained from $(X, Z_i \cup T) = (X, Z_{i-1})$. In this case D_{i-1} is already a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph. Then, we can set $D_i = D_{i-1}$.
On the contrary, suppose that there exist $T, U, W \in \mathcal{P}(\Omega)$ such that $Z_{i-2} = T \cup X_{i-1}$, $Z_{i-1} = U \cup W$ and $Z_k = Z_{k-2} \cup U$. In this case, D_{i-2} is a subgraph of D_{i-i} and in D_{i-2} there are subsets of Ω whose union is Z_{i-2}. Then, we form D_i by adding a vertex labeled U to D_{i-1} and arcs connecting nodes of D_{i-2} labeled by some subsets of Z_{i-2} with U. This ends our inductive construction, so a $\mathcal{D}|(X, Y)$-tail decomposition digraph has been formed.
$(iii) \implies (ii)$: Let us assume that a $\mathcal{D}|(X, Y)$-tail decomposition digraph is given. Let D be a $\mathcal{D}|(X, Y)$-tail decomposition digraph and $D_0, D_1, \ldots, D_k = D$ be a sequence of $\mathcal{D}$-tail decomposition digraphs such that D_i is obtained by D_{i-i} by adding nodes and edges compatibly with Definition 5.4.1. For each $i = 0, \ldots, k$, we assume that Z_i is the union of subsets of Ω labelling some nodes in D_i and, moreover, suppose that $Z_k \supseteq Y$. We will construct a $\mathcal{D}|(X, Y)$-tail resolution $\mathcal{S}$ from D containing $(X, Z_0) = (X, X), (X, Z_1), \ldots, (X, Z_k)$ as a subsequence.
Firstly, set $Z_0 := X$ and D_0 be the digraph whose vertex set is exactly X and $Arc(D_0) = \emptyset$. Let us set $\mathcal{S}_0 := \{(X, X)\}$. Let now $i \geq 1$ and suppose that a there exists $\mathcal{D}|(X, Z_j)$-tail resolution $\mathcal{S}_j$ containing $(X, Z_0) = (X, X), (X, Z_1), \ldots, (X, Z_j)$ as a subsequence, for all $j \leq i-1$. According to Definition 5.4.1, we obtain D_i from D_{i-1} through a pair $(T, U \cup W) \in \mathcal{D}$, where $T, W \in \mathcal{P}(\Omega)$ and $T \subseteq Z_{i-1}$, by adding a vertex labeled by U and edges of the form (S_l, U) for $l = 1, \ldots, m$, where m is a suitable integer and $\bigcup_{l=1}^{m} S_l \subseteq T$. By considering the pairs (X, Z_{i-1}) and $(T, U \cup W)$, we obtain (X, Z_i), where $Z_i = Z_{i-1} \cup U$, by $(C2)$. Therefore, we add both the pairs $(T, U \cup W)$ and (X, Z_i) to $\mathcal{S}_{i-1}$.
In this way we construct a $\mathcal{D}|(X, Z_k)$-tail resolution, where $Y \subseteq Z_k$. It could happen that $Y \subsetneqq Z_k$. In this case, since by $(C1)$ we obtain the pair (X, Y) from (X, Z_k), we can add (X, Y) to $\mathcal{S}_k$. Denote by $\mathcal{S}$ the sequence so constructed. Thus, we obtain a $\mathcal{D}|(X, Y)$-tail resolution $\mathcal{S}$ from D containing $(X, Z_0) = (X, X), (X, Z_1), \ldots, (X, Z_k)$ as a subsequence. Finally, let us observe that we used only conditions $(C1)$ and $(C2)$ to construct the above $\mathcal{D}|(X, Y)$-tail resolution. Hence $(X, Y) \in \mathcal{D}^+$ and the claim has been proved. $\square$

By Theorem 5.4.4 we deduce the following results.

Corollary 5.4.5. *Let $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$. Then, $\mathcal{D} \simeq \mathcal{D}'$ if and only if $\mathcal{D} \rightsquigarrow_\Omega \mathcal{D}'$ and $\mathcal{D}' \rightsquigarrow_\Omega \mathcal{D}$.*

Proof. Let us observe that by Theorem 5.4.4, condition $\mathcal{D} \rightsquigarrow_\Omega \mathcal{D}'$ becomes $\mathcal{D}' \subseteq \mathcal{D}^+$ so, by Proposition 5.2.6. Hence, we easily infer the claim. $\square$

Corollary 5.4.6. *Let $\mathcal{D} \subseteq \mathcal{G}(\mathfrak{P})$. Then, $\mathcal{D} \rightsquigarrow_\Omega \mathcal{G}(\mathfrak{P})$ if and only if $\mathcal{D}^+ = \mathcal{G}(\mathfrak{P})$.*

Proof. Suppose that $\mathcal{D} \rightsquigarrow_\Omega \mathcal{G}(\mathfrak{P})$. By Theorem 5.4.4 it follows that $\mathcal{G}(\mathfrak{p}) \subseteq \mathcal{D}^+$ and, by Proposition 5.2.6, we also have the reverse inclusion. Hence, $\mathcal{D}^+ = \mathcal{G}(\mathfrak{P})$.

On the other hand, if $(X, Y) \in \mathcal{G}(\mathfrak{P}) = \mathcal{D}^+$, by Theorem 5.4.4 we have that $\mathcal{D} \rightsquigarrow_\Omega (X, Y)$. By the arbitrariness of $(X, Y) \in \mathcal{G}(\mathfrak{P})$ we conclude the thesis. $\square$

By means of Theorem 5.4.4 we are also able to establish a deeper relation between tail resolutions and tail decomposition digraphs.

Proposition 5.4.7. *Let $(X, Y) \in \mathcal{P}(\Omega)^2$. If there exists a $\mathcal{D}|(X, Y)$-tail resolution $\mathcal{S}$, then there exists a $\mathcal{D}|(X, Y)$-tail decomposition digraph D such that $\mathcal{U}_\mathcal{D}(D) = \mathcal{R}^\mathcal{S}_\mathcal{D}(X, Y)$.*

Proof. Let $\mathcal{S}$ be a $\mathcal{D}|(X, Y)$-tail resolution and let (X, Z_0), (X, Z_1), $\ldots$, (X, Z_k) be the ordered subsets pairs occurring in $\mathcal{S}$, with $Z_0 = X$ and $Z_k = Y$. Let us construct a collection of $\mathcal{D}$-tail decomposition digraphs D_0, D_1, $\ldots$, D_k such that, for each i, D_i is a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph. When $i = 0$, the graph with just a node X is clearly a $\mathcal{D}|(X, Z_0)$-tail decomposition digraph. Let us suppose, by using induction, that for any k in $0, 1, \ldots, i-1$, D_k is a $\mathcal{D}|(X, Z_k)$-tail decomposition digraph and let us construct a $\mathcal{D}|(X, Z_k)$-tail decomposition digraph by adding suitable vertices and edges. For this let us consider (X, Z_i) in $\mathcal{S}$. If $(X, Z_i) \in \hat{\mathcal{D}}$, then clearly D_{i-1} is still a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph. Let us suppose now that (X, Z_i) is not in $\hat{\mathcal{D}}$ and that there exist T, W, $Z \in \mathcal{P}(\Omega)$ such that $T \subset Z_j$ for some $j < i$, $Z_i = Z_j \cup W$ and $(T, W \cup Z) \in \hat{\mathcal{D}}$, so, by Theorem 5.4.4 and by Theorem 5.3.4, it follows that (X, Z_j), $(T, W \cup Z) \rightsquigarrow_\Omega (X, Z_j \cup W) = (X, Z_i)$. In this case D_j is a subgraph of D_{i-i} and in D_j there are vertices of Ω whose union is Z_j. Then, by adding a vertex W to D_{i-1} and suitable edges we obtain a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph D_i. Finally let us suppose that (X, Z_i) is not in $\mathcal{D}$ and there exists $j < i$ such that $Z_j \subseteq Z_i$, so $(X, Z_j) \rightsquigarrow_\Omega (X, Z_i)$. In this case D_j is already a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph, so is D_{i-1}. Then, we can set $D_i = D_{i-1}$. Finally, it is now easy to prove that $D := D_k$ is a $\mathcal{D}|(X, Z_k)$-tail decomposition digraph and that $\mathcal{U}_\mathcal{D}(D) = \mathcal{R}^\mathcal{S}_\mathcal{D}(X, Y)$. $\square$

We conclude this section with the following result.

Proposition 5.4.8. *Let $\mathcal{D} \subseteq \mathcal{G}(\mathfrak{P})$, $(X, Y) \in \mathcal{P}(\Omega)^2$ and $\mathcal{S}$ be a $\mathcal{D}|(X, Y)$-tail resolution. If $(U, V) \in \mathcal{R}^\mathcal{S}_\mathcal{D}(X, Y)$, then we can obtain a $\mathcal{D}|(X, U)$-tail resolution $\mathcal{S}'$ by adding eventually (X, U) to a subsequence of $\mathcal{S}$.*

Proof. Let $(U, V) \in \mathcal{R}^\mathcal{S}_\mathcal{D}(X, Y)$. If $U = X$, then $(X, U) = (X, X)$ and there's nothing to prove. If $U \neq X$, then $(U, V) = (X_i, Y_i)$ and, by definition of $\mathcal{D}|(X, Y)$-tail resolution $X_{i-1} = X$ and $Y_{i-1} = T \cup X_i = T \cup U$. So by adding (X, U) to the subsequence of $\mathcal{S}$, $((X_0, Y_0), (X_1, Y_1) \ldots, (X_{i-1}, Y_{i-1}))$, we obtain a $\mathcal{D}|(X, U)$-tail resolution $\mathcal{S}'$. $\square$

5.5 Generators

From the above sections, we observe that the investigation of the mathematical foundations concerning an abstract theory of the binary relations between specific types of families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ has an its implicit justification. On the other hand, there also more structural reasons that induce to study these types of binary relations in $\mathcal{P}(\mathcal{P}(\Omega)^2)$. In this perspective we introduce the set map $DP : \mathcal{P}(\mathcal{P}(\Omega)^2) \to \mathcal{P}(\mathcal{P}(\Omega))$ defined by

$$DP(\mathcal{D}) := \{Z \in \mathcal{P}(\Omega) : X \subseteq Z \implies Y \subseteq Z \ \forall (X, Y) \in \mathcal{D}\}. \tag{5.15}$$

Let us notice that $DP(\mathcal{D}) \in CLSY(\Omega)$ for any $\mathcal{D} \in \mathcal{P}(\Omega)^2$. Furthermore, if $\mathcal{D} \subseteq \mathcal{D}'$, it results that

$$DP(\mathcal{D}) \supseteq DP(\mathcal{D}'). \tag{5.16}$$

We now show that the map DP is invariant in the passage from a family $\mathcal{D}$ to its union additive envelope.

Theorem 5.5.1. *$DP(\mathcal{D}^+) = DP(\mathcal{D})$.*

Proof. By (5.16), we have that $DP(\mathcal{D}^+) \subseteq DP(\mathcal{D})$. Conversely, let $Z \in DP(\mathcal{D})$ and let $(X, Y) \in \mathcal{D}^+$ such that $X \subseteq Z$. We have to prove that $Y \subseteq Z$. Clearly, if $(X, Y) \in \mathcal{D}_0^*$, the claim is obvious.
Assume now that $(X, Y) \in \mathcal{D}^+ \setminus \mathcal{D}_0^*$. So, $(X, Y) \in \mathcal{D}_k^*$, for some $k \geq 1$. By Theorem 5.3.1, the pair (X, Y) can be obtained by applying $(A1)$ or $(A2)$ to some pairs belonging to $\mathcal{D}_{k-1}^*$. Thus, we must show our claim inductively. To this regard, let $(X, Y) \in \mathcal{D}_1^*$ and assume that there exists $Y' \in \mathcal{P}(\Omega)$ such that $(X, Y'), (Y', Y) \in \mathcal{D}_0^*$. By our assumptions on Z, it follows that $Y' \subseteq Z$ and, then, that $Y \subseteq Z$. On the other hand, if there exist $\{Y_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ for which $(Y, Y_i) \in \mathcal{D}_0^*$ for any $i \in I$ and $Y = \bigcup_{i \in I} Y_i$, then $Y_i \subseteq Z$ for any $i \in I$, i.e. $Y = \bigcup_{i \in I} Y_i \subseteq Z$. This proves that $DP(\mathcal{D}) = DP(\mathcal{D}_1^*)$. Assume now that $DP(\mathcal{D}_{j-1}^*) = DP(\mathcal{D}_j^*)$ for any $j \leq k$. Let us prove that $DP(\mathcal{D}_k^*) = DP(\mathcal{D}_{k+1}^*)$. By (5.16), we have only to show that $DP(\mathcal{D}_k^*) \subseteq DP(\mathcal{D}_{k+1}^*)$. Let $(X, Y) \in DP(\mathcal{D}_{k+1}^*)$ and $Z \in DP(\mathcal{D}_k^*)$ be such that $X \subseteq Z$. If there exists $Y' \in \mathcal{P}(\Omega)$ such that $(X, Y'), (Y', Y) \in \mathcal{D}_k^*$, then $Y' \subseteq Z$ and, then, $Y \subseteq Z$. On the other hand, suppose that there exist $\{Y_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ for which $(Y, Y_i) \in \mathcal{D}_k^*$ for any $i \in I$ and $Y = \bigcup_{i \in I} Y_i$. Then, $Y_i \subseteq Z$ for any $i \in I$, i.e. $Y = \bigcup_{i \in I} Y_i \subseteq Z$. This proves that $DP(\mathcal{D}_k^*) = DP(\mathcal{D}_{k+1}^*)$ and concludes the proof. $\square$

We consider now the family given by

$$\mathcal{Q}(\mathfrak{P}) := \{(X, y) : X \in MINP(\mathfrak{P}),\, y \in M_{\mathfrak{P}}(X) \setminus X\}. \tag{5.17}$$

Then, through the map DP we can alternative characterize $MAXP(\mathfrak{P})$ in terms of the families $\mathcal{G}(\mathfrak{P})$ and $\mathcal{Q}(\mathfrak{P})$.

Theorem 5.5.2. *$DP(\mathcal{G}(\mathfrak{P})) = DP(\mathcal{Q}(\mathfrak{P})) = MAXP(\mathfrak{P})$.*

Proof. We firstly prove that $DP(\mathcal{Q}(\mathfrak{P})) = MAXP(\mathfrak{P})$. Let $Z \in MAXP(\mathfrak{P})$ and let $(X, y) \in \mathcal{Q}(\mathfrak{P})$ with $X \subseteq Z$. Then, or $X \in [Z]_{\approx_{\mathfrak{P}}}$ and thus, $y \in Z$ or $M_{\mathfrak{P}}(X) \subsetneq Z$ and, also in this case, $y \in Z$. This proves that $Z \in MAXP(\mathfrak{P})$.
On the other hand, let $Z \in DP(\mathcal{Q}(\mathfrak{P}))$. Then, there exists $X \in MINP(\mathfrak{P})$ such that $X \approx_{\mathfrak{P}} Z$ and $X \subseteq Z$. Hence, by our assumption on Z, it results that $M_{\mathfrak{P}}(X) \subseteq Z \subseteq M_{\mathfrak{P}}(X)$, i.e. $Z = M_{\mathfrak{P}}(X)$. Thus, $Z \in MAXP(\mathfrak{P})$ and the claim has been shown.

Now, we show that $DP(\mathcal{G}(\mathfrak{P})) = MAXP(\mathfrak{P})$. Let $Z \in MAXP(\mathfrak{P})$ and $(X, Y) \in \mathcal{G}(\mathfrak{P})$ with $X \subseteq Z$. Then, we have that

$$Z \supseteq M_{\mathfrak{P}}(X) \supseteq M_{\mathfrak{P}}(Y) \supseteq Y,$$

hence $Z \in DP(\mathcal{G}(\mathfrak{P}))$.
On the other hand, let $Z \in DP(\mathcal{G}(\mathfrak{P}))$. Then, $Z \subseteq M_{\mathfrak{P}}(Z)$. Moreover, the pair $(Z, M_{\mathfrak{P}}(Z))$ clearly belongs to $\mathcal{G}(\mathfrak{P})$. Thus, we conclude that $M_{\mathfrak{P}}(Z) \subseteq Z$, i.e. $Z = M_{\mathfrak{P}}(Z)$ and so $Z \in MAXP(\mathfrak{P})$. □

Based on the results given in Theorems 5.5.1 and 5.5.2, it is natural to investigate some particular types of families $\mathcal{D}$ that completely characterize all the semi-symmetries of a given information table. The role that these families have in our context is analogue to that of a generator system in an algebraic structure.

Definition 5.5.3. We say that:

- $\mathcal{D}$ *generates the semi-symmetries* of $\mathfrak{P}$, denoted by $\mathcal{D} \Vdash \mathfrak{P}$, if $\mathcal{D} \subseteq \mathcal{G}(\mathfrak{P})$ and $\mathcal{D}^+ = \mathcal{G}(\mathfrak{P})$.
- $\mathcal{D}$ *minimally generates the semi-symmetries* of $\mathfrak{P}$, denoted by $\mathcal{D} \Vdash_{mgs} \mathfrak{P}$, if $\mathcal{D} \Vdash \mathfrak{P}$ and for any $\mathcal{D}' \subsetneqq \mathcal{D}$ we have that $(\mathcal{D}')^+ \subsetneqq \mathcal{G}(\mathfrak{P})$;
- $\mathcal{D}$ is a *essentially generates of the semi-symmetries* of $\mathfrak{P}$, denoted by $\mathcal{D} \Vdash_{egs} \mathfrak{P}$, if $\mathcal{D} \Vdash \mathfrak{P}$ and $|\mathcal{D}| = \min\{|\mathcal{D}'| : \mathcal{D}' \Vdash \mathfrak{P}\}$.

By virtue of Theorem 5.5.1 and 5.5.2 we obtain the following result.

Corollary 5.5.4. *If* $\mathcal{D} \Vdash \mathfrak{P}$, *then* $DP(\mathcal{D}) = MAXP(\mathfrak{P})$.

A partial converse of the previous corollary can be given.

Proposition 5.5.5. *Let* $\mathcal{D} \subseteq \mathcal{P}(\mathcal{P}(\Omega)^2)$ *be such that* $DP(\mathcal{D}) = MAXP(\mathfrak{P})$. *Then,* $\mathcal{D} \subseteq \mathcal{G}(\mathfrak{P})$.

Proof. Let $(X, Y) \in \mathcal{D}$. Then, $M_{\mathfrak{P}}(X) \supseteq X$ implies $M_{\mathfrak{P}}(X) \supseteq Y$, therefore $M_{\mathfrak{P}}(Y) \subseteq M_{\mathfrak{P}}(X)$, i.e., by (5.4), $X \geq_{\mathfrak{P}} Y$. This shows that $(X, Y) \in \mathcal{G}(\mathfrak{P})$. □

We determine now a specific family of ordered pairs of $\mathcal{P}(\Omega)^2$ that generates the semi-symmetries of $\mathfrak{P}$. To this regard, we set

$$\mathcal{G}_{minp}(\mathfrak{P}) := \{(X, Y) \in MINP(\mathfrak{P}) \times (MINP(\mathfrak{P}) \cup \mathcal{P}(M_{\mathfrak{P}}(\emptyset))\} \cap \mathcal{G}(\mathfrak{P}).$$

We have then the following result.

Theorem 5.5.6. $\mathcal{G}_{minp}(\mathfrak{P}) \Vdash \mathfrak{P}$.

Proof. It is obvious that $\mathcal{G}^+_{minp}(\mathfrak{P}) \subseteq \mathcal{G}(\mathfrak{P})$, since $^+$ is a closure operator and $\mathcal{G}(\mathfrak{P}) \in UAF(\Omega)$. On the other hand, it results that $\mathcal{G}(\mathfrak{P}) \subseteq \mathcal{G}^+_{minp}(\mathfrak{P})$. In fact, let $(X, Y) \in \mathcal{G}(\mathfrak{P})$. Then, there exists $Z \in MINP(\mathfrak{P})$ such that $Z \subseteq X$ and $Z \approx_{\mathfrak{P}} X$. In particular $\Gamma_{\mathfrak{P}}(Z, Y) = U$, so $(Z, Y) \rightsquigarrow_{\Omega} (X, Y)$ by $(I2)$. Similarly, there exists $W_1 \in MINP(\mathfrak{P})$ such that $W_1 \subseteq Y$ and $W_1 \approx_{\mathfrak{P}} Y$. Hence $Y = W_1 \cup W_1'$, where $W_1' = Y \setminus W_1$. Clearly, $\Gamma_{\mathfrak{P}}(Z, W_1) = \Gamma_{\mathfrak{P}}(Z, W_1') = U$. Furthermore, by $(I3)$, we have that $(Z, W_1), (Z, W_1') \rightsquigarrow_{\Omega} (Z, Y)$. Let us consider the pair

(Z, W'_1). Let $W_2 \in MINP(\mathfrak{P})$ such that $W_2 \approx_{\mathfrak{P}} W'_1$ and $W'_1 = W_2 \cup W'_2$, where $W'_2 \cap W_2 = \emptyset$. Again, $\Gamma_{\mathfrak{P}}(Z, W_2) = \Gamma_{\mathfrak{P}}(Z, W'_2) = U$. By iterating this procedure, we obtain a family of subsets $W_1, \ldots, W_r$ belonging in $MINP(\mathfrak{P})$ and such that $\Gamma_{\mathfrak{P}}(Z, W_i) = U$ for any $i = 1, \ldots, r$ and a set $W'_r \in \mathcal{P}(M_{\mathfrak{P}}(\emptyset))$ with $W'_r \cap W_r = \emptyset$. In particular $\Gamma_{\mathfrak{P}}(Z, W'_r) = U$. Therefore, $(Z, W_1), \ldots, (Z, W_r), (Z, W'_r) \rightsquigarrow_\Omega (Z, Y)$, hence $\mathcal{G}_{minp}(\mathfrak{P}) \rightsquigarrow_\Omega (X, Y)$, i.e. $(X, Y) \in \mathcal{G}^+_{minp}(\mathfrak{P})$. □

Given $X, Y \in \mathcal{P}(\Omega)$, we now consider the case in which both the pairs (X, Y) and (Y, X) belong to $\mathcal{D}^+$. Formally, whenever $X, Y \in \mathcal{P}(\Omega)$ are such that $\{(X, Y), (Y, X)\} \subseteq \mathcal{D}^+$, we write $X \rightleftharpoons_{\mathcal{D}} Y$ and we also set

$$[X|\mathcal{D}] := \{X' \in \mathcal{P}(\Omega) : X \rightleftharpoons_{\mathcal{D}} X'\},$$

$$F_{\mathcal{D}}(X) := \{(X', Y') \in \mathcal{D} : \ X \rightleftharpoons_{\mathcal{D}} X'\},$$

and

$$f_{\mathcal{D}}(X) := \{X' \in \mathcal{P}(\Omega) : \ \exists Y' \in \mathcal{P}(\Omega) \text{ such that } (X', Y') \in F_{\mathcal{D}}(X)\}.$$

Proposition 5.5.7. *Let $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$ such that $\mathcal{D} \Vdash_{mgs} \mathfrak{P}$ and $\mathcal{D}' \Vdash_{mgs} \mathfrak{P}$. Then, for any $(X, Y) \in \mathcal{D}$ there exists $(X', Y') \in \mathcal{D}'$ such that $X \rightleftharpoons_{\mathcal{D}} X'$.*

Proof. By Corollary 5.4.5, it holds $\mathcal{D}' \rightsquigarrow_\Omega \mathcal{D}$ and thus $\mathcal{D}' \rightsquigarrow_\Omega (X, Y)$. Let $\mathcal{S}$ be a $\mathcal{D}'|(X, Y)$-tail resolution. Then, because of the minimality of $\mathcal{D}$, there exists $(X', Y') \in \mathcal{R}^{\mathcal{S}}_{\mathcal{D}}(X, Y)$ that can be deduced by $\mathcal{D}$ by using (X, Y). Thus, $\mathcal{D} \rightsquigarrow_\Omega (X', X)$. In a similar way, $\mathcal{D}' \rightsquigarrow_\Omega (X, X')$ since we use (X', Y') to deduce (X, Y). This implies that $\mathcal{D} \rightsquigarrow_\Omega (X, X')$. The proposition is thus proved. □

We prove now that if $\mathcal{D}$ and $\mathcal{D}'$ essentially generate the semi-symmetries of $\mathfrak{P}$, then the two sets $L_{\mathcal{D}}(X)$ and $L_{\mathcal{D}'}(X)$ have the same cardinality, for any $X \in \mathcal{P}(\Omega)$. To this aim, we need a technical notion.

Definition 5.5.8. Let $(X, Y) \in \mathcal{G}(\mathfrak{P})$. We say that X *resolves* Y in $\mathfrak{P}$, denoted by $X|_{\mathfrak{P}}Y$, if there exist $\mathcal{D}$ such that $\mathcal{D} \Vdash_{mgs} \mathfrak{P}$ and a $\mathcal{D}|(X, Y)$-tail decomposition digraph D such that $F_{\mathcal{D}}(X) \cap \mathcal{U}_{\mathcal{D}}(D) = \emptyset$.

Now, we establish a preliminary technical result.

Proposition 5.5.9. *(i) Let $\mathcal{D} \Vdash_{egs} \mathfrak{P}$, $X \in \mathcal{P}(\Omega)$ and $\{(Y, Z), (Y', Z')\} \subseteq \mathcal{D} \cap F_{\mathcal{D}}(X)$ such that $Y|_{\mathfrak{P}}Y'$. Then, $(Y, Z) = (Y', Z')$.*
(ii) Let $\mathcal{D} \Vdash_{mgs} \mathfrak{P}$, $(X, Y) \in \mathcal{D}$ and $X' \in [X|\mathcal{D}]$. Then, $X'|_{\mathfrak{P}}Y'$, for some $Y' \in f_{\mathcal{D}}(X)$.

Proof. (i): Let us assume, by contradiction, that the thesis is false. Let D be a $\mathcal{D}|(Y, Y')$-tail decomposition digraph such that $\mathcal{U}_{\mathcal{D}}(D) \cap F_{\mathcal{D}}(X) = \emptyset$ and let $\mathcal{D}' \subseteq \mathcal{G}(\mathfrak{P})$ obtained from $\mathcal{D}$ by replacing (Y, Z) and (Y', Z') with $(Y', Z \cup Z')$. By $(I4)$ we obtain $(Y', Z \cup Z') \rightsquigarrow_\Omega (Y', Z')$, so $\mathcal{D}' \rightsquigarrow_\Omega (Y', Z')$. Since $(Y, Z), (Y', Z') \notin \mathcal{U}_{\mathcal{D}}(D)$, D is also a $\mathcal{D}'|(Y, Y')$-tail decomposition digraph, by the proof of Theorem 5.4.4, $\mathcal{D}' \rightsquigarrow_\Omega (Y, Y')$ and thus $\mathcal{D}' \rightsquigarrow_\Omega (Y, Z)$, so finally we obtain $\mathcal{D} \rightsquigarrow_\Omega \mathcal{D}'$. On the other side it is easy to prove that $\mathcal{D}' \rightsquigarrow_\Omega \mathcal{D}$, so $\mathcal{D}$ and $\mathcal{D}'$ are equivalent, by contradicting the minimality of $\mathcal{D}$. The claim is thus proved.
(ii): If $X' \in f_{\mathcal{D}}(X)$, then trivially X' resolves itself and $(X', X') \notin \mathcal{D}$ by the minimality of $\mathcal{D}$.

So the claim trivially holds in this case.
Let us suppose by contradiction that the thesis does not hold. Since $(X', Y') \in \mathcal{G}(\mathfrak{P})$ for any choice of $Y' \in f_{\mathcal{D}}(X)$, there exists a $\mathcal{D}|(X', Y')$-tail decomposition digraph for every $Y' \in f_{\mathcal{D}}(X)$. Let Y' be a subset of Ω such that the number of nodes of $\mathcal{D}$ in a $\mathcal{D}|(X', Y')$-tail decomposition digraph is minimum and let D be such a digraph. Let us prove that $F_{\mathcal{D}}(X') \cap \mathcal{U}_{\mathcal{D}}(D) = \emptyset$. For, let $(U, V) \in \mathcal{D} \cap \mathcal{U}_{\mathcal{D}}(D)$. Then, it follows easily by Theorem 5.4.4 and Proposition 5.4.7 that D is a (X', U)-tail decomposition digraph and there exists a pair in $\mathcal{U}_{\mathcal{D}}(D)$ that can be removed and still leave a $\mathcal{D}|(X', U)$-tail decomposition digraph H', contradicting our assumptions. Then, $F_{\mathcal{D}}(X) \cap \mathcal{U}_{\mathcal{D}}(D) = \emptyset$. □

Theorem 5.5.10. *Let $\mathcal{D}$ and $\mathcal{D}'$ such that $\mathcal{D} \Vdash_{egs} \mathfrak{P}$ and $\mathcal{D}' \Vdash_{egs} \mathfrak{P}$. Then, $|F_{\mathcal{D}}(X)| = |L_{\mathcal{D}'}(X)|$ for any $X \in \mathcal{P}(\Omega)$.*

Proof. Let us suppose, by contradiction, that $F_{\mathcal{D}}(X) = \{(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\}$ and $L_{\mathcal{D}'}(X) = \{(X'_1, Y'_1), (X'_2, Y'_2), \ldots, (X'_m, Y_m)\}$, with $n < m$. By (i) of Proposition 5.5.9 there exist $j \in \{1, 2, \ldots, m\}$ such that X'_j is different from any of the X_i's.

By (ii) of Proposition 5.5.9 X'_j resolves X_k for some $k \in \{1, 2, \ldots, n\}$. Without loss in generality we can consider $j = 1$ and $k = 1$. Let $\mathcal{D}''$ obtained from $\mathcal{D}'$ by replacing (X'_1, Y'_1) with (X_1, Y'_1). Then, $\mathcal{D}''$ is a minimum generator of the semi-symmetries of $\mathfrak{P}$ such that $|L_{\mathcal{D}'}(X)| = |L_{\mathcal{D}''}(X)|$.

If $X_1 = X'_j$ for some $j > 1$, then, by $(I3)$, (X_1, Y'_1), $(X'_j, Y'_j) \rightsquigarrow_{\Omega} (X_1, Y'_1 \cup Y'_j)$ and conversely, by $(I6)$, it holds both $(X_1, Y'_1 \cup Y'_j) \rightsquigarrow_{\Omega} (X_1, Y'_1)$ and $(X_1, Y'_1 \cup Y'_j) \rightsquigarrow_{\Omega} (Y_j, Y'_j)$, by contradicting the minimality of $\mathcal{D}''$. Thus $X_1 \neq X'_j$ for all $j > 1$ and $|f_{\mathcal{D}}(X) \cap l_{\mathcal{D}''}(X)| = |f_{\mathcal{D}}(X) \cap l_{\mathcal{D}'}(X)| + 1$. As in the beginning of the proof we still have that some X'_j is different from any of the X_i's. The previous procedure can thus be iterated, obtaining a minimum generator $\mathcal{D}^{(s)}$ of the pairs of $\mathfrak{P}$ such that $f_{\mathcal{D}}(X) \subset l_{\mathcal{D}^{(s)}}(X)$.

Let now $X'_j \in l_{\mathcal{D}^{(s)}}(X) \setminus f_{\mathcal{D}}(X)$. Then, by (ii) of Proposition 5.5.9, there exists $k \in \{1, 2, \ldots, n\}$ such that X'_j resolves X_k. Let $\mathcal{F}$ be the minimum generator of the semi-symmetries of $\mathfrak{P}$ obtained from $\mathcal{D}^{(s)}$ by replacing (X'_j, Y'_j) with (X_k, Y'_j). By (i) of Proposition 5.5.9, $Y'_k = Y'_j$. So $m = n$ and the theorem is proved. □

5.6 Pointed Pair Systems on Ω

Equation (5.17) allows us to investigate the general properties of the pairs $(X, x) \in \mathcal{P}(\Omega) \times \Omega$. In what follows, we assume that Ω is an arbitrary (even non finite) set. We call any ordered pair $(X, x) \in \mathcal{P}(\Omega) \times \Omega$ a *pointed pair* on Ω. Let $\hat{\Omega} := \mathcal{P}(\Omega) \times \Omega$, $\hat{\Omega}_{tr} := \{(X, x) \in \hat{\Omega} : x \in X\}$ and $\hat{\Omega}_{ntr} := \hat{\Omega} \setminus \hat{\Omega}_{tr}$. We call the elements of $\hat{\Omega}_{tr}$ and $\hat{\Omega}_{ntr}$ respectively the *trivial* and the *non-trivial* pointed pairs. Let $SS(\Omega) := \mathcal{P}(\mathcal{P}(\Omega))$ and $PPS(\Omega) := \mathcal{P}(\hat{\Omega})$.

Let us define a map $\partial : PPS(\Omega) \to SS(\Omega)$ such that

$$\partial(\mathcal{D}) := \{A \in \mathcal{P}(\Omega) : (A \setminus \{x\}, x) \notin \mathcal{D} \ \forall x \in A\}, \tag{5.18}$$

for any $\mathcal{D} \in PPS(\Omega)$. We call the set system $\partial(\mathcal{D})$ the *boundary* of $\mathcal{D}$.

We now provide some basic properties of the boundary of a pointed pair family $\mathcal{D}$.

Proposition 5.6.1. *Let $\{\mathcal{D}_i : i \in I\} \subseteq PPS(\Omega)$. Then, the following conditions hold:*
(i) If $\mathcal{D}_i \subseteq \mathcal{D}_j$ then $\partial(\mathcal{D}_i) \supseteq \partial(\mathcal{D}_j)$;
(ii) If $\mathcal{D} := \bigcap_{i \in I} \mathcal{D}_i$, then $\partial(\mathcal{D}) \supseteq \bigcup_{i \in I} \partial(\mathcal{D}_i)$;
(iii) If $\mathcal{D} := \bigcup_{i \in I} \mathcal{D}_i$, then $\partial(\mathcal{D}) = \bigcap_{i \in I} \partial(\mathcal{D}_i)$.

Proof. (i): Let $A \in \partial(\mathcal{D}_j)$. Then, for any $x \in A$ it results that $(A \setminus \{x\}, x) \notin \mathcal{D}_j$ so, a fortiori, $(A \setminus \{x\}, x) \notin \mathcal{D}_i$, i. e. $A \in \partial(\mathcal{D}_i)$.
(ii): It follows immediately by part (i).
(iii): By part (i), it is straightforward to see that $\partial(\mathcal{D}) \subseteq \bigcap_{i \in I} \partial(\mathcal{D}_i)$. On the other hand, let $A \in \bigcap_{i \in I} \partial(\mathcal{D}_i)$. Hence, for any $x \in A$ and any $i \in I$ it follows that $(A \setminus \{x\}, x) \notin \mathcal{D}_i$, i. e. for any $x \in A$ we have that $(A \setminus \{x\}, x) \notin \bigcup_{i \in I} \mathcal{D}_i = \mathcal{D}$ or, equivalently, $A \in \partial(\mathcal{D})$. □

We now introduce two fundamental equivalence relations on $PPS(\Omega)$.

Definition 5.6.2. Let $\mathcal{D}, \mathcal{D}' \in PPS(\Omega)$. We say that $\mathcal{D}$ and $\mathcal{D}'$ are:

- *boundary equivalent*, denoted by $\mathcal{D} \simeq_\partial \mathcal{D}'$, if $\partial(\mathcal{D}) = \partial(\mathcal{D}')$. We denote by $[\mathcal{D}]_{\simeq_\partial}$ the equivalence class of $\mathcal{D}$ in $PPS(\Omega)$ with respect to the equivalence relation $\simeq_\partial$ and we call it the *boundary class* of $\mathcal{D}$.

- *essentially equivalent*, denoted by $\mathcal{D} \doteq \mathcal{D}'$, if $\mathcal{D} \cap \hat{\Omega}_{ntr} = \mathcal{D}' \cap \hat{\Omega}_{ntr}$. We denote by $[\mathcal{D}]_{\doteq}$ the equivalence class of $\mathcal{D}$ in $PPS(\Omega)$ with respect to the equivalence relation $\doteq$ and call it the *essential class* of $\mathcal{D}$.

The basic relation between the essential equivalence and the boundary equivalence is easily established in next result.

Proposition 5.6.3. *Let $\mathcal{D}, \mathcal{D}' \in PPS(\Omega)$. Then, $\mathcal{D} \doteq \mathcal{D}' \implies \mathcal{D} \simeq_\partial \mathcal{D}'$.*

Proof. Just observe that $A \in \partial(\mathcal{D})$ if and only if $(A \setminus \{x\}, x) \notin \mathcal{D}$ for any $x \in A$ and, by essential equivalence, this is equivalent to say that $(A \setminus \{x\}, x) \notin \mathcal{D}'$ for any $x \in A$, i. e. $A \in \partial(\mathcal{D}')$. This proves that $\partial(\mathcal{D}) = \partial(\mathcal{D}')$. □

At this point, we introduce the specific families of pointed pair systems on Ω that are our main element of study.

Definition 5.6.4. Let $\mathcal{D} \in PPS(\Omega)$. Then, we say that $\mathcal{D}$ is:

- *extensive*, if whenever $(X, x) \in \mathcal{D}$, $Y \in \mathcal{P}(\Omega)$ and $X \subseteq Y$, then $(Y, x) \in \mathcal{D}$. We set $PPS_{ext}(\Omega) := \{\mathcal{D} \in PPS(\Omega) : \mathcal{D} \quad \text{is} \quad \text{extensive}\}$;

- *reflexive*, if $\hat{\Omega}_{tr} \subseteq \mathcal{D}$. We set $PPS_{ref}(\Omega) := \{\mathcal{D} \in PPS(\Omega) : \mathcal{D} \quad \text{is} \quad \text{reflexive}\}$;

- *finitely hereditary*, if for any $(X, x) \in \mathcal{D}$ there exists a finite subset F of X such that $(F, x) \in \mathcal{D}$. We set $PPS_{fh}(\Omega) := \{\mathcal{D} \in PPS(\Omega) : \mathcal{D} \quad \text{is} \quad \text{finitely} \quad \text{hereditary}\}$;

- *absorbing*, if for any $(X, x) \in \hat{\Omega}_{ntr}$ there exists $A \in \partial(\mathcal{D})$ such that $A \subseteq X$ and $(A, x) \in \mathcal{D}$. We set $PPS_{abs}(\Omega) := \{\mathcal{D} \in PPS(\Omega) : \mathcal{D} \quad \text{is} \quad \text{absorbing}\}$;

- *matroidal*, if for any $A \in \partial(\mathcal{D})$ and any $x, y \in \Omega$ such that $(A, x) \notin \mathcal{D}$ and $(A \cup \{y\}, x) \in \mathcal{D}$, then $(A \cup \{x\}, y) \in \mathcal{D}$. We set $PPS_{mat}(\Omega) := \{\mathcal{D} \in PPS(\Omega) : \mathcal{D} \quad \text{is} \quad \text{matroidal}\}$.

We provide below some basic properties of the previous families of pointed pair systems and, in particular, investigate their relations with the corresponding boundaries.

Proposition 5.6.5. *The following conditions hold:*

(i) if $\mathcal{D}$ is extensive then $\partial(\mathcal{D}) \in AC(\Omega)$ and, moreover,

$$A \in \partial(\mathcal{D}) \iff (B \subseteq A,\ x \in A \setminus B \implies (B, x) \notin \mathcal{D}); \tag{5.19}$$

(ii) if $\mathcal{D}$ is finitely hereditary then $\partial(\mathcal{D})$ is a finitary set system on Ω;

(iii) if $\mathcal{D}$ is extensive and finitely hereditary then

$$A \in \partial(\mathcal{D}) \iff (F \subseteq_f A,\ x \in A \setminus F \implies (F, x) \notin \mathcal{D}); \tag{5.20}$$

(iv) if $\mathcal{D}$ is extensive and matroidal, $A \in \partial(\mathcal{D})$ and $x \in \Omega \setminus A$, then

$$(A, x) \in \mathcal{D} \iff A \cup \{x\} \notin \partial(\mathcal{D}); \tag{5.21}$$

(v) if for any $A \in \partial(\mathcal{D})$ and any $x \in \Omega \setminus A$, we have that $A \cup \{x\} \notin \partial(\mathcal{D})$ if and only if $(A, x) \in \mathcal{D}$, then $\mathcal{D}$ is matroidal;

(vi) Let $\mathcal{D}, \mathcal{D}' \in PPS(\Omega)$ such that $\mathcal{D} \subseteq \mathcal{D}'$ and $\mathcal{D} \simeq_\partial \mathcal{D}'$. If $\mathcal{D}$ is extensive and matroidal and $\mathcal{D}'$ is absorbing, then $\mathcal{D}$ is absorbing and $\mathcal{D} \doteq \mathcal{D}'$.

Proof. (i): Clearly, $\emptyset \in \partial(\mathcal{D})$. Let $X \subseteq Y$ with $Y \in \partial(\mathcal{D})$. Suppose by contradiction that $X \notin \partial(\mathcal{D})$. Then, there exists $x \in X$ such that $(X \setminus \{x\}, x) \in \mathcal{D}$. Since $\mathcal{D} \in PPS_{ext}(\Omega)$, it results that $(Y \setminus \{x\}, x) \in \mathcal{D}$, contradicting our hypothesis on Y. This shows that $\partial(\mathcal{D}) \in AC(\Omega)$.
Let $A \in \partial(\mathcal{D})$, $B \subseteq A$ and $x \in A \setminus B$. It must be $(B, x) \notin \mathcal{D}$, otherwise $(A \setminus \{x\}, x) \in \mathcal{D}$, contradicting the fact that $A \in \partial(\mathcal{D})$.
On the contrary, let $A \in \mathcal{P}(\Omega)$ and assume that for any $B \subseteq A$ and any $x \in A \setminus B$, it follows that $(B, x) \notin \mathcal{D}$. Let us fix $x \in A$ and set $B := A \setminus \{x\}$. Then, $(A \setminus \{x\}, x) \notin \mathcal{D}$ and, by the arbitrariness of x, we infer the thesis.
(ii): Let $X \in \mathcal{P}(\Omega)$ and assume that $F \in \partial(\mathcal{D})$ for any $F \subseteq_f X$. Suppose by contradiction that $X \notin \partial(\mathcal{D})$, i. e. there exists $x \in X$ for which $(X \setminus \{x\}, x) \in \mathcal{D}$. In particular, there exists $F' \subseteq_f X \setminus \{x\}$ such that $(F', x) \in \mathcal{D}$. Let $G := F' \cup \{x\}$. Then, $G \in \partial(\mathcal{D})$ but this would imply that $(G \setminus \{x\}, x) = (F, x) \notin \mathcal{D}$, that is a contradiction. This proves that $X \in \partial(\mathcal{D})$.
(iii): Let $A \in \mathcal{P}(\Omega)$ such that $(F, x) \notin \mathcal{D}$ for any $F \subseteq_f A$ and any $x \in A \setminus F$. Assume by contradiction that $A \notin \partial(\mathcal{D})$, i. e. $(A \setminus \{x\}, x) \in \mathcal{D}$ for some $x \in A$. Since $\mathcal{D} \in PPS_{fh}(\Omega,$ there exists $F \subseteq_f A \setminus \{x\}$ such that $(F, x) \in \mathcal{D}$, but this contradicts our choice of A. Thus $A \in \partial(\mathcal{D})$. To show the converse, just take $F \subseteq_f A$ instead of B in (5.19).

(iv): Let $A \in \partial(\mathcal{D})$, $x \in \Omega \setminus A$ and assume that $(A,x) \in \mathcal{D}$. It must necessarily be $A \cup \{x\} \notin \partial(\mathcal{D})$, otherwise $(A,x) \notin \mathcal{D}$ by definition of boundary. Vice versa, assume that $B := A \cup \{x\} \notin \partial\mathcal{D}$, i. e. there exists $y \in B$ for which $(B \setminus \{y\}, y) \in \mathcal{D}$. If $y = x$, then $(A,x) \in \mathcal{D}$ and the claim has been proved; on the contrary, assume $y \in A$. By part (i), $\partial(\mathcal{D})$ is an abstract complex, therefore $C := A \setminus \{y\} \in \partial(\mathcal{D})$. Note that $(C \cup \{x\}, y) = (B \setminus \{y\}, y) \in \mathcal{D}$ and that $(C, y) = (A \setminus \{y\}, y) \notin \mathcal{D}$ since $A \in \partial(\mathcal{D})$ and $y \in A$. Now, since $\mathcal{D} \in PPS_{mat}(\Omega)$, it results that $(C \cup \{y\}, x) = (A, x) \in \mathcal{D}$ and this shows the claim.
(v): Let $A \in \partial(\mathcal{D})$ and $x, y \in \Omega$ such that $(A,x) \notin \mathcal{D}$ and $(A \cup \{y\}, x) \in \mathcal{D}$. In particular, it must necessarily be $A \cup \{x\} \cup \{y\} \notin \partial(\mathcal{D})$. So, $(A \cup \{x\}, y) \in \mathcal{D}$, i. e. $\mathcal{D} \in PPS_{mat}(\Omega)$.
(vi): Let $(X,x) \in \mathcal{D}$ be a non-trivial pair. Then, $(X,x) \in \mathcal{D}'$. Since $\mathcal{D}' \in PPS_{abs}(\Omega)$, there exists $A \in \partial(\mathcal{D}')$ such that $A \subseteq X$ and $(A,x) \in \mathcal{D}'$. Moreover, by $\partial(\mathcal{D}) = \partial(\mathcal{D}')$, it results that $A \in \partial(\mathcal{D})$. Now, let us observe that $A \cup \{x\} \notin \partial(\mathcal{D}') = \partial(\mathcal{D})$, otherwise $(A,x) \notin \mathcal{D}'$, that contradicts the fact that $\mathcal{D}' \in \mathcal{D}' \in PPS_{abs}(\Omega)$. By (5.21), it follows that $(A,x) \in \mathcal{D}$, i. e. $\mathcal{D} \in \mathcal{D}' \in PPS_{abs}(\Omega)$.
We now show that $\mathcal{D} \doteq \mathcal{D}'$. Let $(X,x) \in \mathcal{D}'$ be non-trivial. Since $\mathcal{D}' \in PPS_{abs}(\Omega)$, there exists $A \in \partial(\mathcal{D}')$ such that $A \subseteq X$ and $(A,x) \in \mathcal{D}'$. In particular, (5.21), $A \cup \{x\} \notin \partial(\mathcal{D}')$. Thus, $A \in \partial(\mathcal{D})$ and $A \cup \{x\} \notin \partial(\mathcal{D})$ and this implies that $(A,x) \in \mathcal{D}$. We conclude the proof by using the fact that $\mathcal{D} \in PPS_{ext}(\Omega)$. $\square$

In what follows we want to understand in which way the boundary and essential classes of a generic $\mathcal{D} \in PPS(\Omega)$ behave with respect to the various families of specific pointed pair systems introduced in Definition 5.6.4. To this regard, we need some notations. We then set:

$$[\mathcal{D}|ema]_{\simeq_\partial} := [\mathcal{D}]_{\simeq_\partial} \cap PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega) \cap PPS_{abs}(\Omega);$$

$$[\mathcal{D}|efm]_{\simeq_\partial} := [\mathcal{D}]_{\simeq_\partial} \cap PPS_{ext}(\Omega) \cap PPS_{fh}(\Omega) \cap PPS_{mat}(\Omega);$$

$$[\mathcal{D}|refm]_{\simeq_\partial} := [\mathcal{D}|efm]_{\simeq_\partial} \cap PPS_{ref}(\Omega);$$

$$[\mathcal{D}|refma]_{\simeq_\partial} := [\mathcal{D}|refm]_{\simeq_\partial} \cap PPS_{abs}(\Omega);$$

$$[\mathcal{D}|m]_{\doteq} := [\mathcal{D}]_{\doteq} \cap PPS_{mat}(\Omega);$$

$$[\mathcal{D}|em]_{\doteq} := [\mathcal{D}]_{\doteq} \cap PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega);$$

$$[\mathcal{D}|efm]_{\doteq} := [\mathcal{D}|em]_{\doteq} \cap PPS_{fh}(\Omega).$$

In the next results, we will prove that both $[\mathcal{D}|refm]_{\simeq_\partial}$ and $[\mathcal{D}|efm]_{\doteq}$ are complete lattices for each $\mathcal{D} \in PPS(\Omega)$.

Theorem 5.6.6. *For any $\mathcal{D} \in PPS(\Omega)$, $[\mathcal{D}|refm]_{\simeq_\partial}$ is a complete lattice such that if $\{\mathcal{D}_i : i \in I\} \subseteq [\mathcal{D}|refm]_{\simeq_\partial}$ then $\bigvee_{i \in I} \mathcal{D}_i = \bigcup_{i \in I} \mathcal{D}_i$ and $\bigwedge_{i \in I} \mathcal{D}_i = \bigcap_{i \in I} \mathcal{D}_i$ when I is finite.*

Proof. Let us fix $\mathcal{D} \in PPS(\Omega)$ and let $\{\mathcal{D}_i : i \in I\} \subseteq [\mathcal{D}|refm]_{\simeq_\partial}$. We divide the proof in several steps.

I We will show that $\bigcup_{i\in I} \mathcal{D}_i \in [\mathcal{D}|refm]_{\simeq_\partial}$ and $\bigvee_{i\in I} \mathcal{D}_i = \bigcup_{i\in I} \mathcal{D}_i$.

It is obvious that $\bigcup_{i\in I} \mathcal{D}_i$ is reflexive. Moreover, by *(iii)* of Proposition 5.6.1 we have that $\partial(\bigcup_{i\in I} \mathcal{D}_i) = \bigcap_{i\in I} \partial(\mathcal{D}_i) = \partial(\mathcal{D})$, so $\bigcup_{i\in I} \mathcal{D}_i \simeq_\partial \mathcal{D}_i$ for any $i \in I$. Let us prove now that $\bigcup_{i\in I} \mathcal{D}_i$ is extensive. Assume that $(X, x) \in \bigcup_{i\in I} \mathcal{D}_i$ and $X \subseteq Y$. Then, there exists an index $k \in I$ such that $(X, x) \in \mathcal{D}_k$, so $(Y, x) \in \mathcal{D}_k$ and, in particular, $(Y, x) \in \bigcup_{i\in I} \mathcal{D}_i$. This shows that $\bigcup_{i\in I} \mathcal{D}_i \in PPS_{ext}(\Omega)$.

Let us prove now that $\bigcup_{i\in I}$ is finitely hereditary. For this, assume that $(X, x) \in \bigcup_{i\in I} \mathcal{D}_i$. Then, there exists $k \in I$ such that $(X, x) \in \mathcal{D}_k$, so, there exists $F \subseteq_f X$ such that $(F, x) \in \mathcal{D}_k$. A fortiori, $(F, x) \in \bigcup_{i\in I} \mathcal{D}_i$. This shows that $\bigcup_{i\in I} \mathcal{D}_i \in PPS_{fh}(\Omega)$.

Let us show that $\bigcup_{i\in I} \mathcal{D}_i \in PPS_{mat}(\Omega)$. Let $A \in \partial(\bigcup_{i\in I} \mathcal{D}_i)$ and $x, y \in \Omega$ such that $(A, x) \notin \bigcup_{i\in I} \mathcal{D}_i$ and $(A \cup \{y\}, x) \in \bigcup_{i\in I} \mathcal{D}_i$. It is straightforward to verify the existence of an index $k \in I$ such that $(A \cup \{y\}, x) \in \mathcal{D}_k$. In particular, it follows that $A \in \partial(\mathcal{D}_k)$, $(A, x) \notin \mathcal{D}_k$ and $(A\cup\{y\}, x) \in \mathcal{D}_k$. Hence, $(A\cup\{y\}, x) \in \mathcal{D}_k$, hence $(A\cup\{y\}, x) \in \bigcup_{i\in I} \mathcal{D}_i$. This proves that $\bigcup_{i\in I} \mathcal{D}_i \in PPS_{mat}(\Omega)$.

Finally, it is now easy to deduce that $\bigvee_{i\in I} \mathcal{D}_i = \bigcup_{i\in I} \mathcal{D}_i$.

II We prove that

$$\overline{\mathcal{D}} := \hat{\Omega}_{tr} \cup \{(X, x) \in \hat{\Omega}_{ntr} : \exists A \subseteq X \text{ such that } A \in \partial(\mathcal{D}) \text{ and } A \cup \{x\} \notin \partial(\mathcal{D})\} \quad (5.22)$$

is a minimum in $[\mathcal{D}|refm]_{\simeq_\partial}$.

Obviously, $\overline{\mathcal{D}}$ is reflexive and extensive. Furthermore, it is contained in each $\mathcal{D}'$, with $\mathcal{D}' \in [\mathcal{D}|refm]_{\simeq_\partial}$. In fact, if $(X, x) \in \overline{\mathcal{D}}$ is non-trivial, by (5.21) it follows that $(A, x) \in \mathcal{D}'$ for each $\mathcal{D}' \in [\mathcal{D}|refm]_{\simeq_\partial}$, so $(X, x) \in \mathcal{D}'$ by extensivity.

Let us show that $\overline{\mathcal{D}}$ is finetely hereditary. Let $(X, x) \in \overline{\mathcal{D}}$ be non-trivial. Then, there exists $A \subseteq X$ such that $A \in \partial(\mathcal{D})$ and $A \cup \{x\} \notin \partial(\mathcal{D})$. Since $\partial(\mathcal{D})$ is a finitary abstract complex, there exists $F \subseteq_f A$ such that $F \in \partial(\mathcal{D})$ and $F \cup \{x\} \notin \partial(\mathcal{D})$. This shows that $(F, x) \in \overline{\mathcal{D}}$ and the claim follows.

Let us prove now that $\overline{\mathcal{D}} \in PPS_{mat}(\Omega)$. Let $A \in \partial(\overline{\mathcal{D}})$ and $x, y \in \Omega$ such that $(A, x) \notin \overline{\mathcal{D}}$ and $(A \cup \{y\}, x) \in \overline{\mathcal{D}}$. We have to prove that $(A \cup \{x\}, y) \in \overline{\mathcal{D}}$, ore, equivalently, the existence of $B \subseteq A \cup \{x\}$ belonging to $\partial(\mathcal{D})$ and such that $B \cup \{y\} \notin \partial(\mathcal{D})$. By our assumptions, there exists $C \subseteq A \cup \{y\}$ such that $C \in \partial(\mathcal{D})$ and $C \cup \{x\} \notin \partial(\mathcal{D})$. It is straightforward to see that $y \in C$. Now, let $\sigma = \sigma' \cup \{y\}$, where clearly $C' \subseteq A$. We are saying that $C' \cup \{x\} \cup \{y\} \notin \partial(\mathcal{D})$. We must prove that $C' \cup \{x\} \in \partial(\mathcal{D})$. Suppose by contradiction that $C' \cup \{x\} \notin \partial(\mathcal{D})$. Hence $C' \cup \{x\} \notin \partial(\mathcal{D})$ by (5.21) so, since $\mathcal{D}_i \in PPS_{ext}(\Omega)$ for any $i \in I$, it must be $(A, x) \in \mathcal{D}_i$ for any $i \in I$. Again by (5.21), the previous condition is equivalent to say that $A \cup \{x\} \notin \partial(\mathcal{D})$. This contradicts our choice of A. Hence $C' \cup \{x\} \in \partial(\mathcal{D})$ and setting $B := \sigma' \cup \{x\}$, the claim has been proved.

Finally, let us prove now that $\overline{\mathcal{D}} \in [\mathcal{D}]_{\simeq_\partial}$. Let $A \in \partial(\overline{\mathcal{D}})$ and suppose by contradiction that $A \notin \partial(\mathcal{D})$. Since $\partial(\mathcal{D})$ is finitary, there exists $F \subseteq_f A$ such that $F \notin \partial(\mathcal{D})$. Thus, $F \in \partial(\overline{\mathcal{D}})$, otherwise there would be $y \in F$ such that $(F \setminus \{y\}, y) \in \overline{\mathcal{D}}$ and, by extensivity of $\overline{\mathcal{D}}$, $(A \setminus \{y\}, y) \in \overline{\mathcal{D}}$, contradicting our choice of A. Hence, we proved that $\partial(\overline{\mathcal{D}}) \setminus \partial(\mathcal{D})$ is non-empty. Let G be minimal in $\partial(\overline{\mathcal{D}}) \setminus \partial(\mathcal{D})$. Surely, $G \neq \emptyset$ since $\emptyset$ belongs by definition to each boundary. Therefore, we infer the existence of some $x \in G$ for which $G \setminus \{x\} \in \partial(\mathcal{D})$. Since $G = (G \setminus \{x\}) \cup \{x\} \notin \partial(\mathcal{D})$, we have that $(G \setminus \{x\}, x) \in \overline{\mathcal{D}}$ and, by extensivity of $\overline{\mathcal{D}}$, $(A \setminus \{x\}, x) \in \overline{\mathcal{D}}$, contradicting our choice of A. Thus $\partial(\overline{\mathcal{D}}) \subseteq \partial(\mathcal{D})$. Contrariwise, let $A \in \partial(\mathcal{D})$. If $A \notin \partial(\overline{\mathcal{D}})$, there exists $B \subseteq A \setminus \{x\}$ such that $B \in \partial(\mathcal{D})$ and $B \cup \{x\} \notin \partial(\mathcal{D})$. By (5.21), this is equivalent to require that $(B, x) \in \mathcal{D}'$, with $\mathcal{D}' \in [\mathcal{D}]_{\simeq_\partial}$, so $(A \setminus \{x\}, x) \in \mathcal{D}'$ for any $\mathcal{D}' \in [\mathcal{D}]_{\simeq_\partial}$, contradicting the fact that $A \in \partial(\mathcal{D})$. This proves that $\partial(\mathcal{D}) \subseteq \partial(\overline{\mathcal{D}})$ and, hence $\partial(\mathcal{D}) = \partial(\overline{\mathcal{D}})$.

III Let $\mathcal{L}$ be the family of all lower bounds of $\{\mathcal{D}_i : i \in I\}$. By the previous part, $\mathcal{L}$ is non-empty. By the first part of the proof, $\bigcup \mathcal{L} \in PPS_{ref}(\Omega) \cap PPS_{ext}(\Omega) \cap PPS_{fh}(\Omega) \cap PPS_{mat}(\Omega)$. Moreover, it is obvious that $\bigcup \mathcal{L}$ is maximal in $\mathcal{L}$ and that $\partial(\bigcup \mathcal{L}) = \partial(\mathcal{D}_i)$ for any $i \in I$. This shows that $\bigwedge_{i \in I} \mathcal{D}_i = \bigcup \mathcal{L}$.

IV We now prove that $\bigwedge_{i \in I} \mathcal{D}_i = \bigcap_{i \in I} \mathcal{D}_i$ when I is finite. To be more specific, we will prove that $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{ref}(\Omega) \cap PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega)$ also in the infinite case and, in the last part of the proof, that $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{fh}(\Omega)$ only for I finite.

Obviously $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{ref}(\Omega)$. Let us show that $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{ext}(\Omega)$. Assume that $(X, x) \in \bigcap_{i \in I} \mathcal{D}_i$ and $X \subseteq Y$. Then, $(X, x) \in \mathcal{D}_i$ for any $i \in I$ so $(Y, x) \in \mathcal{D}_i$ for any $i \in I$ and, consequently, $(Y, x) \in \bigcap_{i \in I} \mathcal{D}_i$.

V In order to show that $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{mat}(\Omega)$, we need the following technical result. Let $F \subseteq_f \Omega$. Then, $F \in \partial(\mathcal{D})$ if and only if $F \in \partial(\bigcap_{i \in I} \mathcal{D}_i)$.

To this regard, let us observe that, since $\bigcap_{i \in I} \mathcal{D}_i \subseteq \mathcal{D}_i$ for any $i \in I$, by (i) of Proposition 5.6.1, we deduce that if $F \in \partial(\mathcal{D})$ then $F \in \partial(\bigcap_{i \in I} \mathcal{D}_i)$.

On the other hand, we prove the converse by induction on $|F|$. Assume that $F = \{x\} \in \partial(\bigcap_{i \in I} \mathcal{D}_i)$. If $\{x\} \notin \partial(\mathcal{D})$, then $(\emptyset, x) \in \mathcal{D}_i$ for any $i \in I$, thus $(\emptyset, x) \in \bigcap_{i \in I} \mathcal{D}_i$, so $(\{x\}, x) \notin \partial(\bigcap_{i \in I} \mathcal{D}_i)$, contradicting our hypothesis. Suppose now the claim to be true for all finite subsets having cardinality at most n and let $F \in \partial(\bigcap_{i \in I} \mathcal{D}_i)$ such that $|F| = n + 1$. Suppose by contradiction that $F \notin \partial(\mathcal{D})$. By using (5.21), it is immediate to prove the existence of some $y \in F$ for which $(F \setminus \{y\}, y) \in \mathcal{D}_i$, for any $i \in I$. Since $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{ext}(\Omega)$, by (i) of Proposition 5.6.5, $\partial(\bigcap_{i \in I} \mathcal{D}_i)$ is an abstract complex. This implies that $F \setminus \{y\} \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$. By the inductive hypothesis, $F \setminus \{y\} \in \partial(\mathcal{D})$. But, since $(F \setminus \{y\}, y) \in \bigcap_{i\in I} \mathcal{D}_i$, we are contradicting the fact that $F \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$. This shows that $F \in \partial(\mathcal{D})$.

VI We now show that $\bigcap_{i\in I} \mathcal{D}_i \in [\mathcal{D}]_{\simeq_\partial}$ and that $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{mat}(\Omega)$.
By (ii) of Proposition 5.6.5, $\partial(\mathcal{D})$ is finitary on Ω, while by (i) of Proposition 5.6.1, $\partial(\mathcal{D}) \subseteq \partial(\bigcap_{i\in I} \mathcal{D}_i)$. On the contrary, let $A \notin \partial(\mathcal{D})$. Since $\partial(\mathcal{D})$ is finitary on Ω, there exists $F \subseteq_f A$ such that $F \notin \partial(\mathcal{D})$. In particular, by the previous part, $F \notin \partial(\bigcap_{i\in I} \mathcal{D}_i)$. Therefore, there exists $y \in F$ such that $(F\setminus\{y\}, y) \in \bigcap_{i\in I} \mathcal{D}_i$. Since $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{ext}(\Omega)$, we conclude that $(A \setminus \{y\}, y) \in \bigcap_{i\in I} \mathcal{D}_i$, so $A \notin \partial(\bigcap_{i\in I} \mathcal{D}_i)$. This shows that $\partial(\bigcap_{i\in I} \mathcal{D}_i) = \partial(\mathcal{D})$.
Finally, we must prove that $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{mat}(\Omega)$. Let $A \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$ and $x, y \in \Omega$ such that $(A, x) \notin \bigcap_{i\in I} \mathcal{D}_i$ and $(A \cup \{y\}, x) \in \bigcap_{i\in I} \mathcal{D}_i$. By (5.21), it is straightforward to see that $A \cup \{x\} \in \partial(\mathcal{D})$ and, by the same relation, $(A, x) \notin \mathcal{D}_i$ for each $i \in I$. Since $\mathcal{D}_i \in PPS_{mat}(\Omega)$ for any $i \in I$, it follows that $(A \cup \{x\}, y) \in \mathcal{D}_i$ for any $i \in I$, thus it also belongs to $\bigcap_{i\in I} \mathcal{D}_i$.

VII To conclude the proof, let us show that $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{fh}(\Omega)$ when I is finite. We prove the claim for two pointed pair systems $\mathcal{D}', \mathcal{D}'' \in [\mathcal{D}|refm]_{\simeq_\partial}$. To this regard, let $(X, x) \in \mathcal{D}' \cap \mathcal{D}''$. Then, there exist $F, F' \subseteq_f X$ such that $(F, x) \in \mathcal{D}$ and $(F', x) \in \mathcal{D}''$. Thus $(F \cup F', x) \in \mathcal{D}' \cap \mathcal{D}''$ and the claim has been showed. Now, extending by induction the previous argument, we conclude the proof.

$\square$

Remark 5.6.7. The same claim of Theorem 5.6.6 holds even if we consider $[\mathcal{D}|efm]_{\simeq_\partial}$. The main change in this case is in part (**II**), since $\overline{\mathcal{D}}$ can be defined by removing the trivial pairs, i. e.

$$\overline{\mathcal{D}}' := \{(X, x) \in \hat{\Omega}_{ntr} : \exists A \subseteq X \text{ such that } A \in \partial(\mathcal{D}) \text{ and } A \cup \{x\} \notin \partial(\mathcal{D})\}. \tag{5.23}$$

The other modifications are obvious and subsequent.

Theorem 5.6.8. *For any $\mathcal{D} \in PPS(\Omega)$, we have that $[\mathcal{D}|efm]_{\simeq}$ is a complete lattice such that if $\{\mathcal{D}_i\} \subseteq [\mathcal{D}|efm]_{\simeq}$, then $\bigvee_{i\in I} \mathcal{D}_i = \bigcup_{i\in I} \mathcal{D}_i$.*

Proof. Let $\{\mathcal{D}_i : i \in I\} \subseteq [\mathcal{D}|efm]_{\simeq}$. It has been shown in Theorem 5.6.6 that $\bigcup_{i\in I} \mathcal{D}_i \in PPS_{ext}(\Omega) \cap PPS_{fh}(\Omega) \cap PPS_{mat}(\Omega)$. Furthermore, it is straightforward to verify that $\bigcup_{i\in I} \mathcal{D}_i \in [\mathcal{D}]_{\simeq}$ and, hence, that $\bigvee_{i\in I} \mathcal{D}_i = \bigcup_{i\in I} \mathcal{D}_i$.

Let us consider the pointed pair system

$$\hat{\mathcal{D}} := \{(X,x) \in \hat{\Omega} : \exists A \subseteq X \text{ such } \text{ that } x \notin A \text{ and } (A,x) \in \mathcal{D}\}. \tag{5.24}$$

Obviously $\hat{\mathcal{D}} \in PPS_{ext}(\Omega) \cap PPS_{fh}(\Omega)$. We now claim that $\hat{\mathcal{D}}PPS_{mat}(\Omega)$. To this regard, let $A \in \partial(\hat{\mathcal{D}})$ and $x \in \Omega \setminus A$. We argue that $A \cup \{x\} \notin \partial(\hat{\mathcal{D}})$ if and only if $(A,x) \in \hat{\mathcal{D}}$. In fact, assume that $A \cup \{x\} \notin \partial(\hat{\mathcal{D}})$. It must necessarily be $(A,x) \in \hat{\mathcal{D}}$, otherwise $(A,x) \notin \mathcal{D}_i$ for any $i \in I$ and, by (5.21), this is equivalent to say that $A \cup \{x\} \in \partial(\mathcal{D}_i) \subseteq \partial(\hat{\mathcal{D}})$, that is a contradiction. Contrariwise, if $(A,x) \in \hat{\mathcal{D}}$, by definition of boundary, we must necessarily have $A \cup \{x\} \notin \partial(\hat{\mathcal{D}})$. We infer the claim by (v) of Proposition 5.6.5.
Furthermore, $\hat{\mathcal{D}}$ is a minimum in $[\mathcal{D}|efm]_{\doteq}$. In fact, it is immediate to see that if $(X,x) \in \hat{\mathcal{D}}$, then $(X,x) \in \mathcal{D}'$ for any $\mathcal{D}' \in [\mathcal{D}|efm]_{\doteq}$. Finally, if $(X,x) \in \mathcal{D}'$ is a non-trivial pair, then it obviously belongs to $\hat{\mathcal{D}}$. This shows that $\hat{\mathcal{D}} \in [\mathcal{D}]_{\doteq}$.
Let now $\mathcal{L}$ be the family of all lower bounds of $\{\mathcal{D}_i : i \in I\}$. By the above argument, $\mathcal{L}$ is non-empty. Moreover $\bigcup \mathcal{L} \in [\mathcal{D}]_{\doteq} \cap PPS_{ext}(\Omega) \cap PPS_{fh}(\Omega) \cap PPS_{mat}(\Omega)$, belongs to $\mathcal{D}_i$ for any $i \in I$ and it is maximal in $\mathcal{L}$. This shows that $\bigwedge_{i \in I} \mathcal{D}_i = \bigcup \mathcal{L}$ and, hence, that $[\mathcal{D}|efm]_{\doteq}$ is a complete lattice. □

In the next result, we will show that whenever we take $\mathcal{D}' \in PPS(\Omega)$ such that $\mathcal{D}_1 \subseteq \mathcal{D}' \subseteq \mathcal{D}_2$, where $\mathcal{D}_1, \mathcal{D}_2 \in [\mathcal{D}|efm]_{\doteq}$, then $\mathcal{D}'$ inherits only matroidality and essential equivalence.

Proposition 5.6.9. *For any $\mathcal{D}, \mathcal{D}' \in PPS(\Omega)$ and $\mathcal{D}_1, \mathcal{D}_2 \in [\mathcal{D}|efm]_{\doteq}$ such that $\mathcal{D}_1 \subseteq \mathcal{D}' \subseteq \mathcal{D}_2$, we have that $\mathcal{D}' \in [\mathcal{D}|m]_{\doteq}$.*

Proof. It is immediate to show that $\mathcal{D}' \in [\mathcal{D}]_{\doteq}$ We now prove that $\mathcal{D} \in PPS_{mat}(\Omega)$. To this regard, let $A \in \partial(\mathcal{D}')$ and $x \in \Omega \setminus A$. Let us show that $A \cup \{x\} \notin \mathcal{D}'$ if and only if $(A,x) \in \partial(\mathcal{D}')$. Assume that $(A,x) \in \mathcal{D}'$. Then, $(A,x) \in \mathcal{D}_1$ so, by (5.21), $A \cup \{x\} \notin \partial(\mathcal{D}_1)$. In particular, $A \cup \{x\} \notin \partial(\mathcal{D}')$. On the contrary, if $A \cup \{x\} \notin \partial(\mathcal{D}')$, then $A \cup \{x\} \notin \partial(\mathcal{D}_1)$. This entails that $(A,x) \notin \mathcal{D}_1$ and, hence, that $(A,x) \notin \mathcal{D}'$. We conclude by (vi) of Proposition 5.6.5. □

We show now that $[\mathcal{D}|ema]_{\simeq_\partial}$ is a complete lattice.

Theorem 5.6.10. *For any $\mathcal{D} \in PPS(\Omega)$, we have that $[\mathcal{D}|ema]_{\simeq_\partial}$ is a complete lattice such that if $\{\mathcal{D}_i\} \subseteq [\mathcal{D}|ema]_{\simeq_\partial}$, then $\bigvee_{i \in I} \mathcal{D}_i = \bigcup_{i \in I} \mathcal{D}_i$ and $\bigwedge_{i \in I} \mathcal{D}_i = \bigcap_{i \in I} \mathcal{D}_i$.*

Proof. Let $\{\mathcal{D}_i : i \in I\} \subseteq [\mathcal{D}|ema]_{\simeq_\partial}$. It has been shown in the proof of Theorem 5.6.6 that $\bigcup_{i \in I} \mathcal{D}_i \in [\mathcal{D}]_{\simeq_\partial} \cap PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega)$. We now show that $\bigcup_{i \in I} \mathcal{D}_i \in PPS_{abs}(\Omega)$. Let $(X,x) \in \bigcup_{i \in I} \mathcal{D}_i$ be a non-trivial pair. Then, there exists $k \in I$ such that $(X,x) \in \mathcal{D}_k$, so there exists $A \in \partial(\mathcal{D}_k)$ with $A \subseteq X$ and $(A,x) \in \mathcal{D}_k$. We conclude by using the fact that $\bigcup_{i \in I} \mathcal{D}_i \in [\mathcal{D}]_{\simeq_\partial}$. This shows that $\bigcup_{i \in I} \mathcal{D}_i \in PPS_{abs}(\Omega)$. In particular, it is straightforward to deduce that $\bigvee_{i \in I} \mathcal{D}_i = \bigcup_{i \in I} \mathcal{D}_i$.
Let us prove that $\bigwedge_{i \in I} \mathcal{D}_i = \bigcap_{i \in I} \mathcal{D}_i$. It has been shown in Theorem 5.6.6 that $\bigcap_{i \in I} \mathcal{D}_i \in PPS_{ext}(\Omega)$.

Now, we have to prove that $\bigcap_{i\in I} \mathcal{D}_i \in [\mathcal{D}]_{\simeq_\partial} \cap PPS_{abs}(\Omega) \cap PPS_{mat}(\Omega)$. By (i) of Proposition 5.6.1, we have $\partial(\mathcal{D}) \subseteq \partial(\bigcap_{i\in I} \mathcal{D}_i)$. Contrariwise, let $A \in \partial(\bigcap_{i\in I} \mathcal{D}_i) \setminus \partial(\mathcal{D})$. Then, there exists $y \in A$ such that $(A \setminus \{y\}, y) \in \mathcal{D}_j$ for some $j \in I$. Since $\mathcal{D}_j \in PPS_{abs}(\Omega)$, there exists $B \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$ such that $B \subseteq A \setminus \{y\}$ and $(B, y) \in \mathcal{D}_j$. By (5.21), it follows that $B \cup \{y\} \notin \partial(\mathcal{D})$ and, again by (5.21) and since $\mathcal{D}_i \in [\mathcal{D}]_{\simeq_\partial}$ for any $i \in I$, we deduce that $(B, y) \in \mathcal{D}_i$ for any $i \in I$, i. e. $(B, y) \in \bigcap_{i\in I} \mathcal{D}_i$. Since $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{ext}(\Omega)$, we have that $(A \cup \{y\}, y) \in \bigcap_{i\in I} \mathcal{D}_i$, i. e. $A \notin \partial(\bigcap_{i\in I} \mathcal{D}_i)$, contradicting our choice of A. This shows that $\partial(\bigcap_{i\in I} \mathcal{D}_i) = \partial(\mathcal{D})$.
We now prove that $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{mat}(\Omega)$. Let $A \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$ and $x, y \in \Omega$ such that $(A, x) \notin \bigcap_{i\in I} \mathcal{D}_i$ and $(A \cup \{y\}, x) \in \bigcap_{i\in I} \mathcal{D}_i$. Thus, $(A, x) \notin \mathcal{D}_i$ for each $i \in I$ and, by (5.21), it follows that $A \cup \{x\} \in \partial(\mathcal{D}_i)$ for any $i \in I$. Since $\mathcal{D}_i \in PPS_{mat}(\Omega)$ for any $i \in I$, it results that $(A \cup \{x\}, y) \in \mathcal{D}_i$ for any $i \in I$, thus it also belongs to $\bigcap_{i\in I} \mathcal{D}_i$.
Finally, we show that $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{abs}(\Omega)$. To this regard, let $(X, x) \in \bigcap_{i\in I} \mathcal{D}_i$ be non-trivial. Suppose by contradiction that for each $A \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$, with $A \subseteq X$, it results that $(A, x) \notin \bigcap_{i\in I} \mathcal{D}_i$. Since $\bigcap_{i\in I} \mathcal{D}_i \in PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega)$, by (5.21) we have that $A \cup \{x\} \in \partial(\bigcap_{i\in I} \mathcal{D}_i)$. Then, $A \cup \{x\} \in \partial(\mathcal{D})$, so, again by (5.21), $(A, x) \notin \mathcal{D}_j$ for some $j \in I$. In other terms, we are saying that $(X, x) \in \mathcal{D}_j$, $x \in \Omega \setminus X$ and for each $A \in \partial(\mathcal{D}_j)$ with $A \subseteq X$, it results that $(A, x) \notin \mathcal{D}_j$, contradicting the fact that $\mathcal{D}_j \in PPS_{abs}(\Omega)$. It is now straightforward to show that $\bigwedge_{i\in I} \mathcal{D}_i = \bigcap_{i\in I} \mathcal{D}_i$. □

In the next result we assert that boundary equivalence, together with extensivity, matroidality and absorbency entails essential equivalence.

We now prove that $[\mathcal{D}|refma]_{\simeq_\partial}$ is an ideal of $[\mathcal{D}|refm]_{\simeq_\partial}$.

Theorem 5.6.11. *For any $\mathcal{D} \in PPS(\Omega)$ we have that $[\mathcal{D}|refma]_{\simeq_\partial}$ is an ideal of the lattice $[\mathcal{D}|refm]_{\simeq_\partial}$.*

Proof. First of all, we have to show that $[\mathcal{D}|refma]_{\simeq_\partial}$ is non-empty. To this regard, let us consider the pointed pair system $\overline{\mathcal{D}}$ defined in (5.22). By (5.21), it is straightforward to see that $\overline{\mathcal{D}}$ is absorbing. Hence $[\mathcal{D}|refma]_{\simeq_\partial}$ is non-empty.
Let $\mathcal{D}' \subseteq \mathcal{D}$, where $\mathcal{D} \in [\mathcal{D}|refma]_{\simeq_\partial}$ and $\mathcal{D}' \in [\mathcal{D}|refm]_{\simeq_\partial}$. By (vi) of Proposition 5.6.5, it follows that $\mathcal{D}' \in [\mathcal{D}|refma]_{\simeq_\partial}$.
Finally, by Theorem 5.6.6, Corollary 5.6.11 and Theorem 5.6.10, given $\mathcal{D}', \mathcal{D}'' \in [\mathcal{D}|refma]_{\simeq_\partial}$, consider $\mathcal{D}' \cup \mathcal{D}'' \in [\mathcal{D}|refma]_{\simeq_\partial}$. Then, $\mathcal{D}', \mathcal{D}'' \subseteq \mathcal{D}' \cup \mathcal{D}''$ and this proves the claim. □

To conclude this section, we consider the map $\eth : SS(\Omega) \to PPS(\Omega)$ such that

$$\eth(\mathcal{F}) := \hat{\Omega}_{tr} \cup \{(X, x) \in \hat{\Omega}_{ntr} : X \cup \{x\} \notin \mathcal{F}\}, \tag{5.25}$$

for any $\mathcal{F} \in SS(\Omega)$. We now show that ∂ and $\eth$ are inverse each other.

Proposition 5.6.12. *Let $\mathcal{F} \in SS(\Omega)$. Then:*
(i) $\mathcal{F} = \partial(\eth(\mathcal{F}))$.
(ii) Let $X \in \mathcal{P}(\Omega)$ and $x, y \notin X$. Assume that $(X, x) \notin \eth(\mathcal{F})$ but $(X \cup \{y\}, x) \in \eth(\mathcal{F})$. Then, $(X \cup \{x\}, y) \in \eth(\mathcal{F})$.

Proof. (i): Let $X \in \partial(\mathcal{D}_{\mathcal{F}})$. This is equivalent to say that for each $x \in X$ it must be $(X \setminus \{x\}, x) \notin \eth(\mathcal{F})$, i. e. $X \setminus \{x\} \cup \{x\} = X \in \mathcal{F}$.
(ii): By our assumptions, we have that $X \cup \{x\} \in \mathcal{F}$ but $X \cup \{x\} \cup \{y\} \notin \mathcal{F}$. It is now obvious that $(X \cup \{x\}, y) \in \mathcal{F}$ and the claim has been shown. □

Summarizing suitably the results of this section, we can state the following result.

Theorem 5.6.13. *Let $\mathcal{D} \in PPS(\Omega)$ extensive, finitely hereditary and matroidal. Then, $\partial(\mathcal{D}) \in FAC(\Omega)$. Moreover, if $\mathcal{F} \in FAC(\Omega)$ then the $[\eth(\mathcal{F})|efm]_{\simeq_\partial}$ is a complete lattice.*

5.7 Pre-Closure Operators and Pointed Pair Systems

In this final section we study the interrelations between pre-closure operators on Ω and pointed pair systems $\mathcal{D} \in PPS(\Omega)$ that are extensive, finitely hereditary and matroidal.

Let $\mathcal{D} \in PPS(\Omega)$, then we consider the set operator $\sigma_{\mathcal{D}} \in OP(\Omega)$ given by

$$\sigma_{\mathcal{D}}(X) := \{x \in \Omega : \, x \in X \, \vee \, (X, x) \in \mathcal{D}\}, \tag{5.26}$$

Contrariwise to any set operator $\sigma \in OP(\Omega)$ we associate the pointed pair system

$$\mathcal{D}_\sigma := \{(X, x) \in \hat{\Omega} : \, x \in \sigma(X)\} \tag{5.27}$$

In the next result, we prove that, whenever σ is an extensive set operator, then $\sigma = \sigma_{\mathcal{D}_\sigma}$.

Proposition 5.7.1. *If $\sigma \in OP(\Omega)$ is extensive, then $\sigma = \sigma_{\mathcal{D}_\sigma}$.*

Proof. Let $X \in \mathcal{P}(\Omega)$. Then, by both (5.26) and (5.27), it results that $x \in \sigma_{\mathcal{D}_\sigma}(X)$ if and only if $x \in X$ or $(X, x) \in \mathcal{D}_\sigma$, i. e. if and only if $x \in X$ or $x \in \sigma(X)$. By extensiveness of σ, we conclude that the previous condition is equivalent to say that $x \in \sigma(X)$. We conclude by the arbitrariness of $X \in \mathcal{P}(\Omega)$. □

We have the following result.

Theorem 5.7.2. *Let $\mathcal{D} \in PPS(\Omega)$ extensive, finitely hereditary an matroidal. Then, $\sigma_{\mathcal{D}}$ is a WMLS, finitary pre-closure operator on Ω. On the contrary, let σ be a WMLS, finitary pre-closure operator on Ω. Then, $\mathcal{D}_\sigma$ is a reflexive, extensive, finitely hereditary and matroidal pointed pair system on Ω and, moreover, $[\mathcal{D}_\sigma|efm]_{\simeq}$ is a complete lattice.*

Proof. We divide the proof in two parts.

(I) Let $\mathcal{D} \in PPS(\Omega)$ be extensive, finitely hereditary and matroidal. We associate the $\sigma_{\mathcal{D}}$ given in (5.26). It is easy to show that it is a pre-closure operator. Let us prove that

$$\Theta(\sigma_{\mathcal{D}}) = \partial(\mathcal{D}). \tag{5.28}$$

Let $X \in \Theta(\sigma_{\mathcal{D}})$ and assume that $X \notin \partial(\mathcal{D})$. Thus, there exists $x \in X$ such that $(X \setminus \{x\}, x) \in \mathcal{D}$. In particular, we have that $X \setminus \{x\} \nsubseteq X \subseteq \sigma_{\mathcal{D}}(X \setminus \{x\})$, obviously a contradiction with our choice of X.

Contrariwise, let $A \notin \Theta(\sigma_{\mathcal{D}})$. This means that there exists $B \subsetneqq A \subseteq \sigma_{\mathcal{D}}(B)$. In particular, there exists $x \in A \setminus B$ such that $(B, x) \in \mathcal{D}$. Since $\mathcal{D} \in PPS_{ext}(\Omega)$, it follows that $(A \setminus \{x\}, x) \in \mathcal{D}$, i. e. $A \notin \partial(\mathcal{D})$. This proves (5.28).

Let us prove that $\sigma_{\mathcal{D}}(X)$ is finitary, i. e. $\sigma_{\mathcal{D}}(X) = \bigcup\{\sigma_{\mathcal{D}}(F) : F \subseteq_f X\}$ for any $X \in \mathcal{P}(\Omega)$. Let $x \in \sigma_{\mathcal{D}}(X)$. If $x \in X$, then $x \in \sigma_{\mathcal{D}}(\{x\})$. On the contrary, if $x \notin X$, then $(X, x) \in \mathcal{D}$ and, since $\mathcal{D} \in PPS_{fh}(\Omega)$, there exists $F \subseteq_f X$ such that $(F, x) \in \mathcal{D}$. Hence, $x \in \sigma_{\mathcal{D}}(F)$. This proves that $\sigma_{\mathcal{D}}(X) \subseteq \bigcup\{\sigma_{\mathcal{D}}(F) : F \subseteq_f X\}$. The reverse inclusion is obvious. This proves that $\sigma_{\mathcal{D}}$ is finitary.

We have to show that $\sigma_{\mathcal{D}}$ is WMLS. Let $A \in \Theta(\sigma_{\mathcal{D}})$ and $x \in \sigma_{\mathcal{D}}(A \cup \{y\}) \setminus \sigma_{\mathcal{D}}(A)$. This means that $x \notin A$, $(A, x) \notin \mathcal{D}$ and $(A \cup \{y\}, x) \in \mathcal{D}$. Moreover, $A \in \partial(\mathcal{D})$ by (5.28). Hence, by the fact that $\mathcal{D} \in PPS_{ext}(\Omega)$, it follows that $(A \cup \{x\}, y) \in \mathcal{D}$, i. e. $y \in \sigma_{\mathcal{D}}(A \cup \{x\})$. This shows that $\sigma_{\mathcal{D}}$ is WMLS.

(II) Let $\sigma \in OP(\Omega)$ be a WMLS, finitary pre-closure operator. We associate with it the pointed pair system $\mathcal{D}_\sigma$ defined in (5.27). Obviously, $\mathcal{D}_\sigma \in PPS_{ref}(\Omega)$. Moreover, we have that $\mathcal{D}_\sigma \in PPS_{ext}(\Omega)$, in fact if $(X, x) \in \mathcal{D}_\sigma$ and $X \subseteq Y$, then, $x \in \sigma(X) \subseteq \sigma(Y)$, i. e. $(Y, x) \in \mathcal{D}_\sigma$.

It is also straightforward to see that $\mathcal{D}_\sigma \in PPS_{fh}(\Omega)$. As a matter of fact, let $(X, x) \in \mathcal{D}_\sigma$, then $x \in \sigma(X) = \bigcup\{\sigma(F) : F \subseteq_f X\}$, i. e. there exists $F \subseteq_f X$ such that $x \in \sigma(F)$, i. e. $(F, x) \in \mathcal{D}_\sigma$. Let us observe that

$$\Theta(\sigma) = \partial(\mathcal{D}_\sigma) \tag{5.29}$$

by the same argument used to show (5.28). We now prove that if $\mathcal{D} \doteq \mathcal{D}_\sigma$, then $\sigma_{\mathcal{D}} = \sigma$. Given $(X, x) \in \hat{\Omega}_{ntr}$, we have that $(X, x) \in \mathcal{D}$ if and only if $(X, x) \in \mathcal{D}_\sigma$. This implies that $\sigma_{\mathcal{D}}(X) = \sigma_{\mathcal{D}_\sigma}(X) = \sigma(X)$ and, by the arbitrariness of $X \in \mathcal{P}(\Omega)$, that $\sigma_{\mathcal{D}} = \sigma$.

Finally, $\mathcal{D}_\sigma \in PPS_{mat}(\Omega)$. In fact, let $A \in \partial(\mathcal{D}_\sigma)$ and $x, y \in \Omega$ such that $(A, x) \notin \mathcal{D}_\sigma$ and $(A \cup \{y\}, x) \in \mathcal{D}_\sigma$. This implies that $x \in \sigma_{\mathcal{D}_\sigma}(A \cup \{y\}) \setminus \sigma_{\mathcal{D}_\sigma}(A)$, i. e. $x \in \sigma(A \cup \{y\}) \setminus \sigma(A)$. By (5.29), $A \in \Theta(\sigma)$. Since σ is WMLS, then $y \in \sigma(A \cup \{x\})$, i. e. $(A \cup \{x\}, y) \in \mathcal{D}_\sigma$ and the thesis follows.

□

Let $\mathcal{F} \in FAC(\Omega)$. By Theorem 5.6.13, $[\mathfrak{D}(\mathcal{F})|efm]_{\simeq_\partial}$ is a complete lattice. In particular, for any $\mathcal{D} \in [\mathfrak{D}(\mathcal{F})|efm]_{\simeq_\partial}$, by Theorem 5.7.2 we can consider the WMLS, finitary pre-closure operator $\sigma_{\mathcal{D}}$. In what follows, we denote by $WFPCO(\mathcal{F}, \Omega)$ the family of all WMLS, finitary pre-closure operators $\sigma_{\mathcal{D}}$, when $\mathcal{D} \in [\mathfrak{D}(\mathcal{F})|efm]_{\simeq_\partial}$.

In the next result, we show that $(WFPCO(\mathcal{F}, \Omega), \sqsubseteq)$ is a complete lattice and we also determine its minimum element.

Theorem 5.7.3. *If σ be a WMLS, finitary pre-closure operator on Ω then $\partial(\mathcal{D}_\sigma) \in FAC(\Omega)$. Moreover, if $\mathcal{F} \in FAC(\Omega)$ then $(WFPCO(\mathcal{F}, \Omega), \sqsubseteq)$ is a complete lattice whose minimum is*

$$\sigma_{\mathcal{F}}(X) := \begin{cases} X \cup \bigcup\{x : X \cup \{x\} \notin \mathcal{F}\} & \text{if} \quad X \in \mathcal{F} \\ \bigcup_{A \in \mathcal{F}} \{\sigma_{\mathcal{F}}(A) : A \subseteq X\} & \text{otherwise} \end{cases} \tag{5.30}$$

Proof. Let σ be a WMLS, finitary pre-closure operator on Ω. By Theorem 5.7.2, $[\sigma_{\mathcal{D}}|efm]_{\doteq}$ is a complete lattice. By Proposition 5.6.3, for each $\mathcal{D}, \mathcal{D}' \in [\sigma_{\mathcal{D}}|efm]_{\doteq}$, we have that $\partial(\mathcal{D}) = \partial(\mathcal{D}')$. In particular, $\partial(\mathcal{D}_\sigma) \in FAC(\Omega)$ by Theorem 5.6.13.
On the contrary, let $\mathcal{F} \in FAC(\Omega)$. Let us prove that $(WFPCO(\mathcal{F}, \Omega), \sqsubseteq)$ forms a complete lattice. To this aim, let $\{\sigma_i : i \in I\} \subseteq WFPCO(\mathcal{F}, \Omega)$ and let us consider $\sigma := \bigvee_{i \in I} \sigma_i$, where the join has been taken in $(OP, \sqsubseteq)$. We have to prove that σ is a WMLS, finitary pre-closure operator. First of all, let us observe that

$$\sigma(X) := \left\{ x \in \Omega : x \in X \ \vee \ (X, x) \in \bigcup_{i \in I} D_i \right\}$$

for any $X \in \mathcal{P}(\Omega)$. In other terms, σ is the set operator associated with $\bigcup_{i \in I} D_i \in [\eth(\mathcal{F})|efm]_{\simeq_\partial}$.
By Theorem 5.7.2, σ is a WMLS, finitary pre-closure operator and, by (i) of Proposition 5.6.12, $\partial(\bigcup_{i \in I} D_i) = \mathcal{F}$, i. e. $\sigma \in WFPCO(\mathcal{F}, \Omega)$.
On the other hand, let $\{\sigma_i : i \in I\} \subseteq WFPCO(\mathcal{F}, \Omega)$ and $\mathcal{D}^*$ be the meet of the family $\{[\mathcal{D}_{\sigma_i}|efm]_{\doteq} : i \in I\}$ in $[\eth(\mathcal{F})|efm]_{\simeq_\partial}$. Moreover, let us consider the set operator

$$\sigma_{\mathcal{D}^*} := \{x \in \Omega : x \in X \ \vee \ (X, x) \in \mathcal{D}^*\}.$$

Since it is the set operator associated with $\mathcal{D}^*$, we deduce it is a WMLS, finitary pre-closure operator such that $\partial(\bigcup_{i \in I} D_i) = \mathcal{F}$. Furthermore, it is straightforward to show that $\sigma_{\mathcal{D}^*} \sqsubseteq \sigma_i$ for any $i \in I$. This proves that $\sigma_{\mathcal{D}^*} \in WFPCO(\mathcal{F}, \Omega)$, i. e. $(WFPCO(\mathcal{F}, \Omega), \sqsubseteq)$ is a complete lattice.
Let us consider $\sigma_{\overline{\mathcal{D}'}}$, where $\overline{\mathcal{D}'}$ is the minimum of $[\eth(\mathcal{F})|efm]_{\simeq_\partial}$ defined as in (5.23). Hence, by (5.26) it follows that $\sigma_{\overline{\mathcal{D}'}}(X) = \{x \in \Omega : x \in X \ \vee \ (X, x) \in \overline{\mathcal{D}'}\}$. Obviously, $\sigma_{\overline{\mathcal{D}'}}$ is a WMLS, finitary pre-closure operator since it is associated with $\overline{\mathcal{D}}' \in [\eth(\mathcal{F})|efm]_{\simeq_\partial}$, so $\sigma_{\overline{\mathcal{D}'}} \in WFPCO(\mathcal{F}, \Omega)$.
Furthermore, we observe that if $X \in \partial(\overline{\mathcal{D}'}) = \mathcal{F}$, then it results that $\sigma_{\overline{\mathcal{D}'}}(X) = X \cup \bigcup\{x : X \cup \{x\} \notin \mathcal{F}\}$; on the other hand, assume $X \notin \mathcal{F}$. We will show that $\sigma_{\overline{\mathcal{D}'}}(X) := \bigcup_{A \in \mathcal{F}} \{\sigma_{\overline{\mathcal{D}'}}(A) : A \subseteq X\}$. Let $x \in \sigma_{\overline{\mathcal{D}'}}(X)$. Then, or $x \in A$ for some $A \subseteq X$ such that $A \in \mathcal{F}$ or, on the contrary, $A \cup \{x\} \notin \mathcal{F}$, so $x \in \bigcup_{A \in \mathcal{F}} \{\sigma_{\overline{\mathcal{D}'}}(A) : A \subseteq X\}$. Vice versa, let $x \in \bigcup_{A \in \mathcal{F}} \{\sigma_{\overline{\mathcal{D}'}}(A) : A \subseteq X\}$. This means that or $x \in A$ for some $A \subseteq X$ or there exists $A \subseteq X$ such that $A \in \mathcal{F}$ and $A \cup \{x\} \notin \mathcal{F}$, i. e. $(X, x) \in \overline{\mathcal{D}}'$. Thus, $x \in \sigma_{\overline{\mathcal{D}'}}(X)$. In this way, we showed that $\sigma_{\overline{\mathcal{D}'}}$ coincides with the set operator $\sigma_{\mathcal{F}}$ defined in (5.30).
Finally, by (iii) of Proposition 1.5.1, it results that $\sigma_{\mathcal{F}} \sqsubseteq \sigma$ for any $\sigma \in WFPCO(\mathcal{F}, \Omega)$. This concludes the proof. $\square$

Chapter 6

Representation Theorems

In this chapter, we deal with representation theorems. In particular, we will show that an arbitrary closure system $\mathfrak{S}$ on a finite set Ω are represented by a pairing $\mathfrak{P} \in PAIR(\Omega)$ whose $MAXP(\mathfrak{P})$ coincides with $\mathfrak{S}$ and, by using a classical result by Birkhoff and Frink [19], that for any finite lattice $\mathbb{L}$ there exist a finite set $\Omega_{\mathbb{L}}$ and a pairing $\mathfrak{P} \in PAIR(\Omega_{\mathbb{L}})$ whose maximum partitioner lattice is exactly $\mathbb{L}$. In other terms, we can consider closure system theory and finite lattice theory as sub-theories of that of pairings. Finally, we introduce the notion of symmetry transmission average, that provides a numerical information between all the nodes of the finite lattice $\mathbb{L}$. Thus, we are able to compare through the set map $\Gamma_{\mathfrak{P}}$ also two elements of the lattice $\mathbb{L}$ that are non-comparable with respect to the order $\geq_{\mathbb{L}}$.

6.1 Representation Theorems for Closure Systems and Finite Lattices

We now assume that Ω is a finite set and in the next result we provide a representation theorem for arbitrary closure systems on Ω.

Theorem 6.1.1. *The map $MAXP : \mathfrak{P} \in PAIR(\Omega) \mapsto MAXP(\mathfrak{P}) \in CLSY(\Omega)$ is surjective.*

Proof. Let $\mathfrak{S} \in CLSY(\Omega)$, $E \in \mathfrak{S}$ be the minimum element in $\mathfrak{S}$ and let $C_1, C_2, \ldots, C_k$ be all maximal chains from E to the top Ω, where C_i is the chain in $\mathfrak{S}$ given by $A_{i_0} = E \subset A_{i_1} \subset \cdots A_{i_{l_i}} = \Omega$, for $i = 1, 2, \ldots, k$. Furthermore, we set $m_0 = 0$, $m_i := \sum_{s=1}^{i} l_s$, and $m := m_k$. We now define a pairing

$$\mathfrak{P} = \mathfrak{P}(\mathfrak{S}) := \langle U_{\mathfrak{S}}, \Omega, F_{\mathfrak{S}}, \mathbb{N} \rangle \in PAIR(\Omega) \tag{6.1}$$

whose maximum partitioner lattices is exactly $\mathfrak{S}$. First of all, we consider the set $U_{\mathfrak{S}} = \{u_1, u_2, \ldots, u_m\}$. On the other hand, we must define the function $F_{\mathfrak{S}} : U_{\mathfrak{S}} \times \Omega \longrightarrow \mathbb{N}$. To this regard, we set $F_{\mathfrak{S}}(u_s, a_j) := 1$ if $a_j \in E$, for any $u_s \in U_{\mathfrak{S}}$. Let now $a_j \in \Omega \setminus E$. For each $i \in \{1, 2, \ldots, k\}$ we define t_{ij} be the smallest positive integer such that $a_j \in A_{i_{t_{ij}}} \setminus A_{i_{t_{ij}-1}}$.

Finally, we set:

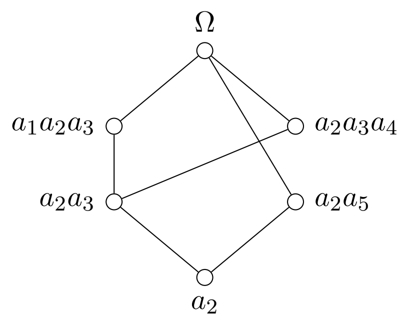
$$F_{\mathfrak{S}}(u_s, a_j) := \begin{cases} 1 & \text{if } s = 1, \\ F_{\mathfrak{S}}(u_{s-1}, a_j) & \text{if } m_{i-1} + 2 \leq s \leq m_i - t_{ij} + 1, \\ F_{\mathfrak{S}}(u_{s-1}, a_j) + 1 & \text{otherwise.} \end{cases} \tag{6.2}$$

Let us firstly show that if $A \in \mathfrak{S}$ then $A \in MAXP(\mathfrak{P})$. If $A = \Omega$ there is nothing to prove. Hence, assume $A \neq \Omega$. Let $C_i : A_{i_0} = E \subset A_{i_1} \subset \cdots A_{i_{l_i}} \subset \Omega$ be a chain from E to Ω and $t \in \{1, 2, \ldots, l_i\}$ be such that $A = A_{i_t}$. Set moreover $s := m_i - t$. Let us prove now that $u_s \equiv_A u_{s+1}$ but, for each choice of $a_j \in \Omega \setminus A$, it holds $F_{\mathfrak{S}}(u_s, a_j) \neq F_{\mathfrak{S}}(u_{s+1}, a_j)$. Let us notice that if $a_j \in A$, then $t_{ij} \leq t$, so $m_{i-1} + 1 \leq s \leq m_i - t_{ij}$. By (6.2), it follows that $F_{\mathfrak{S}}(u_s, a_j) = F_{\mathfrak{S}}(u_{s+1}, a_j)$, therefore we have shown that $u_s \equiv_A u_{s+1}$. On the other hand, if $a_j \notin A$, then $t_{ij} > t$, so $s = m_i - t > m_i - t_{ij}$. Again by (6.2), it follows that $F_{\mathfrak{S}}(u_{s+1}, a_j) = F_{\mathfrak{S}}(u_s, a_j) + 1$, i.e. $u_s \not\equiv_A u_{s+1}$. This proves that $A \in MAXP(\mathfrak{P})$.
Let now $A \in MAXP(\mathfrak{P})$. We will now prove that $A \in \mathfrak{S}$. To this regard, let us consider the minimum element B in $\mathfrak{S}$ such that $A \subseteq B$. We now prove that $A \approx_{\mathfrak{P}} B$. Let us notice that, for any $B' \in \mathfrak{S}$ such that $B' \subset B$, it results that $B' \subset A$. In fact, let $a_k \in B'$ and $a_h \in A \setminus B' \subseteq B \setminus B'$. Let s be such that $u_{s-1} \equiv_A u_s$. Thus $F_{\mathfrak{S}}(u_s, a_h) = F_{\mathfrak{S}}(u_{s-1}, a_h)$, so $m_{i-1} + 2 \leq s \leq m_i - t_{ih} + 1$. But both the conditions $a_h \in B \setminus B'$ and $a_k \in B'$ imply that $t_{ih} > t_{ik}$, so $m_{i-1} + 2 \leq s \leq m_i - t_{ik} + 1$ and, then, $F_{\mathfrak{S}}(u_s, a_k) = F_{\mathfrak{S}}(u_{s-1}, a_k)$. Since $A \in MAXP(\mathfrak{P})$, it holds that $a_k \in A$ and thus that $B' \subset A$. Without loss in generality, we can assume that B' is maximal among the proper subsets of B. Let now C_i be a chain through B. Then, there exists t such that $B' := A_{i_{t-1}} \subset A \subseteq A_{i_t} = B$ and let $s \in \{1, 2, \ldots, m\}$ such that $u_{s-1} \equiv_A u_s$. If $a_k \in B$ and $a_h \in A \setminus B'$, then $t_{ik} < t_{ih}$, so, since $u_{s-1} \equiv_A u_s$, it holds that $F_{\mathfrak{S}}(u_s, a_k) = F_{\mathfrak{S}}(u_{s-1}, a_k)$, thus $a_k \in M_{\mathfrak{P}}(A) = A$. This proves that $A = B \in \mathfrak{S}$ and the thesis has been shown. □

Example 6.1.2. *Let us consider* $\Omega := \{a_1, a_2, a_3, a_4, a_5\}$ *and let*

$$\mathfrak{S} := \{\Omega, \{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_2, a_3\}, \{a_2, a_5\}, \{a_2\}\}$$

be a closure system on Ω*. The minimum in* $\mathfrak{S}$ *is* $E = \{a_2\}$*. The lattice associated with* $\mathfrak{S}$ *has the following Hasse diagram*



	a_1	a_2	a_3	a_4	a_5
u_1	1	1	1	1	1
u_2	1	1	1	2	2
u_3	2	1	1	3	3
u_4	3	1	2	4	4
u_5	4	1	2	4	5
u_6	5	1	2	5	6
u_7	6	1	3	6	7
u_8	7	1	4	7	7

Figure 6.1: The pairing $\mathfrak{P}(\mathfrak{S})$.

There are three chains from E to Ω:

$$\begin{aligned} &C_1 : \{a_2\} \subset \{a_2,\, a_3\} \subset \{a_1,\, a_2,\, a_3\} \subset \Omega \\ &C_2 : \{a_2\} \subset \{a_2,\, a_3\} \subset \{a_2,\, a_3,\, a_4\} \subset \Omega \\ &C_3 : \{a_2\} \subset \{a_2,\, a_5\} \subset \Omega \end{aligned} \tag{6.3}$$

Thus $m_0 = 0$, $m_1 = 3$, $m_2 = 6$ and $m_3 = 8$. Moreover $t_{11} = 2$, $t_{21} = 3$, $t_{31} = 2$, $t_{13} = 1$, $t_{23} = 1$, $t_{33} = 2$, $t_{14} = 3$, $t_{24} = 2$, $t_{34} = 2$, $t_{15} = 3$, $t_{25} = 3$ and $t_{35} = 1$. Then, $U_{\mathfrak{S}} = \{u_1,\, u_2,\, u_3,\, u_4,\, u_5,\, u_6,\, u_7,\, u_8\}$, and, by (6.2) *and* (6.1) *we can construct the pairing in Figure 6.1.*

Moreover, in Figure 6.2 we represent the Hasse diagram of $\mathbb{G}(\mathfrak{P}(\mathfrak{S}))$ (we use string notation in the figure).

Corollary 6.1.3. *Let $\mathfrak{S} \in CLSY(\Omega)$ and let $\mathfrak{P} \in PAIR(\Omega)$ such that $MAXP(\mathfrak{P}) = \mathfrak{S}$. Then, the lattices $PSYM(\mathfrak{P})$, $\mathbb{M}(\mathfrak{P})$ and $(\mathfrak{S}, \subseteq^*)$ are order isomorphic.*

Proof. The result is a direct consequence of Theorem 6.1.1 and part (*ii*) of Corollary 2.2.2. □

Another consequence of Theorem 6.1.1 is that any union additive relation on Ω satisfies a condition similar to the definition of the relation $\geq_{\mathfrak{P}}$. Therefore, this means that in the finite case, it is not restrictive to consider the union additive relations induced by pairing systems.

Theorem 6.1.4. *Let $\geq$ be a union additive relation on Ω. Then, there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $\geq$ coincides with $\geq_{\mathfrak{P}}$.*

Proof. Let $\mathcal{D} := \mathcal{D}_{\geq}$ be the union additive family associated with union additive relation $\geq$. By Proposition 5.2.8, we can associate with $\mathcal{D}$ a closure operator $c_{\mathcal{D}}$ such that $c_{\mathcal{D}}(X) := \bigcup\{Y \in \mathcal{P}(\Omega) : (X, Y) \in \mathcal{D}\}$ for any $X \in \mathcal{P}(\Omega)$. By Theorem 1.5.7, to the closure operator $c_{\mathcal{D}}$ corresponds a closure system $\mathfrak{S}$. Moreover, by Theorem 6.1.1, there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $MAXP(\mathfrak{P}) = \mathfrak{S}$. As a direct consequence, it is easy to show that $c_{\mathcal{D}}(X) = M_{\mathfrak{P}}(X)$ for any $X \in \mathcal{P}(\Omega)$. Indeed, $X \subseteq M_{\mathfrak{P}}(X)$ and since $c_{\mathcal{D}}(X) \in MAXP(\mathfrak{P})$, we conclude that $M_{\mathfrak{P}}(X) \subseteq c_{\mathcal{D}}(X)$. Vice versa, since $M_{\mathfrak{P}}(X) \in \mathfrak{S}$, then $c_{\mathcal{D}}(X) \subseteq M_{\mathfrak{P}}(X)$.
Now, we have to prove that $X \geq Y$ if and only if $X \geq_{\mathfrak{P}} Y$. Assume that $X \geq Y$. Then,

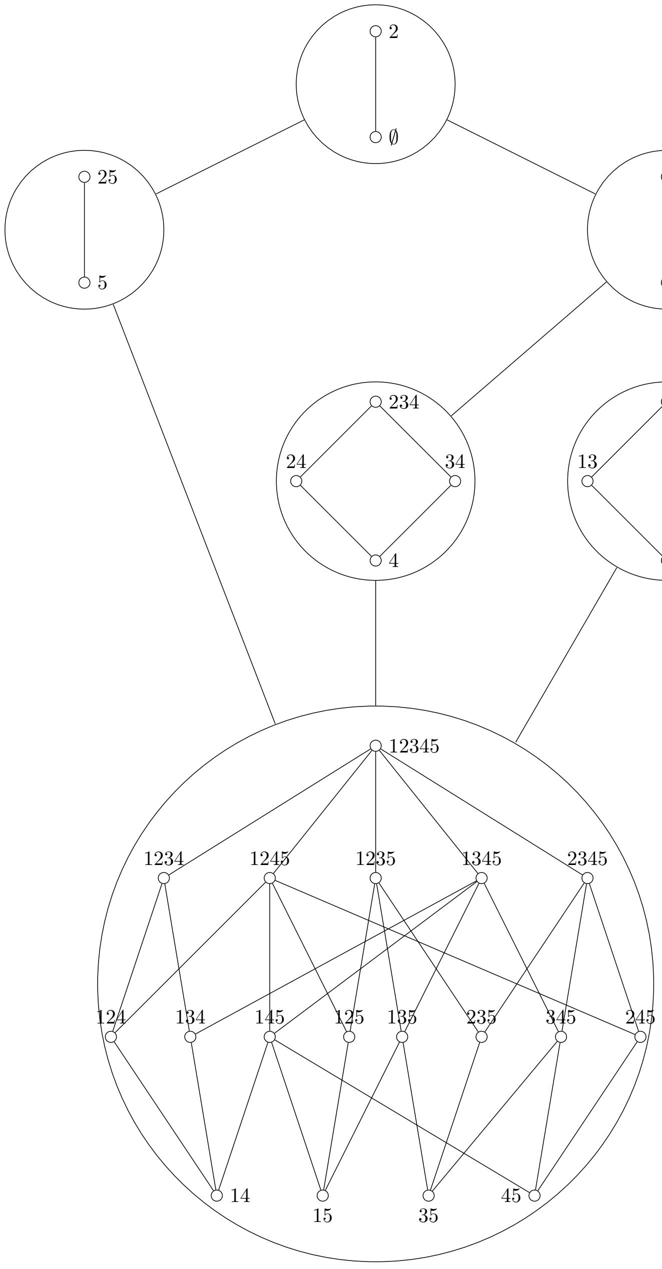


Figure 6.2: Hasse Diagram of $\mathbb{G}(\mathfrak{P}(\mathfrak{S}))$.

$Y \subseteq c_{\mathcal{D}}(X) = M_{\mathfrak{P}}(X)$, so $M_{\mathfrak{P}}(Y) \subseteq M_{\mathfrak{P}}(X)$ and, by definition of $\geq_{\mathfrak{P}}$, it follows that $X \geq_{\mathfrak{P}} Y$. On the other hand, if $X \geq_{\mathfrak{P}} Y$, it follows that $Y \subseteq M_{\mathfrak{P}}(X) = c_{\mathcal{D}}(X)$, so $(X, Y) \in \mathcal{D}$ and, thus, $X \geq Y$. This concludes the proof. □

At this point, based on a classical representation theorem of Birkhoff and Frink (see [19]), we can establish the more important theoretical result of this section, that we can consider a pairing representation theorem for finite lattices.

Theorem 6.1.5. *Let $\mathbb{L} = (L, \leq_{\mathbb{L}})$ be a finite lattice. Then, there exist a finite set Ω, a positive integer n such that $|\Omega| = n$ and a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that the lattices $\mathbb{L}$, $\mathbb{M}(\mathfrak{P})$, $\mathbb{G}(\mathfrak{P})$ and $\mathbb{P}(\mathfrak{P})$ are order isomorphic. Therefore, if $\eta_{\mathfrak{P}} : L \to MAXP(\mathfrak{P})$ is the map that induces such an isomorphism between $\mathbb{L}$ and $\mathbb{M}(\mathfrak{P})$, we have that*

$$x \leq_{\mathbb{L}} x' \iff \eta_{\mathfrak{P}}(x) \subseteq^* \eta_{\mathfrak{P}}(x') \iff \pi_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x)) \preceq \pi_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x')) \iff \gamma_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x), \eta_{\mathfrak{P}}(x')) = 1 \tag{6.4}$$

for any $x, x' \in L$.

Proof. In [19], Birkhoff and Frink proved that any finite lattice is order isomorphic to the corresponding lattice induced from some closure system. So that there exist a finite set Ω, a closure system $\mathfrak{S} \in CLSY(\Omega)$ and a map $\eta : L \to \mathfrak{S}$ that is an order isomorphism between the dual lattice $\mathbb{L}^*$ and the lattice $(\mathfrak{S}, \subseteq)$. Then, by Theorem 6.1.1, we can construct a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $MAXP(\mathfrak{P}) = \mathfrak{S}$. Therefore $\eta_{\mathfrak{P}} := \eta$ is an order isomorphism between $\mathbb{L}$ and $\mathbb{M}(\mathfrak{P})$, and (6.4) becomes a direct consequence of (2.17). □

The relation established in (6.4) has deep theoretical consequences: it says us that the study of the symmetry transmission measure between subsets of finite sets is equivalent to the study of order relations on finite lattices. Therefore, we now investigate the basic theoretical properties of the symmetry transmission measure between subsets of finite sets and the direct interrelation of this notion with other classical notions of lattice theory.

Definition 6.1.6. We call *pairing characteristic* of the lattice $\mathbb{L}$, denoted by $pc(\mathbb{L})$, the minimum positive integer N for which the thesis of Theorem 6.1.5 is verified.

Let $N = pc(\mathbb{L})$. We set then

$$PAIR(\mathbb{L}) := \{\mathfrak{P} \in PAIR(\Omega) : \mathbb{M}(\mathfrak{P}) \cong \mathbb{L}\}.$$

Now, if $\mathfrak{P} \in PAIR(\mathbb{L})$, with the same notations introduced in the statement of Theorem 6.1.5, we can consider the inverse order isomorphism $\eta_{\mathfrak{P}}^{-1} : MAXP(\mathfrak{P}) \to L$ and the closure operator $M_{\mathfrak{P}} : \mathcal{P}(\Omega) \to MAXP(\mathfrak{P})$, so that we obtain the surjective map

$$\xi_{\mathfrak{P}} := \eta_{\mathfrak{P}}^{-1} \circ M_{\mathfrak{P}} : \mathcal{P}(\Omega) \to L$$

Let us note that the map $\xi_{\mathfrak{P}}$ is not canonically determined. In fact, it depends from the order isomorphism $\eta_{\mathfrak{P}}$, which in turn depends on the not uniquely determined closure system $\mathfrak{S}$ given in the proof of Theorem 6.1.5. However, by means of the map $\xi_{\mathfrak{P}}$ we can formally describe the following equivalences.

Theorem 6.1.7. *Let $\mathbb{L} = (L, \leq_{\mathbb{L}})$ be a finite lattice, $N = pc(\mathbb{L})$ and $X, Y \in \mathcal{P}(\Omega)$. Then, the following conditions are equivalent:*
(i) for any $\mathfrak{P} \in PAIR(\mathbb{L})$ we have $\xi_{\mathfrak{P}}(X) \leq_{\mathbb{L}} \xi_{\mathfrak{P}}(Y)$;
(ii) for any $\mathfrak{P} \in PAIR(\mathbb{L})$ we have $\gamma_{\mathfrak{P}}(X, Y) = 1$;
(iii) there exists $\mathfrak{P} \in PAIR(\mathbb{L})$ such that $\gamma_{\mathfrak{P}}(X, Y) = 1$.

Proof. $(i) \implies (ii)$: Let $x := \xi_{\mathfrak{P}}(X) \leq_{\mathbb{L}} y := \xi_{\mathfrak{P}}(Y)$. Let $\mathfrak{P}' \in PAIR(\mathbb{L})$ and $\eta_{\mathfrak{P}'} : L \to MAXP(\mathfrak{P}')$. By (6.4), $\eta_{\mathfrak{P}'}(x) \subseteq^* \eta_{\mathfrak{P}'}(y)$, i.e.

$$\pi_{\mathfrak{P}'}(\eta_{\mathfrak{P}'}(x)) \preceq \pi_{\mathfrak{P}'}(\eta_{\mathfrak{P}'}(x')) \iff \gamma_{\mathfrak{P}'}(\eta_{\mathfrak{P}'}(x), \eta_{\mathfrak{P}'}(x')) = 1$$

and the claim has been shown.
$(ii) \implies (iii)$: Obvious.
$(iii) \implies (i)$: By (6.4), we have that $\gamma_{\mathfrak{P}}(X, Y) = \gamma_{\mathfrak{P}}(M_{\mathfrak{P}}(X), M_{\mathfrak{P}}(Y)) = 1$ if and only if $\pi_{\mathfrak{P}}(X) \preceq \pi_{\mathfrak{P}}(Y)$ or, equivalently, $M_{\mathfrak{P}}(Y) \subseteq M_{\mathfrak{P}}(X)$, that is equivalent to $\eta_{\mathfrak{P}}^{-1}(M_{\mathfrak{P}}(X)) \preceq \eta_{\mathfrak{P}}^{-1}(M_{\mathfrak{P}}(Y))$, i.e. $\xi_{\mathfrak{P}}(X) \leq_{\mathbb{L}} \xi_{\mathfrak{P}}(Y)$. □

6.2 Pairing Representation of Finite Complementary Involution Lattices

In this section, we prove a second representation theorem for finite lattices. For, we assume that $M_{\mathfrak{P}}(\emptyset) = \emptyset$ and define a set operator $I_{\mathfrak{P}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ that behaves as the topological interior part of a set. To be more specific, we set

$$I_{\mathfrak{P}}(A) := (M_{\mathfrak{P}}(A^c))^c \,. \tag{6.5}$$

Definition 6.2.1. We call $I_{\mathfrak{P}}(A)$ the *interior* of A. We say that $A \in \mathcal{P}(\Omega)$ is *regularly closed* if $A = M_{\mathfrak{P}}(I_{\mathfrak{P}}(A))$ and denote by $\mathcal{R}(\mathfrak{P})$ the family of all regularly closed subsets.

Similarly to what happens to its topological counterpart, the complement of the interior of any subset belongs to $MAXP(\mathfrak{P})$.

Proposition 6.2.2. *Let $A \in \mathcal{P}(\Omega)$. Then:*
(i) $I_{\mathfrak{P}}(A)$ is the greatest subset of A whose complement is a maximum partitioner.
(ii) If $A \subseteq B$ then $I_{\mathfrak{P}}(A) \subseteq I_{\mathfrak{P}}(B)$.
(iii) $I_{\mathfrak{P}}(I_{\mathfrak{P}}(A)) = I_{\mathfrak{P}}(A)$.
(iv) If $Z \in \mathcal{P}(\Omega)$ is the complement of some maximum partitioner, then $M_{\mathfrak{P}}(Z) \in \mathcal{R}(\mathfrak{P})$.
(v) If A and A^c both belong to $MAXP(\mathfrak{P})$ then A is regularly closed.

Proof. (i): Let $Z \subseteq A$ such that $Z = \Omega \setminus M_{\mathfrak{P}}(B)$ for some $B \in \mathcal{P}(\Omega)$. Then, $Z = \Omega \setminus M_{\mathfrak{P}}(B) \subseteq A$ implies that $M_{\mathfrak{P}}(B) \supseteq A^c$, i.e. $M_{\mathfrak{P}}(B) \supseteq M_{\mathfrak{P}}(A^c)$, so $Z \subseteq \Omega \setminus M_{\mathfrak{P}}(A^c) = I_{\mathfrak{P}}(A)$.
(ii): Straightforward.
(iii): $I_{\mathfrak{P}}(I_{\mathfrak{P}}(A)) = \Omega \setminus M_{\mathfrak{P}}(\Omega \setminus I_{\mathfrak{P}}(A)) = \Omega \setminus M_{\mathfrak{P}}(\Omega \setminus (\Omega \setminus M_{\mathfrak{P}}(A^c))) = \Omega \setminus M_{\mathfrak{P}}(A^c) = I_{\mathfrak{P}}(A)$.
(iv): Let $Z \in \mathcal{P}(\Omega)$ be the complement of some maximum partitioner. By (i), it readily follows that $Z \subseteq I_{\mathfrak{P}}(M_{\mathfrak{P}}(Z)) \subseteq M_{\mathfrak{P}}(Z)$, therefore $M_{\mathfrak{P}}(I_{\mathfrak{P}}(M_{\mathfrak{P}}(Z))) = M_{\mathfrak{P}}(Z)$, i.e. $M_{\mathfrak{P}}(Z) \in \mathcal{R}(\mathfrak{P})$.
(v): Suppose that A and A^c both belong to $MAXP(\mathfrak{P})$. Then, $I_{\mathfrak{P}}(A) = (M_{\mathfrak{P}}(A^c))^c = A$, thus $M_{\mathfrak{P}}(I_{\mathfrak{P}}(A)) = A$, i.e. A is regularly closed. □

Let us set $\mathfrak{R}(\mathfrak{P}) := (\mathcal{R}(\mathfrak{P}), \subseteq)$. In next result we will show that $\mathfrak{R}(\mathfrak{P})$ is a complete lattice.

Theorem 6.2.3. *$\mathfrak{R}(\mathfrak{P})$ is a complete lattice with $\hat{0}_{\mathfrak{R}(\mathfrak{P})} = \emptyset$ and $\hat{1}_{\mathfrak{R}(\mathfrak{P})} = \Omega$.*

Proof. It is evident that $\emptyset$ and Ω are both regularly closed subsets. Let Σ be a subfamily of $\mathcal{R}(\mathfrak{P})$ and set

$$A := M_{\mathfrak{P}}\left(I_{\mathfrak{P}}\left(\bigcap_{B\in\Sigma} B\right)\right). \tag{6.6}$$

Since $I_{\mathfrak{P}}\left(\bigcap_{B\in\Sigma} B\right)$ is the complement of a maximum partitioner, by (iv) of Proposition 6.2.2, we deduce that $A \in \mathcal{R}(\mathfrak{P})$. We now prove that $A = \inf \Sigma$. If there exists $A' \in \mathcal{R}(\mathfrak{P})$ such that $A' \subseteq B$ for any $B \in \Sigma$, then $A' \subseteq \bigcap_{B\in\Sigma} B$, so $I_{\mathfrak{P}}(A') \subseteq I_{\mathfrak{P}}\left(\bigcap_{B\in\Sigma} B\right)$ by (ii) of Proposition 6.2.2. Applying $M_{\mathfrak{P}}$ to both sides of the previous relation, we obtain $A' \subseteq A$. This implies that A is the g.l.b. of Σ and our claim has been proved.
On the other hand, set

$$C := M_{\mathfrak{P}}\left(\bigcup_{B\in\Sigma} I_{\mathfrak{P}}(B)\right). \tag{6.7}$$

By (iv) of Proposition 6.2.2, it is clear that $C \in \mathcal{R}(\mathfrak{P})$, since $\bigcup_{B\in\Sigma} I_{\mathfrak{P}}(B) = \Omega \setminus \bigcap_{B\in\Sigma} M_{\mathfrak{P}}(B^c)$ is the complement of a maximum partitioner. We now show that $C = \sup \Sigma$. Let $C' \in \mathcal{R}(\mathfrak{P})$ such that $B \subseteq C'$ for any $B \in \Sigma$, then, by (ii) of Proposition 6.2.2, it follows that $\bigcup_{B\in\Sigma} I_{\mathfrak{P}}(B) \subseteq I_{\mathfrak{P}}(C')$. Applying $M_{\mathfrak{P}}$ to both sides of the previous relation, we obtain $C \subseteq C'$. This implies that A is the l.u.b. of Σ and our claim has been shown. □

Let us define a map $\psi : \mathcal{R}(\mathfrak{P}) \longrightarrow MAXP(\mathfrak{P})$ by setting $\psi(A) := M_{\mathfrak{P}}(A^c)$.

Theorem 6.2.4. *The map ψ satisfies the following properties:*

(i) $\psi(\mathcal{R}(\mathfrak{P})) \subseteq \mathcal{R}(\mathfrak{P})$;

(ii) *If* $A \subseteq B$, *then* $\psi(A) \geq_{\mathfrak{P}} \psi(B)$;

(iii) ψ *is an involution;*

(iv) $\psi(A) \wedge A = \emptyset$ *and* $\psi(A) \vee A = \Omega$.

Proof. (i): Let $A \in \mathcal{R}(\mathfrak{P})$. Since $A \in MAXP(\mathfrak{P})$, then, by (iv) of Proposition 6.2.2, $M_{\mathfrak{P}}(A^c) \in \mathcal{R}(\mathfrak{P})$.
(ii): ψ is clearly order-reversing, in fact if $A \subseteq B$, then $\Omega\setminus B \subseteq \Omega\setminus A$, i.e. $M_{\mathfrak{P}}(\Omega\setminus B) \subseteq M_{\mathfrak{P}}(\Omega\setminus A)$ or, in terms of union additive relation, $M_{\mathfrak{P}}(\Omega \setminus A) \geq_{\mathfrak{P}} M_{\mathfrak{P}}(\Omega \setminus B)$, i.e. $\psi(A) \geq_{\mathfrak{P}} \psi(B)$.
(iii): Let $A \in \mathcal{R}(\mathfrak{P})$. Then

$$\psi(\psi(A)) = \psi(M_{\mathfrak{P}}(A^c)) = M_{\mathfrak{P}}(\Omega \setminus M_{\mathfrak{P}}(A^c)) = M_{\mathfrak{P}}(I_{\mathfrak{P}}(A)) = A.$$

Thus, $\psi^2 = Id$, i.e. ψ is an involution.
(iv): Let us prove that $\psi(A) \wedge A = \emptyset$. Firstly, by (6.6), we have that

$$\psi(A) \wedge A = M_{\mathfrak{P}}(I_{\mathfrak{P}}(\psi(A) \cap A)) = M_{\mathfrak{P}}(I_{\mathfrak{P}}(M_{\mathfrak{P}}(A^c) \cap A)).$$

We now prove that the unique subset contained in $M_{\mathfrak{P}}(A^c) \cap A$ and whose complement is a maximum partitioner is the empty set. As a matter of fact, let $Z \subseteq M_{\mathfrak{P}}(A^c) \cap A$, then $Z \subseteq A$ and, in particular, by (i) of Proposition 6.2.2, $Z \subseteq I_{\mathfrak{P}}(A) = \Omega \setminus M_{\mathfrak{P}}(A^c)$. But it happens if and only if $Z = \emptyset$. Thus, $I_{\mathfrak{P}}(M_{\mathfrak{P}}(A^c) \cap A)$ and, since $M_{\mathfrak{P}}(\emptyset) = \emptyset$, we conclude that $\psi(A) \wedge A = M_{\mathfrak{P}}(\emptyset) = \emptyset$.
On the other hand, let us now compute $\psi(A) \vee A$. By (6.7), it results that

$$\psi(A) \vee A = M_{\mathfrak{P}}(I_{\mathfrak{P}}(A) \cup I_{\mathfrak{P}}(\psi(A))) = M_{\mathfrak{P}}(\Omega \setminus M_{\mathfrak{P}}(A^c)) \cup ((\Omega \setminus M_{\mathfrak{P}}(\psi(A)^c)))$$

but it follows that

$$M_{\mathfrak{P}}(M_{\mathfrak{P}}(\Omega \setminus M_{\mathfrak{P}}(A^c)) \cup M_{\mathfrak{P}}(A^c)) = \Omega.$$

by using (b) of Corollary 2.2.2 and the definition of ψ. This shows that $\psi(A) \vee A = \Omega$ and the proof concludes here. □

In next result, we prove that for any finite complementary involution lattice there exists a pairing on a certain finite set $\Omega_{\mathbb{L},\psi}$ whose regularly closed family is isomorphic to $\mathbb{L}$ itself.

Theorem 6.2.5. *Let $(\mathbb{L}, \psi)$ be a finite complementary involution lattice. Then, there exist a finite set $\Omega_{\mathbb{L},\psi}$ and a pairing $\mathfrak{P} \in PAIR(\Omega_{\mathbb{L},\psi})$ such that $\mathfrak{R}(\mathfrak{P})$ is order-isomorphic to $(\mathbb{L}, \leq_{\mathbb{L}})$.*

Proof. We divide the proof in three steps:

I Let us show that given an order-preserving ψ-complementary map $f : \mathbb{L} \to \mathcal{F}$, where $\mathcal{F}$ is a complement-closed family on a certain finite set Ω_n, there exist a finite set Ω, a complement-closed family $\mathcal{G}$ on it and, fixed two elements $\alpha, \beta \in \mathbb{L}$, there exists an order-preserving ψ-complementary map $\hat{g} : \mathbb{L} \to \mathcal{G}$ $\{\alpha, \beta\}$-preserving such that:

(1) $\Omega := \Omega_n$ and $\mathcal{F} = \mathcal{G}$ or there exists $\xi \notin \Omega_n$ such that $\Omega := \Omega_{n+1} = \Omega_n \cup \{\xi\}$ or there exist some $\eta, \xi \notin \Omega_n$ such that $\Omega := \Omega_{n+2} = \Omega_n \cup \{\eta, \xi\}$;

(2) for any $\gamma \in \mathbb{L}$, $\hat{g}(\gamma) = f(\gamma)$ or $\hat{g}(\gamma) = f(\gamma) \cup \{\xi\}$ or $\hat{g}(\gamma) = f(\gamma) \cup \{\eta, \xi\}$.

Clearly, if f is $\{\alpha, \beta\}$-preserving, there is nothing to prove since we can set $\hat{g} := f$, $\mathcal{G} := \mathcal{F}$ and $\Omega := \Omega_n$. Hence, we must prove the claim when f is not $\{\alpha, \beta\}$-preserving. For, we have to investigate two cases: in the first case, let us assume that $\alpha <_{\mathbb{L}} \beta$ (or that they are non-comparable) and that $f(\alpha) = f(\beta)$. Let $\xi \notin \Omega_n$ and $\Omega' := \Omega_n \cup \{\xi\}$. We now define a map $h_1 : \mathbb{L} \to h_1(\mathbb{L}) \subseteq \Omega'$ where, for any $\gamma \in \mathbb{L}$,

$$h_1(\gamma) := \begin{cases} f(\gamma) \cup \{\xi\} & \text{if } \beta \leq_{\mathbb{L}} \gamma \\ f(\gamma) & \text{otherwise.} \end{cases}$$

First of all, we observe that h_1 is an order-preserving map. Moreover it holds that, whenever f is $\{\gamma, \delta\}$-preserving for some $\gamma, \delta \in \mathbb{L}$, then also h_1 satisfies the same property.

In fact, if γ and δ are non-comparable, then one of the following situations occur: if $\beta \leq_{\mathbb{L}} \gamma, \delta$, then $h_1(\gamma) = f(\gamma) \cup \{\xi\}$ and $h_1(\delta) = f(\delta) \cup \{\xi\}$ and since $f(\gamma)$ and $f(\delta)$ are non-comparable, it follows that $h_1(\gamma)$ and $h_1(\delta)$ are also non-comparable; if $\gamma, \delta <_{\mathbb{L}} \beta$, then $h_1(\gamma) = f(\gamma)$ and $h_1(\delta) = f(\delta)$, so $h_1(\gamma)$ and $h_1(\delta)$ are also non-comparable; if $\gamma <_{\mathbb{L}} \beta$ and β and δ are non-comparable, then $h_1(\gamma) = f(\gamma)$ and $h_1(\delta) = f(\delta)$, so $h_1(\gamma)$ and $h_1(\delta)$ are also non-comparable and, finally, if $\beta \leq_{\mathbb{L}} \gamma$ and β and δ are non-comparable, then $h_1(\gamma) = f(\gamma) \cup \{\xi\}$ and $h_1(\delta) = f(\delta)$. Since $f(\gamma)$ and $f(\delta)$ are non-comparable, also $h_1(\gamma)$ and $h_1(\delta)$ are.
The other cases to be analyzed are similar. This proves that h_1 is $\{\gamma, \delta\}$-preserving.
Moreover, it results that

$$h_1(\gamma) \cap h_1(\psi(\gamma)) = \emptyset \tag{6.8}$$

for any $\gamma \in \mathbb{L}$. Let us suppose that $\beta \leq_{\mathbb{L}} \gamma$. This implies that $\psi(\gamma)$ and β are non-comparable. Therefore $h_1(\gamma) = f(\gamma) \cup \{\xi\}$ and $h_1(\psi(\gamma)) = f(\psi(\gamma)) = \Omega_n \setminus f(\gamma)$ and (6.8) holds. The other possible cases are similar.
Now, let us consider all possible pairs $\{\gamma, \psi(\gamma)\}$, for each $\gamma \in \mathbb{L}$. We order these pairs in the following way

$$\{\gamma_0, \psi(\gamma_0)\}, \{\gamma_1, \psi(\gamma_1)\}, \ldots, \{\gamma_m, \psi(\gamma_m)\},$$

so that $\gamma_i \not\leq_{\mathbb{L}} \alpha$ for each $i = 0, \ldots, m$.
Let us set $g_0 := h_1$. Suppose that, for each $i = 1, \ldots, m-1$, there exist a complement-closed family $\mathcal{G}_i$ on Ω' and a mapping $\hat{g}_i : \mathbb{L} \to \mathcal{G}_i$ such that:

- (i) $\hat{g}_i(\gamma) = h_1(\gamma)$ or $\hat{g}_i(\gamma) = h_1(\gamma) \cup \{\xi\}$ for any $\gamma \in \mathbb{L}$;
- (ii) if f is $\{\gamma, \delta\}$-preserving, then also $\hat{g}_i$ is;
- (iii) if $i \leq j$, then $\hat{g}_i(\gamma) \subseteq \hat{g}_j(\gamma)$ for any $\gamma \in \mathbb{L}$;
- (iv) $\hat{g}_i(\gamma) \cap \hat{g}_i(\psi(\gamma)) = \emptyset$;
- (v) if $\gamma \leq_{\mathbb{L}} \delta$ and $\xi \in \hat{g}_i(\gamma)$, then $\xi \in \hat{g}_i(\delta)$;
- (vi) $\hat{g}_i(\psi(\gamma_i)) = \Omega' \setminus \hat{g}_i(\gamma_i)$, i.e. $\hat{g}_i$ is an order-preserving ψ-complementary map.

We set, for any $\gamma \in \mathbb{L}$ and $1 \leq k \leq m-1$,

$$\overline{g}_k(\gamma) := \bigcup_{i=0}^{k-1} \hat{g}_i(\gamma).$$

If $\overline{g}_i(\psi(\gamma_q)) \cup \overline{g}_i(\gamma_q) = \Omega_n$ for some $\gamma_q \in \mathbb{L}$ and $i = 0, \ldots, k-1$, then we set

$$g_i(\gamma) := \begin{cases} \overline{g}_i(\gamma) \cup \{\xi\} & \text{if } \gamma_q \leq_{\mathbb{L}} \gamma \\ \overline{g}_i(\gamma) & \text{otherwise.} \end{cases}$$

whereas, if $\overline{g}_i(\psi(\gamma)) = \Omega' \setminus \overline{g}_i(\gamma)$ for any $\gamma \in \mathbb{L}$, we set $g_i := \overline{g}_i$. Clearly, if $\overline{g}_i$ satisfies $(i) - (vi)$, then also g_i does. Moreover, in both cases, we set

$$\mathcal{G}_i := \{g_i(\gamma) : \gamma \in \mathbb{L}\}.$$

Analogously, we set:

$$g_m(\gamma) := \bigcup_{i=0}^{m-1} g_i(\gamma)$$

and

$$\mathcal{G}_m := \{g_m(\gamma) : \gamma \in \mathbb{L}\}.$$

Let us prove $(i) - (vi)$ for g_m. The properties (i), (iii) and (v) are obvious. Suppose now that f is $\{\gamma, \delta\}$-preserving for some $\gamma, \delta \in \mathbb{L}$. Suppose moreover that $\gamma <_{\mathbb{L}} \delta$. Then, $f(\gamma) \subsetneqq f(\delta)$ so, by the definition of g_m and h_1, we necessarily have $g_m(\gamma) \subsetneqq g_m(\delta)$. The other cases are similar. This proves (ii).
In order to prove (iv), firstly assume by contradiction that $g_{m-1}(\gamma) \cap g_{m-1}(\psi(\gamma)) \neq \emptyset$ for some $\gamma \in \mathbb{L}$. Then, there exist two indices $i \leq j$ such that $\emptyset \neq g_i(\gamma) \cap g_j(\psi(\gamma)) \subseteq g_j(\gamma) \cap g_j(\psi(\gamma))$, that is an absurd. On the other hand, if we assume by contradiction that $g_{m-1}(\gamma_q) \cap g_{m-1}(\psi(\gamma_q)) \neq \emptyset$ for some $q \leq m-1$, then $\{\xi\} = g_{m-1}(\gamma_q) \cap g_{m-1}(\psi(\gamma_q))$, i.e. $\gamma_q = 0_{\mathbb{L}}$, that is an absurd. This proves (iv). As a direct consequence of (iv), we obtain (vi), i.e. g_m is an order-preserving ψ-complementary map.
Assume now that $\xi \in g_m(\alpha)$ and let j be the minimum index so that $\xi \in g_j(\alpha)$. This implies that $\gamma_q \leq_{\mathbb{L}} \alpha$, that is an absurd. So $g_m(\alpha) \subsetneqq g_m(\beta)$. If $\alpha <_{\mathbb{L}} \beta$, we set $\hat{g} := g_m$, $\mathcal{G} := \mathcal{G}_m$ and $\Omega := \Omega'$ and this shows that $\hat{g}(\gamma) = f(\gamma) \cup \{\xi\}$. Otherwise, if α and β are non-comparable, let $\eta \notin \Omega'$ and set $\Omega'' := \Omega' \cup \{\eta\}$. Moreover, for any $\gamma \in \mathbb{L}$, we set

$$h_2(\gamma) := \begin{cases} g_m(\gamma) \cup \{\eta\} & \text{if } \alpha \leq_{\mathbb{L}} \gamma \\ g_m(\gamma) & \text{otherwise.} \end{cases}$$

By reasoning as above, we can construct g'_m through g_m and h_2 using η instead of ξ. Clearly, since $\xi \in g'_m(\beta) \setminus g'_m(\alpha)$ and $\eta \in g'_m(\alpha) \setminus g'_m(\beta)$, we deduce that g'_m is $\{\alpha, \beta\}$-preserving. Therefore, we can set $\hat{g} := g'_m$ and $\mathcal{G} := \{\hat{g}(\gamma) : \gamma \in \mathbb{L}\}$. This concludes the proof in the case $\alpha <_{\mathbb{L}} \beta$ (or they are non-comparable) and $f(\alpha) = f(\beta)$.
In the second case, we assume α and β non-comparable and $f(\alpha) \subsetneqq f(\beta)$. We can construct $\hat{g}$ by starting from f in the same way we obtained g'_m from g_m, by replacing η with ξ and it holds $\hat{g}(\gamma) = f(\gamma) \cup \{\eta, \xi\}$.

II We now prove that there exist a finite set $\Omega_{\mathbb{L},\psi}$ and an isomorphism $f : \mathbb{L} \to \mathcal{F}$, where $\mathcal{F}$ is a complement-closed family on $\Omega_{\mathbb{L},\psi}$. Firstly, let $g : \mathbb{L} \to \emptyset\}$. Let us consider the family of all 2-subsets of $\mathbb{L}$, i.e. $\{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$. We set $f_0 := \hat{g}$, $\mathcal{F}_0 := \mathcal{G}$ and $G_0 := \Omega$, where $\hat{g}, \mathcal{G}, \Omega$ have been constructed in the previous step, $\hat{g}$ is $\{\alpha_0, \beta_0\}$-preserving and $\mathcal{F}_0$ is a complement-closed family on G_0.
Suppose that for any $0 \leq i \leq m-1$ there exist f_i, $\mathcal{F}_i$ and G_i such that:

- (A) $f_i : \mathbb{L} \to \mathcal{F}_i$ is an order-preserving complementarity $\{x_i, y_i\}$-preserving and $\mathcal{F}_i$ is a complement-closed family on G_i;
- (B) If $i < j < m$ and f_i is $\{\alpha, \beta\}$-preserving, then f_j does. Moreover, for any $\alpha \in \mathbb{L}$, it results that $f_j(\alpha) \cap G_i = f_i(\alpha)$.

Let us set:

$$f_m(\alpha) := \bigcup_{i=0}^{m-1} f_i(\alpha)$$

for any $\alpha \in \mathbb{L}$,

$$\mathcal{F}_m := \{f_m(\alpha) : \alpha \in \mathbb{L}\}$$

and

$$G_m := \bigcup \mathcal{F}_m.$$

Let us prove that f_m satisfies both (A) and (B). It is obvious that f_m is order-preserving since if $\alpha \leq_{\mathbb{L}} \beta$, then $f_i(\alpha) \subseteq f_i(\beta)$ for each $i < m$ and, by the definition of f_m, a fortiori, it follows that $f_m(\alpha) \subseteq f_m(\beta)$. We now prove that f_m is an order-preserving ψ-complementary map. Suppose that $f_m(\psi(\alpha)) \cap f_m(\alpha) \neq \emptyset$ for some $\alpha \in \mathbb{L}$. This implies that there exist two indices $i < j < m$ and $b \in G_m$ such that $b \in f_i(\psi(\alpha)) \cap f_j(\alpha)$ but, by (2), it follows that $f_i(\psi(\alpha)) \subseteq f_j(\psi(\alpha))$, i.e. $b \in f_j(\psi(\alpha)) \cap f_j(\alpha)$, that is a contradiction. Thus, $f_m(\psi(\alpha)) \subseteq G_m \setminus f_m(\alpha)$. Indeed, the previous relation is an equality. In fact, let $b \in G_m$, then there exists $\beta \in \mathbb{L}$ and an index $i < m$ such that $b \in f_i(\beta)$. In particular, $b \in G_i = f_i(\alpha) \cup f_i(\psi(\alpha))$. Therefore it necessarily must be $f_m(\psi(\alpha)) = G_m \setminus f_m(\alpha)$. So, f_m satisfies (A).
Suppose that f_i is $\{\alpha, \beta\}$-preserving. Let us show that f_m is also $\{\alpha, \beta\}$-preserving. In fact, suppose without loss of generality that $\alpha \leq_{\mathbb{L}} \beta$ and let $b \in f_i(\beta) \setminus f_i(\alpha)$. Then, clearly $b \in f_m(\beta)$. If we assume by contradiction that $b \in f_m(\alpha)$, there would be an index $j > i$ such that $b \in f_j(\alpha)$, so $b \in f_j(\alpha) \cap G_i = f_i(\alpha)$, that is a contradiction. Hence $b \in f_m(\beta) \setminus f_m(\alpha)$. The converse is obviously true. Therefore f_m is $\{\alpha, \beta\}$-preserving.
Furthermore, it is immediate that $f_m(\alpha) \cap G_i = f_i(\alpha)$ for any $i = 0, \ldots, m-1$. So f_m satisfies (B). It is also obvious that $\mathcal{F}_m$ is a complement-closed family on G_m.
Therefore we have constructed an order-preserving ψ-complementary map $f_m : \mathbb{L} \to \mathcal{F}_m$. By starting from f_m, $\mathcal{F}_m$ and G_m, as in the first step of the proof, we can construct a finite set $\Omega_{\mathbb{L},\psi}$, a complement-closed family $\mathcal{F}$ on it and an order-preserving ψ-complementary map $f : \mathbb{L} \to \mathcal{F}$. Obviously, f is an isomorphism since it is $\{\alpha, \beta\}$-preserving for each 2-subset $\{\alpha, \beta\}$ of $\mathbb{L}$. Thus, since $(\mathcal{F}, \subseteq)$ is a poset, the isomorphism $f : \mathbb{L} \to \mathcal{F}$ induces on $(\mathcal{F}, \subseteq)$ a natural lattice structure.

III The last step of the proof consists of proving the existence of pairing $\mathfrak{P}$ on $\Omega_{\mathbb{L},\psi}$ such that $\mathcal{F} = \mathcal{R}(\mathfrak{P})$.
Let us consider the map $\phi : \mathcal{P}(\Omega_{\mathbb{L},\psi}) \to \mathcal{P}(\Omega_{\mathbb{L},\psi})$ defined as follows:

$$\phi(X) = \bigcap \{Y \in \mathcal{F} : X \subseteq Y\}. \tag{6.9}$$

It is immediate to see that $X \subseteq \phi(X)$ and that $\phi(X) \subseteq \phi(Y)$ whenever $X \subseteq Y$. Moreover, it also results that ϕ is idempotent. In fact, by (6.9), if $Z \in \mathcal{F}$ contains X, then it also contains $\phi(X)$. The converse is obvious. This implies that ϕ is a closure operator. Furthermore, it is evident that $\phi(\emptyset) = \emptyset$. Thus, by Theorem 6.1.1, there exists $\mathfrak{P} \in PAIR(\Omega)$ such that $MAXP(\mathfrak{P})$ coincides with the family of all the fixed subsets of ϕ. This means that in next calculations we are allowed to use $M_{\mathfrak{P}}$ instead of ϕ.

We must prove that $\mathcal{F} = \mathcal{R}(\mathfrak{P})$. For, let us observe that if $Z \in \mathcal{F}$, then $M_{\mathfrak{P}}(Z) = Z$. So, $\mathcal{F} \subseteq MAXP(\mathfrak{P})$. In particular, we have that $I_{\mathfrak{P}}(Z) = Z \in \mathcal{F}$, so $Z = M_{\mathfrak{P}}(I_{\mathfrak{P}}(Z))$, i.e. $Z \in \mathcal{R}(\mathfrak{P})$.
On the other hand, let $Z \in \mathcal{R}(\mathfrak{P})$. Since $\mathcal{F}$ is a complete lattice, it follows that

$$Z^* := \inf\{Y \in \mathcal{F} : Z \subseteq Y\} \in \mathcal{F}.$$

Furthermore, we have that $Z^* \subseteq \bigcap\{Y \in \mathcal{F} : Z \subseteq Y\} = M_{\mathfrak{P}}(Z) = Z$. Let $A \subseteq Z$ such that $A^c \in MAXP(\mathfrak{P})$. Let us show that

$$A = \bigcup\{Y \in \mathcal{F} : Y \subseteq A\}. \tag{6.10}$$

In fact,

$$A^c = M_{\mathfrak{P}}(A^c) = \bigcap\{Y \in \mathcal{F} : A^c \subseteq Y\} = \bigcup\{Z \in \mathcal{F} : Z \subseteq A\}.$$

as claimed.
Moreover, let us note that Z^* contains each Y in (6.10). This implies that $A \subseteq Z^*$. Now, since $Z^* \in \mathcal{F}$, then $(Z^*)^c \in MAXP(\mathfrak{P})$. By (i) of Proposition 6.2.2, we conclude that $I_{\mathfrak{P}}(Z) = Z^*$. But since $\mathcal{F} \subseteq MAXP(\mathfrak{P})$, we deduce that $Z = M_{\mathfrak{P}}(I_{\mathfrak{P}}(Z)) = I_{\mathfrak{P}}(Z) \in \mathcal{F}$. Hence $\mathcal{F} = \mathcal{R}(\mathfrak{P})$. This concludes the proof.

$\square$

6.3 Symmetry Transmission Table

Let $\mathbb{L}$ be a finite lattice, $N = pc(\mathbb{L})$ and $\mathfrak{P} \in PAIR(\mathbb{L})$. By means of (6.4) of Theorem 6.1.5, we can note that the function $\gamma_{\mathfrak{P}}$ is a refinement of the partial order $\leq_{\mathbb{L}}$, since it provides a numerical information also between two non-comparable nodes of the lattice $\mathbb{L}$. In fact, for any two elements $x, y \in \mathbb{L}$, we have that $x \leq_{\mathbb{L}} y$ if and only if $\gamma_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x), \eta_{\mathfrak{P}}(y)) = 1$ and, moreover, we can always compute $\gamma_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x), \eta_{\mathfrak{P}}(y))$ even if x and y are non-comparable each other. Therefore, it is natural to investigate all possible values of $\gamma_{\mathfrak{P}}(X, Y)$ when (X, Y) runs over $\mathcal{P}(\Omega_N) \times \mathcal{P}(\Omega_N)$.
To this aim, we introduce the *symmetry transmission table* of a pairing $\mathfrak{P} \in PAIR(\Omega_N)$ within which we collect all values $\gamma_{\mathfrak{P}}(X, Y)$ when $(X, Y) \in \mathcal{P}(\Omega_N) \times \mathcal{P}(\Omega_N)$.

Definition 6.3.1. Let $\mathfrak{P} \in PAIR(\Omega_N)$. We call *symmetry transmission table* of $\mathfrak{P}$, denoted by $T_{dep}(\mathfrak{P})$, the $2^N \times 2^N$ table having as rows and columns all elements of $\mathcal{P}(\Omega_N)$ and in the place corresponding to row X and column Y the value $\gamma_{\mathfrak{P}}(X, Y)$.

The symmetry transmission table is a source of very useful information when one studies the properties of $\mathfrak{P}$. In fact, a complete knowledge of the symmetry transmission table enables us to build the symmetry partition lattice of $\mathfrak{P}$. Hence, we can think the table $T_{dep}(\mathfrak{P})$ as a type of numerical completion of the symmetry partition lattice of $\mathfrak{P}$.

Since the symmetry transmission table provides numerical informations for any pair $(X, Y) \in \mathcal{P}(\Omega_N) \times \mathcal{P}(\Omega_N)$, it is natural to introduce a global symmetry transmission measure as well as it is usually done in many sectors of discrete mathematics.

To this regard, if $A \subseteq \Omega_N$, we set

$$\lambda_r(A) := \frac{1}{2^n} \sum_{X \subseteq \Omega_N} \gamma_{\mathfrak{P}}(A, X)$$

and

$$\lambda_c(A) := \frac{1}{2^n} \sum_{X \subseteq \Omega_N} \gamma_{\mathfrak{P}}(X, A).$$

Definition 6.3.2. We call *symmetry transmission average* of $\mathfrak{P}$ the number

$$\lambda(\mathfrak{P}) := \frac{1}{2^n} \sum_{A \subseteq \Omega_N} \lambda_r(A) = \frac{1}{2^n} \sum_{A \subseteq \Omega_N} \lambda_c(A).$$

We can interpret the number $\lambda(\mathfrak{P})$ as the average capacity to transmit (or to receive) symmetry of a generic subset of Ω_N.

6.4 Symmetry Transmission Tables on Some Examples

In this section, we compute the symmetry transmission average for some basic digraph family, also providing an asymptotic estimate. On the other hand, we consider two cases of pairings on infinite sets Ω and we compute the map $\Gamma_{\mathfrak{P}}$ in this cases.

6.4.1 The Directed n-Cycle

We determine the symmetry transmission table for the pairing $\mathfrak{P}[\vec{C_n}]$.

Proposition 6.4.1. *Let A and B be two distinct vertex subsets of $V = V(\vec{C_n}) = \{v_1, \dots, v_n\}$. Then*

$$\Gamma_{\mathfrak{P}}(A, B) = \begin{cases} V & \text{if } B = \emptyset \text{ or } A = V \text{ or } |A| = n-1 \text{ or } B \subseteq A \\ \emptyset & \text{if } A = \emptyset \text{ and } B \neq \emptyset \\ A_{-1} & \text{otherwise} \end{cases} \tag{6.11}$$

and

$$\gamma_{\mathfrak{P}}(A, B) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = V \text{ or } |A| = n-1 \text{ or } B \subseteq A \\ 0 & \text{if } A = \emptyset \text{ and } B \neq \emptyset \\ \frac{|A|}{n} & \text{otherwise} \end{cases} \tag{6.12}$$

Proof. By (ii) of Proposition 2.3.9, we have that if $B = \emptyset$, then $\pi_{\vec{C_n}}(B) = V$. Therefore if A is any subset of V and $v \in V$ is any vertex, it follows that $[v]_A \subset [v]_B = V$. Similarly, if $A = V$ or if $|A| = n-1$, we have $\pi_{\vec{C_n}}(A) = v_1 | \dots | v_n$. In other words, in these situations $\Gamma_{\mathfrak{P}}(A, B) = V$. Furthermore, by (2.13), if $B \subseteq A$, we deduce that $\Gamma_{\mathfrak{P}}(A, B) = V$. Again by (ii) of Proposition 2.3.9, it is clear that if $A = \emptyset$ and $B \neq \emptyset$, we have $[v]_A = V$ for every $v \in V$, but $[v]_B \subsetneqq V$. So, $\Gamma_{\mathfrak{P}}(A, B) = \emptyset$. Let $|A| = k$ and $|B| = l$. By (ii) of Proposition 2.3.9, we have $\pi_{\vec{C_n}}(A) = v_{i_1-1} | \dots | v_{i_k-1} | A^c_{-1}$ and $\pi_{\vec{C_n}}(B) = v_{j_1-1} | \dots | v_{j_l-1} | B^c_{-1}$. This means that $[v]_A \subseteq [v]_B$ only for $x = v_{i_1-1}, \dots, v_{i_k-1}$, therefore $\Gamma_{\mathfrak{P}}(A, B) \supseteq \{v_{i_1-1}, \dots, v_{i_k-1}\}$. Let $v \in \Gamma_{\mathfrak{P}}(A, B) \setminus A_{-1}$. Then, $[v]_A = A^c_{-1}$. We have two cases: $v \in B$ or $v \notin B$. In the first case, $[v]_B$ has a single element, so $[v]_A$ can't be contained in $[v]_B$ unless A^c_{-1} consists of a single element, but this happens if $|A| = n-1$ and we have excluded this situation. In the second case, we have $[v]_A \subseteq [v]_B$ if and only if $A^c_{-1} \subseteq B^c_{-1}$, i.e. if and only if $B_{-1} \subseteq A_{-1}$. This means that $B \subseteq A$, but this situation has been excluded. Hence the proof of (6.11) is complete. Finally, by definition we have $\gamma_{\mathfrak{P}}(A,B) = \frac{|\Gamma_{\mathfrak{P}}(A,B)|}{|V|}$, therefore (6.12) follows directly from (6.11). $\square$

We compute the previous average numbers in the particular case of the pairing $\mathfrak{P}[\vec{C_n}]$ and we also provide an asymptotic estimate for the symmetry transmission average.

Theorem 6.4.2. *(i): Let $A \subseteq V(\vec{C_n})$. Then*

$$\lambda_r(A) = \begin{cases} \frac{1}{2^n} & \text{if } A = \emptyset \\ \frac{1}{2^{n-k}} + \frac{k}{n2^{n-k}}(2^{n-k}-1) & \text{if } k = |A| < n-1 \\ 1 & \text{if } |A| \geq n-1 \end{cases}$$

(ii): Let $B \subseteq V = V(\vec{C_n})$. Then, we have:

$$\lambda_c(B) = \begin{cases} 1 & \text{if } B = \emptyset \\ \frac{3n-1}{4n} + \frac{1}{n2^n} & \text{if } |B| = 1 \\ \frac{1}{2^k} + \frac{k}{2^n} + \frac{1}{2^n}\left[\sum_{l=1}^{k-1}\binom{n}{l}\frac{l}{n} + \sum_{l=k}^{n-2}\left(\binom{n}{l} - \binom{n-k}{l-k}\right)\frac{l}{n}\right] & \text{if } 1 < k = |B| < n \\ \frac{1}{2^n} + \frac{1}{2} & \text{if } |B| = n \end{cases}$$

(iii): $\lambda(\vec{C_n}) = \frac{1}{4^n}\left\{\sum_{k=0}^{n-2}\binom{n}{k}\left[2^k + k\frac{2^n-2^k}{n}\right] + 2^n(n+1)\right\}$ and $\lim_{n\to\infty}\lambda(\vec{C_n}) = \frac{1}{2}$.

Proof. (i): If $A = \emptyset$, by (6.12), in the first row of symmetry transmission table of $\vec{C_n}$ we have all the entries 0 except the first, which is 1, so $\lambda_r(\emptyset) = \frac{1}{2^n}$. Otherwise, if $|A| = r$, where $r = 1, \ldots, n-2$, we have exactly 2^r entries equal to 1, corresponding to all its possible subsets, and the other are equal to $\frac{r}{n}$. finally, when $|A| = n-1$ or $A = V$, the entries of the corresponding rows in the symmetry transmission table are all 1.
(ii): We use the numbers $\gamma_{\mathfrak{P}}(A,B)$ determined in (6.12) of Proposition 6.4.1. The first column, corresponding to $B = \emptyset$, has all entries 1, therefore, $\lambda_c(\emptyset) = 1$. Let B be a singleton. Then, if $B \subseteq A$, the corresponding entries of the symmetry transmission table are 1; this happens 2^{n-1} times. Moreover, it is true also if $|A| = n-1$. Now, we compute the other values. There are exactly $\binom{n-1}{k}$ $k-$subsets of V not containing B and the corresponding value of the symmetry transmission table is $\frac{k}{n}$. Therefore we have

$$\lambda_c(\{v_i\}) = 2^{n-1} + 1 + \sum_{k=1}^{n-2}\binom{n-1}{k}\frac{k}{n}.$$

Since $\sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k}{n} = \frac{1}{n} \sum_{k=0}^{n-1} k\binom{n-1}{k} - \frac{n-1}{n} = \frac{n-1}{n}(2^{n-2}-1)$, after algebraic manipulation we obtain

$$\lambda_c(\{v_i\}) = \frac{3n-1}{4n} + \frac{1}{n2^n}.$$

Let B be a generic subset with $1 < |B| < n$ elements. First of all, we observe that there are exactly 2^{n-k} subsets of V containing it and there are $\binom{n}{n-1} - \binom{n-k}{n-k-1} = k$ subsets of cardinality $n-1$ not containing B, to which correspond the value 1. Furthermore there are no $l-$subsets containing B for $l = 1, \ldots, k-1$ and, for each of them, the corresponding value in the symmetry transmission table is $\frac{l}{n}$, i.e. we can sum the quantity $\sum_{l=1}^{k-1} \binom{n}{l} \frac{l}{n}$. Now, suppose that $k \le l \le n-2$. There are exactly $\binom{n}{l} - \binom{n-k}{l-k}$ subsets of cardinality l not containing B and in correspondence of them, the value taken in the symmetry transmission table is $\frac{l}{n}$. Therefore we have:

$$\lambda_c(B) = \frac{1}{2^k} + \frac{k}{2^n} + \frac{1}{2^n}\left[\sum_{l=1}^{k-1}\binom{n}{l}\frac{l}{n} + \sum_{l=k}^{n-2}\left(\binom{n}{l} - \binom{n-k}{l-k}\right)\frac{l}{n}\right].$$

Finally, when $|B| = n$, we have $n+1$ entries equal to 1. Furthermore, there are no $l-$subsets containing B for $l = 1, \ldots, n-2$ and, for each of them, the corresponding value in the symmetry transmission table is $\frac{l}{n}$. So $\lambda_c(V) = \frac{n+1}{2^n} + \frac{1}{2^n}\sum_{l=1}^{n-2}\binom{n}{l}\frac{l}{n}$. Adding and subtracting the quantities $n(n-1)+n$, using the fact that $\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1}$, with algebraic manipulations we obtain $\lambda_c(V) = \frac{1}{2^n} + \frac{1}{2}$.

(iii): Setting $k = |A|$, we have 2^{2n} elements in the table, while, by (i), the sum of elements in the same row is $2^k + \frac{k2^k}{n}(2^{n-k}-1)$ for $k = 1, \ldots, n-2$. We observe that if $k = 0$, then the previous formula gives the result 1, in agreement with part (i). Therefore we have

$$\lambda(\vec{C_n}) = \frac{1}{4^n}\left\{\sum_{k=0}^{n-2}\binom{n}{k}\left[2^k + k\frac{2^n - 2^k}{n}\right] + 2^n(n+1)\right\}. \tag{6.13}$$

Now, let us note that $\lim_{n\to\infty} \frac{2^n(n+1)}{4^n} = 0$ and $\lim_{n\to\infty}\frac{1}{4^n}\sum_{k=0}^{n-2}\binom{n}{k}2^k = 0$, since $\sum_{k=0}^{n-2}\binom{n}{k}2^k = \sum_{k=0}^{n}\binom{n}{k}2^k - n2^{n-1} - 2^n = 3^n - n2^{n-1} - 2^n$. Moreover

$$\sum_{k=0}^{n-2}\binom{n}{k}\frac{k}{n}(2^n - 2^k) \le \sum_{k=0}^{n-2}\binom{n}{k}\frac{k2^n}{n} = \frac{2^n}{n}\left[\sum_{k=0}^{n} k\binom{n}{k} - n(n-1) - n\right] = 2^{2n-1} - n2^n.$$

Hence

$$\lim_{n\to\infty}\lambda(\vec{C_n}) \le \lim_{n\to\infty}\frac{2^{2n-1} - n2^n}{4^n} = \frac{1}{2}.$$

Similarly, since $\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1}$, by (6.13) we obtain

$$\lambda(\vec{C_n}) = \frac{1}{4^n}\left\{\sum_{k=0}^{n-2}\binom{n}{k}\left[k\frac{2^n}{n} + 2^k\left(1 - \frac{k}{n}\right)\right] + 2^n(n+1)\right\} \ge \frac{1}{4^n}[2^{2n-1} + 2^n(1+n)].$$

Therefore

$$\lim_{n\to\infty} \lambda(\vec{C}_n) \geq \lim_{n\to\infty} \frac{1}{4^n}[2^{2n-1} + 2^n(1+n)] = \frac{1}{2}$$

and the thesis follows. $\square$

6.4.2 The directed n-path

Now we focus our attention to the case of the directed n-path $\vec{P}_n$.

Proposition 6.4.3. *Let A and B be two distinct vertex subsets of $V = V(\vec{P}_n) = \{v_1, \ldots, v_n\}$. Then*

$$\Gamma_{\mathfrak{P}}(A, B) = \begin{cases} V & \text{if } \; B = \emptyset \vee B = \{v_1\} \vee A = V \vee A = \{v_2, \ldots, v_n\} \vee B \subseteq A \\ \emptyset & \text{if } \; (A = \emptyset \vee A = \{v_1\}) \wedge (B \neq \emptyset \wedge B \neq \{v_1\}) \\ A_{-1} & \text{otherwise} \end{cases} \tag{6.14}$$

Moreover

$$\gamma_{\mathfrak{P}}(A, B) = \begin{cases} 1 & \text{if } \; B = \emptyset \vee B = \{v_1\} \vee A = V \vee A = \{v_2, \ldots, v_n\} \vee B \subseteq A \\[2ex] 0 & \text{if } \; (A = \emptyset \vee A = \{v_1\}) \wedge (B \neq \emptyset \wedge B \neq \{v_1\}) \\[2ex] \frac{k}{n} & \text{if } \quad |A| = k \wedge v_1 \notin A \\[2ex] \frac{k-1}{n} & \text{if } \quad |A| = k \wedge v_1 \in A \end{cases} \tag{6.15}$$

Proof. By Proposition 2.3.9, we have that if $B = \emptyset$ or $B = \{v_1\}$, then $\pi_{\vec{P}_n}(B) = V$. Therefore if A is any subset of V and $v \in V$ is any vertex, it follows that $[v]_A \subseteq [v]_B = V$. Similarly, if $A = V$ or if $A = \{v_2, \ldots, v_n\}$, we have $\pi_{\vec{P}_n}(A) = v_1|\ldots|v_n$. Moreover, by (2.13), if $B \subseteq A$, then $\pi_{\vec{P}_n}(A) \preceq \pi_{\vec{P}_n}(B)$. In other words, in these cases $\Gamma_{\vec{P}_n}(A, B) = V$. A fortiori, even if $A = \emptyset$ or $A = \{v_1\}$, then $\Gamma_{\vec{P}_n}(A, B) = V$. This is false when $B \neq \emptyset, \{v_1\}$, in fact in this case we would have $[v]_A = V$ contained in one of its subsets, that is an absurd. Let $|A| = k$ and $|B| = l$ vertex subsets different from the previous. We have respectively $\pi_{\vec{P}_n}(A) = v_{i_1-1}|\ldots|v_{i_k-1}|A^c_{-1}$ and $\pi_B = v_{j_1-1}|\ldots|v_{j_l-1}|B^c_{-1}$. it is clear that the classes of vertices which form single blocks with respective to A are contained in their classes with respect to B, so $A_{-1} \subseteq \Gamma_{\vec{P}_n}(A, B)$. Let $v \in \Gamma_{\vec{P}_n}(A, B) \setminus A_{-1}$. Then, $[v]_A = A^c_{-1}$. We have two cases: $v \in B$ or $v \notin B$. In the first case, $[v]_B$ has a single element, so $[v]_A$ can't be contained in $[v]_B$ unless A^c_{-1} consists of a single element, but this happens if $A = \{v_2, \ldots, v_n\}$ and we have excluded this situation. In the second case, we have $[v]_A \subseteq [v]_B$ if and only if $A^c_{-1} \subseteq B^c_{-1}$, i.e. if and only if $B_{-1} \subseteq A_{-1}$. This means that $B \subseteq A$, but this situation has been excluded. Thus $\Gamma_{\vec{P}_n}(A, B) = \{v_{i_1-1}, \ldots, v_{i_k-1}\}$. This complete the proof of (6.14). Finally, by definition, we have $\gamma_{\vec{P}_n}(A, B) = \frac{|\Gamma_{\vec{P}_n}(A,B)|}{|V|}$, therefore (6.15) follows directly from (6.14). $\square$

Lemma 6.4.4. *Let $A \subseteq V = V(\vec{P_n})$. Then, we have*

$$\lambda_r(A) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } A = \emptyset \vee A = \{v_1\} \\ 1 & \text{if } \{v_2, \ldots, v_n\} \subseteq A \\ \frac{1}{n} + \frac{3}{2^n}(1 - \frac{1}{n}) & \text{if } |A| = 1 \\ \frac{1}{2^{n-k}}(1 + \frac{k-1}{n}(2^{n-k} - 1)) & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \\ \frac{1+2^k+\frac{k}{n}(2^n-2^k-1)}{2^n} & \text{if } |A| = k \neq n-1 \wedge v_1 \notin A \end{cases}$$

Proof. By (6.15) of Lemma 6.4.3, in the rows corresponding to $A = \emptyset$ and $A = \{v_1\}$ we have only two entries 1 and the others are all 0, so $\lambda_r(\emptyset) = \lambda_r(\{v_1\}) = \frac{1}{2^{n-1}}$. When $A = V$, we have that $\gamma_A^+(B) = 1$ for every $B \subseteq V = V(\vec{P_n})$, therefore $\lambda_r(V) = 1$. If A is singleton $\{v_i\} \neq \{v_1\}$, by (6.15) only in three entries we have 1, in the other entries we have $\frac{1}{n}$. Therefore we have $\lambda_r(A) = \frac{1}{2^n}\left(\frac{2^n}{n} - \frac{3}{n} + 3\right)$, i.e. $\frac{1}{n} + \frac{3}{2^n}(1 - \frac{1}{n})$. Let $A = \{v_1, v_{i_2}, \ldots, v_{i_k}\}$. By (6.15), the entries are 1 when the sets in the columns of the symmetry transmission table are contained in A, therefore there are exactly $\binom{k}{0} + \cdots + \binom{k}{k} = 2^k$ entries equal to 1, i.e. the number of all vertex subsets of a set of cardinality k, while the other entries are equal to $\frac{k-1}{n}$. If A doesn't contain the vertex v_1 and has cardinalitiy $|A| = k \neq n-1$, there are $2^k + 1$ entries equal to 1, i.e. those corresponding to all its subsets and one corresponding to the second column. By (6.15), we deduce that the remaining entries have value $\frac{k}{n}$. □

Theorem 6.4.5. *The quantity $\lambda(\vec{P_n})$ is equal to:*

$$\frac{1}{4^n}\left\{4 + 2^{n+1} + (n-2)\left(3 + \frac{2^n - 3}{n}\right) + \sum_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1 + \frac{k-1}{n}(2^{n-k} - 1)\right]\right\} +$$

$$\frac{1}{4^n}\left\{\sum_{k=2}^{n-2}\binom{n-1}{k}\left[1 + 2^k + \frac{k}{n}(2^n - 2^k - 1)\right]\right\}.$$

and $\lim\limits_{n\to\infty} \lambda(\vec{P_n}) = \frac{1}{2}$.

Proof. We must add the numbers $\lambda_r(A)$ determined in Lemma 6.4.4, when A varies among the subsets of $V(\vec{P_n})$. So, if $A = \emptyset$ or $A = \{v_1\}$, we have to add 2, while if $A = \{v_2, \ldots, v_n\}$ or $A = v$ we have to add 2^n. If A is a singleton v_i, then we have exactly three entries equal to 1, while the others take the value $\frac{1}{n}$. Since there are $n-2$ of these singletons, we can add the quantity $(n-2)\left(3 + \frac{2^n-3}{n}\right)$. Since, for $k = 2, \ldots, n-1$ the number of sets of cardinality k containing the vertex v_1 is $\binom{n-1}{k-1}$, we have to add the quantity $\sum\limits_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1 + \frac{k-1}{n}(2^{n-k} - 1)\right]$. Moreover, since for $k = 2, \ldots, n-2$ the number of sets of cardinality k not containing the vertex

v_1 is $\binom{n-1}{k}$, we also have to add the quantity $\sum\limits_{k=2}^{n-2}\binom{n-1}{k}2^k\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]$. Thus, we conclude

$$\lambda(\vec{P_n})=\frac{1}{4^n}\left\{4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+\sum_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1+\frac{k-1}{n}(2^{n-k}-1)\right]\right\}+$$
$$\frac{1}{4^n}\left\{\sum_{k=2}^{n-2}\binom{n-1}{k}\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]\right\}.$$

We set $R=\sum\limits_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1+\frac{k-1}{n}(2^{n-k}-1)\right]$ and $S=\sum\limits_{k=2}^{n-2}\binom{n-1}{k}\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]$. Then, by means of both the Stifel formula and the identities $\sum\limits_{k=0}^{n}\binom{n}{k}2^k=3^n$, $\sum\limits_{k=0}^{n}\binom{n}{k}k=n2^{n-1}$, $\sum\limits_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$, with algebraic manipulations we obtain

$$R+S\le 3^n+2^{2n-1}-2-3\cdot 2^n-3n-\frac{n-1}{n}2^n.$$

Then, we have

$$\lambda(\vec{P_n})=\frac{1}{4^n}\left[4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+R+S\right]\le$$
$$\le\frac{1}{4^n}\left[4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+3^n+2^{2n-1}-2-3\cdot 2^n-3n-\frac{n-1}{n}2^n\right],$$

hence

$$\lim_{n\to\infty}\lambda(\vec{P_n})\le\frac{1}{2}.$$

In a similar way, again by means of both the Stifel formula and the identities $\sum\limits_{k=0}^{n}\binom{n}{k}k=n2^{n-1}$, $\sum\limits_{k=0}^{n}\binom{n}{k}k2^k=2n\cdot 3^{n-1}$, $\sum\limits_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$, $\sum\limits_{k=0}^{n-1}\binom{n-1}{k}2^k=3^{n-1}$, $\sum\limits_{k=0}^{n-1}\binom{n-1}{k}k=(n-1)2^{n-2}$, we get

$$\begin{aligned}R+S\ge\frac{1}{n}[2^{n+1}-2^n+2^{n-1}(n-1)^2+2n-2-2(n-1)(2^{n-3}+1)]+\\ \left(1-\frac{1}{n}\right)2^{2n-1}-(2\cdot 3^{n-1}+2^{n-1})\left(1-\frac{1}{n}\right)\end{aligned}\tag{6.16}$$

Adding the term $Q=4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)$, dividing by 4^n and passing to the limit as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\lambda(\vec{P_n})\ge\frac{1}{2}.$$

Hence $\lim\limits_{n\to\infty}\lambda(\vec{P_n})=\frac{1}{2}$. □

6.4.3 The Case of $(V, \varphi, \mathbb{K})$-VBP

Let us analyze the case of $(V, \varphi, \mathbb{K})$-VBP and that of pairings associated with group actions on Ω.

Proposition 6.4.6. *Let V be a $(V, \varphi, \mathbb{K})$-VBP and $A, B \subseteq V$. Then*

$$\Gamma_V(A, B) = \begin{cases} V & \text{if } B \subseteq A \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. Since $\pi_{\mathfrak{P}}(A) = \{u + A^{\perp} : u \in V\}$, we deduce that $u + A^{\perp} \subseteq u + B^{\perp}$ if and only if $A^{\perp} \subseteq B^{\perp}$ or, equivalently, if $B \subseteq A$. Hence, $\Gamma_V(A, B) = V$ if and only if $B \subseteq A$, otherwise $\Gamma_V(A, B) = \emptyset$. $\square$

6.4.4 The Case of Pairings Arising from Group Actions on Ω

Let us analyze the case of pairings associated with group actions on Ω.

Proposition 6.4.7. *Let (G, ψ) be the pairing associated with the action ψ of the group G on Ω and $A, B \subseteq \Omega$. Then*

$$\Gamma_{G,\psi}(A, B) = \begin{cases} G & \text{if } \operatorname{Stab}_{G,\psi}(A) \leq \operatorname{Stab}_{G,\psi}(B) \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. Since $\pi_{\mathfrak{P}}(A) = \{g \operatorname{Stab}_{G,\psi}(A) : g \in G\}$, we deduce that $g \operatorname{Stab}_{G,\psi}(A) \subseteq g \operatorname{Stab}_{G,\psi}(B)$ if and only if $\operatorname{Stab}_{G,\psi}(A) \leq \operatorname{Stab}_{G,\psi}(B)$. Hence, for any $g' \in G$, it results that $g' \operatorname{Stab}_{G,\psi}(A) \subseteq g' \operatorname{Stab}_{G,\psi}(B)$, so $\Gamma_{G,\psi}(A, B) = G$. Otherwise, we must have $\Gamma_{G,\psi}(A, B) = \emptyset$. $\square$

Chapter 7

Matroidal Properties of $MINP(\mathfrak{P})$

The result provided in Theorem 6.1.1 can be linked in a deeper way with the classical notion of matroid. As a matter of fact, Theorem 1.6.5 gives a necessary and sufficient condition so that a rank-symmetry operator of a matroid $\mathcal{M}$ on a finite set Ω coincides with a specific closure operator on Ω.

This means that we can pass from the rank-symmetry operator σ of a given matroid on Ω to the closure operator associated with some pairing $\mathfrak{P}$ on Ω itself by means of Theorem 6.1.1. To be more specific, in [20] it has been showed that

$$MINP(\mathfrak{P}) = \{A \in \mathcal{P}(\Omega) : a \notin M_{\mathfrak{P}}(A \setminus \{a\}) \, \forall a \in A\}. \tag{7.1}$$

Then, the identity provided in (7.1) tells us that $MINP(\mathfrak{P})$ behaves as the family of independent sets of a potential matroid associated with the closure operator $M_{\mathfrak{P}}$. In other terms, $MINP(\mathfrak{P})$ is related to $MAXP(\mathfrak{P})$ in a similar way in which the indipendent set family of a matroid is related to its closed set family.
Therefore, the representation Theorem 6.1.1 leads us to conclude that we do not lose in generality whenever we study closure systems or abstract simplicial complexes through pairings and, furthermore, we also obtain explicit constructive criteria in order to study matroids as particular cases of abstract simplicial complexes and finite lattice theory as a subtheory of closure system theory.

In this chapter, we assume that Ω is a finite fixed set. We will establish new links between matroid theory and the maximum partitioners, the minimal ones and the symmetry bases.
To this regard, our first basic result is a new representation theorem, where we prove that any matroid $\mathcal{M}$ on Ω can be represented through a pairing whose $MINP(\mathfrak{P})$ coincides with $\mathcal{M}$.

Theorem 7.0.8. *Let $\mathcal{M}$ be the independent set family of a matroid on Ω. Then, there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $MINP(\mathfrak{P}) = \mathcal{M}$.*

Proof. Let σ be the closure operator associated with $\mathcal{M}$ (see [182]). Then, it is well known that

$$\mathcal{M} = \{A \in \mathcal{P}(\Omega) : a \notin \sigma(A \setminus \{a\}) \, \forall a \in A\} \tag{7.2}$$

By Theorem 6.1.1 there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $M_{\mathfrak{P}} = \sigma$. Then, the thesis follows by (7.2) and by (iv) of Theorem 2.6.1. $\square$

We now study the two maps $rk_{\mathfrak{P}} := rk_{MINP(\mathfrak{P})}$ and $Drk_{\mathfrak{P}} := Drk_{MINP(\mathfrak{P})}$. To this regard, we firstly note that

$$rk_{\mathfrak{P}}(A) = \max\{|X| : X \in MINP_{\mathfrak{P}}(A)\} = \max\{|X| : X \in \max(MINP_{\mathfrak{P}}(A))\} \tag{7.3}$$

and

$$Drk_{\mathfrak{P}}(A) := \{x \in \Omega : rk_{\mathfrak{P}}(A \cup \{x\}) = rk_{\mathfrak{P}}(A)\}. \tag{7.4}$$

Example 7.0.9. *In reference to the pairing $\mathfrak{P}$ given in Example 3.4.5, it results that:*

$$rk_{\mathfrak{P}}(\emptyset) = 0;\ rk_{\mathfrak{P}}(\{r\}) = rk_{\mathfrak{P}}(\{c\}) = rk_{\mathfrak{P}}(\{m\}) = rk_{\mathfrak{P}}(\{b\}) = rk_{\mathfrak{P}}(\{m, c\}) = 1;$$

$$rk_{\mathfrak{P}}(\{r, m\}) = rk_{\mathfrak{P}}(\{r, c\}) = rk_{\mathfrak{P}}(\{r, b\}) = rk_{\mathfrak{P}}(\{b, c\}) = rk_{\mathfrak{P}}(\{m, b\}) = rk_{\mathfrak{P}}(\{r, m, c\}) =$$
$$= rk_{\mathfrak{P}}(\{r, m, b\}) = rk_{\mathfrak{P}}(\{m, c, b\}) = rk_{\mathfrak{P}}(\{r, c, b\}) = rk_{\mathfrak{P}}(\{r, m, c, b\}) = 2$$

Therefore, we deduce that

$$Drk_{\mathfrak{P}}(\emptyset) = \emptyset;\ Drk_{\mathfrak{P}}(\{r\}) = \{r\};\ Drk_{\mathfrak{P}}(\{b\}) = \{b\};$$

$$Drk_{\mathfrak{P}}(\{c\}) = Drk_{\mathfrak{P}}(\{m\}) = Drk_{\mathfrak{P}}(\{m, c\}) = \{m, c\};$$
$$Drk_{\mathfrak{P}}(\{r, m\}) = Drk_{\mathfrak{P}}(\{r, c\}) = Drk_{\mathfrak{P}}(\{r, b\}) = Drk_{\mathfrak{P}}(\{m, b\}) = Drk_{\mathfrak{P}}(\{b, c\}) =$$
$$= Drk_{\mathfrak{P}}(\{r, m, c\}) = Drk_{\mathfrak{P}}(\{r, c, b\}) = Drk_{\mathfrak{P}}(\{m, c, b\}) = Drk_{\mathfrak{P}}(\{r, m, b\}) =$$
$$= Drk_{\mathfrak{P}}(\{r, m, c, b\}) = \{r, m, c, b\}.$$

Proposition 7.0.10. *The function $rk_{\mathfrak{P}}$ satisfies the following properties:*

(i) $rk_{\mathfrak{P}}(\emptyset) = 0$*;*

(ii) If $A \subseteq B$, then $rk_{\mathfrak{P}}(A) \leq rk_{\mathfrak{P}}(B)$;

(iii) $0 \leq rk_{\mathfrak{P}}(A) \leq |A|$ for any $A \in \mathcal{P}(\Omega)$;

(iv) $rk_{\mathfrak{P}}(A) \leq rk_{\mathfrak{P}}(A \cup \{x\}) \leq rk_{\mathfrak{P}}(A) + 1$.

Proof. All these properties are direct consequences of (7.3). □

In [20]. it has been proved the next characterization for set system $MINP(\mathfrak{P})$ in terms of $rk_{\mathfrak{P}}$:

$$MINP(\mathfrak{P}) = \{A \in \mathcal{P}(\Omega) :\ rk_{\mathfrak{P}}(A) = |A|\}. \tag{7.5}$$

Remark 7.0.11. Let us observe that the map $rk_{\mathfrak{P}}$ behaves as the rank function of a matroid whose independent set coincides with $MINP(\mathfrak{P})$ and whose bases are the elements of $\max(MINP(\mathfrak{P}))$. However, in general, $MINP(\mathfrak{P})$ is not a matroid, even if it has many matroidal-like properties.

We can establish now two basic links between some properties of the symmetry bases and the matroidal properties of the set system $MINP(\mathfrak{P})$.

Proposition 7.0.12. *(i) If $MINP(\mathfrak{P})$ is a matroid, then $BAS_{\mathfrak{P}}(A)$ has uniform cardinality for any $A \in \mathcal{P}(\Omega)$.*
(ii) If $M_{\mathfrak{P}}$ is a MLS closure operator on Ω then $MINP(\mathfrak{P})$ is a matroid.

Proof. (i): It follows immediately by Theorem 1.6.4 and by (iii) of Theorem 3.2.2.
(ii): Since $M_{\mathfrak{P}}$ is a MLS closure operator on Ω, the family

$$\mathcal{F} := \{A \in \mathcal{P}(\Omega) : a \notin M_{\mathfrak{P}}(A \setminus \{a\}) \ \forall a \in A\}$$

coincides with the indipendent set family of a matroid on Ω (see [182]). By virtue of (7.1), we deduce that $\mathcal{F} = MINP(\mathfrak{P})$, hence the claim has been showed. □

We now ask which relation there is between the set operators $Drk_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$. In general, the identity $Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ does not hold, as we can see in the next example.

Example 7.0.13. *Let $\mathfrak{P}$ be the pairing given in Example 3.2.3 and let $A = \{1,2\}$. It is immediate to see that $rk_{\mathfrak{P}}(\{1,2\}) = rk_{\mathfrak{P}}(\{1,2,5\}) = 2$ and that $5 \notin M_{\mathfrak{P}}(\{1,2\}) = \{1,2\}$. Hence $M_{\mathfrak{P}}(A) \neq Drk_{\mathfrak{P}}(A)$. Furthermore, let us observe that $BAS(\mathfrak{P}) \neq \max(MINP(\mathfrak{P}))$. In fact, we observe by Figure 3.2 that $BAS(\mathfrak{P}) = \{\{2,5\},\{1,2,4\},\{1,3,4\},\{1,3,5\},\{1,2,3\},\{2,3,4\}\}$, whilst $\max(MINP(\mathfrak{P})) = \{\{2,5\},\{4,5\}\{1,2,4\},\{1,3,4\},\{1,3,5\},\{1,2,3\},\{2,3,4\}\}$. Finally, $MINP(\mathfrak{P})$ is not a matroid: let $A = \{2,5\}$ and $B = \{1,2,4\}$. Then, for any $b \in B \setminus A$, it follows that $A \cup \{b\} \notin MINP(\mathfrak{P})$. As a matter of fact, neither $\{1,2,5\}$ nor $\{2,4,5\}$ belong to $MINP(\mathfrak{P})$, as Figure 3.2 shows.*

However, we show now that $MINP(\mathfrak{P})$ is a matroid on Ω when the set operators $Drk_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$ coincide.

Theorem 7.0.14. *If $Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ for any $A \in \mathcal{P}(\Omega)$, then $MINP(\mathfrak{P}) \in MATR(\Omega)$.*

Proof. Let us prove that $rk_{\mathfrak{P}}$ satisfies $(Rk1), (Rk2)$ and $(Rk3)$. By (i) and (ii) of Proposition 7.0.10, we have that $rk_{\mathfrak{P}}$ satisfies $(Rk1)$ and $(Rk2)$. Let us assume that $rk_{\mathfrak{P}}(A \cup \{a\}) = rk_{\mathfrak{P}}(A \cup \{b\}) = rk_{\mathfrak{P}}(A)$. This means that $a, b \in Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$. Hence, $A \cup \{a\} \cup \{b\} \approx_{\mathfrak{P}} A$. Moreover, it clearly results that $M_{\mathfrak{P}}(A \cup \{a\}) = Drk_{\mathfrak{P}}(A \cup \{a\}) = Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ and $M_{\mathfrak{P}}(A \cup \{b\}) = Drk_{\mathfrak{P}}(A \cup \{b\}) = Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$. Thus, $b \in M_{\mathfrak{P}}(A \cup \{a\}) = Drk_{\mathfrak{P}}(A \cup \{a\})$, therefore we conclude that $rk_{\mathfrak{P}}(A \cup \{a\} \cup \{b\}) = rk_{\mathfrak{P}}(A \cup \{a\}) = rk_{\mathfrak{P}}(A)$. This shows that $rk_{\mathfrak{P}}$ satisfies $(Rk3)$ and that $rk_{\mathfrak{P}}$ is the rank function of a matroid on Ω. By Theorem 1.6.2 and (7.5), we deduce that $MINP(\mathfrak{P})$ is a matroid. □

The reverse theorem does not hold, as we can see in next example.

Example 7.0.15. *Let us consider the pairing $\mathfrak{P}$ of Example 3.4.5. It is easy to see that $MINP(\mathfrak{P})$ is a matroid. In fact, for the empty set, Property $(M2)$ holds trivially. Moreover, if we take A, B such that $|A| = 1$ and $|B| = 2$, two cases occur: if $A \subsetneq B$, $(M2)$ is trivially verified, otherwise, we clearly have that $A \cap B = \emptyset$. Let us consider, for example $A = \{r\}$ and $B = \{b,c\}$. Since both $\{r,c\}$ and $\{r,b\}$ belong to $MINP(\mathfrak{P})$, we conclude that $(M2)$ is satisfied for these two particular sets. A similar argument holds for any pair of subsets as A and B. Therefore $MINP(\mathfrak{P})$ is a matroid. Neverthless, by Example 7.0.9, we have that $Drk_{\mathfrak{P}}(\{r,b\}) = \{r,m,c,b\} \neq M_{\mathfrak{P}}(\{r,b\}) = \{r,b\}$.*

Nevertheless, the following partial reverse of Theorem 7.0.14 can be provided.

Theorem 7.0.16. *If $MINP(\mathfrak{P}) \in MATR(\Omega)$, then $M_{\mathfrak{P}}(A) \subseteq Drk_{\mathfrak{P}}(A)$ for any $A \in \mathcal{P}(\Omega)$.*

Proof. Let $A \in \mathcal{P}(\Omega)$ and $a \notin Drk_{\mathfrak{P}}(A)$. Then, $rk_{\mathfrak{P}}(A \cup \{a\}) = rk_{\mathfrak{P}}(A) + 1$. In particular, by Theorem 1.6.4, it results that $||BAS_{\mathfrak{P}}(A)|| = rk_{\mathfrak{P}}(A)$ and $||BAS_{\mathfrak{P}}(A \cup \{a\})|| = rk_{\mathfrak{P}}(A) + 1$. If a were in $M_{\mathfrak{P}}(A)$, we would have that $BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ has not uniform cardinality. But, $BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \subseteq \max(MINP_{\mathfrak{P}}(M_{\mathfrak{P}}(A)))$ so, by Theorem 1.6.4, $BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ must have uniform cardinality too. Therefore, $a \notin M_{\mathfrak{P}}(A)$. $\square$

In the last part of this chapter, we first recall the notion of maxp-symmetry base uniform pairing introduced in [20] and we show the link between such pairings and symmetry bases, minimal partitioners and the coincidence of the two set operators $Drk_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$.

Definition 7.0.17. We say that $\mathfrak{P} \in PAIR(\Omega)$ is *maxp-symmetry base uniform* if the following two conditions are satisfied:

- (i) for any $A \in \mathcal{P}(\Omega)$, $BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ has uniform cardinality;
- (ii) if $A, B \in \mathcal{P}(\Omega)$, $A \subsetneqq B$ and $A \not\approx_{\mathfrak{P}} B$, then $||BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A))|| < ||BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(B))||$.

We denote by $MSBUP(\Omega)$ the set of all maxp-symmetry base uniform pairings.

In the next fundamental result, we characterize the maxp-symmetry base uniform pairings in terms of equality of the two operators $Drk_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$ and of the two set systems $BAS_{\mathfrak{P}}(A)$ and $\max(MINP_{\mathfrak{P}}(A))$ for any $A \in \mathcal{P}(\Omega)$.

Theorem 7.0.18. *The following conditions are equivalent:*
(i) $\mathfrak{P} \in MSBUP(\Omega)$;
(ii) $\max(MINP_{\mathfrak{P}}(A)) = BAS_{\mathfrak{P}}(A)$ for any $A \in \mathcal{P}(\Omega)$;
(iii) $Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ for any $A \in \mathcal{P}(\Omega)$.

Proof. $(i) \implies (ii)$: By (iii) of Theorem 3.2.2, we have that $BAS_{\mathfrak{P}}(A) \subseteq \max(MINP_{\mathfrak{P}}(A))$ for any $A \in \mathcal{P}(\Omega)$. Let now $B \in \max(MINP_{\mathfrak{P}}(A)) \setminus BAS_{\mathfrak{P}}(A)$. Therefore $\pi_{\mathfrak{P}}(B) \neq \pi_{\mathfrak{P}}(A)$, so there exists $a \in A \setminus M_{\mathfrak{P}}(B)$ such that $B \cup \{a\} \not\approx_{\mathfrak{P}} B$. Since $B \in \max(MINP_{\mathfrak{P}}(A))$, it follows that $BAS_{\mathfrak{P}}(B) = \{B\}$. Let us prove that $a \in C$ for any $C \in BAS_{\mathfrak{P}}(B \cup \{a\})$. Suppose the claim false. Then, $C \subseteq B \subseteq B \cup \{a\}$, but this implies $\pi_{\mathfrak{P}}(B \cup \{a\}) \preceq \pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(B \cup \{a\})$, i.e. $\pi_{\mathfrak{P}}(B \cup \{a\}) = \pi_{\mathfrak{P}}(B)$, contradiction. Thus, $a \in C$ for any $C \in BAS_{\mathfrak{P}}(B \cup \{a\})$. Since $\mathfrak{P}$ is a maxp-symmetry base uniform pairing, $||BAS_{\mathfrak{P}}(B \cup \{a\})|| > ||BAS_{\mathfrak{P}}(B)||$. Moreover, if $C \in MINP_{\mathfrak{P}}(B)$, then $C \in MINP_{\mathfrak{P}}(B \cup \{a\})$ hence, in particular, $B \in MINP_{\mathfrak{P}}(B \cup \{a\})$. So, necessarily, $BAS_{\mathfrak{P}}(B \cup \{a\}) = \{B \cup \{a\}\}$. In other terms, we have found a subset containing B and belonging to $\max(MINP_{\mathfrak{P}}(A))$, i.e. $B \notin \max(MINP_{\mathfrak{P}}(A))$, contradiction. This shows the thesis.
$(ii) \implies (iii)$: Let $a \notin M_{\mathfrak{P}}(A)$. Hence $A \cup \{a\} \not\approx_{\mathfrak{P}} A$. Suppose by contradiction that $a \in Drk_{\mathfrak{P}}(A)$, that is $rk_{\mathfrak{P}}(A) = rk_{\mathfrak{P}}(A \cup \{a\})$. Let $B \in MINP_{\mathfrak{P}}(A)$ such that $|B| = rk_{\mathfrak{P}}(A)$. It must necessarily be $B \in \max(MINP_{\mathfrak{P}}(A \cup \{a\})) = BAS_{\mathfrak{P}}(A \cup \{a\})$, so $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A \cup \{a\})$, contradicting the fact that $a \notin M_{\mathfrak{P}}(A)$. Thus, $a \notin Drk_{\mathfrak{P}}(A)$.

	1	2	3	4	5
u_1	1	0	0	1	0
u_2	0	1	0	0	1
u_3	0	0	0	0	1
u_4	1	0	0	1	0
u_5	1	0	0	0	1

Figure 7.1: Functional Table $\mathfrak{P}$.

Vice versa, let $a \in M_{\mathfrak{P}}(A)$, then $A \cup \{a\} \approx_{\mathfrak{P}} A$. Let us consider $B \in BAS_{\mathfrak{P}}(A \cup \{a\})$. We have two possibilities: if also $B \in BAS_{\mathfrak{P}}(A)$, then $rk_{\mathfrak{P}}(A) = rk_{\mathfrak{P}}(A \cup \{a\})$ by the definition of $rk_{\mathfrak{P}}$; otherwise $a \in B$. In the latter case, we have $B \neq C \cup \{a\}$ for each symmetry base C of A, otherwise $\pi_{\mathfrak{P}}(B \setminus \{a\}) = \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$ but, by (ii) of Theorem 2.6.1, it must result $\pi_{\mathfrak{P}}(B \setminus \{a\}) \neq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$. It must necessarily result $B = B' \cup \{a\}$ for some $B' \subsetneqq C$, where C is a symmetry base of A. This implies that $rk_{\mathfrak{P}}(A \cup \{a\}) \leq rk_{\mathfrak{P}}(A)$. Thus, by monotonicity of $rk_{\mathfrak{P}}$, we conclude that $rk_{\mathfrak{P}}(A) = rk_{\mathfrak{P}}(A \cup \{a\})$, i.e. $a \in Drk_{\mathfrak{P}}(A)$.
$(iii) \implies (i)$: By Theorem 7.0.14, we deduce that $MINP(\mathfrak{P})$ is a matroid. This implies that $||BAS_{\mathfrak{P}}(A)||$ has uniform cardinality for any $A \in \mathcal{P}(\Omega)$, so, in particular, $||BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A))||$ has uniform cardinality for any $A \in \mathcal{P}(\Omega)$. Moreover, if $A, B \in \mathcal{P}(\Omega)$, $A \subsetneqq B$ and $A \not\approx_{\mathfrak{P}} B$, we conclude by (ii) of Proposition 7.0.10 that $rk_{\mathfrak{P}}(A) < rk_{\mathfrak{P}}(B)$, so $||BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(A))|| < ||BAS_{\mathfrak{P}}(M_{\mathfrak{P}}(B))||$ and the thesis follows. □

In the following example, we provide an example of maxp-symmetry base uniform pairing.

Example 7.0.19. *Let us consider the pairing $\mathfrak{P}$ represented in Figure 7.1.*
The indistinguishability lattice of $\mathbb{I}(\mathfrak{P})$ has been drawn in Figure 7.2.
It is easy to verify that $Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ for any $A \in \mathcal{P}(\Omega)$. Thus, we conclude that $MINP(\mathfrak{P})$ is a matroid. As it can be easily seen by Figure 7.2, $\mathfrak{P}$ is maxp-symmetry base uniform, whence it follows that $BAS_{\mathfrak{P}}(A) = \max(MINP_{\mathfrak{P}}(A))$ for any $A \in \mathcal{P}(\Omega)$.

As immediate consequences of the previous theorem, we obtain the following results.

Corollary 7.0.20. *Let $\mathfrak{P} \in MSBUP(\Omega)$. Then, $M_{\mathfrak{P}}$ is a MLS closure operator.*

Proof. By Theorem 7.0.18, it results that $Drk_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ for any $A \in \mathcal{P}(\Omega)$. Furthermore, by Theorem 7.0.14, it follows that $MINP(\mathfrak{P})$, therefore $Drk_{\mathfrak{P}}$ is a MLS closure operator. A fortiori, $M_{\mathfrak{P}}$ is a MLS closure operator. □

Let us note that the converse of Corollary 7.0.20 does not hold as we see in next example.

Example 7.0.21. *In reference to the pairing $\mathfrak{P}$ given in Example 3.4.5, it results that $M_{\mathfrak{P}}$ is a MLS closure operator, but $\mathfrak{P}$ is not a maxp-symmetry base uniform pairing.*

Corollary 7.0.22. *Let $\mathfrak{P} \in MSBUP(\Omega)$. Then, $MINP(\mathfrak{P})$ is a matroid whose basis family is $BAS(\mathfrak{P})$.*

Proof. It follows immediately by Theorems 7.0.18 and 7.0.14. □

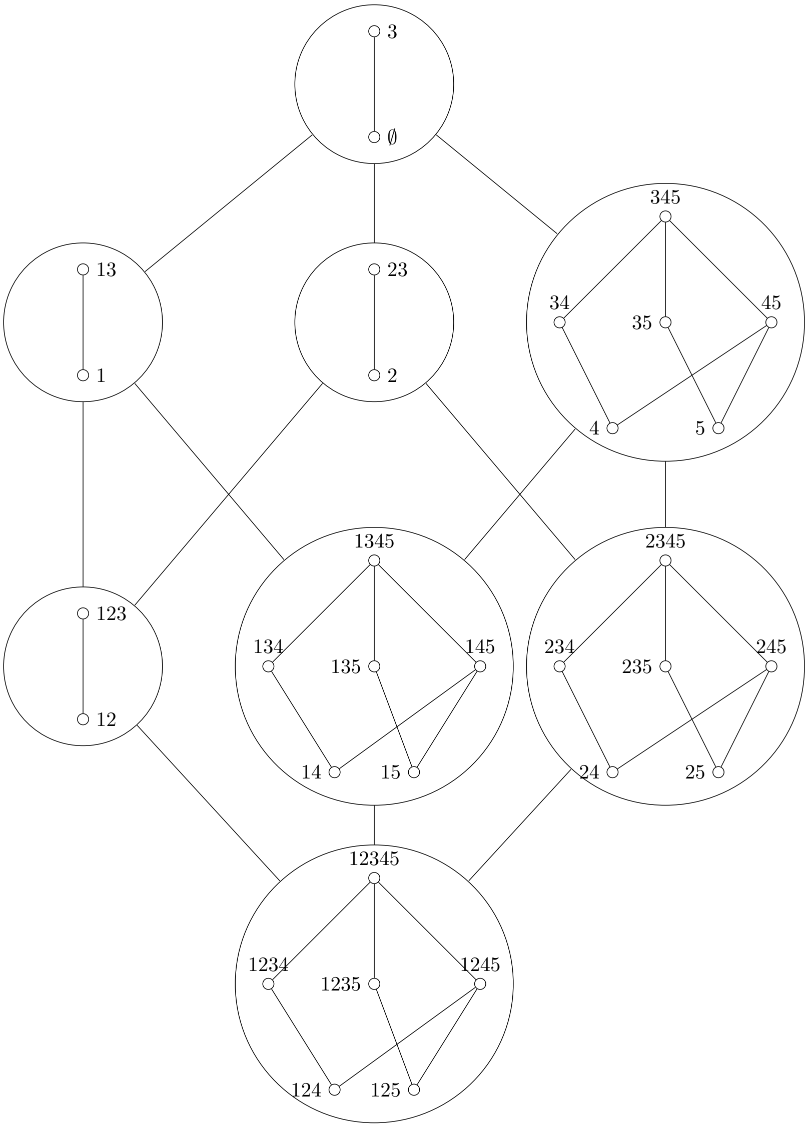


Figure 7.2: The indistinguishaibility lattice $\mathbb{I}(\mathfrak{P})$.

Chapter 8

Symmetry Approximations and Indistinguishability Linear Systems

In this chapter, we introduce the notions of lower and upper symmetry approximations and determine in specific cases, namely graphs and digraphs, the corresponding sets. Furthermore, we link these approximations to a class of pairings that gives rise to an analogy with linear systems and that we call *indistinguishability linear systems.*

8.1 Symmetry Approximations

We provide now the basic definitions concerning symmetry approximation.

Definition 8.1.1. Let $\mathfrak{P} = (U, \Lambda, F) \in PAIR(\Omega)$, $A \subseteq \Omega$ and $X \subseteq U$.

- The A-*lower symmetry approximation* of X is the following subset of U:
$$\mathbf{l}_A(X) := \{x \in U : [x]_A \subseteq X\}.$$
- The A-*upper symmetry approximation* of X is the following subset of U:
$$\mathbf{u}_A(X) := \{x \in U : [x]_A \cap X \neq \emptyset\}.$$
- The subset X is called A-*symmetry exact* if and only if $\mathbf{l}_A(X) = \mathbf{u}_A(X)$ and A-*rough* otherwise.
- The subset
$$BN_A(X) := \mathbf{u}_A(X) \setminus \mathbf{l}_A(X)$$
is called the A-*boundary region* of X.
- If $X \neq \emptyset$, the A-*symmetry approximation accuracy* of X is the quantity $\alpha_A(X) := \frac{|\mathbf{l}_A(X)|}{|\mathbf{u}_A(X)|}$.

Remark 8.1.2. It is immediate to verify that both the following relations hold:

(i) $\mathbf{l}_A(X) = \bigcup\{C \in \pi_{\mathfrak{P}}(A) : C \subseteq X\}$;

(ii) $\mathbf{u}_A(X) = \bigcup\{C \in \pi_{\mathfrak{P}}(A) : C \cap X \neq \emptyset\}$.

8.2 Symmetry Approximation in Some Examples

In this section we compute the A-upper and the A-lower symmetry approximations for some specific cases of pairings.

8.2.1 The A-upper and the A-lower Approximations for C_n

In this subsection, we compute the $A-$upper and the $A-$lower approximation functions for the $n-$cycle C_n. However, we interpret in geometric terms what the A-lower and A-upper approximation represent in graph context.

Proposition 8.2.1. *Let $G = (V(G), E(G))$ be a simple undirected graph and let A and Y be two subsets of $V(G)$. Then:*
(i) $\boldsymbol{u}_A(Y) = \{v \in V(G) : \exists u \in Y : N_G(u) \cap A = N_G(v) \cap A\}$.
Therefore, $v \in \boldsymbol{u}_A(Y)$ if and only if v is an A-symmetric vertex of some $u \in Y$.
(ii) $\boldsymbol{l}_A(Y) = \{v \in V(G) : (u \in V(G) \wedge N_G(u) \cap A = N_G(v) \cap A) \Longrightarrow u \in Y\}$.
Therefore, $v \in \boldsymbol{l}_A(Y)$ if and only if all A-symmetric vertices of v are in Y.

Proof. It is an immediate consequence of (2.2) and of the definitions of lower and upper symmetry approximations. □

Hence the lower symmetry approximation of a vertex set Y represents a subset of Y such that there are no elements outside Y with the same connections of any vertex in $\mathbf{l}_A(Y)$ (relatively to A). The upper symmetry approximation of Y is the set of vertices with the same connections (with respect to A) of at least one element in Y. By the previous proposition it is natural to call $\mathbf{l}_A(Y)$ the A-*symmetry kernel* of Y and $\mathbf{u}_A(Y)$ the A-*symmetry closure* of Y.

In order to determine the general form of the A-upper approximation function of C_n, we must examine all possible relations between the vertex subset Y and the three subsets $B_{C_n}(A)$, $K_{C_n}(A)$ and $S_{C_n}(A)$. Next, we also show that any possible choice of the vertex subsets A and Y is included in the cases we examined.

If A and Y are two vertex subsets of C_n we set

$$Q_A(Y) := \bigcup \{N_{C_n}(v) : v \in C_{C_n}(A) \wedge N_{C_n}(v) \cap Y \neq \emptyset\}.$$

Theorem 8.2.2. *Let A and Y be two vertex subsets of C_n. The map $\boldsymbol{u} : (A, Y) \in \mathcal{P}(V(C_n)) \times \mathcal{P}(V(C_n)) \mapsto \boldsymbol{u}_A(Y) \in \mathcal{P}(V(C_n))$ is completely described from the cases listed in the following table:*

CONDITIONS	$\boldsymbol{u}_A(Y)$
$Y = V(C_n)$	$V(C_n)$
$Y = \emptyset$	$\emptyset$
$Y \neq \emptyset \wedge Y \subseteq B_{C_n}(A)$	$B_{C_n}(A)$
$Y \neq \emptyset \wedge Y \subseteq K_{C_n}(A)$	$Q_A(Y)$
$Y \neq \emptyset \wedge Y \subseteq S_{C_n}(A)$	Y
$Y \cap B_{C_n}(A) \neq \emptyset \wedge Y \cap K_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) = \emptyset$	$Q_A(Y) \cup B_{C_n}(A)$
$Y \cap B_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) \neq \emptyset \wedge Y \cap K_{C_n}(A) = \emptyset$	$B_{C_n}(A) \cup Y$
$Y \cap K_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) \neq \emptyset \wedge Y \cap B_{C_n}(A) = \emptyset$	$Q_A(Y) \cup Y$
$Y \cap B_{C_n}(A) \neq \emptyset \wedge Y \cap K_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) \neq \emptyset$	$B_{C_n}(A) \cup Y \cup Q_A(Y)$

Proof. 1): Let $Y = V(C_n)$ and A any vertex subset of $V(C_n)$. Obviously $[v]_A \cap Y \neq \emptyset$ for every vertex v, so $\mathbf{u}_A(V(C_n)) = V(C_n)$.
2): Let $Y = \emptyset$ and A any vertex subset of $V(C_n)$. Obviously $[v]_A \cap \emptyset = \emptyset$ for every vertex v, so $\mathbf{u}_A(\emptyset) = \emptyset$.
3): Let A and Y be two vertex subsets such that $Y \neq \emptyset \wedge Y \subseteq B_{C_n}(A)$. This means that the indiscernibility block intersecting Y is exactly $B_{C_n}(A)$, therefore $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$.
4): Let A and Y be two vertex subsets such that $Y \neq \emptyset \wedge Y \subseteq K_{C_n}(A)$. Recalling that $K_{C_n}(A) = N_{C_n}(C_{C_n}(A))$, we deduce that Y intersects only the neighbourhoods of some points $v_{j_1}, \dots, v_{j_m} \in C_{C_n}(A)$, therefore $Q_A(Y) \neq \emptyset$ and $\mathbf{u}_A(Y) = Q_A(Y)$.
5): Let A and Y be two vertex subsets such that $Y \neq \emptyset \wedge Y \subseteq S_{C_n}(A)$. Since the elements of $S_{C_n}(A)$ form single blocks in the $A-$indiscernibility partition, we have that $Y \cap [v]_A \neq \emptyset$ if and only if $v \in S_{C_n}(A) \cap Y = Y$. Hence $\mathbf{u}_A(Y) = Y$.
6): Let A and Y be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \wedge Y \cap K_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) = \emptyset$. In other words, Y is transversal only to $B_{C_n}(A)$ and $K_{C_n}(A)$; therefore we have that $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$ or $\exists w \in C_{C_n}(A) : \ v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq Q_A(Y)$. Thus $\mathbf{u}_A(Y) = B_{C_n}(A) \cup Q_A(Y)$.
7): Let A and Y be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) \neq \emptyset \wedge Y \cap K_{C_n}(A) = \emptyset$. In this case, Y is transversal only to $B_{C_n}(A)$ and $S_{C_n}(A)$, hence $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$ or $v \in Y \cap S_{C_n}(A)$. Thus $\mathbf{u}_A(Y) = B_{C_n}(A) \cup (Y \cap S_{C_n}(A)) = Y \cup B_{C_n}(A)$.
8): Let A and Y be two vertex subsets such that $Y \cap K_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) \neq \emptyset \wedge Y \cap B_{C_n}(A) = \emptyset$. Then, $[v]_A \cap Y \neq \emptyset$ if and only if $v \in Y \cap S_{C_n}(A)$ or $\exists w \in C_{C_n}(A) : \ v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq Q_A(Y)$, since Y is transversal to both $K_{C_n}(A)$ and $S_{C_n}(A)$. So, we conclude that $\mathbf{u}_A(Y) = Q_A(Y) \cup (Y \cap S_{C_n}(A)) = Q_A(Y) \cup Y$.
9): Let A and Y be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \wedge Y \cap K_{C_n}(A) \neq \emptyset \wedge Y \cap S_{C_n}(A) \neq \emptyset$. It means that Y is transversal to the three sets. Therefore we have $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$ or $v \in Y \cap S_{C_n}(A)$ or $\exists w \in C_{C_n}(A) : \ v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq Q_A(Y)$. Therefore, $\mathbf{u}_A(Y) = B_{C_n}(A) \cup Q_A(Y) \cup (Y \cap S_{C_n}(A)) = Y \cup B_{C_n}(A) \cup Q_A(Y)$ and we are done.
At this point, we prove that the previous cases are all disjoint each other and they are all possible cases that can occur. Let Y be a proper vertex subset of $V = V(C_n)$, then since $V(C_n) = B_{C_n}(A) \cup K_{C_n}(A) \cup S_{C_n}(A)$, we deduce that Y can be a subset of one of these three sets, as we have said writing down the conditions 3), 4) and 5), or it can be transversal to two of them, without containing none of them and without intersecting the third, as we have said

writing down the conditions 6), 7) and 8). Finally, Y can be transversal to every set, without containing none of them, as written in the last condition. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. In this way we have shown the theorem. $\square$

Let us compute now the $A-$lower approximation function for the $n-$cycle C_n. Also to compute the A-lower approximation function of C_n we will use the previous technique, namely to study all possible relations between the vertex subset Y and the three subsets $B_{C_n}(A)$, $K_{C_n}(A)$ and $S_{C_n}(A)$ determining the A-symmetry partition.

If A and Y are two vertex subsets of C_n we set

$$T_A(Y) := \bigcup_{v \in C_{C_n}(A)} \{N_{C_n}(v) : N_{C_n}(v) \subseteq Y\}.$$

Theorem 8.2.3. *Let A and Y be two vertex subsets of C_n. The map $\boldsymbol{l}\colon (A, Y) \in \mathcal{P}(V(C_n)) \times \mathcal{P}(V(C_n)) \mapsto \boldsymbol{l}_A(Y) \in \mathcal{P}(V(C_n))$ is completely described from the cases listed in the following table:*

CONDITIONS	$\boldsymbol{l}_A(Y)$
$Y = V(C_n)$	$V(C_n)$
$Y \subsetneqq B_{C_n}(A)$	$\emptyset$
$Y \supseteq B_{C_n}(A) \wedge T_A(Y) = \emptyset$	$B_{C_n}(A) \cup (Y \cap S_{C_n}(A))$
$Y \supseteq B_{C_n}(A) \wedge T_A(Y) \neq \emptyset$	$B_{C_n}(A) \cup (Y \cap S_{C_n}(A)) \cup T_A(Y)$
$Y \cap B_{C_n}(A) \neq \emptyset \wedge B_{C_n}(A) \not\subseteq Y \wedge T_A(Y) = \emptyset$	$Y \cap S_{C_n}(A)$
$Y \cap B_{C_n}(A) \neq \emptyset \wedge B_{C_n}(A) \not\subseteq Y \wedge T_A(Y) \neq \emptyset$	$(Y \cap S_{C_n}(A)) \cup T_A(Y)$

Proof. 1): Let $Y = V(C_n)$ and A be any vertex subset of $V(C_n)$. Obviously $[v]_A \subseteq V(C_n)$ for every vertex v, so $\mathbf{l}_A(V(C_n)) = V(C_n)$.
2): Let $Y = \emptyset$ and A be any vertex subset of $V(C_n)$. Obviously no indiscernibility class is contained in the empty set, so $\mathbf{l}_A(\emptyset) = \emptyset$.
3): Let A and Y be two vertex subsets such that $Y \subsetneqq B_{C_n}(A)$. Since $B_{C_n}(A)$ forms a single block and since Y is disjoint from $K_{C_n}(A)$ and $S_{C_n}(A)$, we deduce that there is no vertex whose indiscernibility class is contained in Y, thus $\mathbf{l}_A(Y) = \emptyset$.
4): Let A and Y be two vertex subsets such that $Y \supseteq B_{C_n}(A)$ and $T_A(Y) = \emptyset$. We observe that Y may or not intersect the vertex subset $K_{C_n}(A)$ and, in the first case, in such a way that there not exists any vertex $w \in C_{C_n}(A)$ whose neighbourhood is contained in Y. Furthermore, Y may or not intersect $S_{C_n}(A)$. Thus we conclude that $[v]_A \subseteq Y$ if and only if $v \in B_{C_n}(A)$ or, possibly, $v \in Y \cap S_{C_n}(A)$, i.e. $\mathbf{l}_A(Y) = B_{C_n}(A) \cup (Y \cap S_{C_n}(A))$.
5): Let A and Y be two vertex subsets such that $Y \supseteq B_{C_n}(A)$ and $T_A(Y) \neq \emptyset$. We observe that Y may or not intersect the vertex subset $S_{C_n}(A)$. Moreover, there exists at least one vertex $w \in C_{C_n}(A)$ whose neighbourhood is contained in Y. Hence we conclude that $[v]_A \subseteq Y$ if and only if $v \in B_{C_n}(A)$ or $v \in N_{C_n}(w) \subseteq T_A(Y)$ for some $w \in C_{C_n}(A)$ or, possibly, $v \in Y \cap S_{C_n}(A)$, i.e. $\mathbf{l}_A(Y) = B_{C_n}(A) \cup T_A(Y) \cup (Y \cap S_{C_n}(A))$.
6): Let A and Y be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \wedge B_{C_n}(A) \not\subseteq Y \wedge T_A(Y) = \emptyset$. Since $B_{C_n}(A)$ forms a single block, we conclude that $B_{C_n}(A) \not\subseteq \mathbf{l}_A(Y)$. We also observe that Y

may or not intersect the vertex subset $S_{C_n}(A)$. Moreover, there is no vertex $w \in C_{C_n}(A)$ such that $v \in N_{C_n}(w)$ and $N_{C_n}(w) \subseteq Y$. This means that $[v]_A \subseteq Y$ if and only if $v \in Y \cap S_{C_n}(A)$, i.e. $\mathbf{l}_A(Y) = Y \cap S_{C_n}(A)$.
7): Let A and Y be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \wedge B_{C_n}(A) \not\subseteq Y \wedge T_A(Y) \neq \emptyset$. Since $B_{C_n}(A)$ forms a single block, we conclude that $B_{C_n}(A) \not\subseteq \mathbf{l}_A(Y)$. We also observe that Y may or not intersect the vertex subset $S_{C_n}(A)$. Furthermore, there exists at least a vertex $w \in C_{C_n}(A)$ such that $v \in N_{C_n}(w)$ and $N_{C_n}(w) \subseteq Y$. In other words, we are saying that $[v]_A \subseteq Y$ if and only if $v \in Y \cap S_{C_n}(A)$ or $\exists w \in C_{C_n}(A):\ v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq T_A(Y)$, i.e. $\mathbf{l}_A(Y) = (Y \cap S_{C_n}(A)) \cup T_A(Y)$.
At this point, we show that we have studied all the occurring cases. Let Y be a proper vertex subset of $V = V(C_n)$. In case 3) we have that $Y \subsetneqq B_{C_n}(A)$ while in cases 4) and 5) we are requiring that Y contains $B_{C_n}(A)$ and it may (or not) be transversal to both $K_{C_n}(A)$ and $S_{C_n}(A)$. Finally, Y may only transversal to $B_{C_n}(A)$, without containing it, as we have seen in the last two cases. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. Hence, the theorem is proved. □

To conclude this section, we recall that A-symmetry exactness has been investigated in [52].

8.2.2 Symmetry Approximation in Some Basic Digraphs

Let $D = (V(D), Arc(D))$ be a digraph and let A, Y be two vertex subsets of D. We denote by $\mathbf{l}^+_A(Y) := \{v \in V(D) : [v]_A \subseteq Y\}$ the A-lower approximation of Y and by $\mathbf{u}^+_A(Y) := \{v \in V(D) : [v]_A \cap Y \neq \emptyset\}$ the A-upper approximation of Y.

In the next results we completely determine A-lower approximation, A-upper approximation and A-symmetry exactness for the four basic digraph families $\vec{K}_{p,q}$, $\vec{P}_n$, $\vec{C}_n$ and $\vec{T}_n$.

We begin with the case of the complete bipartite digraph $\vec{K}_{p,q}$.

Proposition 8.2.4. *Let A and Y be two subsets of $V(\vec{K}_{p,q}) = \{x_1, \dots, x_p, y_1, \dots, y_q\}$, where $Y \neq \emptyset$ and $Y \neq V$. Then:*
(i) the A-lower approximation of Y is

$$\mathbf{l}^+_A(Y) = \begin{cases} Y & \text{if } A = V \\ \emptyset & \text{if } A = \emptyset \vee \\ & A \subseteq B_1 \vee (Y \cap B_1 \neq \emptyset \wedge Y \cap B_2 \neq \emptyset \wedge B_1 \not\subseteq Y \wedge B_2 \not\subseteq Y) \\ B_1 & \text{if } B_1 \subseteq Y \\ B_2 & \text{if } B_2 \subseteq Y \end{cases}$$

(ii) The A-upper approximation of Y is

$$\mathbf{u}^+_A(Y) = \begin{cases} B_1 & \text{if } Y \subseteq B_1 \wedge A \not\subseteq B_1 \\ B_2 & \text{if } Y \subseteq B_2 \wedge A \not\subseteq B_1 \\ V & \text{otherwise} \end{cases}$$

(iii) If $A = V$, then Y any vertex subset is A-symmetry exact. In the other cases, Y is A-symmetry exact if and only if $Y = B_1$ or $Y = B_2$.

Proof. (i): The statement in the case $A = V$ is obvious. Moreover, let $A = \emptyset \vee A \subseteq B_1$. Then $[v]_A = V$, whence $\mathbf{l}_A^+(Y) = \emptyset$. Let A be a non-empty vertex subset not contained in B_1. By Proposition 2.3.9, $\pi_{\vec{K}_{p,q}}(A) = B_1|B_2$, therefore if Y is any vertex subset containing B_1 or B_2, it is clear that, respectively, $\mathbf{l}_A^+(Y) = B_1$ or $\mathbf{l}_A^+(Y) = B_2$; otherwise, if Y is transversal to both B_1 and B_2 without containing none of them, then $[v]_A \not\subseteq Y$, hence $\mathbf{l}_A^+(Y) = \emptyset$.
(ii): By Proposition 2.3.9, if we suppose that $A = \emptyset$ or $A \subseteq B_1$, we conclude easily that $\mathbf{u}_A^+(Y) = V$. Let A be a non-empty vertex subset such that $A \not\subseteq B_1$ and let Y be another vertex subset. We have three cases: if $Y \subseteq B_1$, clearly it results $\mathbf{u}_A^+(Y) = B_1$, if $Y \subseteq B_2$, clearly it results $\mathbf{u}_A^+(Y) = B_2$, otherwise Y is transversal to both B_1 and B_2, so it intersects both the symmetry blocks of $\pi_{\vec{K}_{p,q}}(A)$. This means that, in this case, $\mathbf{u}_A^+(Y) = V$.
(iii): It follows immediately by the previous parts. □

We can now examine the digraph $\vec{P_n}$.

Proposition 8.2.5. *Let A and Y be two subsets of $V(\vec{P_n}) = \{v_1, \ldots, v_n\}$, where $Y \neq \emptyset$ and $Y \neq V$. Then:*
(i) the A-lower approximation of Y is

$$\mathbf{l}_A^+(Y) = \begin{cases} Y & \text{if } A = V \vee A = \{v_2, \ldots, v_n\} \vee (|A| = k \wedge Y \supseteq A_{-1}^c) \\ \emptyset & \text{if } A = \emptyset \vee A = \{v_1\} \\ Y \cap A_{-1} & \text{otherwise.} \end{cases}$$

(ii) The A-upper approximation of Y is

$$\mathbf{u}_A^+(Y) = \begin{cases} V & \text{if } A = \emptyset \vee A = \{v_1\} \\ Y \cup A_{-1}^c & \text{if } |A| = k \wedge Y \cap A_{-1}^c \neq \emptyset \\ Y & \quad \text{otherwise.} \end{cases}$$

(iii) If $A = V$ or $A = \{v_2, \ldots, v_n\}$ then Y is always A-symmetry exact. In the other cases, Y is A-symmetry exact if and only if $A \neq \emptyset \wedge A \neq \{v_1\} \wedge (|A| = k \wedge (Y \supseteq A_{-1}^c \vee Y \subseteq A_{-1}))$.

Proof. (i) By Proposition 2.3.9, if $A = V$ or $A = \{v_2, \ldots, v_n\}$, then $\pi_{\vec{P_n}}(A) = v_1|\ldots|v_n$ and therefore $\mathbf{l}_A^+(Y) = Y$ for any choice of the vertex subset Y. Again, by Proposition 2.3.9, it is obvious that if $A = \emptyset$ or if $A = \{v_1\}$, then $\mathbf{l}_A^+(Y) = \emptyset$, since the only subset Y containing V is V itself, but by hypothesis $Y \neq V$. Let $|A| = k$; then, by Proposition 2.3.9, $\pi_{\vec{P_n}}(A) = v_{i_1-1}|\ldots|v_{i_k-1}|A_{-1}^c$ or $\pi_{\vec{P_n}}(A) = v_{i_2-1}|\ldots|v_{i_k-1}|A_{-1}^c$ depending on whether $v_1 \notin A$ or $v_1 \in A$. In both cases, if $Y \supseteq A_{-1}^c$, then $[v]_A \subseteq Y$ if and only if $v \in A_{-1}^c$ or $v \in Y \cap A_{-1}$ and $\mathbf{l}_A^+(Y) = A_{-1}^c \cup (Y \cap A_{-1}) = Y$. If $|A| = k$ and $Y \not\supseteq A_{-1}^c$, then $[v]_A \subseteq Y$ if and only if $v \in Y \cap A_{-1}$, so $\mathbf{l}_A^+(Y) = Y \cap A_{-1}$.
(ii) By Proposition 2.3.9, if $A = \emptyset$ or $A = \{v_1\}$, then $\pi_{\vec{P_n}}(A) = V$ and therefore all the vertices of the digraph are contained in a single block that obviously intersects Y. Let $A = V$, then by Proposition 2.3.9, $\pi_{\vec{P_n}}(A) = v_1|\ldots|v_n$. If $|A| = k$ and $Y \cap A_{-1}^c \neq \emptyset$, then $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A_{-1}^c$ or $v \in Y \cap A_{-1}$ and thus in this case $\mathbf{u}_A^+(Y) = Y \cup A_{-1}^c$. If $|A| = k$ and $Y \subseteq A_{-1}$, then $[v]_A \cap Y \neq \emptyset$ if and only if $v \in Y \cap A_{-1} = Y$, so $\mathbf{u}_A^+(Y) = Y$. If $A = V$ or $A = \{v_2, \ldots, v_n\}$, then by using Proposition 2.3.9, it is easy to see that $[v]_A \cap Y \neq \emptyset$ if and

only if $v \in Y$. This completes the proof of (ii).
(iii) By the previous parts, it is clear that if $A = V$ or if $A = \{v_2, \dots, v_n\}$ then Y is always A-symmetry exact. Moreover if $A \neq \emptyset \wedge A \neq \{v_1\} \wedge (Y \supseteq A^c_{-1} \vee Y \subseteq A_{-1})$ then Y is A-symmetry exact. On the other hand, it is obvious that if $A = \emptyset$ or $A = \{v_1\}$ then no proper vertex subset Y is A-symmetry exact. Now, let $|A| = k$, where $A \neq \{v_2, \dots, v_n\}$. A vertex subset Y is A-symmetry exact if and only if $\mathbf{l}^+_A(Y) = Y = \mathbf{u}^+_A(Y)$. By part (i), $\mathbf{l}^+_A(Y) = Y$ if and only if $Y \supseteq A^c_{-1}$ or $Y \cap A^c_{-1} = \emptyset$ which is equivalent to say that $Y \subseteq A_{-1}$. Finally, by part (ii), $Y = \mathbf{u}^+_A(Y)$ if and only if $Y \cap A^c_{-1} \neq \emptyset$ and $Y \cup A^c_{-1} = Y$ (equivalent to say that $Y \supseteq A^c_{-1}$) or $Y \cap A^c_{-1} = \emptyset$ (equivalent to say that $Y \subseteq A_{-1}$). The proposition is thus proved. $\square$

We determine now the rough approximation functions and the corresponding exactness for the n-directed cycle.

Proposition 8.2.6. *Let A and Y be two subsets of $V(\vec{C_n}) = \{v_1, \dots, v_n\}$, where $Y \neq \emptyset$ and $Y \neq V$. Then:*
(i) the A-lower approximation of Y is

$$\mathbf{l}^+_A(Y) = \begin{cases} Y & \text{if } (A = V) \vee (1 \leq |A| \leq n-2 \wedge Y \supseteq A^c_{-1}) \vee |A| = n-1 \\ \emptyset & \text{if } A = \emptyset \\ A_{-1} \cap Y & \text{otherwise} \end{cases}$$

(ii) The A-upper approximation of Y is

$$\mathbf{u}^+_A(Y) = \begin{cases} V & \text{if } A = \emptyset \\ Y \cup A^c_{-1} & \text{if } 1 \leq |A| \leq n-2 \wedge Y \cap A^c_{-1} \neq \emptyset \\ Y & \text{otherwise} \end{cases}$$

(iii) If $A = V$ or $|A| = n-1$, then Y is always A-symmetry exact. In the other cases, Y is A-symmetry exact if and only if $1 \leq |A| \leq n-2 \wedge (Y \supseteq A^c_{-1} \vee Y \subseteq A_{-1})$.

Proof. (i) : By Proposition 2.3.9, if $A = \emptyset$ then $\mathbf{l}^+_A(Y) = \emptyset$, since the only subset Y containing V is V itself, but by hypothesis $Y \neq V$. Suppose that $|A| \geq n-1$, then by Proposition 2.3.9, it results that $\pi_{\vec{C_n}}(A) = v_1|\dots|v_n$, therefore it is clear that $\mathbf{l}^+_A(Y) = Y$ for any choice of the vertex subset Y. Let $1 \leq |A| = k \leq n-2$; by Proposition 2.3.9 we deduce that if $Y \supseteq A^c_{-1}$, then $[v]_A \subseteq Y$ if and only if $v \in A^c_{-1}$ or $v \in Y \cap A_{-1}$ and $\mathbf{l}^+_A(Y) = A^c_{-1} \cup (Y \cap A_{-1}) = Y$. If $1 \leq |A| \leq n-2$ and $Y \supsetneq A^c_{-1}$, then $[v]_A \subseteq Y$ if and only if $v \in Y \cap A_{-1}$, so $\mathbf{l}^+_A(Y) = Y \cap A_{-1}$.
(ii) : By Proposition 2.3.9, if $|A| \geq n-1$, it results that $\pi_{\vec{C_n}}(A) = v_1|\dots|v_n$, therefore it is clear that $\mathbf{u}^+_A(Y) = Y$ for any choice of the vertex subset Y. On the contrary, if $A = \emptyset$ then $\mathbf{u}^+_A(Y) = V$, since for every vertex v, we have $[v]_A = V$ and every vertex subset Y intersects V. Let $1 \leq |A| = k \leq n-2$ and $Y \cap A^c_{-1} \neq \emptyset$, then $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A^c_{-1}$ or $v \in Y \cap A_{-1}$, therefore $\mathbf{u}^+_A(Y) = Y \cup A^c_{-1}$. Finally, if $1 \leq |A| \leq n-2$ and $Y \subseteq A_{-1}$, then $[v]_A \cap Y \neq \emptyset$ if and only if $v \in Y \cap A_{-1} = Y$, hence we conclude that $\mathbf{u}^+_A(Y) = Y$. We have thus shown (ii).
(iii) : By the previous parts, it is clear that if $|A| \geq n-1$ then Y is always A-symmetry exact. it is also clear that if $1 \leq |A| \leq n-2 \wedge (Y \supseteq A^c_{-1} \vee Y \subseteq A_{-1})$ then Y is A-symmetry exact. Moreover, it is obvious that if $A = \emptyset$ then no proper vertex subset Y is A-symmetry

exact. Now, let $1 \leq |A| \leq n-2$. A vertex subset Y is A-symmetry exact if and only if $\mathbf{l}_A^+(Y) = Y = \mathbf{u}_A^+(Y)$. By part (i), $\mathbf{l}_A^+(Y) = Y$ if and only if $Y \supseteq A_{-1}^c$ or $Y \cap A_{-1}^c = \emptyset$, i.e. if and only if $Y \subseteq A_{-1}$. Furthermore, by part (ii), $Y = \mathbf{u}_A^+(Y)$ if and only if $Y \cap A_{-1}^c \neq \emptyset$ and $Y \cup A_{-1}^c = Y$ or $Y \cap A_{-1}^c = \emptyset$, i.e. if and only if $Y \supseteq A_{-1}^c$ or $Y \subseteq A_{-1}$. This completes the proof. □

Finally, we focus our attention on the n-transitive tournament $\vec{T_n}$. Let $A = \{v_{i_1}, \dots, v_{i_k}\} \subseteq V(\vec{T_n})$. We set then, for each choice of $j \in \{0, 1, \dots, k\}$:

$$C_A^j := \begin{cases} \{v_1, \dots, v_{i_1-1}\} & \text{if} \quad i_1 \neq 1 \wedge j = 0, \\ \{v_{i_j}, \dots, v_{i_{j+1}-1}\} & \text{if} \quad 1 \leq j < k, \\ \{v_{i_k}, \dots, v_n\} & \text{if} \quad j = k. \end{cases}$$

These sets are the blocks of the partition $\pi_{\vec{T_n}}(A)$ (see Propostion 2.3.9). Thus clearly it holds:

$$\mathbf{l}_A^+(Y) = \begin{cases} Y & \text{if } A = V \vee A = \{v_2, \dots, v_n\} \\ \emptyset & \text{if } A = \emptyset \vee A = \{v_1\} \\ \bigcup_{v_{i_j} \in A} \{C_A^j : C_A^j \subseteq Y\} & \text{otherwise.} \end{cases}$$

and

$$\mathbf{u}_A^+(Y) = \begin{cases} Y & \text{if } A = V \vee A = \{v_2, \dots, v_n\} \\ V & \text{if } A = \emptyset \vee A = \{v_1\} \\ \bigcup_{v_{i_j} \in A} \{C_A^j : C_A^j \cap Y \neq \emptyset\} & \text{otherwise.} \end{cases}$$

8.3 Exactness as a Type of Dependency

Exactness with respect to a fixed subset can be characterized through symmetry approximation accuracy, in fact, an element subset X is A-symmetry exact if and only if $\alpha_A(X) = 1$. In the sequel, we introduce the notion of *indistinguishability linear system* and then we see in which way this notion is linked to symmetry approximation accuracy.

Definition 8.3.1. We call *indistinguishability linear system* a structure $\mathcal{S} = \langle U_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}, F_{\mathcal{S}}, \Lambda_{\mathcal{S}} \rangle$, where $U_{\mathcal{S}}, C_{\mathcal{S}}, \Lambda_{\mathcal{S}}$ are non-empty sets, $D_{\mathcal{S}}$ is empty or consists of a single element $d_{\mathcal{S}}$ such that $d_{\mathcal{S}} \notin C_{\mathcal{S}}$. $F_{\mathcal{S}}$ is a map having domain $U_{\mathcal{S}} \times (C_{\mathcal{S}} \cup D_{\mathcal{S}})$ and codomain $\Lambda_{\mathcal{S}}$. We also call:

- the elements of $U_{\mathcal{S}}$ the *equations* of $\mathcal{S}$;
- the elements of $C_{\mathcal{S}}$ the *variables* of $\mathcal{S}$;
- the element $d_{\mathcal{S}}$ the *constant* of $\mathcal{S}$;
- the set $D_{\mathcal{S}}$ the *constant set* of $\mathcal{S}$;
- the elements of $\Lambda_{\mathcal{S}}$ the *coefficients* of $\mathcal{S}$;
- the map $F_{\mathcal{S}}$ the *coefficient map* of $\mathcal{S}$.

We say that $\mathcal{S}$ is a *homogeneous indistinguishability linear system* when $D_{\mathcal{S}} = \emptyset$. If the sets $U_{\mathcal{S}}$ and $\Omega_{\mathcal{S}}$ are both finite we say that $\mathcal{S}$ is a *finite indistinguishability linear system.* In what follows we will set $\Omega_{\mathcal{S}} := C_{\mathcal{S}} \cup D_{\mathcal{S}}$.

In what follows we denote by $\mathcal{S} = \langle U_{\mathcal{S}}, \Omega_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}, F_{\mathcal{S}}, \Lambda_{\mathcal{S}} \rangle$ a given indistinguishability linear system.

If $W \in \mathcal{P}(U_{\mathcal{S}})$ and $A \in \mathcal{P}(C_{\mathcal{S}})$ we denote by $\mathcal{S}_{W,A}$ the indistinguishability linear system having equation set W, variable set A, constant set $D_{\mathcal{S}}$, coefficient map the restriction of $F_{\mathcal{S}}$ to the subset $W \times (A \cup D_{\mathcal{S}}) \subseteq U_{\mathcal{S}} \times \Omega_{\mathcal{S}}$, coefficient set $\Lambda_{\mathcal{S}}$. In particular, we write $\mathcal{S}_A$ instead of $\mathcal{S}_{U_{\mathcal{S}},A}$ and $\mathcal{S}_W$ instead of $\mathcal{S}_{W,C_{\mathcal{S}}}$.

Remark 8.3.2. Let us observe that indistinguishability linear systems are a particular case of pairings. Hence, the notion of A-symmetry partition can be given for them, recalling that $\Omega = C_{\mathcal{S}} \cup D_{\mathcal{S}}$. At an interpretative level, given a linear system, we fix a subset of variables (and possibly also the constant term) and we consider all the equations restricted to the previous set of variables. At this point, there may be some identical equations that we identify. By means of this equivalence relation we reinterpret many concepts of local and global compatibility relative to the indistingushability linear system $\mathcal{S}$. In what follows, we denote by $\pi_{\mathcal{S}}(A)$ the set partition on $U_{\mathcal{S}}$ induced by $\equiv_A$.
On the other hand, if we restrict our attention only to variable subsets, we can introduce the indstinguishability relation $\approx_{\mathcal{S}}$ and the notion of maximum partitioner (as in the pairing context) in the case of indistinguishability linear systems and, hence, the other structures related to $\approx_{\mathcal{S}}$. Therefore, we can extend to indistinguishability linear systems all the results we proved in the previous chapters.

Let $\mathcal{S}$ be a finite indistinguishability linear system having Boolean values 0 and 1 in the column corresponding to the constant d_j, for some $1 \leq j \leq p$. Then, there is only a subset $X \subseteq U_{\mathcal{S}}$ such that $F_{\mathcal{S}}(u, d_j) = 1$ if $u \in X$ and $F_{\mathcal{S}}(u, d_j) = 0$ if $u \in U_{\mathcal{S}} \setminus X$.

Definition 8.3.3. Let $\mathcal{S}$ and X as above. We say that:

- $\mathcal{S}$ is a decision table with *j-th decision* X, and we set $d_j := d^X$;
- X is a *decision* of $\mathcal{S}$ if $\mathcal{S}$ is a decision table with j-th decision X, for some $1 \leq j \leq p$.

If $\mathcal{S}$ is a simple decision table that such $d_{\mathcal{S}} = d^X$, for some $X \subseteq U_{\mathcal{S}}$, we say that S is a simple *X-decision table.*

We have then the following result.

Proposition 8.3.4. *Let $X \subseteq U_{\mathcal{S}}$ and $A \subseteq C_{\mathcal{S}}$. Then,*

$$\gamma_{\mathcal{S}}(A, \{d^X\}) = 1 - \frac{|BN_A(X)|}{|U_{\mathcal{S}}|} \tag{8.1}$$

Hence, in particular, X is A-symmetry exact if and only if $\gamma(A, \{d^X\}) = 1$.

Proof. We have

$$\Gamma_{\mathcal{S}}(A, \{d^X\})) = \mathbf{l}_A(X) \cup \mathbf{l}_A(X^c) = \mathbf{l}_A(X) \cup (U_{\mathcal{S}} \setminus \mathbf{u}_A(X)),$$

therefore

$$\gamma_{\mathcal{S}}(A, \{d^X\}) = \frac{|\Gamma_{\mathcal{S}}(A, \{d^X\}))|}{|U_{\mathcal{S}}|} = \frac{|\mathbf{l}_A(X)| + |U_{\mathcal{S}}| - |\mathbf{u}_A(X)|}{|U_{\mathcal{S}}|} = \frac{|U_{\mathcal{S}}| - |\mathbf{u}_A(X) \setminus \mathbf{l}_A(X)|}{|U_{\mathcal{S}}|},$$

and the thesis follows. □

Remark 8.3.5. The result established in Proposition 8.3.4 is simple but important, since it connects the A-symmetry exactness of the subset X to the value of the symmetry transmission measure $\gamma(A, \{d^X\})$. In other terms, we can see that X is A-symmetry exact if and only if the single element d^X totally depends by the subset A in the indistinguishability linear system $\mathcal{S}$.

Definition 8.3.6. If $X \subseteq U_{\mathcal{S}}$ and $A \subseteq C_{\mathcal{S}}$, we set

$$\beta_A^{\mathcal{S}}(X) := \gamma_{\mathcal{S}}(A, \{d^X\})$$

We call $\beta_A(X)$ the (A, d^X)-*approximation symmetry measure* of X.

The identity established in (8.1) can also be reformulated in the following two equivalent ways:

$$\beta_A(X) = 1 - (1 - \alpha_A(X))\frac{|\mathbf{u}_A(X)|}{|U_{\mathcal{S}}|} \tag{8.2}$$

and

$$\alpha_A(X) = 1 - (1 - \beta_A(X))\frac{|U_{\mathcal{S}}|}{|\mathbf{u}_A(X)|} \tag{8.3}$$

By (8.2) we immediately deduce the following result

Proposition 8.3.7. *If $\alpha_A(X) = \beta_A(X)$ then $\alpha_A(X) = \beta_A(X) = 1$ or $\boldsymbol{u}_A(X) = U_{\mathcal{S}}$.*

Here we simply recall the following differences between α_A and β_A:

(α_1) $\alpha_A(X) = 0 \iff \mathbf{l}_A(X) = \emptyset$,

(α_2) for a fixed $\mathbf{l}_A(X) \neq \emptyset$, $\alpha_A(X)$ strictly monotonically decreases with $|\mathbf{u}_A(X)|$,

and

(β_1) $\beta_A(X) = 0 \iff (\mathbf{l}_A(X) = \emptyset \ \wedge \ \mathbf{u}_A(X) = U_{\mathcal{S}})$,

(β_2) for a fixed $\mathbf{l}_A(X)$, $\beta_A(X)$ strictly monotonically decrease with $|\mathbf{u}_A(X)|$.

8.4 Compatibility for Indistinguishability Linear Systems

Let $X \in \mathcal{P}(U_{\mathcal{S}})$ and $Y \in \mathcal{P}(C_{\mathcal{S}})$. We set

$$\Theta_{\mathcal{S}}(X, Y) := \{v \in X : [v]_Y \cap X \subseteq [v]_{D_{\mathcal{S}}} \cap X\} \tag{8.4}$$

and

$$\theta_{\mathcal{S}}(X, Y) := \frac{|\Theta_{\mathcal{S}}(X, Y)|}{|X|} \tag{8.5}$$

if $\mathcal{S}$ is finite.

Moreover, if $Y, Z \in \mathcal{P}(\Omega_{\mathcal{S}})$ we also set

$$\Gamma_{\mathcal{S}}(Y, Z) := \{v \in U_{\mathcal{S}} : [v]_Y \subseteq [v]_Z\}$$

and

$$\gamma_{\mathcal{S}}(Y, Z) := \frac{|\Gamma_{\mathcal{S}}(Y, Z)|}{|U_{\mathcal{S}}|}$$

if $\mathcal{S}$ is finite.

For $A \in \mathcal{P}(C_{\mathcal{S}})$ and $u \in U_{\mathcal{S}}$ we denote by $F^u_{\mathcal{S},A}$ the map from A to $\Lambda_{\mathcal{S}}$ defined by $F^u_{\mathcal{S},A}(a) := F_{\mathcal{S}}(u, a)$ for any $a \in A$. Moreover, we denote by $\eta_{\mathcal{S},A}$ the map from $U_{\mathcal{S}}$ to $\Lambda^A_{\mathcal{S}}$ defined by $\eta_{\mathcal{S},A}(u) := F^u_{\mathcal{S},A}$ for any $u \in U_{\mathcal{S}}$. We will denote by $\eta^W_{\mathcal{S},A}$ the restriction of $\eta_{\mathcal{S},A}$ to $W \in \mathcal{P}(U_{\mathcal{S}})$. By means of these notations we establish (also in the non finite case) an equivalence between the notion of compatibility $(ILS1)$, discussed in the introductory section, and one of its formulations expressed in terms of symmetry classes.

Proposition 8.4.1. *Let $W \in \mathcal{P}(U_{\mathcal{S}})$ and $A \in \mathcal{P}(C_{\mathcal{S}})$. Then, the following conditions are equivalent.*

$(ILS1)$ There exists a map $f : \Lambda^A_{\mathcal{S}} \to \Lambda_{\mathcal{S}}$ such that $F_{\mathcal{S}}(u, d_{\mathcal{S}}) = f(\eta_{\mathcal{S},A}(u))$ for any $u \in W$.

$(ILS2)$ $\Theta_{\mathcal{S}}(W, A) = W$.

Proof. Suppose that $(ILS1)$ holds and let $u \in W$ and $u' \in W$ such that $u' \equiv_A u$. Thus $F_{\mathcal{S}}(u, a) = F_{\mathcal{S}}(u', a)$ for any $a \in A$. In other terms it results that $F^u_{\mathcal{S},A} = F^{u'}_{\mathcal{S},A}$, so $\eta^W_{\mathcal{S},A}(u) = \eta^W_{\mathcal{S},A}(u')$. By hypothesis, we conclude that $F_{\mathcal{S}}(u, d_{\mathcal{S}}) = F_{\mathcal{S}}(u', d_{\mathcal{S}})$, therefore $u' \in [u]_{D_{\mathcal{S}}} \cap W$. This means that $\Theta_{\mathcal{S}}(W, A) = W$.
On the other hand, suppose that $(ILS2)$ holds. Let us consider an application $f : \Lambda^A_{\mathcal{S}} \to \Lambda_{\mathcal{S}}$ such that $F_{\mathcal{S}}(u, d_{\mathcal{S}}) := (f \circ \eta_{\mathcal{S},A})(u)$ for any $u \in W$. Let $u, u' \in W$ such that $u \equiv_A u'$. Hence, by our assumption, $F_{\mathcal{S}}(u, d_{\mathcal{S}}) = F_{\mathcal{S}}(u', d_{\mathcal{S}})$ and, in particular, $f(\eta_{\mathcal{S},A}(u)) = f(\eta_{\mathcal{S},A}(u'))$, so f is well defined and the claim has been shown. □

Remark 8.4.2. (i) When $A = \{a_{i_1}, \ldots, a_{i_p}\}$ is a finite set of $C_{\mathcal{S}}$ the condition *$(ILS1)$* is equivalent to the following:

$(ILS1')$ There exists a map $f : \Lambda^p_{\mathcal{S}} \to \Lambda_{\mathcal{S}}$ such that $F_{\mathcal{S}}(u, d_{\mathcal{S}}) = f(F_{\mathcal{S}}(u, a_{i_1}), \ldots, F_{\mathcal{S}}(u, a_{i_p}))$ for any $u \in W$.

(ii) When $\mathcal{S}$ is a finite indistinguishability linear system the condition *(ILS2)* is equivalent to the condition $\theta_{\mathcal{S}}(W, A) = 1$.

Definition 8.4.3. If the condition *(ILS1)*, or equivalently *(ILS2)*, is satisfied we say that $\mathcal{S}$ is (W, A)-*compatible*, otherwise we say that $\mathcal{S}$ is (W, A)-*incompatible*. In particular, we say that $\mathcal{S}$ is W-*equation compatible* if $\mathcal{S}$ is $(W, C_{\mathcal{S}})$-compatible and A-*variable compatible* if it is $(U_{\mathcal{S}}, A)$-compatible. Finally, we say that $\mathcal{S}$ is *compatible* if it is $(U_{\mathcal{S}}, C_{\mathcal{S}})$-compatible.

Remark 8.4.4. (i) Let us note that when $\mathcal{S}$ is a homogeneous indistinguishability linear system the condition $(ILS2)$ is trivially satisfied for any $W \in \mathcal{P}(U_{\mathcal{S}})$ and $A \in \mathcal{P}(C_{\mathcal{S}})$. On the other hand, as we will see in next sections, several constructions depend exclusively by the variable set $C_{\mathcal{S}}$, therefore, mantaining the analogy with classical linear systems, we can assert that these constructions depend by the homogeneous indistinguishability linear system associated with $\mathcal{S}$. Therefore, the investigation of homogeneous indistinguishability linear systems is also important in the analysis of compatibility; in fact we restrict our attention to the variable set $C_{\mathcal{S}}$ and then "trasle" to the non-homogeneous case our results.
(ii) When we have a classical compatible linear system, if we keep all variables fixed and take a subset of equations, the compatibility of the system is preserved. This is true also in our context, in fact it is immediate to verify that $\mathcal{S}$ is compatible if and only if $\mathcal{S}$ if W-equation compatible for any $W \in \mathcal{P}(U_{\mathcal{S}})$.

Based on the terminology introduced in Definition 8.4.3, we note that $\Theta_{\mathcal{S}}(X, Y)$ is the set of all equations of X that form a compatible system relatively to the variables of Y. Therefore we call $\Theta_{\mathcal{S}}(X, Y)$ the (X, Y)-*compatibility region* and $\theta_{\mathcal{S}}(X, Y)$ the (X, Y)-*compatibility measure* (when $\mathcal{S}$ is finite).
Analogously, $\Gamma_{\mathcal{S}}(Y, Z)$ is the set of all equations whose symmetry with respect to the variables in Y implies also their symmetry with respect to the variables in Z.

8.5 Compatibility Operators

In this section we investigate the notion of local compatibility of $\mathcal{S}$ in terms of compatibility operators. In fact, we can introduce two operators $\phi_A : \mathcal{P}(U_{\mathcal{S}}) \to \mathcal{P}(U_{\mathcal{S}})$ and $\varphi_W : \mathcal{P}(C_{\mathcal{S}}) \to \mathcal{P}(U_{\mathcal{S}})$ by setting respectively

$$\phi_A(X) = \Theta_{\mathcal{S}}(X, A) \tag{8.6}$$

and

$$\varphi_W(Y) := \Theta_{\mathcal{S}}(W, Y). \tag{8.7}$$

We call ϕ_A the A-*equation compatibility operator* of $\mathcal{S}$ and φ_W the W-*variable compatibility operator* of $\mathcal{S}$. Moreover, let us set $\hat{\phi}_A(X) := \theta_{\mathcal{S}}(X, A)$ and $\hat{\varphi}_W(Y) := \theta_{\mathcal{S}}(W, Y)$.

The compatibility operators are directly related to the local compatibility of $\mathcal{S}$ by means of the next result.

Proposition 8.5.1. *(i)* $\mathcal{S}_{W,A}$ *is compatible if and only if* $\phi_A(W) = W$.
(ii) If $\varphi_W(A) = \emptyset$ *then* $\mathcal{S}_{W,A}$ *is incompatible.*

Proof. (i): $\mathcal{S}_{W,A}$ is compatible if and only if for any pair of ponits $u, u' \in W$ such that $u \equiv_A u'$ we have that $d_{\mathcal{S}}(u) = d_{\mathcal{S}}(u')$. But this is equivalent to require that $[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$ for any $u \in W$, i.e. $\Theta_{\mathcal{S}}(W, A) = W$ or, equivalently, $\phi_A(W) = W$.
(ii): By our assumptions, $\varphi_W(A) = \emptyset$, i.e. $\Theta_{\mathcal{S}}(W, A) = \emptyset$. This means that

$$[u]_A \cap W \not\subseteq [u]_{D_{\mathcal{S}}} \cap W$$

for any $u \in W$. Thus, for any $u \in W$ there exists $u' \in W$ such that $u \equiv_A u'$ and $u \not\equiv_{d_{\mathcal{S}}} u'$. Hence $d_{\mathcal{S}}(u) \neq d_{\mathcal{S}}(u')$, therefore $\mathcal{S}_{W,A}$ is incompatible. □

We state now some basic properties of the A-equation compatibility operator.

Proposition 8.5.2. *The A-equation compatibility operator ϕ_A satisfies the following properties:*

(i) $\phi_A(W) \subseteq W$ *for any* $W \in \mathcal{P}(U_{\mathcal{S}})$*;*

(ii) $\phi_A^2(W) = \phi_A(W)$ *for any* $W \in \mathcal{P}(U_{\mathcal{S}})$*;*

(iii) *if* $W' \subseteq W$ *then* $\phi_A(W) \cap W' \subseteq \phi_A(W')$*;*

(iv) $\phi_A(W) = W$ *if and only if* $\phi_A(W'') \subseteq \phi_A(W')$ *for any* $W', W'' \in \mathcal{P}(W)$ *such that* $W'' \subseteq W'$*.*

Proof. (i): It follows directly by the definition of $\phi_A(W)$.
(ii): By (i) it's clear that $\phi_A^2(W) \subseteq \phi_A(W)$. On the other hand, let $v \in \phi_A(W) = \Theta_{\mathcal{S}}(W, A)$. Hence $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$. By (i), we have that $\Theta_{\mathcal{S}}(W, A) \subseteq W$; therefore, if $v \in \phi_A(W)$, then $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$, thus

$$[v]_A \cap \phi_A(W) = [v]_A \cap W \cap \phi_A(W) \subseteq [v]_{D_{\mathcal{S}}} \cap W \cap \phi_A(W) = [v]_{D_{\mathcal{S}}} \cap \phi_A(W),$$

so $v \in \Theta_{\mathcal{S}}(\phi_A(W), A) = \phi_A^2(W)$. Hence $\phi_A(W) \subseteq \phi_A^2(W)$ and (ii) follows.
(iii): Let $v \in \phi_A(W) \cap W' = \Theta_{\mathcal{S}}(W, A) \cap W'$. Then $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$, and this relation is preserved if we both intersect members with W', thus $[v]_A \cap W' \subseteq [v]_{D_{\mathcal{S}}} \cap W'$, i.e. $v \in \Theta_{\mathcal{S}}(W', A) = \phi_A(W')$.
(iv): Suppose that $\Theta_{\mathcal{S}}(W, A) = W$ and let $W'' \subseteq W' \subseteq W$. Firstly we observe that $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$, for any $v \in W$, hence, obviously, $[v]_A \cap W' \subseteq [v]_{D_{\mathcal{S}}} \cap W'$ for any $W' \subseteq W$. A fortiori, if $v \in \phi_A(W'')$, then it results that $v \in \phi_A(W')$ and the claim follows.
Conversely, since $\phi_A(\{v\}) = \{v\}$ for any $v \in W$, we deduce that $\phi_A(W) = \Theta_{\mathcal{S}}(W, A) = W$.
□

Corollary 8.5.3. *ϕ_A is a kernel operator if and only if $\phi_A(U_{\mathcal{S}}) = U_{\mathcal{S}}$.*

Proof. The thesis follows immediately by $(i), (ii), (iv)$ of Proposition 8.5.2. □

In the next result we establish the first basic properties of the W-variable compatibility operator.

Proposition 8.5.4. *The W-variable compatibility operator φ_W satisfies the following properties:*

(i) *If* $A' \in \mathcal{P}(C_{\mathcal{S}})$ *and* $A \subseteq A'$ *then* $\varphi_W(A) \subseteq \varphi_W(A')$.

(ii) *Let* $A, B \in MAXP(\mathcal{S})$. *Then,*

$$\varphi_W(A \vee B) \subseteq \varphi_W(A) \cap \varphi_W(B) \subseteq \varphi_W(A) \cup \varphi_W(B) \subseteq \varphi_W(A \wedge B). \tag{8.8}$$

Moreover, if $\varphi_W(A \cap B) = \varphi_W(A) \cap \varphi_W(B)$ *it results that*

$$\hat{\varphi}_W(A \wedge B) + \hat{\varphi}_W(A \vee B) \geq \hat{\varphi}_W(A) + \hat{\varphi}_W(B). \tag{8.9}$$

(iii) *If* $A, B \in \mathcal{P}(C_{\mathcal{S}})$ *and* $A \approx_{\mathcal{S}} B$ *then* $\varphi_W(A) = \varphi_W(B)$.

(iv) $Im_{\mathcal{S}}(\varphi_W) = \{\varphi_W(A) : A \in MAXP(\mathcal{S})\}$.

Proof. *(i)*: Let $u \in \varphi_W(A) = \Theta_{\mathcal{S}}(W, A)$. Since $A \subseteq A'$, it results that $[u]_{A'} \subseteq [u]_A$, therefore

$$[u]_{A'} \cap W \subseteq [u]_A \cap W, \tag{8.10}$$

But the condition $u \in \Theta_{\mathcal{S}}(W, A)$ is equivalent to require that $[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$. By (8.10), we conclude that $[u]_{A'} \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$, i.e. $u \in \varphi_W(A) = \Theta_{\mathcal{S}}(W, A')$.
(ii): We have that $A \vee B = A \cap B$ and $A \wedge B = M_{\mathcal{S}}(A \cup B)$. Therefore, by *(i)*, we deduce that

$$\varphi_W(A \vee B) = \phi_{A \cap B}(W) \subseteq \phi_A(W) \cap \phi_B(W) = \varphi_W(A) \cap \varphi_W(B)$$

and

$$\varphi_W(A \wedge B) = \phi_{M_{\mathcal{S}}(A \cup B)}(W) \supseteq \phi_A(W) \cup \phi_B(W) = \varphi_W(A) \cup \varphi_W(B).$$

This proves (8.8).
Let now $A, B \in MAXP(\mathcal{S})$ satisfying the condition $\varphi_W(A \cap B) = \varphi_W(A) \cap \varphi_W(B)$. We prove the following equivalent version of (8.9):

$$\hat{\phi}_{A \wedge B}(W) + \hat{\phi}_{A \vee B}(W) \geq \hat{\phi}_A(W) + \hat{\phi}_B(W). \tag{8.11}$$

Since

$$|\Theta_{\mathcal{S}}(W, A) \cup \Theta_{\mathcal{S}}(W, B)| = |\Theta_{\mathcal{S}}(W, A)| + |\Theta_{\mathcal{S}}(W, B)| - |\Theta_{\mathcal{S}}(W, A) \cap \Theta_{\mathcal{S}}(W, B)|$$

and $A \vee B = A \cap B$, from our hypothesis on A and B it follows that

$$|\Theta_{\mathcal{S}}(W, A) \cup \Theta_{\mathcal{S}}(W, B)| + |\Theta_{\mathcal{S}}(W, A \vee B)| = |\Theta_{\mathcal{S}}(W, A)| + |\Theta_{\mathcal{S}}(W, B)|. \tag{8.12}$$

On the other hand,we have that $A \wedge B = M_{\mathcal{S}}(A \cup B)$, therefore, since $A, B \subseteq M_{\mathcal{S}}(A \cup B)$ it follows that

$$\Theta_{\mathcal{S}}(W, A) \cup \Theta_{\mathcal{S}}(W, B) \subseteq \Theta_{\mathcal{S}}(W, M_{\mathcal{S}}(A \cup B)) = \Theta_{\mathcal{S}}(W, A \wedge B). \tag{8.13}$$

Then, by (8.12) and (8.13) we have that

$$|\Theta_{\mathcal{S}}(W, A \wedge B)| + |\Theta_{\mathcal{S}}(W, A \vee B)| \geq |\Theta_{\mathcal{S}}(W, A)| + |\Theta_{\mathcal{S}}(W, B)|,$$

that is equivalent to (8.11).
(iii) : Clearly, $A \approx_{\mathcal{S}} B$ if and only if $\pi_{\mathcal{S}}(A) = \pi_{\mathcal{S}}(B)$. Thus it is clear that $\Theta_{\mathcal{S}}(W, A) = \Theta_{\mathcal{S}}(W, B)$, i.e. $\varphi_W(A) = \phi_A(W) = \phi_B(W) = \varphi_W(B)$.
(iv): It follows directly by *(iii)*, since $\varphi_W(A) = \varphi_W(M_{\mathcal{S}}(A))$ for any $A \in \mathcal{P}(\mathcal{S})$. □

Remark 8.5.5. By (*iii*) and (*iv*) of Proposition 8.5.4, we can identify the W-variable compatibility operator φ_W with its restriction on $MAXP(\mathcal{S})$. Therefore in what follows we will assume that

$$\varphi_W : MAXP(\mathcal{S}) \to \mathcal{P}(U_{\mathcal{S}}).$$

We have so far undertaken a local study of compatibility. We now want to focus our attention to the global counterpart of the theory.

For any $W \in \mathcal{P}(U_{\mathcal{S}})$ we set

$$\Phi_{\mathcal{S}}(W) := \{(A, \varphi_W(A)) : A \in MAXP(\mathcal{S})\}, \quad \mathbb{PR}_{\mathcal{S}}(W) := (\Phi_{\mathcal{S}}(W), \sqsubseteq)$$

and

$$\hat{\Phi}_{\mathcal{S}}(W) := \{(A, \theta_{\mathcal{S}}(W, A)) : A \in MAXP(\mathcal{S})\}, \quad \widehat{\mathbb{PR}}_{\mathcal{S}}(W) := (\hat{\Phi}_{\mathcal{S}}(W), \sqsubseteq),$$

if $\mathcal{S}$ is finite, where

$$(A, \varphi_W(A)) \sqsubseteq (A', \varphi_W(A')) :\iff (A, \theta_{\mathcal{S}}(W, A)) \sqsubseteq (A', \theta_{\mathcal{S}}(W, A')) :\iff A \subseteq^* A',$$

so that $\mathbb{PR}_{\mathcal{S}}(W), \widehat{\mathbb{PR}}_{\mathcal{S}}(W)$ and $\mathbb{M}_{\mathcal{S}}(\mathcal{S})$ are three order isomorphic lattices.

We can interpret, respectively, the lattices $\mathbb{PR}_{\mathcal{S}}(W)$ and $\widehat{\mathbb{PR}}_{\mathcal{S}}(W)$ as a type of W-*projection* and W-*numerical projection* of the maximum partitioner lattice $\mathbb{M}_{\mathcal{S}}(\mathcal{S})$ on the corresponding equation subset W.

Definition 8.5.6. We call $\mathbb{PR}_{\mathcal{S}}(W)$ the W-*projection lattice* and $\widehat{\mathbb{PR}}_{\mathcal{S}}(W)$ the W-*numerical projection lattice* of $\mathcal{S}$.

At this point we can provide an interpretation of W-equation compatibility and A-variable compatibility in terms of the previous set operator language.

Proposition 8.5.7. (*i*) *$\mathcal{S}$ is W-equation compatible* $\iff (C_{\mathcal{S}}, W) \in \Phi_{\mathcal{S}}(W) \iff \varphi_W(C_{\mathcal{S}}) = W$.
(*ii*) *$\mathcal{S}$ is A-variable compatible* $\iff$ *ϕ_A is a kernel operator.*

Proof. (*i*): By (*iv*) of Proposition 8.5.4, it results that $(A, W) \in \Phi_{\mathcal{S}}(W)$ if and only if there exists $A \in MAXP(\mathcal{S})$ such that $\varphi_W(A) = \Theta_{\mathcal{S}}(W, A) = W$. By (*i*) of the same Proposition, we deduce that $\varphi_W(C_{\mathcal{S}}) = W$. By Proposition 8.5.1 it follows immediately that $\mathcal{S}_W$ is compatible if and only if $\Theta_{\mathcal{S}}(W, C_{\mathcal{S}}) = W$.
(*ii*): Straightforward. □

We can relate from a global perspective the local incompatibilities of an indistinguishability linear system $\mathcal{S}$ and the W-variable compatibility operator. In order to analyze this interrelation, let $\pi_{\mathcal{S}}(C_{\mathcal{S}}) = \{B_i : i \in I\}$ and $\pi_{\mathcal{S}}(d_{\mathcal{S}}) = \{C_j : j \in J\}$. For any index $i \in I$ we set then

$$(SB)_i := \{B' \preccurlyeq B_i : B' \not\subseteq C_j \ \forall j \in J\}$$

and

$$N_{\mathcal{S}} := \bigcup_{i \in I} (SB)_i.$$

We have then the following result.

Proposition 8.5.8. *Let $\emptyset \neq W \subseteq U_{\mathcal{S}}$. Then, $\Phi_{\mathcal{S}}(W) = \{(A, \emptyset),\, A \in MAXP(\mathcal{S})\}$ if and only if $W = \bigcup_{k \in K} B'_k$, for some sub-family $\{B'_k : k \in K\} \subseteq N_{\mathcal{S}}$.*

Proof. By (i) of Proposition 8.5.4 the condition $\{(A, \emptyset),\, A \in MAXP(\mathcal{S})\}$ is equivalent to $\varphi_W(C_{\mathcal{S}}) = \emptyset$. Let us note that $\varphi_W(C_{\mathcal{S}}) = \emptyset$ if and only if for any $u \in W$ it results that $[u]_{C_{\mathcal{S}}} \cap W \not\subseteq [u]_{D_{\mathcal{S}}} \cap W$. This is equivalent to require that, for any $u \in W$, there exists at least a equation $u' \in W$ such that $u \equiv_{U_{\mathcal{S}}} u'$ and $u \not\equiv_{d_{\mathcal{S}}} u'$. In other terms, we are saying that there exist some $C_{\mathcal{S}}$-symmetry sub-blocks $B'_k \in N_{\mathcal{S}}$, for some index set $k \in K$, such that $\bigcup_{k \in K} B'_k = W$. Hence the thesis follows. $\square$

We set now

$$DEP_{\mathcal{S}}(A) := \{W \in \mathcal{P}(U_{\mathcal{S}}) : \phi_A(W) = W\}, \tag{8.14}$$

that is equivalent to the following

$$DEP_{\mathcal{S}}(A) = \{W \in \mathcal{P}(U_{\mathcal{S}}) : \forall v \in W,\, [v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W\}. \tag{8.15}$$

We observe that if we consider the operator $DEP_{\mathcal{S}} : \mathcal{P}(C_{\mathcal{S}}) \to SS(U_{\mathcal{S}})$, this is constant over each global symmetry class. Therefore it is sufficient to consider its restriction

$$DEP_{\mathcal{S}} : MAXP(\mathcal{S}) \to SS(U_{\mathcal{S}}).$$

We now prove three basic properties holding for the above operator $DEP_{\mathcal{S}}$.

Proposition 8.5.9. *The following properties hold.*

- (i) *$\emptyset \in DEP_{\mathcal{S}}(A)$ and $\{v\} \in DEP_{\mathcal{S}}(A)$ for any $v \in U_{\mathcal{S}}$;*
- (ii) *$\phi_A(W) = \varphi_W(A) \in DEP_{\mathcal{S}}(A)$ for any $W \in \mathcal{P}(U_{\mathcal{S}})$;*
- (iii) *If $W \in DEP_{\mathcal{S}}(A)$ and $W' \subseteq W$ then $W' \in DEP_{\mathcal{S}}(A)$.*

Proof. (i): It is immediate to see that (8.15) holds for $W = \emptyset$. Moreover, let $W = \{v\}$. Then

$$[v]_A \cap \{v\} = \{v\} = [v]_{D_{\mathcal{S}}} \cap \{v\}$$

and the claim is proved.
(ii): It is a direct consequence of (ii) of Proposition 8.5.2 and (8.14).
(iii): Let $W \in DEP_{\mathcal{S}}(A)$ and $W' \subseteq W$. Then, by (8.15), we have

$$[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W, \;\; \forall v \in W.$$

In particular, it results that

$$[v]_A \cap W \cap W' = [v]_A \cap W' \subseteq [v]_{D_{\mathcal{S}}} \cap W \cap W' = [v]_{D_{\mathcal{S}}} \cap W' \;\; \forall v \in W'. \tag{8.16}$$

So (iii) follows. $\square$

In next result we state some important properties of the operator $DEP_{\mathcal{S}}$. We will use the notation

$$MDEP_{\mathcal{S}}(A) := \max(DEP_{\mathcal{S}}(A)).$$

Theorem 8.5.10. *The following conditions hold.*

(i) *If $v \in W \setminus \varphi_W(A)$, then $[v]_A \cap W \subseteq W \setminus \varphi_W(A)$.*

(ii) *We have that*

$$\varphi_{U_{\mathcal{S}}}(A) = \bigcap_{Z \in MDEP_{\mathcal{S}}(A)} Z. \tag{8.17}$$

(iii) $\varphi_{U_{\mathcal{S}}}(A) = U_{\mathcal{S}} \iff DEP_{\mathcal{S}}(A) = \mathcal{P}(U_{\mathcal{S}})$.

(iv) *If $\varphi_{U_{\mathcal{S}}}(A) \neq U_{\mathcal{S}}$ and $W \in DEP_{\mathcal{S}}(A)$, then $\emptyset \neq \varphi_{U_{\mathcal{S}}}(A)^c \not\subseteq W$.*

(v) *If $W \in DEP_{\mathcal{S}}(A)$ then $[v]_A \not\subseteq W$ for any $v \in \varphi_{U_{\mathcal{S}}}(A)^c$.*

(vi) *Suppose that $[v]_A \cap W = \emptyset$ for any $v \in \varphi_{U_{\mathcal{S}}}(A)^c$, whence $W \in DEP_{\mathcal{S}}(A)$.*

(vii) *Let $A \subseteq B$. Then $DEP_{\mathcal{S}}(A) \subseteq DEP_{\mathcal{S}}(B)$.*

Proof. (i): Let $v \in W \setminus \varphi_W(A)$ and let $u' \in [v]_A \cap W$. Suppose by contradiction that $u' \in \varphi_W(A)$, i.e. $[u']_A \cap W \subseteq [u']_{D_{\mathcal{S}}} \cap W$. Then, we have $v \in [v]_A \cap W = [u']_A \cap W \subseteq [u']_{D_{\mathcal{S}}} \cap W$, hence $[v]_{D_{\mathcal{S}}} \cap W = [u']_{D_{\mathcal{S}}} \cap W$, therefore $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$, i.e. $v \in \varphi_W(A)$, contradiction. Then, $u' \in W \setminus \varphi_W(A)$ and the claim follows.
(ii): Let $v \in \varphi_{U_{\mathcal{S}}}(A) = \Gamma_{\mathcal{S}}(A, \{d_{\mathcal{S}}\})$. Then $[v]_A \subseteq [v]_{D_{\mathcal{S}}}$. Let us prove now that, if $W \in DEP_{\mathcal{S}}(A)$, then $W \cup \{v\} \in DEP_{\mathcal{S}}(A)$. This implies that, for all $Z \in MDEP_{\mathcal{S}}(A)$, $v \in Z$. If $u \in W$, then, by definition of $DEP_{\mathcal{S}}$, $[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$. If $w \not\equiv_A u$, then $[u]_A \cap (W \cup \{w\}) = [u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap (W \cup \{w\})$. If $w \equiv_A u$, then $w \equiv_{d_{\mathcal{S}}} u$, so $[u]_A \cap (W \cup \{w\}) \subseteq [u]_{D_{\mathcal{S}}} \cap (W \cup \{w\})$. Moreover $[u]_A \subseteq [u]_{D_{\mathcal{S}}}$ implies $[u]_A \cap (W \cup \{w\}) \subseteq [u]_{D_{\mathcal{S}}} \cap (W \cup \{w\})$.
Conversely, let us suppose that $v \in Z$, $\forall Z \in MDEP_{\mathcal{S}}(A)$. Then, by Proposition 8.5.9, part (ii), we obtain:

$$\bigcup_{Z \in MDEP_{\mathcal{S}}(A)} Z = U_{\mathcal{S}},$$

thus

$$[v]_A = \bigcup_{Z \in MDEP_{\mathcal{S}}(A)} (Z \cup [v]_A) \subseteq \bigcup_{Z \in MDEP_{\mathcal{S}}(A)} (Z \cup [v]_{D_{\mathcal{S}}}) = [v]_{D_{\mathcal{S}}},$$

so $v \in \Gamma_{\mathcal{S}}(A, \{d_{\mathcal{S}}\}) = \varphi_{U_{\mathcal{S}}}(A)$.
(iii): Since $v \in \varphi_{U_{\mathcal{S}}}(A) = \Gamma_{\mathcal{S}}(A, \{d_{\mathcal{S}}\}) = U_{\mathcal{S}}$, then $[v]_A \subseteq [v]_{D_{\mathcal{S}}}$, so it is obvious that $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$ for any $W \in \mathcal{P}(U_{\mathcal{S}})$, i.e. $DEP_{\mathcal{S}}(A) = \mathcal{P}(U_{\mathcal{S}})$.
Conversely, suppose that $DEP_{\mathcal{S}}(A) = \mathcal{P}(U_{\mathcal{S}})$. Hence, $U_{\mathcal{S}} \in DEP_{\mathcal{S}}(A)$, so, by (8.15), we have

$$[v]_A \cap U_{\mathcal{S}} = [v]_A \subseteq [v]_{D_{\mathcal{S}}} \cap U_{\mathcal{S}} = [v]_{D_{\mathcal{S}}}$$

for any $v \in U_{\mathcal{S}}$. Thus $v \in \Gamma_{\mathcal{S}}(A, \{d_{\mathcal{S}}\})$ for any $v \in U_{\mathcal{S}}$.
(iv): Let $X \in DEP_{\mathcal{S}}(A)$. Since $\varphi_{U_{\mathcal{S}}}(A) \neq U_{\mathcal{S}}$, we clearly have that $\varphi_{U_{\mathcal{S}}}(A)^c \neq \emptyset$. Now, suppose

by contradiction that $\varphi_{U_{\mathcal{S}}}(A)^c \subseteq X$ and let $u \in \varphi_{U_{\mathcal{S}}}(A)^c$. By (i), we have that if $u \in \varphi_{U_{\mathcal{S}}}(A)^c$ and $u' \equiv_A u$ then $u' \in \varphi_{U_{\mathcal{S}}}(A)^c$. An immediate consequence is that

$$[v]_A \cap W = [v]_A. \tag{8.18}$$

Moreover, since $u \in \varphi_{U_{\mathcal{S}}}(A)^c$, it results that $[v]_A \not\subseteq [v]_{D_{\mathcal{S}}}$, hence, a fortiori, $[v]_A \not\subseteq [v]_{D_{\mathcal{S}}} \cap W$ and the thesis follows.

(v): Let $W \in DEP_{\mathcal{S}}(A)$ and suppose by contradiction that there exists $u \in \varphi_{U_{\mathcal{S}}}(A)^c$ such that $[v]_A \subseteq W$. By (i) we have that if $u' \equiv_A u$, then $u' \in \varphi_{U_{\mathcal{S}}}(A)^c$. Therefore $u' \in W$ for any $u' \in [v]_A$. In particular, we have

$$[v]_A \cap W = [v]_A. \tag{8.19}$$

Moreover, since $u \in \varphi_{U_{\mathcal{S}}}(A)(D_{\mathcal{S}})^c$, we have that $[v]_A \not\subseteq [v]_{D_{\mathcal{S}}}$, therefore $[v]_A \not\subseteq [v]_{D_{\mathcal{S}}} \cap W$ and the thesis follows.

(vi): Suppose that $[v]_A \cap W = \emptyset$ for any $v \in \varphi_{U_{\mathcal{S}}}(A)^c$. Then $v \notin W$ for any $v \in \varphi_{U_{\mathcal{S}}}(A)^c$, i.e. $\varphi_{U_{\mathcal{S}}}(A)^c \subseteq W^c$ or, equivalently, $W \subseteq \varphi_{U_{\mathcal{S}}}(A)$. Therefore, for any $v \in W$ it results $[v]_A \subseteq [v]_{D_{\mathcal{S}}}$, so that

$$[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W.$$

Hence, by (8.15), we have that $W \in DEP_{\mathcal{S}}(A)$.

(vii): Let $A \subseteq B$. We want to show that any equation of $DEP_{\mathcal{S}}(A)$ belongs also to $DEP_B(\mathcal{S})$. Let $W \in DEP_{\mathcal{S}}(A)$, thus $[v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W$ for any $v \in W$. Moreover, by (2.13) it follows that $\pi_{\mathcal{S}}(B) \preceq \pi_{\mathcal{S}}(A)$ or, equivalently, $[v]_B \subseteq [v]_A$. In particular, it results that $[v]_B \cap W \subseteq [v]_A \cap W$, hence we obtain

$$[v]_B \cap W \subseteq [v]_A \cap W \subseteq [v]_{D_{\mathcal{S}}} \cap W,$$

i.e. $W \in DEP_B(\mathcal{S})$. □

8.6 Reduction of Indistinguishability Linear Systems

In this section we investigate the question concerning the *reduction* of an indistinguishability linear system $\mathcal{S}$. In other terms, we fix an equation subset W and a variable subset A. The reduction technique consists of the choice of a criterion in order to reduce the complexity of a linear system. Clearly, this criterion refers to the choice of an operator or function associated with the property we want to preserve. In what follows, we provide two different reduction criteria. In fact, we can reduce the linear system by taking the smallest variable sets providing the same set of constants relative to all the equations of a given equation subset W. On the other hand, the second kind of reduction we consider here consists of taking the smallest variable set B of A such that any equation compatible with respect to A is also compatible with respect to B.

We set now

$$\Delta^{d_{\mathcal{S}}}_{W,A}(u,u') := \begin{cases} \Delta_{W,A}(u,u') & \text{if } u \not\equiv_{d_{\mathcal{S}}} u', \\ \emptyset & \text{otherwise.} \end{cases} \tag{8.20}$$

for all $u, u' \in W$. In particular, we set $\Delta^{d_{\mathcal{S}}} := \Delta^{d_{\mathcal{S}}}_{U_{\mathcal{S}},C_{\mathcal{S}}}$.

We call $\Delta^{d_{\mathcal{S}}}_{W,A}(u,u')$ the $(d_{\mathcal{S}},W,A)$-*discernibility neighborhood* of u and u' in $\mathcal{S}$ and $(d_{\mathcal{S}},W,A)$-*discernibility set system* of $\mathcal{S}$ the system $\mathfrak{D}(W,A,d_{\mathcal{S}}) := (A, DIS_{\mathcal{S}}(d_{\mathcal{S}},W,A))$, where

$$DIS_{\mathcal{S}}(d_{\mathcal{S}},W,A) := \{\Delta^{d_{\mathcal{S}}}_{W,A}(u,u') : u,u' \in W \text{ and } \Delta^{d_{\mathcal{S}}}_{W,A}(u,u') \neq \emptyset\}.$$

In particular, we set $DIS(d_{\mathcal{S}},\mathcal{S}) := DIS_{\mathcal{S}}(d_{\mathcal{S}},U_{\mathcal{S}},C_{\mathcal{S}})$ and $\mathfrak{D}(d_{\mathcal{S}},\mathcal{S}) := (U_{\mathcal{S}}, DIS(d_{\mathcal{S}},\mathcal{S}))$, moreover we call $\Delta^{d_{\mathcal{S}}}(u,u')$ the $d_{\mathcal{S}}$-*discernibility neighborhood* of u and u' and $\mathfrak{D}(d_{\mathcal{S}},\mathcal{S})$ the $d_{\mathcal{S}}$-*discernibility set system* of $\mathcal{S}$. If W and A are both finite sets with $W = \{u_1,\dots,u_m\}$, we denote by $DT_{\mathcal{S}}[d_{\mathcal{S}},W,A]$ the $m \times m$ table having the equations $u_1,\dots,u_m$ on both its rows and columns and the subsets $\Delta^{d_{\mathcal{S}}}_{W,A}(u_i,u_j)$ in the (u_i,u_j)-entry and we call $DT_{\mathcal{S}}[d_{\mathcal{S}},W,A]$ the $(W,A,d_{\mathcal{S}})$-*discernibility table* of $\mathcal{S}$. In particular, we set $DT_{\mathcal{S}}[d_{\mathcal{S}}] := DT_{\mathcal{S}}[U_{\mathcal{S}},C_{\mathcal{S}},d_{\mathcal{S}}]$ and we call it the $d_{\mathcal{S}}$-*discernibility table* of $\mathcal{S}$.

In what follows we fix a variable set A and an equation u. We associate with u the set of all constants relative to all the equations that are A-symmetric to u.

We introduce the following function $\xi_A : W \to \mathcal{P}(\Lambda_{\mathcal{S}})$:

$$\xi_A(u) = \{F_{\mathcal{S}}(u',d_{\mathcal{S}}) : u' \in [u]_A\}. \tag{8.21}$$

Remark 8.6.1. (i): If A and B are two variable subsets such that $A \subseteq B$, then $\xi_A(w) \subseteq \xi_B(w)$ for any $w \in W$.
(ii): If $\mathcal{S}_{W,A}$ is compatible, then $\xi_A(w) = \{F_{\mathcal{S}}(w,d_{\mathcal{S}})\}$.

We now introduce the (ξ,W,A)-essentials.

Definition 8.6.2. We say that a subset $B \subseteq A$ is a (ξ,W,A)-*essential* if:

- there exists $w \in W$ such that $\xi_{A\setminus B}(w) \neq \xi_A(w)$,
- for any $B' \subsetneqq B$ and for any $w \in W$, we have $\xi_{A\setminus B'}(w) = \xi_A(w)$.

We denote by $ESS_{\mathcal{S}}(\xi,W,A)$ the family of all (ξ,W,A)-essentials. We denote by $ESS(\xi,\mathcal{S}) := ESS_{\mathcal{S}}(\xi,U_{\mathcal{S}},C_{\mathcal{S}})$ the family of all ξ-*essentials* of $\mathcal{S}$.

We introduce the (ξ,W,A)-reducts.

Definition 8.6.3. We say that a subset $B \subseteq A$ is a (ξ,W,A)-*reduct* if:

- $\xi_A(w) = \xi_B(w)$ for any $w \in W$,
- for any $B' \subsetneqq B$ there exists $w_{B'} \in W$ such that $\xi_{B'}(w_{B'}) \neq \xi_A(w_{B'})$.

We denote by $RED_{\mathcal{S}}(\xi,W,A)$ the family of all (ξ,W,A)-reducts. We denote by $RED(\xi,\mathcal{S}) := RED_{\mathcal{S}}(\xi,U_{\mathcal{S}},C_{\mathcal{S}})$ the family of all ξ-*reducts* of $\mathcal{S}$.

In next result we relate the (ξ,W,A)-reducts to the compatibility of $\mathcal{S}_{W,A}$.

Theorem 8.6.4. *Let $\mathcal{S}_{W,A}$ be a compatible indistinguishability linear system. Then $RED_{\mathcal{S}}(\xi,W,A)$ coincides with the family of variable sets B such that:*

(*i*) $\Theta_{\mathcal{S}}(W,B) = W$;

(*ii*) $\Theta_{\mathcal{S}}(W,B') \neq W$ *for any* $B' \subsetneqq B$.

Proof. Let $B \in RED_{\mathcal{S}}(\xi, W, A)$, then $\xi_B(w) = \xi_A(w) = \{F_{\mathcal{S}}(w, d_{\mathcal{S}})\}$ for any $w \in W$. We must prove that $[w]_B \cap W \subseteq [w]_{D_{\mathcal{S}}} \cap W$. Let $w' \in [w]_B \cap W$ and set $t = F_{\mathcal{S}}(w', d_{\mathcal{S}})$. Then, $t \in \xi_B(w) = \xi_A(w)$, so that $t = F_{\mathcal{S}}(w, d_{\mathcal{S}})$ or, equivalently, $w' \in [w]_{D_{\mathcal{S}}} \cap W$. So $\Theta_{\mathcal{S}}(W,B) = W$. Let us consider $B' \subsetneqq B$. Then, $B' \notin RED_{\mathcal{S}}(\xi, W, A)$, hence there exists $w \in W$ such that $\xi_B(w) \neq \xi_A(w)$, so $\xi_A(w) \subsetneqq \xi_{B'}(w)$. This means that there exists $w' \in W$ such that $w' \not\equiv_A w$, $w' \equiv_{B'} w$ and $t := F_{\mathcal{S}}(w', d_{\mathcal{S}}) \neq F_{\mathcal{S}}(w, d_{\mathcal{S}})$. Hence, $[w]_{B'} \cap W \nsubseteq [w]_{D_{\mathcal{S}}} \cap W$, i.e. $w \in W \setminus \Theta_{\mathcal{S}}(W, B')$.
On the other hand, let $B \in \mathcal{P}(C_{\mathcal{S}})$ satisfying (*i*) and (*ii*). By Proposition 8.5.7, $\mathcal{S}_{W,B}$ is compatible, hence $\xi_B(w) = \xi_A(w) = \{F_{\mathcal{S}}(w, d_{\mathcal{S}})\}$. Now, let $B' \subsetneqq B$. Then $\Theta(W, B') \subsetneqq W$, so there exists $w \in W$ such that $[w]_{B'} \cap W \nsubseteq [w]_{D_{\mathcal{S}}} \cap W$. This means that there exists $w' \in W$ such that $w' \equiv_{B'} w$ but $F_{\mathcal{S}}(w, d_{\mathcal{S}}) \neq F_{\mathcal{S}}(w', d_{\mathcal{S}})$, so $\xi_A(w) \neq \xi_{B'}(w)$. We conclude that $B \in RED_{\mathcal{S}}(\xi, W, A)$. □

We now provide a geometric property for the set system $ESS_{\mathcal{S}}(\xi, W, A)$.

Theorem 8.6.5. *Let $\mathcal{S}_{W,A}$ be a compatible indistinguishability linear system. Then,*

$$ESS_{\mathcal{S}}(\xi, W, A) = \min(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)).$$

Proof. Let $B \in ESS_{\mathcal{S}}(\xi, W, A)$. Then, there exists $u \in W$ such that $\xi_A(u) \neq \xi_{A \setminus B}(u)$. By compatibility of $\mathcal{S}_{W,A}$, $\xi_A(u)$ consists of the single element $F_{\mathcal{S}}(u, d)$. Since $\pi_{\mathcal{S}}(A) \preceq \pi_{\mathcal{S}}(A \setminus B)$, there exists $u' \in W$ such that:

(*a*) $u \equiv_{A \setminus B} u'$;

(*b*) $u \equiv_A u'$;

(*c*) $F_{\mathcal{S}}(u, d) \neq F_{\mathcal{S}}(u', d)$.

Hence $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \subseteq B$. Now, let $b \in B$ and set $C = B \setminus \{b\}$. By (*ii*) of Definition 8.6.2, it results $\xi_{A \setminus C}(u) = \xi_A(u)$ for any $u \in W$. Let u, u' as before. It must results

$$u \not\equiv_{A \setminus C} u',$$

otherwise, we would have $F_{\mathcal{S}}(w, d), F_{\mathcal{S}}(u, d) \in \xi_{A \setminus C}(u)$, so $\xi_{A \setminus C}(u) \neq \xi_A(u)$, that is an absurd. In other terms, we have shown that $B \in \min(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A))$.
Conversely, let $B \in \min(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A))$. Hence there are two equations $u, u' \in W$ such that $B = \Delta^{d_{\mathcal{S}}}_{W,A}(u, u')$. By (8.20), we have $u \not\equiv_{d_{\mathcal{S}}} u'$ and $u \not\equiv_A u'$. It follows immediately that $u \equiv_{A \setminus B} u'$, therefore $\xi_A(u) \neq \xi_{A \setminus B}(u)$. Let $C \subsetneqq B$. We firstly observe that by minimality of B, we must have $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \nsubseteq C$ for any $u, u' \in W$. We show that $\pi_{\mathcal{S}}(A \setminus C) = \pi_{\mathcal{S}}(A)$. In fact, if $u \equiv_{A \setminus C} u'$ and $u \not\equiv_A u'$, we would have $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \subseteq C$, that is an absurd. Hence $u \equiv_A u' \iff u \equiv_{A \setminus C} u'$, so $\pi_{\mathcal{S}}(A \setminus C) = \pi_{\mathcal{S}}(A)$. So B satisfies also condition (*ii*) of Definition 8.6.2 and the theorem is proved. □

Theorem 8.6.6. *Let $\mathcal{S}_{W,A}$ be a compatible indistinguishability linear system. Then:*
(i) $\xi_B(u) = \xi_A(u)$ for any $u \in W$ if and only if B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$.
(ii) $RED_{\mathcal{S}}(\xi, W, A) = Tr(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A))$.

Proof. Assume that $\xi_A(u) = \xi_B(u)$ for any $u \in W$. We show that B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Let then $E \in DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Then, there exists two distinct equations $u, u' \in W$ such that $u \not\equiv_{d_{\mathcal{S}}} u'$ and $E = \{a \in A : (F_{\mathcal{S}}(u, a) \neq F_{\mathcal{S}}(u', a))\}$. Since $\mathcal{S}_{W,A}$ is compatible, we deduce that $u \not\equiv_A u'$ and so $E \neq \emptyset$. If $u \equiv_B u'$, by hypothesis we would have $\xi_A(u) \neq \xi_B(u)$, that is an absurd. So, we have thus found a variable $a' \in B$ such that $F_{\mathcal{S}}(u, a') \neq F_{\mathcal{S}}(u', a')$, i.e. $a' \in B \cap E$. Therefore B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$.
Conversely, let B be a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Moreover, let $u, u' \in W$ such that $u \equiv_B u'$. Suppose by contradiction that $F_{\mathcal{S}}(u, d) \neq F_{\mathcal{S}}(u', d)$. Then $u \not\equiv_{d_{\mathcal{S}}} u'$ and, by consistency, $u \not\equiv_A u'$. Therefore $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \neq \emptyset$ but $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \cap B = \emptyset$, that is an absurd. Hence, $\xi_B(u) = \xi_A(u)$ for any $u \in W$.
(ii) : Let $B \in RED_{\mathcal{S}}(\xi, W, A)$. Hence $\xi_A(u) = \xi_B(u)$ for any $u \in W$, and by (i) this implies that B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Now, if $c \in B$, again by Definition 8.6.3, there exists some $u \in W$ such that $\xi_{B\setminus\{c\}}(u) \neq \xi_A(u)$, therefore by (i) it follows that $B \setminus \{c\}$ is not a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Hence B is a minimal transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. On the other hand, let B be a minimal transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$, then by (i) it follows that $\xi_A(u) = \xi_B(u)$ for any $u \in W$. Moreover, for any $c \in B$, the subset $B\setminus\{c\}$ is not a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$, therefore, again by (i), there exists some $u \in W$ such that $\xi_{B\setminus\{c\}}(u) \neq \xi_A(u)$. Hence $B \in RED_{\mathcal{S}}(\xi, W, A)$. □

In what follows we provide another criterion of reduction relied on the operator φ_W.

Definition 8.6.7. We say that a subset $B \subseteq A$ is a (φ, W, A)-*essential* if:

- $\varphi_W(A \setminus B) \subsetneqq \varphi_W(A)$,
- for any $B' \subsetneqq B$ we have $\varphi_W(A \setminus B') = \varphi_W(A)$,

that are equivalent to the following conditions,

- $\hat{\varphi}_W(A \setminus B) < \hat{\varphi}_W(A)$;
- for any $B' \subsetneqq B$ we have $\hat{\varphi}_W(A \setminus B') = \hat{\varphi}_W(A)$.

We denote by $ESS_{\mathcal{S}}(\varphi, W, A)$ the family of all (φ, W, A)-essentials and we denote by $ESS(\varphi, \mathcal{S}) := ESS_{\mathcal{S}}(\varphi, U_{\mathcal{S}}, C_{\mathcal{S}})$ the family of all φ-*essentials* of $\mathcal{S}$.

We now introduce the notion of (φ, W, A)-reducts.

Definition 8.6.8. We say that a subset $B \subseteq A$ is a (φ, W, A)-*reduct* if:

- $\varphi_W(A) = \varphi_W(B)$,
- for any $B' \subsetneqq B$ we have $\varphi_W(B') \subsetneqq \varphi_W(A)$,

that are equivalent to the following conditions,

$\mathcal{S}_2$	a_1	a_2	a_3	a_4	a_5	$d_{\mathcal{S}_6}$
u_1	1	0	0	0	0	1
u_2	0	1	1	0	0	2
u_3	0	1	1	0	0	3
u_4	1	0	0	0	0	4
u_5	0	1	1	1	1	5

Figure 8.1: Indistinguishability linear system $\mathcal{S}_2$.

- $\hat{\varphi}_W(A) = \hat{\varphi}_W(B)$,
- for any $B' \subsetneqq B$ we have $\hat{\varphi}_W(B') < \hat{\varphi}_W(A)$.

We denote by $RED_{\mathcal{S}}(\varphi, W, A)$ the family of all the (φ, W, A)-reducts. We denote by $RED(\varphi, \mathcal{S}) := RED_{\mathcal{S}}(\varphi, U_{\mathcal{S}}, C_{\mathcal{S}})$ the family of all the *φ-reducts* of $\mathcal{S}$.

In the next example we see that the $ESS_{\mathcal{S}}(\varphi, W, A)$ does not coincide with the minimals of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$ and that $RED_{\mathcal{S}}(\varphi, W, A)$ is not the family of the minimal transversals of $DIS(d_{\mathcal{S}}, \mathcal{S})$.

Example 8.6.9. *Let us consider the indistinguishability linear system $\mathcal{S}_2$ represented in Figure 8.1.*

Then,

$$\pi_{\mathcal{S}_2}(d_{\mathcal{S}_2}) = u_1|u_2|u_3|u_4|u_5$$

and

$$\pi_{\mathcal{S}_2}(C_{\mathcal{S}_2}) = u_1u_4|u_2u_3|u_5.$$

Furthermore, it is easy to see that

$$DIS(d_{\mathcal{S}_2}, \mathcal{S}_2) = \{\{a_1, a_2, a_3\}, \{a_4, a_5\}, \{a_1, a_2, a_3, a_4, a_5\}\},$$

and

$$\min(DIS(d_{\mathcal{S}_2}, \mathcal{S}_2)) = \{\{a_1, a_2, a_3\}, \{a_4, a_5\}\}.$$

By Definition 8.6.7, we have that

$$ESS(\varphi, \mathcal{S}_2) = \{\{a_4, a_5\}\}.$$

Thus, we conclude that, in general, $ESS(\varphi, \mathcal{S}_2)$ does not coincide with the family of all minimal hyper-edges of $DIS(d_{\mathcal{S}_2}, \mathcal{S}_2)$. Finally, by Definition 8.6.8, we see that

$$RED(\varphi, \mathcal{S}_2) = \{\{a_4\}, \{a_5\}\},$$

so the φ-reducts family is not the transversal of $DIS(d_{\mathcal{S}_2}, \mathcal{S}_2)$.

The next proposition asserts that we can focus our attention only on the maximum partitioners in order to study the family of all φ-reducts of an global symmetry class.

Proposition 8.6.10. *Let $A \approx_{\mathcal{S}} B$ and $A \subseteq B$, then*

$$RED_{\mathcal{S}}(\varphi, W, A) \subseteq RED_{\mathcal{S}}(\varphi, W, B). \tag{8.22}$$

In particular,

$$RED_{\mathcal{S}}(\varphi, W, A) \subseteq RED_{\mathcal{S}}(\varphi, W, M_{\mathcal{S}}(A)). \tag{8.23}$$

Proof. We firstly observe that since $A \approx_{\mathcal{S}} B$, then $\phi_A(W) = \phi_B(W)$. By Definition 8.6.8 it follows that, if $C \in RED_{\mathcal{S}}(\varphi, W, A)$, then $\phi_B(W) = \phi_C(W)$ and $\phi_{C'}(W) \subsetneqq \phi_B(W)$ for any $C' \subsetneqq C$. Thus, $C \in RED_{\mathcal{S}}(\varphi, W, B)$. □

In next result we show the interrelation between (φ, W, A)-reducts and the (φ, W, A)-essentials.

Theorem 8.6.11.

$$RED_{\mathcal{S}}(\varphi, W, A) = Tr(ESS_{\mathcal{S}}(\varphi, W, A))$$

Proof. Let $B \subseteq A$ such that $\varphi_W(A) = \varphi_W(B)$. This means that

$$\Theta_{\mathcal{S}}(W, A) = \Theta_{\mathcal{S}}(W, B) \tag{8.24}$$

i.e.

$$[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W \iff [u]_B \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W. \tag{8.25}$$

We claim that B is a transversal of $ESS_{\mathcal{S}}(\varphi, W, A)$. For, let $E \in ESS_{\mathcal{S}}(\varphi, W, A)$. Hence

$$\Theta_{\mathcal{S}}(W, A \setminus E) \subsetneqq \Theta_{\mathcal{S}}(W, A). \tag{8.26}$$

In other terms, there exists an equation $u \in W$ such that

$$[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W \tag{8.27}$$

and an equation $u' \in W$ such that

(a) $u' \equiv_{A \setminus E} u$;

(b) $u' \not\equiv_{d_{\mathcal{S}}} u$.

This implies by (8.27) that $u \not\equiv_A u'$ and, in particular, by (8.25), it follows that

$$u' \not\equiv_B u. \tag{8.28}$$

Thus, there exists $b \in B$ such that $F_{\mathcal{S}}(u, b) \neq F_{\mathcal{S}}(u', b)$. Therefore, we have that $b \in \Delta^{d_{\mathcal{S}}}_{W,A}(u, u')$. By (a) and (b) it is easy to see that $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \subseteq E$, in fact for any $a \in A \setminus E$ it results that

$$F_{\mathcal{S}}(u, a) = F_{\mathcal{S}}(u', a),$$

hence $A \setminus E \subseteq A \setminus \Delta^{d_{\mathcal{S}}}_{W,A}(u, u')$, i.e. $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \subseteq E$. So $b \in E$ and B is a transversal of E and, since the choice of $E \in ESS_{\mathcal{S}}(\varphi, W, A)$ is arbitrary, B is a transversal of $ESS_{\mathcal{S}}(\varphi, W, A)$.

On the other hand, let B be a transversal of $ESS_{\mathcal{S}}(\varphi, W, A)$ and fix $E \in ESS_{\mathcal{S}}(\varphi, W, A)$. By Definition 8.6.7, there exists $u \in \Theta_{\mathcal{S}}(W, A)$ such that $u \notin \Theta_{\mathcal{S}}(W, A \setminus E)$. Hence

$$[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$$

and there exists $u' \in W$ satisfying (a) and (b). In other terms, it results that $u' \in [u]_A^c \cap W \cap [u]_{A \setminus E}$. Let $b \in B \cap E$. We have that $F_{\mathcal{S}}(u, b) \neq F_{\mathcal{S}}(u', b)$, otherwise

$$F_{\mathcal{S}}(u, a) = F_{\mathcal{S}}(u', a) \tag{8.29}$$

for any $a \in (A \setminus E) \cup \{b\}$. If we set $E' = E \setminus \{b\}$, we are saying that (8.29) holds for any $a \in A \setminus E'$, thus $u \equiv_{A \setminus E'} u'$, so $u \notin \Theta_{\mathcal{S}}(W, A \setminus E')$, contradicting Definition 8.6.7. Therefore, $F_{\mathcal{S}}(u, b) \neq F_{\mathcal{S}}(u', b)$, i.e. $u \not\equiv_B u'$ or, equivalently, $u' \in [u]_B^c \cap W \cap [u]_{A \setminus E}$. Since the previous argument can be extended to any $u' \in [u]_A^c \cap W \cap [u]_{A \setminus E}$, we conclude that

$$[u]_A^c \cap W \cap [u]_{A \setminus E} \subseteq [u]_B^c \cap W \cap [u]_{A \setminus E},$$

i.e.

$$[u]_B \cap W \cap [u]_{A \setminus E} \subseteq [u]_A \cap W \cap [u]_{A \setminus E}$$

or, equivalently,

$$[u]_B \cap W \subseteq [u]_A \cap W.$$

Since $u \in \Theta_{\mathcal{S}}(W, A)$, we deduce that $u \in \Theta_{\mathcal{S}}(W, B)$ or, equivalently, that $\Theta_{\mathcal{S}}(W, A) \subseteq \Theta_{\mathcal{S}}(W, B)$. The reverse inclusion holds by Proposition 8.5.4. So, $\varphi_W(B) = \varphi_W(A)$. We have thus shown that $B \subseteq A$ is a transversal of $ESS_{\mathcal{S}}(\varphi, W, A)$ if and only if $\varphi_W(B) = \varphi_W(A)$. We can now conclude the proof. Let $c \in B$ and $B' = B \setminus \{c\}$. By (ii) of Definition 8.6.8, we have that

$$\Theta_{\mathcal{S}}(W, B') \subsetneq \Theta_{\mathcal{S}}(W, A)$$

therefore, by the above argument, B' is not a transversal of $ESS_{\mathcal{S}}(\varphi, W, A)$. Thus

$$RED_{\mathcal{S}}(\varphi, W, A) \subseteq Tr(ESS_{\mathcal{S}}(\varphi, W, A)).$$

Conversely, if $B \in Tr(ESS_{\mathcal{S}}(\varphi, W, A))$, by the above argument it results that

$$\Theta_{\mathcal{S}}(W, B) = \Theta_{\mathcal{S}}(W, A)$$

and by minimality, (ii) of Definition 8.6.8 follows immediately. This shows the claim. □

In what follows, we want to define a new class of indistinguishability linear systems such that the (φ, W, A)-essentials are exactly the minimal elements of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$ and the (φ, W, A)-reducts are the minimal transversal of the same set system. In order to do this, we introduce the following terminology.

Definition 8.6.12. We say that $\mathcal{S}$ is (φ, W, A)-*disjunctive* if for any pair of equations $u, v \in W$ the following condition holds:

$$\Delta_{W,A}^{d_{\mathcal{S}}}(u, v) \neq \emptyset \implies u \in \varphi_W(A) \vee v \in \varphi_W(A). \tag{8.30}$$

In particular, if $W = U_{\mathcal{S}}$ and $A = C_{\mathcal{S}}$, we say that $\mathcal{S}$ is φ-*disjunctive*.

$\mathcal{S}_3$	a_1	a_2	a_3	a_4	a_5	$d_{\mathcal{S}_3}$
u_1	1	0	0	1	0	2
u_2	0	1	0	0	1	1
u_3	0	0	0	0	1	0
u_4	1	0	0	1	0	3
u_5	1	0	0	0	1	0

Figure 8.2: Indistinguishability linear system $\mathcal{S}_3$.

In the next example we show that a φ-disjunctive indistinguishability linear system fails to be a (φ, W, A)-disjunctive indistinguishability linear system for any $A \subseteq C_{\mathcal{S}}$ and any $W \subseteq U_{\mathcal{S}}$.

Example 8.6.13. *Let us consider the indistinguishability linear system $\mathcal{S}_3$ given in Figure 8.2.*

It is immediate to see that $\mathcal{S}_3$ is a φ-disjunctive indistinguishability linear system. On the other hand, let $W = \{u_1, u_2, u_3, u_4\}$ and $A = \{a_1\}$. It is easy to verify that

$$[u]_A \cap W = \begin{cases} \{u_1, u_4\} & \text{if } u = u_1 \vee u = u_4 \vee u = u_5 \\ \{u_2, u_3\} & \text{if } u = u_2 \vee u = u_3 \end{cases}$$

and

$$[u]_{d_{\mathcal{S}_3}} \cap W = \begin{cases} \{u_1\} & \text{if } u = u_1 \\ \{u_2\} & \text{if } u = u_2 \\ \{u_3\} & \text{if } u = u_3 \vee u = u_5 \\ \{u_4\} & \text{if } u = u_4 \end{cases}$$

Thus

$$\Theta_{\mathcal{S}}(W, A) = \emptyset.$$

This shows that $\mathcal{S}_3$ is not a (φ, W, A)-disjunctive indistinguishability linear system.

As an immediate result, we prove that all the compatible indistinguishability linear systems are also φ-disjunctive.

Proposition 8.6.14. *If $\mathcal{S}$ is compatible, then it is also a φ-disjunctive indistinguishability linear system.*

Proof. By Proposition 8.4.1, compatibility of $\mathcal{S}$ is equivalent to the condition $\Theta_{\mathcal{S}}(U_{\mathcal{S}}, C_{\mathcal{S}}) = U_{\mathcal{S}}$. □

Remark 8.6.15. Let us note that there exist incompatible indistinguishability linear systems that are φ-disjunctive, as we can see taking the indistinguishability linear system $\mathcal{S}_3$ of Figure 8.2.

In the next result we show that for any (φ, W, A)-disjunctive indistinguishability linear system the essentials $ESS_{\mathcal{S}}(\varphi, W, A)$ are exactly the minimal elements of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$.

Theorem 8.6.16. *Let $\mathcal{S}$ be a (φ, W, A)-disjunctive indistinguishability linear system. Then,*

$$ESS_{\mathcal{S}}(\varphi, W, A) = \min(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)). \tag{8.31}$$

Proof. We will prove that a variable subset $B \subseteq A$ belongs to $ESS_{\mathcal{S}}(\varphi, W, A)$ if and only if it satisfies the following two conditions:

(a) $B \supseteq \Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v) \neq \emptyset$, for some $u,\, v \in W$;

(b) if there exists $u, v \in W$ such that $\emptyset \neq \Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v) \subseteq B$, then $\Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v) = B$.

Let $ESS_{\mathcal{S}}(\varphi, W, A)$. By Definition 8.6.7 and by Proposition 8.5.4, this means that there exists $u \in W$ such that $[u]_A \cap W \subseteq [u]_{D_{\mathcal{S}}} \cap W$ and $[u]_{A \setminus B} \cap W \not\subseteq [u]_{D_{\mathcal{S}}} \cap W$. This means that there exists $v \in W$ such that $u \equiv_{A\setminus B} v$ and $u \not\equiv_{d_{\mathcal{S}}} v$, so, $F_{\mathcal{S}}(u, a) = F_{\mathcal{S}}(v, a)$ for any $a \in A \setminus B$. By (8.20), we deduce that $A \setminus B \subseteq A \setminus \Delta^{d_{\mathcal{S}}}_{W,A}(u, v)$, so B satisfies condition (a). Now, suppose that there exists $u, v \in W$ such that $\emptyset \neq \Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v) \subseteq B$. Let $b \notin \Delta^{d_{\mathcal{S}}}_{W,A}(u, v)$; hence $F_{\mathcal{S}}(u, b) = F_{\mathcal{S}}(v, b)$. Let $B' := B \setminus \{b\}$. Since $F_{\mathcal{S}}(u, a) = F_{\mathcal{S}}(v, a)$ for all $a \in A \setminus B$ and $F_{\mathcal{S}}(u, b) = F_{\mathcal{S}}(v, b)$, we have $F_{\mathcal{S}}(u, a) = F_{\mathcal{S}}(v, a)$ for all $a \in (A \setminus B) \cup \{w\} = A \setminus B'$. In other terms, we have $u \equiv_{A\setminus B'} v$. Then, since $u \not\equiv_{d_{\mathcal{S}}} v$, we deduce that $u \notin \Theta_{\mathcal{S}}(W, A \setminus B')$. But, we have that $u \in \Theta_{\mathcal{S}}(W, A)$, so $b \notin B$, by Definition 8.6.7. Hence part (b) holds.

Assume now that $B \subseteq A$ satisfies (a) and (b). Since $\Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v) \neq \emptyset$, it holds that $u \not\equiv_A v$ and $u \not\equiv_{d_{\mathcal{S}}} v$. We may assume that $u \in \Theta_{\mathcal{S}}(W, A)$. Moreover $B \supseteq \Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v)$ implies that $u \equiv_{A\setminus B} v$ and $u \not\equiv_{d_{\mathcal{S}}} v$, thus $u \notin \Theta_{\mathcal{S}}(W, A \setminus B)$, so ($i$) of Definition 8.6.7 holds.

Let now $B' \subsetneqq B$. Condition (b) implies that, for all $u,\, v \in W$ such that $\emptyset \neq \Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v)$, it holds that $\Delta^{d_{\mathcal{S}}}_{W,A}(u,\, v) \not\subseteq B'$. This is equivalent to say that $\pi_{\mathcal{S}}(A \setminus B') = \pi_{\mathcal{S}}(A)$. In fact, it is obvious that $u \equiv_A u'$ implies $u \equiv_{A\setminus B'} u'$; moreover, suppose that $u \equiv_{A\setminus B'} u'$ and assume by contradiction that $u \not\equiv_A u'$. Then, we have $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \subseteq B'$, that is an absurd. So (ii) of Definition 8.6.7 holds too and $B \in ESS_{\mathcal{S}}(\varphi, W, A)$. □

The next result shows that the reducts of $\mathcal{S}$ are exactly the minimal transversals of the essentials family $ESS(\mathcal{S})$ for any φ-disjunctive indistinguishability linear system $\mathcal{S}$.

Theorem 8.6.17. *Let $\mathcal{S}$ be a (φ, W, A)-disjunctive indistinguishability linear system and let $B \subseteq A$. Then:*
(i) $\theta_{\mathcal{S}}(W, B) = \theta_{\mathcal{S}}(W, A)$ if and only if B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$.
(ii) $RED_{\mathcal{S}}(\varphi, W, A) = Tr(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)) = Tr(ESS_{\mathcal{S}}(\varphi, W, A))$.

Proof. (i) : Assume that $\theta(W, B) = \theta(W, A)$. We show that B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Let then $E \in DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Then, there exists two distinct equations $u, u' \in W$ such that $u \not\equiv_{d_{\mathcal{S}}} u'$ and $E = \{a \in A : (F_{\mathcal{S}}(u, a) \neq F_{\mathcal{S}}(u', a))\}$. We deduce that $u \not\equiv_A u'$ and we can assume that $u \in \Theta_{\mathcal{S}}(W, A)$. By hypothesis, $u \in \Theta_{\mathcal{S}}(W, B)$, hence $u \not\equiv_B u'$. Thus, there exists $b \in B$ such that $F_{\mathcal{S}}(u, b) \neq F_{\mathcal{S}}(u', b)$, i.e. $b \in B \cap E$. Therefore B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. On the other hand, we suppose now that B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Hence, there exist $u, u' \in W$ such that $\Delta^{d_{\mathcal{S}}}_{W,A}(u, u') \neq \emptyset$. We clearly have that $u \not\equiv_A u'$. We set now $E := \Delta^{d_{\mathcal{S}}}_{W,A}(u, u')$. Let $b \in B \cap E$, then $F_{\mathcal{S}}(u, b) \neq F_{\mathcal{S}}(u', b)$, so $u \not\equiv_B u'$. We are saying that for any $u, u' \in W$ such that $u \not\equiv_{d_{\mathcal{S}}} u'$ and $u \not\equiv_A u'$, then $u \not\equiv_B u'$. In other terms, we have shown that

$$[u]_A^c \cap W \subseteq [u]_B^c \cap W$$

or, equivalently, that

$$[u]_B \cap W \subseteq [u]_A \cap W.$$

Since $u \in \Theta_{\mathcal{S}}(W, A)$, we deduce that $u \in \Theta_{\mathcal{S}}(W, B)$, i.e. $\Theta_{\mathcal{S}}(W, A) \subseteq \Theta_{\mathcal{S}}(W, B)$. The reverse inclusion follows by Proposition 8.5.4. So $\theta_{\mathcal{S}}(W, B) = \theta_{\mathcal{S}}(W, A)$.
(ii) : Let $B \in RED_{\mathcal{S}}(\varphi, W, A)$. Hence $\theta_{\mathcal{S}}(W, B) = \theta(W, A)$, and by (i) this implies that B is a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Now, let $b \in B$ and set $B' = B \setminus \{b\}$. By Definition 8.6.8 we have that $\theta_{\mathcal{S}}(W, B') < \theta_{\mathcal{S}}(W, A)$, therefore by (i) it follows that B' is not a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. Hence B is a minimal transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$. On the other hand, let B be a minimal transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$, then by (i) it follows that $\theta_{\mathcal{S}}(W, B) = \theta_{\mathcal{S}}(W, A)$. Now, if $b \in B$, the subset $B' := B \setminus \{b\}$ is not a transversal of $DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)$ by virtue of the minimality of B, therefore, again by (i) we obtain $\theta_{\mathcal{S}}(W, B') < \theta_{\mathcal{S}}(W, A)$. Hence $B \in RED_{\mathcal{S}}(\varphi, W, A)$. Thus, $RED_{\mathcal{S}}(\varphi, W, A) = Tr(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A))$. By Theorem 8.6.16, it results $Tr(DIS_{\mathcal{S}}(d_{\mathcal{S}}, W, A)) = Tr(ESS_{\mathcal{S}}(\varphi, W, A))$, hence (ii) has been proved. $\square$

Bibliography

[1] H. Aghabozorgi, B. Davvaz, M. Jafarpour, Nilpotent groups derived from hypergroups, *Journal of Algebra*, 382 (2013), 177–184.

[2] M.O. Albertson, K. L. Collins, Symmetry Breaking in Graphs, *The Electronic Journal of Combinatorics*, 3 (1996), 1–17.

[3] J.A. Aledo, S. Martínez, J. C. Valverde, Parallel Dynamical Systems over Graphs and Related Topics: A Survey. *Journal of Applied Mathematics* Volume 2015 (2015), Article ID 594294, 14 pages.

[4] J.A. Aledo, S. Martínez, J. C. Valverde, Graph Dynamical Systems with General Boolean States, *Applied Mathematics and Information Sciences*, 9, No.4, 1803–1808 (2015).

[5] J.A. Aledo, S. Martínez, J. C. Valverde, Parallel Dynamical Systems over Graphs and Related Topics: A Survey. *Journal of Applied Mathematics* Volume 2015 (2015), Article ID 594294, 14 pages.

[6] S.M. Anvariyeh, S. Mirvakili, B. Davvaz, Combinatorial aspects of n-ary polygroups and n-ary color schemes, *European Journal of Combinatorics*, 34 (2013), 207–216.

[7] N. Apollonio, M. Caramia, P. G. Franciosa, On the Galois lattice of bipartite distance hereditary graphs, *Discrete Applied Mathematics*, 190 (2015), 13–23.

[8] N. Apollonio, B. Simeone, Improved Approximation of Maximum Vertex Coverage Problem on Bipartite Graphs, *Siam Journal Discrete Mathematics*, 28(3) (2014), 1137–1151.

[9] N. Apollonio, B. Simeone, The maximum vertex coverage problem on bipartite graphs, *Discrete Applied Mathematics*, 165 (2014), 37–48.

[10] N. Apollonio, M. Caramia, Recognizing Helly Edge Path Tree Graphs and their Clique Graphs, *Discrete Applied Mathematics*, 159 (2011), 1166–1175.

[11] R. A. Bayley, Orthogonal partitions in designed experiments, *Designs, Codes and Cryptography*, Volume 8, Issue 3 (1996), 45–77.

[12] R. A. Bayley, Association Schemes: Designed Experiments, Algebra and Combinatorics, Cambridge University Press, Cambridge 2004, 387pp.

[13] I. Balla, B. Bollobás, T. Eccles, Union-closed families of sets. *Journal of Combinatorial Theory Ser. A* Volume 120, Issue 3, April 2013, 531–544.

[14] J. Bang-Jensen, G. Gutin, *Digraphs. Theory, algorithms and applications.* Second edition. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2009. xxii+795 pp.

[15] C. Berge, Hypergraphs: Combinatorics of Finite Sets, Elsevier, Amsterdam, 1984.

[16] A. Berry, A. Sigayret, Representing a concept lattice by a graph, *Discrete Applied Mathematics*, 144 (2004), 27–42.

[17] D. Bianucci, G. Cattaneo, Information Entropy and Granulation Co-Entropy of Partitions and Coverings: a Summary, in: Special Issue on Foundations of Rough Sets, J.F. Peter and A. Skowron Eds., *Transactions on Rough Sets X, Lecture Notes in Computer Science*, vol. 5656, Springer-Verlag, 2009, pp. 15-66.

[18] G. Birkhoff, Lattice Theory, American Mathematical Society, Providence, Rhode Island, Third Edition, 1967.

[19] G. Birkhoff, O. Frink, Representations of lattices by sets, *Transactions of the American Mathematical Society*, 64 (1948), 299–316.

[20] C. Bisi, G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Micro and Macro Models of Granular Computing induced by the Indiscernibility Relation. *Information Sciences*, Volumes 388–389, May 2017, Pages 247–273.

[21] C. Bisi and G. Chiaselotti, A class of lattices and boolean attributes related to the Manickam-Miklös-Singhi conjecture, *Advances in Geometry* 13, no.1, (2013), 1–27.

[22] C. Bisi, G. Chiaselotti, P.A.Oliverio; Sand Piles Models of Signed Partitions with d Piles, ISRN Combinatorics, vol. 2013, Article ID 615703, 7 pages, 2013. doi:10.1155/2013/615703.

[23] C. Bisi, G. Chiaselotti, G. Marino, P.A. Oliverio, A natural extension of the Young partition lattice. *Advances in Geometry* 15, no.3, (2015), 263–280.

[24] C. Bisi, G. Chiaselotti, T. Gentile, P.A. Oliverio, Dominance Order on Signed Partitions. *Advances in Geometry* Volume 17, Issue 1 (2017), 5–29.

[25] T. Boykett, Orderly Algorithm to Enumerate Central Groupoids and Their Graphs, *Acta Mathematica Sinica, English Series*, 2007, Vol.23, Issue 2, 249–264.

[26] T. Boykett, Rectangular groupoids and related structures, *Discrete Mathematics*, 313 (2013), 1409–1418.

[27] N. Caspard, B. Monjardet, The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey, *Discrete Applied Mathematics*, 127, (2003), 241–269.

[28] G. Cattaneo, Generalized Rough Sets (Preclusivity Fuzzy-Intuitionistic (BZ) Lattices), *Studia Logica*, 01, 1997, pp. 47–77.

[29] Cattaneo, G.: Abstract approximation spaces for rough theories, in: Rough Sets in Knowledge Discovery 1: Methodology and Applications (L. Polkowski, A.Skowron (eds.)). Studies in Fuzziness and Soft Computing, Physica, Heidelberg (1998) 59–98

[30] G. Cattaneo, D. Ciucci, D. Bianucci, Entropy and Co-entropy of Partitions and Coverings with Applications to Roughness Theory, in: *Granular Computing: At the Junction of Rough Sets and Fuzzy sets*. Series: Studies on Fuzziness and Soft Computing. R. Bello, R. Falcon, W Pedrycz and J. Kacprzyk, Eds., vol. 224, Springer-Verlag, 2008, pp. 55-77.

[31] G. Cattaneo, An investigation about rough set theory: some foudational and mathematical aspects, *Fundamenta Informaticae* 108, 2011, pp. 197–221.

[32] G. Cattaneo, G. Chiaselotti, A. Dennunzio, E. Formenti, L. Manzoni, Non Uniform Cellular Automata Description of Signed Partition Versions of Ice and Sand Pile Models. *Cellular Automata. Lecture Notes in Computer Science.* Volume 8751, pp. 115–124, 2014.

[33] G. Cattaneo, G. Chiaselotti, T. Gentile and P. A. Oliverio, The lattice structure of equally extended signed partitions. A generalization of the Brylawski approach to integer partitions with two possible models: ice piles and semiconductors. *Fundamenta Informaticae*, vol. 141, no.1, pp.1–36, 2015.

[34] G. Cattaneo, G. Chiaselotti, D. Ciucci, T. Gentile, On the connection of Hypergraph Theory with Formal Concept Analysis and Rough Set Theory. *Information Sciences*, 330 (2016), 342–357.

[35] G. Cattaneo, G. Chiaselotti, P.A. Oliverio, F. Stumbo, A New Discrete Dynamical System of Signed Integer Partitions. *European Journal of Combinatorics* 55 (2016) 119–143.

[36] B. Chen, The Incidence Algebra of Polyhedra over the Minkowski Algebra, *Advances in Mathematics*, 118, 1996, 337–365.

[37] B. Chen, M. Yan, Eulerian Stratification of Polyhedra, *Advances in Applied Mathematics*, 21, (1998), 22–57.

[38] B. Chen, V. Turaev, Counting lattice points of rational polyhedra, *Advances in Mathematics*, 155, 2000, 84–97.

[39] J. Chen, J. Li, An application of rough sets to graph theory, *Information Sciences*, 201, 114–127 (2012).

[40] G. Chiaselotti, On a problem concerning the weight functions, *European Journal of Combinatorics*, 23 (2002), 15–22.

[41] G. Marino, G. Chiaselotti, A method to count the positive 3-subsets in a set of real numbers with non-negative sum, *European Journal of Combinatorics*, 23 (2002), 619–629.

[42] G. Chiaselotti, G. Marino, P.A. Oliverio, D. Petrassi, A Discrete Dynamical Model of Signed Partitions, *Journal of Applied Mathematics*, Volume 2013, Article ID 973501, 10 pages.

[43] G. Chiaselotti, T. Gentile, G. Marino, P.A. Oliverio, Parallel Rank of Two Sandpile Models of Signed Integer Partitions, *Journal of Applied Mathematics*, Volume 2013, Article ID 292143, 12 pages.

[44] G. Chiaselotti, G. Infante, G. Marino, New results related to a conjecture of Manickam and Singhi, *European Journal of Combinatorics* 29 (2008), n. 2, 361–368.

[45] G. Chiaselotti, G. Marino, C. Nardi, A minimum problem for finite sets of real numbers with nonnegative sum, *Journal of Applied Mathematics*, Volume 2012, Article ID 847958, 15 pages.

[46] G. Chiaselotti, T. Gentile, Intersection Properties of Maximal Directed Cuts in Digraphs. *Discrete Mathematics* 340 (2017), 3171–3175.

[47] G. Chiaselotti, D. Ciucci, T. Gentile, Simple Graphs in Granular Computing. *Information Sciences*, Volumes 340–341, 1 May 2016, 279–304.

[48] G. Chiaselotti, D. Ciucci, T. Gentile, Simple Undirected Graphs as Formal Contexts. *Proc. ICFCA, Lecture Notes in Computer Science*, Volume 9113, pp. 287–302, Springer, 2015.

[49] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Rough Set Theory Applied to Simple Undirected Graphs. *Proc. RSKT 2015, Lecture Notes in Computer Science*, Vol. 9436, pp. 423–434, Springer 2015.

[50] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Preclusivity and Simple Graphs. *Proc. RSFDGrC 2015, Lecture Notes in Computer Science*, Vol. 9437, 127–137, Springer 2015.

[51] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Preclusivity and Simple Graphs: the n–cycle and n–path Cases. *Proc. RSFDGrC 2015, Lecture Notes in Computer Science*, Vol. 9437, 138–148, Springer 2015.

[52] G. Chiaselotti, T. Gentile, F. Infusino, P.A. Oliverio, Rough Sets for n-Cycles and n-Paths. *Applied Mathematics and Information Sciences*, 10, No.1, 117–124 (2016).

[53] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, The Granular Partition Lattice of an Information Table. *Information Sciences* 373 (2016), 57–78.

[54] G. Chiaselotti, T. Gentile, F. Infusino, P. A. Oliverio, The Adjacency Matrix of a Graph as a Data Table. A Geometric Perspective. *Annali di Matematica Pura e Applicata*, Vol. 196, No. 3, (2017), 1073–1112.

[55] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Generalizations of Rough Set Tools inspired by Graph Theory. *Fundamenta Informaticae*, 148 (2016), 207–227.

[56] G. Chiaselotti, T. Gentile, F. Infusino, Knowledge Pairing Systems in Granular Computing. *Knowledge Based Systems*, 124 (2017), 144–163.

[57] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Rough Set Theory and Digraphs. *Fundamenta Informaticae*, 153 (2017), 291–325.

[58] G. Chiaselotti, T. Gentile, F. Infusino, Granular Computing on Information Tables: Families of Subsets and Operators; *Information Sciences* (Submitted).

[59] G. Chiaselotti, T. Gentile, F. Infusino, Some Classifications of Graphs with respect to a Set Adjacency Relation, *Annali di Matematica Pura e Applicata*, (Submitted).

[60] G. Chiaselotti, T. Gentile, F. Infusino, Granular Computing on Basic Digraphs. *Discrete Applied Mathematics* (Submitted).

[61] G. Chiaselotti, T. Gentile, F. Infusino, P. A. Oliverio Some Mathematical Structures induced from Function Sets. *Journal of Discrete Mathematics and Cryptography* (Accepted for Publication).

[62] Decision Systems in Rough Set Theory. A Set Operatorial Perspective, *Journal of Algebra and its Applications* (Accepted for publication).

[63] G. Chiaselotti, T. Gentile, F. Infusino, Pairings and Related Symmetry Notions, *Annali Dell'Universita' di Ferrara* (Accepted for publication).

[64] G. Chiaselotti, T. Gentile, F. Infusino, Union Additive Relations Induced by Pairings. *International Journal of Algebra and Computation* (Submitted).

[65] G. Chiaselotti, T. Gentile, F. Infusino, Pairings and Related Type-Topological Structures. *Annali di Matematica Pura e Applicata* (Submitted).

[66] G. Chiaselotti, T. Gentile, F. Infusino, Sequentiality of Subset Ordered Pair Families on Finite Sets. *Discrete Applied Mathematics* (Submitted).

[67] G. Chiaselotti, T. Gentile, F. Infusino, Lattices and Set Operators induced by Pointed Subset Families. *Geometriae Dedicata* (Submitted).

[68] G. Chiaselotti, T. Gentile, F. Infusino, Simplicial Complexes and Closure Systems induced by Indistinguishability Relations. *C. R. Acad. Sci. Paris, Ser. I*, 355 (2017), 991–1021.

[69] G. Chiaselotti, T. Gentile, F. Infusino, Dependency Structures for Decision Tables. *International Journal of Approximate Reasoning* 88 (2017), 333–370.

[70] G. Chiaselotti, T. Gentile, F. Infusino, P.A. Oliverio, Dependency and Accuracy Measures for Directed Graphs, *Applied Mathematics and Computation*, 320 (2018), 781–794.

[71] G. Chiaselotti, T. Gentile, F. Infusino, P.A. Oliverio, Local Dissymmetry on Graphs and Related Algebraic Structures. *Journal of Pure and Applied Algebra*, (Submitted).

[72] G. Chiaselotti, T. Gentile, F. Infusino, F. Tropeano, Dependency and Accuracy Measures for Some Basic Graphs, *Applied Mathematics and Computation* (Submitted).

[73] D. E. Ciucci, Classification of Dynamics in Rough Sets, RSCTC 2010, *Lecture Notes in Computer Science*, vol. 6086, pp. 257–266, Springer-Verlag 2010.

[74] D. E. Ciucci, Temporal Dynamics in Information Tables, *Fundamenta informaticae* 115 (2012), n. 1, 57–74.

[75] F. Comellas, M. Sampels, Deterministic small-world networks; *Physica A*, 309, pp.231–235, 2002.

[76] G. Czédli, On averaging Frankl's conjecture for large union-closed sets, *Journal of Combinatorial Theory Series A*, 116, Issue 3, (2009) 724–729.

[77] B. A. Davey, H. A. Priestley, *Introduction to lattices and order*. Second edition. Cambridge University Press, New York, (2002).

[78] B. Davvaz, P. Corsini and T. Changphas, Relationship between ordered semihypergroups and ordered semigroups by using pseudorders, *European Journal of Combinatorics*, 44 (2015), 208–217.

[79] R. Diestel, Graph Theory (4th edition), Graduate Text in Mathematics, Springer 2010.

[80] T. Eiter and G. Gottlob, Identifying the Minimal Transversals of a Hypergraph and Related Problems. *Siam Journal on Computing*, 24, 1995, 1278–1304.

[81] T. Eiter and G. Gottlob, Hypergraph transversal computation and related problems in logic and AI, *Proc. JELIA 2002, Lecture Notes in Computer Science*, Volume 2424, 2002, 549–564.

[82] K. Elbassioni; On the Complexity of Monotone Dualization and Generating Minimal Hypergraph Transversals, *Discrete Applied Mathematics*, 32 (2), pp. 171–187, 2008.

[83] P. Erdös, A. Rényi; Asymmetric Graphs, *Acta Mathematica Hungarica*, Vol.14, Issue 3–4, pp. 295–315, 1963.

[84] W. L. Fouché, Symmetry and the Ramsey Degree of Posets, *Discrete Mathematics* 167/168 (1997), 309–315.

[85] P. Frankl, Extremal Set Systems, Chapter 24 of the Handbook of Combinatorics, MIT Press, 1995.

[86] Z. Frolik, M. Katetov, V. Ptak, General Topology and its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, Academic Press, New York, 1966.

[87] B. Ganter, R. Wille, Formal Concept Analysis. Mathematical Foundations, Springer-Verlag, 1999.

[88] B. Gaume, E. Navarro, H. Prade, A Parallel between Extended Formal Concept Analysis and Bipartite Graphs Analysis In: E. Hüllermeier, R. Kruse, F. Hoffmann(eds.) Proc. IPMU'10, *LNCS*, vol. 6178, pp. 270–280. Springer (2010).

[89] G. Glauberman; Central Elements in Core-Free Groups, *Journal of Algebra*, Vol.4, pp. 403–420, 1966.

[90] A. V. Goldberg, S. A. Plotkin, G. E. Shannon, Parallel Symmetry-Breaking in Sparse Graphs, *SIAM Journal of Discrete Mathematics*, Vol. 1, No. 4, 1988, pp. 434–446.

[91] M. Golubitsky, I. Stewart, Hopf Bifurcation in the Presence of Symmetry, *Bulletin (New Series) of the American Mathematical Society*, Volume 11, Number 2, 1984, 339–343.

[92] M. Golubitsky, I. Stewart, Hopf Bifurcation in the Presence of Symmetry, *Archive for Rational Mechanics and Analysis*, 1985, Volume 87, Issue 2, pp. 107–165.

[93] M. Golubitsky, I. Stewart, An Algebraic Criterion for Symmetric Hopf Bifurcation, *Proceedings of the Royal Society*, London, A 440 (1993), 727–732.

[94] L. Guo, F. Huang, Q. Lia, G. Zhang, Power contexts and their concept lattices, *Discrete Mathematics*, 311 (2011), 2049–2063.

[95] V. Gutev, Selections and Topological Convexity, *Topology and its Applications*, 155 (2008), 814–823.

[96] V. Gutev, Selections and hyperspaces of finite sets, *Topology and its Applications*, 157 (2010), 83–89.

[97] A. Gyárfás; J. Lehel; Hypergraph families with bounded edge cover or transversal number; *Combinatorica*, Vol. 3, Issue 3–4, pp. 351–358, 1983.

[98] S. Greco et Al., Rough Sets methodology for sorting problems in presence of multiple attributes and criteria, *European Journal of Operational Research*, 138, pp. 247–259, 2002.

[99] M. Hagen; Lower Bounds for Three Algorithms for Transversal Hypergraph Generation; *Discrete Applied Mathematics*, Vol.157, pp. 1460–1469, 2009.

[100] P. W. Harley III, Metric and Symmetric Spaces. *Proceedings of the American Mathematical Society*, Vol.43, no.2 (1974), 428–430.

[101] F. Harary, D. Ranjan, Breaking Symmetry in Complete Graphs by Orienting Edges: Asymptotic Bounds, *Information Processing Letters*, 67 (1998), 227–230.

[102] F. Harary, W. Slany, O. Verbitsky, On the lengths of symmetry breaking-preserving games on graphs, *Theoretical Computer Science* 313 (2004) 427–446.

[103] G. Hahn and G. Sabidussi (Editors); Graph Symmetry. Algebraic Methods and Applications; *NATO ASI Series*, Vol.497, 1997, Springer.

[104] P. Hońko, Description and classification of complex structured objects by applying similarity measures, *International Journal of Approximate Reasoning*, 49(3): 539–554 (2008).

[105] P. Hońko, Relational pattern updating, *Information Sciences*, 189, 2012, 208–218.

[106] P. Hońko, Association discovery from relational data via granular computing, *Information Sciences*, 10, 2013, 136–149.

[107] A. Huang, Z. Lin, W. Zhu, Matrix approaches to rough sets through vector matroids over fields, *IJGCRSIS*, 3(3), 179–194 (2014).

[108] A. Huang, H. Zhao, W. Zhu, Nullity-based matroid of rough sets and its application to attribute reduction, *Information Sciences*, 263, 153–165 (2014).

[109] B. Huang, Y. Zhang, H. Li, D. Wei, A Dominance Intuitionistic Fuzzy-Rough Set Approach and Its Applications, Applied Mathematical Modelling, 37, 2013, 7128–7141.

[110] X. Kang, D. Li, S. Wang, K. Qu, Formal concept analysis based on fuzzy granularity base for different granulations, *Fuzzy Sets and Systems*, 203, 2012, 33–48.

[111] K. Kashiwabara, M. Nakamura, Y. Okamoto, The affine representation theorem for abstract convex geometries, *Computational Geometry*, 30(2), 129–144 (2005).

[112] M. Hachimori, K. Kashiwabara, On the topology of the free complexes of convex geometries, *Discrete Mathematics*, 307(2), 274–279 (2007).

[113] K. Kashiwabara, M. Nakamura, Characterizations of the convex geometries arising from the double shellings of posets, *Discrete Mathematics* 310 (15–16) (2010), 2100–2112.

[114] K. Kashiwabara, M. Nakamura, The Prime Stems of Rooted Circuits of Closure Spaces and Minimum Implicational Bases, *The Electronic Journal of Combinatorics*, 20 (1), (2013), 1–22.

[115] A. Kelarev, S. J. Quinn, Directed graphs and combinatorial properties of semigroups. *Journal of Algebra*, 251 (1) (2002), 16–26.

[116] A. Kelarev, C. E. Praeger, On transitive Cayley graphs of groups and semigroups. *European Journal of Combinatorics*, 24(1) (2003), 59–72.

[117] A. Kelarev, J. Ryan, J. Yearwood; Cayley graphs as classifiers for data mining: The influence of asymmetries. *Discrete Mathematics*, 309 (2009), 5360–5369.

[118] L. Khachiyan, E. Boros, K. Elbassioni, V. Gurvich; A global parallel algorithm for the hypergraph transversal problem; *Information Processing Letters*, Vol.101, Issue 4, pp. 148–155, 2007.

[119] D. Kozlov, Combinatorial Algebraic Topology, *Springer-Verlag Berlin*, Heidelberg 2008, vol. 21, 2001.

[120] V Kreinovich, Interval Computation as an Important Part of Granular Computing: An Introduction, in *Handbook of Granular Computing*, Wiley, 2008, 3–32.

[121] R. Ksontini, The fundamental group of the Quillen complex of the symmetric group, *Journal of Algebra*, 282 (2004), 33–57.

[122] H. Li, W. Zhu, Connectedness of Graph and Matroid by Covering-Based Rough Sets, *LNCS*, RSFDGrC 2015: 149–160.

[123] Y. Liu, W. Zhu, The Matroidal Structures of the Second Type of Covering-Based Rough Set, *LNCS*,RSKT 2015: 231–242.

[124] Y. Liu, W. Zhu, On the matroidal structure of generalized rough set based on relation via definable sets, *Int. J. Machine Learning and Cybernetics*, 7(1): 135–144 (2016).

[125] T. Y. Lin, Data Mining: Granular Computing Approach, in Methodologies for Knowledge Discovery and Data Mining *Lecture Notes in Computer Science*, 1574, 1999, 24–33.

[126] T. Y. Lin, Data Mining and Machine Oriented Modeling: A Granular Approach, *Applied Intelligence*, 13, 2000, 113–124.

[127] J. Xie, T. Y. Lin, W. Zhu, Granular and Rough computing on Covering, *IEEE International Conference on Granular Computing*, 2012.

[128] T. Y. Lin, Y. Liu, W. Huang, Unifying Rough Set Theories via Large Scaled Granular Computing, *Fundamenta Informaticae*, 127(1-4), 413–428, 2013.

[129] T. Y. Lin, Y.-R. Syau, Unifying Variable Precision and Classical Rough Sets: Granular Approach; *Rough Sets and Intelligent Systems*, (2), 2013, 365–373.

[130] I. G. Macdonald, *Symmetric functions and Hall polynomials*. Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995. x+475 pp.

[131] A. Magner, S. Janson, G. Kollias, W. Szpankowski, On Symmetry of Uniform and Preferential Attachment Graphs, *Electronic Journal of Combinatorics*, 21 (3) 2014.

[132] H. W. Martin, Metrization of symmetric spaces and regular maps. *Proceedings of the American Mathematical Society*, 35 (1972), 269–274.

[133] F. Merkl, H. Wagner, Recurrent Random Walks and the Absence of Continuous Symmetry Breaking on Graphs, *Journal of Statistical Physics*, Vol. 75, Nos. 1/2, 1994, 153–165.

[134] D.Q. Miao, Y. Zhao, Y.Y. Yao, H.X. Li, F.F. Xu, Relative reducts in consistent and inconsistent decision tables of the Pawlak rough set model, *Information Sciences*, 179, 2009, 4140–4150

[135] S. Mirvakili, B. Davvaz, Applications of the α^*-relation to Krasner hyperrings, *Journal of Algebra*, 362 (2012), 145–156.

[136] L. Molnár, Orthogonality Preserving Transformations on Indefinite Inner Product Spaces: Generalization of Uhlhorn's Version of Wigner'sTheorem. *Journal of Functional Analysis*, 194, 248–262 (2002).

[137] R. Morris, FC-Families and Improved Bounds for Frankl's Conjecture, *European Journal of Combinatorics*, Issue 2, 27, (2006), 269–282.

[138] L. Nachbin, Topology and Order, *Van Nostrand Mathematical Studies* 1965.

[139] T. Qiu, X. Chen, Q. Liu, H. Huang, Granular Computing Approach to Finding Association Rules in Relational Database, *International Journal of Intelligent Systems*, 25, 2010, 165–179.

[140] Z. Pawlak, Information systems theoretical foundations, *Information Sciences* 6, Issue 3, 1981, pp. 205–218

[141] Z. Pawlak, Rough Sets, *International Journal of Computer and Information Sciences*, vol. 11, No. 5, 1982, pp. 341–356

[142] Z. Pawlak, Rough sets – Theoretical aspects of reasoning about data, Kluwer Academic Publishers, Dordrecht, 1991.

[143] D. Petrassi, On Some Numbers Related to Extremal Combinatorial Sum Problems, *Journal of Discrete Mathematics*, Volume 2014, Article ID 979171, 6 pages.

[144] Z. Pawlak, A. Skowron, Rudiments of rough sets, *Information Sciences* 177, 2007, pp. 3–27

[145] Z. Pawlak, A. Skowron, Rough sets: Some extensions, *Information Sciences* 177, 2007, pp. 28–40

[146] Z. Pawlak, A. Skowron, Rough sets and Boolean reasoning, *Information Sciences* 177, 2007, pp. 41–73

[147] W. Pedrycz, Granular Computing: An Emerging Paradigm, Springer-Verlag, Berlin (2001).

[148] W. Pedrycz, A. Skowron, V. Kreinovich, Handbook of Granular Computing, Wiley, 2008.

[149] Y. Qian, J. Liang, W. Pedrycz, C. Dang, Positive approximation: An accelerator for attribute reduction in rough set theory, *Artificial Intelligence*, 174, 2010, 597–618.

[150] W. Pedrycz, K. Hirota, W. Pedrycz, F. Dong, Granular representation and granular computing with fuzzy sets, *Fuzzy Sets and Systems*, 203, 2012, 17–32.

[151] W. Pedrycz, Granular computing : analysis and design of intelligent systems, CRC Press, 2013.

[152] B. Poonen, Union-Closed Families. *Journal of Combinatorial Theory Ser. A*, 59, 253–268 (1992).

[153] P. Ribemboimb, Algebraic Structures on Graphs, *Algebra Universalis*, 16 (1983) 105–123.

[154] B. E. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*, Second edition. Graduate Texts in Mathematics, 203. Springer-Verlag, New York, 2001. xvi+238 pp.

[155] M. Sampels, Vertex-symmetric generalized Moore graphs ; *Discrete Appl. Math.*, Vol.138, pp. 195–202, 2004.

[156] S. M. Sanahuja, A Computational Tool for Some Boolean Partial Maps, *Applied Mathematics and Information Sciences*, 9, No. 3, 1139–1145 (2015).

[157] S. M. Sanahuja, New rough approximations for n-cycles and n-paths, *Applied Mathematics and Computation*, 276 (2016), 96–108.

[158] D. Schattschneider, The Plane Symmetry Groups: Their Recognition and Notation, *The American Mathematical Monthly*, Vol. 85, No. 6 (1978), pp. 439–450.

[159] Y. She, G. Wang, An Axiomatic Approach of Fuzzy Rough Sets Based on Residuated Lattices, Computers and Mathematics with Applications, 58, 2009, 189–201.

[160] R. Simion, R.P. Stanley, Flag-symmetry of the poset of shuffles and a local action of the symmetric group, *Discrete Mathematics* 204 (1999), 369–396.

[161] D. A. Simovici, C. Djeraba, Mathematical Tools for Data Mining. Springer-Verlag, London 2014.

[162] A. Skowron, P. Wasilewski, Information systems in modeling interactive computations on granules, *Theoretical Computer Science*, 412, 2011, 5939–5959.

[163] A. Skowron, P. Wasilewski, Interactive information systems: Toward perception based computing, Theoretical Computer Science, 454, 2012, 240–260.

[164] D. Ślezak, Approximate Entropy Reducts, *Fundamenta Informaticae*, 53 (2002), 365–390.

[165] J. Spencer, Extremal Problems, Partition Theorems, Symmetric Hypergraphs, *Combinatorica* 1 (3), (1981), 303–307.

[166] D. Ślezak, Rough Sets and Bayes Factor, in Transactions on Rough Sets III *Lecture Notes in Computer Science*, Volume 3400, 2005, 202–229.

[167] R.P. Stanley, Flag-symmetric and locally rank-symmetric partially ordered sets *Electronic Journal of Combinatorics* 3 R6 (1996), 22pp.

[168] J. G. Stell, Granulation for Graphs, Sp. Inf. Th., *Lecture Notes in Computer Science*, Volume 1661, 1999, 417–432.

[169] J. G. Stell, Relations in Mathematical Morphology with Applications to Graphs and Rough Sets, Sp. Inf. Th., *Lecture Notes in Computer Science*, Volume 4736, 2007, 438–454.

[170] J. G. Stell, Relational Granularity for Hypergraphs, RSCTC, *Lecture Notes in Computer Science*, Volume 6086, 2010, 267–276.

[171] J. G. Stell, Relational on Hypergraphs, RAMiCS, *Lecture Notes in Computer Science*, Volume 7560, 2012, 326–341.

[172] J. G. Stell, Formal concept analysis over graphs and hypergraphs, GKR2013, *Lecture Notes in Computer Science*, Volume 8323, 2014, 165–179.

[173] J. G. Stell, Symmetric Heyting relation algebras with applications to hypergraphs, *Journal of Logical and Algebraic Methods in Programming*, 84 (2015), 440–455.

[174] J. Stepaniuk, Rough - Granular Computing in Knowledge Discovery and Data Mining, 2008 Springer-Verlag Berlin-Heidelberg.

[175] B. Szegedy, The Symmetry Preserving Removal Lemma, *Proceedings of the American Mathematical Society*, Volume 138, No. 2, (2010) pp. 405–408.

[176] W. Tholen, Ordered Topological Structures, *Topology and its Applications*, 156 (2009), 2148–2157.

[177] J. Tkadlec, Partially additive measures and set representations of orthoposets, *Journal of Pure and Applied Algebra*, Volume 86, Issue 1(1993), 79–94.

[178] W. T. Trotter, Combinatorics and Partially Ordered Sets: Dimension Theory. The Johns Hopkins University Press, Baltimore, Maryland, 1992.

[179] P. M. Van den Broek, Symmetry transformations in indefinite metric spaces: A generalization of Wigner's theorem. *Physica A: Statistical Mechanics and its Applications*, Volume 127, Issue 3, October 1984, 599–612.

[180] T. P. Vaughan, Families Implying the Frankl Conjecture. *European Journal of Combinatorics*, Volume 23, Issue 7, (2002), 851–860.

[181] H. Weyl; Symmetry; Princeton University Press, 1952

[182] D. J. A. Welsh; Matroid Theory. Academic Press, 1976.

[183] E. P. Wigner; Group Theory and Its Application to the Quantum Theory of Atomic Spectra. Academic Press Inc., New York, 1959.

[184] R. Wille, Subdirect product construction of concept lattices, *Discrete Mathematics*, Volume 63, Issues 2–3, 1987, 305–313.

[185] R. Wille, Concept lattices and conceptual knowledge systems, *Computers and Mathematics with Applications*, Volume 23, Issues 6-9, March-May 1992, 493–515.

[186] R. Wille, The Basic Theorem of triadic concept analysis, *Order*, Volume 12, Issues 2, 1995, 149–158.

[187] P. Wóycik, Density of Union-Closed Families, *Discrete Mathematics*, 105, (1992), 259–267.

[188] W. Z. Wu, Y. Leung, J. S. Mi, Granular Computing and Knowledge Reduction in Formal Contexts, *IEEE Trans. on Knowledge and Data Engineering*, 21, 2009, 1461–1474.

[189] W. Xiao, B. Parhami; Cayley graphs as models of deterministic small-world networks; *Inform. Process. Lett.*, Vol. 97, pp. 115–117, 2006.

[190] H. Yang, I. King, M. R. Lyu, The Generalized Dependency Degree Between Attributes, *Journal of the American Society for Information Science And Technology*, 2007 Vol. 58, No. 14, 2280–2294.

[191] Y. Y. Yao, On modeling data mining with granular computing, COMPSAC 2001. IEEE, 2001, pp. 638–643.

[192] Y. Y. Yao, Information granulation and rough set approximation, *International Journal of Intelligent Systems*, 2001 Vol. 16, No. 1, 87–104.

[193] G. Chen, N. Zhong, Y. Yao, A Hypergraph Model of Granular Computing, *The 2008 IEEE, Int. Conf. Gran. Comput.*, GrC2008, pp. 26–28.

[194] Y. Yao, Y. Zhao, Discernibility matrix simplification for constructing attribute reducts, *Information Sciences*, 179, 2009, 867–882.

[195] Y. Yao, B. X. Yao, Covering based rough set approximations, *Information Sciences*, 200, 2012, 91–107.

[196] Yao, Y.Y. The two sides of the theory of rough sets, *Knowledge-based Systems* 80, 67–77, 2015.

[197] J. T. Yao, Y. Y. Yao, A Granular Computing Approach to Machine Learning, in Proceedings of the 1st International Conference on Fuzzy Systems and Knowledge Discovery, 2002, 732–736

[198] L.A. Zadeh, Fuzzy sets and information granularity, in: *Advances in Fuzzy Set Theory and Applications*, Gupta, N., Ragade, R. and Yager, R. (Eds.), North- Holland, Amsterdam, 3–18, 1979.

[199] L. A. Zadeh, Towards a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, 19, 1997, 111–127.

[200] W. Zhu, S. Wang, Rough matroids based on relations, *Information Sciences*, 232: 241–252 (2013).