Università della Calabria
Department of Mathematics and Computer Science

Ph.D. in Mathematics and Computer Science
XXX Cycle

PH.D. THESIS

Pairings and Symmetry Notions. A New Unifying Perspective in Mathematics and Computer Science

Scientific Disciplinary Sector: MAT/03 - Geometry

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Abstract

In this thesis we introduce two types of symmetry notions, local and global, that are applicable to many different mathematical contexts. To be more specific, let Ω be a fixed set. We call a triple \( \mathfrak{P} = (U, F, \Lambda) \), where \( U \) and \( \Lambda \) are non-empty sets and \( F : U \times \Omega \to \Lambda \) is a map, a pairing on \( \Omega \). Then, for any subset \( A \subseteq \Omega \) we define a local symmetry relation on \( U \) given by \( u \equiv_A u' \) if \( F(u, a) = F(u', a) \) for any \( a \in A \), and a global symmetry relation on the power set \( \mathcal{P}(\Omega) \) given by \( A \approx_{\mathfrak{P}} A' \) if \( \equiv_A \) coincides with \( \equiv_{A'} \).

We use the very general notion of pairing to investigate the above two types of symmetry notions for a broad spectrum of mathematical theories very far apart, such as simple undirected graphs, digraphs, vector spaces endowed of bilinear forms, information tables, metric spaces and group actions.

The relation \( \approx_{\mathfrak{P}} \) induces a closure operator \( M_{\mathfrak{P}} \) on \( \Omega \) and we prove that any finite lattice is order isomorphic to the closure system induced by an appropriate operator \( M_{\mathfrak{P}} \). On the other hand, the relation \( \approx_{\mathfrak{P}} \) also induces a set operator \( C_{\mathfrak{P}} \) on \( \Omega \) whose fixed point set \( MINP(\mathfrak{P}) \) is an abstract simplicial complex, substantially dual to the closure system induced by \( M_{\mathfrak{P}} \). In this way, we obtain a closure system and an abstract simplicial complex mutually interacting by means of three set systems having relevance in both theoretical computer science and discrete mathematics. As a matter of fact, any independent set family of a matroid on \( \Omega \) can be represented as the set system of the minimal partitioners of some pairing \( \mathfrak{P} \) on \( \Omega \). Hence, we are enabled to investigate \( MINP(\mathfrak{P}) \) in relation to classical operators derived from matroid theory.

Moreover, by means of the set operator \( M_{\mathfrak{P}} \), it is possible to introduce a preorder \( \geq_{\mathfrak{P}} \) on \( \mathcal{P}(\Omega) \) by setting \( A \geq_{\mathfrak{P}} B \) if and only if \( M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(B) \). This preorder (and the associated pair family \( \mathcal{D} \subseteq \mathcal{P}(\Omega)^2 \)) satisfies the so-called union additive property according to which if \( A'' \in \mathcal{P}(\Omega) \) and \( A \geq_{\mathfrak{P}} A', A \geq_{\mathfrak{P}} A'' \) then \( A \geq_{\mathfrak{P}} A' \cup A'' \) and whose induced equivalence relation coincides exactly with \( \approx_{\mathfrak{P}} \). In particular, there exists a bijection between closure operators on \( \Omega \) and any preorder on \( \mathcal{P}(\Omega) \) having also the above property.

Actually, the preorder \( \geq_{\mathfrak{P}} \) represents a local version of a more general preorder on \( \mathcal{P}(\mathcal{P}(\Omega)^2) \), denoted by \( \sim_{\Omega} \), depending uniquely on the starting set \( \Omega \) and mutually interrelating the preorders \( \geq_{\mathfrak{P}} \), for each \( \mathfrak{P} \in PAIR(\Omega) \).

In this context, the study of the preorder \( \geq_{\mathfrak{P}} \) is relevant since it is possible to show that any finite lattice \( \mathcal{L} \) is order-isomorphic to the symmetrization of a preorder induced by \( \sim_{\Omega_{\mathcal{L}}} \) on the power set \( \mathcal{P}(\Omega_{\mathcal{L}}) \), for a suitable finite set \( \Omega_{\mathcal{L}} \) and a pairing \( \mathfrak{P} \) on it. On the other hand, we will see that the above preorder has an interpretation in terms of symmetry transmission. As a matter of fact, we define a set map \( \Gamma_{\mathfrak{P}} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U) \) and, in the finite case, an associated numerical
quantity, thanks to which it is possible to evaluate the transmission of symmetry between two subsets \( A, B \in \mathcal{P}(\Omega) \). In this way, the previous finite lattices representation theorem provides a refinement of order theory enabling us to connect order properties to topological, matroidal or set combinatorial properties. The epistemological consequence of all the aforementioned representation results is that we can consider closure system, finite lattice and matroid theories as sub-theories that are parts of the more general pairing paradigm.

In such a perspective, we relate the preorder \( \geq_{\mathcal{P}} \) with the closure system \( MAXP(\mathcal{P}) \) from an operatorial standpoint by considering the map \( DP : \mathcal{P}(\mathcal{P}(\Omega)^2) \rightarrow \mathcal{P}(\mathcal{P}(\Omega)) \) and proving that \( DP(D^+) = DP(D) \) for any \( D \subseteq \mathcal{P}(\mathcal{P}(\Omega)^2) \) and that \( DP(\mathcal{G}(\mathcal{P})) = DP(\mathcal{Q}(\mathcal{P})) = MAXP(\mathcal{P}) \), where \( \mathcal{G}(\mathcal{P}) := \{(A,B) \in \mathcal{P}(\Omega)^2 : A \geq_{\mathcal{P}} B\} \) and \( \mathcal{Q}(\mathcal{P}) := \{(X,y) : X \in MINP(\mathcal{P}), y \in M_{\mathcal{P}}(X) \setminus X\} \). Thus, it follows that the study of specific subset pair families \( \mathcal{D} \subseteq \mathcal{P}(\Omega)^2 \) is strictly related to the set systems and to the set operators characterizing pairings. This explains the need to study the main properties of the specific families satisfying the same properties as the above two previous models.

Finally, the investigation of the set map \( \Gamma_{\mathcal{P}} \) leads us to the analysis of a particular structure, namely the indistinguishability linear systems. In particular, we focus our attention on the corresponding concept of compatibility from both a local and a global standpoint. The term compatibility is here used in analogy with the study of the compatibility of classical finite system of linear equations with coefficients in any field \( K \). An indistinguishability linear system \( S \) is a very general structure \( \langle U_S, C_S, D_S, F_S, \Lambda_S \rangle \) where \( U_S \) can be seen as the equation set, \( C_S \) as the variable set, \( D_S \) consists of at most an element \( d_S \) (constant term) and \( F_S : U_S \times (C_S \cup D_S) \rightarrow \Lambda_S \) is a map that plays the role of the coefficients matrix in a classic linear system. Then, from a local perspective, the study of the compatibility in \( S \) means to fix \( W \subseteq U_S, A \subseteq C_S \) and to determine in what cases all equations \( u \in W \) satisfy a specific condition related to local symmetry. On the other hand, from a global perspective, we try to determine the conditions to be satisfied by some operators formalizing the compatibility notion.
Acknowledgements

I am extremely grateful to my supervisors Prof. Paolo A. Oliverio, Dr. Giampiero Chiaselotti and Prof. Francesco Polizzi and to my research colleague Dr. Tommaso Gentile who have accompanied me patiently and affectionately during these three years of PhD, giving me the opportunity to grow from both the human and the professional level. I am also grateful to Prof. Enrico Formenti, who supervised me during my stay at the University of Nice-Sophia Antipolis and stimulated me with his interesting and accurate remarks, suggestions and explanations. Therefore, I dedicate to them and, clearly, also to my family the efforts and the joys of this period that lead me to conclude my PhD and to write this work.
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Introduction

The study of the symmetry has always fascinated mathematicians. In his classical book [181], Hermann Weyl discusses the symmetry concept in geometry from several points of view. Nevertheless, currently, in literature there is no paradigm concerning symmetry that is sufficiently unifying and general enough to be applied to mathematical structures apparently very far apart. Therefore, it happens that in each specific research field, a characteristic notion of symmetry is given and there is no a common paradigm connecting them. The following contexts provide some of the most natural structures where specific symmetry questions can be investigated.

In graph theory, the symmetry notion is classically related to the properties of the automorphism group $\text{Aut}(G)$ of a graph $G$ (see [83]) and it is also object of deep algebraic investigations (see [103]). However, even within graph theory itself, many notions of symmetry, different from the classical one proposed in [83] have been introduced and well studied. For example, in [115, 116, 117, 131, 155] several types of symmetry notions has been investigated for Cayley graphs, Moore graphs, attachment graphs, graph algebras and directed graphs. In fact, Cayley graphs of groups play an important role in symmetric graphs. In [36, 37, 38] it have been introduced some types of homotopy relations based on the symmetry of graphs, that are strictly related with database theory. Moreover, there is also a wide literature concerning the study of symmetry breaking [2, 90, 101, 102, 133].

In hypergraph theory some questions concerning the hyperedge symmetry have been investigated in [165, 173, 175].

In a more algebraic context, the symmetry is a basic notion when one studies the symmetric group [121, 154], the symmetric functions [130], the plane symmetry groups [158], the interrelations with Hopf bifurcations [91, 92, 93] and the association schemes [11, 12].

In order theory, the symmetry notion has been studied in relation to a specific types of symmetric functions and representations of the symmetric group [160, 167] and in relation to Ramsey theory [84]. In [21, 23, 24, 35] some classes of lattices with high symmetry degree have been investigated in terms of sequential and discrete dynamical systems.

In metric space theory, some types of symmetry transformations have been related in [136, 179] to the Wigner’s theorem, that is a classical result in mathematical foundation of quantum mechanics [183]. From a more general perspective, if $X$ is any set a map $d : X \times X \rightarrow [0, \infty]$ such that $d(u, v) = 0$ if and only if $u = v$ and such that $d(u, v) = d(v, u)$ has been studied in [100, 132] exactly for its symmetry properties.

In this work we introduce the unifying notion of pairing (see [20, 53, 55, 56, 58, 63, 64, 65]) or, equivalently, function system [61, 68]), that serves as paradigm to link each other, from an interdisciplinary outlook, several different mathematical structures, such as matroids, group
actions, vector spaces endowed with bilinear forms, metric spaces, simple undirected graphs and digraphs. By means of this notion, we provide a new perspective to investigate a type of generalized symmetry in the above structures and in all other mathematical structures where it is natural to introduce an appropriate pairing.

Let us fix an arbitrary set Ω (finite or infinite). Our basic notion is given in the next definition.

**Definition 0.0.1.** A pairing on Ω is a triple \( P = (U, F, \Lambda) \), where \( U, \Lambda \) are non-empty sets and \( F : U \times \Omega \to \Lambda \) is a map having domain \( U \times \Omega \) and codomain \( \Lambda \). We denote by \( \text{PAIR}(\Omega) \) the set of all pairings on Ω. When both the sets \( U = \{u_1, \ldots, u_m\} \) and \( \Omega = \{a_1, \ldots, a_n\} \) are finite, we say that \( P \) is a finite pairing. For a finite pairing, we denote by \( T[P] \) the \( m \times n \) rectangular table having on the \( i \)-th row the element \( u_i \), on the \( j \)-th column the element \( a_j \) and \( F(u_i, a_j) \) in the place \((i, j)\). In the finite case, we identify \( P \) with \( T[P] \) and call \( T[P] \) the functional table of \( P \).

By starting from the very general notion of pairing, we can develop new refined mathematical notions without additional hypotheses. In particular, through the comparison of specific partition symmetries induced on \( U \) by the subsets of \( \Omega \), we will construct a type of symmetrization geometry on \( \mathcal{P}(\Omega) \).

In this thesis, we are going to present several new and original results obtained during our researches spread over the last three years, in relation to the aforementioned topic. In the next subsection we introduce the basic relations, operators and corresponding set systems which are naturally associated with any pairing \( \mathcal{P} \in \text{PAIR}(\Omega) \).

**Local and Global Symmetry Relations on Pairings**

Let \( \mathcal{P} = (U, F, \Lambda) \in \text{PAIR}(\Omega) \) and \( A, A' \in \mathcal{P}(\Omega) \). For any \( u, u' \in U \) we set

\[
 u \equiv_A u' : \iff F(u, a) = F(u', a), \quad \forall a \in A.
\]

It easily follows that \( \equiv_A \) is an equivalence relation on \( U \) that we call A-symmetry relation of \( \mathcal{P} \). For any \( u \in U \) we denote by \( [u]_A \) the equivalence class of \( u \) with respect to \( \equiv_A \) and we call \( [u]_A \) the A-symmetry class of \( u \). Finally, we call the set partition \( \pi_{\mathcal{P}}(A) := \{[u]_A : u \in U\} \)
induced by \( \equiv_A \) on \( U \) the A-symmetry partition of \( \mathcal{P} \).

The terminology symmetry relation stems by the fact that in graph context (i.e. when we identify a graph with its adjacency matrix considered as a pairing), the previous equivalence relation takes actually account of a local type of symmetry (see [47, 49, 52, 54, 63, 71]). Consider for example the usual Petersen graph (briefly Pet) with vertex set \( V(\text{Pet}) = \{1, \ldots, 10\} \) and let us take \( A = \{1, 6\} \). Then, it is easy to verify that there are exactly the following three symmetry blocks of \( A \) in \( \text{Pet} \): \( \{1, 8, 9\} \), \( \{2, 5, 6\} \) and \( \{3, 4, 7, 10\} \). Hence \( \pi_{\text{Pet}}(A) = 189|256|347(10) \). In fact, we can note that the vertices in each one of the previous blocks have all the same behavior with respect to the vertices 1 and 6. For example 1, 8 and 9 are all vertices of Pet adjacent only to 6; the vertices 2, 5 and 6 are all vertices adjacent only to 1 and the vertices 3, 4, 7 and 10 are all vertices non adjacent neither 1 nor 6. Finally, we note that there is no vertex
adjacent to both 1 and 6. We can see better the symmetric position of the previous blocks with respect to $A$ if we colour them with three different colours, as we did in Figure 1.

Therefore, from our perspective, symmetry with respect to $A \subseteq V(G)$ can be formally translated as the equivalence relation $\equiv_A$ and its equivalence classes can be seen as symmetry blocks relatively to $A$. In general, note that when we fix a vertex subset $A$, we implicitly assume $A$ as an observation point (or, equivalently, a reference system), with respect to which observe the behavior of all other vertices in $G$. Then, if $v, v'$ are two not adjacent vertices of $G$ such that $v \equiv_A v'$, it results (see Corollary 2.1.4) that the induced subgraph by $A \cup \{v, v'\}$ is symmetric in the sense of Erdős-Rényi [83].

Now, let us note that $\pi_G(A)$ depends on both $G$ and $A$, so we can not provide a general way to determine $\pi_G(A)$, except for some simple graph structures (for example the complete graph and the complete bipartite one). Therefore, given a finite undirected simple graph $G$, we must study (in some way to determine) the whole family of vertex partitions induced by the vertex subsets $A \in \mathcal{P}(\Omega)$. In this way we try to establish some type of parameters, depending only on the structure of $G$, which provide useful informations concerning the links among all the $A$-symmetry partitions.

We observe now that the above discussed $A$-symmetry relation can be considered a type of local symmetry on the set $U$, because it depends on the choice of the subset $A$. Then, by starting from the above $A$-symmetry relation, we introduce a type of global symmetry relation relatively to a given pairing. To this regard, we consider the following equivalence relation on $\mathcal{P}(\Omega)$:

$$A \cong_\mathcal{P} A' : \iff \pi_\mathcal{P}(A) = \pi_\mathcal{P}(A') \iff (u \equiv_A u' \iff u \equiv_{A'} u')$$

for all $u, u' \in U$. We call $\cong_\mathcal{P}$ the global symmetry relation of $\mathcal{P}$, and we denote by $[A]_{\cong_\mathcal{P}}$ the equivalence class of $A$ with respect to $\cong_\mathcal{P}$, that we call global symmetry class. When the pairing is clear in the context, we omit the subscript $\mathcal{P}$ when denoting the global symmetry relation.

Hence the global symmetry class $[A]_{\cong_\mathcal{P}}$ contains all subsets of $\Omega$ which induce the same local symmetry on $U$. Moreover, a basic property of any such class is that it is a union-closed family.
(Theorem 2.2.1, see [13, 76, 137, 152, 180, 187] for some studies on union-closed families arisen mainly from the famous Frankl’s conjecture [85]), therefore it contains a maximum element (the union of all its members), that we denote by $M_{\Psi}(A)$.

Then, $M_{\Psi}(A)$ is the largest subset of $\Omega$ which induces on $U$ the same local symmetry of $A$. Now, if we consider the set operator $M_{\Psi} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ such that $A \mapsto M_{\Psi}(A)$ it results (see Theorem 2.2.1) that $M_{\Psi}$ is a closure operator on $\Omega$. As a direct consequence, the set system

$$\text{MAXP}(\Psi) := \{ M_{\Psi}(A) : A \in \mathcal{P}(\Omega) \} = \{ C \in \mathcal{P}(\Omega) : M_{\Psi}(C) = C \}$$

is a closure system on $\Omega$. On the other hand, we also consider the set partition family given by

$$\text{PSYM}(\Psi) := \{ \pi_{\Psi}(A) : A \in \mathcal{P}(\Omega) \}.$$ 

Then, it results that the two posets $\text{M}(\Psi) := \text{MAXP}(\Psi)$ and $\text{P}(\Psi) := \text{PSYM}(\Psi)$ (where $\preceq$ is the usual partial order on set partitions) are two order isomorphic complete lattice (see (ii) of Corollary 2.2.2). We call $\text{M}(\Psi)$ the maximum partitioner lattice of $\Psi$.

Again, we also associate with the pairing $\Psi$ the family of all minimal partitioners, defined as follows:

$$\text{MINP}(\Psi) := \bigcup \{ \min([A]_{\approx_{\Psi}}) : A \in \text{MAXP}(\Psi) \}.$$ 

Hence, it results that $\text{MINP}(\Psi)$ is an abstract simplicial complex [119] (see Theorem 2.6.1).

In this thesis, we are going to establish a representation result (Theorem 6.1.1), through which we prove that for any closure operator $\sigma$ on an arbitrary finite set $\Omega$ it is possible to construct a pairing $\Psi \in \text{PAIR}(\Omega)$ such that $\sigma = M_{\Psi}$. By means of the aforementioned theorem and of a classical result of matroid theory (Theorem 1.6.5), we deduce that any independent set family of a matroid on $\Omega$ can be represented as the set system of the minimal partitioners of some pairing $\Psi$ on $\Omega$.

Therefore, by virtue of what we said before, we can study by means rectangular tables (in the finite case) or pairings (in the general case) both closure systems and matroids through the study of the local and global symmetry. In other terms, we can assert that we are going to develop a symmetrization geometry on pairings through the study of a closure system and an abstract simplicial complex, that are naturally associated with each pairing. In the remaining part of the introduction, we denote by $\Psi$ a fixed pairing in $\text{PAIR}(\Omega)$.

### The Global Symmetry Lattice

The deep link between maximum partitioners and minimal ones is provided by the following structure. Consider the poset $(\text{GSYM}(\Psi), \subseteq)$, where

$$\text{GSYM}(\Psi) := \{ [A]_{\approx_{\Psi}} : A \in \text{MAXP}(\Psi) \}$$

is endowed with the partial order

$$[A]_{\approx_{\Psi}} \subseteq [B]_{\approx_{\Psi}} : \iff A \subseteq^* B.$$ 

It can be shown (Theorem 2.2.8) that $\text{G}(\Psi)$ is a complete lattice order isomorphic to $\text{M}(\Psi)$, that we call global symmetry lattice of $\Psi$. 

x
Let us note that each global symmetry class contains all the subsets of \( \Omega \) inducing the same symmetry partition, and these subsets have a natural poset structure with respect to the usual set inclusion. Therefore these posets can be studied with the usual techniques derived from the order theory. Obviously, a relevant question is to understand if, and how, the inner order structure of each class \([A]_{\approx_p}\) interacts with the global order structure of the global symmetry lattice \(G(\mathfrak{P})\).

To this regard, we provide the following three type of results as a consequence of our study.

1. results concerning the properties of the global symmetry classes.
2. results concerning the global symmetry lattice.
3. results that concern the possible interactions between the inner structure of any global symmetry class and the maximum partitioner lattice.

A basic property relative to (3) is the global-local regularity, which can be expressed as follows

\[(GLR) \; M_p(A) \subsetneq M_p(A') \implies Y \nsubseteq X \text{ for any } X \in [A]_{\approx_p} \text{ and } Y \in [A']_{\approx_p}.\]

The global-local regularity property tells us that the inclusion between the maximum elements of any two global symmetry classes preserves the same type of inclusion between any two members of these classes.

**Example 0.0.2.** Let us consider the pairing \(\mathfrak{P}\) whose functional table is given in Figure 2. In Figure 3 we represent the maximum partitioner lattice of \(\mathfrak{P}\). In Figure 4 we represent the global symmetry lattice of \(\mathfrak{P}\).

### Symmetry Set Systems Related to the Global Symmetry Lattice

The subset \(M_p(A)\) is the largest subset of \(\Omega\) that induces on \(U\) the same local symmetry of \(A\). On the other hand, given \(A \in \mathcal{P}(\Omega)\), if one studies the \(A\)-symmetry in relation to the subsets of \(A\) there are two natural types of subsets that we can consider.

(A) The subsets \(C\) of \(A\) providing the same symmetry of \(A\) (i.e. \(\pi_p(C) = \pi_p(A)\)) and which are also \(\subseteq\)-minimals with respect to this property. Formally, we have the set system

\[BAS_p(A) := \{C \in \mathcal{P}(A) : \pi_p(A) = \pi_p(B) \text{ and } \pi_p(A) \neq \pi_p(B') \forall B' \subsetneq B\},\]
and call any member of $\text{BAS}_{\Psi}(A)$ an $A$-symmetry base of $\Psi$.

(B) The subsets $C$ of $A$ which are ”essential” to provide a maximum level of symmetry when we partition $A$ (that is $\pi_{\Psi}(A \setminus C) \neq \pi_{\Psi}(A)$) and that are $\subseteq$-minimal with respect to this property.

In such a case we have the set system

$$ESS_{\Psi}(A) := \{ C \in \mathcal{P}(A) : \pi_{\Psi}(A \setminus C) \neq \pi_{\Psi}(A) \text{ and } \pi_{\Psi}(A \setminus C') = \pi_{\Psi}(A) \forall C' \subseteq C \},$$

and call any member of $ESS_{\Psi}(A)$ an $A$-symmetry essential of $\Psi$.

In [54], the basic properties of the previous set systems in relation to simple undirected graphs have been studied in relation to the notion of local dissymmetry. To this regard, we set

$$DIS(G) := \{ \Delta_G(v_i, v_j) : v_i, v_j \in V(G) \},$$

where $\Delta_G(v_i, v_j) := N_G(v_i) \Delta N_G(v_j)$ is the set symmetric difference between the adjacency neighborhoods of $v_i$ and $v_j$. We call the vertex subset family $DIS(G)$ and the subset $\Delta_G(v_i, v_j)$ respectively the local dissymmetry set system of $G$ and the dissymmetry neighborhood of $v_i$ and $v_j$. In fact, $\Delta_G(v_i, v_j)$ contains those vertices which cut off a potential 2-path between $v_i$ and $v_j$ (another kind of symmetric difference among neighborhoods of vertices has been introduced recently in [131] in order to study some types of symmetries in random graphs).

In this case, we can put within a table $\Delta[G]$ the members of the dissymmetry set system, i.e. if we denote by $N_G(v)$ the open neighborhood of all adjacent vertices to $v \in V(G)$, then in the place corresponding at the intersection between the $v_i$-row and the $v_j$-column there is the set symmetric difference $\Delta_G(v_i, v_j)$. Now, because $G$ has no loops and multiple edges, it is immediate to see that $G$ is uniquely characterized from this table. In other terms, if $G$ and $G'$ be two graphs on the same vertex set. Then

$$\Delta[G] = \Delta[G'] \iff E(G) = E(G'),$$

Figure 3: The Maximum Partitioner Lattice $M(\Psi)$ of Example 0.0.2
Figure 4: Global Symmetry Lattice $\mathbb{G}(\mathcal{P})$ of Example 0.0.2.
where $E(G)$ denotes the edge set of $G$ or, equivalently, two graphs $G$ and $G'$ coincide if and only if $\Delta_G(v_i, v_j) = \Delta_G(v_i, v_j)$ for all $v_i, v_j \in V(G)$. An immediate and intuitive interpretation for the previous identities is that if $v_i \neq v_j$ then $v \in \Delta_G(v_i, v_j)$ if and only if $v$ is adjacent to exactly one vertex between $v_i$ and $v_j$, in other terms we can think any vertex in $\Delta_G(v_i, v_j)$ as a local dissymmetry vertex between $v_i$ and $v_j$. Therefore, it is natural to think the entries of $\Delta[G]$ as local dissymmetry neighborhoods of $G$ and to call $G$ a locally dissymmetric graph when it happens that $\Delta_G(v_i, v_j) \neq \emptyset$ for all $v_i \neq v_j$. In particular, in [59] it has been provided an interpretation of local dissymmetry in graph context within the so-called formal context analysis (briefly FCA) [16, 27, 87, 88, 94, 184, 185, 186] (for the relation between graphs and FCA, see also [7, 8, 9, 10, 16]).

However, let us note that the condition $DIS(G) = DIS(G')$ is weaker than $\Delta[G] = \Delta[G']$, in fact there are non isomorphic simple graphs $G$ and $G'$ having the same local dissymmetry set system, for example the following:

```
   1  4
  2  3
and
   1  4
  2  3
```

The above example contains in itself the following potential research development: to classify the simple graphs by means of the local dissimmetry, that is with respect to the condition $DIS(G) = DIS(G')$.

Moreover, in [71], it has been studied a generalization of the local dissymmetry set system, in fact it has been analyzed the behavior of the symmetric differences of vertex subsets. Again in [71], the local symmetry relation $\equiv_A$ has been studied in its interrelations with a specific type of binary operation $\circ$ defined on the power set $\mathcal{P}(V(G))$ and whose automorphism group is isomorphic to a subgroup of $\text{Aut}(G)$ (for other works on similar topics see also [25, 26, 103, 153]). In Section 4.2, we will investigate the main algebraic properties of the operation $\circ$ and see that it satisfies many group-like properties, even if it is not associative (see Proposition 4.2.3 and Theorems 4.2.18, 4.2.21 and 4.2.26).

Again in [54], it has been proved that $BAS(G)$ is the minimal transversal set system of $DIS(G)$, whereas $ESS(G)$ is the set system of all minimal (with respect to the usual inclusion) subsets of $DIS(G)$. The problem of compute the symmetry base set system is thus equivalent to that of finding the minimal transversal of a given hypergraph [80, 81, 82, 97, 99, 118] and it is a very difficult task that has many applications in mathematics and computer science [80, 81]. In Subsections 3.2.1 and 3.2.2, we treat the aforementioned problem for two graph models, namely the Petersen graph, for which we completely classified the subgraphs corresponding to the elements of $BAS(Pet)$, and the cycle on $n$ vertices, for whose symmetry bases we provide an algebraic characterization (see Theorem 3.2.27).

We can generalize the local dissymmetry for graphs introducing the notion of dissymmetry...
in the more general pairing context as follows. For any $A \in \mathcal{P}(\Omega)$ we consider the map $\Delta^A : U \times U \to \mathcal{P}(\Omega)$ defined by

$$\Delta^A(u, u') := \{a \in A : F(u, a) \neq F(u', a)\},$$

for any $u, u' \in U$, and the following set system on $\Omega$:

$$DIS^A := \{\Delta^A(u, u') : u, u' \in U \text{ and } \Delta^A(u, u') \neq \emptyset\}.$$

We call $DIS^A$ the $A$-dissymmetry set system of $\mathfrak{A}$. Then, the map $\Delta := \Delta^\Omega$ satisfies the following three basic properties:

1. $\Delta(u, u') = \emptyset$ for each $u \in U$;
2. $\Delta(u, u') = \Delta(u', u)$ for all $u, u' \in U$;
3. $\Delta(u, u') = \bigcap_{u'' \in U} [\Delta(u, u'') \cup \Delta(u', u'')]$.

On the other hand, we prove (see Theorem 3.1.7) that if any map $D : U \times U \to \mathcal{P}(\Omega)$ satisfies the above properties, then there exists a pairing $\mathfrak{A} \in \text{PAIR}(\Omega)$ such that $\Delta = D$. Therefore we can say that the properties (i), (ii) and (iii) can be considered as abstract axioms to define an abstract notion of dissymmetry space on $U$. In other terms, we call dissymmetry space (see [55]) on $\Omega$ a pair $\mathfrak{D} := \langle U, D \rangle$, where $U$ is a non-empty set and $D : U \times U \to \mathcal{P}(\Omega)$ is a map which satisfies (i), (ii) and (iii).

Theorem 3.1.7 is the first of a series of representation theorems by pairings that are aimed to unify in a broader and general perspective several different theories, such as dissymmetry spaces, finite lattices, closure systems and matroids.

Now, given a pairing $\mathfrak{A}$, for any $A \in \mathcal{P}(\Omega)$, based on the computation of the $A$-dissymmetry set system of $\mathfrak{A}$, it is possible also determine the $A$-symmetry bases and the $A$-symmetry essentials. In fact, as in the graph case [54], we can prove (Theorems 3.2.13 and 3.2.15) respectively that the $A$-symmetry essentials are the minimal elements of the set system $DIS^A$ and that the $A$-symmetry bases are the minimal transversal of the same set system. Finally, in our treatment, a relevant role will be played by the subset of $A$ given by

$$C^A := \{a \in A : \pi(A) \neq \pi(A \setminus \{a\})\},$$

that we call core of $A$. This subset can be identified with all singletons of the set system $ESS^A$ and, it determines a set operator $C^A$ on $\Omega$ that we can substantially to consider a type of dual version of the set operator $M^A$ (see Proposition 3.2.8 and Theorem 3.2.9). Moreover, we also prove (see Theorem 3.2.9) that the set system $MIN^A$ is exactly the fixed point set of $C^A$.

**Comparison Between A-Symmetry Relations**

Relatively to the symmetry induced on $U$ by the subsets of $\Omega$, we want to specificate how these subsets are mutually related. In particular, we are going to compare the various symmetry partitions on $U$ and, in the finite case, to determine a measure of transmission of symmetry. As a matter of fact, we consider the map $\Gamma : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U)$ so defined:

$$\Gamma(A, B) := \{u \in U : [u]_A \subseteq [u]_B\}.$$
Furthermore, in the finite case, we can also define the *symmetry transmission measure* $\gamma_P$ as follows

$$
\gamma_P(A, B) := \frac{|\Gamma_P(A, B)|}{|U|}.
$$

Let us note (see [20, 53] for details) that

$$
M_P(A) \supseteq M_P(A') \iff \pi_P(A) \preceq \pi_P(A') \iff \Gamma_P(A, A') = U,
$$

for all $A, A' \in \mathcal{P}(\Omega)$.

Then, in an operatorial perspective, which is based on the above three equivalences established, we introduce the binary relation $\geq_P$ on $\mathcal{P}(\Omega)$ defined by

$$
A \geq_P A' : \iff M_P(A) \supseteq M_P(A'),
$$

for any $A, A' \in \mathcal{P}(\Omega)$. We say that $A$ is $\Psi$-symmetrically finer than $A'$ whenever $A \geq_P A'$. We can restate the notion of $\Psi$-symmetrically finer by noting that $A \geq_P B \iff \Gamma_P(A, B) = U$. Since

$$
A \approx_P A' \iff A \geq_P A' \text{ and } A' \geq_P A,
$$

we can interpret the global symmetry relation $\approx_P$ as the equivalence relation on $\mathcal{P}(\Omega)$ induced by the preorder $\geq_P$.

The preorder $\geq_P$ has also the following property of union additivity: if $A'' \in \mathcal{P}(\Omega)$ and $A \geq_P A', A \geq_P A''$ then $A \geq_P A' \cup A''$. Then, it is easy to see that any preorder on $\mathcal{P}(\Omega)$ having also the above union additive property (thus named *union additive relation* on $\Omega$, and *union additive family* its corresponding version as subset family) is uniquely associated with a closure operator on $\Omega$ and vice versa. Hence the study of specific closure operators on $\Omega$ is equivalent to the study of corresponding types of union additive families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$. Then, we will show (Theorem 5.2.8) that there is a bijective correspondence between closure operators on $\Omega$ and union additive relations on $\mathcal{P}(\Omega)$. Moreover, with respect to this correspondence, the set operator $M_P$ can be identified with the union additive relation $\geq_P$. Let us note that the study of the preorder $\geq_P$ can also be seen as the abstract mathematical study of the *functional dependency relation* (for details see [161]) induced by any concrete model of data table associated with a pairing $\Psi$ on the set $\Omega$ and that has recently been analyzed in relation to closure systems, abstract simplicial complexes [68] and to the corresponding convex geometries [111, 112, 113, 114].

Based on the previous considerations, given any family $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$, it is natural to ask how to characterize the smallest union additive family, denoted by $\mathcal{D}^+$, that contains $\mathcal{D}$ as a subfamily. To this regard, we provide three distinct recursive way to build $\mathcal{D}^+$ by starting from the elements of $\mathcal{D}$ (Theorem 5.3.1). Next, we address the links between order properties and subset structural properties from a perspective similar to that of model theory. In other terms, if $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$, we say that a pairing $\Psi \in PAIR(\Omega)$ is a *model* of $\mathcal{D}$, denoted by $\Psi \models \mathcal{D}$, if $X \geq_P Y$ for any ordered pair $(X, Y) \in \mathcal{D}$. At this point, we introduce a preorder on $\mathcal{P}(\mathcal{P}(\Omega)^2)$: if $\mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$ we say that the ordered pair $(\mathcal{D}, \mathcal{D}')$ is a *sequentiality* (relatively to the given set $\Omega$, [66]), denoted by $\mathcal{D} \prec_{\Omega} \mathcal{D}'$, if for any pairing $\Psi \in PAIR(\Omega)$ such that $X \geq_P Y$ for all pairs $(X, Y) \in \mathcal{D}$ we also have that $X' \geq_P Y'$ for all pairs $(X', Y') \in \mathcal{D}'$. In this way we obtain a preorder...
between subset ordered pairs families of $\Omega$ that can be considered a global version of the preorder $\geq_P$ on $\mathcal{P}(\Omega)$. In fact, the preorder $\geq_P$ is related to the subsets of the starting set $\Omega$, but it can be considered a type of local preorder because it depends on the choice of the particular model $\Psi$. On the other hand, the preorder $\sim_\Omega$ depends only on the starting set $\Omega$, therefore we can consider the study of $\sim_\Omega$ as an abstract global order study depending on the mutual interrelations between the preorders $\geq_P$, for each $\Psi \in \operatorname{PAIR}(\Omega)$.

In this way, we link order theory with specific subset properties (for example the above discussed union additivity). In such a perspective, we relate the preorder $\geq_P$ with the closure system $\operatorname{MAXP}(\Psi)$ from an operatorial standpoint as follows. We consider the map $DP : \mathcal{P}(\Omega)^2 \to \mathcal{P}(\mathcal{P}(\Omega))$ defined by

$$DP(D) := \{ Z \in \mathcal{P}(\Omega) : X \subseteq Z \implies Y \subseteq Z \ \forall (X,Y) \in D \}$$

and we show that $DP(D^+) = DP(D)$ for any $D \subseteq \mathcal{P}(\mathcal{P}(\Omega)^2)$ (Theorem 5.5.1) and that $DP(\mathcal{G}(\Psi)) = \operatorname{MAXP}(\Psi)$ (Theorem 5.5.2), where

$$\mathcal{G}(\Psi) := \{(A,B) \in \mathcal{P}(\Omega)^2 : A \geq_\Psi B \}.$$

Therefore it is natural to investigate the closure system $\operatorname{MAXP}(\Psi)$ and its associated abstract complex $\operatorname{MINP}(\Psi)$ from the perspective of the union additive relations.

Again by Theorem 5.5.2, it results that $DP(\mathcal{Q}(\Psi)) = \operatorname{MAXP}(\Psi)$, where

$$\mathcal{Q}(\Psi) := \{(X,y) : X \in \operatorname{MINP}(\Psi), y \in M_\Psi(X) \setminus X\}.$$

In the pairing context, starting from the identities $DP(\mathcal{G}(\Psi)) = DP(\mathcal{Q}(\Psi)) = \operatorname{MAXP}(\Psi)$, it can be seen that the study of specific subset pair families $D$ is strictly related to the set systems and to the set operators characterizing pairings. Therefore, in this thesis, we feel the need to include as an additional part the investigation of the specific properties of such families, namely the pointed pair system on $\Omega$ [67], that is a family of ordered subset pairs whose second component is a singleton. In particular, in Subsections 5.6 and 5.7 we see that some classical notions that are usually investigated for several classes of set systems on $\Omega$, such as, for instance, extensiveness and finitely inheritance, can be translated in the context of pointed pair systems and, furthermore, several links between specific classes of pointed pair systems, finitary abstract complexes (an example in pairing context has been given on the euclidean line, see Theorem 2.7.5) and weak types of MacLane-Steinitz set operators have been found.

**Representation Theorems: Closure Systems, Finite Lattices and Matroids as Pairings**

In this thesis we establish a representation result (Theorem 6.1.1), according to which it is possible to associate with any closure operator $\sigma$ on an arbitrary finite set $\Omega$ a pairing $\Psi \in \operatorname{PAIR}(\Omega)$ such that $\sigma = M_\Psi$. In other terms, for any closure operator $\sigma$ on a finite set $\Omega$ we can find a pairing $\Psi \in \operatorname{PAIR}(\Omega)$ such that $\gamma_\Psi$ uniquely characterize $\sigma$ and vice versa. This equivalence implies that the symmetry transmission measure for pairings can provide to closure operator theory all possible concrete models of investigation. Then, as a consequence of such
a result, we deduce some relevant facts concerning union additive relations, finite lattices and matroid theory. Firstly, we also show (Theorem 6.1.4) that any union additive relation on $\mathcal{P}(\Omega)$, when $\Omega$ is finite, is of the type $\geq_{\mathfrak{P}}$, for some pairing $\mathfrak{P} \in PAIR(\Omega)$. Hence, the study of a union additive relation becomes an abstract mathematical study of the functional dependency relation induced by any concrete model of data table associated with a pairing $\mathfrak{P}$ on $\Omega$.

Secondly, in Theorem 6.1.5 and Theorem 6.1.7, we show that the partial order $\leq_{L}$ of any finite lattice $L$ can be described in terms of the symmetry transmission measure, for an appropriate finite set $\Omega_L$ and a pairing $\mathfrak{P} \in PAIR(\Omega_L)$. Moreover, the function $\gamma_{\mathfrak{P}}$ provides more information with respect to the partial order $\leq_{L}$, because $\gamma_{\mathfrak{P}}$ can be computed on all pairs of $\mathcal{P}(\Omega_L) \times \mathcal{P}(\Omega_L)$ that are in bijective correspondence with the pairs in $L \times L$, whereas the partial order $\leq_{L}$ provides information only for the comparable pairs of $L \times L$. Hence the order relation of any finite lattice can be studied as a quotient relation of the preorder $\geq_{\mathfrak{P}}$ induced by some appropriate pairing. The theoretical consequence of this fact is that the whole finite lattice theory can be developed in a pairing perspective. To this regard, the real refinement that we can carry out in finite lattice theory by means of the study of the preorder $\geq_{\mathfrak{P}}$ is that we can connect order properties with set family properties. For example, the above described union additive property is explicitly a type of condition that naturally connects order theory with topological, matroidal or set combinatorial properties. More in general, it is natural to ask in which way a joint study of order properties and subset structural properties induced by pairings interact each other. We recall that there is a wide literature dealing with studies on the links between order theory, topological structures [95, 96, 138, 176], combinatorial and algebraic structures on subset families [1, 6, 78, 135, 177] and discrete dynamical systems [23, 24, 35].

Another consequence of Theorem 6.1.1 is that any independent set family of a matroid on $\Omega$ can be represented as the set system of the minimal partitioners of some pairing $\mathfrak{P}$ on $\Omega$. Hence, we are enabled to investigate $MINP(\mathfrak{P})$ in relation to classical operators derived from matroid theory (see Theorems 7.0.14 and 7.0.16). To this regard, we relate $MINP(\mathfrak{P})$, the set operator $M_{\mathfrak{P}}$ and the classical rank-symmetry operator of a matroid. Finally (Theorem 7.0.18) we establish two equivalent conditions so that the rank-symmetry operator associated with $MINP(\mathfrak{P})$ coincides with $M_{\mathfrak{P}}$.

The epistemological consequence of the aforementioned results is that we can consider closure system, finite lattice and matroid theories as sub-theories that are parts of the more general pairing paradigm.

Applications in Computer Science

The notion of pairing generalizes the classical notion of Pawlak’s information table [142, 144, 145, 146].

**Definition 0.0.3.** An information table is a table whose rows are labeled with the elements (called objects) of a finite set $U$ and whose columns are labeled with the elements (called attributes) of a finite set $Att$ and having at the entry $(i,j)$ the value $F(u_i,a_j) \in Val$, where $F : U \times Att \rightarrow Val$ is the so-called information map and $Val$ the set of values.
Remark 0.0.4. Our notion of pairing is actually a generalization of Pawlak’s information table for two reasons. First of all, we can consider both \( U \) and \( \Omega \) even infinite; on the other hand, Pawlak’s notion of information table implicitly assumes a distinction between the nature of the objects and that of the attributes. In our case (see for example the pairing arising in graph context), we do not make this distinction.

Information tables arise from the necessity to study knowledge representation and data extraction. As a matter of fact, in database theory there is a very frequent need to study finite tables having a very large quantity of data, therefore many researches have been directed towards the purpose of reducing and simplifying the interpretation of these data. With such an aim, Pawlak [140, 141, 142, 144, 145, 146] developed the so called rough set theory (abbreviated RST). RST is an elegant and powerful methodology in extracting and minimizing rules from data tables. In 1979 the concept of information granularity was introduced by Zadeh [198] and it was related to the research on fuzzy sets. Next, the term granular computing (briefly GrC) was introduced again by Zadeh in 1997 (see [199]). Roughly speaking, information granules are collections of entities arranged together due to their similarity, functional or physical adjacency, indistinguishability, and so on. Since 1979, granular computing has become a very developed area of research in the scope of both applied and theoretical information science [148, 174]. GrC deals with representing and processing information in the form of some type of aggregates. These aggregates are generally called information granules or simply granules and they arise in the process of data abstraction and knowledge derivation from data. From a methodological perspective, GrC can be considered as an important attempt to investigate several research fields by means of the unifying granularity paradigm: rough set theory [17, 30, 74, 127, 128, 129, 144, 145, 146, 147, 148, 149, 151, 174, 190, 192, 194, 196] and its generalizations [28, 29, 31, 164, 166, 195], mathematical morphology [169], interval analysis [120], temporal dynamics [23, 24, 32, 33, 35, 73, 74], machine learning [197], preclusivity spaces [28], formal concept analysis [110, 172, 188], database theory [104, 105, 139], data mining [106, 125, 126, 191], fuzzy set theory [109, 150, 159, 199], interactive computing [162, 163], matroid theory [107, 108, 122, 123, 124, 200], hypergraph theory [34, 170, 171, 193], graph theory [39, 46, 47, 55, 156, 168], operative research [98], discrete dynamical systems [3, 4, 5, 21, 22, 40, 42, 43, 41, 44, 45, 143, 157].

The notion of \( A \)-symmetry represents the so-called indiscernibility relation [142]. The terminology indiscernibility relation derives by the classical Leibniz’ indiscernibility principle, according to which two objects are indiscernible with respect to a given knowledge provided by a subset of attributes if and only if they share the same properties on these attributes. Formalizing, we obtain the following equivalence relation: let \( A \in \mathcal{P}(\text{Att}) \) and \( u, u' \in U \). Then

\[
u \equiv_A u' : \iff F(u, a) = F(u', a) \quad \forall a \in A.
\]

### Indistinguishability Linear Systems

In the last part of this thesis, we deal with symmetry approximations. Suppose we have a fixed set \( Y \subseteq U \). We ask whether it is constituted of symmetry blocks with respect to a given
A ∈ P(Ω) or it also contains part of a block. In the first case, we say that Y is A-symmetry exact, while in the second case, we can approximate it through two sets, namely the lower symmetry approximation and upper symmetry approximation. The lower symmetry approximation of a non-empty Y (briefly 1_A(Y)) represents the set of elements for which it can be said that they surely belong to Y in relation to the symmetry provided by A; the upper symmetry approximation of an element subset Y (briefly u_A(Y)) for which it can be said that they surely or possibly belonging to Y, even in relation to the symmetry provided by A. Furthermore, it can be seen that Y is A-symmetry exact if 1_A(Y) = u_A(Y).

In particular, in Subsections 8.2.1 and 8.2.2 we compute A-lower and A-upper approximations for some basic graphs and digraphs families [47, 49, 54, 55, 57, 60, 70, 72] and recall that the complete determination of A-lower and A-upper approximations represents a difficult task, even if the graphic structures at issue are very simple.

Some natural links of RST with both graph and hypergraph theory have recently been founded. In fact, in [47, 48, 49, 50, 51] the idea to study any simple undirected graph G as if it were a pairing was developed (in [34] this idea has been also extended to hypergraph theory). The basic tool to connect graphs and Boolean pairings is the adjacency matrix of G, which in [47, 48, 49, 50, 51] has been interpreted as the Boolean table of a particular information system. In particular, it can be shown that any Boolean pairing is induced from a bipartite graph [56].

Nevertheless, for finite sets it has been shown that the A-symmetry exactness of a subset X ⊆ U is related to the value of the symmetry transmission measure γ(A, {d^X}), where d^X is a special element depending on A.

The choice of a special element allows us to introduce a new class of structures, namely the indistinguishability linear systems [62], that can be seen as a generalization of the so-called decision tables [69, 134, 144, 145, 146].

We call a structure S = ⟨U_S, C_S, D_S, F_S, Λ_S⟩, where U_S, C_S, Λ_S are non-empty sets, D_S is empty or consists of a single element d_S such that d_S /∈ C_S. F_S is a map having domain U_S × (C_S ∪ D_S) and codomain Λ_S an indistinguishability linear system.

In the last part of this work, we deal with the investigation of the main properties of indistinguishability linear systems. Indeed, we will study from an operatorial standpoint the notion of compatibility for indistinguishability linear systems.

Let us consider a system of linear equations S with coefficients in some field K_S. Let U_S := {u_1, ..., u_m} be the set of the m linear equations u_1, ..., u_m and C_S = {x_1, ..., x_n} the set of the n unknowns x_1, ..., x_n of S. Let Q_S = (q_ij) the coefficient matrix and d_S = (d_1, ..., d_m)^T the column of the constant terms of S. If W ⊆ U_S and A ⊆ Ω_S, we denote by S_W,A the sub-system of S determined by the equations contained in W, with their corresponding constant terms, and by the unknowns contained in A. Then, it is well known that the compatibility of S do not imply the compatibility of S_W,A. In order to know the compatibility of any single sub-system S_W,A, it is sufficient to find the rank of the matrix and of the complete matrix associated with this sub-system, but what can we say about the interrelations between the compatibilities among all possible local sub-systems of S? In Subsection 8.4, we reformulate the above problem in a combinatorial context. We say that S_W,A is compatible if there exists a map f : K^p → K such that F_S(u, d_S) = f(F_S(u, x_{i_1}), ..., F_S(u, x_{i_p})) for any u ∈ W.
Hence, we propose a detailed investigation of the previous condition as a natural study of the notion of *local compatibility* for an indistinguishability linear system. We will see that the previous condition is logically equivalent to another condition relating an appropriate family of partitions of $U_\Sigma$ induced by the subsets of $C_\Sigma$ to the family of partitions induced by $d_\Sigma$. Since set partitions are essential for compatibility, we can interpret it as a kind of *partition compatibility*. In fact, the aforementioned partitions are induced by the $A$-symmetry partition on the equation set $U_\Sigma$. Then, it is easy to verify that the above condition is equivalent to the following inclusion:

$$[u]_A \cap W \subseteq [u]_{D_\Sigma} \cap W$$

for any $u \in W$. Therefore the investigation of the compatibility condition for $S_{W,A}$ ultimately becomes an analysis of the combinatorial properties of the partitions $\pi_\Sigma(A)$ of $U_\Sigma$ induced by $\equiv A$.

The study of compatibility has been undertaken by defining some operators that we call *compatibility operators* (see Subsection 8.5). In particular, given $W \in \mathcal{P}(U_\Sigma)$ and $A \in \mathcal{P}(C_\Sigma)$, we set $\phi_A(X) = \Theta_\Sigma(X, A)$, where

$$\Theta_\Sigma(X, Y) := \{v \in X : [v]_Y \cap X \subseteq [v]_{D_\Sigma} \cap X\}.$$

Then, if we fix a variable set $A$, it follows that $\phi_A(U_\Sigma) = U_\Sigma$, that is $A$ gives rise to compatibility on $U_\Sigma$ if and only if $\phi_A$ is a *kernel operator* [86]. In this way, we have that specific technical properties of the set operator at issue assumes a relevant interpretative role.
Chapter 1

Preliminaries

In this chapter we introduce notations we will use within the text and show some basic results.

1.1 Notations

In what follows, $\Omega$ will be an arbitrary (even infinite) set and $\mathcal{P}(\Omega)$ denotes its power set. We also set $SS(\Omega) := \mathcal{P}(\mathcal{P}(\Omega))$, $S(\Omega) := (SS(\Omega), \subseteq)$. If we want to specify that $Y$ is a finite subset of $\Omega$, we will write $Y \subseteq f_{\Omega}$; moreover if $n$ is a non-negative integer, we write $Y \subseteq \leq n_{\Omega}$ if $Y \subseteq \Omega$ and $|Y| \leq n$. If $|\Omega| = n$, we set $\Omega_n := \Omega$. If $k$ is a positive integer, $Y \subseteq \Omega$ and $|Y| = k$, we say that $Y$ is a $k$-subset of $\Omega$. We write $\hat{n}$ to denote the set $\{1, \ldots, n\}$. We will denote by $I_n$ the $n \times n$ identity matrix and by $J_n$ the $n \times n$ matrix having 1 in all its entries. If $X$ and $\Omega$ are two sets and we have two maps $\psi : X \rightarrow X$ and $f : X \rightarrow \mathcal{P}(\Omega)$ such that $f(\psi(x))) = \Omega \setminus f(x)$ for any $x \in X$, we say that $f$ is a $\psi$-complementary map.

1.2 Posets and Lattices

A partially ordered set (abbreviated poset) is a pair $P = (\Omega, \leq)$, where $\Omega$ is a set and $\leq$ is a binary, reflexive, antisymmetric and transitive relation on $\Omega$. If $P = (\Omega, \leq)$ is a partially ordered set and $x, y \in \Omega$, we also write $x < y$ if $x \leq y$ and $x \neq y$. If $x, y$ are two distinct elements of $\Omega$, we say that $y$ covers $x$, denoted by $x \downarrow y$ if $x \leq y$ and there exists no element $z \in \Omega$ such that $x \leq z \leq y$. Moreover, we set $[x|P \uparrow] := \{z \in \Omega : x \downarrow z\}$ and call covers of $x$ all its elements. Analogously, we set $[x|P \downarrow] := \{y \in \Omega : y \downarrow x\}$ and call co-covers of $x$ its elements. We also set $\mathcal{L}_\downarrow(P) = \{x \in P : ||x|P \downarrow|| \leq 1\}$ and $\mathcal{L}_\uparrow(P) = \{x \in P : ||x|P \uparrow|| \leq 1\}$. Through the covering relation, it is possible to represent the so called Hasse diagram of $P$ (see [18]): draw a small circle for any element of $P$ and a segment connecting $x$ to $y$ whenever $x$ covers $y$. An element $x \in \Omega$ is called minimal in $P$ if $z \leq x$ implies $z = x$, and in a similar way one defines a maximal element in $P$. If there is an element $\hat{0}_\Omega \in \Omega$ such that $\hat{0}_\Omega_\Omega \leq x$ then $\hat{0}_\Omega$ is unique and it is called the minimum of $P$. Analogously the maximum of $P$, usually denoted by $\hat{1}_\Omega$ is defined (if it exists). We call upper bound of a subset $\Omega$ of $P$ an element $y \in P$ such that $x \leq y$ for any $x \in \Omega$. We call least upper bound the minimum of all the upper bounds. The notions of lower bound and greatest lower bound are dual. We call lattice a poset any two
of whose elements has both the least upper bound that the greatest lower bound. A lattice is complete when each of its subsets $\Omega$ has a least upper bound and greatest lower bound in the lattice (however a finite lattice is always complete). We say that a subset $I \subseteq \Omega$ is an ideal of the lattice $P = (\Omega, \leq)$ if the following conditions hold:

(a) $I \neq \emptyset$;
(b) for any $x \in I$ and $y \leq x$, it follows that $y \in I$;
(c) for any $x, y \in I$ there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

A chain $C$ of $P$ is a subset $C \subseteq \Omega$ such that for all $x, y \in C$ we have $x \leq y$ or $y \leq x$.

If a chain $C$ has $n + 1$ elements $x_0, \ldots, x_n$ such that $x_0 < \cdots < x_n$, it is said that $C$ is an $n + 1$-chain and has length $n$; in this case often we write $C = \{x_0, \ldots, x_n\}$. If $\Omega$ itself is a chain the poset $P$ is said linearly ordered. If $P$ is not linearly ordered, a maximal chain of $P$ is a chain $C$ of $P$ which is not properly contained in any other chain of $P$.

A map $f : \mathbb{L} \to \mathbb{L}$ is said:

- order-preserving if $x \leq y$ implies $f(x) \leq f(y)$;
- involution if $f(f(x)) = x$.

Let $\mathbb{L}$ be a lattice endowed with an order-preserving involution $\psi : \mathbb{L} \to \mathbb{L}$ such that $\psi(\alpha) \land \alpha = \hat{0}_\mathbb{L}$ and $\psi(\alpha) \lor \alpha = \hat{1}_\mathbb{L}$. In this case we say that the pair $(\mathbb{L}, \psi)$ is a complementary involution lattice.

Two posets $P = (\Omega_1, \leq_1)$ and $P_2 = (\Omega_2, \leq_2)$ are said isomorphic if there exists a bijective map $\phi : \Omega_1 \to \Omega_2$ such that $x \leq_1 y \iff \phi(x) \leq_2 \phi(y)$, for all $x, y \in \Omega_1$. The dual poset of $P$ is the poset $P^* := (\Omega, \leq^*)$, where $\leq^*$ is the partial order on $\Omega$ defined by $x \leq^* y : \iff y \leq x$, for all $x, y \in \Omega$. A poset $P$ is called self-dual if there exists a bijection between $P$ and $P^*$.

Let $\mathbb{L} := (\Omega, \leq)$ and $\mathbb{L}' := (\Omega', \leq')$ be two posets, $\alpha, \beta \in \Omega$ and $f : \Omega \to \Omega'$ a map. We say that $f$ is $(\alpha, \beta)$-preserving if the restriction map $f_{|\{\alpha, \beta\}} : \{\alpha, \beta\} \to \{f(\alpha), f(\beta)\}$ is an order isomorphism.

A poset $P = (\Omega, \leq)$ is said graded of rank $l$ if all the maximal chains in $P$ have length $l$, and in this case $l$ is called rank of $P$.

Let $F_n = \{x_1, \ldots, x_n\}$ with $x_1 > x_2 < x_3 > \cdots x_n$ (and no other comparabilities). A poset $P = (\Omega, \leq)$ which is isomorphic to $F_n$ or to its dual $F^*_n$ is called a $n$-fence (see [77]).

Let $C_n = \{x_1, y_1, \ldots, x_n, y_n\}$ with $x_1 < y_1 > x_2 < y_2 > x_3 < \cdots < y_{n-1} > x_n < y_n > x_1$ (and no other comparabilities). A poset $P = (\Omega, \leq)$ which is isomorphic to $C_n$ or $C^*_n$ is called a $n$-crown (see [178]).

### 1.3 Set Partitions

A set-partition $\pi$ on $\Omega$ is a collection of non-empty subsets $\{B_i : i \in I\}$ of $\Omega$ such that $B_i \cap B_j = \emptyset$ for all $i \neq j$ and such that $\bigcup_{i \in I} B_i = \Omega$. The subsets $B_i$ are called blocks of $\pi$ and, if $I$ is finite, we write $\pi := B_1 | \cdots | B_|I|$ to denote that $\pi$ is a set partition having blocks $B_1, \ldots, B_{|I|}$. If $u \in \Omega$, we denote by $\pi(u)$ the block of $\pi$ which contains the element $u$. We use the standard notation $\pi := B_1 | \cdots | B_{|\pi|}$, where $|\pi|$ is the number of distinct blocks of $\pi$. We denote by $\pi_\emptyset(\Omega)$
the set of all set-partitions of \( \Omega \). It is well known that on the set \( \pi(\Omega) \) we can consider a partial order \( \preceq \) defined as follows: if \( \pi, \pi' \in \pi(\Omega) \), then

\[
\pi \preceq \pi' : \iff (\forall B \in \pi) (\exists B' \in \pi') : B \subseteq B' \iff (\forall u \in \Omega) (\pi(u) \subseteq \pi'(u)).
\] (1.1)

We will write \( \pi < \pi' \) when \( \pi \preceq \pi' \) and \( \pi \neq \pi' \). The pair \( (\pi(\Omega), \preceq) \) is a complete lattice which is called partition lattice of the set \( \Omega \).

Let \( \pi_1 = A_1 | \ldots | A_m \) and \( \pi_2 = B_1 | \ldots | B_n \) be two partitions on the same finite universe \( \Omega \), i.e., \( \pi_1, \pi_2 \in \pi(\Omega) \), we firstly set

\[
S_{\pi_1, \pi_2} := \{ C \subseteq \Omega : \text{ if } x \in C, \text{ then } \pi_1(x) \subseteq C \text{ and } \pi_2(x) \subseteq C \}
\]

Then, the meet of \( \pi_1 \) and \( \pi_2 \), denoted by \( \pi_1 \land \pi_2 \), is the set partition of \( \Omega \) whose blocks are given by

\[
\pi_1 \land \pi_2 := \{ A_i \cap B_j : i = 1, \ldots, m; j = 1, \ldots, n \}.
\] (1.2)

On the other hand, the more simple way to describe the join of \( \pi_1 \) and \( \pi_2 \), denoted by \( \pi_1 \lor \pi_2 \), is the following:

\[
\pi_1 \lor \pi_2 := C_1 | \ldots | C_k,
\] (1.3)

where \( C_1, \ldots, C_k \) are the minimal elements of the set family \( S_{\pi_1, \pi_2} \) with respect to the inclusion.

**Example 1.3.1.** Let \( \Omega = \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \), \( \pi_1 = \{ x_1, x_2 \} | \{ x_3 \} | \{ x_4, x_5 \} | \{ x_6 \} \) and \( \pi_2 = \{ x_1, x_3 \} | \{ x_2 \} | \{ x_4 \} | \{ x_5 \} | \{ x_6 \} \) be two set partitions of \( \Omega \). Then, \( \pi_1 \land \pi_2 \) is the partition

\[
\pi_1 \land \pi_2 = \{ x_1 \} | \{ x_2 \} | \{ x_3 \} | \{ x_4 \} | \{ x_5 \} | \{ x_6 \}.
\]

The family \( S_{\pi_1, \pi_2} \) is equal to:

\[
S_{\pi_1, \pi_2} = \{ \{ x_1, x_2, x_3 \}, \{ x_1, x_2, x_3, x_4, x_5 \}, \{ x_1, x_2, x_3, x_5 \}, \{ x_1, x_2, x_3, x_6 \}, \{ x_4, x_5 \}, \{ x_4, x_5, x_6 \}, \{ x_6 \} \}.
\]

Therefore, the meet of \( \pi_1 \) and \( \pi_2 \) is the partition

\[
\pi_1 \lor \pi_2 = \{ x_1, x_2, x_3 \} | \{ x_4, x_5 \} | \{ x_6 \}.
\]

### 1.4 Set Systems

We call an element \( \mathcal{F} \in SS(\Omega) \) a set system on \( \Omega \). We call a set system family \( \mathfrak{F} \subseteq SS(\Omega) \) a 2-set system on \( \Omega \). If \( \mathfrak{F} \) is a 2-set system on \( \Omega \), we consider it as sub-posets of \( S(\Omega) \).

A hypergraph is a pair \( H = (\Omega, \mathcal{F}) \), where \( \Omega = \{ x_1, \ldots, x_n \} \) is a finite set (called vertex set of \( H \)) and \( \mathcal{F} = \{ Y_1, \ldots, Y_m \} \) is a non-empty family of subsets \( Y_1, \ldots, Y_m \) of \( \Omega \). The elements \( x_1, \ldots, x_n \) are called vertices of \( H \) and the subsets \( Y_1, \ldots, Y_m \) of \( \Omega \) are called hyperedges of \( H \).

If \( \mathcal{F} \in SS(\Omega) \) we say that:

- \( \mathcal{F} \) has uniform cardinality if each \( X \in \mathcal{F} \) has the same cardinality. We denote by \( \| \mathcal{F} \| \) the cardinality of each \( X \in \mathcal{F} \).
• $\mathcal{F}$ is \textit{k-uniform} if it is the family of all $k$-subsets of $\Omega$. In this case, we denote it by $\binom{\Omega}{k}$.

• $\mathcal{F}$ is \textit{union-closed} if whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cup \mathcal{F}' \in \mathcal{F}$. We denote by $UCL(\Omega)$ the collection of all union-closed set systems on $\Omega$.

• $\mathcal{F}$ is \textit{chain union-closed} if whenever $\mathcal{F}' \subseteq \mathcal{F}$ is a chain, then $\cup \mathcal{F}' \in \mathcal{F}$.

• $\mathcal{F}$ is a \textit{complement-closed family} if $\Omega \in \mathcal{F}$ and $Z \in \mathcal{F}$ implies $Z^c \in \mathcal{F}$.

• $\mathcal{F}$ is an \textit{abstract simplicial complex} (or simply an \textit{abstract complex}) on $\Omega$ if $\emptyset \in \mathcal{F}$ and whenever $Y \in \mathcal{F}$ and $Z \subseteq Y$, then $Z \in \mathcal{F}$. We denote by $ASC(\Omega)$ the 2-set system on $\Omega$ whose elements are all abstract complexes on $\Omega$.

• $\mathcal{F}$ is \textit{finitary} if whenever $X \in \mathcal{P}(\Omega)$ and $F \in \mathcal{F}$ for any $F \subseteq_f X$, then $X \in \mathcal{F}$;

• a \textit{finitary abstract complex} on $\Omega$ if $\mathcal{F}$ is a finitary set system and also an abstract complex on $\Omega$. We denote by $FAC(\Omega)$ the 2-set system on $\Omega$ whose elements are all finitary abstract complexes on $\Omega$.

• $\mathcal{F}$ is a \textit{closure system} on $\Omega$ if $\Omega \in \mathcal{F}$ and whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cap \mathcal{F}' \in \mathcal{F}$. We denote by $CLSY(\Omega)$ the 2-set system on $\Omega$ whose elements are all closure systems on $\Omega$.

We say that a subset $Y \subseteq \Omega$ is a \textit{transversal} of a set system $\mathcal{F} \in SS(\Omega)$ if $Y \cap A \neq \emptyset$ for each non-empty $A \in \mathcal{F}$. Moreover, a transversal $A$ of $\mathcal{F}$ is \textit{minimal} if no proper subset of $A$ is a transversal of $\mathcal{F}$. We denote by $Tr(\mathcal{F})$ the family of all minimal transversals of $\mathcal{F}$. We denote by $\max(\mathcal{F})$ the set system of all maximal members of $\mathcal{F}$ and by $\min(\mathcal{F})$ the set system of all minimal members of $\mathcal{F}$. If $A \in \mathcal{P}(\Omega)$, we set $\max(\mathcal{F}|A) := \max\{B \in \mathcal{P}(A) : B \in \mathcal{F}\}$ and $\min(\mathcal{F}|A) := \min\{B \in \mathcal{P}(A) : B \in \mathcal{F}\}$, so that $\max(\mathcal{F}|\Omega) = \max(\mathcal{F})$ and $\min(\mathcal{F}|\Omega) = \min(\mathcal{F})$.

We prove now that the 2-set system of all finitary abstract complexes on $\Omega$ is a complete lattice.

**Theorem 1.4.1.** $FAC(\Omega)$ is a complete lattice such that if $\mathcal{F} = \{\mathcal{F}_i\}_{i \in I} \subseteq FAC(\Omega)$ then $\bigwedge_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i$. Moreover, if $I$ is finite we also have $\bigvee_{i \in I} \mathcal{F}_i = \bigcup_{i \in I} \mathcal{F}_i$.

**Proof.** We firstly show that given a collection $\{\mathcal{F}_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ of finitary abstract simplicial complexes on $\Omega$, then $\bigcap_{i \in I} \mathcal{F}_i$ is a finitary abstract simplicial complex on $\Omega$. For, let $X \in \mathcal{P}(\Omega)$.

Suppose that $F \in \bigcap_{i \in I} \mathcal{F}_i$ for any $F \subseteq_f X$. Thus, $F \in \mathcal{F}_i$ for any $i \in I$ and, since each set system $\mathcal{F}_i$ is finitary, it results that $X \in \mathcal{F}_i$ for each index $i \in I$. We conclude that $X \in \bigcap_{i \in I} \mathcal{F}_i$, i.e. $\bigcap_{i \in I} \mathcal{F}_i$ is finitary.

On the other hand, we clearly have that $\emptyset \in \bigcap_{i \in I} \mathcal{F}_i$. Furthermore, let $Y \in \bigcap_{i \in I} \mathcal{F}_i$ and $Z \subseteq Y$.

Since $Y \in \mathcal{F}_i$ for any $i \in I$, we conclude that $Z \in \mathcal{F}_i$ for each index $i \in I$, so $Z \in \bigcap_{i \in I} \mathcal{F}_i$. This shows that $\bigcap_{i \in I} \mathcal{F}_i$ is also an abstract simplicial complex, thus the claim has been shown.

Let $\mathcal{F}$ be such that $\mathcal{F} \subseteq \mathcal{F}_i$ for any $i \in I$. Then, $\mathcal{F} \subseteq \bigcap_{i \in I} \mathcal{F}_i$, so $\bigwedge_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i$. 

4
We now show the existence of the join for an arbitrary collection \( \{ F_i : i \in I \} \subseteq \mathcal{P}(\Omega) \). Since \( \mathcal{P}(\Omega) \in FA(\Omega) \), the family of all upper bounds \( \mathcal{U} \) of \( \{ F_i : i \in I \} \) is non-empty. Let us take \( \bigcap \mathcal{U} \). By the previous part, it is a finitary abstract simplicial complex on \( \Omega \); clearly, it belongs to \( \mathcal{U} \) and is minimal in \( \mathcal{U} \). This proves that \( \bigvee_{i \in I} F_i = \bigcap \mathcal{U} \).

To conclude the proof, we note that the arbitrary union of abstract complexes is also an abstract complex.

On the other hand, given a finite collection \( \{ F_1, \ldots, F_n \} \) of finitary abstract simplicial complexes on \( \Omega \), we have that \( \bigcup_{i=1}^n F_i \) is finitary. In fact, let \( X \in \mathcal{P}(\Omega) \) such that \( F \subseteq \bigcup_{i=1}^n F_i \) for any \( F \subseteq \Omega \). Assume by contradiction that \( X \notin \bigcup_{i=1}^n F_i \). Then, there exists \( F_i \subseteq \Omega \) such that \( F \notin F_i \) for any \( i = 1, \ldots, n \). Let us take \( F := \bigcup_{i=1}^n F_i \). Thus, \( F \subseteq \Omega \), so there exists \( j \in \{1, \ldots, n\} \) such that \( F \in \mathcal{F}_j \) and, in particular, \( F_j \in \mathcal{F}_j \), contradicting our choice of \( F_j \). This shows our claim.

**Remark 1.4.2.** When \( I \) is not a finite set, it turns out to be false in general that \( \bigcup_{i \in I} F_i \) is a finitary abstract complex on \( \Omega \). As an example, just consider the 2-set system \( \mathcal{F} = \{ F_i : i \in I \} \), where \( F_i = \{ F : F \subseteq \bigcup_{i \in I} F_i \} \).

### 1.5 Set Operators

A **set operator** on \( \Omega \) is a map \( \sigma : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega) \). We denote by \( OP(\Omega) \) the set of all the set operators on \( \Omega \) and we consider on \( OP(\Omega) \) the partial order \( \sqsubseteq \) defined by

\[
\sigma \sqsubseteq \sigma' : \iff \sigma(X) \subseteq \sigma'(X) \quad \forall X \in \mathcal{P}(\Omega),
\]

for any \( \sigma, \sigma' \in OP(\Omega) \).

We will consider the maps \( Fix : OP(\Omega) \to SS(\Omega) \) and \( Int, Drk, Imx : SS(\Omega) \to OP(\Omega) \) defined respectively by

\[
Fix(\sigma) := \{ A \in \mathcal{P}(\Omega) : A \in \mathcal{P}(\Omega) \},
\]

\[
Int(\sigma)(C) := \bigcap\{ A \in \mathcal{F} : C \subseteq A \},
\]

\[
Drk(\sigma)(C) := \{ b \in \Omega : rk(\sigma)(C \cup \{ b \}) = rk(C) \},
\]

\[
Imx(\sigma)(C) := \bigcap\{ A \in \max(\mathcal{F}|C) \},
\]

for any \( \sigma \in OP(\Omega), \mathcal{F} \in SS(\Omega), C \in \mathcal{P}(\Omega) \).

We call **fixed points** of \( \sigma \) the members of \( Fix(\sigma) \), **intersecting operator** of \( \mathcal{F} \) the set operator \( Int_{\mathcal{F}} \), **rank-symmetry operator** of \( \mathcal{F} \) the set operator \( Drk_{\mathcal{F}} \) and **intersecting maximal operator** of \( \mathcal{F} \) the set operator \( Imx_{\mathcal{F}} \).

We consider now the operator \( \Theta : OP(\Omega) \to SS(\Omega) \) defined by

\[
\Theta(\sigma) := \{ X \in \mathcal{P}(\Omega) : A \subseteq X \implies X \subseteq \sigma(A) \},
\]

for any \( \sigma \in OP(\Omega) \). We call \( \Theta \) the **minimality operator** on \( \Omega \). The following result easily follows.
Proposition 1.5.1. (i) \((\text{OP}(\Omega), \subseteq)\) is a complete lattice. In particular, given \(\{\sigma_i : i \in I\} \subseteq \text{OP}(\Omega)\), it results that \(\bigvee_{i \in I} \sigma_i = \sigma^+\) and \(\bigwedge_{i \in I} \sigma_i = \sigma^-\), where, respectively, \(\sigma^+(X) := \bigcup_{i \in I} \sigma_i(X)\) and \(\sigma^-(X) := \bigcap_{i \in I} \sigma_i(X)\) for any \(X \in \mathcal{P}(\Omega)\).

(ii) If \(\sigma, \sigma' \in \text{OP}(\Omega)\) and \(\sigma \subseteq \sigma'\) then \(\Theta(\sigma) \supseteq \Theta(\sigma')\).

(iii) Let \(D, D' \subseteq \mathcal{P}(\mathcal{P}(\Omega) \times \Omega)\) such that \(D \subseteq D'\). Then, \(\sigma_D \subseteq \sigma'_D\).

Let \(\sigma \in \text{OP}(\Omega)\). We say that:

- \(\sigma\) is extensive if \(X \subseteq \sigma(X)\);
- \(\sigma\) is intensive if \(\sigma(X) \subseteq X\);
- \(\sigma\) is monotone if \(X \subseteq Y\) implies \(\sigma(X) \subseteq \sigma(Y)\);
- \(\sigma\) is a pre-closure operator on \(\Omega\) if it is extensive and monotone;
- \(\sigma\) is idempotent if \(\sigma(\sigma(X)) = X\);
- \(\sigma\) is a closure operator on \(\Omega\) if it is an idempotent pre-closure operator. We denote by \(\text{CLOP}(\Omega)\) the set of all closure operators on \(\Omega\);
- \(\sigma\) is a kernel operator [86] if it is an intensive, monotone and idempotent set operator;
- \(\sigma\) is pseudo-monotone if, whenever \(A \subseteq B\), then \(A \cap \sigma(B) \subseteq \sigma(A)\);
- \(\sigma\) is a simplicial operator if it is intensive and pseudo-monotone. We denote by \(\text{SOP}(\Omega)\) the family of all simplicial operators on \(\Omega\);
- \(\sigma\) is quasi-regular if \(b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))\) and \(c \in \sigma(A \cup \{c\}) \setminus A\), then \(c \in \sigma(A \cup \{b, c\})\);
- \(\sigma\) is a core operator if it is a simplicial quasi-regular operator. We denote by \(\text{COOP}(\Omega)\) the family of all core operators on \(\Omega\);
- \(\sigma\) is finitary, if \(\sigma(X) = \bigcup \{\sigma(Y) : Y \subseteq f \ X\}\);
- \(\sigma\) is a weak MacLane-Steinitz operator (briefly \(\sigma\) is WMLS), if for any \(A \in \Theta(\sigma)\) and any \(x, y \in \Omega\) such that \(x \in \sigma(A \cup \{y\}) \setminus \sigma(A)\), we have that \(y \in \sigma(A \cup \{x\})\);
- \(\sigma\) is a MacLane-Steinitz operator (briefly \(\sigma\) is WMLS), if for any \(A \in \mathcal{P}(\Omega)\) and any \(x, y \in \Omega\) such that \(x \in \sigma(A \cup \{y\}) \setminus \sigma(A)\), we have that \(y \in \sigma(A \cup \{x\})\).

When the set operator \(\sigma\) is extensive, we have that \(\Theta(\sigma)\) is an abstract simplicial complex.

Proposition 1.5.2. If \(\sigma \in \text{OP}(\Omega)\) is extensive, then \(\Theta(\sigma) \in \text{ASC}(\Omega)\).

Proof. Clearly, \(\emptyset \in \Theta(\sigma)\). Let \(X \in \Theta(\sigma)\) and \(Y \subseteq X\). If were \(Y \notin \Theta(\sigma)\), there would be some \(y \in Y\) belonging to \(\sigma(Y \setminus \{y\})\). In particular, by extensiveness, it must be \(y \in \sigma(X \setminus \{y\})\), contradicting the fact that \(X \in \Theta(\sigma)\). \(\square\)
In the following result we establish the basic links between abstract simplicial complexes and simplicial operators.

**Theorem 1.5.3.** Let $\sigma \in SOP(\Omega)$ and $\mathcal{F} \in ASC(\Omega)$. Then, the following hold:

(i) $Fix(\sigma) \in ASC(\Omega)$;

(ii) $Fix(Imx_3) = \mathcal{F}$;

(iii) if $\mathcal{F} \in ASC(\Omega) \cap UCL(\Omega)$, then $Imx_3 \in SOP(\Omega)$ and $Imx_3$ is idempotent.

**Proof.** (i) : Clearly, $\emptyset \in Fix(\sigma)$. On the other hand, let $B \in Fix(\sigma)$ and $A \subseteq B$. By intensivity, it follows that $\sigma(A) \subseteq A$. On the other hand, by pseudo-monotonicity it follows that $\sigma(A) \supseteq \sigma(B) \cap A = B \cap A = A$, so $A = \sigma(A)$, i.e. $A \in Fix(\sigma)$.

(ii) : Just observe that $Imx_3(A) = A$ if and only if $A \in \mathcal{F}$.

(iii) : Clearly, $Imx_3(A) \subseteq A$ for any $A \in \mathcal{F}(\Omega)$. Assume now that $A \subseteq B$ and let $x \in A \cap Imx_3(B)$. Then, $x \in A \cap C$ for any $C \in max(\mathcal{F}|B)$. Let us prove that there exists $D \in max(\mathcal{F}|A)$ such that $x \in D$. For, consider the set system $\mathcal{G} := \{Z \in \mathcal{F} : A \cap C \subseteq D \subseteq A\}$. It is clearly non-empty, since $A \cap C \subseteq \mathcal{G}$. Take a chain $\{Z_i : i \in I\} \subseteq \mathcal{G}$ and consider $Z := \bigcup_{i \in I} Z_i$. By the fact that $\mathcal{F}$ is union-closed, it follows that $Z \in \mathcal{F}$. Moreover, it is obvious that $A \cap C \subseteq Z \subseteq A$. Hence, we can apply Zorn’s Lemma and prove that there exists an element $D \in max \mathcal{G}$.

Finally, it is also clear that $x \in D$ and it is straightforward to see that $D \in max(\mathcal{F}|A)$.

We now show that $x \in E$ for any $E \in max(\mathcal{F}|A)$. To this regard, assume by contradiction the existence of some $E' \in max(\mathcal{F}|A)$ for which $x \notin E'$. Let us prove that $E' \subseteq C'$ for some $C' \in max(\mathcal{F}|B)$. To this regard, set $\mathcal{G}' := \{Y \in \mathcal{F} : E' \subseteq Y \subseteq B\}$. It is clearly non-empty since $E' \subseteq \mathcal{G}'$. So, take a chain $\{Y_i : i \in I\} \subseteq \mathcal{G}'$ and set $Y := \bigcup_{i \in I} Y_i$. It is immediate to see that $E' \subseteq Y \subseteq B$ and $Y \in \mathcal{F}$. By Zorn’s Lemma, $\mathcal{G}'$ admits a maximal element $C'$, that clearly belongs to max($\mathcal{F}|B$).

Now, we have that $x \notin C'$, otherwise $C' \cap A \supseteq E'$ and $C' \cap A \in max(\mathcal{F}|A)$, contradicting the maximality of $E'$. But $x$ must belong to each subset of max($\mathcal{F}|B$), so we reach to an absurd. This proves pseudo-monotonicity.

Finally, in order to prove that $Imx_3$ is idempotent, just observe that $Imx_3(A) \in \mathcal{F}$ so, by previous part (ii), we have $Imx_3(Imx_3(A)) = Imx_3(A)$.

**Remark 1.5.4.** It is easy to show that (iii) of Theorem 1.5.3 holds when $\Omega$ is a finite set, without assuming that $\mathcal{F} \in UCL(\Omega)$.

We now provide a characterization for $\Theta(\sigma)$ when $\sigma \in CLOP(\Omega)$.

**Proposition 1.5.5.** Let $\sigma \in CLOP(\Omega)$. Then:

$$\Theta(\sigma) = \{A \in \mathcal{P}(\Omega) : x \notin \sigma(A \setminus \{x\}) \forall x \in A\}. \quad (1.6)$$

**Proof.** Let $A \in \Theta(\sigma)$ and $x \in A$. Then, $A \setminus \{x\} \subseteq A$, therefore $\sigma(A \setminus \{x\}) \subseteq \sigma(A)$. If $x \in \sigma(A \setminus \{x\})$, we would have $A \subseteq \sigma(A \setminus \{x\})$, i.e. $\sigma(A) = \sigma(A \setminus \{x\})$, that is a contradiction. Vice versa, let $A \in \mathcal{P}(\Omega)$ such that $x \notin \sigma(A \setminus \{x\})$ for any $x \in A$. Let $B \subseteq A \setminus \{x\} \subseteq A$ for some $x \in A$. This obviously implies that $x \notin \sigma(B)$, i.e. $\sigma(A) \not\subseteq \sigma(B)$. \qed
In the next result, we characterize the condition for a closure operator to be WMLS.

**Theorem 1.5.6.** Let \( \sigma \in \text{CLOP}(\Omega) \). The following conditions are equivalent:

(i) \( \sigma \) is WMLS;

(ii) if \( X \in \mathcal{P}(\Omega) \), \( y \in \Omega \) and \( x \in \sigma(X \cup \{y\}) \setminus \sigma(X) \) are such that \( X \cup \{y\} \in \Theta(\sigma) \), then \( y \in \sigma(X \cup \{x\}) \);

(iii) for any \( X \in \Theta(\sigma) \) and any \( x \in \Omega \setminus \sigma(X) \), \( X \cup \{x\} \in \Theta(\sigma) \);

(iv) if \( X \in \mathcal{P}(\Omega) \) and \( Y \in \max(\Theta(\sigma)|X) \), then \( \sigma(Y) = \sigma(X) \).

**Proof.** (i) \( \implies \) (ii): Let \( X \in \mathcal{P}(\Omega) \), \( y \in \Omega \) and \( x \in \sigma(X \cup \{y\}) \setminus \sigma(X) \) be such that \( X \cup \{y\} \in \Theta(\sigma) \). By Proposition 1.5.2, \( X \in \Theta(\Omega) \), therefore by the fact that \( \sigma \) is WMLS, it follows that \( y \in \sigma(X \cup \{x\}) \).

(ii) \( \implies \) (iii): Let \( X \in \Theta(\sigma) \), \( x \in \Omega \setminus \sigma(X) \) and assume by contradiction that \( X \cup \{x\} \notin \Theta(\sigma) \).

So, there exists \( z \in X \) such that \( z \in \sigma(X \cup \{x\} \setminus \{z\}) \setminus \sigma(X \setminus \{z\}) \) but, by our assumptions, this implies \( x \in \sigma(X) \), that is a contradiction. Hence, \( X \cup \{x\} \in \Theta(\sigma) \).

(iii) \( \implies \) (iv): Let \( X \in \mathcal{P}(\Omega) \) and \( Y \in \max(\Theta(\sigma)|X) \). Since \( \sigma \in \text{CLOP}(\Omega) \), we need only to prove that \( \sigma(Y) \subseteq \sigma(X) \). To this regard, if \( x \in \sigma(X) \setminus \sigma(Y) \), by our assumptions, it would be \( Y \cup \{x\} \in \Theta(\sigma) \), contradicting maximality of \( Y \).

(iv) \( \implies \) (i): Let \( A \in \Theta(\sigma) \) and \( x, y \in \Omega \) with \( x \in \sigma(A \cup \{y\}) \setminus \sigma(A) \). It must necessarily be \( A \cup \{x\} \in \Theta(\sigma) \) otherwise, \( \sigma(A) = \sigma(A \cup \{x\}) \), contradicting the choice of \( x \). On the other hand, \( A \cup \{x\} \cup \{y\} \notin \Theta(\sigma) \), since \( x \in \sigma(A \cup \{y\}) \). This entails that \( \sigma(A \cup \{x\} \cup \{y\}) = \sigma(A \cup \{x\}) \), i.e. \( y \in A \cup \{x\} \).

If \( \mathcal{F} \in \text{CLSY}(\Omega) \), the map \( \sigma_{\mathcal{F}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega) \) defined by

\[
C \in \mathcal{P}(\Omega) \mapsto \sigma_{\mathcal{F}}(C) := \bigcap \{ A \in \mathcal{F} : C \subseteq A \} \in \mathcal{P}(\Omega).
\]  

(1.7)

is a closure operator on \( \Omega \). Moreover, if \( \sigma \in \text{CLOP}(\Omega) \), the set system on \( \Omega \) defined by

\[
\mathcal{F}_{\sigma} := \{ A \in \mathcal{P}(\Omega) : \sigma(A) = A \}.
\]

(1.8)

is a closure system on \( \Omega \).

**Theorem 1.5.7.** The maps \( \mathcal{F} \in \text{CLSY}(\Omega) \mapsto \sigma_{\mathcal{F}} \in \text{CLOP}(\Omega) \) and \( \sigma \in \text{CLOP}(\Omega) \mapsto \mathcal{F}_{\sigma} \in \text{CLSY}(\Omega) \) are inverses each other. In other terms, if \( \mathcal{F} \in \text{CLSY}(\Omega) \) and \( \sigma \in \text{CLOP}(\Omega) \) then \( \mathcal{F}_{\sigma_{\mathcal{F}}} = \mathcal{F} \) and \( \sigma_{\mathcal{F}_{\sigma}} = \sigma \). Moreover, if \( \mathcal{F} \in \text{CLSY}(\Omega) \), then it is a complete lattice under set-inclusion in which meet means intersection.

**Proof.** See [18].

By Theorem 1.5.7, we can consider equivalent the notions of closure system and closure operator on the same set \( \Omega \) (clearly this equivalence is described by means of (1.7) and (1.8)). Therefore, for any closure system \( \mathcal{F} \in \text{CLSY}(\Omega) \), we always associate with \( \mathcal{F} \) the corresponding closure operator \( \sigma_{\mathcal{F}} \in \text{CLOP}(\Omega) \), and vice versa.
Let $A, B, C, D \in \mathcal{P}(\Omega)$. We say that $A \in \mathcal{P}(\Omega)$ is $\sigma$-complemented if $\{A, \Omega \setminus A\} \subseteq \mathcal{F}_\sigma$.

Moreover, if $\{A, B\} \subseteq \mathcal{F}_\sigma$ and $\{A, B\}$ is a covering on $\Omega$, we say that $\{A, B\}$ is a $\sigma$-covering on $\Omega$. If $C \cap D = \emptyset$ and $\{A, B\}$ is a covering on $\Omega$, we say that $\{C, D\}$ splits $\{A, B\}$, denoted by $\{C, D\} \lhd \{A, B\}$, if $C \cap B = \emptyset = D \cap A$. Furthermore, we say that $\{C, D\} \sigma$-splits $\{A, B\}$, denoted by $\{C, D\} \lhd_\sigma \{A, B\}$, if $\{C, D\} \lhd \{A, B\}$ and $\{A, B\} \subseteq \mathcal{F}_\sigma$.

\section{1.6 Matroids on Finite Sets}

For general results on matroids, see [182]. In what follows, we provide the main results on matroid theory that we use in the thesis.

\textbf{Definition 1.6.1.} Let $\Omega$ be a finite set. We say that a set system $F \in \mathcal{SS}(\Omega)$ is a matroid on $\Omega$ if:

\begin{enumerate}[(M1)]
\item $F \in \mathcal{AC}(\Omega)$;
\item for any $U, V \in F$ such that $|U| = |V| + 1$, then there exists $x \in U \setminus V$ such that $V \cup \{x\} \in F$.
\end{enumerate}

In this case, we call independent set any element of $F$. We denote by $\mathcal{MATR}(\Omega)$ the set of all matroids on $\Omega$.

In the next result, the condition for a set system $F$ is a matroid will be characterized in terms of rank function.

\textbf{Theorem 1.6.2.} A map $\rho : \mathcal{P}(\Omega) \to \mathbb{N}$ is the rank function of a matroid on $\Omega$ if and only if for any $A, B \in \mathcal{P}(\Omega)$, $a, b \in \Omega$, it results that:

\begin{enumerate}[(Rk1)]
\item $\rho(\emptyset) = 0$;
\item if $A \subseteq B$, then $\rho(A) \leq \rho(B)$;
\item if $\rho(A \cup \{a\}) = \rho(A \cup \{b\}) = \rho(A)$, then $\rho(A \cup \{a\} \cup \{b\}) = \rho(A)$.
\end{enumerate}

In this case, the corresponding matroid is $F = \{A \in \mathcal{P}(\Omega) : \rho(A) = |A|\}$.

If $F \in \mathcal{MATR}(\Omega)$ and $A \in \mathcal{P}(\Omega)$, an element $B \in \max(F|A)$ is usually called an $A$-basis of $F$. In particular, an element $B \in \max(F)$ is called a basis of $F$.

In the next result, the condition for a subset family $F$ is a matroid will be characterized in terms of basis family.

\textbf{Theorem 1.6.3.} A non-empty set system $\mathcal{B} \in \mathcal{SS}(\Omega)$ is the basis family of a matroid on $\Omega$ if and only if the following holds:

\begin{enumerate}[(B1)]
\item if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.
\end{enumerate}

In this case, the corresponding matroid is $F = \{A \in \mathcal{P}(\Omega) : A \in \bigcup_{B \in \mathcal{B}} \mathcal{P}(B)\}$.

Finally, we recall the following classical results concerning two further different characterizations of matroid.
Theorem 1.6.4. A set system $\mathcal{F} \subseteq SS(\Omega)$ is a matroid on $\Omega$ if and only if max($\mathcal{F}|A$) has uniform cardinality for any $A \in \mathcal{P}(\Omega)$.

Theorem 1.6.5. A set operator $\sigma \in OP(\Omega)$ is the rank-symmetry operator of a matroid on $\Omega$ if and only if $\sigma$ is a MLS closure operator on $\Omega$. In this case, the corresponding matroid is $\mathcal{F} = \{ A \in \mathcal{P}(\Omega) : a \notin \sigma(A \setminus \{a\}) \forall a \in A \}$.

1.7 Graphs

We refer to [79] for any general notion concerning graph theory. Let $G = (V(G), E(G))$ be a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. In this case, we also use the term $n$-graph. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \sim v'$ otherwise. We denote by $Adj(G)$ the usual adjacency matrix of $G$.

We call neighborhood of $v$ in $G$ the set $N_G(v) := \{ w \in V(G) : v \sim w \}$. In particular, if $A \subseteq V(G)$ we call neighborhood of $A$ in $G$ the set

$$N_G(A) := \bigcup_{v \in A} N_G(v)$$

We say that $H = (V(H), E(H))$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $X \subseteq V(G)$, the generated subgraph by $X$ in $G$, denoted by $G[X]$, is the graph having $X$ as vertex set and such that if $v$ and $v'$ are two distinct vertices in $X$, then $\{v, v'\} \in E(G[X])$ if and only if $\{v, v'\} \in E(G)$. If $G_1$ and $G_2$ are two graphs such that with $V(G_1) \cap V(G_2) = \emptyset$, we call disjoint union of $G_1$ and $G_2$, denoted by $G_1 + G_2$, the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

If $v$ and $w$ are two vertices of $G$, we call a sequence of vertices $v_0 \ldots v_k$, where $v_0 = v$ and $v_1 = w$, a path between $v$ and $w$. The number of edges of the path between $v$ and $w$ is called the length of the path. We denote by $d(v, w)$ the distance between $v$ and $w$, i.e. the length of any shortest path between $v$ and $w$. The girth of $G$ is the minimum length of a cycle contained in $G$. A graph is said connected if there exists a path for any pair of vertices $v$ and $w$. If $v$ is a vertex, we call the maximal connected subgraph of $G$ containing $v$ the connected component of $v$ in $G$. Each maximal connected subgraph of $G$ is said connected component of $G$. An automorphism $\phi$ of $G$ is a bijective map $\phi : V(G) \rightarrow V(G)$ such that for all $v, v' \in V(G)$ it results that

$$\{v, v'\} \in E(G) \iff \{\phi(v), \phi(v')\} \in E(G).$$

The set of all automorphisms of $G$ is a group with respect to the composition of maps and it is usually denoted by $Aut(G)$. The graph $G$ is said vertex-transitive if $Aut(G)$ acts transitively on the set of vertices $V(G)$, i.e. for all $v, v' \in V(G)$ there exists $\phi \in Aut(G)$ such that $\phi(v) = v'$. The graph $G$ is said edge-transitive if $Aut(G)$ acts transitively on the set of edges $E(G)$, i.e. for all $\{v, w\}, \{v', w'\} \in E(G)$ there exists $\phi \in Aut(G)$ such that $\phi(\{v, w\}) = \{v', w'\}$. The graph $G$ is said k-transitive if for any two ordered $k$–tuples of vertices $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ such that $d(v_i, v_j) = d(w_i, w_j)$ for all $i, j \in \{1, \ldots, k\}$, there exists $\phi \in Aut(G)$ such that $\phi(v_i) = w_i$ for every $i \in \{1, \ldots, k\}$.

We give now a list of the graphs we use in the sequel:
• **n-null graph.** It is denoted by $N_n$ and is the $n$-graph without edges;

• **Complete graph on $n$ vertices.** It is denoted by $K_n$ and is the graph such that $\{v_i, v_j\}$ is an edge, for each pair of indexes $i \neq j$;

• $(r_1, \ldots, r_s)$—**complete multipartite graph on $n$ vertices.** It is denoted by $K_{r_1, \ldots, r_s}$, where $r_1 + \cdots + r_s = n$ and there exist $s$ non-empty subsets $B_1, \ldots, B_s$ of $V(G)$ such that

  (i) $|B_i| = r_i$;

  (ii) $B_i \cap B_j = \emptyset$ if $i \neq j$;

  (iii) $\bigcup_{i=1}^s B_i = V(G)$;

  (iv) $E(G) = \{(x, y) : x \in B_i, y \in B_j, i \neq j\}$.

In this case, we also denote $K_{r_1, \ldots, r_s}$ by the symbol $(B_1|\ldots|B_s)$. Moreover, if $s = 2$, we say that $K_{r_1, r_2}$ is a complete bipartite graph;

• **$n$-path.** It is denoted by $P_n$ and has edge set $E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\};$

• **$n$-cycle.** It is denoted by $C_n$ and has edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.

In the sequel, all sums between vertex indexes of $C_n$ are implicitly taken mod($n$);

• **$n$-wheel** is denoted by $W_{n+1}$ and has edge set $E(W_{n+1}) = E(C_n) \cup \{(v_i, v_{i+1}) : 1 \leq i \leq n\};$

• **$n$-sticks.** It is denoted by $F_n$, where $n \geq 4$ is even and $E(F_n) = \{(v_i, v_{i+1}) : i = 1, 3, \ldots, n-1\}$.

Let $G$ and $H$ be two graphs and let $V := V(G) \times V(H)$.

- The **Cartesian product** of $G$ and $H$ is the graph $G \square H$ on $V$ whose edge set is
  
  $$E(G \square H) := \{(u, u'), (v, v') \in V(G) \times V(H) : u = v \land u' \sim v' \lor (u' = v' \land u \sim v')\}.$$

- The **tensor product** of $G$ and $H$ is the graph $G \otimes H$ whose vertex set is $V(G \otimes H) := V(G) \times V(H)$ and whose edge set is
  
  $$E(G \otimes H) := \{(u, u'), (v, v') \in V(G) \times V(H) : u \sim v \land u' \sim v'\}.$$

Let us note that if $(u, u') \in V(G) \times V(H)$ then

$$N_{G \square H}(u, u') = (\{u\} \times N_H(u')) \cup (N_G(u) \times \{u'\})$$

and

$$N_{G \otimes H}(u, u') = N_G(u) \times N_H(u').$$

Some well studied Cartesian product graph families are the following:

• Let $m \geq 2$ and $n \geq 3$. The $(m, n)$-**prism graph** is the Cartesian product $P_m \square C_n$.

• Let $m, n \geq 2$. The $(m, n)$-**grid graph** is the Cartesian product $P_m \square P_n$.

• Let $m, n \geq 3$. The $(m, n)$-**rook’s graph** is the Cartesian product $K_m \square K_n$. 
1.8 Digraphs

A **digraph** (see [14]) is a pair $D = (V(D), Arc(D))$, where $V(D) = \{v_1, \ldots, v_n\}$ and $Arc(D) = \{e_1, \ldots, e_m\}$ are both finite sets. The elements of $V(D)$ are called **vertices** of $D$ and the elements of $Arc(D)$ are called **arcs** of $D$. If the ordered pair $(v, w)$ is an arc of $D$ we also write $v \to w$. If $v \in V(D)$, we set

$$N^+_D(v) := \{z \in V(D) : (v, z) \in Arc(D)\}. \quad (1.10)$$

Usually $N^+_D(v)$ is called the **out-neighborhood** of $v$ in $D$. We say that a vertex $v \in V(D)$ is an **initial node** of $D$ if $N^+_D(v) = \emptyset$. We denote by $I(D)$ the set of all initial nodes of $D$. A path $P$ in $D$ is an alternating sequence $v_1e_1 \ldots v_ke_k$, where $v_1, \ldots, v_k \in V(D)$ are distinct vertices and $e_1, \ldots, e_k \in Arc(D)$ are distinct arcs. If $v_1 = v_k$, we say that $P$ is a **cycle**. A digraph is called **acyclic** if it has no cycles. It is easy to prove that if a digraph is acyclic, then $I(D) \neq \emptyset$.

Particular digraphs we will study are:

- **Complete bipartite digraph.** If $p$ and $q$ be are two positive integers, we denote by $\overrightarrow{K}_{p,q}$ the digraph having vertex set $V(\overrightarrow{K}_{p,q}) = \{x_1, \ldots, x_p, y_1, \ldots, y_q\}$ and arc set $Arc(\overrightarrow{K}_{p,q}) = \{(x_i, y_j) : i = 1, \ldots, p, j = 1, \ldots, q\}$. We call $\overrightarrow{K}_{p,q}$ the $(p, q)$-complete bipartite digraph, or, simply complete bipartite digraph if $p$ and $q$ are clear from the context.

- **Direct path.** If $n$ is a positive integer, we denote by $\overrightarrow{P}_n$ the direct path on $n$ vertices, i.e. the digraph having arc set $Arc(\overrightarrow{P}_n) = \{(v_i, v_{i+1}) : i = 1, \ldots, n-1\}$.

- **Direct cycle.** Let $n \geq 2$ we denote by $\overrightarrow{C}_n$ the direct cycle on $n$ vertices, i.e. the digraph having arc set $Arc(\overrightarrow{C}_n) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$.

- **Tournament.** A tournament is a digraph $\overrightarrow{T}$ obtained by assigning a direction to each edge in an undirected complete graph.

- **Transitive tournament.** A $n$-transitive tournament is a tournament $\overrightarrow{T}$ on $n$ vertices such that, if $(v_i, v_j)$ and $(v_j, v_k)$ are in $Arc(\overrightarrow{T})$, then $(v_i, v_k) \in Arc(\overrightarrow{T})$.

**Remark 1.8.1.** It is easy to verify that all $n$-transitive tournaments are isomorphic to the tournament $\overrightarrow{T}_n$ such that

$$V(\overrightarrow{T}_n) = \{v_1, \ldots, v_n\}, \quad Arc(\overrightarrow{T}_n) = \{(v_i, v_j) : 1 \leq i \leq n-1, j = i+1, \ldots, n\}.$$ 

Therefore in the sequel we refer to the above $\overrightarrow{T}_n$ when we use the term $n$-transitive tournament.

1.9 Formal Concept Analysis

We start recalling the general definition of formal contexts and their basic properties (for details see [87]).

**Definition 1.9.1.** A **formal context** is a triple $\mathbb{K} = (Z, M, \mathcal{R})$, where $Z$ and $M$ are sets and $\mathcal{R} \subseteq Z \times M$ is the binary relation involving them. The elements of $Z$ and $M$ are called **objects**
and attributes (or properties) respectively. We write \( g \mathcal{R} m \) instead of \((g, m) \in \mathcal{R} \). If \( O \subseteq Z \) and \( Q \subseteq M \), we set 
\[
O^\uparrow := \{ m \in M : (\forall g \in O) g \mathcal{R} m \} \subseteq M
\]
and 
\[
Q^\downarrow := \{ g \in Z : (\forall m \in Q) g \mathcal{R} m \} \subseteq Z.
\]

In this way the following two mappings are defined: \( \uparrow : \mathcal{P}(Z) \rightarrow \mathcal{P}(M), O \mapsto O^\uparrow \) and 
\( \downarrow : \mathcal{P}(M) \rightarrow \mathcal{P}(Z), Q \mapsto Q^\downarrow \).

By suitable compositions of the above two mappings we are able to construct the two new mappings \( * : \mathcal{P}(Z) \rightarrow \mathcal{P}(Z), O \mapsto O^* := O^\downarrow \uparrow \), which is a transformation on \( \mathcal{P}(Z) \), and 
\( \circ : \mathcal{P}(M) \rightarrow \mathcal{P}(M), Q \mapsto Q^\circ := Q^\downarrow \uparrow \), which is a transformation on \( \mathcal{P}(M) \).

**Definition 1.9.2.** A formal concept of the formal context \( \mathcal{K} = (Z, M, \mathcal{R}) \) is a pair \((O, Q)\), where \( O \subseteq Z, Q \subseteq M, O^\uparrow = Q \) and \( Q^\downarrow = O \). If \((O, Q)\) is a formal concept, \( O \) is called *extent* of \((O, Q)\) and \( Q \) is called *intent* of \((O, Q)\). We denote by \( \mathfrak{B}(\mathcal{K}) \) the set of all formal concepts of the formal context \( \mathcal{K} \).

**Remark 1.9.3.** If \( O \subseteq Z \), it is immediate to verify that:
(i) \( O \subseteq O^* \).
(ii) The ordered pair \((O, O^\uparrow)\) is a formal concept of \( \mathcal{K} \) if and only if \( O = O^* \).
(iii) The ordered pair \((O, O^\downarrow)\) is a formal concept of \( \mathcal{K} \) if and only if there exists an attribute subset \( Q \subseteq M \) such that \( O = Q^\downarrow \). In this case \( O^\uparrow \) coincides with the largest attribute subset \( Q \) such that \( O = Q^\downarrow \).
(iv) Obviously similar conditions to previous the (i), (ii) and (iii) also hold for the intent case.

**Definition 1.9.4.** Let \( \mathcal{K} = (Z, M, \mathcal{R}) \) be a Formal Context. We call *extent set system* of \( \mathcal{K} \) the set system \( \mathcal{E}(\mathcal{K}) := (Z, \text{EXT}(\mathcal{K})) \), where \( \text{EXT}(\mathcal{K}) \) is the family of all extents of \( \mathcal{K} \). Analogously, we call *intent set system* of \( \mathcal{K} \) the set system \( \mathcal{I}(\mathcal{K}) := (M, \text{INT}(\mathcal{K})) \), where \( \text{INT}(\mathcal{K}) \) is the family of all intents of \( \mathcal{K} \).

If \((O_1, Q_1)\) and \((O_2, Q_2)\) are two concepts in \( \mathfrak{B}(\mathcal{K}) \), it is usual to consider the relation \((O_1, Q_1) \sqsubseteq (O_2, Q_2)\) if and only if \( O_1 \subseteq O_2 \) (that is equivalent to \( Q_1 \supseteq Q_2 \)). Then, \( \sqsubseteq \) is a partial order on \( \mathfrak{B}(\mathcal{K}) \) and \((\mathfrak{B}(\mathcal{K}), \sqsubseteq)\) is a complete lattice, called *concept lattice* (or also *Galois lattice*) of the formal context \( \mathcal{K} \), whose meet and join operations on an arbitrary family of formal concepts \( \{(O_\alpha, Q_\alpha) : \alpha \in A\} \) are the following:
\[
\bigwedge_{\alpha \in A} (O_\alpha, Q_\alpha) = \left( \bigcap_{\alpha \in A} O_\alpha, \left( \bigcup_{\alpha \in A} Q_\alpha \right)^* \right) \\
\bigvee_{\alpha \in A} (O_\alpha, Q_\alpha) = \left( \bigcup_{\alpha \in A} O_\alpha \right)^*, \bigcap_{\alpha \in A} Q_\alpha
\]
(1.11)

**Proposition 1.9.5.** Both the set systems \( \mathcal{E}(\mathcal{K}) \) and \( \mathcal{I}(\mathcal{K}) \) are closure systems.

**Proof.** It follows directly by the two identities (1.11). \(\square\)
Chapter 2

Pairings and Symmetry Relations

In this chapter, we provide some basic examples in which pairings arise in a natural way. Furthermore, we study the main properties of the $A$-symmetry relation and of the indistinguishability relation. In particular, we will prove that each global symmetry class $[A]_{\approx_P}$ is union-closed, so it admits a maximum, that we call maximum partitioner. The family of the maximum partitioners is a closure system on $\Omega$ and the set operator $M_P : A \in \mathcal{P}(\Omega) \mapsto M_P(A) \in \mathcal{P}(\Omega)$ is a closure operator characterizing the pairing itself and satisfying many interesting topological properties. Finally, we also consider the family of all minimal elements of each global symmetry class and prove that it is an abstract simplicial complex on $\Omega$.

2.1 Pairings

Let us consider an arbitrary set $\Omega$ (finite of infinite).

**Definition 2.1.1.** A pairing on $\Omega$ is a triple $\mathcal{P} = (U,F,\Lambda)$, where $U, \Lambda$ are non-empty sets and $F : U \times \Omega \to \Lambda$ is a map having domain $U \times \Omega$ and codomain $\Lambda$. We denote by $PAIR(\Omega)$ the set of all pairings on $\Omega$. When both the sets $U = \{u_1, \ldots, u_m\}$ and $\Omega = \{a_1, \ldots, a_n\}$ are finite, we say that $\mathcal{P}$ is a finite pairing. For a finite pairing, we denote by $T[\mathcal{P}]$ the $m \times n$ rectangular table having on the $i$-th row the element $u_i$, on the $j$-th column the element $a_j$ and $F(u_i,a_j)$ in the place $(i,j)$. In the finite case, we identify $\mathcal{P}$ with $T[\mathcal{P}]$ and call $T[\mathcal{P}]$ the functional table of $\mathcal{P}$.

In the whole thesis, when $\Omega$ is finite, we always restrict our attention on finite pairings and use the notation $PAIR(\Omega)$ in order to denote the aforementioned family.

Let $\mathcal{P} = (U,F,A) \in PAIR(\Omega)$ and $A, A' \in \mathcal{P}(\Omega)$. For any $u, u' \in U$ we set

$$u \equiv_A u' :\iff F(u,a) = F(u',a), \forall a \in A. \quad (2.1)$$

It is immediate to show that $\equiv_A$ is an equivalence relation on $U$ and we call it the $A$-symmetry relation of $\mathcal{P}$. For any $u \in U$ we denote by $[u]_A$ the equivalence class of $u$ with respect to $\equiv_A$ and we call $[u]_A$ the $A$-symmetry class of $u$. Finally, we call the set partition $\pi_{\mathcal{P}}(A) := \{[u]_A : u \in U\}$ induced by $\equiv_A$ on $U$ the $A$-symmetry partition of $\mathcal{P}$. 

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We provide here some examples taken by various mathematical context within the notion of pairing arises in a natural way. In the examples we consider in the sequel, we omit the subscript \(\mathfrak{P}\) in all the notations when the context is clear.

2.1.1 Pairings by Graphs

Let \(G\) be a \(n\)-graph. We now consider the pairing \(\mathfrak{P}[G] := (V(G), F, \{0, 1\}) \in \text{PAIR}(V(G))\), where

\[
F(u, v) := \begin{cases} 
1 & \text{if } u \sim v \\
0 & \text{otherwise}
\end{cases}
\]

Hereafter, we write \(G\) instead of \(\mathfrak{P}[G]\). Let \(A \subseteq V(G)\) a vertex subset. It is easy to prove that the \(A\)-symmetry relation \(\equiv_A\) can be translated as follows:

\[
v \equiv_A v' \iff N_G(v) \cap A = N_G(v') \cap A
\] (2.2)

We now prove some basic property of \(A\)-symmetry for graphs.

**Proposition 2.1.2.** Let \(v, v' \in G\) and \(A \subseteq V(G)\). Then:

(i) \(v \equiv_A v'\) if and only if for all \(z \in A\) it results that \(v \sim z\) if and only if \(v' \sim z\).

(ii) If \(v \sim v'\) then \(v \notin A v'\) or \(\{v, v'\} \cap A = \emptyset\).

(iii) If \(v \equiv_A v'\) and \(\{v, v'\} \cap A \neq \emptyset\), then \(v \sim v'\).

**Proof.** (i): It follows immediately by (2.2).

(ii): We suppose that \(v \sim v'\) and \(v \equiv_A v'\). We must show that \(\{v, v'\} \cap A = \emptyset\). We suppose by contradiction that \(\{v, v'\} \cap A \neq \emptyset\). Without loss of generality, we can assume \(v \in A\). Hence, we deduce that \(F(v, v) = F(v', v)\), but \(F(v, v) = 0\) since there are no loops in \(G\) while by our assumption \(F(v', v) = 1\). So the equality \(F(v, v) = F(v', v)\) does not hold, that is an absurd. The case \(v' \in A\) is analogous.

On the other hand, a similar argument shows that if \(v \sim v'\) and \(\{v, v'\} \cap A \neq \emptyset\) then \(v \notin A v'\). In fact we just observe that if \(v \in A\), then \(F(v, v) \neq F(v', v)\), hence we conclude by (2.2). This proves (ii).

(iii): It is the contra-nominal version of (ii). 

We now explain formally why the equivalence relation \(\equiv_A\) is actually a symmetry relation with respect to the vertex subset \(A\). Let \(A \subseteq V(G)\) and let \(v, v'\) be two distinct vertices of \(G\). We set \(X := A \cup \{v, v'\}\) and we define a map \(\phi_{v,v'}^A : X \to X\) as follows: if \(x \in X\) then

\[
\phi_{v,v'}^A(x) := \begin{cases} 
x & \text{if } x \in A \setminus \{v, v'\} \\
v' & \text{if } x = v \\
v & \text{if } x = v'.
\end{cases}
\] (2.3)

**Proposition 2.1.3.** With the above notations, if \(v \sim v'\) or \(A \cap \{v, v'\} = \emptyset\), then

\[
v \equiv_A v' \iff \phi_{v,v'}^A \in \text{Aut}(G[X])
\]
Proof. We set $\psi := \phi^A_{v,v'}$. Firstly we assume that $v \equiv_A v'$. In order to prove that $\psi \in \text{Aut}(G[X])$ we observe, by (2.3), that the map $\psi$ is bijective. Therefore we only need to show that for all $x, y \in X$ it results that

$$x \sim y \iff \psi(x) \sim \psi(y).$$

(2.4)

Let $x, y \in X$. Let us note that if $x, y \in A \setminus \{v, v'\}$ by (2.3) we have $\psi(x) = x$ and $\psi(y) = y$, therefore (2.4) is satisfied. We can assume then that $\{x, y\} \cap \{v, v'\} \neq \emptyset$. We distinguish two cases. Firstly, let us suppose that $x \in \{v, v'\}$ and $y \notin \{v, v'\}$ (the condition $x \notin \{v, v'\}$ and $y \in \{v, v'\}$ is similar. In this case we can assume without loss of generality that $x = v$. Since $y \in X$ and $y \notin \{v, v'\}$, it follows that $y \in A$. By (i) of Proposition 2.1.2 we obtain that

$$v \sim y \iff v' \sim y.$$  

(2.5)

On the other hand, since $y \neq v$ and $y \neq v'$, by (2.3) we deduce that $\psi(y) = y$, and this implies that (2.5) is equivalent to (2.4). The other case that we have to examine is $\{x, y\} = \{v, v'\}$. We can assume without loss of generality that $x = v$ and $y = v'$. By (2.3) it follows that $\psi(x) = y$ and $\psi(y) = x$, and due to the symmetry of the relation $\sim$ we obtain (2.4).

We suppose now that $\psi \in \text{Aut}(G[X])$ and we prove that for all $z \in A$ it results

$$z \sim v \iff z \sim v'.$$  

(2.6)

By (i) of Proposition 2.1.2, the equation (2.6) is equivalent to show that $v \equiv_A v'$. Let therefore $z \in A$. Firstly we assume that $z \notin \{v, v'\}$. By (2.3) we have then $\psi(z) = z$. From this last identity and since $\psi \in \text{Aut}(G[X])$ we deduce that

$$v \sim z \iff \psi(v) \sim \psi(z) \iff v' \sim z,$$

that is exactly (2.6). Now, if $A \cap \{v, v'\} = \emptyset$, since $z \in A$, it cannot be $z \in \{v, v'\}$. Hence, if $A \cap \{v, v'\} = \emptyset$ the proof is completed. We can suppose then that $v \sim v'$ and $z \in \{v, v'\}$. Without loss of generality we can assume that $z = v$. Since $G$ is a simple graph, $(z = v) \sim v$, and by hypothesis $(z = v) \sim v'$, hence (2.6) is satisfied. This complete the proof.

Corollary 2.1.4. In the same hypotheses of the previous theorem, if $v \equiv_A v'$ then $|\text{Aut}(G[X])| > 1$. Thus, the graph $G[X]$ is symmetric in the Erdős-Rényi terminology [83].

The result described in Proposition 2.1.3 tells us that the relation $\equiv_A$ can be considered as a type of symmetry relation with respect to the fixed vertex subset $A$. In other terms, we can think the blocks of the set partition $\pi_G(A)$ as if they were symmetry blocks with respect to $A$.

2.1.2 Pairings by Digraphs

To any digraph $D$ we can associate the pairing $\mathfrak{P}[D] := (V(D), F, \{0, 1\}) \in \text{PAIR}(\Omega)$ where $F(v_i, v_j) := 1$ if $v_i \rightarrow v_j$ and $F(v_i, v_j) := 0$ otherwise. Hereafter, we write $D$ instead of $\mathfrak{P}[D]$.

We now provide a basic property of $A$-symmetry for digraphs.

Theorem 2.1.5. Let $A \subseteq V(D)$ and $v, v' \in V(D)$. The following conditions are equivalent:

(i) $v \equiv_A v'$.

(ii) For all $z \in A$ it results that $v \rightarrow z$ if and only if $v' \rightarrow z$.

(iii) $N^+_D(v) \cap A = N^+_D(v') \cap A$. 

16
Definition 2.1.6. Let us start by defining a pairing from a hypergraph.

2.1.3 Pairings by Hypergraphs

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have all the same out-adjacency relation with respect all vertices in

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Proposition 2.1.7. Let

therefore

a /

A

A

= F

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v,z

v,a

Proof. (i) \implies (ii): Let \( z \in A \) and \( v \rightarrow z \), we show that \( v' \rightarrow z \). By (i) we have that \( F_D(v,a) = F_D(v',a) \) for all \( a \in A \), therefore \( F_D(v,z) = F_D(v',z) \). Since \( v \rightarrow z \) it follows that \( F_D(v,z) = 1 \), and hence also \( F_D(v',z) = 1 \), that is \( v' \rightarrow z \). By symmetry of the relation \( \equiv_A \), if we assume that \( v' \rightarrow z \), we obtain \( v \rightarrow z \). This proves (ii)

(ii) \implies (iii): By symmetry of the condition (ii), it is sufficient to prove that \( N_D^+(v) \cap A \subseteq N_D^+(v') \cap A \). Let therefore \( z \in N_D^+(v) \cap A \), then \( v \rightarrow z \) and \( z \in A \). By (ii) we have then that \( v' \rightarrow z \), that is \( z \in N_D^+(v') \). Hence \( z \in N_D^+(v') \cap A \).

(iii) \implies (i): We must show that \( F_D(v,a) = F_D(v',a) \) for all \( a \in A \). Let \( a \in A \) such that \( F_D(v,a) = 1 \). Then \( v \rightarrow a \), i.e. \( a \in A \cap N_D^+(v) \). Since by hypothesis \( N_D^+(v) \cap A = N_D^+(v') \cap A \), we have \( v' \rightarrow a \), i.e. \( F_D(v',a) = 1 \). On the other hand, let \( a \in A \) such that \( F_D(v,a) = 0 \). This means that \( v \not\rightarrow a \), i.e. \( a \notin A \cap N_D^+(v) \). Therefore, \( a \notin N_D^+(v') \cap A \) and thus \( v' \not\rightarrow a \), i.e. \( F_D(v',a) = 0 \).

Thus, if \( v \in V(D) \), it results by Theorem 2.1.5 that \( v' \in [v]_A \) if and only if \( v \) and \( v' \) “out-see” all the vertices \( z \in A \) in the same way, that is, \( v \rightarrow z \) if and only if \( v' \rightarrow z \) for all \( z \in A \). So that an \( A \)-symmetry class for the pairing \( \mathcal{F} \) is exactly a vertex subset of \( V(D) \) whose vertices have all the same out-adjacency relation with respect all vertices in \( A \). Therefore, if we fix a vertex subset \( A \) of \( V(D) \), we can interpret \( A \) as if it were a type of symmetry block against which to examine the out behavior of all vertices in \( D \).

2.1.3 Pairings by Hypergraphs

Let us start by defining a pairing from a hypergraph.

Definition 2.1.6. Let \( H \) be a hypergraph on a set \( \Omega = \{x_1, \ldots, x_n\} \), with hyperedges \( Y_1, \ldots, Y_m \). We associate with \( H \) the pairing \( \Gamma(H) \) on \( \Omega \) defined as follows. The object set of \( \Gamma(H) \) is \( \{Y_1, \ldots, Y_m\} \), \( A = \{0,1\} \) and \( F \) is

\[
F(Y_i, x_j) := \begin{cases} 
1 & \text{if } x_j \in Y_i \\
0 & \text{otherwise.}
\end{cases}
\]

Proposition 2.1.7. Let \( H \) be a hypergraph on a set \( \Omega = \{x_1, \ldots, x_n\} \), with hyperedge family \( \mathcal{F} = \{Y_1, \ldots, Y_m\} \) and \( \Gamma(H) = (\mathcal{F}, \{0,1\}, F) \). Let \( A \subseteq \Omega \). Then, if \( \equiv_A \) is the \( A \)-symmetry relation in \( \Gamma(H) \), we have that

\( Y_i \equiv_A Y_j \iff Y_i \cap A = Y_j \cap A \)

for all \( i, j \in \{1, \ldots, m\} \).

Proof. We assume that \( Y_i \equiv_A Y_j \). If \( a \in A \cap Y_i \), then \( a \in Y_i \), whence \( F(Y_i, a) = 1 \). Therefore, we also have \( F(Y_j, a) = 1 \), i.e. \( a \in Y_j \) and therefore \( A \cap Y_i \subseteq A \cap Y_j \). The proof of other inclusion is similar.

On the other hand, suppose that \( A \cap Y_i = A \cap Y_j \). Let \( a \in A \). If \( F(Y_i, a) = 1 \), then \( a \in A \cap Y_i = A \cap Y_j \) and hence \( F(Y_j, a) = 1 \). If \( F(Y_i, a) = 0 \), then \( a \in A \) but \( a \notin Y_i \), hence \( a \notin A \cap Y_i = A \cap Y_j \), therefore \( a \notin Y_j \), and this implies that \( F(Y_j, a) = 0 \). This proves that \( Y_i \equiv_A Y_j \) because of the arbitrariness of \( a \).

\[\square\]
Let us note that, by Proposition 2.1.7, if \( Y \) is a hyperedge of \( H \) then
\[
[Y]_A = \{ Y_i \in \mathcal{F} : Y_i \cap A = Y \cap A \}. \tag{2.7}
\]

### 2.1.4 Pairings by Metric Spaces

Let \((X,d_X)\) be a metric space and \( \Omega = X \). We can consider the pairing \( p[X,d_X] := (X,F,\{0,1\}) \in \text{PAIR}(\Omega) \) where \( F : X \times X \to \{0,1\} \) defined by \( F(u,u') := 1 \) if \( d_X(u,u') = 1 \) and \( F(u,u') := 0 \) otherwise. It is immediate to see that the A-symmetry relation can be translated as follows: if \( u,u' \in X \), then
\[
u \equiv_A u' : \iff N_X(u) \cap A = N_X(u') \cap A, \tag{2.8}
\]
where \( N_X(u) = \{ u' \in X : d_X(u,u') = 1 \} \). Therefore, also in this case, \( \equiv_A \) can be naturally interpreted as a symmetry relation with respect to the fixed subset \( A \).

### 2.1.5 Pairings by Vector Spaces

If \( V \) is a vector space and \( H \subseteq V \), we write \( H \leq V \) when \( H \) is a vector subspace of \( V \).

Let \( V \) be a vector space over a field \( K \) and \( \varphi : V \times V \to K \) a \( K \)-bilinear form on \( V \). We call it the \((V,\varphi,K)\)-vector bilinear pairing, abbreviated \((V,\varphi,K)\)-VBP. Hereafter, we write \( V \) instead of \( p[V,\varphi] \).

Let \( A \) be an arbitrary fixed subset of \( V \). We denote by \( A^\perp \) the orthogonal subspace of \( A \) with respect to \( \varphi \), i.e. \( A^\perp := \{ v \in V : \varphi(v,a) = 0 \ \forall a \in A \} \). In particular, the subspace \( V^\perp \) is the classical (left) radical of \( V \) and the form is said (left) non degenerate if \( V^\perp = \{0\} \). We denote by \( V^* := \text{Hom}(V,K) \) the dual vector space of \( V \). In particular, if \( V \) has finite dimension, then \( \varphi \) is non degenerate if and only if the map \( d_\varphi : v \in V \to \varphi(v,\cdot) \in V^* \) is an isomorphism, that is if \( \text{Ker}(d_\varphi) = \{ v \in V : \varphi(v,w) = 0 \ \forall w \in V \} = V^\perp = \{0\} \).

Now, we examine the \( A \)-symmetry relation relative to the \((V,\varphi,K)\)-VBP. For any \( u,u' \in V \), we set
\[
u \equiv_A u' \iff \varphi(u,a) = \varphi(u',a) \ \forall a \in A \iff \varphi(u-u',a) = 0 \ \forall a \in A. \tag{2.9}
\]
In other terms,
\[
u \equiv_A u' \iff u \equiv u' \mod A^\perp \iff u + A^\perp = u' + A^\perp.
\]
Therefore \( [u]_A = u + A^\perp \) for any \( u \in V \) and
\[
\pi_p(A) = \{ u + A^\perp : u \in V \} = V/A^\perp.
\]
So that, in this case, the set partition \( \pi_p(A) \) has itself a vector space structure that coincides with that of the quotient vector space \( V/A^\perp \).
2.1.6 Pairings by Group Actions

Let $\Omega$ be a generic set and $G$ a group acting on it. Let us denote by $\psi : (g,x) \in G \times \Omega \mapsto \psi((g,x)) = gx \in \Omega$ this action. Then, we can associate with $G$ the pairing $\mathcal{P}[G,\psi] := (G,\psi,\Omega) \in \text{PAIR}(\Omega)$. Hereafter, we write $(G,\psi)$ instead of $\mathcal{P}[G,\psi]$.

Let $A \subseteq \Omega$. We recall that the stabilizer of $A$ is the following subgroup of $G$:

$$\text{Stab}_{G,\psi}^A := \{ g \in G : ga = a \ \forall a \in A \} = \bigcap_{a \in A} \text{Stab}_{G,\psi}^a.$$ 

In this context, the $A$-symmetry relation becomes

$$g \equiv_A g' \iff ga = g'a \ \forall a \in A \iff (g')^{-1}g \in \text{Stab}_{G,\psi}^a \ \forall a \in A$$

so that

$$g \equiv_A g' \iff (g')^{-1}g \in \text{Stab}_{G,\psi}^A.$$ 

This means that

$$[g]_A = g \text{ Stab}_{G,\psi}^A$$

and

$$\pi_{G,\psi}^A = \{ g \text{ Stab}_{G,\psi}^A : g \in G \}.$$ 

2.2 Global Symmetry and Maximum Partitioners

In this section we investigate the main properties of the global symmetry relation. To this regard, we first introduce the map $\pi_{\mathcal{P}} : \mathcal{P}(\Omega) \to \pi_{\mathcal{P}}(U)$ defined by

$$A \in \mathcal{P}(\Omega) \mapsto \pi_{\mathcal{P}}(A),$$

and we denote by $\text{PSYM}(\mathcal{P})$ the image of the operator $\pi_{\mathcal{P}}$, i.e.

$$\text{PSYM}(\mathcal{P}) := \{ \pi_{\mathcal{P}}(A) : A \in \mathcal{P}(\Omega) \},$$

$$\mathcal{P}(\mathcal{P}) := (\text{PSYM}(\mathcal{P}), \preceq),$$

so that $\mathcal{P}(\mathcal{P})$ is a sub-poset of the partition lattice $(\Pi(U), \preceq)$.

Let us note that

$$A \subseteq A' \implies \pi_{\mathcal{P}}(A') \preceq \pi_{\mathcal{P}}(A).$$

By starting from the $A$-symmetry relation, we introduce a type of global symmetry relation relatively to a given pairing. To this regard, let us consider the following equivalence relation on $\mathcal{P}(\Omega)$:

$$A \approx_{\mathcal{P}} A' : \iff \pi_{\mathcal{P}}(A) = \pi_{\mathcal{P}}(A'),$$

that is equivalent to the condition

$$u \equiv_A u' \iff u \equiv_{A'} u',$$

for all $u, u' \in U$. We call $\approx_{\mathcal{P}}$ the global symmetry relation of $\mathcal{P}$, and we denote by $[A]_{\approx_{\mathcal{P}}}$ the equivalence class of $A$ with respect to $\approx_{\mathcal{P}}$, that we call global symmetry class.

The basic properties of the global symmetry relation are established in the following result.
Theorem 2.2.1. (i) If $A \approx \Omega A'$ and $D \subseteq \Omega$, then $A \cup D \approx \Omega A' \cup D$.

(ii) The global symmetry class $[A]_{\approx \Omega}$ is a union-closed family.

(iii) The subset family $[A]_{\approx \Omega}$ has a maximum $M_{\Omega}(A)$ that coincides with $\bigcup [A]_{\approx \Omega}$.

(iv) $M_{\Omega}(A) = \{ b \in \Omega : A \cup \{ b \} \approx \Omega A \}$. 

(v) $M_{\Omega}(A) = \{ a \in \Omega : (u, u' \in \Omega \cap u \equiv A u') \implies F(u, a) = F(u', a) \}$. 

(vi) $M_{\Omega}(A) \supseteq M_{\Omega}(A') \iff \pi_{\Omega}(A) \preceq \pi_{\Omega}(A')$. 

(vii) The set operator $M_{\Omega} : B \in \mathcal{P}(\Omega) \mapsto M_{\Omega}(B) \in \mathcal{P}(\Omega)$ is a closure operator on $\Omega$.

(viii) The family

$$MAXP(\Omega) := \{ M_{\Omega}(B) : B \in \mathcal{P}(\Omega) \} = \{ C \in \mathcal{P}(\Omega) : M_{\Omega}(C) = C \}$$

is a closure system on $\Omega$.

(ix) $A \approx \Omega A' \iff M_{\Omega}(A) = M_{\Omega}(A')$.

Proof. (i): Let $u \equiv_{A \cup \{ b \}} u'$, i.e. $F(u, a) = F(u', a)$ for any $a \in A \cup D$. This means that $u \equiv_{A} u'$ and, since $A \approx \Omega A'$, that $u \equiv_{A'} u'$. But, in particular, if follows that $F(u, d) = F(u', d)$ for any $d \in A' \cup D$, i.e. $u \equiv_{A \cup \{ b \}} u'$. The same argument holds when we start with $u \equiv_{A \cup \{ b \}} u'$.

(ii): Let $u, u' \in U$ such that $u \equiv_{A} u'$ and let $\{ A_j : j \in J \} \subseteq [A]_{\approx \Omega}$. Hence, $u \equiv_{A_j} u'$ for all $j \in J$ by definition of the relation $\equiv_{\Omega}$. If $z \in \bigcup_{j \in J} A_j$ there exists some index $j \in J$ such that $z \in A_j$, so that $F(u, z) = F(u', z)$ because $u \equiv_{A_j} u'$. Hence $u \equiv_{\bigcup_{j \in J} A_j} u'$. This implies that $\pi_{\Omega}(A) \preceq \pi_{\Omega}(\bigcup_{j \in J} A_j)$. On the other hand, by (2.13) we have that $\pi_{\Omega}(\bigcup_{j \in J} A_j) \preceq \pi_{\Omega}(A_j) = \pi_{\Omega}(A)$ because $A_j \approx \Omega A$. Since $\preceq$ is a partial order on $PSYM(\Omega)$ we deduce that $\pi_{\Omega}(A) = \pi_{\Omega}(\bigcup_{j \in J} A_j)$, that is equivalent to the condition $\bigcup_{j \in J} A_j \in [A]_{\approx \Omega}$.

(iii): Let $M_{\Omega}(A) := \bigcup \{ B : B \in [A]_{\approx \Omega} \}$. By part (ii), it follows that $M_{\Omega}(A) \in [A]_{\approx \Omega}$, moreover we also have that $B \subseteq M_{\Omega}(A)$ for all $B \in [A]_{\approx \Omega}$. Uniqueness is obvious.

(iv): Let $b \in M_{\Omega}(A)$. Then, there exists $A' \in [A]_{\approx \Omega}$ such that $b \in A'$. Let now $u, u' \in U$ such that $u \equiv_{A \cup \{ b \}} u'$, then $u \equiv_{A} u'$ because $A$ is a subset of $A \cup \{ b \}$. On the other hand, let assume that $u \equiv_{A} u'$. Since $A' \in [A]_{\approx \Omega}$, we have $u \equiv_{A'} u'$ and therefore $F(u, b) = F(u', b)$ because $b \in A'$. Hence $u \equiv_{A \cup \{ b \}} u'$, so we deduce that $A \approx \Omega A \cup \{ b \}$.

Let now $b \in \Omega$ such that $A \approx \Omega A \cup \{ b \}$. Then, we have that $b \in A \cup \{ b \} \in [A]_{\approx \Omega}$ and we deduce that $b \in M_{\Omega}(A)$.

(v): Let $B := \{ b \in \Omega : (u, u' \in U \cap u \equiv_{A} u') \implies F(u, b) = F(u', b) \}$. We show that $B \approx \Omega A$.

In fact, let $u, u' \in U$ such that $u \equiv_{A} u'$ and let $b \in B$. By definition of $B$ we have that $F(u, b) = F(u', b)$, so that $u \equiv_{B} u'$. Hence $\pi_{\Omega}(A) \preceq \pi_{\Omega}(B)$. Let now $C \in [A]_{\approx \Omega}$ and $c \in C$. Then, for all $u, u' \in U$ such that $u \equiv_{A} u'$, by definition of $\approx_{\Omega}$ we have that $u \equiv_{C} u'$, so that $F(u, c) = F(u', c)$. It follows that $C \subseteq B$ for all $C \in [A]_{\approx \Omega}$, in particular $A \subseteq M_{\Omega}(A) \subseteq B$. By (2.13) we obtain then $\pi_{\Omega}(B) \preceq \pi_{\Omega}(A)$. Therefore $\pi_{\Omega}(B) = \pi_{\Omega}(A)$, i.e. $B \in [A]_{\approx \Omega}$. By part (iii) we deduce then that $B \subseteq M_{\Omega}(A)$. Hence $B = M_{\Omega}(A)$.

(vi): If $A$ and $A'$ are two subsets of $\Omega$ such that $M_{\Omega}(A') \subseteq M_{\Omega}(A)$ then $\pi_{\Omega}(M_{\Omega}(A)) \preceq \pi_{\Omega}(M_{\Omega}(A'))$ by virtue of (2.13), which is equivalent to $\pi_{\Omega}(A) \preceq \pi_{\Omega}(A')$ by virtue of part (ii).

We assume now that $\pi_{\Omega}(M_{\Omega}(A)) \preceq \pi_{\Omega}(M_{\Omega}(A'))$. Let $v \in M_{\Omega}(A')$ and let us suppose by contradiction that $v \not\in M_{\Omega}(A)$, so that $M_{\Omega}(A) \not\subseteq M_{\Omega}(a \cup \{ v \})$. By (2.13) we obtain then $\pi_{\Omega}(M_{\Omega}(A) \cup \{ v \}) \not\preceq \pi_{\Omega}(M_{\Omega}(A))$. On the other hand, by maximality of $M_{\Omega}(A)$ proved in part (ii) we also deduce that $M_{\Omega}(A) \not\preceq M_{\Omega}(A) \cup \{ v \}$, i.e. $\pi_{\Omega}(M_{\Omega}(A)) \neq \pi_{\Omega}(M_{\Omega}(A) \cup \{ v \})$.

Then, there exist $u, u' \in U$ such that $u \equiv_{M_{\Omega}(A)} u'$ and $u \not\equiv_{M_{\Omega}(A) \cup \{ v \}} u'$, and this is possible
only if \( F(u, v) \neq F(u', v) \). Now, since \( \pi_\Psi(M_\Psi(A)) \leq \pi_\Psi(M_\Psi(A')) \) and \( u \equiv_{M_\Psi(A)} u' \), we obtain \( u \equiv_{M_\Psi(A')} u' \), therefore \( F(u, v) = F(u', v) \) because \( v \in M_\Psi(A') \). This shows the contradiction and conclude the proof.

(vii): We now prove that the set operator \( M_\Psi \) is a closure operator. It is obvious that \( A \subseteq M_\Psi(A) \). Let now \( A' \in P(\Omega) \) such that \( A \subseteq A' \). Let \( b \in M_\Psi(A) \). Thus \( A \equiv_\Psi A \cup \{b\} \). Let \( v, v' \in \Omega \) such that \( v \equiv_\Psi A' \equiv_{\Psi}(v') \). Since \( A \) is a subset of \( A' \), we have that \( v \equiv_\Psi A \) and this implies \( F(v, b) = F(v', b) \) because \( A \equiv_\Psi A \cup \{b\} \). This shows that \( v \equiv_{A' \cup \{b\}} v' \), that is \( v \in M_\Psi(A') \). Hence \( M_\Psi(A) \subseteq M_\Psi(A') \). Finally, we must prove that \( M_\Psi(M(A)) = M_\Psi(A) \), for this it is sufficient to show that \( M_\Psi(M(A)) \subseteq M_\Psi(A) \). Let then \( b \in M_\Psi(M(A)) \) and \( v, v' \in \Omega \) such that \( v \equiv_\Psi v' \). Since \( A \equiv_\Psi M_\Psi(A) \), it follows that \( v \equiv_{M_\Psi(A)} v' \), therefore \( F(v, b) = F(v', b) \) by part (v), because \( b \) belongs to \( M_\Psi(M(A)) \). Hence \( b \in M_\Psi(A) \) again by part (v).

(viii): \( MAXP(\Psi) \) is the family of all closed sets for the closure operators \( M_\Psi \). Therefore the result follows by Theorem 1.5.7.

(ix) : If \( A \equiv_\Psi A' \), by part (ii), it follows that \( M_\Psi(A) = M_\Psi(A') \). On the other hand, let \( M_\Psi(A) = M_\Psi(A') \). Then \( A \equiv_\Psi M_\Psi(A) \) and \( A' \equiv_\Psi M_\Psi(A') \) and we conclude that \( A \equiv_\Psi A' \). □

We set now

\[
\mathcal{M}(\Psi) := (MAXP(\Psi), \subseteq^*)
\]

By Theorem 2.2.1 we deduce then the following result.

**Corollary 2.2.2.** (i) \( \mathcal{M}(\Psi) \) is a complete lattice such that if \( \{A_j : j \in J\} \subseteq MAXP(\Psi) \) then:

(a) \( \bigvee_{j \in J} A_j = \bigcap_{j \in J} A_j \);

(b) \( \bigwedge_{j \in J} A_j = M_\Psi(\bigcup_{j \in J} A_j) \).

(ii) The map \( \pi_\Psi \) induces an order isomorphism between the posets \( \mathcal{M}(\Psi) \) and \( P(\Psi) \). Moreover, since \( \mathcal{M}(\Psi) \) is a complete lattice, the same holds also for \( P(\Psi) \).

(iii) If \( A, A' \in MAXP(\Psi) \) then

\[
A \supseteq A' \iff \pi_\Psi(A) \preceq \pi_\Psi(A')
\]

**Proof.** (i): It follows by (vii) of Theorem 2.2.1 and by Theorem 2, p. 112 of [18].

(ii): It is an immediate consequence of (vi) of Theorem 2.2.1.

(iii): It is a special case of part (vi) of Theorem 2.2.1. □

**Definition 2.2.3.** We call the elements of \( MAXP(\Psi) \) the **maximum partitioners** of \( \Psi \) and \( \mathcal{M}(\Psi) \) the **symmetry partition lattice of \( \Psi \)**. We call the set operator \( M_\Psi \) the \( \Psi \)-maximum partitioner operator.

For the sake of completeness, we introduce a relativization of the notion of maximum partitioner. For any \( A \in P(\Omega) \), we set

\[
MAXP_\Psi(A) := \{B \cap A : B \in MAXP(\Psi)\}, \quad M_\Psi(A) := (MAXP_\Psi(A), \subseteq^*)
\]

therefore we obtain a map \( MAXP_\Psi : P(\Omega) \rightarrow P(\Psi(\Omega)) \) such that \( MAXP_\Psi(\Omega) = MAXP(\Psi) \) and

\[
MAXP_\Psi(A) \in CLSY(A),
\]

for any \( A \in P(\Omega) \).
**Definition 2.2.4.** We call an *A-relative maximum partitioner* of $\mathcal{P}$ any element of $\text{MAXP}_\mathcal{P}(A)$.

We now prove that the family $\text{MAXP}_\mathcal{P}(A)$ coincides with the family of all maximum partitioners contained in $A$ whenever $A \in \text{MAXP}(\mathcal{P})$.

**Proposition 2.2.5.** If $A \in \text{MAXP}(\mathcal{P})$, then

$$\text{MAXP}_\mathcal{P}(A) = \{ B \in \text{MAXP}(\mathcal{P}) : B \subseteq A \}$$

**Proof.** Let $B \in \text{MAXP}_\mathcal{P}(A)$. Then, there exists $C \in \text{MAXP}(\mathcal{P})$ such that $B = A \cap C$. Therefore, $B \subseteq A$ and $B \in \text{MAXP}(\mathcal{P})$ because $\text{MAXP}(\mathcal{P})$ is a closure system. On the other hand, let $B \in \text{MAXP}(\mathcal{P})$ and $B \subseteq A$. Then $B = A \cap B \in \text{MAXP}_\mathcal{P}(A)$. $\square$

### 2.2.1 Characterization of the Symmetry Partitions

In what follows, we want to understand when a given set partition $\pi$ of $U$ is an $A$-symmetry partition of $U$, for some $A \in \mathcal{P}(\Omega)$.

If $W \subseteq U$ we set

$$\Xi_\mathcal{P}(W) := \{ a \in \Omega : \forall u, u' \in W, F(u, a) = F(u', a) \}.$$  \hfill (2.20)

Let us observe that $\Xi_\mathcal{P} : \mathcal{P}(U) \to \mathcal{P}(\Omega)$. Moreover, if $W' \in \mathcal{P}(U)$ it is clear that

$$W \subseteq W' \implies \Xi_\mathcal{P}(W) \supseteq \Xi_\mathcal{P}(W').$$  \hfill (2.21)

We have the following basic result.

**Proposition 2.2.6.** $\Xi_\mathcal{P}(W) = A$ if and only if the following conditions hold:

(i) $W \preceq \pi_\mathcal{P}(A)$;

(ii) if $W \preceq \pi_\mathcal{P}(A')$, then $A' \subseteq A$.

**Proof.** Straightforward. $\square$

If $\pi = \{ B_i : i \in I \} \in \pi_\mathcal{P}(U)$ we set now

$$\text{Max}(\pi) := \bigcap_{i \in I} \Xi_\mathcal{P}(B_i).$$  \hfill (2.22)

By means of the set operator $\Xi_\mathcal{P}$ we can characterize now the symmetry partitions of $\mathcal{P}$.

**Theorem 2.2.7.** Let $\pi = \{ B_i : i \in I \} \in \pi(U)$ and $A = \text{Max}(\pi)$. Then:

(i) $A \in \text{MAXP}(\mathcal{P})$;

(ii) $\pi \preceq \pi_\mathcal{P}(A)$;

(iii) $\pi \in \text{PSYM}(\mathcal{P}) \iff \pi = \pi_\mathcal{P}(A)$;
Let \( \pi(A') = \{ C_l : l \in I' \} \). Then
\[
M_\mathbb{P}(A') = \bigcap_{i \in I'} \Xi_\mathbb{P}(C_i).
\]

**Proof.**

(i): Since \( \text{MAXP}(\mathbb{P}) \) is a closure system and \( \Xi_\mathbb{P}(B_i) \in \text{MAXP}(\mathbb{P}) \), the thesis holds.

(ii): Let \( v, v' \in V(\mathbb{P}) \) belonging to a same block \( B_k \) of \( \pi \). We must show that \( v \equiv_A v' \). Therefore, let \( a \in A \). Then one has \( a \in \Xi_\mathbb{P}(B_k) \), so \( F(v, a) = F(v', a) \) by definition of \( \Xi_\mathbb{P}(B_k) \).

(iii): The implication \( \Leftarrow \) is obvious. We assume therefore that \( \pi \in \text{PSYM}(\mathbb{P}) \). Let \( C = \text{Max}(\pi) \). Then we have \( \pi = \pi(\mathbb{P})(C) \), and this implies that \( F(v, c) = F(v', c) \) for all \( v, v' \in B_i \), for each \( i = 1, \ldots, s \) and each \( c \in C \). By definition of \( \Xi_\mathbb{P}(B_i) \) we deduce then that \( C \subseteq \Xi_\mathbb{P}(B_i) \) for every \( i = 1, \ldots, s \). Hence \( C \subseteq A \).

Let now \( a \in A \). By definition of \( A \) we have
\[
F(v, a) = F(v', a), \quad \forall v, v' \in B_i, \quad \text{for each } i \in I.
\]  

Let us suppose now by contradiction that \( a \notin C \). Since \( C \in \text{MAXP}(\mathbb{P}) \), by part (i) it follows that \( \pi(C \cup \{ a \}) \prec \pi(C) = \pi \). Therefore there exists at least an index \( r \in \{ 1, \ldots, s \} \) and two distinct vertices \( v, v' \in B_r \) such that \( F(v, a) \neq F(v', a) \), and this in contrast with (2.23). This shows that \( A \subseteq C \). Thus \( A = C = \pi(\mathbb{P})(C) = \pi(\mathbb{P})(A) \). This concludes the proof of (iii).

(iv): We prove only the statement for \( A' \). We set \( \pi := \pi_\mathbb{P}(A') \) and \( A := \bigcap_{i \in I''} \Xi_\mathbb{P}(C_i) \), so that \( \pi \in \text{PSYM}(\mathbb{P}) \). By part (iii), it follows that \( \pi_\mathbb{P}(A') = \pi = \pi_\mathbb{P}(A) \), that is \( A \approx_\mathbb{P} A' \). Hence, by (vi) of Theorem 2.2.1 and again by part (iii), we obtain \( M_\mathbb{P}(A') = M_\mathbb{P}(A) = A \). \( \square \)

### 2.2.2 Global Symmetry Lattice

In this subsection we introduce the global symmetry lattice of a pairing \( \mathbb{P} \). Let \( \text{MAXP}(\mathbb{P}) = \{ C_j : j \in J \} \). We set now
\[
\text{GSYM}(\mathbb{P}) = \{ [C_j]_{\approx_\mathbb{P}} : j \in J \}.
\]  

We introduce the following partial order \( \sqsubseteq \) on \( \text{GSYM}(\mathbb{P}) \). If \( [C_i]_{\approx_\mathbb{P}}, [C_j]_{\approx_\mathbb{P}} \in \text{GSYM}(\mathbb{P}) \) we set
\[
C_i \subseteq^* C_j \iff [C_i]_{\approx_\mathbb{P}} \sqsubseteq [C_j]_{\approx_\mathbb{P}}
\]
and
\[
\mathbb{G}(\mathbb{P}) := (\text{GSYM}(\mathbb{P}), \sqsubseteq).
\]

We have then the following immediate result

**Theorem 2.2.8.** \( \mathbb{G}(\mathbb{P}) \) is a lattice that is order isomorphic to the symmetry partition lattice \( \mathbb{M}(\mathbb{P}) \).

**Proof.** It follows immediately by (2.25). \( \square \)

We call \( \mathbb{G}(\mathbb{P}) \) the global symmetry lattice of \( \mathbb{P} \) and any \( ([C_j]_{\approx_\mathbb{P}}, \sqsubseteq) \) a local symmetry poset of \( \mathbb{P} \).

In the next result we show that the knowledge of \( \text{MAXP}(\mathbb{P}) \) uniquely determine the knowledge of the whole global symmetry lattice \( \mathbb{G}(\mathbb{P}) \).

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Theorem 2.2.9. Let $\mathcal{P}, \mathcal{P}' \in PAIR(\Omega)$. Then, the following are equivalent:

(i) $\approx_{\mathcal{P}}$ coincides with $\approx_{\mathcal{P}'}$;

(ii) $\mathbb{G}(\mathcal{P}) = \mathbb{G}(\mathcal{P}')$;

(iii) $MAXP(\mathcal{P}) = MAXP(\mathcal{P}')$.

Proof. (i) $\implies$ (ii): Obvious.

(ii) $\implies$ (iii): Since $\mathbb{G}(\mathcal{P}) = \mathbb{G}(\mathcal{P}')$, for any $A \in \mathcal{P}(\Omega)$ we have that $[A]_{\mathbb{G}(\mathcal{P})} = [A]_{\mathbb{G}(\mathcal{P}')}$. Therefore, the maximum partitioners $M_{\mathcal{P}}(A)$ and $M_{\mathcal{P}'}(A)$ coincide for any $A \in \mathcal{P}(\Omega)$, so $MAXP(\mathcal{P}) = MAXP(\mathcal{P}')$.

(iii) $\implies$ (i): Since $M_{\mathcal{P}}$ and $M_{\mathcal{P}'}$ are the closure operators associate respectively to the closure systems $MAXP(\mathcal{P})$ and $MAXP(\mathcal{P}')$, by part (viii) of Theorem 2.2.1 we obtain (i). \qed

2.3 Global Symmetry on Several Examples

In this section we provide some computations on the concrete pairings defined in Section 2.1.

2.3.1 Some Examples Taken From Graph Context

We now compute the maximum partitioner of some basic graph families.

Complete Multipartite Graphs

In what follows, we firstly determine the general form of the $A$-symmetry for $K_{r_1,\ldots,r_s}$ and then its symmetry partition lattice.

Proposition 2.3.1. Let $G := K_{r_1,\ldots,r_s} = (B_1|\ldots|B_s)$. Then $A \in MAXP(G)$ if and only if $A = V(G)$ or $A = \bigcup_{q=1}^{t} B_q$, where $1 \leq t \leq s - 2$.

Proof. We need to compute the $A$-symmetry partition of $K_{r_1,\ldots,r_s}$. For, let $A \subseteq V(G)$, then we set

$$Q_G(A) := \{B_i : A \cap B_i \neq \emptyset\}.$$  

We denote by $r$ the quantity $|Q_G(A)|$. We claim that

$$\pi_G(A) = B_{i_1}|\ldots|B_{i_r}|Q_G(A)^c$$

(2.27)

For, just observe that two vertices $u, u'$ belonging to the same subset $B_i$ are not adjacent to the vertices of the same subset but they are adjacent to the other vertices. In particular, we deduce that $u \equiv_A u'$ if and only if $u, u' \in B_i$ for some $i = 1, \ldots, s$ or $u, u' \in Q_G(A)^c$. Therefore, we have $\pi_G(A) = B_{i_1}|\ldots|B_{i_r}|Q_G(A)^c$ and this proves (2.27). Let us also observe that if $r = n - 1$, then $\pi_G(A) = B_1|\ldots|B_s = \pi_G(V(G))$. It is now straightforward to deduce that $A \in MAXP(G)$ if and only if $A = V(G)$ or $A = \bigcup_{q=1}^{t} B_q$, where $1 \leq t \leq s - 2$. \qed

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As a particular subcase, if $G$ is a complete bipartite graph, then $MAXP(G)$ is a 2-chain. We now show that the only graph $G$ for which $MAXP(G)$ is a 2-chain is the complete bipartite one.

**Proposition 2.3.4.** $MAXP(G)$ is a 2-chain if and only if $G$ is a complete bipartite graph.

*Proof.* Let $G$ be a graph such that $M(G)$ is a 2-chain. If $G$ is the $n$-null graph, then obviously $M(G)$ has only the trivial partition and this contradicts our hypothesis. Therefore let $v$ be a non isolated vertex of $G$. The partition induced by the singleton set $\{v\}$ has only two blocks: the neighborhood $N_G(v)$ of $v$ in $G$ and its complement in $V(G)$. Since $v \sim v$ and $N_G(v) \neq \emptyset$, this partition is different from $\pi_G(\emptyset) = \hat{1}_G$. Moreover to say that $M(G)$ is a 2-chain implies that $\pi_G(\{v\}) = \pi_G(V(G))$.

Let $u, u'$ be vertices of $G$. If $u, u' \in N_G(v)$ or $u, u' \notin N_G(v)$, then $u \equiv_{\{v\}} u'$ and thus $u \equiv_{V(G)} u'$. By part $(iii)$ of Proposition 2.1.2 we have therefore $u \sim u'$. If $u \in N_G(v)$ and $u' \notin N_G(v)$, then $u \not\equiv_{\{v\}} u'$, which is equivalent to $u \not\equiv_{V(G)} u'$. In this case, since $v \in N_G(u)$, $u$ is a non isolated vertex of $G$ and so, as before, $\pi_G(\{u\}) = \pi_G(V(G))$. Thus $u \not\equiv_{\{u\}} u'$ and $u \sim u$ implies $u \sim u'$. It follows that $G = (N_G(v), (V(G) \setminus N_G(v))$ is a complete bipartite graph.

Let $G = (V_1|V_2)$ be a complete bipartite graph. Let $A$ be a non-empty subset of $V(G)$ and $u, u' \in V(G)$. If $u, u' \in V_1$ or $u, u' \in V_2$, then for each $v \in A$ we have $F(v, u) = F(v, u')$, so $u \equiv_A u'$. If $u \in V_1$ and $u' \in V_2$, then for each $v \in A$ we have $F(v, u) \neq F(v, u')$, so $u \not\equiv_A u'$. Thus $\pi_G(A) = V_1|V_2$. It follows that $PSYM(G)$ has only two elements: the trivial partition $\pi_G(\emptyset) = \hat{1}_G$ and the partition $\pi_G(B) = V_1|V_2$, induced by all non-empty subsets $B$ of $V(G)$, with $\pi_G(B) \leq \hat{1}_G$, so $M(G)$ is a 2-chain. \hfill $\square$

**The Complete Graph and The $n$-Cycle**

In the next results we give the $A$-symmetry partition for the complete graph $K_n$ and its symmetry partition lattice.

**Proposition 2.3.3.** Let $n \geq 1$ and let $A = \{w_1, \ldots, w_k\}$ be a generic subset of $V(K_n) = \{v_1, \ldots, v_n\}$. Then

$$\pi_{K_n}(A) = w_1|w_2|\ldots|w_k|A^c.$$  

*Proof.* Let $v, v' \in V(K_n)$, with $v \neq v'$. By $(iii)$ of Proposition 2.1.2, since $v \sim v'$, it holds that if $v \equiv_A v'$, then $v, v' \in A^c$. On the other hand, if $v, v' \in A^c$, then $\forall z \in A, F(z, v) = F(z, v') = 1$, namely $v \equiv_A v'$. The proposition is proved. \hfill $\square$

We now provide an explicit expression for $MAXP(K_n)$. To this aim, we first introduce the following notation: if $n \geq k$ we set

$$\mathcal{V}(n,k) := \{K \subseteq \hat{n} : |K| \leq n-k \} \cup \{\hat{n}\} \quad (2.28)$$

**Proposition 2.3.4.** Let $n \geq 2$, $G = K_n$ and $V(G) = \hat{n}$. Then $MAXP(G) = \mathcal{V}(n,2)$.

*Proof.* By virtue of $(viii)$ of Theorem 2.2.1 the identity $\mathcal{V}(n,2) = MAXP(G)$ is equivalent to the following:

$$\mathcal{V}(n,2) = \{K \subseteq V(G) : K = M(K)\}. \quad (2.29)$$

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Let $K \in \mathcal{V}(n, 2)$. We must prove that $K = M(K)$. By definition of $M(K)$ we have that $K \subseteq M(K)$, therefore it remains to show that $M(K) \subseteq K$. By (v) of Theorem 2.2.1 we have that

$$M(K) = \{v \in V(G) : (u, u' \in V(G) \land u \equiv_K u') \implies F(u, v) = F(u', v)\}. \quad (2.30)$$

If $K = \hat{n}$, then obviously $K = M(K)$. We can assume therefore $K \neq \hat{n}$. Let now $v \in \hat{n} \setminus K$. Since $\left| K \right| \leq n - 2$, there exists a vertex $z \in \hat{n} \setminus K$ such that $v \neq z$. Since $0 = F(v, v) \neq F(v, z) = 1$ and $v \equiv_K z$ by (2.30) we have that $v \notin M(K)$. It follows that $\hat{n} \setminus K \subseteq \hat{n} \setminus M(K)$, and thus $M(K) \subseteq K$. Hence if $K \in \mathcal{V}(n, 2)$ then $K = M(K)$.

We assume now that $K \subseteq V(G)$ and $K = M(K)$. We must prove that $K = \hat{n}$ or $\left| K \right| \leq n - 2$. Let $K \neq \hat{n}$ and suppose by contradiction that $\left| K \right| = n - 1$. Then $K^c$ is a singleton and therefore we have that $\pi_G(K) = v_1|\cdots|v_n = \pi_G(V(G))$, by Proposition 2.3.3. Thus $K \approx V(G)$, and this implies $K = M(K) = V(G)$ by virtue of part (ix) of Theorem 2.2.1, which gives a contradiction because $K \neq \hat{n} = V(G)$. This shows that $\left| K \right| \leq n - 2$, i.e. $K \in \mathcal{V}(n, 2)$. Hence the identity in (2.29) is proved. Since $\mathcal{V}(n, 2) = \text{MAXP}(G)$, we have that $M(G) = (\mathcal{V}(n, 2), \subseteq^*)$, therefore the claim follows by (ii) of Corollary 2.2.2.

We provide below the $A$-symmetry partition of the cycle $C_n$ on $n$ vertices.

**Proposition 2.3.5.** (i) Let $V = V(C_n) = \{v_1, \ldots, v_n\}$. Fix a subset $A \subseteq V$, $A = \{v_{i_1}, \ldots, v_{i_k}\}$, and let $v_i, v_j \in V$, with $i < j$. Then $v_i \equiv_A v_j$ if and only if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \emptyset$ or $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$.

(ii) Let us set $B_{C_n}(A) := (N_{C_n}(A))^c$, $C_{C_n}(A) := \{v_i \in A : v_{i-2} \notin A \land v_{i+2} \notin A\}$ and $K_{C_n}(A) := N_{C_n}(C_{C_n}(A))$. Then, if $v_{s_1}, \ldots, v_{s_l}$ are the vertices in $S_{C_n}(A) := V(C_n) \setminus [B_{C_n}(A) \cup K_{C_n}(A)]$ and $v_{j_1}, \ldots, v_{j_h}$ are the vertices in $C_{C_n}(A)$, we have

$$\pi_{C_n}(A) = B_{C_n}(A)|v_{j_1-1}v_{j_1+1}v_{j_2-1}v_{j_2+1}v_{j_3}v_{j_4}v_{j_3}v_{j_4}|.$$  

**Proof.** (i): Let us observe that for each $i \in \{1, \ldots, n\}$, $N_{C_n}(v_i) = \{v_{i-1}, v_{i+1}\}$, where the index sums are all taken mod($n$). The proof follows easily by observing that, since $v_i \neq v_j$, then $|N_{C_n}(v_i) \cap N_{C_n}(v_j)| \leq 1$ and the equality holds if and only if $j = i + 2$. It follows that, if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A$, then $N_{C_n}(v_i) \cap A = (N_{C_n}(v_i) \cap N_{C_n}(v_j)) \cap A \subseteq N_{C_n}(v_i) \cap N_{C_n}(v_j)$. By (2.2), $v_i \equiv_A v_j$ if and only if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A$. Thus $|N_{C_n}(v_i) \cap A| = |N_{C_n}(v_j) \cap A| \leq 1$ and the equality holds if and only if $j = i + 2$ and $v_{i+1} = v_{j-1} \in A$. This proves the thesis.

(ii): The proof follows directly by part (i). In fact, let $v_i, v_j \in V(C_n)$, with $i < j$ and $v_i \equiv_A v_j$. Then, either (a) $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \emptyset$ or (b) $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = v_{i+1} = v_{j-1}$. But (a) is equivalent to say that $v_i, v_j \in B_A$, (b) that $\{v_i, v_j\} = N_{C_n}(v)$, for some $v \in C_A$. The proposition is thus proved.

In the next result we provide a characterization for the maximum partitioners of $C_n$.

**Theorem 2.3.6.** Let $A \subseteq V(C_n)$. Then $A \in \text{MAXP}(C_n)$ if and only if for any $v \notin A$ it holds one of the following conditions:

(i) $v_{i-2} \notin A$ or $v_{i+2} \notin A$ and $(B_{C_n}(A) \neq \{v_{i-1}, v_{i+1}\})$;

(ii) $(v_{i-2} \in A$ and $v_{i-4} \notin A$) or $(v_{i+2} \in A$ and $v_{i+4} \notin A)$.  

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Proof. If $A \subseteq V(C_n)$ is a maximum partitioner, then, for all $v_i \notin A$ there exist $v_j, v_k \in V(C_n)$ such that $v_j \equiv_A v_k$ and $F(v_i, v_j) \neq F(v_i, v_k)$. This holds if and only if it holds one of the two conditions: (1) $v_j, v_k$ are both in $B_{C_n}(A)$ and $|\{v_j, v_k\} \cap B_{C_n}(A \cup \{a\})| = 1$, or (2) $|k - j| = 2$, $v_{j+1} \in C_{C_n}(A)$ and either $v_i = v_{j-1}$ or $v_i = v_{k+1}$. Case (1) is equivalent to say that $v_{i-2} \notin A$ or $v_{i+2} \notin A$ and if both $v_{i-2}$ and $v_{i+2}$ are not in $A$, there exists another vertex $v_l \in B_{C_n}(A) \setminus \{v_{i-1}, v_{i+1}\}$. This is clearly equivalent to condition (i) of the Theorem. Condition (2) holds if and only if $v_{j+1} \in C_{C_n}(A) \setminus C_{C_n}(A \cup \{a\})$. This is equivalent to say that $v_{j+1} \in A$, $v_{j-1} \notin A$, $v_{j+3} = v_{k+1} \notin A$ and $v_i \in \{v_{j-1}, v_{k+1}\}$. This means that $v_{i+2} \in A$ and $v_{i+4} \notin A$ or $v_{i-2} \in A$ and $v_{i-4} \notin A$, that is condition (ii). The theorem is thus proved.

Example 2.3.7. In Figure 2.1 we represent the symmetry partition lattice of $C_5$, while in Figure 2.2 that of $C_6$. Let us observe that $\mathbb{P}_{sym}(C_5)$ is self-dual, but that of $C_6$ is not. So far, there exists no geometric characterization for the symmetry partition lattice of $C_n$.

2.3.2 Some Examples Taken From Digraph Context

In the next results we determine the form of any $A$-symmetry partition for the three digraph families, $\vec{P}_n$, $\vec{C}_n$ and $\vec{T}_n$. In order to simplify the exposure form of all the following statements we will use a convention.
Figure 2.2: Symmetry Partition Lattice of $C_6$
Remark 2.3.8. Let $D$ be one of the three digraphs $\tilde{P}_n$, $\tilde{C}_n$ or $\tilde{T}_n$. If $A \subseteq V(D)$ is such that $|A| = k$, we implicitly assume that $1 \leq k < n$. Therefore, if $|A| = k$, we always have exactly one of the following two cases:

(i) $A = \{v_1, \ldots, v_k\}$,

(ii) $A = \{v_1, v_{i_2}, \ldots, v_{i_k}\}$, where $1 < i_1 < i_2 < \cdots < i_k \leq n$.

Then, in the case (i) we will set $A_{i_1-1} := \{v_{i_1-1}, \ldots, v_k\}$. In the case (ii) we will set $A_{i_1-1} := \{v_{i_1-1}, \ldots, v_{i_k-1}\}$ if $D = \tilde{P}_n$, whereas $A_{i_1-1} := \{v_n, v_{i_2-1}, \ldots, v_{i_k-1}\}$ if $D = \tilde{C}_n$.

Finally, only when $D = \tilde{P}_n$ or $D = \tilde{T}_n$, if we take a subset $A \subseteq V(D)$ such that $|A| = 1$ we implicitly assume that $A \neq \{v_1\}$.

Proposition 2.3.9. We have that:

(i) $\pi_{\tilde{P}_n}(A) = \begin{cases} V & \text{if } A = \emptyset \text{ or } A \subseteq \{v_1\} \\ v_{i_1-1} \ldots v_{i_k-1} | A_{i_1-1} & \text{if } A = \{v_1, \ldots, v_k\} \text{ and } 1 \leq |A| < n-1 \\ v_{i_1-1} \ldots v_{i_k-1} | A_{i_1-1} & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \text{ and } 1 < |A| < n \\ v_1 \ldots v_{i_1-1} & \text{if } A = V \text{ or } A = \{v_2, \ldots, v_n\} \end{cases}$

(ii) $\pi_{\tilde{C}_n}(A) = \begin{cases} V & \text{if } A = \emptyset \\ v_{i_1-1} \ldots v_{i_k-1} | A_{i_1-1} & \text{if } A = \{v_1, \ldots, v_k\} \text{ and } 1 \leq |A| < n-1 \\ v_n v_{i_2-1} \ldots v_{i_k-1} | A_{i_1-1} & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \text{ and } 1 < |A| < n-1 \\ v_1 \ldots v_n & \text{if } A = V \text{ or } |A| = n-1 \end{cases}$

(iii) $\pi_{\tilde{T}_n}(A) = \begin{cases} V & \text{if } A = \emptyset \text{ or } A = \{v_1\} \\ v_{i_1} v_{i_2} \ldots v_{n-1} | v_1 \ldots v_n & \text{if } A = \{v_{i_1}, \ldots, v_k\} \text{ and } 1 \leq |A| < n-1 \\ v_{i_1} v_{i_2} \ldots v_{n-1} | v_1 \ldots v_n & \text{if } A = \{v_1, v_{i_2}, \ldots, v_k\} \text{ and } 2 \leq |A| < n \\ v_1 \ldots v_n & \text{if } A = V \text{ or } A = \{v_2, \ldots, v_n\} \end{cases}$

(iv) Let $\tilde{K}_{p,q} = (B_1 | B_2)$, where $B_1 = \{x_1, \ldots, x_p\}$ and $B_2 = \{y_1, \ldots, y_q\}$ and let $V = V(\tilde{K}_{p,q}) = \{x_1, \ldots, x_p, y_1, \ldots, y_q\}$. Then

$$\pi_A(\tilde{K}_{p,q}) = \begin{cases} V & \text{if } A = \emptyset \vee A \subseteq B_1 \\ B_1 | B_2 & \text{otherwise} \end{cases}$$

Proof. (i): By the characterization of the symmetry relation given in Theorem 2.1.5, we have only to study the intersections $N^+_{\tilde{P}_n}(v) \cap A$, when $A$ and the vertex $v$ vary. Therefore, if $A = \emptyset$, the result is trivial. Even if $A = V(\tilde{P}_n)$, computations are trivial. Let $A = \{v_1\}$, no neighborhood $N^+_{\tilde{P}_n}(v)$ contains the element $v_1$, so every intersection $N^+_{\tilde{P}_n}(v) \cap A$ is empty and $v_1, \ldots, v_n$ form a single block. Now, let $A$ be a singleton different from $\{v_1\}$. The only neighborhood whose intersection with $A$ is non-empty is $N^+_{\tilde{P}_n}(v_{i_1-1})$. This means that, in this case, $\pi_{\tilde{P}_n}(A) = v_{i_1-1} | \{v_{i_1-1}\}^c$. On the other hand, let $A = \{v_1, \ldots, v_k\}$ and $2 \leq k < n$. 

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it is clear that the neighborhoods intersecting $A$ are $N^+_F(v_{i_j-1})$, for $j = 1,\ldots,k$. Since $N^+_F(v_{i_j-1}) \neq N^+_F(v_{i_l-1})$ if $j \neq l$, every vertex $v_{i_1-1},\ldots,v_{i_k-1}$ form a single block and that the remaining vertices form the last block, hence $\pi^F_n(A) = v_{i_1-1}|\ldots|v_{i_k-1}|A^c_{-1}$. In particular, if $A = \{v_2,\ldots,v_n\}$, the set $A^c_{-1}$ consists of a single element, namely $v_n$, thus $\pi^F_n(A) = v_1|\ldots|v_n$. Furthermore, suppose $A = \{v_1,v_{i_2},\ldots,v_{i_k}\}$, with $1 < |A| < n$. In this case, the neighborhoods intersecting $A$ are $N^+_F(v_{i_j-1})$, for $j = 2,\ldots,k$, because $v_1$ is not contained in any neighborhood. So $\pi^F_n(A) = v_{i_2-1}|\ldots|v_{i_k-1}|A^c_{-1}$.

(ii): If $A = \emptyset$ or if $A = V(C_n^r)$ computations are obvious. Now, let $A = \{v_{i_1},\ldots,v_{i_k}\}$ be a generic proper subset of $V(C_n^r)$ not containing $v_1$. Thus we have that the neighborhoods intersecting $A$ are $N^+_C(v_{i_j-1})$, with $j = 1,\ldots,k$; furthermore, $N^+_C(v_{i_l-1}) \neq N^+_C(v_{i_k-1})$ if $j \neq l$, so every vertex $v_{i_1-1},\ldots,v_{i_k-1}$ form a single block and the remaining vertices form the last block, since the intersections of their neighborhoods with $A$ are empty. Let $A = \{v_{i_1},v_{i_2},\ldots,v_{i_k}\}$ be a proper subset of $V(C_n^r)$. As before, $N^+_C(v_{i_j-1})$, with $j = 2,\ldots,k$ and $N^+_C(v_{i_n})$ are the neighborhoods intersecting $A$. Since $N^+_C(v_j) \neq N^+_C(v_l)$ if $j,l \in \{i_2-1,\ldots,i_k-1,n\}$ are two distinct indices, we deduce that the elements $v_{i_1-1},\ldots,v_{i_k-1},v_n$ form a single block, while the others stay in another unique block. Finally, by the previous argument, if $|A| = n - 1$, we observe that $A^c_{-1}$ consists of a single element, therefore $\pi^C_n(A) = v_1|\ldots|v_n$.

(iii): It is clear that if $A = \emptyset$, then $\pi^T_n(A)$ consists of a unique block whereas if $A = V(T_n)$, every vertex form a single block of $\pi^T_n(A)$. Moreover, if $A = \{v_1\}$, we have that $N^+_T(v_1)$ doesn’t contain $v_1$ for all indices $i$, so $A \cap N^+_T(v_1) = \emptyset$. Let $A = \{v_i\}$, where $i \neq 1$. Since $N^+_T(v_2) = \{v_{i+1},\ldots,v_n\}$, clearly $N^+_T(v_j) \cap A = \{v_i\}$ if and only if $A \subseteq N^+_T(v_j)$. This happens if and only if $j = 1,\ldots,i - 1$, otherwise, this intersection is empty. Now, we can provide the proof for the general case. Let $A = \{v_{i_1},\ldots,v_{i_k}\}$ where $1 < i_1 < \cdots < i_k \leq n$ and $2 \leq |A| < n - 1$; we have to study the intersections $N^+_T(v_j) \cap A$. Proceeding exactly as before, we deduce that the first block consists of $v_{i_1},\ldots,v_{i_{l-1}}$; the second block contains all the vertices such that $N^+_T(v_j) \cap A\{v_{i_1}\} \neq \emptyset$, i.e. it contains exactly the $i_2 - i_1$ elements $v_{i_1},\ldots,v_{i_{l-1}}$ and so on. In particular, if $A = \{v_{i_2},\ldots,v_n\}$, $A^c_{-1}$ consists of a single element, therefore $\pi^C_n(A) = v_1|\ldots|v_n$. On the other hand, if $A = \{v_{i_1},v_{i_2},\ldots,v_{i_k}\}$ where $1 < i_2 < \cdots < i_k \leq n$ and $2 \leq |A| < n$, the argument is exactly the same as before, starting from the index $i_2$, since there is no vertex $v_j$ whose neighborhood $N^+_T(v_j)$ contains $v_1$. Thus $\pi^T_n(A) = v_1|\ldots|v_{i_{l-1}}|\ldots|v_{i_k} \ldots v_n$ and the thesis follows.

(iv): We observe that
\[
N^+_K(v) = \begin{cases} 
B_2 & \text{if } v \in B_1 \\
\emptyset & \text{if } v \in B_2
\end{cases}
\]
Therefore, if $A \subseteq B_1$, then $N^+_K(v) \cap A = \emptyset$ for every vertex $v \in V$. Hence, $\pi^T_K(K_{p,q}) = V$.
In the other cases, it is easy to see that $N^+_K(v) \cap A$ is the same subset of $B_2$ for every $v \in B_1$ and this intersection is empty for every $v \in B_2$. Hence, we conclude that $\pi^T_K(A) = B_1|B_2$ in the other cases.

In next result we provide the general form for the symmetry partition lattices of $T_n$, $\overline{P}_n$.
and $\bar{C}_n$.

**Theorem 2.3.10.** (i) Let $D = T_n$ or $D = \bar{T}_n$. Then

$$\text{MAXP}(D) = \{\{v_1, v_{i_1}, \ldots, v_{k}\} : 1 < i_1 < \cdots < i_k \leq n\}$$

and $(\mathcal{M}(D), \subseteq^*) \cong (\mathcal{P}([n-1]), \subseteq^*)$ by means of the isomorphism $\{v_1, v_{i_1}, \ldots, v_{i_k}\} \mapsto \{i_1, \ldots, i_k\}$.

(ii) If $V = V(\bar{C}_n)$ we have

$$\text{MAXP}(\bar{C}_n) = \{A \subseteq V : |A| \leq n - 2\} \cup \{V\}$$

and $(\mathcal{M}(\bar{C}_n), \subseteq^*) \cong (\mathcal{P}(\mathbb{N}, 2), \subseteq^*)$ by means of the isomorphism $\{v_1, v_{i_1}, \ldots, v_{i_k}\} \mapsto \{i_1, \ldots, i_k\}$.

**Proof.** (i): Let $D = T_n$ or $D = \bar{T}_n$, $A = \{v_1, \ldots, v_k\} \subseteq V = V(D)$, where $1 < i_1 < \cdots < i_k \leq n$ and $B = \{v_1\} \cup A$. By Proposition 2.3.9, it is immediate that $A \cong B$. In particular, a set $B$ is the maximum partitioner of $\pi_D(A)$ if and only if it is of the form $B = A \cup \{v_1\}$. In fact, let $B$ be a maximum partitioner. Then, it must to contain the vertex $v_1$. On the other hand, let $B = \{v_1, \ldots, v_k\}$, for $1 < i_1 < \cdots < i_k \leq n$. By definition of maximum partitioner, $B \subseteq M_{\mathbb{P}}(B)$. Suppose by contradiction that $B \not\subseteq M_{\mathbb{P}}(B)$ and let $C = M_{\mathbb{P}}(B)$. This means that $C$ contains at least another vertex distinct from those of $B$. So $\pi_D(C) \cong \pi_D(B) = \pi_D(A)$ but $\pi_D(C) \neq \pi_D(B)$; therefore $C \not\cong B$, and this in contrast with $C = M_{\mathbb{P}}(B)$. Therefore $\text{MAXP}(D) = \{\{v_1, v_{i_1}, \ldots, v_{i_k}\}\}$, where $1 < i_1 < \cdots < i_k \leq n$. Moreover, it results that $A \subseteq B$ if and only if $B \subseteq A$. We define the bijective map $\phi : \mathcal{M}(D) \mapsto \mathcal{P}([n-1])$ such that $B = \{v_1, v_{i_1}, \ldots, v_{i_k}\} \mapsto \phi(B) = \{i_1, \ldots, i_k\}$. Let $A \subseteq B$, i.e. $B \subseteq A$; it’s obvious that $\phi(B) \subseteq \phi(A)$, that’s to say $\phi(A) \subseteq^* \phi(B)$. This means that $(\mathcal{M}(T_n), \subseteq^*) \cong (\mathcal{P}([n-1]), \subseteq^*)$.

(ii): Let $A \subseteq V = V(\bar{C}_n)$. By Proposition 2.3.9, it is then immediate that $[A]_{\cong B} = \{A\}$ if $|A| < n - 1$. Hence if $|A| > n - 2$, then $\pi_{\mathcal{C}_n}(A) = \pi_{\mathcal{C}_n}(V(\bar{C}_n))$, otherwise $[A]_{\cong B} = \{A\}$, i.e. the maximum partitioner of $[A]_{\cong B}$ is $A$ itself. This means that $\text{MAXP}(\bar{C}_n) = \{A \subseteq V : |A| \leq n - 2\} \cup \{V\}$. Define the bijective map $\phi : \mathcal{M}(\bar{C}_n) \mapsto \mathcal{P}(\mathbb{N}, 2)$ such that $A = \{v_{i_1}, \ldots, v_{i_k}\} \mapsto \phi(A) = \{i_1, \ldots, i_k\}$. Let $A \subseteq B$, i.e. $B \subseteq A$; hence $\phi(B) \subseteq \phi(A)$, i.e. $\phi(A) \subseteq^* \phi(B)$. This means that $(\mathcal{M}(\bar{C}_n), \subseteq^*) \cong (\mathcal{P}(\mathbb{N}, 2), \subseteq^*)$.

### 2.3.3 Global Symmetry in Hypergraphs

By using Proposition 2.1.7 we obtain the following useful characterization of the $A$-symmetry classes in $\Gamma((\mathbb{N})^k)$, where $A$ is a subset of $\mathbb{N}$.

**Proposition 2.3.11.** Let $\mathfrak{P}$ be the pairing $\Gamma((\mathbb{N})^k)$ where $n \geq k$ are positive integer and let $A \subseteq \mathbb{N}$. Then, there exists a bijection between the set $\pi_{\mathfrak{P}}(A)$ of all $A$-symmetry classes in $\Gamma(H)$ and the set

$$\Sigma_A := \{S \subseteq A : \max\{0, k + a - n\} \leq |S| \leq \min\{a, k\}\},$$

where $a := |A|$.

**Proof.** Let $S \in \Sigma_A$ and set $s := |S|$. By Proposition 2.1.7 all $k$-subsets of $\mathbb{N}$ whose intersection with $A$ is equal to $S$, if non-empty, is an $A$-symmetry class in $\mathfrak{P}$. But if $\max\{0, k + a - n\} \leq s$ then $|S| \geq \min\{a, k\}\).
$s \leq \min\{a, k\}$ then there exists $(n-a)_{k-s}$ $k$-subsets of $\hat{n}$ such that their intersection with $A$ is $S$. On the other hand if $x \in \binom{\hat{n}}{k}$, then the cardinality $s$ of $S := x \cap A$ satisfies $\max\{0, k + a - n\} \leq s \leq \min\{a, k\}$. In fact $s \geq 0$, $s \leq a$ and $s \leq k$ by definition of $S$. Moreover it holds that $S = x \cap A = x \setminus (\hat{n} \setminus A)$ and so $s \geq k - (n - a) = k + a - n$. We proved that the function that associates with each $S \in \Sigma_A$ the $A$-symmetry class $C := \{x \in \binom{\hat{n}}{k} : x \cap A = S\}$ is bijective with inverse that associates with each $C \in \pi_{\mathcal{B}}(A)$ the subset $S := A \cap (\bigcap_{x \in C} x)$.

**Corollary 2.3.12.** Let $\mathcal{B}$ be the Boolean information system $\Gamma\left(\binom{\hat{n}}{k}\right)$ where $n \geq k$ are positive integers and let $A \subseteq \hat{n}$ and $x \in \binom{\hat{n}}{k}$. By setting $a := |A|$, $S := x \cap A$ and $s := |S|$ we have $\max\{0, k + a - n\} \leq s \leq \min\{a, k\}$ and $|x|/A| = \binom{n-a}{k-s}$.

**Proof.** As we have seen in the first part of the proof of Proposition 2.3.11 the $A$-symmetry class $[x]_A$ is the set of all $k$-subsets of $\hat{n}$ whose intersection with $A$ is equal to $S$, $\max\{0, k + a - n\} \leq s \leq \min\{a, k\}$ and $|x|/A| = \binom{n-a}{k-s}$.

**Remark 2.3.13.** Let $H = (\hat{n}, \mathcal{F})$ be a simple $k$-uniform hypergraph on $n$ vertices (i.e. $\mathcal{F}$ is a sub-family of $\binom{\hat{n}}{k}$). Let $\mathcal{P} := \Gamma(H)$ and $A$ a subset of the attribute set $\hat{n}$.

In order to compute the number of distinct equivalence classes in the set $\pi_{\mathcal{B}}(A)$ and the complete list of all the objects of any symmetry class $[x]_A \in \pi_{\mathcal{B}}(A)$ it is convenient to see $\mathcal{B}$ as a binary matrix $T(\mathcal{B})$. With the subset $A \subseteq \Omega$, we can associate in a natural way a $\binom{n}{k} \times |A|$ submatrix $T_A(\mathcal{B})$ of $T(\mathcal{B})$ by choosing the columns corresponding to the elements of $A$. By using standard topological sort algorithms we can easily find the number of distinct rows of $T_A(\mathcal{B})$ and partition the rows of $T_A(\mathcal{B})$ with respect to the equality relation.

In the following result we determine the number of elements in any symmetry partition $\pi_{\mathcal{B}}(A)$ when $\mathcal{B} = \Gamma\left(\binom{\hat{n}}{k}\right)$.

**Proposition 2.3.14.** Let $\mathcal{B} = \Gamma\left(\binom{\hat{n}}{k}\right)$. If $A$ is a subset of attributes of $\mathcal{B}$ such that $|A| = l$, then:

(i) if $l \leq k$ we have $|\pi_{\mathcal{B}}(A)| = \sum_{i=0}^{\min\{l, n-k\}} \binom{l}{i}$;

(ii) if $l > k$ we have $|\pi_{\mathcal{B}}(A)| = \sum_{i=0}^{\min\{k, n-l\}} \binom{k}{i}$.

**Proof.** Without loss of generality we can assume that $A = \{1, 2, \ldots, l\}$. By Proposition 2.1.7 we have that, given two $k$-subsets $S_1$, $S_2$ of $\{n\}$, $S_1 \equiv_A S_2$ if and only if $S_1 \cap A = S_2 \cap A$. Then we can associate with each class in $\pi_{\mathcal{B}}(A)$ a subset of $A$. In particular for each subset $B$ of $A$ we can consider the $k$-subsets $S$ of $\{n\}$ such that $S \cap A = B$. This set is either empty or it is an equivalence class in $\pi_{\mathcal{B}}(A)$. Such a class has $\binom{n-l}{k-|B|}$ elements that can be obtained by choosing $k - |B|$ elements in $\{n\} \setminus A$.

Let us suppose now $l \leq k$. The $k$-subsets of $\{n\}$ containing $A$ form a class in $\pi_{\mathcal{B}}(A)$. More generally however we fix a subset $B$ of $A$, the set of the $k$-subsets of $\{n\}$ containing $B$, if it is not empty, it matches an equivalence class of $\equiv_A$ with $\binom{n-l}{k-|B|}$ elements. Such a set is empty if and only if $k - |B| \leq n - l$ which is equivalent to $l - |B| \leq n - k$. Thus it follows that the number of classes in $\pi_{\mathcal{B}}(A)$ is equal to

$$|\pi_{\mathcal{B}}(A)| = \sum_{j=0}^{n-k} \binom{l}{j} = \sum_{i=0}^{n-k} \binom{l}{l-i}$$
and thus (i) holds.

If \( l > k \) then obviously for each subset \( B \) of \( A \) such that \( |B| > k \) there exists no \( k \)-subset \( S \) of \([n]\) such that \( S \cap A = B \). When \( |B| = k \) the unique \( k \)-subset \( S \) of \([n]\) containing \( B \) is \( B \). So each \( k \)-subset in \( A \) uniquely identifies a class in \( \pi_{\mathbb{P}}(A) \). As before if \( |B| < k \) and \( k - |B| \leq n - l \) then the equivalence class of \( \equiv_A \) of all the \( k \)-subsets \( S \) such that \( S \cap A = B \) contains \( \binom{n - l}{k - |B|} \) elements. Finally if \( |B| < k \) and \( k - |B| > n - l \) then there is not an element \( S \in \binom{[n]}{k} \) such that \( S \cap A = B \). Then when \( l > k \) the number of classes in \( \pi_{\mathbb{P}}(A) \) is thus equal to

\[
|\pi_{\mathbb{P}}(A)| = \sum_{j=l-n+k}^{k} \binom{l}{j} = \sum_{i=0}^{n-l} \binom{l}{k-i}
\]

and the proposition is proved. \( \square \)

By virtue of the result established in Proposition 2.3.14 it is convenient to set

\[
c(n, l, k) := \begin{cases} 
\sum_{i=0}^{\min\{l, n-k\}} \binom{l}{i} & \text{if } l \leq k \\
\sum_{i=0}^{\min\{k, n-l\}} \binom{l}{i} & \text{if } l > k 
\end{cases}
\]

(2.31)

when \( n, l \) and \( k \) are three integers such that \( 0 \leq l, k \leq n \).

We determine now the symmetry partition lattice of \( \binom{[n]}{k} \) for all \( n \geq k \geq 1 \).

**Theorem 2.3.15.** The lattice \( (\mathbb{P}(\binom{[n]}{k}), \subseteq) \) is isomorphic to the dual lattice of \( (\mathcal{V}(n, n-2), \subseteq) \).

**Proof.** It is immediate to notice that \( \pi_{\mathbb{P}}(A) = \pi_{\mathbb{P}}(\hat{n}) \) when \( A \subseteq \hat{n} \) has cardinality \( |A| > n - 2 \) and that \( \pi_{\mathbb{P}}(A) \neq \pi_{\mathbb{P}}(\hat{n}) \) otherwise. Consider now the function \( \phi : \mathbb{P}(\binom{[n]}{k}) \rightarrow \mathcal{V}(n, n-2) \) defined by:

\[
\phi(\pi_{\mathbb{P}}(A)) = \begin{cases} 
A & \text{if } |A| \leq n - 2, \\
\hat{n} & \text{if } |A| > n - 2.
\end{cases}
\]

Let us prove now that \( \phi \) is well defined. For this we have to prove that, if \( \pi_{\mathbb{P}}(A) = \pi_{\mathbb{P}}(B) \) are two subsets of \( \hat{n} \), then \( \phi(\pi_{\mathbb{P}}(A)) = \phi(\pi_{\mathbb{P}}(B)) \). Let \( A, B \subseteq \hat{n} \), with \( |A| \leq n - 2 \), \( B \leq n - 2 \) and \( \phi(\pi_{\mathbb{P}}(A)) = A \neq B = \phi(\pi_{\mathbb{P}}(B)) \). Since \( A \neq B \), we have \( A \Delta B = (A \setminus B) \cup (B \setminus A) \neq \emptyset \). Without loss of generality we can assume \( A \setminus B \neq \emptyset \). Let \( a \in A \setminus B \) and let \( K \) be a \( k \)-subset of \( \hat{n} \) such that \( a \in K \) and \( |B \cap K| > |B| + k - n \). Note that such a \( K \) exists in this case. Set \( S := B \cap K \) and let \( K' \in \binom{[n]}{k} \) such that \( B \cap K' = S \) and \( a \notin K' \). By Proposition 2.1.7, \( K \equiv_B K' \) but \( K \neq_A K' \) and thus \( \pi_{\mathbb{P}}(A) \neq \pi_{\mathbb{P}}(B) \).

The map \( \phi \) is clearly both injective and surjective. Let us prove it is an isomorphism. For this let \( A, B \subset \hat{n} \) such that \( \pi_{\mathbb{P}}(A) \preceq \pi_{\mathbb{P}}(B) \). If \( |A| \geq n - 1 \), then \( \phi(\pi_{\mathbb{P}}(A)) = \hat{n} \supseteq \phi(\pi_{\mathbb{P}}(B)) \), for each \( B \). Let us suppose \( |A| < n - 1 \). In this case, since \( \pi_{\mathbb{P}}(A) \preceq \pi_{\mathbb{P}}(B) \), we have \( |B| < n - 1 \). Thus \( \phi(\pi_{\mathbb{P}}(A)) = A \) and \( \phi(\pi_{\mathbb{P}}(B)) = B \). Since \( \pi_{\mathbb{P}}(A) \neq \pi_{\mathbb{P}}(\hat{n}) \), there exist \( K_1, K_2 \in \binom{[n]}{k} \) such that \( K_1 \neq K_2 \). By Proposition 2.3.11 it holds \( k + a - n < |K_1 \cup A| = |K_2 \cup A| < k \), where \( a = |A| \). Let \( b \in B \setminus A \). Then there exist \( K_1', K_2' \in \binom{[n]}{k} \) such that \( |K_1 \cup A| = |K_2 \cup A| = |K_1' \cup A| = |K_2' \cup A| \) and \( b \in K_1' \setminus K_2' \). This implies \( K_1' \equiv_A K_2' \) but \( K_1' \neq_B K_2' \), and this contradicts the condition \( \pi_{\mathbb{P}}(A) \leq \pi_{\mathbb{P}}(B) \). Thus \( B \subseteq A \) and the theorem is proved. \( \square \)
2.3.4 Euclidean Real Line

Let us consider now the pairing \( \mathcal{P} := \mathcal{P}[\mathbb{R}, d_E] \), where \( d_E \) is the usual real euclidean distance.

Let \( A \subseteq \mathbb{R} \) be fixed. We set

\[
A^* := (A - 1) \cup (A + 1), \quad A_* := (A - 1) \cap (A + 1)^c \cap (A - 3)^c
\]  

(2.32)

In the next result we provide the general form for an \( A \)-symmetry partition of \( \mathbb{R} \).

**Proposition 2.3.16.** We have that

\[
\pi\{A\} = \mathbb{R} \setminus A^* \{x, x + 2\}, y \in A, |\{y\} \in A \cup (A + 1)^c + 2)
\]

**Proof.** Let \( x, y \in \mathbb{R} \) such that \( x < y \). It is easy to prove that \( x \equiv_A y \) in \( \mathbb{R} \) if and only if \( x, y \in \mathbb{R} \setminus A^* \) or \( x \in A_* \) and \( y = x + 2 \). Moreover, in this second case, since \( A_* = \{x \in \mathbb{R} : x - 1 \notin A, x + 3 \notin A, x + 1 \in A\} \), there exist no other point equivalent to \( x \) beside \( y \).

We now characterize the maximum partitioners of \( \mathbb{R} \).

**Proposition 2.3.17.** \( A \in \text{MAXP}(\mathcal{P}) \) if and only if for all \( a \notin A \)

\[1'. (a - 2 \notin A \text{ or } a + 2 \notin A) \text{ and } |\mathbb{R} \setminus A^*| \geq 2; \]

\[2'. (a + 2 \in A \text{ and } a + 4 \notin A) \text{ or } (a - 2 \in A \text{ and } a - 4 \notin A). \]

**Proof.** A subset \( A \subseteq \mathbb{R} \) is a maximum partitioner in \( \mathbb{R} \) if and only if, for all \( a \notin A \), there exist \( x, y \in \mathbb{R} \) such that \( x \equiv_A y \) but \( x \notin A \cup A(a) \) \( y \). By (2.3.16) it is equivalent to say that, for each choice of \( a \notin A \), it holds at least one of the following conditions:

1. \( \exists x, y \in \mathbb{R} \setminus A^* \text{ such that } x < y \) and \( \{x, y\} \cap (A \cup \{a\})^* \neq \emptyset \);
2. \( \exists x \in A_* \setminus (A \cup \{a\})_*. \)

These conditions are equivalent respectively to 1' and 2'.

**Proposition 2.3.18.** We have that

\[
M_{\mathcal{P}}(A) = A \cup [(A - 2) \cap (A + 2) \cap (A - 4) \cap (A + 4)].
\]

(2.33)

**Proof.** It is an immediate consequence of Proposition 2.3.17.

**Corollary 2.3.19.** \( A \in \text{MAXP}(\mathbb{R}) \) if and only if \( (A - 2) \cap (A + 2) \cap (A - 4) \cap (A + 4) \subseteq A \).

As a consequence of Proposition 2.3.17 we obtain the following result.

**Proposition 2.3.20.** If \( A \) is a bounded interval of \( \mathbb{R} \) then \( A \in \text{MAXP}(\mathbb{R}) \).

**Proof.** Let \( A \) be a bounded interval of the real line. Let us consider \( A = [\alpha, \beta] \), the other cases are similar. Let \( a \notin A \). If \( a \in (-\infty, \alpha + 2) \) or \( a \in [\beta - 2, +\infty) \), then \( a - 2 \notin A \) or \( a + 2 \notin A \), so condition 1', is satisfied. If \( a \in (-\infty, \alpha + 2) \cup [\beta - 2, +\infty) \), then \( \beta - \alpha < 2, \{a - 2, a + 2\} \cap A \neq \emptyset \) and thus if \( a - 2 \in A \), then \( a - 4 \notin A \) and if \( a + 2 \in A \), then \( a + 4 \notin A \). It follows the thesis.
2.3.5 Global Symmetry in \((V, \varphi, \mathbb{K})\)-VBP

If \(A, B \subseteq V\), then
\[
A \approx B \iff V/A^\perp = V/B^\perp \iff A^\perp = B^\perp, \tag{2.34}
\]
so that
\[
M_\varphi(A) = \{b \in V : (A \cup \{b\})^\perp = A^\perp\}. \tag{2.35}
\]

We denote by \(\text{Span}(A)\) the vector subspace of \(V\) spanned by \(A\). In next result, we study the relation between \(M_\varphi(A)\) and \(\text{Span}(A)\) and give a sufficient condition on \(\varphi\) to ensure that in finite dimensional vector spaces \(\text{Span}(A) = M_\varphi(A)\).

**Proposition 2.3.21.** (i) \(M_\varphi(A)\) is a vector subspace of \(V\);
(ii) \(\text{Span}(A) \subseteq M_\varphi(A)\).
Moreover, let \(V\) be a finite dimensional vector space. Then:
(iii) if \(\varphi\) is (left) non degenerate, \(\text{Span}(A) = M_\varphi(A)\);
(iv) If \(\varphi\) is (left) non degenerate, then \(\text{MAXP}(V) = \{H : H \leq V\}\);
(v) if \(\varphi\) is (left) degenerate and \(V^\perp \setminus \text{Span}(A) \neq \emptyset\), we have \(\text{Span}(A) \not\subseteq M_\varphi(A)\).

**Proof.** (i): Let \(v, w \in M_\varphi(A)\) and let us consider \(\lambda v + \mu w\). It is obvious that \((A \cup \{\lambda v + \mu w\})^\perp \subseteq A^\perp\). On the other hand, let \(x \in A^\perp\). Then \(\varphi(x, a) = 0\) for any \(a \in A\). Moreover, \(\varphi(x, \lambda v + \mu w) = \lambda \varphi(x, v) + \mu \varphi(x, w) = 0\) since \((A \cup \{v\})^\perp = (A \cup \{w\})^\perp = A^\perp\), so \(x \in (A \cup \{\lambda v + \mu w\})^\perp\) and \((A \cup \{v + w\})^\perp = A^\perp\). This means that \(\lambda v + \mu w \in M_\varphi(A)\) and \(M_\varphi(A)\) is a vector space.
(ii): Since \(A \subseteq M_\varphi(A)\) and \(M_\varphi(A)\) is a vector space, we have that \(\text{Span}(A) \subseteq M_\varphi(A)\).
(iii): Since \(V\) is a finite dimensional vector space and \(\varphi\) is (left) non degenerate, for any \(H \leq V\) it results that \(H = (H^\perp)^\perp\). Suppose now that \(a \in M_\varphi(A)\), i.e. \((A \cup \{a\})^\perp = A^\perp\). Let \(W = \text{Span}\{\{a\}\}\), then \((\text{Span}(A) + W)^\perp = \text{Span}(A)^\perp\), so that \((\text{Span}(A) + W)^\perp = (\text{Span}(A)^\perp)^\perp\). Hence \(\text{Span}(A) + W = \text{Span}(A)\), i.e. \(a \in \text{Span}(A)\). Thus \(M_\varphi(A) \subseteq \text{Span}(A)\) and \(M_\varphi(A) = \text{Span}(A)\).
(iv): It follows immediately by part (iii).
(v): Let \(b \in V^\perp \setminus \text{Span}(A)\). In such a case, \(\text{Span}(A \cup \{b\})^\perp = A^\perp\), i.e. \(\text{Span}(A) \not\subseteq M_\varphi(A)\). \(\square\)

2.3.6 Global Symmetry by Group Actions

If \(A, B \subseteq \Omega\) and \(\Psi = \Psi[G, \psi]\), we have that
\[
A \approx_{G, \psi} B \iff \text{Stab}_{G, \psi}(A) = \text{Stab}_{G, \psi}(B).
\]
Let us consider the pairing \((S_n, \psi) := \Psi[S_n, \psi]\) induced by the action of the symmetric group \(S_n\) on a set \(\Omega\) having \(n\) elements. Then we have the following result:

**Theorem 2.3.22.** \(\text{MAXP}((S_n, \psi)) = \{A \subseteq \Omega : |A| \leq n - 2\} \cup \{\Omega\}\).

**Proof.** Let \(A \subseteq \Omega\) such that \(|A| = k\). Hence, there exists exactly \((n-k)!\) permutations of \(S_n\) fixing all elements of \(A\). Hence \(|\text{Stab}_{S_n, \psi}(A)| = (n-k)!|\). Moreover, since the index of \(\text{Stab}_{S_n, \psi}(A) = n!/(n-k)!\), we deduce that there are \(n!/(n-k)!\) classes of \(A\)-symmetry for any \(A\). We conclude that if \(0 \leq |A| \neq |B| \leq n - 2\), then \(\pi_{S_n, \psi}(A) \neq \pi_{S_n, \psi}(B)\). Furthermore, if \(1 \leq |A| = |B| \leq n - 2\), we clearly have \(|\text{Id}_{A}| = \text{Stab}_{S_n, \psi}(A) \neq \text{Stab}_{S_n, \psi}(B) = |\text{Id}_{B}|, so \(\pi_{S_n, \psi}(A) \neq \pi_{S_n, \psi}(B)\). Finally, we have that \(\pi_{S_n, \psi}(A) = \pi_{S_n, \psi}(\Omega)\) for any \(A \subseteq \Omega\) such that \(|A| = n - 1\). This gives \(\text{MAXP}((S_n, \psi)) = \{A \subseteq \Omega : |A| \leq n - 2\} \cup \{\Omega\}\). \(\square\)
2.4 Separators and Related Notions

In this section we assume a topological point of view for our investigation of the maximum
partitioner family. To be more detailed, the maximum partitioners can be seen as a generaliza-
tion of the closed sets of a topology on \( \Omega \). In this perspective, it is convenient to undertake an
analysis of the family of all corresponding generalized open sets, i.e. the complement of each
maximum partitioner. Therefore, if \( A \subseteq \Omega \), we set
\[
S_P(A) := \Omega \setminus M_P(A)
\]
where
\[
SEP(\mathfrak{P}) := \{ S_P(A) : A \in \mathcal{P}(\Omega) \}.
\]
We call a separator of \( A \) any element \( s \in S_P(A) \). The following characterization is immediate.

Proposition 2.4.1. Let \( s \in \Omega \). Then \( s \) is a separator of \( A \) if and only if there exist two point \( u, u' \in U \) such that:
\[
(i) \quad F(u, a) = F(u', a) \quad \forall a \in A;
(ii) \quad F(u, s) \neq F(u', s).
\]

If \( A = \{ A_i : i \in I \} \in SS(\Omega) \), we set
\[
S_P(A) := \bigcap_{i \in I} S_P(A_i).
\]

Proposition 2.4.2. \( S_P(\cup A) \subseteq S_P(A) \).

Proof. Let \( s \in S_P(\cup A) \), then there exist \( u, u' \in U \) with the property that for any \( a \in \bigcup_{i \in I} A_i \),
\[
F(u, a) = F(u', a) \quad \forall a \in A;
\]
and
\[
F(u, s) \neq F(u', s).
\]

In other words, for any fixed \( i \in I \) and for any choice of \( a \in A_i \subseteq \bigcup_{i \in I} A_i \), it follows that \( F(u, a) = F(u', a) \) and \( F(u, s) \neq F(u', s) \), that is
\[
s \in S_P(A_i) \]. It readily follows that \( S_P(A) \) for any \( i \in I \) and hence \( s \in S_P(A) \).

Remark 2.4.4. Any finite pairing is UCP.

In the following result, we provide other alternative characterizations for UCP pairings.

Theorem 2.4.5. The following three conditions are equivalent:
\[
(i) \quad \mathfrak{P} \text{ is UCP};
(ii) \quad \text{MAXP}(\mathfrak{P}) \text{ is a chain union-closed family};
\]

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(iii) For any $A \in \mathcal{P}(\Omega)$ we have

$$M_\Psi(A) = \bigcup \{M_\Psi(F) : F \subseteq_f A\}. \quad (2.38)$$

Proof. (i) $\implies$ (ii): Let $\Psi$ be a UCP pairing and let $\mathcal{B} = \{B_i : i \in I\}$ be a chain of maximum partitioners of $\Omega$. Let $B := \bigcup \mathcal{B}$ and $s \in \Omega \setminus B$. Then $s \in \Omega \setminus B_i = \Omega \setminus M_\Psi(B_i) = S(B_i)$ for all $i \in I$, because any $B_i$ is a maximum partitioner and therefore $M_\Psi(B_i) = B_i$. Hence $s \in \bigcap_{i \in I} S(B_i) := S(\mathcal{B})$, and $S(\mathcal{B}) \subseteq S(\bigcup \mathcal{B})$ by hypothesis, so that $s \in S(B)$. This shows that $\Omega \setminus B \subseteq S(B) = \Omega \setminus M_\Psi(B)$, i.e. $M_\Psi(B) \subseteq B$, and hence $B \in MAXP(\Psi)$. Therefore $MAXP(\Psi)$ is a chain union-closed family.

(ii) $\implies$ (i): Let $MAXP(\Psi)$ be a chain union-closed family and let $A = \{A_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ be a chain. Since $A_i \subseteq A_j$ implies that $M_\Psi(A_i) \subseteq M_\Psi(A_j)$, it follows that $A_M := \{M_\Psi(A_i) : i \in I\}$ is a chain of maximum partitioners. Then by hypothesis we have that $K := \bigcup_{i \in I} M_\Psi(A_i) \in MAXP(\Psi)$. Let us note now that

$$S(A) := \bigcap_{i \in I} S(A_i) = \bigcap_{i \in I} \Omega \setminus M_\Psi(A_i) = \Omega \setminus K. \quad (2.39)$$

Then since $K = M_\Psi(K)$, by (2.39) we deduce that

$$S(A) = \Omega \setminus M_\Psi(K) = S(K). \quad (2.40)$$

Now, since $A_i \subseteq M_\Psi(A_i)$ for all $i \in I$, we have $\bigcup A := \bigcup_{i \in I} A_i \subseteq K$, therefore

$$S(K) \subseteq S(\bigcup A). \quad (2.41)$$

Hence, by (2.40) and (2.41) we deduce that $S(A) \subseteq S(\bigcup A)$. This shows that $\Psi$ is an UCP.

(ii) $\implies$ (iii): If $\Psi$ is a pairing on a finite set $\Omega$, then the thesis is obvious. Let now $\Psi$ be a pairing on an infinite set $\Omega$ and let $A \subseteq \Omega$. We endow $A$ with a well-order $\prec$. Set

$$C(a) := \{b \in A : b \prec a\}$$

for any $a \in A$. Let us assume that the thesis holds for sets whose cardinality is less than $|A|$ and, furthermore, that for any $a \in A$, the set $C(a)$ has cardinality strictly less than $|A|$. In particular, the thesis holds for $C(a)$, for any $a \in A$. Let $\Omega := \{M_\Psi(C(a)) : a \in A\}$. It is clearly a chain in $MAXP(\Psi)$. Since $MAXP(\Psi)$ is a chain union-closed family, we have that the union

$$D := \bigcup \Omega$$

belongs to $\Omega$. Let $K := \bigcup_{a \in A} C(a)$. Hence, we have that $M_\Psi(C(a)) \subseteq M_\Psi(K)$ for any $a \in A$, so $D \subseteq M_\Psi(K)$. On the other hand, it is obvious that $M_\Psi(K) \subseteq D$, thus $D = M_\Psi(K)$. By transfinite inductive hypothesis, the thesis holds for $C(a)$, for any $a \in A$. We observe that for any $F \subseteq_f A$, there exists $a \in A$ such that $F \subseteq_f C(a)$. This shows the claim.

(iii) $\implies$ (ii): Let $\mathcal{C}$ be a non-empty chain of maximum partitioners in $MAXP(\Psi)$ and let $B := \bigcup \mathcal{C}$. We will show that $B \in MAXP(\Psi)$. To this regard, assume by contradiction that there exists $F \subseteq_f \bigcup \mathcal{C}$ such that $F \nsubseteq_f C$ for any $C \in \mathcal{C}$. Let us fix $a \in F$. Then, just take
\(C_1, C_2 \in \mathcal{C}\) such that \(x \in C_1\) and \(F \setminus \{a\} \subseteq C_2\). Therefore, if \(C_1 \subseteq C_2\), we would have \(F \subseteq C_2\), that is an absurd and, analogously, if \(C_2 \subseteq C_1\), then \(F \subseteq C_1\), once again a contradiction. Thus, we conclude that for any \(F \subseteq f \cup \mathcal{C}\), there must be \(C \in \mathcal{C}\) containing it. Furthermore, we must have \(M_\mathcal{P}(F) \subseteq C \subseteq B\) so, by our hypothesis, the claim has been showed.

In the next result, we provide some sufficient condition in order to verify the existence of a maximal element for any non-empty subfamily \(A\) of \(\text{MAXP}(\mathcal{C})\).

**Theorem 2.4.6.** If \(\text{MAXP}(\mathcal{C})\) is a chain union-closed family and for any \(C \in \text{MAXP}(\mathcal{C})\) there exists \(F \subseteq f C\) such that \(F \approx_{\mathcal{P}} C\), then any non-empty subfamily \(A\) of \(\text{MAXP}(\mathcal{C})\) admits a maximal element.

**Proof.** Let \(A\) be a non-empty subfamily of \(\text{MAXP}(\mathcal{C})\). Since \(\text{MAXP}(\mathcal{C})\) is a chain union-closed family, then \(\bigcup \mathcal{K} \in \text{MAXP}(\mathcal{C})\) for any \(\mathcal{K}\) chain in \(\text{MAXP}(\mathcal{C})\). Therefore, let us take a chain \(\mathcal{K}\) in \(A\), so \(C := \bigcup \mathcal{K} \in \text{MAXP}(\mathcal{C})\). By our assumption, there exists \(F \subseteq C\) such that \(F \approx_{\mathcal{P}} C\). Let us define the map \(\phi : a \in F \mapsto A_a \in \mathcal{K}\), where \(A_a\) is a maximum partitioner in \(\mathcal{K}\) containing \(a\). Then, we obtain a finite chain \(\{A_a : a \in F\}\) in \(A\) clearly having a maximal element, that we call \(A_b\), for some \(b \in F\). Thus, \(A_b \in \text{MAXP}(\mathcal{C})\). Since \(F \subseteq A_b \subseteq C\), we deduce that \(M_\mathcal{P}(A_b) = A_b = C\). But since \(\mathcal{K}\) is a maximal chain, we conclude that \(A_b\) is a maximal element in \(A\) and this completes the proof.

As a consequence, we show that \(\mathcal{R}\) is a UCP pairing.

**Theorem 2.4.7.** \(\mathcal{R}\) is a UCP pairing.

**Proof.** Let us prove that \(\text{MAXP}(\mathcal{R})\) is a chain union-closed family. Let \(A = \{A_i : i \in I\}\) be a chain of maximum partitioners of \(\mathcal{R}\) and let \(A := \bigcup_{i \in I} A_i\). Let \(a \notin A\) and suppose \(a - 2, a + 2, a + 4 \in A\). Since \(A\) is a chain, there exists \(i \in I\) such that \(a - 2, a + 2\) and \(a + 4\) are all in \(A_i\). It follows that \(a - 4 \notin A_i\) and thus for all \(j \in I\) such that \(A_j \subseteq A_i\), \(a - 4 \notin A_j\). Moreover if \(j \in I\) is such that \(A_i \subseteq A_j\), then \(a - 2, a + 2, a + 4 \in A_j\) thus, since \(A_j \in \text{MAXP}(\mathcal{R})\), \(a - 4 \notin A_i\). We conclude that \(a - 4 \notin A\) and \(A \in \text{MAXP}(\mathcal{R})\).

Let us suppose now \(|\mathcal{R} \setminus A^*| \leq 1\) and that \(a + 2 \notin A\) and \(a - 2 \notin A\). We have to prove that \(a - 4 \notin A\) if \(a - 4 \in A\) then for some \(i \in I, \mathcal{R} \setminus A_i^* \leq 1, a + 2 \notin A_i, a - 2 \notin A_i \) and \(a - 4 \notin A_i\) this contradicts the fact that \(A_i\) is a maximum partitioner. Moreover it can not hold simultaneously that both \(a + 2\) and \(a - 2\) do not belong to \(A\), because in this case, since \(a \notin A\), it would hold that \(|\mathcal{R} \setminus A^*| \geq 1\), contradicting our assumption.

Suppose \(a + 4 \in A\). If \(a - 2 \notin A\), then \(a + 2 \in A\) and there exists \(i \in I\) such that \(|\mathcal{R} \setminus A_i^*| \leq 1, a + 2 \notin A_i, a - 2 \notin A_i\) and \(a + 4 \in A_i\), but this is impossible because \(A_i \in \text{MAXP}(\mathcal{R})\). Finally suppose that both \(a - 4\) and \(a + 4\) are in \(A\). Then, in this case for some \(i \in I, |\mathcal{R} \setminus A_i^*| \leq 1, a - 4 \in A_i\) and \(a + 4 \in A_i\), so \(A_i\) is not a maximum partitioner by Proposition 2.3.17, which is a contradiction. It follows that \(A \in \text{MAXP}(\mathcal{R})\).

### 2.4.1 Coverings and UCP Pairings

In this subsection, we analyze which conditions must be satisfied in order to find a covering on \(\Omega\) constituted by maximum partitioners having well specific topological properties.
Let $A, B$ be two non-empty subsets of $\Omega$. We set
\[ \mathcal{E}_\Psi(A, B) := \{ a \in \Omega : A \cap M_\Psi(B \cup \{a\}) \neq \emptyset \} \]
and we call $\mathcal{E}_\Psi(A, B)$ the $B$-expansion of $A$.

In the following proposition we establish some basic properties for the above expansion.

**Proposition 2.4.8.** The following conditions hold:

(i) $A \subseteq \mathcal{E}_\Psi(A, B)$;

(ii) If $A \cap B \neq \emptyset$, then $M_\Psi(A \cup B) \subseteq \mathcal{E}_\Psi(A, B)$;

(iii) If $C \subseteq A$ and $D \subseteq B$, then $\mathcal{E}_\Psi(C, D) \subseteq \mathcal{E}_\Psi(A, B)$;

(iv) $\mathcal{E}_\Psi(A, B) = \bigcup_{a \in A} \mathcal{E}_\Psi(\{a\}, B)$.

**Proof.** Straightforward. \(\Box\)

At this point we provide some sufficient conditions ensuring that $\mathcal{E}_\Psi(A, B) \in M\text{AXP}(\Psi)$.

**Theorem 2.4.9.** Let $\Psi$ be a UCP pairing and $A, B \in M\text{AXP}(\Psi)$ be two non-empty maximum partitioners. Assume that $\mathcal{E}_\Psi(M_\Psi(F), M_\Psi(G)) \in M\text{AXP}(\Psi)$ for any $F \subseteq_f A$ and $G \subseteq_f B$. Then $\mathcal{E}_\Psi(A, B) \in M\text{AXP}(\Psi)$.

**Proof.** Since $\Psi$ is a UCP pairing, we must prove that the maximum partitioner of each finite subset of $\mathcal{E}_\Psi(A, B)$ belongs to $\mathcal{E}_\Psi(A, B)$. For this, let us consider $c_1, \ldots, c_n \in \mathcal{E}_\Psi(A, B)$. By definition of $B$-expansion of $A$, there exist $a_1, \ldots, a_n$ such that $a_i \in A \cap M_\Psi(B \cup \{c_i\})$. Since $\Psi$ is a UCP pairing, for each index $i$ there exists $G_i \subseteq_f B$ such that $a_i \in M_\Psi(G_i \cup \{c_i\})$.

Thus, we set $F := \{a_1, \ldots, a_n\}$ and $G := \bigcup_{i=1}^n G_i$. It is obvious that $c_i \in \mathcal{E}_\Psi(\{a_i\}, M_\Psi(G))$. By (iv) of Proposition 2.4.8, it follows that $c_i \in \mathcal{E}_\Psi(M_\Psi(F), M_\Psi(G))$ for any $i = 1, \ldots, n$ which is a maximum partitioner by our assumption. But by (iii) of Proposition 2.4.8, it follows that $\mathcal{E}_\Psi(M_\Psi(F), M_\Psi(G)) \subseteq \mathcal{E}_\Psi(A, B)$, so $M_\Psi(\{c_1, \ldots, c_n\}) \subseteq \mathcal{E}_\Psi(A, B)$ and, by the arbitrariness of $c_1, \ldots, c_n$, the claim has been proved. \(\Box\)

We now characterize the existence of a $M_\Psi$-split for any pair of disjoint maximum partitioners of a UCP pairing in terms of the expansions of pairs of finite subsets whose maximum partitioners are disjoint from each other.

**Theorem 2.4.10.** Let $\Psi \in PAIR(\Omega)$ be a UCP pairing. Then the following conditions are equivalent:

(i) For each $F, G \subseteq_f \Omega$ such that $M_\Psi(F) \cap M_\Psi(G) = \emptyset$, there exists a covering $\{A, B\}$ on $\Omega$ such that $\{M_\Psi(F), M_\Psi(G)\} \downarrow M_\Psi \{A, B\}$.

(ii) For any $F, G \subseteq_f \Omega$, we have that $\mathcal{E}_\Psi(M_\Psi(F), M_\Psi(G)) \in M\text{AXP}(\Psi)$.

(iii) For any $A, B \in M\text{AXP}(\Psi)$ such that $A \cap B = \emptyset$, there exists a $M_\Psi$-complemented subset $C \in \mathcal{P}(\Omega)$ such that $\{A, B\} \downarrow M_\Psi \{C, C^c\}$.

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We say that a pairing \( P \) consists of a recursive-like action of the closure operator \( M \).

\[ \subseteq \]

**Definition 2.5.1.**

Let \( \Omega \) be the family of all finitary pairings.

In this section, we investigate further properties of the set operator \( M \).

**Definition 2.5.2.**

By maximality, it follows that \( A \subseteq B \) and \( M(F) \subseteq B \setminus A \).

If \( \Omega \) is a set, then \( E(\Omega, F) \) is a decomposable.

**Proof.**

\[ (i) \implies (ii): \] Suppose by contradiction there exist \( F, G \subseteq \Omega \) such that

\[ \exists \in \mathbb{P}(F, G) \notin \mathbb{X}(\mathbb{P}) \]

Let \( C := \exists \mathbb{P}(\exists \mathbb{P}(F, G)) \) and let \( a \in C \setminus \exists \mathbb{P}(\exists \mathbb{P}(F, G)) \). This means that \( \exists \mathbb{P}(F) \cap \exists \mathbb{P}(G) \cup \{a\} = \emptyset \). Let \( A, B \in \mathbb{X}(\mathbb{P}) \) such that \( \{\exists \mathbb{P}(F), \exists \mathbb{P}(G) \cup \{a\}\} \cap A \).

Hence, \( \exists \mathbb{P}(F) \cap A = \emptyset = (\exists \mathbb{P}(G) \cup \{a\}) \cap B \), i.e.

\[ M(G) \cup \{a\} \subseteq A \setminus B \quad \text{and} \quad M(F) \subseteq B \setminus A. \] (2.42)

A fortiori, we have that \( G \cup \{a\} \subseteq A \setminus B \). Nevertheless, it is straightforward that \( C \subseteq B \) since if \( b \in E(\exists \mathbb{P}(F), \exists \mathbb{P}(G)) \), then \( \exists \mathbb{P}(F) \cap \exists \mathbb{P}(G) \cup \{b\} = \emptyset \), so \( b \in B \). In particular, it follows that \( a \in B \), contradicting (2.42).

\[ (ii) \implies (iii): \] Let \( A, B \in \mathbb{X}(\mathbb{P}) \). Surely, we can always find two maximum partitioners \( A' \) and \( B' \) containing respectively \( A \) and \( B \) and maximal with respect to the property of being disjoint. Clearly, if one of them is empty, then \( \Omega \) is the \( \exists \mathbb{P} \)-complemented subset we are searching for. Therefore, let us assume that \( A' \) and \( B' \) are both non-empty and let \( a \in (B')^c \). By maximality, it follows that \( \exists \mathbb{P}(B' \cup \{a\}) \cap A' \neq \emptyset \). This proves that \( B' \subseteq \Omega \setminus \exists \mathbb{P}(A', B') \), which belongs to \( \mathbb{X}(\mathbb{P}) \) by Theorem 2.4.9. Thus, we have found a maximum partitioner \( \exists \mathbb{P}(A', B') \) such that \( A' \subseteq \exists \mathbb{P}(A', B') \) and \( B' \subseteq \Omega \setminus \exists \mathbb{P}(A', B') \). In particular, by maximality, it is immediate to prove that \( B' = \Omega \setminus \exists \mathbb{P}(A', B') \).

Finally, we must prove that \( A' \cup B' = \emptyset \). Suppose by contradiction the claim is false. We firstly observe that \( A' \subseteq \exists \mathbb{P}(A', B') \). In fact, if \( c \in \Omega \setminus (A' \cup B') = (A')^c \cap (B')^c \) then, since \( B' = \Omega \setminus \exists \mathbb{P}(A', B') \), it results that \( A' \subseteq \exists \mathbb{P}(A', B') \). Now, again by maximality, it follows that \( \exists \mathbb{P}(A', B') \cap B' = \emptyset \) or, equivalently, there exists \( b \in B' \) such that \( \exists \mathbb{P}(B' \cup \{b\}) \cap A' \neq \emptyset \), i.e. \( A' \cap B' \neq \emptyset \), contradicting our assumptions. This proves the thesis.

\[ (iii) \implies (i): \] Obvious. \( \square \)

### 2.5 Recursive Properties of \( M \) in UCP Pairings

In this section, we investigate further properties of the set operator \( M \). In particular, we find the conditions needed to express \( M(A \cup \{b\}) \) as the union of the maximum partitioners \( M(\{a, b\}) \) for any \( a \in A \). In other terms, we provide some conditions whose main effect consists of a recursive-like action of the closure operator \( M \) on subset \( A \cup \{b\} \).

**Definition 2.5.1.**

The pair \( (A, b) \) is \( M \)-union decomposable if

\[ M(A \cup \{b\}) = \bigcup_{a \in A} M(\{a, b\}). \] (2.43)

We say that the pairing \( \mathbb{P} \) is \( M \)-union decomposable if any pair \( (A, b) \in \mathbb{X}(\mathbb{P}) \times \Omega \) is \( M \)-union decomposable.

We now define a specific typology of pairing that will be useful in the sequel.

**Definition 2.5.2.**

We say that a pairing \( \mathbb{P} \in \text{PAIR}(\Omega) \) is finitary if \( |M(F)| < \infty \) for any \( F \subseteq \Omega \). We denote by \( \text{PAIR}_{\text{fin}}(\Omega) \) the family of all finitary pairings.
We now show that a finitary pairing $\mathfrak{P}$ is $M_\mathfrak{P}$-union decomposable whenever the property of a maximum partitioner $A$ to be covered by a finite number of maximum partitioner $A_i$ is inherited by $A \cup \{b\}$, where $b \notin A$; in such a way all its covering maximum partitioners are of the form $M_\mathfrak{P}(A_i \cup \{b\})$.

**Proposition 2.5.3.** Let $\mathfrak{P}$ be a finitary UCP pairing on $\Omega$. The following conditions are equivalent:

(i) $\mathfrak{P}$ is $M_\mathfrak{P}$-union decomposable;

(ii) Let $F, F_1, \ldots, F_n \subseteq \mathcal{F} \Omega$, $A := M_\mathfrak{P}(F)$, $A_i := M_\mathfrak{P}(F_i)$. If $A \subseteq \bigcup_{i=1}^n A_i$, for any $b \in \Omega \setminus A$ it results that
\[
M_\mathfrak{P}(A \cup \{b\}) \subseteq \bigcup_{i=1}^n M_\mathfrak{P}(A_i \cup \{b\});
\]

(iii) for any non-empty $F \subseteq \mathcal{F} \Omega$ and $b \notin M_\mathfrak{P}(F)$, it results that
\[
M_\mathfrak{P}(F \cup \{b\}) = \bigcup_{a \in M_\mathfrak{P}(F)} M_\mathfrak{P}(\{a, b\}).
\]

**Proof.** $(i) \implies (ii)$: Let $\mathfrak{P}$ be a $M_\mathfrak{P}$-union decomposable pairing. Let $A, A_1, \ldots, A_n \in \text{MAXP}(\mathfrak{P})$ such that $A \subseteq \bigcup_{i=1}^n A_i$ and $b \in \Omega \setminus A$. By hypothesis, for any $a \in A$ there exists $A_i$ such that $a \in A_i$. Then $\{a, b\} \subseteq A_i \cup \{b\}$. Hence, it results that
\[
\bigcup_{a \in A} M_\mathfrak{P}(\{a, b\}) \subseteq \bigcup_{i=1}^n M_\mathfrak{P}(A_i \cup \{b\});
\]
and, since $\mathfrak{P}$ is $M_\mathfrak{P}$-union decomposable, that
\[
M_\mathfrak{P}(A \cup \{b\}) \subseteq \bigcup_{i=1}^n M_\mathfrak{P}(A_i \cup \{b\}).
\]

$(ii) \implies (iii)$: Let $\emptyset \neq F \subseteq \mathcal{F} \Omega$. We clearly have that
\[
\{a, b\} \subseteq M_\mathfrak{P}(F) \cup \{b\}
\]
for any $a \in M_\mathfrak{P}(F)$. Hence
\[
\bigcup_{a \in M_\mathfrak{P}(F)} M_\mathfrak{P}(\{a, b\}) \subseteq M_\mathfrak{P}(M_\mathfrak{P}(F) \cup \{b\})
\]
or, equivalently, by $(i)$ of Theorem 2.2.1,
\[
\bigcup_{a \in M_\mathfrak{P}(F)} M_\mathfrak{P}(\{a, b\}) \subseteq M_\mathfrak{P}(F \cup \{b\}).
\]
On the other hand, set $F_i = M_{\mathfrak{P}}(\{a_i\})$ for any $a_i \in M_{\mathfrak{P}}(F)$ and let $b \in \Omega \setminus F$. We observe that

$$M_{\mathfrak{P}}(F \cup \{b\}) = M_{\mathfrak{P}}(M_{\mathfrak{P}}(F) \cup \{b\}) \subseteq \bigcup_{i=1}^{n} M_{\mathfrak{P}}(F_i \cup \{b\}) = \bigcup_{i=1}^{n} M_{\mathfrak{P}}(\{a_i, b\}) = \bigcup_{a \in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}).$$

$$(iii) \implies (i):$$ Let $A \in MAXP(\mathfrak{P})$. If $b \in A$, then $M_{\mathfrak{P}}(A \cup \{b\}) = M_{\mathfrak{P}}(A) = A$ therefore it is straightforward to see that $A \subseteq \bigcup_{a \in A} M_{\mathfrak{P}}(\{a, b\})$.

Let now $b \in \Omega \setminus A$. Then

$$M_{\mathfrak{P}}(A \cup \{b\}) = \bigcup_{F \subseteq fA} M_{\mathfrak{P}}(F \cup \{b\})$$

that, by our hypothesis, gives

$$\bigcup_{F \subseteq fA} M_{\mathfrak{P}}(F \cup \{b\}) = \bigcup_{F \subseteq fA} \bigcup_{a \in M_{\mathfrak{P}}(F)} M_{\mathfrak{P}}(\{a, b\}) = \bigcup_{a \in A} M_{\mathfrak{P}}(\{a, b\}).$$

This shows $(i)$. \qed

Based on the results obtained in Proposition 2.5.3, we are also able to prove a more general recursive-like property of the operator $M_{\mathfrak{P}}$.

**Theorem 2.5.4.** Let $\mathfrak{P}$ be a $M_{\mathfrak{P}}$-union decomposable finitary UCP pairing and $A_1, \ldots, A_n \in MAXP(\mathfrak{P})$ are non-empty, then

$$M_{\mathfrak{P}}\left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{a \in A_i} M_{\mathfrak{P}}(\{a_1, \ldots, a_n\}).$$  \hspace{1cm} (2.49)

**Proof.** Let us observe that $M_{\mathfrak{P}}(A \cup B) = M_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \cup M_{\mathfrak{P}}(B))$ for any $A, B \subseteq \Omega$. We shall prove the theorem for two subsets $A, B \in MAXP(\mathfrak{P})$. Let $|A| = k$ and $|B| = l$. We proceed by induction on $k + l$. Fix $k$: if $l = 1$, the thesis follows immediately by (2.43). Let the thesis be true for $|B| \leq l - 1$ and suppose now that $|B| = l$. Let $a \in M_{\mathfrak{P}}(A \cup B)$ and fix $b \in B$. By (2.45), it results that

$$M_{\mathfrak{P}}((A \cup B \setminus \{b\}) \cup \{b\}) \subseteq \bigcup_{c \in M_{\mathfrak{P}}(A \cup B \setminus \{b\})} M_{\mathfrak{P}}(b, c),$$

i.e. there exists $c' \in M_{\mathfrak{P}}(A \cup B \setminus \{b\})$ such that $a \in M_{\mathfrak{P}}(\{b, c'\})$. By what we observed in the first part of the proof, we can use the inductive hypothesis on $A \cup (B \setminus \{b\})$, so there exist $a' \in M_{\mathfrak{P}}(A)$ and $b' \in M_{\mathfrak{P}}(B \setminus \{b\})$ such that $c' \in M_{\mathfrak{P}}(\{a', b'\})$. In other terms, it results that

$$a \in M_{\mathfrak{P}}(\{b, c'\}) = M_{\mathfrak{P}}(\{a', b, b'\}) = M_{\mathfrak{P}}(\{b, b'\} \cup \{a'\}).$$

Again by (2.45), there exists $b'' \in M_{\mathfrak{P}}(\{b, b'\})$ such that $a \in M_{\mathfrak{P}}(\{a', b'', b''\})$. Since $\{b, b'\} \subseteq B$, it follows that $b'' \in B$ and, so, we have shown (2.49). We conclude by induction on the number of maximum partitioners. \qed

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2.6 Minimal Partitioners

In this section, we analyze another basic structures arising from the global symmetry relation and, moreover, we focus our attention to the relations between this new structure and $MAXP(\mathfrak{P})$. To this regard, let us consider the subset family consisting of all the minimal members of each global symmetry class, that is $MINP(\mathfrak{P})$, defined as follows:

$$MINP(\mathfrak{P}) := \bigcup \{ \min([A]_{\approx \mathfrak{P}}) : A \in MAXP(\mathfrak{P}) \}. \quad (2.50)$$

It is clear that the subset family $MINP(\mathfrak{P})$ can be considered a type of dual version of $MAXP(\mathfrak{P})$. We call minimal partitioner of $\mathfrak{P}$ any member of $MINP(\mathfrak{P})$.

In the next result, we show that $MINP(\mathfrak{P})$ is an abstract simplicial complex and, moreover, we provide the so-called global-local regularity, according to which the inclusion between the maximum elements of two any global symmetry classes preserves the same type of inclusion between any two members of these classes.

Theorem 2.6.1. The following hold.

(i) Let $C_i \subset C_j$ be two subsets of $\Omega$. Then, if $X \in [C_i]_{\approx \mathfrak{P}}$ and $Y \in [C_j]_{\approx \mathfrak{P}}$ we have $Y \not\subset X$.

(ii) Let $A \in \mathcal{P}(\Omega)$ and $C \subseteq B \subseteq A$ with $B \in MINP(\mathfrak{P})$. Then $\pi_\mathfrak{P}(C \setminus \{x\}) \neq \pi_\mathfrak{P}(C)$ for any $x \in C$.

(iii) $MINP(\mathfrak{P}) \subseteq ASC(\Omega)$.

(iv) $MINP(\mathfrak{P}) = \{ A \in \mathcal{P}(\Omega) : a \notin M_\mathfrak{P}(A \setminus \{a\}) \forall a \in A \}$.

Proof. (i): Since $C_i$ and $C_j$ are two distinct maximum partitioners such that $C_i \subset C_j$, we have $\pi_\mathfrak{P}(C_j) \prec \pi_\mathfrak{P}(C_i)$. Now, by absurd, let $X \in [C_i]_{\approx \mathfrak{P}}$ and $Y \in [C_j]_{\approx \mathfrak{P}}$ such that $Y \subseteq X$. Then $\pi_\mathfrak{P}(C_i) = \pi_\mathfrak{P}(X) \leq \pi_\mathfrak{P}(Y) = \pi_\mathfrak{P}(C_j)$, that is a contradiction.

(ii) Let us suppose by contradiction that there exists $x \in C$ such that $\pi_\mathfrak{P}(C \setminus \{x\}) = \pi_\mathfrak{P}(C)$. Then, since $C \setminus \{x\} \subseteq B \setminus \{x\}$, we have

$$\pi_\mathfrak{P}(B \setminus \{x\}) \leq \pi_\mathfrak{P}(C \setminus \{x\}) = \pi_\mathfrak{P}(C). \quad (2.51)$$

Therefore, if $v, v' \in \Omega$ by (2.51) it follows that

$$v \equiv_{B \setminus \{x\}} v' \implies v \equiv_{C} v' \implies F(v, x) = F(v', x) \implies v \equiv_{B} v',$$

therefore

$$\pi_\mathfrak{P}(B \setminus \{x\}) \leq \pi_\mathfrak{P}(B). \quad (2.52)$$

On the other hand, since we also have $\pi_\mathfrak{P}(B) \leq \pi_\mathfrak{P}(B \setminus \{x\})$, by (2.52) we deduce that $\pi_\mathfrak{P}(B \setminus \{x\}) = \pi_\mathfrak{P}(B)$, that is in contrast with the hypothesis that $B \in BAS_\mathfrak{P}(A)$. This concludes the proof.

(iii): Let $C \in MINP(\mathfrak{P})$. Then, there exists $B \in MAXP(\mathfrak{P})$ such that $C \in \min([B]_{\approx \mathfrak{P}})$. Let $K \not\subset C$, then there exists $B' \in MAXP(\mathfrak{P})$ such that $K \in [B']_{\approx \mathfrak{P}}$. Suppose by contradiction that $K \notin \min([B']_{\approx \mathfrak{P}})$, then there exists $K' \subset K$ such that $K' \in \min([B']_{\approx \mathfrak{P}})$. Hence, there is $x \in K \setminus K'$. We deduce that

$$K' \subseteq K \setminus \{x\} \subseteq K.$$
i.e. 
\[ \pi_q(K) \preceq \pi_q(K \setminus \{x\}) \preceq \pi_q(K'). \]

But since \( \pi_q(K) = \pi_q(K') = \pi_q(B) \), we conclude that \( \pi_q(K) = \pi_q(K \setminus \{x\}) \), contradicting part (ii). Hence \( K \in MINP(\mathfrak{Q}) \).

(iv): Let \( A \in MINP(\mathfrak{Q}) \) and \( a \in A \). Then \( A \setminus \{a\} \notin [A]_{\sim q} \), so \( M_q(A \setminus \{a\}) \notin [A]_{\sim q} \). Since \( A \setminus \{a\} \subsetneq A \), we have \( \pi_q(A) \preceq \pi_q(A \setminus \{a\}) \). Moreover, suppose by contradiction that \( a \in M_q(A \setminus \{a\}) \). Then \( A \subseteq M_q(A \setminus \{a\}) \) and \( M_q(A) \subseteq M_q(A \setminus \{a\}) \). By (vi) of Theorem 2.7.1, we have \( \pi_q(A \setminus \{a\}) \preceq \pi_q(A) \), so \( \pi_q(A \setminus \{a\}) = \pi_q(A) \) and \( A \setminus \{a\} \notin [A]_{\sim q} \), that is an absurd. So \( a \notin M_q(A \setminus \{a\}) \) and \( MINP(\mathfrak{Q}) \subseteq \{A \in \mathcal{P}(\Omega) : a \notin M_q(A \setminus \{a\}) \forall a \in A \} \). On the other hand, let \( A \in \mathcal{P}(\Omega) \) such that \( a \notin M_q(A \setminus \{a\}) \) for any \( a \in A \). Suppose by contradiction that \( A \in MINP^*(\mathfrak{Q}) \), i.e. there exists \( B \subsetneq A \) such that \( \pi_q(A) = \pi_q(B) \). Then, there exists \( a \in A \) such that \( B \subset A \setminus \{a\} \subset A \). This implies that \( \pi_q(A) \preceq \pi_q(A \setminus \{a\}) \preceq \pi_q(B) \), i.e. \( \pi_q(A \setminus \{a\}) = \pi_q(A) \). In other terms, we have \( M_q(A \setminus \{a\}) = M_q(A) \supseteq A \), contradiction. \( \blacksquare \)

By Theorem 2.6.1, it follows immediately that we obtain a map \( MINP_q : \mathcal{P}(\Omega) \rightarrow SS(\Omega) \) such that \( MINP_q(\Omega) = MINP(\mathfrak{Q}) \) and
\[ MINP_q(A) \in ASC(A), \]  
(2.53)

for any \( A \in \mathcal{P}(\Omega) \).

### 2.7 Minimal Partitioners On Several Examples

In this section, we deal with the characterization of the minimal partitioners of some pairings, namely the simple undirected \( n \)-cycle, a vector space with a bilinear form and the euclidean real line.

#### 2.7.1 Graphs: The \( n \)-Cycle Case

In the next result, we characterize the minimal partitioners of \( C_n \).

**Theorem 2.7.1.** Let \( G = C_n \) and \( A \subseteq V(C_n) \). Then \( A \in MINP(G) \) if and only if for any \( v_i \in A \) it holds one of the following conditions:

(i) \( (v_{i-2} \notin A \text{ or } v_{i+2} \notin A) \) and \( (B_G(A) \neq \emptyset) \);
(ii) \( (v_{i-2} \in A \text{ and } v_{i-4} \notin A) \) or \( (v_{i+2} \in A \text{ and } v_{i+4} \notin A) \).

**Proof.** Let \( A \subseteq V(G) \) and suppose there exists \( v_i \in A \) such that both (i) and (ii) are false. If \( B_G(A) = \emptyset \), then
\[ B_G(A \setminus \{v_i\}) = \begin{cases} \{v_{i-1}\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \in A, \\ \{v_{i+1}\} & \text{if } v_{i-2} \in A \text{ and } v_{i+2} \notin A, \\ \{v_{i-1}, v_{i+1}\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \notin A, \\ \emptyset & \text{otherwise.} \end{cases} \]

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Moreover, since \( v_{i-2} \in A \implies v_{i-4} \in A \) and \( v_{i+2} \in A \implies v_{i+4} \in A \), then

\[
C_G(A \setminus \{v_i\}) = \begin{cases} 
C_G(A) \setminus \{v_i\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \notin A, \\
C_G(A) & \text{otherwise}.
\end{cases}
\]

In all cases we obtain \( A \approx_{\emptyset} A \setminus \{v_i\} \), so \( A \notin MINP(G) \).

Let us suppose now that \( A \notin MINP(G) \) and let \( v_i \in A \) such that \( A \approx_{\emptyset} A \setminus \{v_i\} \). Then, there exist \( v_j, v_k \in V(G) \) such that \( v_j \equiv_{A \setminus \{v_i\}} v_k \) and \( v_j \not\equiv_A v_k \). This happens if and only if the following assertions are false:

1. \( v_j, v_k \in B_G(A \setminus \{v_i\}) \), \( \{v_j, v_k\} \not\subseteq B_G(A) \) and \( B_G(A) \neq \emptyset \);
2. \( v_{i-2} \in C_G(A) \) or \( v_{i+2} \in C_G(A) \).

These conditions are equivalent to conditions (i) and (ii) respectively. This proves the theorem. \( \square \)

### 2.7.2 Vector Spaces

Let \( V \) be a vector space. If \( k \) is a non-negative integer and \( \dim((H) = k \), we write \( H \leq_k V \).

Let us assume that \( V \) is a vector space of dimension \( n \) over \( \mathbb{K} \) and \( \varphi \) is (left) non degenerate. Let \( \mathfrak{P} \) be the \((V, \mathbb{K}, \varphi)\)-VBP. We have the following characterizations.

**Theorem 2.7.2.** (i) \( MAXP_{\mathfrak{P}}(A) = \{H \cap A : H \leq V\} \);

(ii) \( MINP_{\mathfrak{P}}(A) = \{C : C \subseteq_{li} A\} \cup \emptyset \);

(iii) if \( A \in MAXP(\mathfrak{P}) \), \( u, u' \in V \setminus A \) and \( M_{\mathfrak{P}}(A \cup \{u\}) = M_{\mathfrak{P}}(A \cup \{u'\}) \), then \( \text{Span} \{u\} = \text{Span} \{u'\} \).

**Proof.** (i): By (iii) of Proposition 2.3.21, it is immediate to see that \( MAXP(\mathfrak{P}) = \{H : H \leq V\} \). Hence, by the definition of \( MAXP_{\mathfrak{P}}(A) \), it results that \( MAXP_{\mathfrak{P}}(A) = \{H \cap A : H \leq V\} \).

(ii): Again by (iii) of Proposition 2.3.21, it is immediate to see that \( MINP(\mathfrak{P}) = \{C : C \subseteq_{li} V \cup \emptyset\} \), that is the family consisting of all the possible bases of any vector subspace \( H \) of \( V \) and of the null space. Hence, let us consider \( A \subseteq V \). Then \( MINP_{\mathfrak{P}}(A) = \{C : C \subseteq_{li} A\} \cup \emptyset \} \).

(iii): Let \( x \in \text{Span} \{u\} \), then \( x = \lambda u \) for some \( \lambda \in \mathbb{K} \). So, \( x \in \text{Span}(A) + \text{Span}(\{u\}) = \text{Span}(A) + \text{Span}(\{u'\}) \) since \( \varphi \) is (left) non degenerate, therefore \( x \in \text{Span}(A) + \text{Span}(\{u'\}) \), i.e. \( x \in \text{Span}(\{u'\}) \). The converse is analogous. \( \square \)

**Remark 2.7.3.** The property of the operator \( M_{\mathfrak{P}} \) for a VBP established in (iii) of Theorem 3.3.1 can be considered a weak version of the classical exchange property for a closure operator.

### 2.7.3 The Euclidean Real Line

We now characterize the minimal partitioners of the real euclidean line and then show that \( MINP(\mathbb{R}) \) is a finitary abstract complex on \( \mathbb{R} \).

**Proposition 2.7.4.** \( A \in MINP(\mathbb{R}) \) if and only if for any \( a \in A \) it results that \( a - 4 \notin A \) or \( a + 4 \notin A \).
Proof. To say that $A \in MINP(\mathbb{R})$ is equivalent to require that $\pi_0(\partial A \setminus \{a\}) \neq \pi_0(A)$. In other terms, for any $a \in A$ there exist $x, y \in \mathbb{R}$ such that $x \equiv_{A \setminus \{a\}} y$ and $x \not\equiv_A y$. It is straightforward to see that the only possibility is that $\{x, y\} = \{a - 3, a - 1\}$ or $\{x, y\} = \{a + 1, a + 3\}$. Therefore, one of the conditions $a - 4 \notin A$ or $a + 4 \notin A$ must hold.

Theorem 2.7.5. $MINP(\mathbb{R})$ is a finitary abstract complex on $\mathbb{R}$.

Proof. By (iii) of Proposition 2.6.1, $MINP(\mathbb{R})$ is an abstract simplicial complex on $\mathbb{R}$. Let now $A \subseteq \mathbb{R}$. We prove that if $F \in MINP(\mathbb{R})$ for any $F \subseteq f A$ then $A \in MINP(\mathbb{R})$. For, assume by contradiction that $A \notin MINP(\mathbb{R})$. Thus, there exists $a \in A$ such that $a - 4, a + 4 \in A$. Let us consider the finite set $F = \{a - 4, a, a + 4\}$. By Proposition 2.7.4, we conclude that $F \notin MINP(\mathbb{R})$, contradicting our assumption. \qed
Chapter 3

Symmetry Set Systems Induced by a Pairing

In this chapter we will represent by pairings a particular structure on a finite set Ω, called dissymmetry space, that generalizes the symmetry relation since it takes account of the conditions to "discern in Ω" two elements \( x, x' \) of a given set \( X \). Secondly, we introduce the main set systems arising from the symmetry relation, that's to say symmetry bases, core and symmetry essentials. We investigate their interrelations by means of an operatorial standpoint.

3.1 Dissymmetry Spaces

We now introduce the notion of dissymmetry space on \( Ω \).

**Definition 3.1.1.** A dissymmetry space on \( Ω \) is a structure \( \mathcal{D} = \langle X, D \rangle \), where \( X \) is a non-empty set and \( D : X \times X \rightarrow \mathcal{P}(Ω) \) is a map, called local dissymmetry map of \( \mathcal{D} \) having the following properties:

(i) \( D(x, x) = \emptyset \) for each \( x \in X \).
(ii) \( D(x, x') = D(x', x) \) for all \( x, x' \in Ω \).
(iii) \( D(x, x') = \bigcap_{x'' \in X} [D(x, x'') \cup D(x', x'')] \).

We say that \( \mathcal{D} \) is a finite dissymmetry space if both \( X \) and \( Ω \) are finite.

**Remark 3.1.2.** The reader can easily verify that the property \((iii)\) of Definition 3.1.1 is equivalent to the following:

\((iii')\) \( y \in D(x, x') \) if and only if, for all \( x'' \in Ω \), \( y \notin D(x, x'') \implies y \in D(x', x'') \).

In the above definition, the three properties \((i)\)–\((iii)\) are the counterpart of the reflexivity, symmetry and transitivity properties of an symmetry relation. Indeed, \((i)\) says that an element \( x \) is not discernible from itself, \((ii)\) says that what distinguishes \( x \) from \( x' \) is exactly the same of what distinguishes \( x' \) from \( x \) and \((iii)\) expresses the idea that if \( x, x' \) differ on a property \( y \), then there does not exist an element \( x'' \) which is equal to both \( x \) and \( x' \) on property \( y \).

Given a pairing \( \mathcal{P} \in PAIR(Ω) \) and fixed \( A \in \mathcal{P}(Ω) \), let us define the map \( Δ^\mathcal{P}_A : U \times U \rightarrow \mathcal{P}(A) \) as follows

\[
Δ^\mathcal{P}_A(u, u') := \{ a \in A : F(u, a) \neq F(u', a) \},
\]
for any \( u, u' \in U \).

In particular, we set \( \Delta^\Psi := \Delta^\Psi_{\Pi} \), so that \( \Delta^\Psi_A(u, u') = \Delta^\Psi(u, u') \cap A \). Let us observe that \( \Delta^\Psi_A(u, u') \) is the subset of all elements \( a \in \Omega \) which discern \( u \) from \( u' \).

Therefore we call \( \Delta^\Psi_A(u, u') \) the A-dissymmetry neighborhood of \( u \) and \( u' \) and A-dissymmetry set system the subset family

\[
\text{DIS}_\Psi(A) := \{ \Delta^\Psi_A(u, u') : u, u' \in U \text{ and } \Delta^\Psi_A(u, u') \neq \emptyset \}.
\]

In particular, we set \( \text{DIS}(\Psi) := \text{DIS}_\Psi(\Omega) \).

If \( \Psi \in \text{PAIR}(\Omega) \) is a finite pairing, we consider the A-dissymmetry matrix \( \Delta^\Psi \) of \( \Psi \), that is the \( m \times m \) matrix with entries \( (d_{ij}) \), where

\[
d_{ij} := \Delta^\Psi_A(u_i, u_j).
\]

Let us note that \( \Delta^\Psi \) is a symmetric matrix such that \( \Delta^\Psi(u_i, u_i) = \emptyset \) for all \( i = 1, \ldots, m \).

Moreover, in the finite case, if \( k \in \{1, \ldots, n\} \), we set \( \text{DIS}_k(\Psi) := \{ C \in \text{DIS}(\Psi) : |C| = k \} \).

The following result relates the subsets \( \Delta^\Psi_A(u, u') \) to the local symmetry relations.

**Proposition 3.1.3.** Let \( D \subseteq A \) and \( u, u' \in U \). Then:

(i) \( D = \Delta^\Psi_A(u, u') \implies u \equiv_{A \setminus D} u' \);

(ii) \( u \equiv_{A \setminus D} u' \implies \Delta^\Psi_A(u, u') \subseteq D \);

(iii) \( \Delta^\Psi_A(u, u') \cap D = \emptyset \iff u \equiv_D u' \).

**Proof.** (i) : Let \( a \in A \setminus D \). Then, we have \( F(v, a) = F(w, a) \), therefore \( v \equiv_{A \setminus D} w \).

(ii) : By hypothesis, \( F(v, a) = F(w, a) \) for all \( a \in A \setminus D \). This means that the set of elements of \( A \) for which the values of \( F(v, a) \) and \( F(w, a) \) are different, i.e. \( \Delta^\Psi_A(v, w) \), stays in the complement of \( A \setminus D \). Therefore \( \Delta^\Psi_A(v, w) \subseteq D \).

(iii) : Let \( \Delta^\Psi_A(v, w) \cap D = \emptyset \). This is equivalent to require that for all \( c \in D \), \( F(v, c) = F(w, c) \) or, equivalently, \( v \equiv_D w \). \( \square \)

In what follows, we will prove that to each finite dissymmetry space corresponds a pairing. In fact, let us notice that the following result holds.

**Proposition 3.1.4.** Let \( \Psi \in \text{PAIR}(\Omega) \) be a pairing. Then, the map \( \Delta^\Psi : U \times U \to \Psi(\Omega) \) satisfies the following properties:

(i) \( \Delta^\Psi(u, u) = \emptyset \) for all \( u \in U \);

(ii) \( \Delta^\Psi(u, u') = \Delta^\Psi(u', u) \) for all \( u, u' \in U \);

(iii) \( \Delta^\Psi(u, u') = \bigcap_{u'' \in U} [\Delta^\Psi(u, u'') \cup \Delta^\Psi(u', u'')] \).

**Proof.** Parts (i) and (ii) are obvious. Let us prove part (iii). Let \( a \in \Delta^\Psi(u, u') \) and let \( u'' \in U \). If \( a \notin \Delta^\Psi(u, u'') \), then it means that \( F(u, a) = F(u'', a) \). But since \( a \in \Delta^\Psi(u, u') \) we have \( F(u', a) \neq F(u, a) = F(u'', a) \) and thus \( a \in \Delta^\Psi(u', u'') \). This proves the claim. \( \square \)

**Remark 3.1.5.** By Proposition 3.1.4, if \( \Psi \) is a pairing, the structure \( \mathcal{D}[\Psi] := (U, \Delta^\Psi) \) is a dissymmetry space, that we call **dissymmetry space of the pairing** \( \Psi \).
We describe now a method to associate a pairing $\mathcal{P}_D$ to any finite dissymmetry space $(X, D)$ on $\Omega$ in such a way that $\Delta^{\mathcal{P}_D} = D$.

Let $X = \{x_1, \ldots, x_m\}$ be a finite set and let $D : X \times X \rightarrow \mathcal{P}(\Omega)$ be an arbitrary map. Let us construct a pairing $\mathcal{P}_D = (U, \Lambda, F)$ in the following way. We set $U := X$ and define a map $F : U \times \Omega \rightarrow \mathbb{N}$ with the following recursive method:

1. For each $y_j \in \Omega$, set $F(x_1, y_j) := 1$.
2. Let $i \in \{2, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ and we assume to have inductively defined the values $F(x_s, y_j)$, for $s = 1, \ldots, i - 1$.
3. We set $S_{ij} := \{s : s \in \{1, \ldots, i - 1\} \land y_j \notin D(x_s, x_i)\}$ and $t_{ij} := \min S_{ij}$ if $S_{ij} \neq \emptyset$.
4. We set then
   
   $F(x_i, y_j) := \begin{cases} 
   F(x_{t_{ij}}, y_j) & \text{if } S_{ij} \neq \emptyset \\
   1 + \max_{1 \leq s \leq i - 1} F(x_s, y_j) & \text{otherwise}.
   \end{cases}$
5. Finally we set $\Lambda := \operatorname{Im}(F)$.

**Example 3.1.6.** Let $X = \{u_1, u_2, u_3, u_4\}$, $\Omega = \{a_1, a_2, a_3, a_4\}$ and $D : X \times X \rightarrow \mathcal{P}(\Omega)$ represented by the matrix below:

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
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<td>$u_1$</td>
<td>$\emptyset$</td>
<td>${a_1, a_3}$</td>
<td>${a_1, a_2}$</td>
<td>${a_4}$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$\ast$</td>
<td>$\emptyset$</td>
<td>${a_1, a_2, a_4}$</td>
<td>${a_3, a_4}$</td>
</tr>
<tr>
<td>$u_3$</td>
<td>$\ast$</td>
<td>$\ast$</td>
<td>$\emptyset$</td>
<td>${a_2, a_4}$</td>
</tr>
<tr>
<td>$u_4$</td>
<td>$\ast$</td>
<td>$\ast$</td>
<td>$\ast$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Firstly, we set $F(u_1, a_i) = 1$ for $i = 1, 2, 3, 4$. We represent below the table of $S_{i,j}$ for $i = 2, 3, 4$ and $j \in \{1, 2, 3, 4\}$.

<table>
<thead>
<tr>
<th></th>
<th>$S_{2,1}$</th>
<th>$S_{2,2}$</th>
<th>$S_{2,3}$</th>
<th>$S_{2,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{3,1}$</td>
<td>$\emptyset$</td>
<td>${1}$</td>
<td>$\emptyset$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$S_{3,2}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${1, 2}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$S_{1,1}$</td>
<td>${1, 2, 3}$</td>
<td>${1, 2}$</td>
<td>${1, 3}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$S_{1,2}$</td>
<td>${1, 2}$</td>
<td>${1, 2}$</td>
<td>${1, 3}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Therefore, we obtain the following pairing:

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$2$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$u_3$</td>
<td>$3$</td>
<td>$2$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$u_4$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

and, in particular, $\Lambda = \{1, 2, 3\}$.

We have then the following result.
Theorem 3.1.7. Let \( (X, D) \) be a dissymmetry space and let \( \mathcal{P}_D = (U, \Lambda, F) \) be the finite pairing built above. Then \( \Delta_{\mathcal{P}_D} = D \).

Proof. Let \( x_k, x_i \in U \). If \( k = i \), then, by their definitions, both \( \Delta_{\mathcal{P}_D}(x_k, x_i) \) and \( D(x_k, x_i) \) are equal to the empty set. Without loss in generality we can suppose in what follows \( k < i \). Let us recall the definition of the map \( \Delta_{\mathcal{P}_D} \):

\[
\Delta_{\mathcal{P}_D}(x_k, x_i) := \{ y_j \in \Omega : F(x_k, y_j) \neq F(x_i, y_j) \}.
\]

In order to prove that \( \Delta_{\mathcal{P}_D}(x_k, x_i) \subseteq D(x_k, x_i) \), we claim that, \( \forall h \in S_{ij}, S_{hj} = S_{ij} \cap \{1, \ldots, h - 1\} \). In fact, if this holds then clearly, \( \forall h \in S_{ij}, F(x_h, y_j) = F(x_i, y_j) \), so if \( y_j \notin D(x_k, x_i) \), then \( F(x_k, y_j) = F(x_i, y_j) \) and thus \( y_j \notin \Delta_{\mathcal{P}_D}(x_k, x_i) \). This is equivalent to say that \( \Delta_{\mathcal{P}_D}(x_k, x_i) \subseteq D(x_k, x_i) \).

Let us prove the claim. Let \( h \in S_{ij} \). If \( S_{hj} = \emptyset \), then obviously \( S_{hj} \subseteq S_{ij} \cap \{1, \ldots, h - 1\} \). Let us suppose \( S_{hj} \neq \emptyset \) and let \( s \in S_{hj} \). In this case it holds \( y_j \notin D(x_s, x_h) \) and \( y_j \notin D(x_h, x_i) \), so, by condition \((iii')\) of Remark 3.1.2, \( y_j \notin D(x_s, x_h) \) and thus \( s \in S_{ij} \cap \{1, \ldots, h - 1\} \). Conversely if \( S_{ij} \cap \{1, \ldots, h - 1\} = \emptyset \), then clearly \( S_{ij} \cap \{1, \ldots, h - 1\} \subseteq S_{hj} \). Let us suppose then that \( S_{ij} \cap \{1, \ldots, h - 1\} 
eq \emptyset \) and let \( s \in S_{ij} \cap \{1, \ldots, h - 1\} \). As before this means that \( y_j \notin D(x_s, x_i) \) and since \( y_j \notin D(x_s, x_h) \), by \((iii')\) of Remark 3.1.2, \( y_j \notin D(x_s, x_h) \), so \( s \in S_{hj} \) and the claim is proved.

Let \( y_j \in D(x_k, x_i) \). If \( S_{ij} = \emptyset \), by definition we have \( F(x_i, y_j) = 1 + \max_{1 \leq s \leq i - 1} F(x_s, y_j) \neq F(x_k, y_j) \), so \( y_j \in \Delta_{\mathcal{P}_D}(x_k, x_i) \). Let \( S_{ij} \neq \emptyset \). If \( F(x_k, y_j) = F(x_i, y_j) \), by \((iii')\) of Remark 3.1.2 it is straightforward to prove that there exists \( h \) such that \( 1 \leq h \leq k < i, y_j \notin D(x_h, x_k) \) and \( y_j \notin D(x_h, x_i) \) or, equivalently, by \((iii)\) of Definition 3.1.1, that \( y_j \notin D(x_h, x_i) \) and this is a contradiction. Then \( F(x_k, y_j) \neq F(x_i, y_j) \) and thus \( y_j \in \Delta_{\mathcal{P}_D}(x_k, x_i) \). This proves the theorem. \( \square \)

3.2 Symmetry Set Systems Induced by a Pairing

In this section we introduce some new set systems induced by \( A \)-symmetry relation and study their main properties. To this regard, we introduce the notion of \( A \)-symmetry base.

Definition 3.2.1. Let \( B \subseteq A \). We say that \( B \) is an \( A \)-symmetry base of \( \mathcal{P} \) if:

(R1) \( \pi_{\mathcal{P}}(A) = \pi_{\mathcal{P}}(B) \);

(R2) \( \pi_{\mathcal{P}}(A) \neq \pi_{\mathcal{P}}(B') \) for all \( B' \subsetneq B \).

We denote by \( BAS_{\mathcal{P}}(A) \) the family of all \( A \)-symmetry bases of \( \mathcal{P} \) and we set \( BAS(\mathcal{P}) := BAS_{\mathcal{P}}(\Omega) \). We call symmetry bases of \( \mathcal{P} \) the members of \( BAS(\mathcal{P}) \). If \( \mathcal{P} \) is a finite pairing on \( \Omega \) and if \( k \in \{1, \ldots, n\} \), we set \( BAS_k(\mathcal{P}) := \{ C \in BAS(\mathcal{P}) : |C| = k \} \).

We now provide some basic properties of \( A \)-symmetry bases.

Theorem 3.2.2. (i) \( BAS_{\mathcal{P}}(A) = \{ B : B \in \min([A]_{\mathcal{P}}), B \subseteq A \} \) for any \( A \in \mathcal{P}(\Omega) \).

(ii) \( BAS_{\mathcal{P}}(A) \subseteq \max(MINP_{\mathcal{P}}(A)) \) for any \( A \in \mathcal{P}(\Omega) \).
we have that $\pi_\mathcal{P}(A) = \pi_\mathcal{P}(C) \neq \pi_\mathcal{P}(C')$. This means that $C$ is minimal in $[A]_{\approx_\mathcal{P}}$.

Conversely, let $C \in \min([A]_{\approx_\mathcal{P}})$. Then $\pi_\mathcal{P}(C) = \pi_\mathcal{P}(A)$. Moreover, let $C' \subseteq C$. By the minimality of $C$, it follows that $C' \notin [A]_{\approx_\mathcal{P}}$, whence $\pi_\mathcal{P}(C') \neq \pi_\mathcal{P}(A)$. This shows that $C$ satisfies both the conditions of Definition 3.2.1, i.e. $C \in \text{BAS}_{\mathcal{P}}(A)$.

(ii): Let $C \in \text{BAS}_{\mathcal{P}}(A)$, then $C \subseteq A$ and $C \in \min([A]_{\approx_\mathcal{P}}) \subseteq \text{MINP}(\mathcal{P})$ by part (i). This proves that $C \in \text{MINP}_{\mathcal{P}}(A)$. Suppose by contradiction that $\text{BAS}_{\mathcal{P}}(A) \notin \max(\text{MINP}_{\mathcal{P}}(A))$, i.e. there exists $D \in \text{MINP}_{\mathcal{P}}(A)$ such that $C \nsubseteq D$. Then, $\pi_\mathcal{P}(D) \leq \pi_\mathcal{P}(C) = \pi_\mathcal{P}(A)$. On the other hand, by the fact that $D \in \text{MINP}_{\mathcal{P}}(A)$, we deduce that $\pi_\mathcal{P}(A) \leq \pi_\mathcal{P}(D)$. Therefore, we infer that $D \approx_\mathcal{P} A$. It is now straightforward to see that it is in contrast with our choice of $D$.

Let us note here that the inclusion established in (iii) of Theorem (3.2.2) cannot be reversed, as we see in next example.

**Example 3.2.3.** Let $\mathcal{P}$ be the functional table given in Figure 3.1 and $A = \{2, 3, 5\}$. Then, we have that

$\text{BAS}_{\mathcal{P}}(A) = \{\{2, 5\}\} \subseteq \max(\text{MINP}_{\mathcal{P}}(A)) = \{\{2, 3\}, \{2, 5\}, \{3, 5\}\}

We now provide the deep link between relative maximum partitioners and $\text{BAS}_{\mathcal{P}}(A)$.

**Proposition 3.2.4.** Let $B \in \text{MAXP}_{\mathcal{P}}(A)$ such that $B \subseteq A$. Then $A \setminus B$ is a transversal of $\text{BAS}_{\mathcal{P}}(A)$.

**Proof.** Suppose by contradiction that $A \setminus B$ is not a transversal of $\text{BAS}_{\mathcal{P}}(A)$, i.e. there exists $C \in \text{BAS}_{\mathcal{P}}(A)$ such that $C \cap (A \setminus B) = \emptyset$. Thus, $C \subseteq B \subseteq A$. In particular, it results that

$M_{\mathcal{P}}(C) \subseteq M_{\mathcal{P}}(B) \subseteq M_{\mathcal{P}}(A),$

but $M_{\mathcal{P}}(C) = M_{\mathcal{P}}(A)$, so $M_{\mathcal{P}}(A) = M_{\mathcal{P}}(B)$. Since there exists $D \in \text{MAXP}(\mathcal{P})$ such that $B = A \cap D$, we observe that $M_{\mathcal{P}}(B) \subseteq D$, hence

$A \subseteq M_{\mathcal{P}}(A) \subseteq D$

or, equivalently, $B = A \cap D = A$, contradiction. Therefore, $A \setminus B$ is a transversal of $\text{BAS}_{\mathcal{P}}(A)$.

\[ \square \]
Figure 3.2: Diagram of $\mathcal{P}(\mathcal{P})$. 
We now provide the notion of core.

**Definition 3.2.5.** Let \( A \in \mathcal{P}(\Omega) \). An element \( a \in A \) is said *indispensable* if \( \pi_{\mathcal{P}}(A) \neq \pi_{\mathcal{P}}(A \setminus \{a\}) \). The subset of all indispensable elements of \( A \) is called the *core* of \( A \) and it is denoted by \( C_{\mathcal{P}}(A) \). In particular, we set \( \text{CORE}(\mathcal{P}) := C_{\mathcal{P}}(\Omega) \).

Let us prove that the core of \( A \) is given by the intersection of all \( A \)-symmetry bases when \( \Omega \) is finite.

**Proposition 3.2.6.** We have that:

\[
C_{\mathcal{P}}(A) := \bigcap \{ C : C \in \text{BAS}_{\mathcal{P}}(A) \}.
\]  

(3.4)

when \( \Omega \) is finite.

**Proof.** Let \( a \in \bigcap \{ C : C \in \text{BAS}_{\mathcal{P}}(A) \} \) and assume by contradiction that \( a \notin C_{\mathcal{P}}(A) \). This implies that \( \pi_{\mathcal{P}}(A) = \pi_{\mathcal{P}}(A \setminus \{a\}) \) and, in particular, the existence of a subset \( B \subseteq A \setminus \{a\} \) belonging to \( \text{BAS}_{\mathcal{P}}(A \setminus \{a\}) \). But we also have \( B \in \text{BAS}_{\mathcal{P}}(A) \), contradicting the fact that \( a \in \bigcap \{ C : C \in \text{BAS}_{\mathcal{P}}(A) \} \).

On the other hand, let \( a \in C_{\mathcal{P}}(A) \). Let \( B \in \text{BAS}_{\mathcal{P}}(A) \) and assume by contradiction that \( a \notin B \).

This implies \( B \subseteq A \setminus \{a\} \), so \( \pi_{\mathcal{P}}(A) \leq \pi_{\mathcal{P}}(A \setminus \{a\}) \leq \pi_{\mathcal{P}}(B) = \pi_{\mathcal{P}}(A) \), but this would mean that \( a \notin C_{\mathcal{P}}(A) \), leading us to a contradiction. \( \square \)

Actually, the core can be interpreted from an operatorial perspective, therefore we use the following terminology.

**Definition 3.2.7.** We call the set operator \( C_{\mathcal{P}} \) the *\( \mathcal{P} \)-core operator*.

We can now pass to examine the main properties of the core operator. In particular, the \( \mathcal{P} \)-maximum partitioner operator and the \( \mathcal{P} \)-core operator are related in the following way.

**Proposition 3.2.8.** For any \( A \in \mathcal{P}(\Omega) \) we have that:

(i) \( C_{\mathcal{P}}(A) = \{ a \in A : M_{\mathcal{P}}(A \setminus \{a\}) \preceq M_{\mathcal{P}}(A) \} \); 

(ii) \( C_{\mathcal{P}}(A) = \{ a \in A : a \notin M_{\mathcal{P}}(A \setminus \{a\}) \} \); 

(iii) \( C_{\mathcal{P}}(A) = \{ a \in A : A \setminus \{a\} \in \text{MAXP}_{\mathcal{P}}(A) \} \); 

(iv) \( M_{\mathcal{P}}(A) = \{ b \in \Omega : b \notin C_{\mathcal{P}}(A \cup \{b\}) \} \); 

(v) \( C_{\mathcal{P}}(M_{\mathcal{P}}(A)) \subseteq A \). 

**Proof.** Part (i) follows directly from the definitions of \( C_{\mathcal{P}}(A) \) and \( M_{\mathcal{P}}(A) \).

For (ii), let us observe that \( a \notin M_{\mathcal{P}}(A \setminus \{a\}) \) if and only if \( M_{\mathcal{P}}(A \setminus \{a\}) \neq M_{\mathcal{P}}(A) \), i.e. if and only if \( \pi_{\mathcal{P}}(A) \neq \pi_{\mathcal{P}}(A \setminus \{a\}) \).

(iii): Let \( a \in C_{\mathcal{P}}(A) \). We must show that \( A \setminus \{a\} \in \text{MAXP}_{\mathcal{P}}(A) \), i.e. there exists \( B \in \text{MAXP}(\mathcal{P}) \) such that \( A \setminus \{a\} = A \cap B \). Let us assume by absurd that \( A \setminus \{a\} \neq A \cap B \) for any \( B \in \text{MAXP}(\mathcal{P}) \). Since \( \pi_{\mathcal{P}}(A) \neq \pi_{\mathcal{P}}(A \setminus \{a\}) \), then \( A \setminus \{a\} \notin C_{\mathcal{P}}(A) \). Let \( B \) be the maximum
Conversely, let $A \setminus \{a\} \subseteq B$ and we have $A \setminus \{a\} \subseteq A \cap B$ therefore, by our assumption, it follows that $A \setminus \{a\} \subseteq A \cap B$. Hence $A \cap B = A$, i.e. $A \subseteq B$. It follows that $\piq(B) = \piq(A \setminus \{a\}) \leq \piq(A) \leq \piq(B)$, i.e. $\piq(A) = \piq(B)$, that is equivalent to say that $A \approx \piq B$, i.e. $A \approx \piq A \setminus \{a\}$, that is an absurd. Thus $A \setminus \{a\} \in \text{MAXP}_\mathcal{P}(A)$.

Conversely, let $a \in A$ such that $A \setminus \{a\} \in \text{MAXP}_\mathcal{P}(A)$. Hence, there exists $B \in \text{MAXP}(\mathcal{P})$ such that $A \setminus \{a\} = B \cap A$. If $B = A \setminus \{a\}$, then $A \setminus \{a\} \in \text{MAXP}(\mathcal{P})$. In this case, by maximality of $A \setminus \{a\}$, then $A \not\subseteq [A \setminus \{a\}]_{\approx \piq}$, so $\piq(A) \neq \piq(A \setminus \{a\})$, therefore $a \in \text{MAXP}_\mathcal{P}(A) \subseteq C_P(A)$ and the claim is proved. Otherwise, let $A \setminus \{a\} \subseteq B$. Clearly, it results that $a \notin B$. Furthermore, it results $\piq(B) \leq \piq(A \setminus \{a\})$. Suppose by contradiction that $\piq(A) = \piq(A \setminus \{a\})$. Then, we have $\piq(B) \leq \piq(A)$, so

$$u \equiv_B u' \implies u \equiv_A u' \implies F(u, a) = F(u', a) \implies u \equiv_{B \cup \{a\}} u'.$$

In other terms, we have shown that $\piq(B) \leq \piq(B \cup \{a\})$. Nevertheless, we also have $\piq(B \cup \{a\}) \leq \piq(B)$, i.e. $\piq(B) = \piq(B \cup \{a\})$. This contradicts the maximality of $B$, that is an absurd. Thus, $a \in C_P(A)$.

(iv): We simply observe that $M_\mathcal{P}(A) = M_\mathcal{P}(A \cup \{b\} \setminus \{b\})$ so $b \in M_\mathcal{P}(A)$ if and only if $b \notin C_P(A \cup \{b\})$.

(v): Let $a \in C_P(M_\mathcal{P}(A))$ and suppose by contradiction that $a \notin A$. Therefore we have $A \subseteq M_\mathcal{P}(A) \setminus \{a\} \subseteq M_\mathcal{P}(A)$ and, hence, that $\piq(M_\mathcal{P}(A)) \leq \piq(M_\mathcal{P}(A) \setminus \{a\}) \leq \piq(A)$. Since $\piq(M_\mathcal{P}(A)) = \piq(A)$, we conclude that $\piq(M_\mathcal{P}(A) \setminus \{a\}) = \piq(M_\mathcal{P}(A))$, contradicting the fact that $a \in C_P(A)$.

In the next result we establish further new properties for the $\mathcal{P}$-core operator.

**Theorem 3.2.9.** $C_\mathcal{P}$ is an idempotent core operator. Moreover, if $\Omega$ is finite and $A \in \text{MAXP}(\mathcal{P})$, it results that

$$a \in A \setminus C_\mathcal{P}(A) \iff \text{there exists } B \in \mathcal{P}(\Omega) \text{ such that } a \notin B \text{ and } a \in M_\mathcal{P}(B) \subseteq M_\mathcal{P}(A).$$

(3.5)

**Proof.** We clearly have that $C_\mathcal{P}(A) \subseteq A$. The set operator $C_\mathcal{P}$ is pseudo-monotone. In fact, let $a \in C_\mathcal{P}(B) \cap A$, then $a \in A$ and $\piq(B \setminus \{a\}) \neq \piq(A)$. Suppose that $a \in M_\mathcal{P}(A \setminus \{a\})$, then $a \in M_\mathcal{P}(B \setminus \{a\})$, contradicting the fact that $a \in C_\mathcal{P}(B)$.

Let us prove that $C_\mathcal{P}$ is quasi-regular. To this regard, let $A \in \mathcal{P}(\Omega)$ and $b, c \in \Omega \setminus A$. Assume that $b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))$. This means that $b \in M_\mathcal{P}(A \cup \{b\} \setminus \{b\})$, i.e. $b \in M_\mathcal{P}(A) \setminus A$. Moreover, assume that $c \in \sigma(A \cup \{c\})$. This implies that $c \in \Omega \setminus M_\mathcal{P}(A \cup \{c\} \setminus \{c\}) = M_\mathcal{P}(A)$. In particular, it results that $M_\mathcal{P}(A \cup \{b, c\} \setminus \{c\}) = M_\mathcal{P}(A \cup \{b\}) = M_\mathcal{P}(A)$, so $c \in \Omega \setminus M_\mathcal{P}(A)$ or, equivalently, $c \in M_\mathcal{P}(A \cup \{b, c\})$.

Let us prove idempotency. By part (i), we have that $C_\mathcal{P}(C_\mathcal{P}(A)) \subseteq C_\mathcal{P}(A)$. On the other hand, let $a \in C_\mathcal{P}(A)$, then $\piq(A \setminus \{a\}) \neq \piq(A)$, i.e. $M_\mathcal{P}(A \setminus \{a\}) \not\subseteq M_\mathcal{P}(A)$. Now, since $C_\mathcal{P}(A) \subseteq A$, we deduce that $a \in M_\mathcal{P}(C_\mathcal{P}(A) \setminus \{a\}) \subseteq M_\mathcal{P}(A \setminus \{a\})$ but this would imply that $A \subseteq M_\mathcal{P}(A \setminus \{a\})$ or, equivalently, $M_\mathcal{P}(A \setminus \{a\}) = M_\mathcal{P}(A)$, contradicting the fact that $a \in C_\mathcal{P}(A)$.

Finally, we show that (3.5) holds. Let us assume $\Omega$ to be finite. Let $a \in A \setminus C_\mathcal{P}(A)$. Then, just consider $B \in BAS_\mathcal{P}(A)$ such that $a \notin B$. Nevertheless, we have $a \in M_\mathcal{P}(B) = M_\mathcal{P}(A)$.
On the other hand, assume that \( a \in A \) and let \( B \) not containing \( a \) and such that \( a \in M_{\mathcal{Q}}(B) \subseteq M_{\mathcal{Q}}(A) \). Now, set \( C := A \setminus \{a\} \). Clearly, \( M_{\mathcal{Q}}(C) \subseteq M_{\mathcal{Q}}(A) = A \). Moreover, we have that \( B \subseteq C \) and \( a \in M_{\mathcal{Q}}(B) \subseteq M_{\mathcal{Q}}(C) \) so, we conclude that \( A \subseteq M_{\mathcal{Q}}(C) \) and, hence \( A = M_{\mathcal{Q}}(C) \).

In this way, we have shown the existence of a subset \( C \in [A]_{\mathcal{Q}} \) not containing \( a \). Now, let us observe that \( a \not\in D \) for any \( D \in BAS_{\mathcal{Q}}(C) \). A fortiori, since \( BAS_{\mathcal{Q}}(C) \subseteq BAS_{\mathcal{Q}}(A) \), it follows by \((3.4)\) that \( a \in A \setminus C_{\mathcal{Q}}(A) \).

By means of both the operators \( M_{\mathcal{Q}} \) and \( C_{\mathcal{Q}} \) we can also provide the following new characterization for \( BAS_{\mathcal{Q}}(A) \) and \( MINP_{\mathcal{Q}}(A) \).

**Theorem 3.2.10.** Let \( A \in \mathcal{P}(\Omega) \). Then:

(i) \( MINP_{\mathcal{Q}}(A) = \{B \in \mathcal{P}(A) : C_{\mathcal{Q}}(B) = B\} \);

(ii) \( BAS_{\mathcal{Q}}(A) = \{B \in \mathcal{P}(A) : M_{\mathcal{Q}}(B) = M_{\mathcal{Q}}(A) \text{ and } C_{\mathcal{Q}}(B) = B\} \);

(iii) \( BAS_{\mathcal{Q}}(A) = \{B \in \mathcal{P}(A) : M_{\mathcal{Q}}(B) = M_{\mathcal{Q}}(A) \text{ and } C_{\mathcal{Q}}(B) = B\} \);

(iv) \( BAS_{\mathcal{Q}}(A) = \{B \in \mathcal{P}(A) : M_{\mathcal{Q}}(B) = M_{\mathcal{Q}}(A) \text{ and } C_{\mathcal{Q}}(B) = B\} \);

(v) \( M_{\mathcal{Q}}(\mathcal{P}) = \{B \in \mathcal{P}(\Omega) : BAS_{\mathcal{Q}}(B) = \{B\}\} \).

**Proof.** (i): Let \( B \in MINP_{\mathcal{Q}}(A) \). Clearly, \( C_{\mathcal{Q}}(B) \subseteq B \). Vice versa, let \( b \in B \) and assume by contradiction that \( \pi_{\mathcal{Q}}(B \setminus \{b\}) = \pi_{\mathcal{Q}}(B) \). This means that \( B \) is not a minimal partitioner of \( \mathcal{Q} \), contradicting our assumption on \( B \).

On the other hand, let \( B \in \mathcal{P}(A) \) such that \( C_{\mathcal{Q}}(B) = B \) and suppose by contradiction that \( B \not\in MINP_{\mathcal{Q}}(A) \). Then, there exists \( B' \subseteq B \) such that \( B' \approx_{\mathcal{Q}} B \) and \( B' \in MINP_{\mathcal{Q}}(A) \). In particular, \( B' \subseteq B \setminus \{b\} \) for some \( b \in B \). This entails that \( \pi_{\mathcal{Q}}(B \setminus \{b\}) = \pi_{\mathcal{Q}}(B) \), i.e. \( b \not\in C_{\mathcal{Q}}(B) \) or, equivalently, \( B \neq C_{\mathcal{Q}}(B) \), contradicting the choice of \( B \). This proves that \( MINP_{\mathcal{Q}}(A) = \{B \in \mathcal{P}(A) : C_{\mathcal{Q}}(B) = B\} \).

(ii): It follows by part (i) by replacing \( A \) with \( \Omega \).

(iii): Let \( B \in BAS_{\mathcal{Q}}(A) \). This means that \( \pi_{\mathcal{Q}}(B) = \pi_{\mathcal{Q}}(A) \) and \( \pi_{\mathcal{Q}}(B') \neq \pi_{\mathcal{Q}}(A) \) for any \( B' \subseteq B \). In particular, we have that \( M_{\mathcal{Q}}(A) = M_{\mathcal{Q}}(B) \). Furthermore, if we take \( B' = B \setminus \{b\} \), it follows that \( \pi_{\mathcal{Q}}(B \setminus \{b\}) \neq \pi_{\mathcal{Q}}(A) \). By the arbitrariness of \( b \), we deduce that \( C_{\mathcal{Q}}(B) = B \).

On the contrary, let \( B \in \mathcal{P}(A) \) such that \( M_{\mathcal{Q}}(A) = M_{\mathcal{Q}}(B) \) and \( C_{\mathcal{Q}}(B) = B \). Then, \( \pi_{\mathcal{Q}}(B \setminus \{b\}) \neq \pi_{\mathcal{Q}}(B) \) for any \( b \in B \). In particular, if \( B' \subseteq B \), then there exists \( b \in B \) for which \( B' \subseteq B \setminus \{b\} \). So, \( \pi_{\mathcal{Q}}(B') \neq \pi_{\mathcal{Q}}(B) = \pi_{\mathcal{Q}}(A) \) and this proves that \( B \in BAS_{\mathcal{Q}}(A) \).

(iv): It follows by part (iii) by replacing \( A \) with \( \Omega \).

(v): Let \( A \in MINP(\mathcal{P}) \), then \( A \in \min([A]_{\mathcal{Q}}) \). Moreover, the unique subset satisfying both conditions of Definition 3.2.1 is exactly \( A \), that is \( BAS_{\mathcal{Q}}(A) = \{A\} \).

Vice versa, suppose that \( BAS_{\mathcal{Q}}(A) = \{A\} \). Then, we have that \( A \in \max(MINP_{\mathcal{Q}}(A)) \) and, in particular, \( A \in MINP(\mathcal{P}) \).

We now show that any core operator on a finite set \( \Omega \) is the \( \mathcal{Q} \)-core operator of some pairing \( \mathcal{Q} \in PAIR(\Omega) \).

**Theorem 3.2.11.** Let \( \Omega \) be a finite set. If \( \sigma \in COOP(\Omega) \), there exists \( \mathcal{Q} \in PAIR(\Omega) \) such that \( \sigma = C_{\mathcal{Q}} \).

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Proof. Let $\sigma \in COOP(\Omega)$. Let us consider the set operator $\varphi : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ defined as follows:

$$\varphi_\sigma(A) := A \cup A^*,$$

where $A^* := \{b \in \Omega : b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))\}$. We will prove that $\varphi$ is a closure operator. Clearly, $\varphi$ is extensive. Let now $A \subseteq B$. It suffices to show that $A^* \subseteq \varphi_\sigma(B)$. Let $b \in A^*$. Assume that $b \in \Omega \setminus B$. Then, $A \cup \{b\} \subseteq B \cup \{b\}$, therefore, by pseudo-monotonicity, we have that $\sigma(A \cup \{b\}) \supseteq \sigma(B \cup \{b\}) \cap (A \cup \{b\})$. But since $b \in \Omega \setminus \sigma(A \cup \{b\})$, it follows that $b \in \Omega \setminus ((A \cup \{b\}) \cap \sigma(B \cup \{b\}))$. Clearly, it implies that $b \in \Omega \setminus \sigma(B \cup \{b\})$ and this proves that $b \in \Omega \setminus (B \cup \sigma(B \cup \{b\})) = B^*$, i.e. $b \in \varphi_\sigma(B)$. On the other hand, if $b \in B$, it is obvious that $b \in \varphi_\sigma(B)$. This proves monotonicity.

We now show that $\varphi$ is idempotent. By extensiveness of $\varphi_\sigma$, it follows that $\varphi_\sigma(A) \subseteq \varphi_\sigma(\varphi_\sigma(A))$. Clearly, if $(\varphi_\sigma(A))^* = \emptyset$, then $\varphi_\sigma(\varphi_\sigma(A)) \subseteq \varphi_\sigma(A)$. Assume now that $(\varphi_\sigma(A))^* \neq \emptyset$ and let $b \in (\varphi_\sigma(A))^*$. Then, $b \in \Omega \setminus (\varphi_\sigma(A) \cup \sigma(\varphi_\sigma(A) \cup \{b\})) = \Omega \setminus ((A \cup A^*) \cup \sigma(A \cup A^* \cup \{b\}))$. Let us prove that the condition $b \in \Omega \setminus A \cup A^*$ implies $b \in \sigma(A \cup A^* \cup \{b\})$.

Let us observe that since $b \in \Omega \setminus \varphi_\sigma(A)$, then $b \in \sigma(A \cup \{b\})$. Therefore, let us fix an integer $m$ and assume that $b \in \sigma(A \cup B \cup \{b\})$ for any $B \subseteq A^*$ such that $|B| = m - 1$. Let now $B \subseteq A^*$ such that $|B| = m$ and fix $c \in B$. We will prove that $c \in \Omega \setminus \sigma((A \cup B \setminus \{c\}) \cup \{c\})$ and that $b \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$ so that it is possible to use quasi-regularity that, in this case, yields $c \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$, showing the claim for $B$. In particular, the claim holds for $B = A^*$. Then, $b \in \sigma(A \cup A^* \cup \{b\})$, contradicting our choice of $b$. Necessarily, it must be $(\varphi_\sigma(A))^* = \emptyset$.

Hence, let us firstly show that $c \in \Omega \setminus \sigma((A \cup B \setminus \{c\}) \cup \{c\})$. As a matter of fact, we have that $c \in A^*$, so $c \in \Omega \setminus (A \cup \{c\})$. Furthermore, to say that $c \in B$ means that $A \cup \{c\} \subseteq A \cup B$ so, by pseudo-monotonicity, $\sigma(A \cup \{c\}) \supseteq \sigma(A \cup B) \cap (A \cup \{c\})$. In other terms, $c \in \Omega \setminus \sigma((A \cup B \setminus \{c\}) \cup \{c\})$.

On the other hand, we have that $b \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$, in fact we can use the inductive hypothesis on $B \setminus \{c\} \subseteq A^*$ since $|B \setminus \{c\}| = m - 1$. The two hypothesis of the definition of quasi-regular operator are satisfied and this proves that $\varphi_\sigma$ is idempotent, so $\varphi_\sigma \in CLOP(\Omega)$.

By part (i), there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $M_{\mathfrak{P}} = \varphi_\sigma$. To conclude, we have to show that $\sigma = C_{\mathfrak{P}}$.

Let $b \in C_{\mathfrak{P}}(A)$, then $b \in \Omega \setminus M_{\mathfrak{P}}(A \setminus \{b\}) = \varphi_\sigma(A \setminus \{b\})$. This entails that $b \in \Omega \setminus (A \setminus \{b\})^*$ or, equivalently, $b \in \sigma(A \setminus \{b\} \cup \{b\}) = \sigma(A)$.

On the other hand, let $b \in \sigma(A)$. By extensiveness, $b \in A$. Furthermore, assume by contradiction that $b \in M_{\mathfrak{P}}(A \setminus \{b\})$. Then, $b \in (A \setminus \{b\})^*$, i.e. $b \in \Omega \setminus \sigma(A \setminus \{b\} \cup \{b\}) = \Omega \setminus \sigma(A)$, contradicting our choice of $b$. This implies that $C_{\mathfrak{P}}(A) = \sigma(A)$ for any $A \in \mathcal{P}(\Omega)$ and conclude the proof.

We now provide the notion of $A$-symmetry essential.

**Definition 3.2.12.** Let $B \subseteq A$. We say that $B$ is $A$-symmetry essential if:

1. $\pi_\mathfrak{P}(A \setminus B) \neq \pi_\mathfrak{P}(A)$;
2. $\pi_\mathfrak{P}(A \setminus B') = \pi_\mathfrak{P}(A)$ for all $B' \subsetneq B$. 

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We denote by $ESS_\mathcal{P}(A)$ the family of all $A$-symmetry essential subsets of $A$ and we set $ESS(\mathcal{P}) := ESS_\mathcal{P}(\Omega)$. We call symmetry essentials of $\mathcal{P}$ the members of $ESS(\mathcal{P})$. If $\mathcal{P}$ is a finite pairing on $\Omega$ and $k \in \{1, \ldots, n\}$, we set $ESS_k(\mathcal{P}) := \{C \in ESS(\mathcal{P}) : |C| = k\}$

The motivation to introduce the set system $ESS_\mathcal{P}(A)$ is that it is a generalization of the core because $C_\mathcal{P}(A) = \emptyset$ or, otherwise, $\{a\} \in ESS_\mathcal{P}(A)$ for any $a \in C_\mathcal{P}(A)$.

The next result tells us that the $A$-symmetry essentials are exactly the minimal elements of the $A$-dissymmetry set system.

**Theorem 3.2.13.** We have that $ESS_\mathcal{P}(A) = \text{min}(DIS_\mathcal{P}(A))$.

**Proof.** Let $B \subseteq ESS_\mathcal{P}(A)$. By Definition 3.2.12 it results that $\pi_\mathcal{P}(A \setminus B) \neq \pi_\mathcal{P}(A)$. Hence, there exist two distinct elements $v, w \in U$ such that $v \equiv_{A \setminus B} w$ and $v \not\equiv_B w$. Equivalently, we can express the previous condition by saying that $A \setminus B \subseteq A \setminus \Delta_\mathcal{P}(v, w)$, i.e., $\Delta_\mathcal{P}(v, w) \subseteq B$. This shows that any $A$-symmetry essential contains some subset of the $A$-dissymmetry set system. We now claim that $A \setminus \Delta_\mathcal{P}(v, w) = B$. Indeed, if $b \in B$ and $b' := B \setminus \{b\} \varsubsetneq B$, we deduce that $v \not\equiv_{A \setminus B'} w$ by Definition 3.2.12. Therefore $b \in \Delta_\mathcal{P}(v, w)$. By the arbitrariness of $b \in B$, it follows that $\Delta_\mathcal{P}(v, w) = B$. This proves that $B \in DIS_\mathcal{P}(A)$. Moreover, we proved that whenever two elements $v, w \in U$ satisfy the relation $\Delta_\mathcal{P}(v, w) \subseteq B$, then $\Delta_\mathcal{P}(v, w) = B$. This means that $B$ is minimal in $DIS_\mathcal{P}(A)$ with respect to set-theoretical inclusion.

Let now $B = \Delta_\mathcal{P}(v, w) \neq \emptyset$ be minimal in the poset $(DIS_\mathcal{P}(A), \subseteq)$, for some $v, w \in U$. Since $B$ is non-empty, by (iii) of Proposition 3.1.3, it follows that $v \not\equiv_A w$. Moreover, by (i) of Proposition 3.1.3 we also obtain $v \equiv_{A \setminus B} w$. Then, we have $\pi_\mathcal{P}(A \setminus B) \neq \pi_\mathcal{P}(A)$, and thus $B$ satisfies (i) of Definition 3.2.12. Let now $B' \varsubsetneq B$. $B$ minimal in $DIS_\mathcal{P}(A)$ implies that, for all $u, u' \in U$ such that $\Delta_\mathcal{P}(u, u') \neq \emptyset, \Delta_\mathcal{P}(u, u') \not\subseteq B'$. We claim that $\pi_\mathcal{P}(A \setminus B') = \pi_\mathcal{P}(A)$. It is obvious that $u \equiv_A u'$ implies $u \equiv_{A \setminus B'} u'$; furthermore suppose that $u \equiv_{A \setminus B'} u'$ and assume by contradiction that $u \not\equiv_A u'$. Then, we have $\Delta_\mathcal{P}(u, u') \subseteq B'$, that is an absurd. Hence $u \equiv_A u' \iff u \equiv_{A \setminus B'} u'$, so $\pi_\mathcal{P}(A \setminus B') = \pi_\mathcal{P}(A)$. In this way, we have shown that $B$ satisfies also condition (ii) of Definition 3.2.12. Then, $B \in ESS_\mathcal{P}(A)$ and the theorem is proved. $\square$

**Remark 3.2.14.** By Theorem 3.2.13, it is clear that $Tr(DIS_\mathcal{P}(A)) = Tr(ESS_\mathcal{P}(A))$.

The next result shows that the $A$-symmetry bases are exactly the minimal transversals of the family $ESS_\mathcal{P}(A)$.

**Theorem 3.2.15.** Let $B \subseteq A$. Then:

(i) $\pi_\mathcal{P}(B) = \pi_\mathcal{P}(A)$ if and only if $B$ is a transversal of $\text{min}(DIS_\mathcal{P}(A))$.

(ii) $BAS_\mathcal{P}(A) = Tr(DIS_\mathcal{P}(A)) = Tr(ESS_\mathcal{P}(A))$.

**Proof.** (i): Let $B \subseteq A$. Let us note that the equality $\pi_\mathcal{P}(B) = \pi_\mathcal{P}(A)$ is obviously equivalent to the identity $\equiv_B = \equiv_A$. Therefore, we assume first that $\equiv_B = \equiv_A$; we must show that $B$ is a transversal of $DIS_\mathcal{P}(A)$. Let then $D \in DIS_\mathcal{P}(A)$. By definition of $DIS_\mathcal{P}(A)$ it results that $D$ is non-empty and that there exist two distinct elements $u, u' \in U$ such that $D = \{a \in A : F(u, a) \neq F(u', a)\}$. Since $D$ contains at least one element, we deduce that $u \neq_A u'$, and this
also implies \( u \not\equiv_B u' \) from the hypothesis \( \equiv_B = \equiv_A \). Hence, by definition of \( \equiv_B \), we find an element \( b \in B \) such that \( F(u, b) \neq F(u', b) \), i.e. \( b \in B \cap D \). This shows that \( B \cap D \neq \emptyset \), therefore \( B \) is a transversal of \( \text{DIS}_q(A) \).

We suppose now that \( B \) is a transversal of \( \text{DIS}_q(A) \) and let \( u, u' \) be two any distinct elements in \( U \). If \( u \equiv_A u' \) it is obvious that we also have \( u \equiv_B u' \). We can assume therefore that \( u \not\equiv_A u' \).

By definition of \( \equiv_A \) and by (3.1) it follows then that the subset \( \Delta^q_A(u, u') \) is non-empty, so that \( D = \text{DIS}_q(A) \). Since \( B \) is a transversal of \( \text{DIS}_q(A) \) we have that \( B \cap D \neq \emptyset \). Let \( b \in B \cap D \).

Then, there exists \( b \in B \) such that \( F(u, b) \neq F(u', b) \), and this implies that \( u \not\equiv_B u' \). Hence \( \equiv_B = \equiv_A \). This proves that \( \pi_q(B) = \pi_q(A) \) if and only if \( B \) is a transversal of \( \text{DIS}_q(A) \).

Clearly, it suffices to be transversal of \( \min(\text{DIS}_q(A)) \) in order to be also a transversal of \( \text{DIS}_q(A) \).

Hence, we showed that \( \pi_q(A) = \pi_q(B) \) if and only if \( B \) is a transversal of \( \min(\text{DIS}_q(A)) \).

\( (i) \): Let \( B \in \text{BAS}_q(A) \). By Definition 3.2.1 we have then \( \pi_q(B) = \pi_q(A) \), and by part \((i)\) this implies that \( B \) is a transversal of \( \text{DIS}_q(A) \).

Now, if \( b \in B \), again by Definition 3.2.1 we have that \( \pi_q(B \setminus \{b\}) \neq \pi_q(A) \), therefore by \((i)\) it follows that \( B \setminus \{b\} \) is not a transversal of \( \text{DIS}_q(A) \). Hence \( B \) is a minimal transversal of \( \text{DIS}_q(A) \). On the other hand, let \( B \) be a minimal transversal of \( \text{DIS}_q(A) \), then by \((i)\) it follows that \( \pi_q(B) = \pi_q(A) \). Now, if \( b \in B \) the subset \( B \setminus \{b\} \) is not a transversal of \( \text{DIS}_q(A) \) by virtue of the minimality of \( B \), therefore, again by \((i)\) we obtain \( \pi_q(B \setminus \{b\}) \neq \pi_q(A) \). Hence \( B \in \text{BAS}_q(A) \) and the thesis follows. \( \square \)

Based on the relevance of the set system \( \text{ESS}_q(A) \), we now provide a new characterization for its structure in relation to the order structure of the poset \( \text{M}_q(A) \).

**Theorem 3.2.16.** \( \text{ESS}_q(A) = \{B \in \mathfrak{P}(A) : A \setminus B \subseteq \{a | \text{M}_q(A) \uparrow \} \} \).

**Proof.** Let \( B \) be a non-empty subset of \( A \) such that \( A \setminus B \subseteq \{a | \text{M}_q(A) \uparrow \} \). Assume by contradiction that \( \pi_q(A \setminus B) = \pi_q(A) \), i.e. \( \text{M}_q(A \setminus B) = \text{M}_q(A) \). Since \( A \setminus B = A \cap C \) for some \( C \in \text{MAXP}(\mathfrak{P}) \), we have that

\[
\text{M}_q(A \cap C) = \text{M}_q(A \setminus B) = \text{M}_q(A) \subseteq \text{M}_q(A) \cap \text{M}_q(C) = \text{M}_q(A) \cap C,
\]

i.e. \( \text{M}_q(A) \subseteq C \). In other terms, we showed that \( A = A \cap \text{M}_q(A) \subseteq A \cap C = A \setminus B \) or, equivalently \( A = A \setminus B \), that is a contradiction. Hence \( \pi_q(A) \neq \pi_q(A \setminus B) \).

Furthermore, let \( B' \subsetneq B \) and assume that \( \pi_q(A \setminus B') \neq \pi_q(A) \). Then, we have that \( A \setminus B \supsetneq A \setminus B' \supsetneq \text{M}_q(A \setminus B') \cap A \). In particular, we also have that \( \text{M}_q(A \setminus B') \cap A \subseteq A \), otherwise \( A \not\subseteq \text{M}_q(A \setminus B') \), i.e. \( \text{M}_q(A) = \text{M}_q(A \setminus B') \), contradicting our assumption. In this way, we showed that \( A \setminus B \) is not a co-cover of \( A \) in \( \text{M}_q(A) \), that is a contradiction. This shows that \( B \in \text{ESS}_q(A) \).

On the other hand, let \( B \in \text{ESS}_q(A) \). Let us show that \( A \setminus B = \text{M}_q(A \setminus B) \cap A \). Clearly, \( A \setminus B \subseteq \text{M}_q(A \setminus B) \cap A \). 

Vice versa, let \( a \in \text{M}_q(A \setminus B) \cap A \) and assume that \( a \notin A \setminus B \). Hence \( a \in B \). Let \( u \equiv_{A \setminus B} u' \), then \( F(u, c) = F(u', c) \) for any \( c \in A \setminus B \) and, in particular, for any \( c \in \text{M}_q(A \setminus B) \). Thus \( F(u, c) = F(u', c) \) for any \( c \in (A \setminus B) \cup \{a\} \), i.e. for any \( c \in A \setminus B \setminus \{a\} \).

Set \( B' := B \setminus \{a\} \). In other terms, we have shown that \( u \equiv_{A \setminus B} u' \) implies \( u \equiv_{A \setminus B'} u' \). But \( \pi_q(A \setminus B') = \pi_q(A) \), so \( u \equiv_{A \setminus B} u' \) implies \( u \equiv_A u' \) or, equivalently, \( \pi_q(A \setminus B) = \pi_q(A) \), contradicting the fact that \( B \in \text{ESS}_q(A) \). This proves that \( A \setminus B \in \text{M}_q(A) \). We must show that \( A \setminus B \in \{a | \text{M}_q(A) \uparrow \} \). Suppose by contradiction it were false. Then, there exists \( A \setminus B \supsetneq C \supsetneq A \) such that \( C \in \{a | \text{M}_q(A) \uparrow \} \). But, this ensures that \( C = A \setminus B' \) for some non-empty \( B' \subsetneq B \).
Let us prove that \( \pi_\overline{Q}(C) \neq \pi_\overline{Q}(A) \). Assume by contradiction that \( \pi_\overline{Q}(C) = \pi_\overline{Q}(A) \). Since \( C = A \cap M \) for some \( M \in \text{MAXP}(\overline{Q}) \), so \( C \subseteq M \) and \( M_\overline{Q}(C) = M_\overline{Q}(A) \subseteq M \), i.e. \( A \subseteq M \). But, this implies that \( A \cap M = C \supseteq A \) and this is impossible. So, \( \pi_\overline{Q}(C) \neq \pi_\overline{Q}(A) \). But the existence of such a subset \( C \) contradicts the fact that \( B \in \text{ESS}_\overline{Q}(A) \). 

3.2.1 The Petersen Graph

In this section we apply all previous general notions to study the Petersen graph, denoted by \( \text{Pet} \). This graph has the following useful characterization: it is the graph whose vertices can be identified with the 2-subsets of \( \hat{5} = \{1, 2, 3, 4, 5\} \), such that two vertices \( A \) and \( B \) are adjacent if and only if they are disjoint. Therefore, we write \( v_{ij} \) to denote the vertex identified with \( \{i, j\} \).

We recall now, without giving any proofs, some important well known properties of the Petersen graph.

**Proposition 3.2.17.**

1. The valency of each vertex in \( \text{Pet} \) is equal to 3.
2. The girth of \( \text{Pet} \) is equal to 5.
3. \( \text{Pet} \) is 3-transitive, thus both vertex-transitive and edge-transitive.
4. The automorphism group of the Petersen graph is isomorphic to the symmetric group \( S_5 \).

In particular an isomorphism \( \psi \) between the symmetric group \( S_5 \) and the automorphism group of \( \text{Pet}, Aut(\text{Pet}) \), can be given in the following natural way. An element \( \pi \in S_5 \) permutes the element in \( \hat{5} \) and thus induces a bijective map \( \psi(\pi) \) of \( V(\text{Pet}) = \binom{\hat{5}}{2} \) in itself. It is not hard to see that \( \psi: \pi \mapsto \psi(\pi) \) is an isomorphism.

The main aim of this section is to provide a complete description of \( \text{DIS}(\text{Pet}), \text{ESS}(\text{Pet}) \) and \( \text{BAS}(\text{Pet}) \). In what follows the letters \( h, i, j, k, l \) will denote all elements in \( \hat{5} \) in an arbitrary order. In this case, if we consider the vertex \( v_{ij} \), it means an arbitrary vertex in \( V(\text{Pet}) \). We begin to examine \( \text{DIS}(\text{Pet}) \).

**Proposition 3.2.18.** We have that \( \text{DIS}(\text{Pet}) = \text{DIS}_4(\text{Pet}) \cup \text{DIS}_6(\text{Pet}) \), where

\[
\text{DIS}_4(\text{Pet}) = \{ A \subseteq V(\text{Pet}) : A = \Delta^\text{Pet}(v, w) \land v \sim w \}
\]

and

\[
\text{DIS}_6(\text{Pet}) = \{ B \subseteq V(\text{Pet}) : B = \Delta^\text{Pet}(v, w) \land v \sim w \}.
\]

Moreover:

- \( A \in \text{DIS}_4(\text{Pet}) \) if and only if \( \text{Pet}[A] \) is isomorphic to the following graph \( H_1 \):

```
  \[
  H_1
  \]
```

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• $B \in \text{DIS}_6(Pet)$ if and only if $\text{Pet}[B]$ is isomorphic to the following graph $H_2$:

![Graph H2]

Proof. Let $v, w \in V(Pet)$ such that $v \sim w$. In this case there exist distinct elements $i, j, k \in \hat{\delta}$ such that $v = v_{ij}$ and $w = v_{ik}$. If $h, l \in \hat{\delta}$ are such that $\{h, l\} = \hat{\delta} \setminus \{i, j, k\}$, then it holds:

$$\Delta_G(v, w) = \Delta_G(v_{ij}, v_{ik}) = \{v_{hj}, v_{hk}, v_{jl}, v_{kl}\}.$$ 

If $A = \Delta_G(v, w)$, then $\text{Pet}[A]$ is the following graph:

![Graph A]

Conversely, assume that $A = \{x_1, x_2, y_1, y_2\} \subseteq V(Pet)$ is such that the induced subgraph $\text{Pet}[A]$ is the following:

![Graph B]

Since $x_1 \sim x_2$, there exist four distinct indexes $i, j, k, l \in \hat{\delta}$ such that $x_1 = v_{ij}$ and $x_2 = v_{kl}$. Now, $y_1$ and $y_2$ are adjacent to each other and both not adjacent to $v_{ij}$ and $v_{kl}$. The only possibilities are that $\{y_1, y_2\} = \{v_{ik}, v_{jl}\}$ or $\{y_1, y_2\} = \{v_{il}, v_{jk}\}$. It is easy to see that $\{v_{ij}, v_{kl}, v_{ik}, v_{jl}\} = \Delta_{Pet}(v_{ij}, v_{kl})$ and $\{v_{ij}, v_{kl}, v_{il}, v_{jk}\} = \Delta_{Pet}(v_{ij}, v_{kl})$, so $A \in \text{DIS}_4(Pet)$.

Suppose first that $v, w \in V(Pet)$ and $v \sim w$. Then, there exist four distinct indexes $i, j, k, l \in \hat{\delta}$ such that $v = v_{ij}$ and $w = v_{kl}$. If $h$ is the only element in $\hat{\delta} \setminus \{i, j, k, l\}$ then

$$\Delta_{Pet}(v, w) = \Delta_{Pet}(v_{ij}, v_{kl}) = \{v_{ij}, v_{kl}, v_{hl}, v_{hj}, v_{hk}, v_{hl}\}.$$ 

If $B = \Delta_{Pet}(v, w)$, the subgraph $\text{Pet}[B]$ is the following:

![Graph C]

and thus it is isomorphic to $H_2$. Let now $B = \{x_1, x_2, y_1, y_2, y_3, y_4\} \subseteq V(Pet)$ such that $\text{Pet}[B]$ is the following graph:

![Graph D]
Then, it is easy to see that $B = \Delta_{\text{Pet}}(x_1, x_2)$, and thus $B \in \text{DIS}_6(\text{Pet})$. This concludes the proof.

**Corollary 3.2.19.** $\Delta[\text{Pet}] = 4(J_{10} - I_{10}) + 2\text{Adj}(\text{Pet})$.

**Proof.** It follows by Proposition 3.2.18 and by definition of the adjacency matrix.

In order to give a complete description of the symmetry base for the Petersen graph, first we establish some useful results.

**Proposition 3.2.20.** $A \in \text{DIS}_4(\text{Pet})$ if and only if $V(\text{Pet}) \setminus A \in \text{DIS}_6(\text{Pet})$.

**Proof.** Let $A \in \text{DIS}_4(\text{Pet})$. As in the proof of Proposition 3.2.18, there exist three distinct indexes $i, j, k \in \hat{5}$ such that

$$A = \Delta(v_{ij}, v_{ik}) = \{v_{hi}, v_{hl}, v_{ij}, v_{ik}, v_{il}, v_{jk}\} = V(\text{Pet}) \setminus \{v_{hi}, v_{hl}, v_{ij}, v_{il}, v_{jk}\} = V(\text{Pet}) \setminus \Delta(v_{hi}, v_{hl}),$$

where $\{h, l\} = \hat{5} \setminus \{i, j, k\}$. So $V(\text{Pet}) \setminus A \in \text{DIS}_6(\text{Pet})$.

Similarly, if $B \in \text{DIS}_6(\text{Pet})$, then there exist four distinct indexes $i, j, k, l \in \hat{5}$ such that $B = \Delta_{\text{Pet}}(v_{ij}, v_{kl}) = \{v_{ij}, v_{kl}, v_{hi}, v_{hl}, v_{hk}, v_{hl}\} = V(\text{Pet}) \setminus \{v_{ik}, v_{il}, v_{jk}, v_{jl}\} = V(\text{Pet}) \setminus \Delta(v_{hi}, v_{hl})$, where $\{h\} = \hat{5} \setminus \{i, j, k, l\}$. This concludes the proof.

**Corollary 3.2.21.** $|\text{DIS}_4(\text{Pet})| = |\text{DIS}_6(\text{Pet})| = 15$ and $|\text{DIS}(\text{Pet})| = 30$.

**Proof.** It follows by Proposition 3.2.18, Proposition 3.2.20 and by using the fact that $|E(\text{Pet})| = 15$.

**Proposition 3.2.22.** We have that $\text{DIS}(\text{Pet}) = \text{ESS}(\text{Pet})$.

**Proof.** By Theorem 3.2.13 it is sufficient to prove that, if $A \in \text{DIS}_4(\text{Pet})$, then there exists no local dissimmetry subset $B \in \text{DIS}_6(\text{Pet})$ such that $A \subseteq B$. This follows by Proposition 3.2.18 and by using the fact that, if $A \subseteq B \in \text{DIS}_6(\text{Pet})$ has cardinality $|A| = 4$, then $\text{Pet}[A] \cong H_2[A]$ and there exists no 4-subset of $V(H_2)$ such that the subgraph in $H_2$ generated by $A$ is isomorphic to $H_1$. Thus the proposition is proved.

**Proposition 3.2.23.** If $C \subseteq V(\text{Pet})$ is a transversal of $\text{DIS}_4(\text{Pet})$, then either $|C| \geq 5$ or $C \in \text{DIS}_4(\text{Pet})$.

**Proof.** Let $C \subseteq V(\text{Pet})$ such that $C = \{v_{i_1j_1}, v_{i_2j_2}, v_{i_3j_3}, v_{i_4j_4}\} \notin \text{DIS}_4(\text{Pet})$ and $|C| = 4$. In $C$ we have thus 4 vertices associates to distinct pairs of elements in $\hat{5}$. We define the function $\eta_C : \hat{5} \rightarrow \{0, 1, 2, 3, 4\}$ such that, if $i \in \hat{5}$, then $\eta_C(i)$ is the number of time that $i$ occurs in the sequences of number $i_1, i_2, i_3, i_4$. Thus $\sum_{i=1}^5 \eta_C(i) = 2|C| = 8$. Since the pairs are mutually distinct, there exists at most one element $i \in \hat{5}$ such that $\eta_C(i) = 0$. Since $C \notin \text{DIS}_4(\text{Pet})$, there exists $i \in \hat{5}$ such that $\eta(i) = 1$. In fact, if this does not hold, then there would exist $i, j, k, l \in \hat{5}$ such that $\eta_C(i) = \eta_C(j) = \eta_C(k) = \eta_C(l) = 2$. It is easy to see that in such a case Pet[C] $\cong H_1$ and thus $C \in \text{DIS}_4(\text{Pet})$ that contradicts our assumptions. Let $h \in \hat{5}$ such that $\eta_C(h) = 1$. Without loss in generality we can suppose that $v_{hi} \in C$ and
Proof. Note that, if \( \text{Pet} \subseteq A \subseteq \text{Pet} \), then, by Proposition 3.2.18, it is minimal among the transversal of \( \text{DIS} \) and this contradicts the assumption on \( C \).

Let \( \text{DIS} \) be transversals of \( V(\text{Pet}) \). We prove now the main result of this section.

Consider the two graphs:

\[
\begin{array}{ccc}
P_2,2,1 & \bullet & P_4,1 \\
\end{array}
\]

\[
\begin{array}{cc}
P_5 & \bullet \\
\end{array}
\]

The last claim follows directly by definition of symmetry base.

We prove now the main result of this section.

**Theorem 3.2.25.** A subset \( C \subseteq V(\text{Pet}) \) is a symmetry base of \( \text{Pet} \) if and only if \( |C| = 5 \) and \( \text{Pet}[C] \) is isomorphic to one of the following graphs:

\[
\begin{array}{ccc}
P_2,2,1 & \bullet & P_4,1 \\
\end{array}
\]

\[
\begin{array}{cc}
P_5 & \bullet \\
\end{array}
\]

**Proof.** Note first that, if \( C \subseteq V(\text{Pet}) \) is such that \( \text{Pet}[C] \) is isomorphic to \( P_5, P_2,2,1, P_4,1 \) or \( P_5 \), then there is no element \( B \in \text{DIS}_{5}(\text{Pet}) \) such that \( C \subseteq B \). In fact, if \( B \in \text{DIS}_{5}(\text{Pet}) \), then, by Proposition 3.2.18, \( \text{Pet}[B] \) is isomorphic to \( H_2 \) but, however we choose a 5-subset \( C \subseteq V(\text{H}_2) \), \( \text{H}_2[C] \) is not isomorphic to \( P_5, P_2,2,1, P_4,1 \) or \( P_5 \). Thus if \( C \subseteq V(\text{Pet}) \) is such that \( \text{Pet}[C] \) is isomorphic to one of the graphs listed above, then \( C \) is a transversal of \( \text{DIS}(\text{Pet}) \). By Corollary 3.2.24, it is minimal among the transversal of \( \text{DIS}(\text{Pet}) \) with respect to the inclusion relation, so \( C \in \text{Tr}(\text{DIS}(\text{Pet})) = \text{BAS}(\text{Pet}) \) (see Theorem 3.2.15).

Consider the two graphs:

\[
\begin{array}{ccc}
Y & \bullet & P_{3,1,1} \\
\end{array}
\]
We now prove that, if $C \subseteq V(Pet)$ has cardinality $|C| = 5$ and $C \subseteq B \in DIS_6(Pet)$, then $Pet[C]$ is isomorphic to $Y$ or $P_{3,1,1}$. By Proposition 3.2.18, $Pet[B]$ is isomorphic to $H_2$ and if we remove from the graph $H_2$ one vertex (with all the edges incident to it), then we obtain a graph which is isomorphic to $Y$ or $P_{3,1,1}$. On the other hand, if $C \subseteq V(Pet)$ is such that $Pet[C]$ is isomorphic to $Y$ or $P_{3,1,1}$, then there exist exactly one element $B \in DIS_6(Pet)$ such that $C \subseteq B$. To prove that observe that, if $C \subseteq V(Pet)$ is such that $Pet[C] \cong Y$, then there exist $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{kl}, v_{jk}, v_{jl}, v_{hj}\}$ or $C = \{v_{hi}, v_{kl}, v_{jk}, v_{jl}, v_{ij}\}$. In both cases $C$ is included in the unique $B \in DIS_6(Pet)$ given by $B := \{v_{hi}, v_{kl}, v_{jk}, v_{jl}, v_{hj}, v_{ij}\}$. Similarly, if $C \subseteq V(Pet)$ is such that $Pet[C] \cong P_{3,1,1}$, then there exist $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{hj}, v_{hk}, v_{hl}, v_{ij}\}$ and the unique $B \in DIS_6(Pet)$ containing $C$ is $B = \{v_{hi}, v_{hj}, v_{hk}, v_{hl}, v_{ij}, v_{kl}\}$. Thus the number of 5-subsets of $V(Pet)$ that are not included in a local dissymmetry subset of $Pet$ is equal to $b_5 = \binom{10}{5} - \binom{6}{3}|DIS_6(Pet)| = 252 - 90 = 162$. Each of these subsets is not included in any element in $DIS_6(Pet)$, so it is a symmetry base of $Pet$.

We now characterize geometrically such subsets by proving that each of them is isomorphic to one of the graphs listed in the statement of the theorem. For this, we will count the number of 5-subsets of $V(Pet)$ whose corresponding induced subgraph is isomorphic to $C_5$, $P_{2,2,1}$, $P_{4,1}$ and $P_5$, respectively.

If $C \subseteq V(Pet)$, then the orbit of $C$ among the 5-subsets of $V(Pet)$ is equal to

$$|O_{Aut(Pet)}(C)| = \frac{|Aut(Pet)|}{|Aut(Pet)_C|},$$

where $Aut(Pet)_C$ is the stabilizer of $C$ with respect to the action of $Aut(Pet)$ on $V(Pet)$. Moreover, since the Petersen graph is symmetric, the orbit of $C$ is the set of all 5-subsets inducing the same graph (up to isomorphism), and so its cardinality is exactly the number of 5-subsets of $V(Pet)$ such that the induced subgraph is isomorphic to $X$.

The automorphism groups of the graphs $C_5$, $P_{2,2,1}$, $P_{4,1}$ and $P_5$ are isomorphic respectively to the dihedral group $D_5$, the dihedral group $D_4$, $Z_2$ and $Z_2$. We prove now that if $Pet[C]$ is isomorphic to $X$, where $X$ is $C_5$, $P_{4,1}$ or $P_5$, then $Aut(Pet)_C \cong Aut(X)$, while if $X = P_{2,2,1}$, then $Aut(Pet)_C$ is isomorphic to $Z_2 \times Z_2$.

We recall that $D_5$ has order 10 and the following presentation:

$$D_5 = \langle \sigma, \rho | \sigma^5 = \rho^2 = (\sigma \rho)^2 = 1 \rangle$$

To prove that $Aut(Pet)_C \cong Aut(X)$ if $X = C_5$, it is sufficient to prove that each generator of the automorphism group of $C_5$ extends uniquely to an automorphism of $Pet$. If $C \subseteq V(Pet)$ is such that $Pet[C] \cong C_5$, then there exist different $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{jk}, v_{hl}, v_{ij}, v_{kl}\}$. So we can choose the generators $\sigma, \rho \in Aut(C_5)$ defined by

$$\sigma(v_{hi}) = v_{jk}, \sigma(v_{jk}) = v_{hl}, \sigma(v_{hl}) = v_{ij}, \sigma(v_{ij}) = v_{kl}, \sigma(v_{kl}) = v_{hi}$$

and

$$\rho(v_{hi}) = v_{hi}, \rho(v_{jk}) = v_{kl}, \rho(v_{hl}) = v_{ij}, \rho(v_{ij}) = v_{hl}, \rho(v_{kl}) = v_{jk}.$$ 

Then, $\sigma$ extends to the automorphism of $Pet$ defined by the permutation in $S_5$ defined by:

$$\sigma'(h) = j, \sigma'(i) = k, \sigma'(j) = l, \sigma'(k) = h, \sigma'(l) = i,$$
while $\rho$ extends to the automorphism of $Pet$ defined by the permutation in $S_5$ defined by:

$$\rho'(h) = i, \rho'(i) = h, \rho'(j) = l, \rho'(k) = k, \rho'(l) = j.$$ 

Thus $Aut(Pet)_C \cong Aut(C_5)$ and the number of 5-subsets $C \subseteq V(Pet)$ such that $Pet[C] \cong C_5$ is equal to

$$\frac{|Aut(Pet)|}{|Aut(C_5)|} = \frac{120}{10} = 12.$$ 

Let $\tau$ be the generator of the automorphism groups of the graphs $P_{4,1}$ and $P_5$, that are both isomorphic to $\mathbb{Z}_2$. 

A 5-subset $C$ of $V(Pet)$ such that $Pet[C] \cong P_{4,1}$ ($Pet[C] \cong P_5$) is given by

$$C = \{v_{hi}, v_{jk}, v_{hl}, v_{ij}, v_{hj}\} \quad (C = \{v_{hi}, v_{jk}, v_{hl}, v_{ij}, v_{hk}\})$$ 

for a suitable choice of elements $h, i, j, k, l \in \hat{5}$. The automorphism $\tau$ of $P_{4,1}$ ($P_5$) corresponds to the permutation $\tau'$ of the elements in $\hat{5}$ given by:

$$\tau'(h) = j (\tau'(h) = h), \tau'(i) = i (\tau'(i) = k), \tau'(j) = h (\tau'(j) = j),$$

$$\tau'(k) = l (\tau'(k) = i), \tau'(l) = k (\tau'(l) = l).$$

Thus the orbit of $C$ has cardinality $\frac{|Aut(Pet)|}{|\mathbb{Z}_2|} = \frac{120}{2} = 60$.

Now a 5-subset $C$ of $V(Pet)$ satisfies $Pet[C] \cong P_{2,2,1}$ if and only if there exist $h, i, j, k, l \in \hat{5}$ such that $C = \{v_{hi}, v_{jk}, v_{hj}, v_{ik}, v_{hk}\}$. The graph $Pet[C]$ is the following graph:

```
   v_{hi}  v_{hk}
  /    \
v_{jk}  v_{hk}
  \    /```

We now find all automorphisms of $Pet$ fixing $Pet[C]$. An automorphism $\phi$ of $Pet$ that fixes $Pet[C]$ must also fix the vertex $v_{hk}$. It is straightforward to prove that the only permutations of the indices which give such an automorphism are $id, \alpha, \beta, \alpha \beta$ where $\alpha$ is defined by:

$$\alpha(h) = h, \alpha(i) = j, \alpha(j) = i, \alpha(k) = k, \alpha(l) = l.$$ 

and $\beta$ by

$$\beta(h) = k, \beta(i) = i, \beta(j) = j, \beta(k) = h, \beta(l) = l.$$ 

Thus the stabilizer of $C$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the number of the 5-subsets of $Pet$ that induce a subgraph isomorphic to $P_{2,2,1}$ is $\frac{120}{4} = 30$.

The sum of all the numbers found so far is equal to 162, so they are all the possible symmetry bases of $Pet$ having cardinality 5.

Finally, observe that if $A \subseteq V(Pet)$ is such that $|A| > 5$ and $A \notin DIS_5(Pet)$, then $B \subseteq A$, for some $B \in BAS_5(Pet)$ and therefore $A \notin BAS(Pet))$. This proves the theorem.

\[\square\]
3.2.2 The $C_n$ Case

We now provide a general form for the symmetry essentials and the symmetry bases of $C_n$.

**Proposition 3.2.26.** Let $n \geq 5$. Then:

(i) $ESS_2(C_n) = DIS_2(C_n) = \{\{v_i, v_{i+4}\}\}$.

(ii) $ESS_4(C_n) = \{\emptyset, \{v_i, v_{i+2}, v_j, v_{j+2}\}: d(v_i, v_j) \notin \{0, 2, 4, 6, n - 6\}\} \quad \text{if } 5 \leq n \leq 7,$
\[\text{otherwise.}\]

**Proof.** (i): Straightforward.

(ii): Let $G = C_n$ for $5 \leq n \leq 7$. By (iv) of Proposition 3.2.26, we deduce that the symmetry essential subsets of order two of $G$ have the form $\{v_i, v_{i+n-4}\}$. Thus, since a 4-subset of $V(G)$ must contain a pair of the form $\{v_i, v_{i+n-4}\}$, we deduce that $ESS_4(C_n) = \emptyset$ if $5 \leq n \leq 7$.

Let new assume that $n \geq 8$. Firstly, we note that $d(v_i, v_j) = 0$ if and only if $i = j$ and $d(v_i, v_j) = 2$ if and only if $j \equiv i + 2 \mod(n)$ or $j \equiv i - 2 \mod(n)$. In these cases we have $v_i = v_j$, $v_{i+2} = v_j$ or $v_{j+2} = v_i$, thus $|\{v_i, v_{i+2}, v_j, v_{j+2}\}| \neq 4$ and so $\{v_i, v_{i+2}, v_j, v_{j+2}\}$ can not be an element of $ESS_4(C_n)$. By part (i), we have that $ESS_2(C_n) = \{\{v_i, v_{i+n}\}: 1 \leq i \leq n\}$. It is straightforward to prove that a 4-subset of $V(C_n)$ of the form $\{v_i, v_{i+2}, v_j, v_{j+2}\}$ contains a symmetry essential subset of order two if and only if $d(v_i, v_j) \neq 4$, $d(v_i, v_j) \neq 6$ and $d(v_i, v_j) \neq n - 6$. This proves the claim.

We now provide the following characterization for the symmetry bases of $C_n$.

**Theorem 3.2.27.** Let $A \subseteq V(C_n)$. Then, $A$ is a symmetry base of $C_n$ if and only if $C_{C_n}(A) = \emptyset$ and for all $v_i \in A$ one of the following conditions holds:

1. $|B_{C_n}(A)| = 1 \land (v_{i-2} \notin A \lor v_{i+2} \notin A)$;

2. $|B_{C_n}(A)| \leq 1 \land ((v_{i-4} \notin A \land v_{i-2} \in A) \lor (v_{i+2} \in A \land v_{i+4} \notin A))$.

**Proof.** Let $V = V(C_n)$. By Definition 3.2.1 it’s clear that a vertex subset $A \subseteq V(C_n)$ is a symmetry base if and only if $|B_{C_n}(A)| \leq 1$ and $C_{C_n}(A) = \emptyset$. We claim that $A$ is minimal with respect to (i) of Definition 3.2.1 if and only if (1) or (2) holds. Let $A$ be a symmetry base of $C_n$ and $v_i \in A$ such that $|B_{C_n}(A \setminus \{v_i\})| \geq 2$. If $|B_{C_n}(A)| = 0$, we would have $\{v_{i-2}, v_{i+2}\} \cap A = \emptyset$ or, equivalently, $C_{C_n}(A) \neq \emptyset$, contradicting our assumptions. Thus $|B_{C_n}(A)| = 1$ and one of the two vertices $v_{i-2}$ or $v_{i+2}$ is not in $A$, which is condition (1). On the other hand, suppose that $A$ is a symmetry base of $C_n$ and $v_i \in A$ such that $C_{C_n}(A \setminus \{v_i\}) \neq \emptyset$. Hence $v_{i-2} \in C_{C_n}(A \setminus \{v_i\})$ or $v_{i+2} \in C_{C_n}(A \setminus \{v_i\})$. This is equivalent to condition (2). Conversely, suppose that $A \subseteq V$ is a vertex subset satisfying conditions (1) or (2) and such that $C_{C_n}(A) = \emptyset$. It’s obvious that $A$ satisfies (i) of Definition 3.2.1. Let $v_i \in A$ and let us consider the vertex subset $A \setminus \{v_i\}$. Suppose $B_{C_n}(A) = \{v_j\}$. Hence $j \neq i \pm 1$. If $v_{i-2} \notin A$, then $v_{i-1} \equiv A \setminus \{v_i\} v_j$; similarly if $v_{i+2} \notin A$, then $v_{i+1} \equiv A \setminus \{v_i\} v_j$. On the other hand, let $v_i \in A$ such that $v_{i-4} \notin A \lor v_{i-2} \in A$. In this case $v_{i-3} \equiv A \setminus \{v_i\} v_{i-1}$; similarly in the other case. In both cases $\pi_{\Phi}(A \setminus \{v_i\}) \neq \pi_{\Phi}(V)$. This completes the proof. 

**Remark 3.2.28.** So far, there exists no geometric characterization for the symmetry bases of $C_n$. 

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3.3 Symmetry Bases and Symmetry Essentials for \((V, \varphi, K)\)-VBP

Let us consider a finite dimensional vector space \(V\). If \(B, C\) are two non-empty subsets of \(V\), we write \(C \subseteq_i B\) when \(C\) is linearly independent and \(C \subseteq B\). Let \(A\) be a fixed arbitrary subset of \(V\). We denote by \(\text{rank}(A)\) the rank of \(A\), that is \(\text{rank}(A) := \dim(\text{Span}(A))\). We have the following characterizations.

**Theorem 3.3.1.**

(i) \(BAS_p(A) = \{C : C \subseteq_i A, |C| = \text{rank}(A)\}\);

(ii) \(DIS_p(A) = \{A \cap (V \setminus H) : H \leq_{n-1} V\}\);

(iii) \(ESS_p(A) = \{C : C \subseteq A, A^\perp \not\subseteq (A \setminus C)^\perp, C' \not\subseteq A \implies A^\perp = (A \setminus C')^\perp\}\).

**Proof.** (i): Let \(B \in BAS_p(A)\). Since \(BAS_p(A) \subseteq MINP_p(A)\), it results that \(B \subseteq_i A\). Let \(C \subseteq_i A\) such that \(|C| \neq \text{rank}(A)\), then \(\text{rank}(C) \neq \text{rank}(A)\), hence \(\text{Span}(C) \neq \text{Span}(A)\), so \(\text{Span}(C)^\perp \neq \text{Span}(A)^\perp\). Thus \(\pi_p(C) \neq \pi_p(A)\). Therefore, \(C \in BAS_p(A)\) if and only if \(|C| = \text{rank}(A)\), and this proves part (i).

(ii): Let \(a \in A\) such that \(|a| \neq \text{rank}(A)\), then \(\text{rank}(a) \neq \text{rank}(A)\), hence \(\text{Span}(a) \neq \text{Span}(A)\), so \(\text{span}(a)^\perp \neq \text{span}(A)^\perp\). Thus \(\pi_p(a) \neq \pi_p(A)\). Therefore, \(B \in BAS_p(A)\) if and only if \(|a| = \text{rank}(A)\), and this proves part (ii).

(iii): Let \(B \in ESS_p(A)\). Then, \(\pi_p(A) \neq \pi_p(A \setminus B)\) and, therefore, \(A^\perp \neq (A \setminus B)^\perp\), i.e. \(A^\perp \not\subseteq (A \setminus B)^\perp\). Moreover, let \(B' \not\subseteq B\). One has \(\pi_p(A) = \pi_p(A \setminus B')\), i.e. \(A^\perp = (A \setminus B')^\perp\). Therefore \(ESS_p(A) \subseteq \{C : C \subseteq A, A^\perp \not\subseteq (A \setminus C)^\perp, C' \not\subseteq A \implies A^\perp = (A \setminus C')^\perp\}\). The reverse inclusion is obvious.  

\[\Box\]

3.4 Structure of the Global Symmetry Classes

In this subsection, we study the inner structure of each global symmetry class and we relate this inner structure to the partial order of the poset \(\mathbb{L}_p(A)\). The starting point of our analysis is the existence of a twofold link between \(BAS_p(A)\) with the order structure of \(\mathbb{L}_p(A)\) and with the deletion of elements from members of the class \([A]_{\approx_p}\), similarly to what happens to the indispensable elements whenever one defines the classical Pawlak’s core. More in detail, we study a new class of elements preserving an global symmetry class. In this way, we formulate a concept similar to Pawlak’s indispensability notion for the indistinguishability relation, though weaker. It allows us to investigate the effects of deletion of elements in each global symmetry class by means of the following terminology. In this section we assume that \(\Omega\) is finite.

Let \(A\) be a fixed subset of \(\Omega\). We first recall the following notion (introduced in [20]) of **indistinguishability kernel** of \(A\), that is the subset defined by

\[
K_p(A) := \{a \in A : \exists B \in [A]_{\approx_p} \text{ such that } B \setminus \{a\} \notin [A]_{\approx_p}\} \quad (3.8)
\]

Let us note that the indistinguishability kernel can be considered (by analogy) a version of the usual Pawlak’s core relatively to the indistinguishability relation on \(\mathbb{P}\).

We recall then the following result.

**Theorem 3.4.1.** The following conditions hold.
(i) Let $A \in \text{MAXP}(\mathcal{P})$. Then, $a \in K_\mathcal{P}(A)$ if and only if there exists $C \in \text{MAXP}_\mathcal{P}(A)$ such that $a \notin C$ and $M_\mathcal{P}(C \cup \{a\}) = A$.

(ii) Let $A \in \text{MAXP}(\mathcal{P})$. Then, it results that $A \setminus K_\mathcal{P}(A) = \bigcap\{B : B \in [A|\mathcal{M}(\mathcal{P}) \uparrow]\}$.

(iii) $C_\mathcal{P}(A) \subseteq K_\mathcal{P}(A)$.

(iv) If $A \subseteq A'$ and $A \approx_\mathcal{P} A'$, then $K_\mathcal{P}(A) \supseteq K_\mathcal{P}(A')$.

(v) If $A \in \text{MINP}(\mathcal{P})$, then $K_\mathcal{P}(A) = A$.

(vi) $A \approx_\mathcal{P} K_\mathcal{P}(M_\mathcal{P}(A))$.

(vii) $M_\mathcal{P}(A) = M_\mathcal{P}(K_\mathcal{P}(M_\mathcal{P}(A)))$.

Proof. (i): Let $a \in K_\mathcal{P}(A)$. By definition of $K_\mathcal{P}(A)$ there exists $B \in [A]_{\approx_\mathcal{P}}$ such that

$$B \setminus \{a\} \not\approx_\mathcal{P} A.$$  \hspace{1cm} (3.9)

Since $B \approx_\mathcal{P} A$, by (3.9) we have that $a \in B$. Let $C := M_\mathcal{P}(B \setminus \{a\})$. We first show that $a \notin C$. In fact, let us assume by absurd that $a \in C$. In this case $B \subseteq C$ because $B \setminus \{a\} \subseteq M_\mathcal{P}(B \setminus \{a\}) = C$. Therefore $A = M_\mathcal{P}(B) \subseteq M_\mathcal{P}(C) = C$ because $A, C \in \text{MAXP}(\mathcal{P})$ and $B \in [A]_{\approx_\mathcal{P}}$, and this implies that

$$\pi_\mathcal{P}(C) \leq \pi_\mathcal{P}(A).$$  \hspace{1cm} (3.10)

On the other hand, we also have

$$\pi_\mathcal{P}(A) = \pi_\mathcal{P}(B) \leq \pi_\mathcal{P}(B \setminus \{a\}) = \pi_\mathcal{P}(C).$$  \hspace{1cm} (3.11)

Then, by (3.10) and (3.11) we deduce $\pi_\mathcal{P}(A) = \pi_\mathcal{P}(C)$, that is $A \approx_\mathcal{P} C$, and this implies (by definition of $C$) $A \approx_\mathcal{P} B \setminus \{a\}$, that is in contrast with (3.9). Hence $a \notin C$. Let us observe now that

$$C = M_\mathcal{P}(B \setminus \{a\}) \subseteq M_\mathcal{P}(B) = M_\mathcal{P}(A) = A,$$  \hspace{1cm} (3.12)

because $A \in \text{MAXP}(\mathcal{P})$. By (3.12) we have then $C \in \text{MAXP}_\mathcal{P}(A)$. Finally, since $B \setminus \{a\} \subseteq C$, we have $B \subseteq C \cup \{a\} \subseteq A$, so that

$$\pi_\mathcal{P}(A) \preceq \pi_\mathcal{P}(C \cup \{a\}) \preceq \pi_\mathcal{P}(B),$$

and this implies $\pi_\mathcal{P}(C \cup \{a\}) = \pi_\mathcal{P}(A)$ because $\pi_\mathcal{P}(A) = \pi_\mathcal{P}(B)$. Hence $M_\mathcal{P}(C \cup \{a\}) = M_\mathcal{P}(A) = A$ because $A \in \text{MAXP}(\mathcal{P})$. This proves the first implication.

For the other implication, we assume that there exists $C \in \text{MAXP}_\mathcal{P}(A)$ such that $a \notin C$ and $M_\mathcal{P}(C \cup \{a\}) = A$. Then $a \in A$ and we set $B := C \cup \{a\}$. Therefore $B \in [A]_{\approx_\mathcal{P}}$ and $B \setminus \{a\} = C$. Let us note that $C \notin [A]_{\approx_\mathcal{P}}$. In fact, if $C \approx_\mathcal{P} A$ then $C = A$ because $C \in \text{MAXP}_\mathcal{P}(A)$ and $A \in \text{MAXP}(\mathcal{P})$, that is in contrast with the conditions $a \in A$ and $a \notin C$. Hence we obtain an subset $B \in [A]_{\approx_\mathcal{P}}$ such that $B \setminus \{a\} \notin [A]_{\approx_\mathcal{P}}$ and $a \in A$, that is $a \in K_\mathcal{P}(A)$.

(ii): Let $a \in \bigcap\{B : B \in [A|\mathcal{M}(\mathcal{P}) \uparrow]\}$ and suppose by contradiction that $a \in K_\mathcal{P}(A)$. By (i),
there exists $B \in \text{MAXP}_{\mathcal{P}}(A)$ such that $a \notin B$ and $M_{\mathcal{P}}(B \cup \{a\}) = A$. We clearly have that $B \notin [A]\mathcal{P}$.) Therefore, there exists $C \in \text{MAXP}_{\mathcal{P}}(A)$ such that $B \subsetneq C \subsetneq A$. Hence
\begin{equation}
\pi_{\mathcal{P}}(A) \leq \pi_{\mathcal{P}}(C). \tag{3.13}
\end{equation}
We claim that $a \notin C$. In fact, if $a \in C$, we would have $B \cup \{a\} \subseteq C$, hence
\begin{equation}
\pi_{\mathcal{P}}(C) \leq \pi_{\mathcal{P}}(B \cup \{a\}) = \pi_{\mathcal{P}}(A). \tag{3.14}
\end{equation}
Thus, by (3.13) and (3.14), $\pi_{\mathcal{P}}(A) = \pi_{\mathcal{P}}(C)$ and, so, $C = A$, contradicting our assumption on $C$. Proceeding in this way, we will find a subset $D \in [A]\mathcal{P}$ not containing $a$, that is an absurd. Thus $a \in K^c(A).

Conversely, let $a \in K^c(A)$. Suppose by contradiction that there exists $B \in [A]\mathcal{P}$ such that $a \notin B$. Hence $C := B \cup \{a\} \approx_{\mathcal{P}} A$, so we have found an element $C \in [A]\approx_{\mathcal{P}}$ such that $C \setminus \{a\} \neq_{\mathcal{P}} A$, i.e. $a \in K_{\mathcal{P}}(A)$, that is an absurd.

(iii): Let $a \in \text{CORE}(A)$. Hence $\pi_{\mathcal{P}}(A) \neq \pi_{\mathcal{P}}(A \setminus \{a\})$, i.e. $a \in K_{\mathcal{P}}(A).

(iv): Let $a \in K_{\mathcal{P}}(A)$. Hence, for any $B \approx_{\mathcal{P}} A$, it results that $B \setminus \{a\} \approx_{\mathcal{P}} A$. Therefore, $a \in K_{\mathcal{P}}(A').$

(v): Since $A \in \text{MINP}(A)$, we have that $\pi_{\mathcal{P}}(A) \neq \pi_{\mathcal{P}}(A \setminus \{a\})$ for any $a \in A$, hence $K_{\mathcal{P}}(A) = A$.

(vi): Let $A' = K_{\mathcal{P}}(M_{\mathcal{P}}(A)).$ We have that $A' \subseteq M_{\mathcal{P}}(A).$ Let $B \in \text{MINP}(A)$, by part (v) $K_{\mathcal{P}}(B) = B$, so $B \subseteq A'$. We conclude that
$$B \subseteq A' \subseteq M_{\mathcal{P}}(A).$$
Therefore, we have
$$\pi_{\mathcal{P}}(M_{\mathcal{P}}(A)) = \pi_{\mathcal{P}}(A) \leq \pi_{\mathcal{P}}(A') \leq \pi_{\mathcal{P}}(B) = \pi_{\mathcal{P}}(A),$$
so $\pi_{\mathcal{P}}(A) = \pi_{\mathcal{P}}(A')$ and $A \approx_{\mathcal{P}} A' = K_{\mathcal{P}}(M_{\mathcal{P}}(A)).$

(vii): It follows directly by part (vi). \hfill \Box

We establish now the following new results.

**Theorem 3.4.2.** (i) $\bigcup \text{BAS}_{\mathcal{P}}(A) \subseteq K_{\mathcal{P}}(A).$

(ii) Let $A \in \text{MAXP}(A)$. Then $\text{BAS}_{\mathcal{P}}(A) \subseteq \mathcal{L}_+(\mathcal{L}_{\mathcal{P}}(A)).$

**Proof.** (i): Let $a \in \bigcup \text{BAS}_{\mathcal{P}}(A)$, then there exists $B \in \text{BAS}_{\mathcal{P}}(A)$ containing $a$. In particular, $B \setminus \{a\} \notin [A]\approx_{\mathcal{P}}$, therefore $a \in K_{\mathcal{P}}(A).

(ii): Just observe that $B \in \text{BAS}_{\mathcal{P}}(A)$ if and only if $|B| = |\mathcal{L}_{\mathcal{P}}(A)| = 0.$ \hfill \Box

Based on the results given in Theorems 3.4.1 and 3.4.2, we continue with the analysis of the structure of $[A]\approx_{\mathcal{P}}$.

Let us firstly note that the converse of (i) of Theorem 3.4.2 turns out to be false, as it can be seen in the next example.

**Example 3.4.3.** Let $\mathcal{P}$ be the pairing given in Example 3.2.3 and $A = \{2, 4, 5\}$. Then, we have that $K_{\mathcal{P}}(A) = A$ but $\bigcup \text{BAS}_{\mathcal{P}}(A) = \{2, 5\}$. 68
<table>
<thead>
<tr>
<th>$\mathcal{P}$</th>
<th>RAM</th>
<th>Memory</th>
<th>Color</th>
<th>BatteryLife</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>Insufficient</td>
<td>Small</td>
<td>Black</td>
<td>Very Long</td>
</tr>
<tr>
<td>$u_2$</td>
<td>Sufficient</td>
<td>Middle</td>
<td>Blue</td>
<td>Short</td>
</tr>
<tr>
<td>$u_3$</td>
<td>Insufficient</td>
<td>Small</td>
<td>Black</td>
<td>Long</td>
</tr>
<tr>
<td>$u_4$</td>
<td>Sufficient</td>
<td>Big</td>
<td>Grey</td>
<td>Short</td>
</tr>
<tr>
<td>$u_5$</td>
<td>Excellent</td>
<td>Middle</td>
<td>Blue</td>
<td>Long</td>
</tr>
</tbody>
</table>

Table 3.1: Mobile Phone pairing

In the next result we show that the minimum union generator of the class $[A]_{\equiv P}$ coincides with the family of all elements of $[A]_{\equiv P}$ having at most one co-cover in the poset $L_{\equiv P}(A)$.

**Theorem 3.4.4.** $\text{Mug}([A]_{\equiv P}) = \mathcal{L}_{\downarrow}(L_{\equiv P}(A))$.

**Proof.** Let $B \approx_{\equiv P} A$ such that $|[B|L_{\equiv P}(A) \downarrow]| \geq 2$. Firstly, we claim that if $A_1, A_2, A_3$ are three distinct non-empty subsets such that $A_1, A_2 \approx_{\equiv P} A_3$ and $A_1, A_2 \preceq A_3$, then $A_1 \cup A_2 = A_3$. It clearly results that $A_1 \cup A_2 \subseteq A_3$, suppose therefore that $A_1 \cup A_2 \nsubseteq A_3$. Hence, $A_1 \nsubseteq A_1 \cup A_2 \nsubseteq A_3$, so $A_3$ does not cover $A_1$, contradicting our assumption. Thus, if $[B|L_{\equiv P}(A) \downarrow] = \{D, E\}$, we have

$$B = D \cup E.$$ (3.15)

If $D$ (or $E$) covers more than one subset, in (3.15) we can substitute $D$ (or $E$) with the union of subsets it covers; proceeding in this way, since $B$ is finite, we will express $B$ as the union of subsets of $\mathcal{L}_{\downarrow}(L_{\equiv P}(A))$, so $\mathcal{L}_{\downarrow}(L_{\equiv P}(A))$ spans $[A]_{\equiv P}$. In order to prove that $\mathcal{L}_{\downarrow}(L_{\equiv P}(A))$ is minimum, just observe that each element $B$ of $\mathcal{L}_{\downarrow}(L_{\equiv P}(A))$ cannot be written as the union of elements of $[A]_{\equiv P}$ distinct from $B$. 

In the next example, we discuss the result provided in Theorem 3.4.4 in a concrete situation.

**Example 3.4.5.** Let us consider the pairing $\mathcal{P}$ represented in Table 3.1. We indicate by $r, m, c$ and $b$ respectively RAM, memory, color and battery life and, usually, we use string notation to enumerate the various set systems arising during our analysis.

It is easy to see that

$$\text{MAXP}(\mathcal{P}) = \{\emptyset, r, b, mc, rb, rmc, rmcb\}.$$ 

In particular, in Figure 3.3, we represent its maximum partitioner lattice $M(\mathcal{P})$, whereas in Figure 3.4 we represent the diagram $I(\mathcal{P})$.

By seeing Figure 3.4, we have that

$$\text{MINP}(\mathcal{P}) = \{\emptyset, m, c, r, b, rm, rc, rb, cb, mb\}.$$ 

Let now $A = \Omega$. Then, it is easy to see that $\mathcal{L}_{\downarrow}(L_{\equiv P}(\Omega)) = \{\{c, b\}, \{m, b\}, \{r, c, b\}, \{r, m, b\}\}$. As we see, $D := \bigcup_{E \in \text{BAS}(\mathcal{P})} E \neq \Omega$, since $r \notin C$. In particular, $r \in K_{\equiv P}(\Omega)$. By Theorem 3.4.4, it follows that $\text{Mug}(\Omega_{\equiv P}) = \mathcal{L}_{\downarrow}(L_{\equiv P}(\Omega))$. As an example, we could consider the subfamily

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Figure 3.3: Diagram of $M(\mathfrak{P})$.

Figure 3.4: Diagram of $\bar{I}(\mathcal{T})$. 
\{c, b\}, \{m, b\}, \{r, c, b\} \subseteq \mathcal{L}(\Omega). Then, it results that \(\pi_\Omega(\{r, m, b\}) = \pi_\Omega(\Omega)\), but there is no way to obtain \(\{r, m, b\}\) as union of the aforementioned family of subsets. Thus, their union does not span \([\Omega]_{\preceq\Omega}\).

Let us observe that \(|\Omega| < 3\). As we have seen in the proof of Theorem 3.4.4, \(\Omega\) can be obtained as the union of two of its co-cover. In particular, the three co-covers can be obtained by deleting from \(\Omega\) respectively \(m\) or \(r\) or \(c\). It is obvious that the deletion of \(b\) is not allowed, otherwise there exists a symmetry base not containing \(b\), that is an absurd since \(\{b\} = \text{CORE}(\Omega)\).

The preceding discussion enables us to show the following general result.

**Proposition 3.4.6.** If \(|A| = k\), then \(|A|_{\preceq\Omega}(A) \downarrow| \leq k\).

**Proof.** Let \(a_i, a_j \in A\) with \(i \neq j\). Let us set \(A_i = A \setminus \{a_i\}\) and \(A_j = A \setminus \{a_j\}\). If \(|A|_{\preceq\Omega}(A) \downarrow| > k\), then there exist two subsets \(B, C \subseteq A\) for some \(i = 1, \ldots, k\). Thus, \(B \not\subseteq B \cup C \not\subseteq A\), that is an absurd since \(B, C \in [A]_{\preceq\Omega}(A) \downarrow\). \(\square\)

We conclude this subsection by finding find some conditions so that \(\text{BAS}_\Omega(A)\) coincides with the minimum union generator of \([A]_{\approx\Omega}\), when \(A \in \text{MAXP}(\Omega)\).

**Theorem 3.4.7.** Let \(A \in \text{MAXP}(\Omega)\) and \(B, C \in [A]_{\approx\Omega}\) such that

(i) \(B \cap C = \emptyset\),

(ii) \(B, C \in [A]_{\preceq\Omega}(A) \downarrow\).

Then \(\text{BAS}_\Omega(A) = \text{Mug}([A]_{\approx\Omega}) = [A]_{\preceq\Omega}(A) \downarrow = \{B, C\}\) and \([A]_{\approx} = \{A, B, C\}\).

**Proof.** By hypothesis, we have \(\{B, C\} \subseteq [A]_{\preceq\Omega}(A) \downarrow\). Suppose by contradiction that \(\{B, C\} \not\subseteq [A]_{\preceq\Omega}(A) \downarrow\) and let \(D \in [A]_{\preceq\Omega}(A) \downarrow \setminus \{B, C\}\). By the proof of Theorem 3.4.4, we have that the union of any pair of elements of \([A]_{\preceq\Omega}(A) \downarrow\) is exactly \(A\), hence we must have \(B = A \setminus C \subseteq D\) and \(C = A \setminus B \subseteq D\). This implies \(B \cup C = A \subseteq D\), i.e. \(D = A\), contradicting our assumption. Therefore \([A]_{\preceq\Omega}(A) \downarrow = \{B, C\}\).

On the other hand, in order to prove that \(\text{BAS}_\Omega(A) = \{B, C\}\), it only suffices to prove that \(\{B, C\} \subseteq \text{BAS}_\Omega(A)\). Suppose that \(B \not\in \text{BAS}_\Omega(A)\), thus there exists \(D \not\subseteq [A]_{\preceq\Omega}(A)\) such that \(D \in \text{BAS}_\Omega(A)\). Hence, it results

\[C \not\subseteq C \cup D \subsetneq A,\]

contradicting the fact that \(C \in [A]_{\preceq\Omega}(A) \downarrow\). Similarly if \(C \not\in \text{BAS}_\Omega(A)\). In this way, we have that \(\{B, C\} \subseteq \text{BAS}_\Omega(A)\). The reverse inclusion is now obvious. This also proves that \([A]_{\approx} = \{A, B, C\}\), therefore it follows that \(\text{BAS}_\Omega(A) = \text{Mug}([A]_{\approx\Omega}) = [A]_{\preceq\Omega}(A) \downarrow = \{B, C\}\) and the claim has been shown. \(\square\)

To conclude this section, we investigate the geometric properties of the minimal transversal of each global symmetry class \([A]_{\approx\Omega}\).

**Definition 3.4.8.** We say that a subset \(B\) of \(\Omega\) is \(A\)-strongly asymmetric if \(B \subseteq M_\Omega(A)\) and \(C \not\equiv_\Omega M_\Omega(A) \setminus B\), for any non-empty subset \(C \subseteq B\). We denote by \(A_\Omega(A)\) the family of all \(A\)-strongly asymmetric subsets of \(\Omega\).
An $A$-strongly asymmetric subset $B$ of $\Omega$ is any subset no part of which provides the same degree of symmetry of its relative complement with respect to $M_{\mathcal{P}}(A)$. In other terms, we cannot extend $M_{\mathcal{P}}(A) \setminus B$ by means of any point or part of $B$ without changing the symmetry with respect to $M_{\mathcal{P}}(A) \setminus B$ itself.

In the next result we show that any non-empty subset $B$ of $M_{\mathcal{P}}(A)$ which is also a minimal transversal of the set system $[A]_{\mathcal{P}}$ cannot induce the same symmetry partition of $M_{\mathcal{P}}(A) \setminus B$.

**Proposition 3.4.9.** $Tr([A]_{\mathcal{P}}) \cap \mathcal{P}(M_{\mathcal{P}}(A)) \subseteq \mathcal{A}_{\mathcal{P}}(A)$.

**Proof.** Let $B \in Tr([A]_{\mathcal{P}}) \cap \mathcal{P}(M_{\mathcal{P}}(A))$. We now prove that $B \in \mathcal{F}_{\mathcal{P}}(A)$, where

$$\mathcal{F}_{\mathcal{P}}(A) := \{ B \subseteq M_{\mathcal{P}}(A) : \forall C \in \mathcal{P}(B) \setminus \{\emptyset\}, \forall D \in \mathcal{P}(M_{\mathcal{P}}(A)) \text{ with } M_{\mathcal{P}}(D) \supseteq M_{\mathcal{P}}(C), B \cap D \neq \emptyset \}.$$

For assume that $B \notin \mathcal{F}_{\mathcal{P}}(A)$. Then, there exists a proper subset $C$ of $B$ and $D \in \mathcal{P}(M_{\mathcal{P}}(A))$ such that $M_{\mathcal{P}}(D) \supseteq M_{\mathcal{P}}(C)$ and $B \cap D = \emptyset$.

Let us suppose, by contradiction, that there exists $E \in [A]_{\mathcal{P}}$ such that $E \cap (B \setminus C) = \emptyset$ and set $E' := (E \setminus C) \cup D$. It is clearly disjoint from $B$. Because of our assumptions on $D$ and $E$, it readily follows that $M_{\mathcal{P}}(E') \subseteq M_{\mathcal{P}}(E)$. On the other hand, let us prove that $M_{\mathcal{P}}(E) \subseteq M_{\mathcal{P}}(E')$. By the condition $M_{\mathcal{P}}(D) \supseteq M_{\mathcal{P}}(C)$, we have that $C \subseteq M_{\mathcal{P}}(D) \subseteq M_{\mathcal{P}}(E')$ and $E \setminus C \subseteq M_{\mathcal{P}}(E')$, then $E \subseteq M_{\mathcal{P}}(E')$, i.e. $M_{\mathcal{P}}(E) \subseteq M_{\mathcal{P}}(E')$. In this way, we proved that $E' \approx_{\mathcal{P}} E \approx_{\mathcal{P}} A$. But the last claim contradicts the fact that $B \in Tr([A]_{\mathcal{P}})$. This shows that $B \setminus C \in Tr([A]_{\mathcal{P}}) \cap \mathcal{P}(M_{\mathcal{P}}(A))$, contradicting the minimality of $B$. Hence $B \in \mathcal{F}_{\mathcal{P}}(A)$.

We now prove that $\mathcal{F}_{\mathcal{P}}(A) = \mathcal{H}_{\mathcal{P}}(A)$, where

$$\mathcal{H}_{\mathcal{P}}(A) := \{ F \subseteq M_{\mathcal{P}}(A) : M_{\mathcal{P}}(F') \nsubseteq M_{\mathcal{P}}(M_{\mathcal{P}}(A) \setminus F), \forall F' \in \mathcal{P}(F) \setminus \{\emptyset\} \},$$

which is clearly contained in $\mathcal{A}_{\mathcal{P}}(A)$. This will prove the thesis.

Let $F \in \mathcal{F}_{\mathcal{P}}(A)$ and $F' \in \mathcal{P}(F) \setminus \{\emptyset\}$. Then, it is straightforward to see that $F' \nsubseteq M_{\mathcal{P}}(M_{\mathcal{P}}(A) \setminus F)$, otherwise, by the definition of $\mathcal{F}_{\mathcal{P}}(A)$ we should have $F' \cap (M_{\mathcal{P}}(A) \setminus F) \neq \emptyset$, that is an absurd. This implies that $F \notin \mathcal{H}_{\mathcal{P}}(A)$.

On the other hand, let $Z \in \mathcal{H}_{\mathcal{P}}(A)$ and suppose there exist $Z' \in \mathcal{P}(Z \setminus \{\emptyset\})$ and $S \subseteq M_{\mathcal{P}}(A)$ such that $Z' \subseteq M_{\mathcal{P}}(S)$ but $S \cap Z = \emptyset$. Then $S \subseteq M_{\mathcal{P}}(A) \setminus Z$, so

$$Z' \subseteq M_{\mathcal{P}}(Z') \subseteq M_{\mathcal{P}}(S) \subseteq M_{\mathcal{P}}(M_{\mathcal{P}}(A) \setminus Z),$$

contradicting our assumption on $Z$. This shows that $\mathcal{F}_{\mathcal{P}}(A) = \mathcal{H}_{\mathcal{P}}(A)$. The thesis is thus proved. \(\square\)
Chapter 4

A Dissymmetry Model in Graph Theory

For any graph \( G \), it is immediate to verify that the dissymmetry matrix is linked to the neighborhood of a graph as follows:

\[
\forall v_i, v_j \in V(G) \quad \Delta_G(v_i, v_j) := \{a \in V(G) : F(v_i, a) \neq F(v_j, a)\} = N_G(v_i) \Delta N_G(v_j) \quad (4.1)
\]

where \( \Delta_G(v_i, v_j) \) is the dissymmetry map of the pairing associated with \( G \) and \( \Delta \) is the usual set symmetric difference: \( A \Delta B := (A \cup B) \setminus (A \cap B) \).

Hence, the dissymmetry matrix of a graph represents a way to discern between two different vertices \( v_i \neq v_j \) in the sense that \( v \in \Delta_G(v_i, v_j) \) if and only if \( v \) is adjacent to exactly one vertex between \( v_i \) and \( v_j \).

Example 4.0.10. Let \( C_n \) be the cycle on \( n \) vertices. In this case, the indices are taken modulo \( n \). We have

\[
N_{C_n}(v_i) \Delta N_{C_n}(v_j) = \begin{cases} 
\{v_{i-1}, v_{j+1}\} & \text{if } j = i + 2 \\
\{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\} & \text{if } i \leq j \leq n - 1 \land j \neq i + 2 
\end{cases} \quad (4.2)
\]

where \( 1 \leq i < j \leq n \). Moreover, if we fix two vertices \( v_i \) and \( v_j \) such that \( j \neq i + 2 \). It results by (4.1) that \( v_{i-1}, v_{i+1} \) are adjacent to \( v_i \) but not to \( v_j \) and, similarly, \( v_{j-1}, v_{j+1} \) are adjacent only to \( v_j \). Therefore

\[
\Delta_{C_n}(v_i, v_j) = \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\} = N_{C_n}(v_i) \Delta N_{C_n}(v_j).
\]

If \( j = i + 2 \), it is clear that \( v_{i+1} \) is adjacent to both \( v_i \) and \( v_j \), while \( v_{i-1} \) is adjacent only to \( v_i \) and \( v_{j+1} \) only to \( v_j \). Therefore, in this case,

\[
\Delta_{C_n}(v_i, v_j) = \{v_{i-1}, v_{j+1}\} = N_{C_n}(v_i) \Delta N_{C_n}(v_j).
\]

A first immediate result shows that the dissymmetry matrix \( \Delta(G) \) uniquely characterizes its corresponding graph when \( G \) is a simple graph.
Proposition 4.0.11. Let $G_1$ and $G_2$ be two simple graphs defined on the same set of vertices, i.e., $V(G_1) = V(G_2)$. Then:

$$\Delta[G_1] = \Delta[G_2] \iff E(G_1) = E(G_2).$$

Proof. Suppose that $\Delta_{G_1}(v_i, v_j) = \Delta_{G_2}(v_i, v_j)$ for every $v_i, v_j$, then $F_{G_1}(v, w) = F_{G_2}(v, w)$ for every pair of vertices $v$ and $w$, because in $G_1$ and $G_2$ there are no loops. Then, it follows immediately that $E(G_1) = E(G_2)$.

By Proposition 4.0.11, $\Delta$ uniquely determines the edge set of a graph.

In this subsection we deal with the cases of $K_n$ and $K_{p,q}$.

Theorem 4.0.12. Let $G$ be a $n$-graph and $V = V(G)$. Then:
(i) $DIS(G) = \{\{V\}\}$ if and only if $G = K_{p,q}$;
(ii) Let $DIS(G) = \left(\frac{V}{2}\right)$. Then $G = K_n$ if $n$ is odd, while $G = K_n$ or $G = F_n$ if $n$ is even.

Proof. (i): If $G = K_{p,q}$, then it is immediate to show that $DIS(G) = \{\{V\}\}$. Conversely, let $DIS(G) = \{\{V\}\}$ be the discernibility set system of $G$ and let $v_i, v_j \in V(G)$ such that $\Delta_G(v_i, v_j) = V$. Then, in this case, by setting $B_1 := N_G(v_i)$ and $B_2 := N_G(v_j)$, then $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = \Delta_G(v_i, v_j) = V$. Now let $v, w$ of $B_1$ (equivalently $B_2$). They cannot be adjacent, otherwise $\Delta_G(v, w) = V$, contradicting the fact that $v_i \in N_G(v) \cap N_G(w)$ (equivalently $v_j \in N_G(v) \cap N_G(w)$). If $v \in B_1$ and $w \in B_2$, then, since $N_G(v) \neq N_G(w)$, it must result that $\Delta_G(v, w) = V$, hence $v \sim w$. Thus, the claim is proved.

(ii): Firstly, we observe that any pair of vertices $\{v_i, v_j\}$, with $i < j$, occurs just once as $\Delta_G(v_i', v_j')$, with $v_i', v_j' \in V(G)$ and $i' < j'$. If $n = 1$ or $n = 2$, there is nothing to prove. Let $G$ be a $n$-graph such that $DIS(G) = \left(\frac{V}{2}\right)$. Without loss of generality, suppose moreover that $v_{n-1} \sim v_n$. Thus $\Delta_G(v_{n-1}, v_n) = \{v_{n-1}, v_n\}$. Therefore any other vertex must be adjacent to both $v_{n-1}$ and $v_n$ or non-adjacent to none of them. Let $v_i, v_j \in V(G)$ such that $v_i \sim v_{n-1}$, $v_i \sim v_n$, $v_j \sim v_{n-1}$ and $v_j \sim v_n$. In this case, $\Delta_G(v_i, v_j) = \{v_{n-1}, v_n\}$ but this contradicts our preliminary remark. Hence all vertices in $X := \{v_1, \ldots, v_{n-2}\}$ are all adjacent to $v_{n-1}, v_n$ or are all non adjacent to them. Thus, by setting $H := G[X]$, we have that $|V(H)| = n - 2$ and $DIS(H) = \left(\frac{V}{2}\right)$. So, by the inductive hypothesis, $H = K_{n-2}$ or, in the even case, $H = F_{n-2}$ or $H = K_{n-2}$. Let us suppose that $H = K_{n-2}$. If there exists $v_i \in V(G)$ such that $v_i \sim v_{n-1}$ and $v_i \sim v_n$, then $\Delta_G(v_i, v_{n-1}) = \{v_n\} \cup (X \setminus \{v_i\})$. Since $|\Delta_G(v_i, v_{n-1})| = 2$, the only possibility is that $n = 4$ and $G = F_n$. On the other hand, if there exists $v_i \in V(H)$ such that $v_i \sim v_{n-1}$ and $v_i \sim v_n$, then clearly $G = K_n$. Let us suppose that $n$ is even and $H = F_{n-2}$. Let $n \neq 4$ and $v_i, v_j \in V(H)$ such that $v_i \sim v_{n-1}$ and $v_i \sim v_j$. Hence, $\Delta_G(v_i, v_{n-1})$ contains $\{v_i, v_j, v_{n-1}\}$, contradicting our assumption. Therefore, the only possibility in this case is $n = 4$ and $G = K_n$ or $G = F_n$.

4.1 Formal Context Analysis on Graphs

In this section we firstly recall some basic link notions between finite simple graphs and formal context as introduced in [48], next we discuss some new results concerning the formal contexts induced by simple graphs.
Definition 4.1.1. [48] We call formal context of the graph $G$ the formal context $\mathbb{K}[G] := (V(G), V(G), \mathcal{R}_G)$, where $v \mathcal{R}_G v'$ if and only if $(v, v') \in E(G)$ for all $v, v' \in V(G)$.

Hence the object subset and the attribute subset of the formal context $\mathbb{K}[G]$ are both equal to the vertex set $V(G)$, whereas the binary relation which defines this formal context is exactly the incidence relation between vertices of the graph $G$. Let us also note that, since the graph $G$ is undirected, the relation $\mathcal{R}_G$ is symmetric.

Given the above considerations, in the formal context $\mathbb{K}[G]$ induced by a simple undirected graph $G$, the maps $\uparrow : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ and $\downarrow : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ are coincident. Therefore in the sequel we denote with the same symbol $\uparrow$ the map $\uparrow : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ such that $O \mapsto O' := O^{\uparrow} = O^{\downarrow}$, when $O$ is any vertex subset of $G$. This implies obviously that also the two operators $* : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ and $\circ : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ coincide. Therefore in the sequel we set $O \mapsto O'' := O^* = O^\circ$, for all $O \subseteq V(G)$.

If $O$ is a vertex subset of $G$, it is immediate to verify that

$$O' = \bigcap_{v \in O} N_G(v) = \{w \in V(G) : O \subseteq N_G(w)\},$$

moreover, since $G$ has no loops, we also have

$$O' \subseteq V(G) \setminus O.$$  \hspace{1cm} (4.3)

Remark 4.1.2. The identity in (4.3) is also valid when $O = \emptyset$. In fact, in this case, we always have (by convention in FCA) $\emptyset^{\uparrow} = M$ and $\emptyset^{\downarrow} = Z$, that is $O' = O^{\uparrow} = O^{\downarrow} = V(G)$ in the formal context $\mathbb{K}[G]$. On the other hand, it is usual (in elementary set theory) to interpret the intersection $\bigcap_{v \in O} N_G(v)$ as coincident with the whole set $V(G)$ when $O$ is the empty set.

Remark 4.1.3. If $G$ is a finite simple undirected graph, a vertex subset $O \subseteq V(G)$ is the extent [intent] of some concept of the formal context $\mathbb{K}[G]$ if and only if $O'' = O$. In this case, both the pairs $(O, O')$ and $(O', O)$ are concepts of $\mathbb{K}[G]$. Moreover, since $G$ has no loops, the cross table of the formal context $\mathbb{K}[G]$ (that is, the adjacency matrix of $G$) has zeroes in all its diagonal places, and this obviously implies that $V(G)' = \emptyset$. Hence both the pairs $(\emptyset, V(G))$ and $(V(G), \emptyset)$ are always concepts of the formal context $\mathbb{K}[G]$.

Proposition 4.1.4. Let $A \subseteq V$. Then $A$ is the extent [intent] of some concept of the formal context $\mathbb{K}[G]$ if and only if $A = O'$, for some $O \subseteq V$.

By Proposition 4.1.4 and Equation (4.3) it follows that a vertex subset $A$ is an extent [intent] of some concept in $\mathbb{K}[G]$ if and only if $A$ is intersection of open neighborhoods of vertices of $G$.

We explicitly introduce now the notion of concept lattice for a finite simple undirected graph.

Definition 4.1.5. [48] We call concept lattice (or also Galois lattice) of the graph $G$ the concept lattice of the formal context $\mathbb{K}[G]$ and we denote it simply by $(\mathfrak{B}(G), \subseteq)$ instead of $(\mathfrak{B}(\mathbb{K}[G]), \subseteq)$. We also call Galois poset of $G$ the poset $(\mathfrak{B}^\downarrow(G), \subseteq)$, where $\mathfrak{B}^\downarrow(G) := \mathfrak{B}(G) \setminus \{\emptyset, V(G)\}$.
Remark 4.1.6. (i) Since the adjacency matrix of $G$ is a symmetric matrix, we immediately deduce that the concept lattice $(\mathfrak{B}(G), \subseteq)$ is self-dual.

(ii) From FCA general theory it is well known that is sufficient to represent any formal concept $B = (B_1, B_2)$ of $\mathbb{K}[G]$ simply with its extent $B_1$ (or, equivalently, with its intent $B_2$). As a consequence of this premise, in several cases we implicitly identify a formal concepts $B = (B_1, B_2)$ of $\mathbb{K}[G]$ with its extent $B_1$ (or, equivalently, with its intent $B_2$). Obviously, in these cases we can also identify the partial order $\subseteq$ with the usual inclusion relation $\subseteq$ (or, respectively, with the dual inclusion $\subseteq^*$).

In the next two definitions we introduce two basic notions.

**Definition 4.1.7.** Let $G, G'$ be two finite simple graphs.

- We say that $G$ is Galois equivalent with $G'$, denoted by $G \equiv_{gal} G'$, if the corresponding Galois lattices $\mathfrak{B}(G)$ and $\mathfrak{B}(G')$ are isomorphic.

- We call the Galois class of $G$ the graph family $[G]_{gal} = \{G' : G \equiv_{gal} G'\}$.

**Definition 4.1.8.** Let $s$ be a positive integer. We call $s$-Galois class of $G$ the following graph family

$$[G, s]_{gal} := \{G' \in [G]_{gal} : |V(G')| = s\}.$$ 

Moreover, we say that the ordered pair $(G, s)$ is:

- **Galois undetermined** if there exist at least two non-isomorphic locally dissymmetric graphs in $[G, s]_{gal}$;

- **Galois determined** if there is just one locally dissymmetric graph up to isomorphism in $[G, s]_{gal}$;

- **Galois inconsistent** if $[G, q]_{gal}$ is non-empty and there is not any locally dissymmetric graph in $[G, q]_{gal}$.

We also say that the graph $G$ is:

- **Galois undetermined** if the pair $(G, |V(G)|)$ is Galois undetermined;

- **Galois determined** if the pair $(G, |V(G)|)$ is Galois determined;

- **Galois inconsistent** if the pair $(G, |V(G)|)$ is Galois inconsistent.

**Remark 4.1.9.** If $G$ is a locally dissymmetric $n$-graph then it is clear that $G$ is Galois determined if and only if $[G, n]_{gal} = \{G\}$.

In the next propositions of this section we establish some useful properties of the Galois poset of a graph.
Proposition 4.1.10. Let \( G, G' \) be simple graphs. Then:

(i) \( G \equiv_{gal} G' \) if and only if \( \mathfrak{B}^\times(G) = \mathfrak{B}^\times(G') \).

(ii) Let \( G_1, \ldots, G_k \) be the connected components of \( G \). Then, the Galois poset of \( G \) is the direct sum of the Galois posets of its components, \( \mathfrak{B}^\times(G) = \mathfrak{B}^\times(G_1) + \cdots + \mathfrak{B}^\times(G_k) \).

(iii) Let \( v_1, \ldots, v_k \) be some isolated points in \( G \) and let \( G'' := G[V(G) \setminus \{v_1, \ldots, v_k\}] \). Then \( \mathfrak{B}^\times(G) = \mathfrak{B}^\times(G'') \).

Proof. (i) This is trivial and follows directly by the definition of the Galois poset.

(ii) Let \( O \subseteq V(G) \) be a vertex subset. Then \( O' \neq \emptyset \) if and only if \( O' \subseteq V(G_i) \) for some \( i \in \{1, \ldots, k\} \). More in general a vertex subset \( A \) of \( G \), such that \( A \neq V(G) \) and \( A \neq \emptyset \), is an extent [intent] in \( G \) if and only if it is an extent [intent] in a connected component of \( G \). The thesis thus holds.

(iii) This follows directly by part (ii).

Remark 4.1.11. (i) By Proposition 4.1.10, part (i), we can use the Galois poset of \( G \) instead of its Galois lattice in order to study the Galois class of \( G \).

(ii) The notion of local dissymmetry is important in our context because each Galois class is uniquely determined by a locally dissymmetric graph. In fact, if \( G \) be a finite simple graph, then \( G \equiv_{gal} G^\cong \).

(iii) By (ii) of Proposition 4.1.10 we can restrict our analysis to graphs without isolated vertices.

Proposition 4.1.12. Let \( G \) be a \( n \)-graph, \( O \subseteq V(G) \) be an extent and \( v_i \in O \). Then \( O \) contains all the vertices \( v_j \) such that \( v_i \simeq v_j \).

Proof. Let \( v_j \in V(G) \) twin with \( v_i \) and \( v_i \in O \) for some extent \( O \subseteq V(G) \). By Proposition 4.1.4, there exists a vertex subset \( A \subseteq V(G) \) such that \( O = A' = \bigcap_{v \in A} N_G(v) \). Hence \( v_i \in \bigcap_{v \in A} N_G(v) \), so \( v_i \sim v \) for each \( v \in A \). Since \( N_G(v_i) = N_G(v_j) \), we deduce that \( v_j \sim v \) for any \( v \in A \), thus \( v_j \in \bigcap_{v \in A} N_G(v) \), i.e. \( v_j \in O \).

Proposition 4.1.13. Let \( G \) be a graph. Then, if \( O \) is a vertex subset of \( G \) such that \( O' \) is minimal in \( \mathfrak{B}^\times(G) \), then \( |O'| = 1 \). The maximal extents in \( \mathfrak{B}^\times(G) \) are the neighborhoods of the elements which appear in the bottom part.

Proof. Without loss in generality we can suppose \( O = O'' \). Let us suppose that \( |O'| \geq 2 \). Since there are not twin vertices in \( G \), there exists a vertex \( w \in O' \) such that \( N_G(w) \) contains strictly \( O \). Let \( u \in N_G(w) \setminus O \) and set \( U = O \cup \{u\} \). It follows that \( w \in U' \), so \( U' \) is non-empty and contained strictly in \( O' \). This contradicts the assumption that \( O' \) is minimal in \( \mathfrak{B}^\times(G) \). Thus \( |O'| = 1 \). By duality, if \( O' = \{v\} \), then \( O'' = N_G(v) \) is maximal in \( \mathfrak{B}^\times(G) \). This completes the proof.

In the last result of this section we show how the connectedness of the graph is related to the connectedness of the cover graph of the corresponding concept lattice.

Proposition 4.1.14. Let \( G \) be a simple graph. If the cover graph of \( \mathfrak{B}^\times(G) \) is connected, then \( G \) is a connected graph.
Proof. Suppose that the graph $G$ is not connected. Since $G$ has no isolated vertices, every component has at least an edge. This means that we can find two different proper neighborhoods, which are extent in the concept lattice. Hence, $\mathcal{B}(G)$ can be expressed as the disjoint sum of lattices, i.e. its cover graph is not connected.

We will note in the next section that the reverse implication of that established in Proposition 4.1.14 is not true (see the next Remark 4.1.23).

### 4.1.1 Galois classes of $K_n$ and $K_{p,q}$

In this section we describe the Galois classes of the complete graph and of the bipartite complete graph. The complete determination of the Galois class of a graph $G$ is in general a difficult task, even in the case when the structure of $G$ is relatively simple.

In the next result we provide the equivalent conditions that characterize a $n$-graph $G$ that has the same Galois lattice of $K_n$.

**Proposition 4.1.15.** Let $G$ be a $n$-graph and let $V = V(G)$. Then, the following conditions are equivalent:

(i) for any vertex subset $X \subseteq V$ we have $X' = V \setminus X$;
(ii) $G = K_n$;
(iii) $\mathcal{B}(G) = \{(X, V \setminus X) : X \subseteq V\}$;
(iv) $(\mathcal{B}(G), \sqsubseteq) \cong (\mathcal{P}(V), \subseteq)$.

**Proof.**

(i) $\implies$ (ii): Let $v \in V$ and let $X = \{v\}$. By (i) we have that $N_G(v) = \{v\}' = V \setminus \{v\}$, therefore $G = K_n$.

(ii) $\implies$ (iii): Let $X \subseteq V$. Since $G = K_n$, by definition of $X'$ we have that $X' = V \setminus X$, therefore $X'' = (X')' = V \setminus (V \setminus X) = X$. This implies that the ordered pair $(X, V \setminus X)$ is a formal concept of $G$. Hence (iii) is satisfied.

(iii) $\implies$ (iv): It follows directly by the definition of the partial order in the concept lattice.

(iv) $\implies$ (i): Let us suppose there exists $X \subseteq V$ such that $X' \neq V \setminus X$. Since $X'$ is disjoint from $X$, we have that $X' \not\subseteq X^c$ and thus that the cardinality of $X \cup X'$ is strictly less than $n$. By (iv) and by the duality property of the concept lattice associated with a graph, we have that each subset $Y$ of $V$ occurs exactly once as extent and once as intent in suitable formal concepts of $G$. Thus if we sum the cardinalities of all the extents and all the intents in $\mathcal{B}(G)$, then we obtain:

$$\sum_{Y \in \mathcal{P}(V)} |Y \cup Y'| = 2 \sum_{Y \in \mathcal{P}(V)} |Y| = 2n2^{n-1} = n2^n.$$

Since the number of addends in the first sum is equal to $2^n$ and one of addends, $|X \cup X'| = |X| + |X'|$, is strictly less than $n$, then there exists $Y \subseteq V$ such that $|Y \cup Y'| > n$, but this contradicts the fact that $Y$ and $Y'$ are disjoint. This proves that $X' = V \setminus X$.

In the next result we provide the conditions under which a local dissymmetry graph $G$ is Galois equivalent to $K_n$. 78
Proposition 4.1.16. Let $G$ be a locally disymmetric graph. Then, $G \cong_{gal} K_n$ if and only if there exists a vertex subset $U = \{u_1, u_2, \ldots, u_n\} \subseteq V(G)$ such that the following conditions hold:

(i) $G[U] \cong K_n$;
(ii) for each $v \in V(G) \setminus U$, there exist $u_i \in U$ such that $N_G(v) \subseteq N_G(u_i)$;
(iii) if $v, w \in V(G) \setminus U$ are in $V(G)$, then $v \sim w$ if and only if $U \subseteq N_G(v) \cup N_G(w)$.

Proof. By Proposition 4.1.13, since $(\mathcal{B}(G), \subseteq) \cong (\mathcal{P}(V), \subseteq)$, there exist $u_1, \ldots, u_n \in V(G)$ such that the minimal elements in $\mathcal{B}(G)$ are the singletons $\{u_i\}$ and the maximal elements are $N_G(u_i)$, for $i = 1, 2, \ldots, n$. Then, for each $A \subseteq U := \{u_1, \ldots, u_n\}$, there exist just one extent $B_A$ in $\mathcal{B}(G)$ such that $A \subseteq B_A$. Let now $v \in V \setminus U$ and let us consider $N_G(v) = \{v\}'$. By maximality of the extents $N_G(u_i)$ in $\mathcal{B}(G)$, there exists a vertex $u_i \in U$ such that $N_G(v) \subseteq N_G(u_i)$. So (ii) holds.

Let now $v, w \in V(G) \setminus U$ such that $U \subseteq N_G(v) \cup N_G(w)$. In this case $(U \cap N_G(v)) \cup (U \cap N_G(w)) = U$. By setting $A := N_G(v) \cap U$, then $B_A = N_G(v)$. On the other hand, $((U \cap N_G(v)) \setminus N_G(v))'$ is another extent such that its intersection with $U$ is equal to $A$. This implies that $((U \cap N_G(v)) \setminus N_G(v))' = B_A = N_G(v)$ and thus $v \sim w$. Conversely let $v, w \in V(G) \setminus U$, with $v \sim w$. As an immediate consequence, we deduce that $\{w\}' = N_G(w)$ is an extent in $\mathcal{B}(G)$ whose intersection with $U$ is $A := (N_G(v) \cap U)$. Thus it coincides with $(U \cap N_G(v))'$ and if $u \in U \cap N_G(w)$, then, since $v \in \{w\}' = (U \cap N_G(w))'$, we obtain $u \sim v$. It follows (iii).

Let us suppose now that $V(G)$ satisfies the condition (iii). Then, for each $A \subseteq U$, we set $B_A := (U \setminus A)'$. Let us prove now that the $B_A$ are all the possible extents in $\mathcal{B}(G)$. For this, let $O$ be a vertex subset in $G$. Let us prove that $O' = (U \setminus O')'$. For this, let $v \in O'$ and $u \in U \setminus O'$. If $v \in O' \cap U$, then $u \neq v$ and so $u \sim v$ because they are different vertices in $U$ and $G[U] \cong K_n$. Let $v \in O' \setminus U$. In this case, if $u \in O$, then $u \sim v$ because $v \in O'$. If $u \notin O$, since $u \notin O'$, then there exist $v' \in O \setminus U$ such that $u \sim v'$. But since $v' \sim v$, by property (iii), we obtain that $u \sim v$, so $O' \subseteq (U \setminus O')'$. Let now $u \in (U \setminus O')'$. If $u \in U$, then $u \sim v'$, so $v \sim v$, for all $v \in O$. Let us suppose thus $u \notin U$ and let $v \in O$. If $v \in U \cap O$, then $v \in U \setminus O'$, so $v \sim u$. If $u' \in U$ and $u' \sim v$, then $u' \in U \setminus O'$, so $u \sim u'$. This proves that $U \subseteq N_G(v) \cup N_G(w)$ and thus, by property (iii), $u \sim v$. 

At this point we provide our results concerning the complete graph $K_n$ in the following result.

Theorem 4.1.17. (i) The complete graph $K_n$ is Galois determined.
(ii) $[K_n]_{gal}$ is the family of all the finite simple graph $G$ such that $G \cong$ satisfies the conditions (i), (ii) and (iii) of Proposition 4.1.16.
(iii) If $s > n$ then the pair $(K_n, s)$ is Galois undetermined or Galois inconsistent.

Proof. Part (i) follows by Proposition 4.1.15, whereas the parts (ii) and (iii) follow by Proposition 4.1.16.

We devote now our attention to the complete bipartite graphs. We will prove that all these graphs are mutually equivalent to one another and, moreover, that a graph $G$ is Galois equivalent to a complete bipartite graph if and only if it is a complete bipartite graph by itself.
Proposition 4.1.18. The following conditions for a graph $G$ with vertex set $V$ are equivalent:

(i) $G$ is the complete bipartite graph having bipartition $(B_1, B_2)$.

(ii) $\mathcal{B}(G) = \{(\emptyset, V), (B_1, B_2), (B_2, B_1), (V, \emptyset)\}$, where $B_1 \cap B_2$ is a set partition of $V$.

(iii) $(\mathcal{B}(G), \subseteq) \cong (\mathcal{P}(2), \subseteq)$.

Proof. $(i) \implies (ii)$: Let $O \subseteq V$ a non-empty vertex subset. Then, clearly:

$$O' = \begin{cases} B_2 & \text{if } O \subseteq B_1, \\ B_1 & \text{if } O \subseteq B_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

So the only contexts (extents) are $\emptyset, B_1, B_2, V$ and the thesis follows.

$(ii) \implies (iii)$: This is obvious.

$(iii) \implies (i)$: Let $(B_1, B_2)$ and $(B_2, B_1)$ be the two concepts in $\mathcal{B}(G)$ where both $B_1$ and $B_2$ are different from the empty set. Then, by definition of the operator $'$, all vertices in $B_1$ are adjacent to all vertices in $B_2$, so that $\{x, y\} \in E(G)$, for all $x \in B_1$ and $y \in B_2$. Let us suppose now there exist $x_1, x_2 \in B_1$ such that $\{x_1, x_2\} \in E(G)$. Set $O := N_G(x_1) \supseteq \{x_2\} \cup B_2$. In this case $x_1 \in O'$ and $x_2 \notin O'$, so $O' \neq B_1, O' \neq B_2, O' \neq \emptyset$ and $O' \neq V$. As observed in Remark 2.3, this contradicts the fact that the only extents in $\mathcal{B}(G)$ are $\emptyset, B_1, B_2$ and $V$. The proposition is thus proved.

For the complete bipartite graphs we obtain then the following result.

Corollary 4.1.19. (i) The Galois class of any complete bipartite graph is is the family of all the complete bipartite graphs.

(ii) If $s > 2$ then the pair $(K_{p,q}, s)$ is Galois inconsistent.

Proof. It follows directly by Proposition 4.1.18.

4.1.2 Galois Posets of Cycles, Paths and Wheels

In this section we determine the Galois lattices of three well studied graph families: the $n$-cycles $C_n$, the $n$-paths $P_n$ and the $n$-wheels $W_{n+1}$. The Galois posets of these graph families are graded of rank 1 (for $C_n$ and $P_n$) or of rank 2 (for $W_{n+1}$).

In the next result we show that the only $n$-graph in the Galois class of $C_n$ is just $C_n$.

Theorem 4.1.20. Let $G$ be a graph with $|V(G)| = n \geq 5$. Then, $G = C_n$ if and only if the Galois poset $(\mathcal{B}(G), \subseteq)$ is a $n$-crown when $n$ is odd and is a disjoint sum of two $\frac{n}{2}$-crowns when $n$ is even.

Proof. Let $G = C_n$ and let $v_i \in V(G)$. Then, $N_G(v_i) = \{v_{i-1}, v_{i+1}\}$, where the index sums are taken mod($n$). Now let $i, j \in \{1, \ldots, n\}$ such that $i < j$. Then,

$$\{v_i, v_j\}' = N_G(v_i) \cap N_G(v_j) = \begin{cases} \{v_1\} & \text{if } i = 2 \text{ and } j = n \\
\{v_n\} & \text{if } i = 1 \text{ and } j = n - 1 \\
\{v_{i+1}\} & \text{if } j = i + 2 \\
\emptyset & \text{otherwise.} \end{cases}$$
Finally, note that if $O \subseteq V$ has cardinality greater than or equal to 3, then $O' = \emptyset$. Therefore, we conclude that $\mathcal{B}(C_n) = \{\{v_i\}, \{v_{i-1}, v_{i+1}\} : i = 1, \ldots, n\}$, where the index sums are taken mod($n$). Now, let $n$ be an odd integer. We set

$$x_k := \{v_{2k-1}\} \quad \text{for} \quad k = 1, \ldots, n$$

and

$$y_k := \{v_{2k-1}, v_{2k+1}\} \quad \text{for} \quad k = 1, \ldots, n$$

where all index sums are taken mod($n$). It’s immediate to see that, in this way, $\mathcal{B}(C_n)$ is an $n$-crown with respect to set theoretic inclusion.

Let $n$ be an even integer. We set

$$x_k := \{v_{2k-1}\} \quad \text{for} \quad k = 1, \ldots, \frac{n}{2}$$

$$y_k := \{v_{2k-1}, v_{2k+1}\} \quad \text{for} \quad k = 1, \ldots, \frac{n}{2}$$

and

$$x_{k+\frac{n}{2}} := \{v_{2k}\} \quad \text{for} \quad k = 1, \ldots, \frac{n}{2}$$

$$y_{k+\frac{n}{2}} := \{v_{2k}, v_{2k+2}\} \quad \text{for} \quad k = 1, \ldots, \frac{n}{2}$$

where all index sums are taken mod($n$). In this way, it’s obvious that $\mathcal{B}(C_n)$ is the disjoint union of two $\frac{n}{2}$-crowns.

Vice versa, let $(\mathcal{B}(G), \subseteq)$ be the disjoint sum of two $\frac{n}{2}$-crowns. It is obvious that the minimal elements of the first crown are disjoint from those of the second one and hence the maximal elements of the first crown are also disjoint from those of the second one. Arguing as before, it is easy to deduce that, even in this case, $G = C_n$.

**Corollary 4.1.21.** (i) The $n$-cycle $C_n$ is Galois determined, that is equivalent to the condition $[C_n, n]_{gal} = \{C_n\}$ because $C_n$ is locally dissymmetric.

(ii) Let $G$ be a simple finite graph. Then, $G \equiv_{gal} C_n$ if and only if $G \simeq = C_n$.

**Proof.** Both assertions follow directly by Theorem 4.1.20 and by (ii) of Remark 4.1.11.

**Example 4.1.22.** In the figure we represent the concept lattice of $C_5$. 

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Remark 4.1.23. Let us note that the graph $C_n$, where $n$ is even, is connected but the cover graph of its Galois poset is not connected.

We examine now the case $G = P_n$. For this graph we can establish the following result.

Theorem 4.1.24. Let $G$ be a graph with $|V(G)| = n \geq 5$. Then

$$G = P_n \iff (\mathfrak{B}(G), \subseteq) = F_{n-2} \cup_d F^*_n.$$
Proof. Let $G = P_n$ and let $v_i \in V(G)$. Then

$$N_G(v_i) = \{v_i\}' = \begin{cases} \{v_2\} & \text{if } i = 1 \\ \{v_{n-1}\} & \text{if } i = n \\ \{v_{i-1}, v_{i+1}\} & \text{otherwise.} \end{cases} \quad (4.5)$$

Now let $v_i, v_j \in V(G)$ such that $i < j$. Then

$$N_G(v_i) \cap N_G(v_j) = \{v_i, v_j\}' = \begin{cases} \{v_{i+1}\} & \text{if } j = i + 2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover if $O \subseteq V(G)$ is such that $|O| \geq 3$, then $O' = \emptyset$. We conclude that $\mathfrak{B}^\delta(P_n) = \{\{v_i\}, \{v_j, v_{j+2}\} : i = 2, \ldots, n-1, j = 1, \ldots, n-2\}$.

Let $n$ be an even integer. If we set, for $k = 1, \ldots, \frac{n-2}{2},$

$$\{v_{2k}\} = x_{2k-1} \text{ and } \{v_{2k}, v_{2k+2}\} = x_{2k}$$

and

$$\{v_{2k-1}\} = x'_{2k-1} \text{ and } \{v_{2k-1}, v_{2k+1}\} = x'_{2k}$$

it’s immediate to note that $\mathfrak{B}^\delta(P_n)$ is the disjoint union of two $(n-2)$-fences. Similarly, let $n$ be an odd integer. If we set

$$\{v_{2k}\} = x_{2k-1} \text{ for } k = 1, \ldots, \frac{n-1}{2}$$

$$\{v_{2k}, v_{2k+2}\} = x_{2k} \text{ for } k = 1, \ldots, \frac{n-3}{2}$$

and

$$\{v_{2k-1}, v_{2k+1}\} = x'_{2k-1} \text{ for } k = 1, \ldots, \frac{n-1}{2}$$

$$\{v_{2k+1}\} = x'_{2k} \text{ for } k = 1, \ldots, \frac{n-3}{2}$$

and we obtain an $(n-2)$-fence $F_{n-2}$ and the dual $(n-2)$-fence $F^*_{n-2}$. Hence, the Galois poset $\mathfrak{B}^\delta(P_n)$ is the disjoint union of two $(n-2)$-fences.

Vice versa, let $(\mathfrak{B}^\delta(G), \subseteq) = F_{n-2} \cup d F^*_{n-2}$. We observe that the minimal elements of a fence are clearly pairwise disjoint; moreover it is obvious the minimal elements of the first fence are disjoint from those of the second one. Hence the minimal elements are singletons and the maximal elements are pairs of vertices; in particular they are neighbourhoods of some vertex. If $n$ is an even integer, then every fence has exactly $\frac{n-2}{2}$ minimal as many maximal elements, while if $n$ is an odd integer, in the first fence there are $\frac{n-1}{2}$ minimal elements and $\frac{n-3}{2}$ maximal elements and in the second there are $\frac{n-3}{2}$ minimal elements and $\frac{n-1}{2}$ maximal elements. It is immediate to see that the maximal elements of a fence are the neighbourhoods of the minimal elements of the other fence; so, since $G$ is a connected graph, we conclude that the $n-2$ vertices at issue form a $(n-2)$-path. Moreover, the remaining vertices can be linked only to the ends of the previous path and there are no edges between them, otherwise we obtain a $n$-cycle but this is impossible by Theorem 4.1.20. Thus, we deduce that $G = P_n$. \qed
Corollary 4.1.25. (i) The \( n \)-path \( P_n \) is Galois determined, that is equivalent to the condition \([P_n, n]_{\text{gal}} = \{P_n\}\) because \( P_n \) is locally dissymmetric.

(ii) Let \( G \) be a simple finite graph. Then, \( G \equiv_{\text{gal}} P_n \) if and only if \( G \cong P_n \).

Proof. Both assertions follow directly by Theorem 4.1.24 and by (ii) of Remark 4.1.11.

Example 4.1.26. We represent the concept lattice of \( P_6 \) in the figure below.

![Figure 4.3: Concept Lattice of \( P_6 \)](image)

In the following theorem, we will prove that there not exists any graph whose concept lattice is isomorphic to a \( n \)-fence.

Theorem 4.1.27. Let \( F_n \) be a \( n \)-fence. There exists no graph having \( F_n \) as Galois poset.

Proof. Let us assume, by absurd, that there exists a graph \( G \) such that his Galois poset is isomorphic to \( F_n \). Without loss in generality we can assume \( G \) locally dissymmetric. By Proposition 4.1.13 the minimal extents of \( G \) are singletons and the maximal elements in \( \mathfrak{B}(G) \) are the neighborhoods of the vertices that occur in the minimal part. Thus the number of minimal and maximal elements is the same.

Let \( F_n \) be the \( n \)-fence such that \( x_1 < y_1 > x_2 < \cdots < x_n \) and let \( v_1, v_2, \ldots, v_n \in V \) such that, for each \( i \), \( x_i \) corresponds to the singleton \( \{v_i\} \) and \( y_i \) corresponds to \( N_G(v_{\sigma(i)}) \), where \( \sigma \) is a permutation of the index set \( \{1, 2, \ldots, n\} \). Since \( v_i, v_{i+1} \in N_G(v_{\sigma(i)}) \), where \( i = 1, \ldots, n - 1 \), and \( v_n \in N_G(v_{\sigma(n)}) \), the induced subgraph on \( G \) by the vertex subset \( \{v_1, v_2, \ldots, v_n\} \) has one vertex of degree 1 and all the others of degree 2. But there exists no such graph. This concludes the proof.

We close this section by determining the Galois lattice of an \( n \)-wheel.
Proposition 4.1.28. Let $n \geq 5$ and $W_{n+1}$ be the $n$-wheel. Then:

$$(\mathfrak{B}^W(W_{n+1}), \subseteq) = \{\{v_i\}, \{v_j, v_{j+2}\}, \{v_j, v_{n+1}\}, N_{W_{n+1}}(v_i)\},$$

for $i = 1, \ldots, n + 1$, $j = 1, \ldots, n$ and the index sums $j + 2$ are taken mod($n$).

Proof. Let $G = W_{n+1}$ and $v_i \in V(G)$. Then

$$N_G(v_i) = \{v_i\}' = \begin{cases} \{v_2, v_n, v_{n+1}\} & \text{if } i = 1 \\
\{v_1, v_{n-1}, v_{n+1}\} & \text{if } i = n \\
\{v_1, \ldots, v_n\} & \text{if } i = n + 1 \\
\{v_{i-1}, v_{i+1}, v_{n+1}\} & \text{otherwise.} \end{cases}$$

It is easy to see that, if $O \subseteq V(G)$ have cardinality $|O| \geq 2$, then $|O'| \leq 2$ and $|O'| = 2$ if and only if $O' = \{v_j, v_{j+2}\}$ or $O' = \{v_j, v_{n+1}\}$ for some $j = 1, \ldots, n$. Moreover, if $i = 1, \ldots, n + 1$, then:

$$\{v_i\} = \begin{cases} \{v_2, v_n, v_{n+1}\}' & \text{if } i = 1 \\
\{v_{i-1}, v_{i+1}, v_{n+1}\}' & \text{if } i \in \{2, \ldots, n - 1\} \\
\{v_1, v_{n-1}, v_{n+1}\}' & \text{if } i = n \\
\{v_1, v_2\}' & \text{if } i = n + 1. \end{cases}$$

The Proposition is thus proved. \qed

Corollary 4.1.29. The Galois poset of $W_{n+1}$ is graded of rank 2.

Example 4.1.30. In the figure we represent the concept lattice of $W_7$. 

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4.1.3 The Galois Lattice of Products of Cycles and Paths

By using the results of the previous section, we now investigate the concept lattice of graphs obtained by composing cycles and paths by means of two classical graph operations: the Cartesian product and the tensor product.

We firstly establish a preliminary result.

Proposition 4.1.31. Let $G$ and $H$ be two graphs. The Galois poset of the tensor product $G \otimes H$ is the direct product of the two Galois posets $\mathcal{B}^\wedge(G)$ and $\mathcal{B}^\wedge(H)$.

Proof. Let $(u, u') \in V(G \otimes H)$. Then, $N_{G \otimes H}((u, u')) = (u, u')' = N_G(u) \times N_H(u')$. By Equation (4.3) and Proposition 4.1.4 it follows easily that $A \subseteq V(G \otimes H)$ is an extent (intent) if and only if there exist $O \in V(G), Q \in V(H)$ such that $A = O' \times Q'$. The thesis follows by definition of the product order in $\mathcal{B}^\wedge(G \otimes H)$.

In the next result, we determine the extent family of the $(m,n)$-prism and we prove that its Galois poset is graded of rank 2.
Theorem 4.1.32. Let \( m \geq 2 \), \( n \geq 3 \) and \( G \) be the \((m, n)\)-prism \( P_m \sqcup C_n \). Then

\[
\mathcal{B}^5(G) = \{(i, j'), (k, r'), (k + 1, r' - 1)\}, \{(k, r'), (k + 1, r' + 1)\}, N_G((i, j')),
\]  

\[(4.6)\]

where \( i = 1, \ldots, m \), \( j' = 1, \ldots, n \), \( k = 1, \ldots, m - 1 \) and \( r' = 1, \ldots, n \).

Moreover, the Galois poset \( \mathcal{B}^5(G) \) is graded of rank 2.

Proof. All the index sums in this proof are taken mod\((n)\). We have that \( N_G((1, j')) = \{(1, j' - 1), (1, j' + 1), (2, j')\} \) for \( j' = 1, \ldots, n \), \( N_G((i, j')) = \{(i, j' - 1), (i, j' + 1), (i - 1, j'), (i + 1, j')\} \) for any \( j' = 1, \ldots, n \) and \( i = 2, \ldots, m - 1 \) and \( N_G((m, j')) = \{(m, j' - 1), (m, j' + 1), (m - 1, j')\} \) for \( j' = 1, \ldots, n \). In particular, to distinct pairs correspond distinct neighborhoods. We obviously have \( O'' = O \) if \( O = \{(i, j')\} \) or \( O = N_G((i, j')) \). Let \( O \subseteq V(G) \) of the form \( O = \{(i, r'), (i, s')\} \). Since these vertices belong to a \( C_n \), we deduce that \( O' = \emptyset \) unless \( d(r', s') \equiv 2 \) (mod \( n \)). In this case, \( O' = \{(i, r' - 1)\} \) or \( O' = \{(i, r' + 1)\} \), depending on whether \( s' = r' - 2 \) or \( s' = r' + 2 \), but in both situations we have \( O'' \neq O \). Let \( O \subseteq V(G) \) of the form \( O = \{(i, r'), (i + 1, s')\} \) for some \( r', s' \in \{1', \ldots, n'\} \) and \( i \in \{1, \ldots, m - 1\} \). It is easy to see geometrically the different cases occurring: if these two vertices belong to the same lateral face of a prism and are opposite, then \( O' \) consists of the other two vertices of the face; if these two vertices belong to the same lateral face of a prism and are adjacent, then \( O' = \emptyset \) as well as if they belong to the different lateral faces of a prism. Hence \( O'' = O \) if \( O = \{(i, r'), (i + 1, r' - 1)\} \) or \( O = \{(i, r'), (i + 1, r' + 1)\} \).

Now, by \((4.6)\) it is easy to deduce that any maximal chain in \( B^5(G) \) has one of the following forms:

- \( \{(1, j')\} \subseteq \{(1, j'), (2, j' - 1)\} \subseteq N_G((1, j' - 1)); \)
- \( \{(1, j')\} \subseteq \{(1, j'), (2, j' - 1)\} \subseteq N_G((2, j')); \)
- \( \{(1, j')\} \subseteq \{(1, j'), (2, j' + 1)\} \subseteq N_G((1, j' + 1)); \)
- \( \{(1, j')\} \subseteq \{(1, j'), (2, j' + 1)\} \subseteq N_G((2, j')); \)
- \( \{(i, j')\} \subseteq \{(i, j'), (i - 1, j' - 1)\} \subseteq N_G((i - 1, j')); \)
- \( \{(i, j')\} \subseteq \{(i, j'), (i - 1, j' - 1)\} \subseteq N_G((i, j' + 1)); \)
- \( \{(i, j')\} \subseteq \{(i, j'), (i - 1, j' + 1)\} \subseteq N_G((i - 1, j')); \)
- \( \{(i, j')\} \subseteq \{(i, j'), (i - 1, j' + 1)\} \subseteq N_G((i, j' + 1)). \)

where \( i = 2, \ldots, m \) and \( j' \in \{1', \ldots, n'\} \). Hence, any maximal chain has length 2, so the Galois poset \( B^5(G) \) is a graded poset of rank 2.
Example 4.1.33. In the following figure we represent the concept lattice of the (2, 3)-prism.

![Concept Lattice of the (2, 3)-prism](image)

Figure 4.5: Concept Lattice of the (2, 3)-prism.

In the next result we determine the extent family of the \((m, n)\)-grid graph and we prove that its Galois poset is graded of rank 2.

**Theorem 4.1.34.** Let \(m, n \geq 2\) and let \(G = P_m \sqcup P_n\) be the \((m, n)\)-grid graph. Then

\[
\mathfrak{B}^3(G) = \{(i, j'), \{(k, r'), (k + 1, r' - 1)\}, \{(k, s'), (k + 1, s' + 1)\}, N_G((i, j'))\} \tag{4.7}
\]

where \(i = 1, \ldots, n, j' = 1, \ldots, m, k = 1, \ldots, m - 1, r' = 2, \ldots, n\) and \(s' = 1, \ldots, n - 1\). Moreover \(\mathfrak{B}^3(G)\) is a graded poset of rank 2.

**Proof.** We have that \(N_G((i, j')) = \{(1, j' + 1), (2, j')\}\) for \(j' = 1, \ldots, m\), \(N_G((i, j')) = \{(i, j' - 1), (i, j + 1), (i - 1, j'), (i + 1, j')\}\) for any \(j' = 1, \ldots, m\) and \(i = 2, \ldots, n - 1\) and \(N_G((n, j')) = \{(n, j' - 1), (n, j' + 1), (n - 1, j')\}\) for \(j' = 1, \ldots, m\). At this point the remaining part of the proof of (4.1.34) is very similar to that provided in Theorem 4.1.32, therefore we omit it.

For the second part of the statement, by (4.7) we easily deduce that any maximal chain has one of the following form:

- \(\{(1, 1')\} \subsetneq \{(1, 1'), (2, 2')\} \subsetneq N_G((1, 2'))\);
\[
\text{where } i, k = 2, \ldots, m - 1 \text{ and } j', l' = 2', \ldots, n' - 1. \text{ Hence any maximal chain has length 2 and } P_m \sqcup P_n \text{ is a graded poset of rank 2.}
\]
Example 4.1.35. In the following figure we represent the concept lattice of the \((3, 2)\)-grid graph.

![Concept Lattice Diagram](image)

Figure 4.6: Concept Lattice of the \((3, 2)\)-grid graph.

For a rook’s graph we obtain the following result.

**Theorem 4.1.36.** Let \(m, n \geq 5\) and \(G\) be the rook’s graph \(K_m \Box K_n\). Then:

\[
\mathcal{B}^\downarrow(G) = \{\{(i, j'), \{(i, j'_1), \ldots, (i, j'_s)\}, \{(i_1, j'), \ldots, (i_r, j')\}, \{(i, j'), (k, l')\}, N_G((i, j'))\}
\]

where \(i, k = 1, \ldots, m\) and \(j', l' = 1, \ldots, n\), \((i, j') \sim (k, l')\), \(i_1, \ldots, i_r \in \{1, \ldots, m\}\), \(j'_1, \ldots, j'_s \in \{1, \ldots, n\}\) and \(r = m - 2\), \(s = n - 2\).

**Proof.** The rook’s graph is \((m + n - 2)\)-regular. Let \(O \subseteq V(G)\). It is immediate to see that

\[
O' = \begin{cases} 
N_G((i, j')) & \text{if } O = \{(i, j')\} \\
\{(i, j')\} & \text{if } O = N_G((i, j'))
\end{cases}
\]

Now, let \(O\) be a vertex subset of cardinality 2. If the two elements \((i, j')\) and \((k, l')\) are not adjacent, we deduce that \(O'\) contains the pairs respectively \((i, l')\) and \((k, j')\), so we have \(O'' = O\). Moreover, if \(O = \{(i, j'), (i, k')\}\), then their neighborhoods share \(n - 2\) points which are adjacent only to the previous two vertices, i.e. even in this case \(O'' = O\). In particular, this is true for
any vertex subset containing at most \( n - 2 \) vertices of the same row. When we consider \( n - 1 \) elements of the same row, their neighborhoods have in common one vertex, so \( O'' \supseteq O \), while if we consider all the vertices of the row, then \( O' = \emptyset \) and \( O'' = V \). The case in which \( O \) contains elements of the same column is similar. Let \( O \subseteq V(G) \) containing 3 vertices, at least 2 of which non-adjacent. Then, \(|O'| = 1\) when we consider two vertices of the same row (resp. column) and another vertex in one of the remaining columns (rows), so \( O'' \supseteq O \); we have \( O' = \emptyset \) in the other cases, so \( O'' = V \). In the other cases, we have three vertices as before, so \( O'' \neq O \) and we conclude the proof.

**Remark 4.1.37.** In general, the Galois poset of the rook’s graph \( G = K_m \square K_n \) is not graded. In fact, by Proposition 4.1.36 it is immediate to deduce that

\[
\begin{align*}
\bullet & \quad \{(i, j')\} \not\subseteq \{(i, j'), (k, l')\}; \\
\bullet & \quad \{(i_1, l')\} \not\subseteq \{(i_1, l'), (i_2, l')\} \subseteq \cdots \subseteq \{(i_1, l'), \ldots, (i_{m-2}, l')\} \not\subseteq N_G((1, l')); \\
\bullet & \quad \{(1, j')\} \not\subseteq \{(1, j'_1), (1, j'_2)\} \subseteq \cdots \subseteq \{(1, j'_1), \ldots, (1, j'_{n-2})\} \not\subseteq N_G((1, l')),
\end{align*}
\]

where \( (i, j') \sim (k, l') \), are three maximal chains having length respectively 2, \( m - 1 \) and \( n - 1 \).

We conclude this section by determining the extent family of the tensor product of an \( m \)-path and an \( n \)-cycle. We also determine the rank for the corresponding Galois poset.

**Theorem 4.1.38.** Let \( m, n \geq 2 \) and \( G = P_m \otimes C_n \) be the tensor product graph of the \( m \)-path and the \( n \)-cycle. Then

\[
|N_G((i, j'))| = \begin{cases} 1 & \text{if } m=3 \\ 2 & \text{if } m=4 \end{cases}
\]

where \( r = 1, \ldots, m-2, i = 2, \ldots, m-1 \) and \( j' = 1, \ldots, n \).

Moreover, the Galois poset \( \mathfrak{B}(G) \) is graded of rank

\[
\begin{cases} 1 & \text{if } m=3 \\ 2 & \text{if } m=4 \\ 4 & \text{otherwise} \end{cases}
\]

**Proof.** In this proof the index sums in the second component of the tensor product are taken mod(\( n \)). We firstly observe that \( N_G((i, j')) = N_G(i) \times N_G(j') \). This means that

\[
|N_G((i, j'))| = \begin{cases} 2 & \text{if } i = 1 \lor i = m \\ 4 & \text{otherwise} \end{cases}
\]

It is immediate to see that all the singletons \((i, j')\) such that \(|N_G((i, j'))| = 4\) and their neighborhoods are extent of the formal context \( \mathbb{K}[G] \). Let \( O = \{(i, j')\} \) such that \(|O'| = 2\). In order to fix ideas, let \( O = \{(1, j')\} \). Then, \( O' = \{(2, j'-1), (2, j'+1)\} \) and \( O'' = \{(1, j'), (3, j')\} \), hence in this case \( O \) is not an extent. Let \( O \subseteq V(G) \) of the form \( O = \{(i, r'), (i, s')\} \), where \( i = 2, \ldots, m-1 \). Since these vertices belong to the same \( n \)-cycle, we deduce that \( O' = \emptyset \) unless \( d(r', s') \equiv 2 \pmod{n} \). In this case, \( O' = \{(i-1, r'), (i+1, r')\} \). Moreover, let \( O \subseteq V(G) \) of the form \( O = \{(i, r'), (i+2, r')\} \), where \( i = 1, \ldots, m-2 \). Since these two vertices belong to the same \( m \)-path, it is easy to show that \( O' = \{(i+1, r'-1), (i+1, r'+1)\} \), so we deduce that
both the vertex subsets $O = \{(i, r'), (i, s')\}$, where $i = 2, \ldots, m - 1$, and $O = \{(i, r'), (i + 2, r')\}$, where $i = 1, \ldots, m - 2$, are extent of $K[G]$. Let $O \subseteq V(G)$ be a vertex subset of cardinality 2 different from the previous. If the two pairs are adjacent, then their first components different of one, but have their second components are the same, hence it is immediate to see that their neighborhoods do not intersect in any point. If the two pairs are non-adjacent, we have

$$|O'| = \begin{cases} 1 & \text{if } O = \{(i, j'), (i + 2, j' + 2)\} \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1, \ldots, m - 2$. In both cases, it’s clear that $O'' \supseteq O$, i.e. $O$ is not an extent. Let $O \subseteq V(G)$ be any subset of cardinality greater than 2; then we have three cases: (a) there exist a vertex $(i, j')$ such that $O \subseteq N_G((i, j'))$, (b) there exist a vertex $(i, j')$ such that $O = N_G((i, j'))$ or (c) $O$ is not neighborhood of any vertex. In the first situation, it is clear that $O'' \neq O$, while in the third there are always two vertices whose neighborhoods don’t have any point in common. Thus, $O$ is an extent only in case (b) and we can conclude the proof of (4.8).

Let now $m = 3$. By (4.8) we deduce that $\{(2, j')\}$ and pairs $\{(1, j' - 1), (1, j' + 1)\}$, for $j' = 1, \ldots, n$, are disjoint and they are minimal elements for the poset $B^3(P_3 \otimes C_n)$. Moreover, singletons are contained only in the maximal elements $\{(2, j' - 1), (2, j' + 1)\}$, while pairs are contained only in the maximal elements $N_{P_3 \otimes C_n}((2, j'))$, for $j' = 1, \ldots, n$. Thus, if $m = 3$, it results that $B^3(P_3 \otimes C_n)$ is a graded lattice of rank 1.

Let $m \geq 4$. By 4.1.38 one can verify that the maximal chains of $\mathfrak{B}^3(G)$ assume one of the following forms:

- $\{(2, j')\} \not\subseteq \{(2, j'), (2, j' - 1)\} \subseteq N_G((3, j' + 1));$
- $\{(2, j')\} \not\subseteq \{(2, j'), (2, j' + 1)\} \subseteq N_G((3, j' - 1));$
- $\{(2, j')\} \not\subseteq \{(2, j'), (4, j')\} \subseteq N_G((3, j' - 1));$
- $\{(2, j')\} \not\subseteq \{(2, j'), (4, j')\} \subseteq N_G((3, j' + 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i, j' - 1)\} \subseteq N_G((i - 1, j' + 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i, j' - 1)\} \subseteq N_G((i + 1, j' + 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i, j' + 1)\} \subseteq N_G((i - 1, j' - 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i, j' + 1)\} \subseteq N_G((i + 1, j' - 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i - 2, j')\} \subseteq N_G((i - 1, j' - 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i - 2, j')\} \subseteq N_G((i + 1, j' - 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i + 2, j')\} \subseteq N_G((i - 1, j' + 1));$
- $\{(i, j')\} \not\subseteq \{(i, j'), (i + 2, j')\} \subseteq N_G((i + 1, j' + 1));$
- $\{(m - 1, j')\} \not\subseteq \{(m - 1, j'), (m - 1, j' - 1)\} \subseteq N_G((m - 2, j' + 1));$
• \( \{ (m - 1, j') \} \subset \{ (m - 1, j'), (m - 1, j' + 1) \} \subset N_G((m - 2, j' - 1)) \);
• \( \{ (m - 1, j') \} \subset \{ (m - 1, j'), (m - 3, j') \} \subset N_G((m - 2, j' - 1)) \);
• \( \{ (m - 1, j') \} \subset \{ (m - 1, j'), (m - 3, j') \} \subset N_G((m - 2, j' + 1)) \);

where \( i = 3, \ldots, m - 2 \) and \( j' = 1, \ldots, n \). Hence, when \( m \geq 4 \), the Galois lattice of \( P_m \otimes C_n \) is graded of rank 2.

Example 4.1.39. In the following figure we represent the concept lattice of the graph \( P_3 \otimes C_3 \).

\[
\begin{align*}
\{ (1, 1'), (1, 2'), (1, 3'), (2, 1'), (2, 2'), (2, 3'), (3, 1'), (3, 2'), (3, 3') \} \\
\{ (2, 1'), (2, 2') \} & \{ (1, 1'), (1, 2'), (3, 1'), (3, 2') \} \{ (2, 1'), (2, 2') \} \{ (1, 1'), (1, 3'), (3, 1'), (3, 3') \} \{ (2, 2'), (2, 3') \} \{ (1, 2'), (1, 3'), (3, 2'), (3, 3') \} \\
\{ (2, 1') \} & \{ (1, 1'), (1, 3') \} \{ (2, 2') \} \{ (1, 1'), (1, 2') \} \{ (2, 3') \} \{ (1, 2'), (1, 3') \} \\
\emptyset & \\
\end{align*}
\]

Figure 4.7: Concept Lattice of the graph \( P_3 \otimes C_3 \).

4.2 Algebraic Consequences of Dissymmetry and of its Generalizations On Graphs

In this section we study the properties of an algebraic operation arising whenever we consider the symmetric differences between non-empty vertex subsets of a graph \( G \).

4.2.1 The Odd Extension of the Graph \( G \)

In this subsection we introduce the notion of odd extension \( G^\$ \) of a \( n \)-graph \( G \), which stems from the odd number of order pairs between two any non-empty vertex subsets of \( G \). We construct \( G^\$ \) by means of a binary operation having several interesting properties. We will show that \( G^\$ \) turns out connected and locally dissymmetric when \( G \) is connected and the determinant of its adjacency matrix is odd. In the remaining part of this section we assume \( n \geq 3 \).
Definition 4.2.1. Let $G$ be a $n$-graph and let $A, B \in \mathcal{P}^*(V(G))$. We set
\[ c_G(A, B) := |\{(v_i, v_j) \in A \times B : v_i \sim_G v_j\}|. \]
We denote by $\hat{G}$ the graph having vertex set $V(\hat{G}) := \mathcal{P}^*(V(G))$ and the following adjacency relation: if $A, B \in V(\hat{G})$ then
\[ A \sim_{\hat{G}} B : \iff c_G(A, B) \text{ is an odd integer}. \] (4.9)
If $S \subseteq V(\hat{G})$, we denote by $\hat{G}_S$ the subgraph of $\hat{G}$ induced from the vertex set $S$.

We now introduce a binary operation on the vertex set of $\hat{G}$.

Definition 4.2.2. We set
\[ A \circ B := \begin{cases} A & \text{if } A \not\sim_{\hat{G}} B \\ A \Delta B & \text{if } A \sim_{\hat{G}} B, \end{cases} \] (4.10)
for any $A, B \in V(\hat{G})$. It is clear that $\circ$ is a binary operation on $V(\hat{G})$, that we call disparity operation.

In the next result we establish some useful properties of the operation $\circ$.

Proposition 4.2.3. The binary operation $\circ : V(\hat{G}) \times V(\hat{G}) \to V(\hat{G})$ has the following properties:
(i): $A \circ A = A$;
(ii): $(A \circ B) \circ C = (A \circ C) \circ (B \circ C)$;
(iii): $A \circ B \neq A$, then $A \circ B = B \circ A$;
for any $A, B, C \in \mathcal{P}^*(V(G))$.

Proof. Straightforward. \qed

In what follows we will treat with other binary operations having the properties (i) – (iv) established in Proposition 4.2.3, therefore we introduce the following general terminology.

Definition 4.2.4. (i) We call delineated space any pair $(X, \circ)$, where $X$ is a non-empty set and $\circ$ is a binary operation on $X$ that satisfies the properties (i) – (iv) of Proposition 4.2.3.
(ii) If $(X_1, \circ_1)$ and $(X_2, \circ_2)$ are delineated spaces, we say that a map $f : X_1 \to X_2$ is a homomorphism of delineated spaces if $f(x \circ_1 x') = f(x) \circ_2 f(x')$ for all $x, x' \in X_1$. In particular, if $f$ is a bijective homomorphism of delineated spaces, we say that $f$ is an isomorphism of delineated spaces.
(iii) If $f$ is an isomorphism of some delineated space $(X, \circ)$ into itself, we say that $f$ is an automorphism of $(X, \circ)$. We denote by $\text{Aut}(X, \circ)$ the set of all the automorphisms of $(X, \circ)$. Obviously $\text{Aut}(X, \circ)$ is a group with respect to the usual composition of functions.

In this section we will investigate the class of the subsets of $V(\hat{G})$ that are closed with respect to the previous binary operation $\circ$. For such subsets we also consider a corresponding families of subgraphs of $\hat{G}$. We first introduce some terminology.
Definition 4.2.5. (i) If $T$ is a subset of $V(\hat{G})$ that is closed with respect to the binary operation $\circ$ (therefore $(T, \circ)$ is a delineated space) we say that $T$ is a delineated subspace of $\hat{G}$.

(ii) If $S \subseteq V(\hat{G})$, we call delineated closure of $S$ in $V(\hat{G})$, denoted by $S^\circ$, the smallest delineated subspace of $\hat{G}$ that contains $S$.

(iii) We denote by $T_G$ the delineated closure of the subset $V(G) \subset V(\hat{G})$ in $V(\hat{G})$, and we denote by $t_G$ the number of elements of $T_G$. We call odd extension of $G$ the graph $G^\circ := \hat{G}_{T_G}$.

Hence $T_G$ is the vertex subset of the graph $G^\circ$ and $t_G$ is its vertex number.

(iv) A delineated subgraph of $\hat{G}$ is a subgraph of the form $\hat{G}_T$, for some delineated subspace $T$ of $\hat{G}$.

We now show that the algebraic automorphisms of any delineated subspace $T$ of $\hat{G}$ coincide with the geometric automorphisms of the corresponding delineated subgraph $\hat{G}_T$.

Proposition 4.2.6. Let $T$ be a delineated subspace of $\hat{G}$, then $\text{Aut}(T, \circ) = \text{Aut}(\hat{G}_T)$. In particular, $\text{Aut}(T_G, \circ) = \text{Aut}(\hat{G}_T)$.

Proof. Let $\sigma \in \text{Aut}(T, \circ)$ and let $A, B \in T$ such that $A \sim_{\hat{G}_T} B$. Then, $\sigma(A) \neq \sigma(A \circ B) = \sigma(A) \circ \sigma(B)$, hence $\sigma(A) \sim_{\hat{G}_T} \sigma(B)$. On the other hand, suppose that $\sigma(A) \sim_{\hat{G}_T} \sigma(B)$, then $\sigma(A) \neq \sigma(A) \circ \sigma(B) = \sigma(A \circ B)$, so $A \neq A \circ B$ or, equivalently, $A \sim_{\hat{G}_T} B$. The case $A \sim_{\hat{G}_T} B$ is similar. Hence $\sigma \in \text{Aut}(\hat{G}_T)$.

Conversely, let $\sigma \in \text{Aut}(\hat{G}_T)$ and suppose that $A \sim_{\hat{G}_T} B$. Then,

$$\sigma(A \circ B) = \sigma(A) \circ \sigma(B).$$

On the other hand, if $A \sim_{\hat{G}_T} B$, we have that $A \circ B = A \Delta B$. Without loss of generality, let $x \in A \setminus B$. Hence $\sigma(x) \in \sigma(A) \setminus \sigma(B)$, therefore

$$\sigma(x) \in \sigma(A) \Delta \sigma(B)$$

or, equivalently,

$$\sigma(A \circ B) = \sigma(A) \circ \sigma(B).$$

Therefore, $\sigma \in \text{Aut}(T, \circ)$ and the claim follows.

Remark 4.2.7. (i) Let us note that $G$ can be identified with a subgraph of $\hat{G}$ because the adjacency between two vertices of $G$ coincides with the particular case of the adjacency between the singletons of $V(\hat{G})$.

(ii) By Definition of $\hat{G}$, since $G$ is simple also $\hat{G}$ is a simple graph.

(iii) $\hat{G}$ has $2^{|G|} - 1 = 2^n - 1$ vertices.

(iv) Since we treat the elements of $\mathcal{P}^*(V(G))$ as vertices of the graph $\hat{G}$, often we also use the letters $u, v, w, \ldots$ instead of $A, B, \ldots$ in order to denote these elements.

Example 4.2.8. Let $G = P_4$. It is easy to verify that the delineated closure of $V(G) = \{v_1, v_2, v_3, v_4\}$ in $V(\hat{G})$ is the subset

$$T_G = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\}$$

In Figure 4.8, we give a representation for the odd extension of $P_4$:
Figure 4.8: Odd Extension of $P_4$. 
We denote by $\text{Sym}(V(\hat{G}))$ the permutation group of $V(\hat{G})$ and by $\text{Id}$ the identity map on $V(\hat{G})$. We introduce now some specific elements of this group. If $u \in V(\hat{G})$ we set
\[
\xi_u(v) := v \circ u,
\]
for any $v \in V(\hat{G})$.

We have then the following result.

**Proposition 4.2.9.** (i) $\xi_u \in \text{Sym}(V(\hat{G}))$ for any $u \in V(\hat{G})$.
(ii) $\xi_u^2 = \text{Id}$ for any $u \in V(\hat{G})$.
(iii) If $T$ is a delineated subspace of $\hat{G}$ then $\xi_{u|T} \in \text{Aut}(T, \circ)$ for any $u \in T$.
(iv) If $u, w \in T_G$ and $\sigma \in \text{Aut}(\hat{G})$, it results
\[
\sigma^{-1}\xi_u\sigma = \xi_{\sigma^{-1}(u)}.
\]
In particular, we have that
\[
\xi_w^{-1}\xi_u\xi_w = \xi_{u\circ w}.
\]

**Proof.** (i) : Since $V(\hat{G})$ is a finite set it is sufficient to prove that $\xi_u$ is surjective. By (ii) of Proposition 4.2.3, if $v \in V(\hat{G})$, then $v = (v \circ u) \circ u = \xi_u(v \circ u)$.

(ii) : The thesis follows immediately by (ii) of Proposition 4.2.3.

(iii) : By (iii) of Proposition 4.2.3, we deduce immediately the thesis.

(iv) : Let us consider $\sigma^{-1}\xi_u\sigma(v)$. Then,
\[
\sigma^{-1}\xi_u\sigma(v) = \sigma^{-1}(\sigma(v) \circ u) = v \circ \sigma^{-1}(u)
\]
by Proposition 4.2.6. But the last term is exactly $\xi_{\sigma^{-1}(u)}(v)$ and the thesis has been shown. \(\square\)

## 4.2.2 Regularity properties of $\hat{G}$

In what follows, we always assume that $G$ is a connected graph. In the next result we relate the presence of isolated vertices in $\hat{G}$ to the parity of the determinant of the adjacency matrix of $G$.

**Proposition 4.2.10.** $\hat{G}$ has at least an isolated vertex if and only if $\det(\text{Adj}(G))$ is even.

**Proof.** Let us suppose $\hat{G}$ has an isolated vertex $A \in V(\hat{G}) = \mathcal{P}(V(G))$ and let $v_A = (x_1, \ldots, x_n)$ be the vector in $\mathbb{R}^n$ defined by
\[
x_i := \begin{cases} 1 & \text{if } v_i \in A, \\ 0 & \text{otherwise}. \end{cases}
\]
The vector $\text{Adj}(G)v_A = (y_1, \ldots, y_n)$ is the vector in $\mathbb{R}^n$ such that $y_i$ is the number of vertices in $A$ that are adjacent to $v_i$, for all $i = 1, \ldots, n$. Thus $y_i = c_G(\{v_i\}, A) \equiv 0 \pmod{2}$, for all $i = 1, \ldots, n$ because $\{v_i\} \sim_G A$. By considering $A$ as a square matrix of order $n$ with entries in $\mathbb{Z}_2$, then $v_A \in \mathbb{Z}_2^n \setminus \{0\}$ is in the kernel of the matrix $A$. So $\det(A) \equiv 0 \pmod{2}$. For the reserve implication let us note that all of the previous arguments are reversible. In fact if $\det(\text{Adj}(G)) \equiv 0 \pmod{2}$, then, by considering $\text{Adj}(G)$ as a matrix in $\text{Mat}_n(\mathbb{Z}_2)$, it has non-trivial kernel. So there exist $v \in \mathbb{Z}_2^n$ such that $\text{Adj}(G) \cdot v$ is the null vector in $\mathbb{Z}_2^n$. We can interpret $v$ as a vertex subset of $G$. $v \in \ker(\text{Adj}(G))$ implies that, for all $i = 1, \ldots, n$, $c_G(\{v_i\}, v) \equiv 0 \pmod{2}$. It follows that $v$ is an isolated vertex of $\hat{G}$. This proves the proposition. \(\square\)
In the next result we prove several regularity properties of the graph $G^\$ when $\det(Adj(G))$ is odd. In particular, under this hypothesis we will show that $G^\$ is always a connected and locally dissymmetric graph.

**Theorem 4.2.11.** Let $\det(Adj(G))$ be odd. Then:
(i) $G^\$ is connected and locally dissymmetric.
(ii) There exists $\lambda$ such that $\lambda = |N_{G^\}(v) \cap N_{G^\}(w)|$ for any two adjacent vertices $v, w$ of $G^\$.
(iii) $G^\$ is not a complete graph.
(iv) $G^\$ is a $2\lambda$-regular graph.
(v) There exists $\mu \equiv 0 \pmod{4}$ such that $\mu = |N_{G^\}(v) \cap N_{G^\}(w)|$ for any two non-adjacent vertices $v, w$ of $G^\$.

**Proof.** (i): Suppose by contradiction that $G^\$ is not connected. Let $S_G$ the connected component containing $G$ and fix a vertex $v \notin S_G$. Then, $c_G({v_i}, v) \equiv 0 \pmod{2}$, for all $i = 1, \ldots, n$ because $\{v_i\} \sim_G v$. Thus, for all $w \in \mathcal{P}^*(V(G))$, it holds
$$c_G(w, v) = \sum_{v_i \in w} c_G({v_i}, v) \equiv 0 \pmod{2},$$
so $w \sim \hat{G} v$. Then, $w$ is an isolated vertex in $\hat{G}$, but this is impossible by Proposition 4.2.10.

Let us prove that $G^\$ is locally dissymmetric. Suppose by contradiction that $G^\$ is not locally dissymmetric. Hence there exist two distinct elements $v, w \in T_G$ such that $\Delta_{G^\}(v, w) = \emptyset$. Thus, by setting $u := v \Delta w$, then, for all $z \in T_G$, $c_G(u, z) \equiv 0 \pmod{2}$. This contradicts the previous part of the proof.

(ii): Let $v$ and $w$ be two adjacent vertices in $G^\$. The application $\xi_v$ defined in (4.11) induces a bijection between $N_{G^\}(v) \cap N_{G^\}(w)$ and $N_{G^\}(v) \setminus N_{G^\}(w)$. In fact, if $u \in N_{G^\}(v) \cap N_{G^\}(w)$, then $u \circ v = u \Delta v$ and thus
$$c_G(u \circ v, v) = c_G(u, v) + c_G(v, v) - 2c_G(u \cap v, v) \equiv 1 \pmod{2},$$
and
$$c_G(u \circ v, w) = c_G(u, w) + c_G(v, w) - 2c_G(u \cap v, w) \equiv 0 \pmod{2},$$
so $\xi_v(u) = u \circ v \in N_{G^\}(v) \setminus N_{G^\}(w)$. On the other side, if $u \in N_{G^\}(v) \setminus N_{G^\}(w)$, then in this case it results that
$$c_G(u \circ v, v) = c_G(u, v) + c_G(v, v) - 2c_G(u \cap v, v) \equiv 1 \pmod{2},$$
and
$$c_G(u \circ v, w) = c_G(u, w) + c_G(v, w) - 2c_G(u \cap v, w) \equiv 1 \pmod{2},$$
so $\xi_w(u) = u \circ v \in N_{G^\}(v) \cap N_{G^\}(w)$. In the same way, $\xi_w$ induces a bijection between $N_{G^\}(v) \cap N_{G^\}(w)$ and $N_{G^\}(w) \setminus N_{G^\}(v)$, hence $|N_{G^\}(v) \cap N_{G^\}(w)| = |N_{G^\}(w) \setminus N_{G^\}(v)|$. Therefore, the following relation holds for any two adjacent vertices $v, w \in S^\$:
$$|N_{G^\}(v)| = |N_{G^\}(v) \cap N_{G^\}(w)| = |N_{G^\}(w)|. \tag{4.14}$$
So, there exists \( \lambda \) such that \( \lambda = \left| N_{G^5}(v) \cap N_{G^5}(w) \right| \) for any two adjacent vertices \( v, w \) of \( G^5 \).

\((iii)\): If \( G \) is not complete, then there are two non-adjacent vertices \( v_i, v_j \in V(G) \), hence this holds also in \( G^5 \). On the other hand, if \( G \) is the complete graph on \( n \) vertices, it is immediate to see that if \( u, v \) and \( w \) are three distinct vertices in \( V(G) \), then \( u, v \cup w \) are vertices in \( G^5 \) such that \( c_G(u, v \cup w) \equiv 0 \pmod{2} \), hence \( G^5 \) is not complete.

\((iv)\): Since \( G^5 \) is a connected graph, for any two non-adjacent vertices \( v \) and \( w \) we can take a path between them and iterate the equality \((4.14)\) for any two adjacent vertices in the path. Hence, we conclude by part \((ii)\).

\((v)\): Let \( v, w \) be two non-adjacent vertices of \( G^5 \). Note that, if \( u \in N_{G^5}(v) \cap N_{G^5}(w) \), then \( v \circ u, w \circ u, v \circ (w \circ u) \in N_{G^5}(v) \cap N_{G^5}(w) \) and \( \left| \{u, v \circ u, w \circ u, v \circ (w \circ u)\} \right| = 4 \). Moreover, if \( u' \in \{v \circ u, w \circ u, v \circ (w \circ u)\} \), then \( \{u, v \circ u, w \circ u, v \circ (w \circ u)\} = \{u', v \circ u', w \circ u', v \circ (w \circ u')\} \).

In fact, it is straightforward to prove that the vertices \( v \circ u \) and \( w \circ u \) are adjacent to both \( v \) and \( w \). Let us prove now that \( v \circ (w \circ u) \in N_{G^5}(v) \cap N_{G^5}(w) \). It holds

\[
c_G(v \circ (w \circ u), v) = c_G(v, v) + c_G(w \circ u, v) - 2c_G(v \cap (w \circ u), v) \equiv 1 \pmod{2}
\]

and

\[
c_G(v \circ (w \circ u), w) = c_G(v, w) + c_G(w \circ u, w) - 2c_G(v \cap (w \circ u), w) \equiv 1 \pmod{2},
\]

so \( v \circ (w \circ u) \sim_{G^5} v \) and \( v \circ (w \circ u) \sim_{G^5} w \).

Let now \( u' = v \circ u \). Then, we have

\[
u' \circ v = (v \bigtriangleup u) \bigtriangleup v = v,
\]

\[
u' \circ w = (v \bigtriangleup u) \bigtriangleup w = v \bigtriangleup (w \bigtriangleup u) = v \circ (w \circ u)
\]

and

\[
u' \circ (v \circ (w \circ u)) = (v \bigtriangleup u) \bigtriangleup (v \circ (w \circ u)) = w,
\]

because \( v \circ u \sim_{G^5} (v \circ (w \circ u)) \). This prove that \( \left| N_{G^5}(v) \cap N_{G^5}(w) \right| \equiv 0 \pmod{4} \).

We prove that \( \left| N_{G^5}(v) \cap N_{G^5}(w) \right| \) is constant for any two distinct non-adjacent vertices \( v, w \in G \). Let \( v', w' \in G \) two non-adjacent vertices distinct from \( v, w \). First of all, we prove that \( \left| N_{G^5}(v) \cap N_{G^5}(w) \right| = \left| N_{G^5}(v') \cap N_{G^5}(w') \right| \) can be reduced to the case \( \left| N_{G^5}(v) \cap N_{G^5}(y) \right| = \left| N_{G^5}(v) \cap N_{G^5}(y) \right| \) for some \( y \in T_G \) non-adjacent to \( v \). By Proposition 4.2.9, if \( u, v \in V(G) \), then

\[
\xi_u(N_G(y)) = N_G(\xi_v(u)). \tag{4.15}
\]

Let \( P = s_0, s_1, \ldots, s_{l-1}, s_l \) be a path between \( v' = s_0 \) and \( v = s_n \). Since \( s_i \sim s_{i-1} \), for each \( i \in \{1, \ldots, l\} \), by applying sequentially \( \xi_{s_i} \circ s_{i-1} =: \xi_{s_i} \) and by using Equation \((4.15)\), it is easy to prove that

\[
\xi_{s_0} \cdots \xi_{s_{l-1}} \xi_{u_1}(N_G(v) \cap N_G(w)) = N_{G^5}(v) \cup N_{G^5}(y),
\]

where \( y := (\ldots ((w' \circ u_1) \circ u_2) \cdots \circ u_n) \). Moreover, by Proposition 4.2.9 again, \( v \sim_{G^5} y \).

Let us note that \( \left| N_{G^5}(v) \cap N_{G^5}(w) \right| \neq 0 \neq \left| N_{G^5}(v) \cap N_{G^5}(y) \right| \) since any path between two non-adjacent vertices can be shortened to a 2-path by replacing two adjacent vertices by their
symmetric difference. It is easy to prove that \( \xi_w \) induces a bijection between \( N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(y) \) and \( (N_{G^5}(v) \cap N_{G^5}(w)) \setminus N_{G^5}(y) \). Similarly \( \xi_y \) induces a bijection between \( N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(y) \) and \( (N_{G^5}(v) \cap N_{G^5}(w)) \setminus N_{G^5}(y) \). Thus \( |(N_{G^5}(v) \cap N_{G^5}(w)) \setminus N_{G^5}(y)| = |(N_{G^5}(v) \cap N_{G^5}(y)) \setminus N_{G^5}(w)| \). By a simple computation, it is immediate to see that

\[
|N_{G^5}(v) \cap N_{G^5}(w)| = |N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(y)| + |N_{G^5}(w) \cap N_{G^5}(w) \setminus N_{G^5}(y)| = \\
|N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(y)| + |N_{G^5}(v) \cap N_{G^5}(y) \setminus N_{G^5}(w)| = |N_{G^5}(v) \cap N_{G^5}(w) \setminus N_{G^5}(y)|.
\]

So the theorem holds in the first case. Suppose that \( w \approx y \). Then, there exists \( x \in \Delta_{G^5}(v, w) \) because \( G^5 \) is a locally disymmetric graph. Without loss in generality we can assume \( y \sim x \). Let us observe that \( (N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(y)) \setminus N_{G^5}(y) \neq \emptyset \). In fact, we know that \( |N_{G^5}(v) \cap N_{G^5}(w)| \neq 0 \) so, there exists \( z \in (N_{G^5}(v) \cap N_{G^5}(w)) \setminus N_{G^5}(y) \). Furthermore, it is easy to show that

\[
\xi_x : (N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(x)) \setminus N_{G^5}(y) \rightarrow (N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(x)) \setminus N_{G^5}(v) \\
\xi_x : (N_{G^5}(w) \cap N_{G^5}(x)) \setminus N_{G^5}(v) \setminus N_{G^5}(y) \rightarrow (N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(y)) \setminus N_{G^5}(x) \\
\xi_y : (N_{G^5}(v) \cap N_{G^5}(y)) \setminus N_{G^5}(x) \setminus N_{G^5}(v) \rightarrow (N_{G^5}(w) \cap N_{G^5}(y) \cap N_{G^5}(x)) \setminus N_{G^5}(v) \\
\xi_y : (N_{G^5}(w) \cap N_{G^5}(y) \cap N_{G^5}(x)) \setminus N_{G^5}(v) \setminus N_{G^5}(y) \rightarrow (N_{G^5}(w) \cap N_{G^5}(y) \cap N_{G^5}(x)) \setminus N_{G^5}(v)
\]

are all bijections. Therefore, by some simple computations we deduce that

\[
|N_{G^5}(v) \cap N_{G^5}(w)| = |N_{G^5}(v) \cap N_{G^5}(y)| = 4|(N_{G^5}(v) \cap N_{G^5}(w) \cap N_{G^5}(x)) \setminus N_{G^5}(y)|.
\]

\[\square\]

### 4.2.3 Two Algebraic Characterizations for the Odd Extension of \( G \)

In this section we associate with \( G \) two groups, denoted respectively by \( D_G \) and \( K_G \). In terms of these groups we provide two algebraic characterizations related to the graph structure of \( G^5 \).

Let \( T \) be a delineated subspace of \( G \). We set then

\[
F_T := \{ \xi_{u|T} : u \in T \},
\]

and we will consider the following two subgroups of \( Sym(T) \) (the permutation group on \( T \)):

\[
K_T := \langle \xi_{u|T} \xi_{w|T} : u, w \in T \rangle \subseteq D_T := \langle F_T \rangle.
\]

However, when \( T \) is specified and \( u \in T \), we write simply \( \xi_u \) instead of \( \xi_{u|T} \).

**Proposition 4.2.12.** Let \( w \in T \) be a fixed element. Then, \( K_T = \langle \xi_u \xi_w : u \in T \rangle \).

**Proof.** It is sufficient to observe that \( \xi_u \xi_v = (\xi_u \xi_w)(\xi_u \xi_w)^{-1} \) for any \( u, v \in T \). \[\square\]
Definition 4.2.13. We call $D_G := D_{TG}$ the first normalization group of $G$ and $K_G := K_{TG}$ the second normalization group of $G$.

Remark 4.2.14. \( (i) \): By Proposition 4.2.9 and by (4.12), it is immediate to see that $D_G$ is a normal subgroup of $Aut(T_G)$.
\( (ii) \): Since $D_G$ is a subgroup of $Sym(V(\tilde{G}))$, it induces a natural group action on $T_G$.

In what follows, if $H$ is a group acting on a set $X$ and $h \in H$, we denote $hX := \{hx : x \in X\}$.

Definition 4.2.15. Let $X$ be a finite set and $H$ a group acting on $X$. We say that $S \subseteq X$ is a block for $H$ if for any $h \in H$ we have $hS = S$ or $hS \cap S = \emptyset$. We say that $H$ is primitive over $X$ if its action on $X$ is transitive and all its blocks are only singletons or the whole $X$. A group acting on a set $X$ which is not primitive over $X$ is called imprimitive over $X$.

Proposition 4.2.16. Let $T$ be a delineated subspace of $\tilde{G}$ and let us consider the natural action of $D_T$ over $T$. If $S \subseteq T$ is a block for $D_T$, then $S$ is a delineated subspace of $T$.

Proof. Let $S \subseteq T$ be a block for $D_T$ and let $u, v \in S$. Then, since $v = v \circ v \in \xi_v S \cap S$, then $\xi_v S = S$, so $u \circ v \in \xi_v S = S$.

Proposition 4.2.17. $D_G$ is primitive over $T_G$ if and only if for any delineated subspace $S \subsetneq T_G$ such that $|S| \geq 2$ it results
\[ \emptyset \subsetneq \tau S \cap S \subsetneq S, \] (4.16)
for some $\tau \in D_G$.

Proof. If $D_G$ is primitive over $T_G$, the claim is obvious. Vice versa, suppose that (4.16) holds and let $S \subsetneq T_G$ be a block for $D_G$. By Proposition 4.2.16 $S$ is a delineated subspace of $T_G$. Thus, by (4.16), $S$ is a singleton.

Let us prove now that the action of $D_G$ over $T_G$ is transitive. Let us show that, for any $v \in T_G$, $D_Gv := \{\tau(v) : \tau \in D_G\} = T_G$. Let $v \in V(G) \subseteq T_G$, then, since $G$ is connected, there exists $w \in V(G)$ such that $v \sim w$. It follows $|D_Gv| \geq 2$. By (4.16) it holds $D_Gv = T_G$. If $v \in T_G$, then $v \in D_Gw$ for some $w \in V(G)$. Thus $D_Gw = D_Gv = T_G$, so we proved that the action is transitive and the proposition holds.

In the next result we provide an algebraic characterization for the condition that the odd extension of $G$ is a graph connected and locally dissymmetric.

Theorem 4.2.18. The following properties are equivalent:
\( (a) \) The odd extension $G^\delta$ is connected and locally dissymmetric.
\( (b) \) $D_G$ is primitive over $T_G$.

Proof. \( (a) \implies (b) \): Let us suppose that $G^\delta$ is connected and locally dissymmetric and let $v, w$ in $T_G$. Let us prove, by using induction on the distance between $v$ and $w$, that there exists $\tau \in D_G$ such that $v = \tau(w)$. If $n = 1$, then $v \sim_{G^\delta} w$, so $v \circ w \neq v$. By $(iv)$ of Proposition 4.2.3, it holds $v \circ w = w \circ v$, so
\[ v = (v \circ w) \circ w = (w \circ v) \circ v = \tau(w), \]
where $\tau := \xi_v \xi_w$. Let now $n > 1$. In this case $v \sim_{G^\delta} w$ and there exists $u \in T_G$ such that $v \sim_{G^\delta} u$ and $d(u, w) = n - 1$. By using the inductive hypothesis there exists $\tau_1, \tau_2 \in D_G$ such that $\tau_1(w) = u$ and $\tau_2(u) = v$. Thus $v = \tau_2(u) = \tau_2(\tau_1(w))$. This proves transitivity. Let now $S$ be a block for $D_G$. Then, by Proposition 4.2.16, $S$ is a delineated subspace of $T_G$. Let us assume now that $S$ is a proper delineated subspace of $T_G$ such that $|S| \geq 2$ and let $x, y \in S$. Since $S$ is a delineated subspace, $x \circ y \in S$. Let us consider the case $x \circ y \neq x$. If $v \in T_G$, then let us note that, if both $x \circ v$ and $y \circ v$ are not in $S$, then $(x \circ y) \circ v = x \circ y \in S$. By definition of block for the action of $D_G$, it holds $\xi_v S = S$. By transitivity of the action of $D_G$, it follows that $S = T_G$, so $D_G$ is primitive over $T_G$. Consider now $x \circ y = x$. Since $G^\delta$ is locally dissymmetric, then there exists $u \in T_G$ such that either $x \circ u = x$ and $y \circ u = y$ or $y \circ u \neq y$ and $x \circ u \neq x$. Still, by definition of block for $D_G$ over $T_G$, in both cases we obtain $\xi_u S = S$. If $x \circ u \neq x$, then $u \in S$ and $u \neq x$, so the thesis follows as before. Similarly in the case $y \circ u \neq y$.

(b) $\implies$ (a): Let us assume now that $D_G$ is primitive over $T_G$. Let $v, w \in T_G$ with $v \neq w$. Since $D_G$ is primitive, its action over $T_G$ is transitive. So there exist $\tau \in D_G$ such that $\tau(v) = w$. Suppose first that $\tau = \xi_v$, for some $u \in T_G$. In this case $w = v \circ u$, so $w \circ u = (v \circ u) \circ u = v$. Moreover, since $v \circ u \neq w$ and $w \circ u \neq w$, it holds $v \sim_{G^\delta} u$ and $w \sim_{G^\delta} u$. The connectedness of $G^\delta$ follows by using induction over the minimum length of $\tau$ as reduced word in $D_G$. Let us prove now that $G^\delta$ is locally dissymmetric. Let $v, w \in T_G$. If $N_{G^\delta}(v) = N_{G^\delta}(w)$, then $v \sim_{G^\delta} w$. Let us prove now that $S := \{v, w\}$ is a block for $D_G$. For this, let $u \in T_G$ such that $v \sim_{G^\delta} u$ and $w \sim_{G^\delta} u$. Then, in this case, $v \circ u \neq v$ and $w \circ u \neq w$. Moreover, since $v \circ u \sim_{G^\delta} v$ and $w \circ u \sim_{G^\delta} w$, then $v \circ u \neq w$ and $w \circ u \neq v$. Then, $\xi_u S \cap S = \emptyset$. Now let $w \in T_G$ such that $v \sim_{G^\delta} u$ and $w \sim_{G^\delta} u$. In this case $v \circ u = v$ and $w \circ u = w$, so $\xi_u S = S$. This proves that $S$ is a block for $D_G$. Since $D_G$ is primitive over $T_G$, then $|S| = 1$ and thus $v = w$.

\[\square\]

Usually, $K_G$ is not a commutative group. In the next result we provide a necessary and sufficient condition so that $K_G$ is abelian.

**Theorem 4.2.19.** Let $G^\delta$ be a connected and locally dissymmetric graph. Then, $K_G$ is an abelian group if and only if there exists $w \in T_G$ such that $K_G = \{\xi_w \xi_u : u \in T_G\}$.

**Proof.** Suppose that $K_G = \{\xi_u \xi_w : u \in T_G\}$, where $w$ is a fixed element of $T_G$. We will show that $K_G$ is an abelian group. Let $\xi_u \xi_w \in K_G$. Then, it results that the map $\xi_v \xi_w \mapsto \xi_w^{-1} \xi_u \xi_w \xi_w = (\xi_w \xi_u)^{-1}$ is an automorphism of $K_G$ mapping an element to its inverse. This ensures that $K_G$ is an abelian group.

On the other hand, suppose that $K_G$ is an abelian group. Let us consider the following map

\[F : T_G \to F_G\]

\[u \mapsto \xi_u.\]

It is obviously surjective. We show now that $F$ is also injective. In fact, let $u, v \in T_G$ such that $\xi_u(w) = \xi_v(w)$, that is $w \circ u = w \circ v$, for any $w \in T_G$. Then, $u \notin N_{G^\delta}(v)$ if and only if $w \notin N_{G^\delta}(v)$, so $N_{G^\delta}(u) = N_{G^\delta}(v)$. This implies that $u = v$ because $G^\delta$ is locally dissymmetric and $F$ is thus injective.
Let now $v \in T_G$ be a fixed element. Then, for any $u \in T_G$ we define
\[
u^k_v := \begin{cases} (\xi_u \xi_v)^i(v) & \text{if } k = 2i, \\ (\xi_u \xi_v)^i(u) & \text{if } k = 2i + 1. \end{cases} (4.18)
\]
Since $\xi_u \xi_v$ has finite order in $K_G$, there exists a least positive integer $n$ such that $\nu^n_v = v$.
We also observe that
\[
\xi_v (\xi_u \xi_v)^{-i} = (\xi_u \xi_v)^i \xi_v.
\]

In fact, the subgroup of $D_G$ spanned by the elements $\xi_u, \xi_v$ is dihedral since $\langle \xi_u, \xi_v \rangle = \langle \xi_v, \xi_u \xi_v \rangle$ and it results
- $\xi_v^2 = Id$;
- There exists an integer $j$ such that $(\xi_u \xi_v)^j = Id$;
- $\xi_v (\xi_u \xi_v) \xi_v = (\xi_u \xi_v)^{-1}$.

Hence, in particular, $(4.19)$ holds.

By using $(4.18)$, $(4.19)$ and $(4.12)$, it is easy to prove that
\[
\xi_{u^k_v} = (\xi_u \xi_v)^k \xi_v
\]
and, by means of the latter relation, we also conclude that, if $j < 2k$, then
\[
\xi_{u^k_v}(\nu^j_v) = \nu^{2k-j}_v. (4.21)
\]

Now we prove that the smallest integer $n$ such that $\nu^n_v = v$ is an odd integer for any $u \in T_G$.
Suppose by contradiction that there exists an element $u \in T_G$ such that $n$ is the least positive integer for which $\nu^n_v = v$, with $n = 2k$. Let $l$ be the order of $\xi_u \xi_v$ in $K_G$, then by $(4.20)$ $\xi_{u^l_v} = (\xi_u \xi_v)^l \xi_v$, hence $\xi_{u^l_v} = \xi_v$ so, by injectivity of $F$, $\nu^l_v = v$ and thus $n \leq l$. We also observe that, again by $(4.20)$, we also have $(\xi_u \xi_v)^n = \xi_u v \xi_v = \xi_v \xi_v = Id$, thus $l|n$ and this forces $l = n$. Therefore there exists $w \in T_G$ such that $(\xi_u \xi_v)^k (w) \neq w$.

Let us prove, by using that $K_G$ is an abelian group, that $\xi_{\xi_u \xi_v (w)} \xi_w = (\xi_u \xi_v)^2$. In fact, by $(4.12)$,
\[
\xi_{\xi_u \xi_v (w)} = (\xi_u \xi_v) \xi_w (\xi_u \xi_v)^{-1} = (\xi_u \xi_v) \xi_w (\xi_u \xi_u),
\]
so, by using the commutative property on $K_G$,
\[
\xi_{\xi_u \xi_v (w)} \xi_w = \xi_u \xi_v (\xi_w \xi_v) (\xi_u \xi_w) = \xi_u \xi_v (\xi_u \xi_w) (\xi_u \xi_v) = (\xi_u \xi_v)^2.
\]

It follows that $k$ is the order of $\xi_{\xi_u \xi_v (w)} \xi_w$ in $K_G$. Moreover, it is easy to see that
\[
(\xi_u \xi_v)^j (w) = \begin{cases} (\xi_{\xi_u \xi_v (w)} \xi_w)^j (w) & \text{if } j = 2i, \\ (\xi_{\xi_u \xi_v (w)} \xi_w)^j (\xi_u \xi_v (w)) & \text{if } j = 2i + 1. \end{cases}
\]
Thus it follows that $(\xi_u \xi_v)^j (w)$ is $(\xi_u \xi_v)^j (w)$. By $(4.20)$, it holds that
\[
\xi_{\xi_u \xi_v (w)} = (\xi_u \xi_v)^k \xi_w = \xi_w.
\]
By the injectivity of $F$, we observe that $(\xi_u \xi_v)^k(w) = (\xi_u \xi_v(w))^k = w$, by contradicting our assumption. Hence $n$ cannot be an even integer.

We finally show that $K_G = \{\xi_u \xi_w : u \in T_G\}$, where $w \in T_G$ is a fixed element. Let $u, v \in T_G$ two fixed elements such that $u^w_v = v$ and set $n = 2k + 1$. Then, $(\xi_u \xi_v)^k(u) = v$, hence $(\xi_u \xi_v)^{k+1}(v) = u$. As direct consequence, we have $u^{2k+2} = u$. In order to show the claim, we must see that any composition and any inverse can be expressed as elements of $\{\xi_u \xi_w : u \in T_G\}$.

Let $w \in T_G$. Then, there exists $z \in T_G$ such that

$$\xi_w \xi_v \xi_u = \xi_z$$  \hspace{1cm} (4.22)

In fact, $\xi_w \xi_v \xi_u = \xi_w \xi_v \xi_{u^{2k+2}}$ which, by means of (4.12) and (4.19), becomes $(\xi_u \xi_v)^{-(k+1)}(\xi_w \xi_u \xi_v)$, i.e., $(\xi_u \xi_v)^{k+1}(w)$. Therefore, we set $z = (\xi_u \xi_v)^{k+1}(w)$. In particular, $\xi_w \xi_v \xi_u \xi_v = \xi_z \xi_v$. Moreover, it results that

$$(\xi_u \xi_v)^{-1} = \xi_{u^{m-1}} \xi_v,$$  \hspace{1cm} (4.23)

where $m$ is the order of $\xi_u \xi_v$ in $K_G$. So $K_G = \{\xi_u \xi_w : u \in T_G\}$, where $w \in T_G$ is a fixed element. The theorem is thus proved.

In reference to the formalism introduced in (4.18) we use the following terminology.

**Definition 4.2.20.** Let $u, v \in T_G$. We call order of $u$ with respect to $v$ the smallest positive integer $n$ such that $u^n_v = v$.

In next result, we establish some structural properties of the group $K_G$ when $G^\xi$ is a connected locally dissymmetric graph.

**Theorem 4.2.21.** Let $G^\xi$ be connected and locally dissymmetric. Then:

(a) $K_G$ is the unique subgroup of $D_G$ that satisfies the two following properties:

(i) $K_G \leq D_G$;

(ii) For any $\{Id_{T_G}\} \not\subseteq H \leq D_G$ it results $K_G \subseteq H$.

(b) $K_G$ is either a simple group or it is the direct product of two of its non-trivial normal conjugate subgroups.

**Proof.** (a): For the part (i), since $\xi^{-1}_w(\xi_u \xi_v)\xi_w = \xi^{-1}_w \xi_u \xi_v \xi^{-1}_w \xi_v = \xi_{u^{m-1}} \xi_v$, we deduce that $K_G$ is a normal group. We prove now part (ii). Let $H \neq \{Id_{T_G}\}$ be a normal subgroup of $D_G$. For any $v \in T_G$ we consider the $H$-orbit $Hv := \{h(v) : h \in H\}$ of $v$ in $T_G$. Then, since $H \neq \{Id_{T_G}\}$ there exists a vertex $v_0 \in T_G$ such that $|Hv_0| > 1$. We show now that the $H$-orbit

$Hv_0$ is a block for the group $D_G$.

To this aim, we assume that $\sigma \in D_G$ and $\sigma(Hv_0) \cap Hv_0 \neq \emptyset$, so that there is an element

$$w_0 = h_1(v_0) = \sigma h_2(v_0) \in Hv_0 \cap \sigma(Hv_0),$$

for some $h_1, h_2 \in H$. We must prove then that $\sigma(Hv_0) = Hv_0$.

Let first $v = h_3(v_0) \in Hv_0$, for some $h_3 \in H$. Then,

$$v = h_3 h_1^{-1}(w_0) = h_3 h_1^{-1} \sigma h_2(v_0) = \sigma(\sigma^{-1} h_3 \sigma)(\sigma^{-1} h_1^{-1} \sigma) h_2(v_0) \in \sigma(Hv_0)$$

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because $H$ is normal in $D_G$. Hence $Hv_0 \subseteq \sigma(Hv_0)$.

On the other hand, let now $v = \sigma h_4(v_0) \in \sigma(Hv_0)$, for some $h_4 \in H$. Then

$$v = \sigma h_4 h_2^{-1} \sigma^{-1}(w_0) = (\sigma(h_4 h_2^{-1})\sigma^{-1})h_1(v_0) \in Hv_0$$

because $H$ is normal in $D_G$. Hence $\sigma(Hv_0) \subseteq Hv_0$. This shows that $Hv_0$ is a block for $D_G$.

By Theorem 4.2.18, $D_G$ is primitive over $T_G$, so any block for $D_G$ is a singleton or the whole $T_G$. In our case, it must results that $Hv_0 = T_G$. In particular, we observe that $Hv_0 = Hv$ for any $v \in T_G$. Thus $H$ is transitive over $T_G$. In other terms, if $u, v \in T_G$, there exists $\rho \in H$ such that $\rho(v) = u$. By (4.12), we have:

$$\xi_u \xi_v = \xi_u^{-1} \rho^{-1} \xi_u \rho \in H. \quad (4.24)$$

So $K_G \subseteq H$, i.e. $K_G$ is the minimum non-trivial normal subgroup of $D_G$. Clearly, $K_G$ is the unique subgroup of $D_G$ satisfying both (i) and (ii).

(b): Suppose that $K_G$ is not a simple group, we want to show that it can be expressed as the direct product of two simple normal subgroups $N, M \neq \{Id_{T_G}\}$ conjugate in $D_G$, i.e.

$$K_G \cong M \times N.$$  

Since $K_G$ is not simple, there exists $\{Id_{T_G}\} \subsetneq N \triangleleft K_G$. For any $u, v \in T_G$, we have that $\xi_u, \xi_v \in D_G$ and $\xi_u \xi_v \in K_G$. Since $N$ is normal in $K_G$, we have $(\xi_v \xi_u)^{-1} N (\xi_v \xi_u) = N$, therefore we deduce that

$$\xi_u N \xi_u = \xi_v N \xi_v, \quad (4.25)$$

since $\xi_u^2 = \xi_v^2 = Id_{T_G} = Id_{T_G}$. Let $M := \xi_z N \xi_z$ for some, and thus for all, $z \in T_G$. We have to verify that $M \cap N = \{Id_{T_G}\}$ and that $MN$ is a normal subgroup of $D_G$ contained in $K_G$. In order to prove that $M \cap N = \{Id_{T_G}\}$, we firstly show that $M \cap N$ is a normal subgroup of $D_G$. Let then $\phi \in M \cap N$ and $\xi_v \in D_G$. Therefore, there exists $\phi' \in N$ such that $\phi = \xi_z \phi' \xi_z$. Then,

$$\xi_v^{-1} \phi \xi_v = (\xi_z \xi_u)^{-1} \phi' (\xi_z \xi_u) \in N$$

because $N$ is normal in $K_G$. On the other hand, by taking $u = z$ in (4.25), it follows that $\xi_z \phi \xi_z = \xi_z \phi \xi_z = \xi_z \phi' \xi_z \in M$, where $\phi' \in N$. Thus $M \cap N$ is normal in $D_G$. Now, if $M \cap N \neq \{Id_{T_G}\}$, by (ii) of the previous part (a), we would have $K_G \subseteq M \cap N$, that is, $K_G = M \cap N$, that is in contrast with the hypothesis $N \subsetneq K_G$. So, it must be necessarily $M \cap N = \{Id_{T_G}\}$. We observe that $M \subseteq K_G$: in fact $K_G$ consists of all the automorphisms of $D_G$ given by the product of an even number of elements of $F_G$, and any element of $M$ is also given by products of an even number of elements of $F_G$. So, we conclude that $M \subseteq K_G$.

We now prove that $M \subseteq K_G$. In fact, $\xi_u \xi_v \xi_u \xi_v = \xi_u \xi_v \xi_u N \xi_u \xi_v \xi_u = \xi_u N \xi_u = M$. We clearly deduce that $MN$ is a subgroup contained in $K_G$. We claim that $MN \triangleleft D_G$. In fact, $\xi_u MN \xi_v = \xi_u \xi_v N \xi_u N \xi_v = N \xi_v N \xi_u = NM$. Hence, by (ii) of previous part (a), we have that $K_G \subseteq MN$, but since $MN \subseteq K_G$, we deduce that $MN = K_G$. So, the thesis follows. \hfill\Box

### 4.2.4 Solvability of $K_G$

In this subsection, we provide a sufficient condition on $T_G$ in order to ensure that $K_G$ is a solvable group when the odd extension $G^5$ is connected and locally dissymmetric.

**Definition 4.2.22.** We say that a delineated subspace $T$ of $\hat{G}$ is **transitively spanned** if for any $u, v \in T$ there exists $\xi_w \in F_T$, for some $w \in T$, such that $\xi_w(u) = v$.

Any subset of a transitively spanned is also transitively spanned.
Lemma 4.2.23. Let $T_1 \subseteq T_2 \subseteq V(\hat{G})$ be two delineated subspaces of $\hat{G}$. If $T_2$ is transitively spanned, then also $T_1$ is.

Proof. Let $u, v \in T_1$. If $u = v$, then by taking $w = u$, it holds $\xi_w(u) = v$ and clearly $w \in T_1$. Let $u \neq v$. Since $u, v \in T_2$, there exists $x \in T_2$ such that $\xi_x(u) = v$. Thus, $x \sim_\hat{G} u$, so $u \circ x = x \circ u$. If we set $w = v \circ u \in T_1$, then we obtain

$$u \circ w = u \circ (v \circ u) = u \circ ((x \circ u) \circ u) = u \circ x = v.$$

Now, we want to find an algebraic condition on the group $K_G$ ensuring that $T_G$ is transitively spanned. In order to do this, we need some preliminary classical notions and results:

Definition 4.2.24. [89] Let $H$ be a group. We denote by $M_H$ the core of $H$, i.e. the maximal normal subgroup of odd order in $H$. Furthermore, we denote by $Z^*(H)$ the subgroup of $H$ containing $M_H$ such that $Z^*(H)/M_H = Z(H/M_H)$, where $Z(H/M_H)$ denotes the center of $H/M_H$.

Theorem 4.2.25. [89] Let $\tau$ be an involution of a group $H$. Then, $\tau \in Z^*(H)$ if and only if for any $g \in H$ the element $g^{-1} \tau^{-1} g \tau$ has odd order.

In next result we provide a necessary and sufficient on $K_G$ condition so that $T_G$ is transitively spanned.

Proposition 4.2.26. Let $G^\#$ be connected and locally dissymmetric. Then, following conditions are equivalent:

(i) For any $u, v \in T_G$, $u$ has odd order with respect to $v$;

(ii) $T_G$ is transitively spanned;

(iii) $K_G$ has odd order.

Proof. (i) $\iff$ (ii): Suppose that for any $u, v \in T_G$, the order of $u$ with respect to $v$ is odd. Fix two vertices $u, v \in T_G$ such that the order of $u$ with respect to $v$ is $n = 2k + 1$. Hence, if we take $u_v^{k+1}$, by (4.21), we have $\xi_{u_v^{k+1}}(u) = v$. Hence $u$ is mapped to $v$. By the arbitrariness of $u$ and $v$, we conclude that $T_G$ is transitively spanned.

On the other hand, assume that $T_G$ is transitively spanned and suppose by contradiction that there are two elements $u, v \in T_G$ such that the order of $u$ with respect to $v$ is even, i.e. $n = 2k$. Let us note that the delineated subspace $T$ consisting of the distinct elements of \{\(v, u, \ldots, u_{v}^{n-1}\)\} is transitively spanned by Lemma 4.2.23. By (4.21), we have $\xi_{u_v^{k}}(u_{v}^{n-j}) = u_{v}^{n-j}$, for any $0 \leq j \leq n - 1$. Now, since the order $n$ coincides with the order of $\xi_{u_v^{k}} \xi_{u_{v}}$ in $K_G$, it is immediate to see that $\xi_{u_v^{k}}(u_{v}^{n-j}) = u_{v}^{n-j} = u_{v}^{j} = \xi_{v}(u_{v}^{j})$. Since the restriction to $T$ of the map $F$ defined in (4.17) is also bijective, we deduce that $\xi_{u_v^{k}} = \xi_{v}$ and $u_v^{k} = v$ and this contradicts the minimality of $n$.

(ii) $\iff$ (iii): Suppose that $K_G$ has odd order and let $u, v \in T_G$. In the proof of Theorem 4.2.19 we showed that the order $m$ of $\xi_{u} \xi_{v}$ in $K_G$ coincides with the order of $u$ with respect to
Let \( m = 2k + 1 \). Then, by (4.21), it follows that \( \xi_{u^{k+1}}(u) = v \). By the arbitrariness of \( u \) and \( v \), we conclude that \( T_G \) is transitively spanned.

On the other hand, let \( T_G \) be transitively spanned, hence, for any \( u, v \in T_G \), there exists \( w \in T_G \) such that \( \xi_w(u) = v \). For any \( \xi_u \in F_G \) we have \( F_G \cap S_{DG}(\xi_u) = \{ \xi_u \} \), where \( S_{DG}(\xi_u) \) denotes the stabilizer of \( \xi_u \) in \( D_G \). Now, since the group spanned by the single element \( \xi_u \) fixes only \( u \in T_G \), we deduce that \( |F_G| = |T_G| = 2k+1 \), for some integer \( k \). Let \( H := \langle \xi_u, \xi_v \rangle \) be the group spanned by two fixed elements for some \( u, v \in T_G \). Let us consider the set \( \xi_H^\rho := \{ \rho^{-1} \xi_u \rho, \rho \in H \} \). The group \( H \) acts by conjugation on \( \xi_H \) and the only element in \( \xi_H \) fixed by \( \xi_u \) is \( \xi_u \), then its cardinality must be odd. In particular, also \( \xi_v \) fixes only \( v \), hence we obtain

\[
\xi_v = (\xi_u \xi_v)^{-j} \xi_u (\xi_u \xi_v)^{j},
\]

for some integer \( j \). But, since \( H \) is dihedral, we deduce that

\[
\xi_v = (\xi_u \xi_v)^{-j} \xi_u (\xi_u \xi_v)^{j} = \xi_u (\xi_u \xi_v)^{2j}, \tag{4.26}
\]

i.e. \( (\xi_u \xi_v)^{2j-1} = Id \). Thus \( \xi_u \xi_v \) has odd order. It is immediate to see that \( [D_G : K_G] = 2 \).

Let \( D_G' \) be the derived subgroup of \( D_G \). We claim that \( K_G = D_G' \). Since \( T_G \) is transitively spanned, for any \( u, v \in T_G \) there exists \( \xi_w \in F_G \) such that \( \xi_v = \xi_{uw} = \xi_w \xi_u \xi_w \), so \( \xi_u \xi_v \) is a commutator in \( D_G \) and \( K_G \subseteq D_G' \). On the other hand, it is easy to prove that \( D_G' \subseteq K_G \) since any commutator in \( D_G \) is an element of \( K_G \). In particular, for any \( \sigma \in D_G \), it results that the element \( \sigma^{-1} \xi_u^{-1} \xi_v \) is in \( K_G \), then it has odd order and, by Theorem 4.2.25, we deduce that \( \xi_u \in Z^*(D_G) \) for any \( u \in T_G \), i.e. \( D_G = Z^*(D_G) \). Therefore, by Definition 4.2.24, \( D_G/M_G \) is an abelian group and since \( D_G' = K_G \) is the smallest normal subgroup such that \( D_G/K_G \) is abelian, we deduce that \( M_G \) contains \( K_G \). Thus, we have shown that \( K_G \) has odd order.

**Theorem 4.2.27.** Let \( G^\# \) be a connected and locally dissymmetric graph such that \( T_G \) is transitively spanned. Then, \( K_G \) is a solvable group.

**Proof.** It is an immediate consequence of Proposition 4.2.26 and of the classical theorem of Feit-Thompson. \( \square \)
Chapter 5

The Pre-Order Induced by Symmetry Relation

By means of the aforementioned $\mathfrak{P}$-maximum partitioner operator, in this chapter we investigate the set map $\Gamma_{\mathfrak{P}}$ as the preorder $\geq_{\mathfrak{P}}$ (reflexive and transitive binary relation) on $\mathcal{P}(\Omega)$ which is exactly the symmetrization of the above indistinguishability relation $\approx_{\mathfrak{P}}$.

5.1 Symmetry Transmission on Pairings

In this section, we focus our attention to the map $\Gamma_{\mathfrak{P}} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U)$ so defined:

$$\Gamma_{\mathfrak{P}}(A, B) := \{ u \in U : [u]_A \subseteq [u]_B \}.$$  \hspace{2cm} (5.1)

Furthermore, in the finite case, we can also define the symmetry transmission measure $\gamma_{\mathfrak{P}}$ as follows

$$\gamma_{\mathfrak{P}}(A, B) := \frac{|\Gamma_{\mathfrak{P}}(A, B)|}{|U|}.$$ \hspace{2cm} (5.2)

Since we have that

$$M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A') \iff \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A') \iff \Gamma_{\mathfrak{P}}(A, A') = U,$$

for all $A, A' \in \mathcal{P}(\Omega)$, we introduce the binary relation $\geq_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ defined by

$$A \geq_{\mathfrak{P}} A' \iff M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(A'),$$ \hspace{2cm} (5.3)

for any $A, A' \in \mathcal{P}(\Omega)$. We say that $A$ is $\mathfrak{P}$-symmetrically finer than $A'$ whenever $A \geq_{\mathfrak{P}} A'$. Let us observe that

$$A \geq_{\mathfrak{P}} B \iff \Gamma_{\mathfrak{P}}(A, B) = U.$$ \hspace{2cm} (5.5)

Some basic properties of the map $\Gamma_{\mathfrak{P}}$ are described in the next proposition.

**Proposition 5.1.1.** Let $A, A', B, B' \in \mathcal{P}(\Omega)$. Then, the following hold:

(i) If $A \subseteq A'$ and $B \subseteq B'$, then

$$\Gamma_{\mathfrak{P}}(A, B) \subseteq \Gamma_{\mathfrak{P}}(A', B) \quad \text{and} \quad \Gamma_{\mathfrak{P}}(A, B) \supseteq \Gamma_{\mathfrak{P}}(A, B').$$

(ii) If $\mathfrak{P}$ is a finite pairing, then $A \geq_{\mathfrak{P}} A' \iff \gamma_{\mathfrak{P}}(A, A') = 1$. 

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Proof. (i): Let \( A \subseteq A' \), then \( \pi_{\mathcal{P}}(A') \leq \pi_{\mathcal{P}}(A) \). This proves that \( \Gamma_{\mathcal{P}}(A, B) \subseteq \Gamma_{\mathcal{P}}(A', B) \). Let \( u \in \Gamma_{\mathcal{P}}(A, B) \), then \([u]_{A} \subseteq [u]_{B} \) but since \([u]_{A'} \subseteq [u]_{A} \), we deduce that \([u]_{A'} \subseteq [u]_{B} \), i.e. \( u \in \Gamma_{\mathcal{P}}(A', B) \).

On the other hand, let \( B \subseteq B' \) and \( u \in \Gamma_{\mathcal{P}}(A, B) \). Then, \([u]_{A} \subseteq [u]_{B'} \subseteq [u]_{B} \), hence \( u \in \Gamma_{\mathcal{P}}(A, B) \).

(ii): It follows immediately by (5.2) and by (5.3).

We can interpret the global symmetry relation \( \approx_{\mathcal{P}} \) as the equivalence relation induced by the preorder \( \geq_{\mathcal{P}} \). Indeed, by (viii) of Theorem 2.2.1 and (5.4) it follows that

\[
A \approx_{\mathcal{P}} A' \iff A \geq_{\mathcal{P}} A' \text{ and } A' \geq_{\mathcal{P}} A,
\]

consequently, if we consider the preordered set \( \mathbb{H}(\mathcal{P}) := (\mathcal{P}(\Omega), \geq_{\mathcal{P}}) \), we have that

\[
\mathbb{H}(\mathcal{P}) / \approx_{\mathcal{P}} = \mathbb{G}(\mathcal{P}) \cong \mathcal{M}(\mathcal{P}). \tag{5.6}
\]

We will show that, in addition to the reflexivity and transitivity, there is a property that uniquely characterize the above binary relation \( \geq_{\mathcal{P}} \). We call union additivity this property, which can be expressed as follows:

\[
A \geq_{\mathcal{P}} A' \text{ and } A \geq_{\mathcal{P}} A'' \implies A \geq_{\mathcal{P}} A' \cup A'' \tag{5.7}
\]

In order to investigate in a deeper way the link between the closure operator \( M_{\mathcal{P}} \) and the preorder \( \geq_{\mathcal{P}} \) we introduce the following ordered subset pair family

\[
\mathcal{G}(\mathcal{P}) := \{(A, B) \in \mathcal{P}(\Omega)^{2} : \Gamma_{\mathcal{P}}(A, B) = U \} \tag{5.8}
\]

and we call any pair \((A, B) \in \mathcal{G}(\mathcal{P})\) a semi-symmetry of \( \mathcal{P} \).

In the next result we establish eight basic properties that are satisfied by the family \( \mathcal{G}(\mathcal{P}) \). Next, in order to show a deep link between closure operators and specific families of ordered pairs of elements, we use seven of this properties as potential properties for generic families \( \mathcal{D} \subseteq \mathcal{P}(\Omega)^{2} \).

**Theorem 5.1.2.** Let \( X, Y, Z, W, T \in \mathcal{P}(\Omega) \). The family \( \mathcal{G}(\mathcal{P}) \) satisfies the following properties:

1. (I1) if \( X \supseteq Y \) then \((X, Y) \in \mathcal{G}(\mathcal{P})\);
2. (I2) if \((X, Y) \in \mathcal{G}(\mathcal{P})\) then \((X \cup Z, Y) \in \mathcal{G}(\mathcal{P})\);
3. (I3) if \((X, Y) \in \mathcal{G}(\mathcal{P})\) and \((X, Z) \in \mathcal{G}(\mathcal{P})\) then \((X, Y \cup Z) \in \mathcal{G}(\mathcal{P})\);
4. (I4) if \((X, Y \cup Z) \in \mathcal{G}(\mathcal{P})\) then \((X, Y) \in \mathcal{G}(\mathcal{P})\);
5. (I5) if \((X, Y) \in \mathcal{G}(\mathcal{P})\) and \((Y, Z) \in \mathcal{G}(\mathcal{P})\) then \((X, Z) \in \mathcal{G}(\mathcal{P})\);
6. (I6) if \((X, Y) \in \mathcal{G}(\mathcal{P})\) and \((Y \cup Z, W) \in \mathcal{G}(\mathcal{P})\) then \((X \cup Z, W) \in \mathcal{G}(\mathcal{P})\);
7. (I7) if \((X, Y \cup Z) \in \mathcal{G}(\mathcal{P})\) and \((Z, T \cup W) \in \mathcal{G}(\mathcal{P})\) then \((X, Y \cup Z \cup T) \in \mathcal{G}(\mathcal{P})\).
Proof. (I1): Since $X \supseteq Y$, for any $\mathcal{P} \in \text{PAIR}(\Omega)$ we have that $\Gamma_\mathcal{P}(X, Y) = U$.
(I2): Let $\mathcal{P} \in \text{PAIR}(\Omega)$ be such that $\Gamma_\mathcal{P}(X, Y) = U$. Since $X \subseteq X \cup Z$, we deduce that $\{u\}_X \subseteq \{u\}_Y \cup \{u\}_Z$ for any $u \in U$, therefore $\Gamma_\mathcal{P}(X \cup Z, Y) = U$.
(I3): Let $\mathcal{P} \in \text{PAIR}(\Omega)$ be such that $\Gamma_\mathcal{P}(X, Y) = \Gamma_\mathcal{P}(X, Z) = U$. Hence, $\{u\}_X \subseteq \{u\}_Y$ and $\{u\}_X \subseteq \{u\}_Z$ for any $u \in U$. Therefore, $\{u\}_X \subseteq \{u\}_Y \cap \{u\}_Z$ for any $u \in U$ or, equivalently, $\pi_\mathcal{P}(X) \leq \pi_\mathcal{P}(Y) \cap \pi_\mathcal{P}(Z)$ and the latter coincides with $\pi_\mathcal{P}(M_\mathcal{P}(Y \cup Z)) = \pi_\mathcal{P}(Y \cup Z)$. Therefore, $\{u\}_X \subseteq \{u\}_{Y \cup Z}$ for any $u \in U$, i.e. $\Gamma_\mathcal{P}(X, Y \cup Z) = U$.
(I4): Let $\mathcal{P} \in \text{PAIR}(\Omega)$ be such that $\Gamma_\mathcal{P}(X, Y \cup Z) = U$. Then, $\{u\}_X \subseteq \{u\}_{Y \cup Z} \subseteq \{u\}_Y$ for any $u \in U$. Therefore $\Gamma_\mathcal{P}(X, Y) = U$.
(I5): Let $\mathcal{P} \in \text{PAIR}(\Omega)$ be such that $\Gamma_\mathcal{P}(X, Y) = \Gamma_\mathcal{P}(Y, Z) = U$. Then, $\{u\}_X \subseteq \{u\}_Y \subseteq \{u\}_Z$ for any $u \in U$.
(I6): Let $\mathcal{P} \in \text{PAIR}(\Omega)$ be such that $\Gamma_\mathcal{P}(X, Y) = \Gamma_\mathcal{P}(Y \cup Z, W) = U$. Then, $\{u\}_X \subseteq \{u\}_Y$ and $\{u\}_{Y \cup Z} \subseteq \{u\}_W$ for any $u \in U$. Thus, we have that $\{u\}_{X \cup Z} = \{u\}_X \cup \{u\}_Z$ and $\{u\}_{Y \cup Z} = \{u\}_Y \cup \{u\}_Z$ for any $u \in U$. Since $\{u\}_X \cup \{u\}_Z \subseteq \{u\}_Y \cup \{u\}_Z = \{u\}_{Y \cup Z} \subseteq \{u\}_W$, so $\Gamma_\mathcal{P}(X \cup Z, W) = U$.
(I7): Let $\mathcal{P} \in \text{PAIR}(\Omega)$ be such that $\Gamma_\mathcal{P}(X, Y \cup Z) = \Gamma_\mathcal{P}(Z, T \cup W) = U$. This implies that, for any $u \in U$, the following holds:

$$\{u\}_X \subseteq \{u\}_{Y \cup Z} \subseteq \{u\}_Z \subseteq \{u\}_{T \cup W} \subseteq \{u\}_T.$$ 

In other terms, if $u \equiv_X u'$ then $u \equiv_{Y \cup Z} u'$ and $u \equiv_T u'$ and this is equivalent to require that $F(u, a) = F(u, a')$ for any $a \in Y \cup Z \cup T$, i.e. $u \equiv_{Y \cup Z \cup T} u'$. By the arbitrariness of $u \in U$, we conclude that $\Gamma_\mathcal{P}(X, Y \cup Z \cup T) = U$.

5.2 Union Additive Relations

We consider now the conditions established in Proposition 5.1.2 as specific properties that can be satisfied by suitable families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$. To this regard, let $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ be a given family on $\Omega$.

Some Conditions for a Generic Family $\mathcal{D}$ on $\mathcal{P}(\Omega)^2$. Let $X, Y, Z, W, T \in \mathcal{P}(\Omega)$. We give the following possible axioms for $\mathcal{D}$:

(D1) if $X \supseteq Y$ then $(X, Y) \in \mathcal{D}$;
(D2) if $(X, Y) \in \mathcal{D}$ then $(X \cup Z, Y) \in \mathcal{D}$;
(D3) if $(X, Y) \in \mathcal{D}$ and $(X, Z) \in \mathcal{D}$ then $(X, Y \cup Z) \in \mathcal{D}$;
(D4) if $(X, Y \cup Z) \in \mathcal{D}$ then $(X, Y) \in \mathcal{D}$;
(D5) if $(X, Y) \in \mathcal{D}$ and $(Y, Z) \in \mathcal{D}$ then $(X, Z) \in \mathcal{D}$;
(D6) if $(X, Y) \in \mathcal{D}$ and $(Y \cup Z, W) \in \mathcal{D}$ then $(X \cup Z, W) \in \mathcal{D}$;
(D7) if $(X, Y \cup Z) \in \mathcal{D}$ and $(Z, T \cup W) \in \mathcal{D}$ then $(X, Y \cup Z \cup T) \in \mathcal{D}$;
(D8) $(X, Y) \in \mathcal{D}$ if and only if $(M_\mathcal{P}(X), M_\mathcal{P}(Y)) \in \mathcal{D}$.
We have then the following result.

**Proposition 5.2.1.** The following conditions are equivalent:

(i) $\mathcal{D}$ satisfies the properties (D1), (D3) and (D5);

(ii) $\mathcal{D}$ satisfies the properties (D1), (D2) and (D6);

(iii) $\mathcal{D}$ satisfies the properties (D1), (D4) and (D7);

(iv) $\mathcal{D}$ satisfies all properties (D1)–(D7).

**Proof.** (i) $\implies$ (ii): We have to derive the properties (D1), (D2) and (D6) starting from (D1), (D3) and (D5). For, let us consider $(X, Y) \in \mathcal{D}$. By (D1), also $(X \cup Z, X) \in \mathcal{D}$ and, by (D5), it follows that $(X \cup Z, X), (X, Y) \in \mathcal{D}$ implies $(X \cup Z, Y) \in \mathcal{D}$.

On the other hand, let us prove that (D6) derives by (D1), (D2), (D3), (D5). To this regard, assume that $(X, Y), (Y \cup Z, W) \in \mathcal{D}$. By (D2), it results that $(X \cup Z, Y) \in \mathcal{D}$ and, by (D1), this turns out to be true also for the pairs $(Z, Z)$ and $(X \cup Z, Z)$. Thus, by (D3) applied to the pairs $(X \cup Z, Y), (X \cup Z, Z)$, we have that $(X \cup Z, Y \cup Z) \in \mathcal{D}$ and, by (D5) applied to the pairs $(X \cup Z, Y \cup Z), (Y \cup Z, W)$ we have that $(X \cup Z, W) \in \mathcal{D}$. So, the claim has been shown.

(ii) $\implies$ (iii): Let us derive the properties (D1), (D4) and (D7) by using (D1), (D2) and (D6). To this regard, let $(X, Y \cup Z) \in \mathcal{D}$. By (D1), $(Y \cup Z, Y) \in \mathcal{D}$. Now, by (D6) applied to the pairs $(X, Y \cup Z), ((Y \cup Z) \cup \emptyset, Y)$ we have that $(X, Y) \in \mathcal{D}$. This shows that (D4) can be derived by (D1), (D2) and (D6).

We now prove the claim for (D7). Let now assume that $(X, Y \cup Z), (Z, T \cup W) \in \mathcal{D}$. By (D2), $(Y \cup Z, T \cup W) \in \mathcal{D}$. Moreover, applying (D6) to the pairs $(Y \cup Z, T \cup W), (T \cup W, T)$, we have that $(Y \cup Z, T) \in \mathcal{D}$. Again, we apply (D6) to the pair $(X, Y \cup Z)(Y \cup Z, T)$, so $(X, T) \in \mathcal{D}$ and, by (D6) applied to the pairs $(X, Y \cup Z, Y \cup Z \cup T, Y \cup Z \cup T)$, we deduce $(X \cup T, Y \cup Z \cup T) \in \mathcal{D}$. Finally, starting from the pairs $(X, T), (X \cup T, Y \cup Z \cup T)$, by (D6) we obtain that $(X, Y \cup Z \cup T)$.

(iii) $\implies$ (iv): Let us derive the properties (D2), (D3), (D5) and (D6) by using (D1), (D4) and (D7). For, let $(X, Y) \in \mathcal{D}$. We will prove (D2). By (D1), $(X \cup Z, X \cup Z) \in \mathcal{D}$, so, by applying (D7) to the pairs $(X \cup Z, X \cup Z), (X, Y)$ we have that $(X \cup Z, X \cup Y \cup Z) \in \mathcal{D}$. Finally, by applying (D4) to the latter pair, we deduce that $(X \cup Z, Y) \in \mathcal{D}$.

We now show that (D6) can be derived by (D1), (D4) and (D7). Let $(X, Y), (Y \cup Z, W) \in \mathcal{D}$. By (D1), $(X \cup Z, X \cup Z) \in \mathcal{D}$ so, by (D7) applied to the pairs $(X \cup Z, X \cup Z), (X, Y)$ we deduce that also $(X \cup Z, X \cup Y \cup Z) \in \mathcal{D}$. Again by (D7), this time applied to $(X \cup Z, X \cup Y \cup Z)$ and $(Y \cup Z, W)$, we have that $(X \cup Z, X \cup Y \cup Z \cup W) \in \mathcal{D}$. Then, by (D4), $(X \cup Z, W) \in \mathcal{D}$. Furthermore, (D5) is nothing but (D6) with $Z = \emptyset$.

We prove the claim for (D3). We will use (D1), (D2) and (D6) that have been already derived by (D1), (D4) and (D7). Assume $(X, Y), (X, Z) \in \mathcal{D}$. By (D1), we also have $(Y \cup Z, Y \cup Z)$ and, by (D6) applied to the pairs $(X, Y), (Y \cup Z, Y \cup Z)$, we deduce that $(X \cup Z, Y \cup Z) \in \mathcal{D}$. By applying again (D6) to the pairs $(X, Z), (X \cup Z, Y \cup Z)$, it results that $(X, Y \cup Z) \in \mathcal{D}$. This shows the thesis.

(iv) $\implies$ (i): Obvious. 

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Definition 5.2.2. If $\mathcal{D}$ satisfies one of the equivalent conditions established in Proposition 5.2.1 we say that $\mathcal{D}$ is a union additive family on $\Omega$. We denote by $UAF(\Omega)$ the set of all union additive families on $\Omega$.

Since any $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ can be uniquely identified with the binary relation that it induces on $\mathcal{P}(\Omega)$, the conditions established in Proposition 5.1.2 can also be expressed in terms of conditions on binary relations on $\mathcal{P}(\Omega)$ as follows.

Some Conditions for a Generic Binary Relation $\geq$ on $\mathcal{P}(\Omega)$. Let $\geq$ be a binary relation on $\mathcal{P}(\Omega)$ and $X,Y,Z,W,T \in \mathcal{P}(\Omega)$. We give the following possible axioms for $\geq$:

(R1) if $X \supseteq Y$ then $X \geq Y$;
(R2) if $X \geq Y$ then $X \cup Z \geq Y$;
(R3) if $X \geq Y$ and $X \geq Z$ then $X \geq Y \cup Z$;
(R4) if $X \geq Y \cup Z$ then $X \geq Y$;
(R5) if $X \geq Y$ and $Y \geq Z$ then $X \geq Z$;
(R6) if $X \geq Y$ and $Y \cup Z \geq W$ then $X \cup Z \geq W$;
(R7) if $X \geq Y \cup Z$ and $Z \geq T \cup W$ then $X \geq Y \cup Z \cup T$;
(R8) $X \geq Y$ if and only if $M_\mathcal{P}(X) \geq M_\mathcal{P}(Y)$.

Then, Proposition 5.2.1 can be reformulated in terms of equivalences between conditions on binary relations on $\Omega$ as follows.

Proposition 5.2.3. The following conditions are equivalent for any binary relation $\geq$ on $\Omega$.

(i) $\geq$ satisfies the properties (R1), (R3) and (R5);
(ii) $\geq$ satisfies the properties (R1), (R2) and (R6);
(iii) $\geq$ satisfies the properties (R1), (R4) and (R7);
(iv) $\geq$ satisfies all properties (R1)–(R7).

Definition 5.2.4. We call union additive relation on $\Omega$ a binary relation $\geq$ on $\mathcal{P}(\Omega)$ that satisfies one of the equivalent conditions established in Proposition 5.2.3. We denote by $UAR(\Omega)$ the set of all union additive relations on $\Omega$.

It is clear that the notions of union additive family and union additive relation are equivalent. In fact, if $\geq$ is a union additive relation on $\Omega$ then $\mathcal{D}_{\geq} := \{(X,Y) \in \mathcal{P}(\Omega)^2 : X \geq Y\}$ is a union additive family on $\Omega$. On the other hand, if $\mathcal{D}$ is a union additive family on $\Omega$, the binary relation $X \geq_D Y : \iff (X,Y) \in \mathcal{D}$ is a union additive relation on $\Omega$. Moreover, the maps $\mathcal{D} \in UAF(\Omega) \mapsto \geq_{\mathcal{D}} \in UAR(\Omega)$ and $\geq \in UAR(\Omega) \mapsto \mathcal{D}_{\geq} \in UAF(\Omega)$ are inverses each other. So that, in what follows we use the terms union additive relation and union additive.
family in equivalent way. Thus, by Proposition 5.1.2 it follows that \( \geq \mathfrak{P} \) is a union additive relation on \( \Omega \), or, equivalently, \( \mathfrak{U}(\mathfrak{P}) \) is a union additive family on \( \Omega \).

In the following result we establish some first basic properties of the union additive relations.

**Proposition 5.2.5.** (i) If \( \{ D_i : i \in I \} \subseteq UAF(\Omega) \), then \( \bigcap_{i \in I} D_i \in UAF(\Omega) \).

(ii) Let \( \geq \in UAR(\Omega) \) and \( A, B, C, D \in \mathcal{P}(\Omega) \). Then, the following facts hold:

(a) If \( A \geq B, C \supseteq A \) and \( B \supseteq D \), then \( C \supseteq D \);

(b) If \( A \geq B \) and \( C \supseteq D \), then \( A \cup C \supseteq B \cup D \).

**Proof.** (i): Let \( Y \subseteq X \in \mathcal{P}(\Omega) \). Then, \( (X,Y) \in D \) for any \( D \in UAF(\Omega) \), so \( (X,Y) \in \bigcap_{i \in I} D_i \). Let \( (X,Y), (Y,Z) \in \bigcap_{i \in I} D_i \). Then, \( (X,Y), (Y,Z) \in D_i \) for any \( i \in I \). Therefore \( (X,Z) \in D_i \) for any \( i \in I \), i.e. \( (X,Z) \in \bigcap_{i \in I} D_i \). Finally, let \( (X,Y), (X,Z) \in \bigcap_{i \in I} D_i \), hence \( (X,Y), (X,Z) \in D_i \) for any \( i \in I \). So, \( (X,Y \cup Z) \in D_i \) for any \( i \in I \), i.e. \( (X,Y \cup Z) \in \bigcap_{i \in I} D_i \). This shows that \( \bigcap_{i \in I} D_i \in UAF(\Omega) \).

(ii): (a): By (R1), it follows that \( C \supseteq A \) and \( B \supseteq D \). Moreover, by applying (R5) on the three conditions \( C \supseteq A, A \supseteq B \) and \( B \supseteq D \), we conclude that \( C \supseteq D \).

(b): Let us observe that \( A \cup C \supseteq A \supseteq B \) and \( A \cup C \supseteq C \supseteq D \). Moreover, we have that \( B \cup D \supseteq B \) by (R1). So, by applying (R3) and then (R5), we conclude that \( A \cup C \supseteq B \cup D \). \( \Box \)

It is natural to define the smallest union additive family on \( \Omega \) containing \( D \) by setting

\[
D^+ := \bigcap \{ \mathcal{E} \in UAF(\Omega) : D \subseteq \mathcal{E} \} \tag{5.9}
\]

Then, the following result holds.

**Proposition 5.2.6.** \( D^+ \) is the smallest union additive family on \( \Omega \) containing \( D \) as a subfamily and the set operator \( + : \mathcal{D} \in \mathcal{P}(\Omega)^2 \mapsto D^+ \in \mathcal{P}(\Omega)^2 \) is a closure operator on \( \mathcal{P}(\Omega)^2 \).

**Proof.** Let us observe that the set \( \{ \mathcal{E} \in UAF(\Omega) : D \subseteq \mathcal{E} \} \) is non-empty since \( \mathcal{P}(\Omega)^2 \subseteq UAF(\Omega) \). Thus, by (i) of Proposition 5.2.5, it follows that \( D^+ \in UAF(\Omega) \). Moreover, it is obvious that \( D^+ \) is the smallest union additive family on \( \Omega \) containing \( D \).

We now prove that the set operator \( + : D \in \mathcal{P}(\Omega)^2 \mapsto D^+ \in \mathcal{P}(\Omega)^2 \) is a closure operator on \( \mathcal{P}(\Omega)^2 \). Let \( D \in \mathcal{P}(\Omega)^2 \). Then, \( D \subseteq D^+ \). On the other hand, if \( D \subseteq F \), it is immediate to see that \( D^+ \subseteq F^+ \) since the union additive families containing \( F \) also contain \( D \). Finally, since \( D^+ \) is a union additive family, we have \( (D^+)^+ = D^+ \) and the claim has been proved. \( \Box \)

We call \( D^+ \) the union additive envelope of \( D \). If \( D, D' \subseteq \mathcal{P}(\Omega)^2 \), we say that \( D \) and \( D' \) are union additive equivalent (denoted by \( D \simeq D' \)) if \( D^+ = (D')^+ \).
5.2.1 Union Additive Families and Closure Operators

In this subsection, given a fixed finite set $\Omega$, we show (Theorem 5.2.8) that the notions of union additive family and closure operator are equivalent. To this aim we consider the set operator $c_D : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ defined by

$$c_D(X) := \bigcup \{Y \in \mathcal{P}(\Omega) : (X, Y) \in D\}. \quad (5.10)$$

So that, for any $z \in \Omega$ we have that

$$z \in c_D(X) \iff \exists Z \in \mathcal{P}(\Omega) : z \in Z \text{ and } (X, Z) \in D \quad (5.11)$$

When $D$ satisfies the conditions $(D1)$ and $(D3)$, another equivalent way to characterize the subset $c_D(X)$ is the following.

**Proposition 5.2.7.** If $D$ satisfies the conditions $(D1)$ and $(D3)$, then $c_D(X)$ is the largest subset $W$ of $\Omega$ such that $(X, W) \in D$.

**Proof.** By $(D1)$, the subset family $\mathcal{F} := \{Y \in \mathcal{P}(\Omega) : (X, Y) \in D\}$ is non-empty because it contains at least $X$. On the other hand, by means of repeated applications of $(D3)$ we deduce that $W := \bigcup \mathcal{F} \in \mathcal{F}$. Since $c_D(X) = W$, the thesis follows.

On the other hand, for any set operator $\sigma \in OP(\Omega)$ we also consider the subset family

$$\mathcal{C}_\sigma := \{(X, Y) \in \mathcal{P}(\Omega)^2 : \sigma(X) \supseteq Y\}. \quad (5.12)$$

Then, we obtain the following result.

**Theorem 5.2.8.** (i) If $D \in UAF(\Omega)$ then $c_D \in CLOP(\Omega)$.

(ii) If $\sigma \in CLOP(\Omega)$ then $\mathcal{C}_\sigma \in UAF(\Omega)$.

(iii) The map $D \in UAF(\Omega) \mapsto c_D \in CLOP(\Omega)$ is bijective and the map $\sigma \in CLOP(\Omega) \mapsto \mathcal{C}_\sigma$ is its inverse. That is, if $D \in UAF(\Omega)$ and $\sigma \in CLOP(\Omega)$ we have that $\mathcal{C}_c_D = D$ and $c_{\mathcal{C}_\sigma} = \sigma$.

**Proof.** (i): Let $D \in UAF(\Omega)$ and $X \in \mathcal{P}(\Omega)$. By $(D1)$ we have $(X, X) \in D$, so that $X \subseteq c_D(X)$.

Let now $Y \in \mathcal{P}(\Omega)$ and $X \subseteq Y$. Again by $(D1)$ we have $(Y, X) \in D$. Let now $a \in c_D(X)$. By $(5.11)$ there exists $Z \in \mathcal{P}(\Omega)$ such that $(X, Z) \in D$ and $a \in Z$. Then, by $(D5)$ we deduce that $(Y, Z) \in D$, therefore by $(5.11)$ it follows that $a \in c_D(Y)$. Hence $c_D(X) \subseteq c_D(Y)$.

In order to show that $c_D(c_D(X)) = c_D(X)$, let $W := c_D(X)$ and $a \in c_D(W)$. By $(5.11)$ there exists $Z \in \mathcal{P}(\Omega)$ such that $(W, Z) \in D$ and $a \in Z$. By Proposition 5.2.7 we have that $(X, W) \in D$, therefore, by $(D5)$, it follows that $(X, Z) \in D$. Then, by $(5.11)$ we have that $a \in c_D(X)$. This shows that $c_D(W) \subseteq W$. Finally, by the first part of this proof, we also have $W \subseteq c_D(W)$. Hence $c_D(W) = W$.

This proves that $c_D$ is a closure operator on $\Omega$.

(ii): If $Y \subseteq X \in \mathcal{P}(\Omega)$, then $Y \subseteq X \subseteq \sigma(X)$, so $(X, Y) \in \mathcal{C}_\sigma$. Let now $(X, Y), (Y, Z) \in \mathcal{C}_\sigma$, then $Z \subseteq \sigma(Y)$ and $Y \subseteq \sigma(X)$. Moreover, we have $Z \subseteq \sigma(Y) \subseteq \sigma(\sigma(X)) = \sigma(X)$. Finally, if $(X, Y), (X, Z) \in \mathcal{C}_\sigma$, it results that $Y \subseteq \sigma(X)$ and $Z \subseteq \sigma(X)$, so $Y \cup Z \subseteq \sigma(X)$, i.e. $(X, Y \cup Z) \in \mathcal{C}_\sigma$. Thus we showed that $\mathcal{C}_\sigma \in UAF(\Omega)$. 

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(iii): Let us prove that $C_{c\sigma} = D$. Let $(X, Y) \in D$, then clearly $Y \subseteq c_D(X)$ by definition of $c_D(X)$, i.e. $(X, Y) \in C_{c\sigma}$. On the other hand, if $(X, Y) \in C_{c\sigma}$, it is obvious, again by definition of $c_D(X)$, that $(X, Y) \in D$. Vice versa, let us prove that $C_{c\sigma} = \sigma$. Clearly, if $X \in P(\Omega)$ is a fixed subset, we have

$$c_{c\sigma}(X) = \bigcup \{Y \in P(\Omega) : Y \subseteq \sigma(X)\} = \sigma(X),$$

and the claim follows. \qed

If $\sigma \in CLOP(\Omega)$ and $D \in UAF(\Omega)$ are such that $D = C_{c\sigma}$ (and therefore equivalently $\sigma = c_D$, by Theorem 5.2.8), we say that $C_{\sigma}$ is the union additive family induced by $\sigma$ and $c_D$ is the closure operator induced by $D$.

As a direct consequence, we prove that $\Psi(\Psi)$ coincides exactly with the union additive family on $\Omega$ induced by the closure operator $M_\Psi$.

**Proposition 5.2.9.** $C_{M_\Psi} = \Psi(\Psi)$ and $c_M(\Psi) = M_\Psi$. That is, $M_\Psi$ is the closure operator on $\Omega$ induced by $\Psi(\Psi)$ and $\Psi(\Psi)$ is the union additive family on $\Omega$ induced by $M_\Psi$.

**Proof.** Let $(X, Y) \in C_{M_\Psi}$. Then, $Y \subseteq M_\Psi(X)$, so we have $\pi_\Psi(X) = \pi_\Psi(M_\Psi(X)) \leq \pi_\Psi(Y)$. Therefore, $[u]_X \subseteq [u]_Y$ for any $u \in U$ or, equivalently, $\Gamma_\Psi(X, Y) = U$. This implies that $C_{M_\Psi} \subseteq \Psi(\Psi)$. Vice versa, let us suppose that $(X, Y) \in \Psi(\Psi)$, so $\Gamma_\Psi(X, Y) = U$. Therefore, $[u]_X = [u]_{M_\Psi(X)} \subseteq [u]_Y = [u]_{M_\Psi(Y)}$, i.e. $\pi_\Psi(M_\Psi(X)) \leq \pi_\Psi(M_\Psi(Y))$ and this is equivalent to require that $M_\Psi(Y) \subseteq M_\Psi(X)$ hence, a fortiori, $Y \subseteq M_\Psi(X)$. This shows that $C_{M_\Psi} = \Psi(\Psi)$. On the other hand, by (5.10), it is immediate to verify that $c_M(\Psi) = M_\Psi$ \qed

### 5.3 Union Additive Families from an Abstract Mathematical Perspective

Starting now from $D$, we inductively define three sequences of different subset ordered pair families and next we show that all their unions coincide with the above additive envelope $D^+$. In this way, we provide a new interpretation for (D1) – (D7). First of all, let us set $D^+ := D \cup \{(X, Y) \in P(\Omega)^2 : Y \subseteq X\}$.

**I)** Set $D^+_0 := \hat{D}$. Moreover, if $k \geq 1$, we inductively define $D^+_k$ as follows. A subset ordered pair $(X', Y')$ belongs to $D^+_k$ if one of the following holds:

(A1) there exist $Y, Z \in P(\Omega)$ such that $(X', Y), (X', Z) \in D^+_{k-1}$ and $Y' = Y \cup Z$;

(A2) there exists $Y \in P(\Omega)$ such that $(X', Y), (Y, Y') \in D^+_{k-1}$.

**II)** Set $D^8_0 := \hat{D}$. Moreover, if $k \geq 1$, we inductively define $D^8_k$ as follows. A subset ordered pair $(X', Y')$ belongs to $D^8_k$ if one of the following holds:

(B1) there exists $X \subseteq X'$ such that $(X, Y') \in D^8_{k-1}$;
(B2) there exist \(X, Y, Z \in \mathcal{P}(\Omega)\) such that \((X, Y) \in \mathcal{D}_k^\delta\), \((Y \cup Z, Y') \in \mathcal{D}_{k-1}^\delta\) and \(X' = X \cup Z\).

III) Set \(\mathcal{D}^*_0 := \mathcal{D}\). Moreover, if \(k \geq 1\), we inductively define \(\mathcal{D}^*_k\) as follows. A subset ordered pair \((X', Y')\) belongs to \(\mathcal{D}^*_k\) if one of the following holds:

(C1) there exists \(Z \in \mathcal{P}(\Omega)\) such that \((X', Y' \cup Z) \in \mathcal{D}_{k-1}^\delta\);

(C2) there exist \(Y, Z, W, T \in \mathcal{P}(\Omega)\) such that \((X', Y \cup Z), (Z, T \cup W) \in \mathcal{D}_{k-1}^\delta\) and \(Y' = Y \cup Z \cup T\).

We have then the following result.

**Theorem 5.3.1.** We have that

\[
\bigcup_{k \geq 0} \mathcal{D}_k^* = \bigcup_{k \geq 0} \mathcal{D}_k^\delta = \bigcup_{k \geq 0} \mathcal{D}_k = \mathcal{D}^+.
\]

**Proof.** Let us prove that \(\bigcup_{k \geq 0} \mathcal{D}_k^* \in \mathcal{UAF}(\Omega)\). Let \(X \supseteq Y\). Then, \((X, Y) \in \mathcal{D}_0^*\) and so \((X, Y) \in \bigcup_{k \geq 0} \mathcal{D}_k^*\). By (A1), it can be readily proved that \(\mathcal{D}_k^* \subseteq \mathcal{D}_k^*\) for any \(k \in \mathbb{N}\). Hence, if \((X, Y), (X, Z) \in \bigcup_{k \geq 0} \mathcal{D}_k^*\), then there always exists an index \(j\) such that \((X, Y), (X, Z) \in \mathcal{D}_j^*\). Thus \((X, Y \cup Z) \in \mathcal{D}_{j+1}^*\) hence \((X, Y \cup Z) \in \bigcup_{k \geq j} \mathcal{D}_k^*\). Finally, if \((X, Y), (Y, Z) \in \bigcup_{k \geq 0} \mathcal{D}_k^*\), there exists an index \(i\) such that \((X, Y), (Y, Z) \in \mathcal{D}_i^*\). Thus, by (A2), \((X, Z) \in \bigcup_{k \geq 0} \mathcal{D}_k^*\). This shows that \(\bigcup_{k \geq 0} \mathcal{D}_k^* \in \mathcal{UAF}(\Omega)\). Moreover, \(\mathcal{D} \subseteq \bigcup_{k \geq 0} \mathcal{D}_k^*\) and it is immediate to see that any \((X, Y) \in \bigcup_{k \geq 0} \mathcal{D}_k^*\) also belongs to any \(\mathcal{E} \in \mathcal{UAF}(\Omega)\) containing \(\mathcal{D}\). So \(\mathcal{D}^+ = \bigcup_{k \geq 0} \mathcal{D}_k^*\).

Let us prove now that
\[
\bigcup_{k \geq 0} \mathcal{D}_k^* = \bigcup_{k \geq 0} \mathcal{D}_k^\delta.
\]

Obviously, \(\mathcal{D}_k^* \subseteq \bigcup_{k \geq 0} \mathcal{D}_k^\delta\). Fix an integer \(n\) and assume that \(\mathcal{D}_i^* \subseteq \bigcup_{k \geq 0} \mathcal{D}_k^\delta\) for any \(l \leq n - 1\). We will prove that \(\mathcal{D}_n^* \subseteq \bigcup_{k \geq 0} \mathcal{D}_k^\delta\). For, let \((X', Y') \in \mathcal{D}_n^*\). If (A1) holds, there exist \(Y, Z \in \mathcal{P}(\Omega)\) such that \((X', Y), (X', Z) \in \mathcal{D}_n^\delta\) and \(Y' = Y \cup Z\). By the inductive hypothesis, \((X', Y) \in \mathcal{D}_i^\delta\) and \((X', Z) \in \mathcal{D}_j^\delta\), for two integers \(i\) and \(j\). Since \(\mathcal{D}_k^\delta \subseteq \mathcal{D}_{k+1}^\delta\) for any \(k \in \mathbb{N}\), then \((X', Y), (X', Z) \in \mathcal{D}_d^\delta\) for some \(\mathcal{D}_d^\delta\). Moreover, since \((Y \cup Z, Y \cup Z) \in \mathcal{D}_0^\delta\), it also belongs to \(\mathcal{D}_d^\delta\). By (B2), the pair \((X' \cup Z, Y \cup Z) \in \mathcal{D}_{j+1}^\delta\) can be obtained starting from the pairs \((X', Y)\) and \((Y \cup Z, Y \cup Z)\) and, again by (B2), we obtain the pair \((X', Y') = (X', Y') \in \mathcal{D}_{j+2}^\delta\) from to the pairs \((X', Z)\) and \((X' \cup Z, Y \cup Z)\). Thus, \((X', Y') \in \bigcup_{k \geq 0} \mathcal{D}_k^\delta\). If (A2) holds, then there exists \(Y \in \mathcal{P}(\Omega)\) such that \((X', Y), (Y, Y') \in \mathcal{D}_{n-1}^\delta\). Hence, by inductive hypothesis and by the previous remark, it results that \((X', Y), (Y, Y') \in \mathcal{D}_j^\delta\) for some integer \(j\). By (B2), we obtain the pair \((X', Y') \in \mathcal{D}_{j+1}^\delta\) from the pairs \((X', Y), (Y, Y')\), so \((X', Y') \in \bigcup_{k \geq 0} \mathcal{D}_k^\delta\). This shows that \(\bigcup_{k \geq 0} \mathcal{D}_k^* \subseteq \bigcup_{k \geq 0} \mathcal{D}_k^\delta\).
Moreover, again by (B1) holds, then there exists $X \subseteq X'$ such that $(X, Y') \in D^i_{n-1}$ and, by the inductive hypothesis, $(X, Y') \in D^i_k$ for some integer $i$. Since $(X', X) \in D^i_0$, it also belongs to $D^i_i$. By (A2), the pairs $(X', X)$ and $(X, Y')$ yield $(X', Y') \in D^i_{i+1}$, so $(X', Y') \in \bigcup_{k \geq 0} D^i_k$. If (B2) holds, then there exist $X, Y, Z \in \mathcal{P}(\Omega)$ such that $(X, Y), (Y \cup Z, Y') \in D^i_{k-1}$ and $X' = X$. By the inductive hypothesis, $(X, Y) \in D^i_i$ and $(Y \cup Z, Y') \in D^i_{i'}$, for some integers $i$ and $i'$. In particular, there exists $j$ such that $(X, Y) \in D^j_k$. Moreover, since $(X \cup Z, X), (X \cup Z, Z) \in D^i_0$, they belong to $D^i_{i'}$ for any integer $i$. Now, by (A2), the pair $(X \cup Z, Y) \in D^i_{i+1}$ can be obtained from the pairs $(X \cup Z, X)$ and $(X, Y)$ and, by (A1), we obtain the pair $(X \cup Z, Y \cup Z) \in D^i_{i+2}$ from the pairs $(X \cup Z, Y)$ and $(X \cup Z, Z)$. Finally, again by (A2), the pairs $(X \cup Z, Y \cup Z)$ and $(Y \cup Z, Y')$ yield $(X \cup Z, Y') = (X', Y') \in D^i_{i+3}$, thus $(X', Y') \in \bigcup_{k \geq 0} D^i_k$. This implies that $\bigcup_{k \geq 0} D^i_k = \bigcup_{k \geq 0} D^i_k$. Let us prove that

$$\bigcup_{k \geq 0} D^i_k = \bigcup_{k \geq 0} D^i_k.$$  

Obviously, $D^i_0 \subseteq \bigcup_{k \geq 0} D^i_k$. Fix an integer $n$ and assume that $D^i_l \subseteq \bigcup_{k \geq 0} D^i_k$ for any $l \leq n - 1$. We will prove that $D^i_n \subseteq \bigcup_{k \geq 0} D^i_k$. For, let $(X', Y') \in D^i_n$. If (B1) holds, then there exists $X \subseteq X'$ such that $(X', Y') \in D^i_{n-1}$. By the inductive hypothesis, $(X, Y') \in D^i_l$ for some integer $i$. Since $(X', X) \in D^i_0$, then $(X', X) \in D^i_l$. Let $Z = X' \setminus X$. By (C2), by starting from the pairs $(X', X \cup Z), (X, Y')$ we obtain that $(X', X \cup Z \cup Y') \in D^i_{l+1}$. Moreover, by (C1), it follows that $(X', Y') \in D^i_{l+2}$, so $(X', Y') \in \bigcup_{k \geq 0} D^i_k$. If (B2) holds, then there exist $X, Y, Z \in \mathcal{P}(\Omega)$ such that $(X, Y), (Y \cup Z, Y') \in D^i_{k-1}$ and $X' = X \cup Z$. By the inductive hypothesis, $(X, Y) \in D^i_l$ and $(Y \cup Z, W) \in D^i_{i'}$, for some integers $i, i'$. Since $D^i_k \subseteq D^i_{k+1}$ for any $k \in \mathbb{N}$, we have that $(X, Y) \in D^i_j$ for some $j \in \mathbb{N}$. Moreover, since $(X \cup Z, X \cup Z) \in D^i_0$, then $(X \cup Z, X \cup Z) \in D^i_j$. Then, by (C2), the pairs $(X \cup Z, X \cup Z)$ and $(X, Y')$ yield $(X \cup Z, X \cup Y \cup Z) \in D^i_{j+1}$. Moreover, again by (C2), we obtain that $(X \cup Z, X \cup Y \cup Z \cup W) \in D^i_{j+2}$ from the pairs $(X \cup Z, X \cup Y \cup Z)$ and $(Y \cup Z, W)$. Finally, by (C1), by starting from $(Y \cup Z, W)$, we obtain that $(X \cup Z, W) = (X', Y') \in D^i_{j+3}$, so $(X', Y') \in \bigcup_{k \geq 0} D^i_k$. This shows that $\bigcup_{k \geq 0} D^i_k \subseteq \bigcup_{k \geq 0} D^i_k$.

On the other hand, $D^i_0 \subseteq \bigcup_{k \geq 0} D^i_k$. Fix an integer $n$ and suppose that $D^i_l \subseteq \bigcup_{k \geq 0} D^i_k$ for any $l \leq n - 1$. Let $(X', Y') \in D^i_n$. If (C1) holds, then there exists $Z \in \mathcal{P}(\Omega)$ such that $(X', Y' \cup Z) \in D^i_{n-1}$ and, by the inductive hypothesis, $(X', Y' \cup Z) \in D^i_l$ for some integer $i$. Since $(Y' \cup Z, Y') \in D^i_0$, it is contained in $D^i_l$, therefore, from the pairs $(X', Y' \cup Z)$ and $(Y' \cup Z, Y')$ and by using (B2), we obtain that $(X', Y') \in D^i_{l+1}$, thus $(X', Y') \in \bigcup_{k \geq 0} D^i_k$. If (C2) holds, then there exist $Y, Z, W, T \in \mathcal{P}(\Omega)$ such that $(X', Y \cup Z), (Z, T \cup W) \in D^i_{n-1}$.
and $Y' = Y \cup Z \cup T$. By the inductive hypothesis we have that $(X', Y \cup Z) \in D_i^k$ and $(Y \cup Z, T \cup W) \in D_j^k$ for some integers $i$ and $i'$. Since $D_i^k \subseteq D_j^k$ whenever $i < j$, we can assume that $(X', Y \cup Z) \in D_i^k$. By applying $(B1)$, we obtain that $(Y \cup Z, T \cup W) \in D_{j+1}^k$ from the pair $(Z, T \cup W)$. Moreover, the pair $(T \cup W, T) \in D_0^k$ also belongs to $D_{j+1}^k$. Hence, by $(B2)$ we obtain that $(Y \cup Z, T) \in D_{j+2}^k$ from the pairs $(Y \cup Z, T \cup W), (T \cup W, T)$. Again by $(B2)$, we also obtain that $(X, T) \in D_{j+3}^k$ from the pairs $(X, Y \cup Z), (Y \cup Z, T)$. Now, $(Y \cup Z \cup T, Y \cup Z \cup T) \in D_0^k$ also belongs to $D_{j+3}^k$. Therefore, by $(B2)$ we obtain $(X \cup T, Y \cup Z \cup T) \in D_{j+4}^k$ from $(X, Y \cup Z)$ and $(Y \cup Z \cup T, Y \cup Z \cup T)$. Finally, again by $(B2)$, we have the pair $(X, Y \cup Z \cup T) \in D_{j+5}^k$ from $(X, T)$ and $(X \cup T, Y \cup Z \cup T)$. This proves that $\bigcup_{k \geq 0} D_k^k = \bigcup_{k \geq 0} D_k^k$ and the thesis holds. □

At this point, we introduce the notion of model for $\mathcal{D}$.

**Definition 5.3.2.** We say that a pairing $\mathcal{P} \in PAIR(\Omega)$ is a *model* for $\mathcal{D}$, denoted by $\mathcal{P} \vdash \mathcal{D}$, if for any pair $(X, Y) \in \mathcal{D}$ we have that $X \geq_{\mathcal{P}} Y$, i.e. if $\mathcal{D}^+ \subseteq \mathcal{P}(\mathcal{P})$.

In particular, if $\mathcal{D} = \{(X_1, Y_1), \ldots, (X_k, Y_k)\}$, we write simply $\mathcal{P} \vdash (X_1, Y_1), \ldots, (X_k, Y_k)$ instead of $\mathcal{P} \vdash \{(X_1, Y_1), \ldots, (X_k, Y_k)\}$, that is therefore equivalent to say that $X_1 \geq_{\mathcal{P}} Y_1, \ldots, X_k \geq_{\mathcal{P}} Y_k$.

Let us introduce now a preorder between families of $\mathcal{P}(\Omega)^2$.

**Definition 5.3.3.** Let $\mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$. We say that the ordered pair $(\mathcal{D}, \mathcal{D}')$ is a **sequentiality** on $\Omega$, in symbols $\mathcal{D} \sim_{\Omega} \mathcal{D}'$, if, for any $\mathcal{P} \in PAIR(\Omega)$ such that $X \geq_{\mathcal{P}} Y$ for all $(X, Y) \in \mathcal{D}$, we also have $X' \geq_{\mathcal{P}} Y'$ for all $(X', Y') \in \mathcal{D}'$. In particular, if $\mathcal{D} = \emptyset$, then $\emptyset \sim_{\Omega} \emptyset$ if and only if $X' \geq_{\mathcal{P}} Y'$ for any $\mathcal{P} \in PAIR(\Omega)$ and for any $(X', Y') \in \mathcal{D}'$.

Moreover, if $\mathcal{D} = \{(X_1, Y_1), \ldots, (X_k, Y_k)\}$ and $\mathcal{D}' = \{(X_1', Y_1'), \ldots, (X_l', Y_l')\}$, we will write

$$(X_1, Y_1), \ldots, (X_k, Y_k) \sim_{\Omega} (X_1', Y_1'), \ldots, (X_l', Y_l')$$

instead of $\{(X_1, Y_1), \ldots, (X_k, Y_k)\} \sim_{\Omega} \{(X_1', Y_1'), \ldots, (X_l', Y_l')\}$. Let us note that

$$\mathcal{D} \sim_{\Omega} \mathcal{D}' \iff \mathcal{D} \sim_{\Omega} (X', Y'), \forall (X', Y') \in \mathcal{D}'$$

and that $\sim_{\Omega}$ is a preorder between ordered subset pairs families. In the next result, we establish some basic properties concerning this preorder.

**Theorem 5.3.4.** Let $X, Y, Z, W \in \mathcal{P}(\Omega)$. Then, the following sequentiality on $\Omega$ hold:

1. If $X \geq Y$, then $\emptyset \sim_{\Omega} (X, Y)$ and, in particular, $\emptyset \sim_{\Omega} (X, X)$;
2. $(X, Y) \sim_{\Omega} (X \cup Z, Y)$;
3. $(X, Y), (X, Z) \sim_{\Omega} (X, Y \cup Z)$;
4. $(X, Y \cup Z) \sim_{\Omega} (X, Y)$;

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(I5) \((X, Y), (Y, Z) \leadsto_{\Omega} (X, Z)\);

(I6) \((X, Y), (Y \cup Z, W) \leadsto_{\Omega} (X \cup Z, W)\);

(I7) \((X, Y \cup Z), (Z, T \cup W) \leadsto_{\Omega} (X, Y \cup Z \cup T)\).

Proof. It is the same of that of Theorem 5.1.2. □

In the next section we use some properties established in Theorem 5.3.4 to connect our investigation to particular types of acyclic digraphs.

5.4 Tail Decomposition Digraphs

In this section we link the study of the preorder \(\leadsto_{\Omega}\) to the existence of some particular kinds of acyclic digraphs having as vertex sets suitable subsets of \(\mathcal{P}(\Omega)\).

Let \(\mathcal{D} \subseteq \mathcal{P}(\Omega)^2\).

**Definition 5.4.1.** We call \(\mathcal{D}\)-tail decomposition digraph an acyclic digraph \(D = (V(D), Arc(D))\) such that \(V(D) \subseteq \mathcal{P}(\Omega)\) and for any \((W, Z) \in Arc(D)\), there exist \(X_1, X_2, \ldots, X_k, Z' \in \mathcal{P}(\Omega)\) such that:

(i) \(X_1 = W\) and \(Z \subseteq Z'\);

(ii) \((X_1 \cup X_2 \cup \cdots \cup X_k, Z') \in \mathcal{D}\);

(iii) \((X_i, Z) \in Arc(D)\) for all \(i = 1, \ldots, k\).

Let \(D\) be a \(\mathcal{D}\)-tail decomposition digraph. We call a pair \((A, B) \in \mathcal{D}\) that satisfies the above conditions a \(\mathcal{D}\)-extension of the arc \((W, Z)\) and we denote by \(Ext_{\mathcal{D}}(W, Z)\) the set of all \(\mathcal{D}\)-extensions of \((W, Z)\). Moreover, we also set

\[
\cup_{\mathcal{D}}(D) := \bigcup\{Ext_{\mathcal{D}}(W, Z) : (W, Z) \in Arc(D)\}.
\]

If \((X, Y) \in \mathcal{P}(\Omega)^2\), we say that \(D\) is a \(\mathcal{D}((X, Y)\)-tail decomposition digraph if \(X = \bigcup\{W : W \in I(D)\}\) and \(Y = \bigcup\{Z : Z \in \mathcal{F}\}\), for some \(\mathcal{F} \subseteq V(D)\).

**Example 5.4.2.** Let \(\mathcal{D} = \{(X_1 \cup X_2, Z_1 \cup Z_2), (Z_2, Z_3), (Z_1 \cup Z_2 \cup Z_3, Y)\} \subseteq \mathcal{P}(\Omega)^2\). In Figure 6.2, we illustrate an example of \(\mathcal{D}(X_1 \cup X_2, Y)\)-tail decomposition digraph. All the edges have been introduced by using Definition 5.4.1.

In the next Theorem 5.4.4, we show that the condition \(\mathcal{D} \leadsto_{\Omega} (X, Y)'\) is equivalent to require both that \((X, Y) \in \mathcal{D}^+\) and also that there exists a \(\mathcal{D}((X, Y)\)-tail decomposition digraph. In order to prove this equivalence, we firstly introduce the following notion.

**Definition 5.4.3.** We call \(\mathcal{D}((X, Y)\)-tail resolution a sequence of pairs

\[
S = ((X_0, Y_0), (X_1, Y_1) \ldots, (X_r, Y_r))
\]

such that:
Figure 5.1: $\mathcal{D}(X_1 \cup X_2 | Y)$-tail decomposition digraph

(i) $(X_0, Y_0) = (X, X)$;
(ii) $(X_1, Y_1) \in \hat{\mathcal{D}}$;
(iii) if $2 \leq k \leq r - 1$, $(X_k, Y_k)$ satisfies one of the following conditions:
   (a) $(X_k, Y_k) \in \hat{\mathcal{D}}$;
   (b) $X_k = X_{k-2} = X$ and there exist $T, W, Z \in \mathcal{P}(\Omega)$ such that $Y_{k-2} = T \cup X_{k-1}$,
       $Y_{k-1} = Z \cup W$ and $Y_k = Y_{k-2} \cup Z$.
(iv) $(X_r, Y_r) = (X, Y)$ and satisfies one of the following conditions:
   (a) there exist $T, W, Z \in \mathcal{P}(\Omega)$ such that $Y_{r-2} = T \cup X_{r-1}$, $Y_{r-1} = Z \cup W$,
       $X_r = X$ and $Y = Y_{r-2} \cup Z$;
   (b) there exists $Z \in \mathcal{P}(\Omega)$ such that $Y_{r-1} = Y \cup Z$.
(v) No other pair $(X_k, Y_k)$, for all $k = 0, \ldots, r - 1$, is $(X, Y)$.

We denote by $\mathcal{R}_\mathcal{D}^S(X, Y)$ the set of all pairs in $\mathcal{D}$ occurring in the $\mathcal{D}$-tail resolution $S$ of $(X, Y)$.

We have then the following result.

**Theorem 5.4.4.** Let $(X, Y) \in \mathcal{P}(\Omega)^2$. The following conditions are equivalent:
(i) $\mathcal{D} \rightsquigarrow_\Omega (X, Y)$;
(ii) $(X, Y) \in \mathcal{D}^+$;
(iii) there exists a $\mathcal{D}(X, Y)$-tail decomposition digraph.

**Proof.** (i) $\implies$ (ii): Let us suppose that $\mathcal{D} \rightsquigarrow_\Omega (X, Y)$ and assume by contradiction that
$(X, Y) \in \mathcal{P}(\Omega)^2 \setminus \mathcal{D}^+$. We will prove that there exists $\Psi \in PAIR(\Omega)$ such that $\Gamma_\Psi(W, Z) = U$
for any $(W, Z) \in \mathcal{D}^+$ but $\Gamma_\Psi(X, Y) \neq U$. Since $(X, X) \in \mathcal{D}^+ = \bigcup_{k \geq 0} \mathcal{D}_k^+$, we can consider

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\( \overline{X} := \max\{Y \in \mathcal{P}(\Omega) : (X, Y) \in \mathcal{D}^+\} \). Let us consider the pairing \( \mathcal{P} \in \text{PAIR}(\Omega) \) with \( U := \{u, u'\}, \Lambda := \{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n\} \) and \( F \) so defined:

\[
F(v, a_i) = \begin{cases} 
\lambda_i & \text{if } v = u \lor (v = u' \land a_i \in \overline{X}), \\
\mu_i & \text{otherwise.} 
\end{cases}
\]

(5.14)

Firstly, let us show that \( \Gamma_{\mathcal{P}}(W, Z) = U \) for any \( (W, Z) \in \mathcal{D}^+ \). For, we distinguish two cases, namely \( W \not\subseteq \overline{X} \) and \( W \subseteq \overline{X} \).

Let \( W \not\subseteq \overline{X} \). Then, \( u \not\equiv_W u' \), i.e. \( \pi_{\mathcal{P}}(W) = u \not\equiv u' \), therefore \( \Gamma_{\mathcal{P}}(W, Z) = U \) for any \( Z \in \mathcal{P}(\Omega) \) and, in particular \( \Gamma_{\mathcal{P}}(W, Z) = U \) for any pair \( (W, Z) \in \mathcal{D}^+ \).

Suppose now that \( W \subseteq \overline{X} \). Hence, \( u \equiv_W u' \). Since \( (W, Z) \in \mathcal{D}^+ \), there exists an index \( i \) such that \( (W, Z) \in D_i^+ \). Moreover, we have that \( (X, W) \in D_0^+ \). Thus \( (X, W) \in D_k \), for any integer \( k \).

By applying (A2) on the pairs \((X, W)\) and \((W, Z)\), it follows that \( (X, Z) \in D_{i+1}^+ \). On the other hand, by (A1), we obtain \( (X, X \cup Z) \in D_{i+2}^+ \) from the pairs \((X, X)\) and \((X, Z)\). Furthermore, by (A2) we construct the pair \((X, W)\) starting from the pairs \((X, X \cup Z)\) and \((X \cup Z, W)\). By the definition of \( \overline{X} \), it results that \( (X, X) \in \mathcal{D}^+ \), hence we can find a \( D_j^+ \), for some \( j > i \), such that \( (X, X), (X, W) \in D_j^+ \). In particular, by applying (A2) on the latter pairs, we obtain the pair \((X, W) \in D_j^+ \). Finally, since \( (W, Z) \in D_i^+ \subseteq D_{j+1}^+ \), by (A2) we obtain \((X, Z) \in D_{j+2}^+ \) starting from the pairs \((X, W)\) and \((W, Z)\). This means that \( (X, Z) \in \mathcal{D}^+ \). In other terms, we showed that \( Z \subseteq \overline{X} \), thus \( u \equiv_Z u' \), i.e. \( \Gamma_{\mathcal{P}}(W, Z) = U \).

At this point, we must prove that \( \Gamma_{\mathcal{P}}(X, Y) = \emptyset \). By (5.14), we can deduce that \( u \equiv_X u' \). Suppose by contradiction that \( u \equiv_Y u' \); it must necessarily result that \( Y \subseteq \overline{X} \). Since \( (X, \overline{X}) \in \mathcal{D}^+ \), then \( (X, \overline{X}) \in D_i^+ \) for some \( i \in \mathbb{N} \). Therefore, we can obtain the pair \((X, Y) \in D_{i+1}^+ \) starting from the pairs \((X, X)\) and \((X, Y)\). This means that \( (X, Y) \in \mathcal{D}^+ \), contradicting our choice of \((X, Y)\). Therefore, \( u \not\equiv_Y u' \), so \( \Gamma_{\mathcal{P}}(X, Y) = \emptyset \). This shows the claim.

\begin{itemize}
    \item \((ii) \implies (i)\): Let us suppose that \( (X, Y) \subseteq \mathcal{D}^+ = \bigcup_{k \geq 0} D_k \) and let \( \mathcal{P} \) be a pairing such that \( \Gamma_{\mathcal{P}}(W, Z) = U \), for any \( (W, Z) \in \mathcal{D} \). We will prove that \( \Gamma_{\mathcal{P}}(X, Y) = U \). If \((X, Y) \in \mathcal{D} \), then obviously \( \Gamma_{\mathcal{P}}(X, Y) = U \). Let \( n \) be a fixed integer and suppose that for any \( k \leq n - 1 \) we have that \( \Gamma_{\mathcal{P}}(W', Z') = U \) for any \((W', Z') \in D_k \). We now show that for any pair \((X', Y') \in D_n^+ \), it results that \( \Gamma_{\mathcal{P}}(X', Y') = U \). Let \( (X', Y') \in D_n^+ \). If \( (C1) \) holds, there exists \( Z \in \mathcal{P}(\Omega) \) such that \( (X', Y', Z') \in D_{n-1}^+ \). By the inductive hypothesis, \( \Gamma_{\mathcal{P}}(X', Y' \cup Z) = U \), then, by (I4), \( (X', Y' \cup Z) \sim_{\Omega} (X', Y') \), so \( \Gamma_{\mathcal{P}}(X', Y') = U \).

On the other hand, if \( (C2) \) holds, there exist \( Y, Z, W, T \in \mathcal{P}(\Omega) \) such that \((X', Y \cup Z), (Z, T \cup W) \in D_{n-1}^+ \) and \( Y' = Y \cup Z \cup T \). By the inductive hypothesis, \( \Gamma_{\mathcal{P}}(X', Y \cup Z) = \Gamma_{\mathcal{P}}(Z, T \cup W) = U \) hence by (I7), it follows that \((X', Y \cup Z), (Z, T \cup W) \sim_{\Omega} (X', Y') \), i.e. \( \Gamma_{\mathcal{P}}(X', Y') = U \) and the claim has been showed.

\item \((ii) \implies (iii)\): Let us suppose that \((X, Y) \subseteq \mathcal{D}^+ \). We divide the proof of \((iii)\) in two steps. Firstly, we show that there exists a \( \mathcal{D} \)\((X, Y)\)-tail resolution \( S = ((X_0, Y_0), (X_1, Y_1), \ldots, (X_r, Y_r)) \).

By virtue of the identity \( \mathcal{D}^+ = \bigcup_{k \geq 0} D_k \) and by Theorem 5.3.1, there exists a sequence of ordered subsets pairs \((X_0, Y_0), \ldots, (X_s, Y_s) \) such that \((X_0, Y_0) \in \mathcal{D} \) and \((X_s, Y_s) = (X, Y) \). In particular, since the pair \((X, X) \in \mathcal{D} \), we can always assume that \((X_0, Y_0) = (X, X) \).

\end{itemize}
Suppose that a pair \((W, Z) \neq (X, Y)\) in the sequence has been obtained from some pair \((W, Z \cup Z')\) by \((C1)\). If \((W, Z)\) is not used elsewhere to obtain some other pair of sequence, we can remove it from the sequence. On the contrary, suppose that \((W, Z)\) is used to yield another pair. If this second pair has been obtained through \((C1)\), we can suppose directly that it has been obtained by the pair \((W, Z \cup Z')\), hence \((W, Z)\) can be removed from the sequence. Nevertheless, the second pair can be obtained from \((W, Z)\) through \((C2)\). In this case, two possibilities are allowed:

(a) there exist \(S, T, U \in \mathcal{P}(\Omega)\), with \(S \subseteq Z\), such that the pairs \((W, Z), (S, T \cup U)\) yield the pair \((W, T \cup Z)\);

(b) there exist \(U, V \in \mathcal{P}(\Omega)\), with \(V \supseteq W\), such that the pairs \((U, V), (W, Z)\) yield the pair \((U, W \cup V)\).

In case (a) we can use the ordered pair \((W, Z \cup Z')\) to obtain \((W, T \cup Z \cup Z')\) instead of \((W, T \cup Z)\). This implies that we can remove \((W, Z)\) in the above sequence replacing all the pairs derived from \((W, T \cup Z)\) with the ones obtained by starting from \((W, T \cup Z \cup Z')\). In case (b), we can directly replace \((W, Z)\) with \((W, Z \cup Z')\) without changing the resulting pair \((U, W \cup V)\). In other terms, in both cases we replaced a pair \((W, Z)\) with a pair \((W, Z \cup Z')\). Let us observe that with these replacements, it can happen that we derive the pair \((X, Y \cup Y')\) instead of \((X, Y)\). The latter can be yielded by applying \((C1)\) to the pair \((X, Y \cup Y')\).

However, by iterating the previous procedure, we find a sequence \((X_0, Y_0), \ldots, (X_m, Y_m)\), with \((X_0, Y_0) = (X, X)\) and \((X_m, Y_m) = (X, Y)\) and where each intermediate pair can be derived only through \((C2)\). Let us prove now that, if \((X, Z)\) can be obtained by \((C2)\) from some pairs of form \((X, V')\) and \((S, T \cup U)\) of the above sequence, then we can obtain a pair of the form \((X, W)\), with \(Z \subseteq W\), starting from a pair of the form \((X, V)\) and belonging to the previous sequence and a pair \((S, R) \in \mathcal{D}\).

For, let us assume that the pair \((X, Z)\) has been yielded by \((X, V), (S, T \cup U)\) by applying \((C2)\). Firstly, let us note that by the previous argument we can suppose that \((S, T \cup U)\) cannot be obtained by using some application of \((C1)\). We distinguish two cases: if \(S \supseteq T \cup U\), then we can add the pair \((S, T \cup U)\) in the sequence; otherwise, \((S, T \cup U)\) has been obtained by means of \((C2)\) from two pairs \((S, R), (R', T' \cup U')\), both belonging to the above sequence. In this case, we have that \(R \cup T' = T \cup U\) and, thus, another application of \((C2)\) to the pairs \((X, V \cup T'), (S, R)\) yields \((X, Z \cup U)\). The only problem that may arise is that \((S, R) \notin \mathcal{D}\) but, if so, just repeat the same argument for it. Hence, we can remove \((S, T \cup U)\) and all the pairs in our sequence not belonging in \(\mathcal{D}\) and not having the form \((X, Z)\) for some \(Z \in \mathcal{P}(\Omega)\). This proves the existence of a \(\mathcal{D}[(X, Y)\)-tail resolution.

The second step of our proof consists of constructing a \(\mathcal{D}[(X, Y)\)-tail decomposition digraph starting from the above \(\mathcal{D}[(X, Y)\)-tail resolution. Let \(S\) be a \(\mathcal{D}[(X, Y)\)-tail resolution and let \((X, Z_0), (X, Z_1), \ldots, (X, Z_k)\) be the ordered subset pairs occurring in \(S\), with \(Z_0 = X\) and \(Z_k = Y\). We will construct a collection \(D_i\) of \(\mathcal{D}[(X, Z_i)\)-tail decomposition digraphs, for \(i = 0, \ldots, k\). When \(i = 0\), the graph with just a node \(X\) is clearly a \(\mathcal{D}[(X, Z_0)\)-tail decomposition digraph, where \((X, Z_0) = (X, X)\). Let \(i \leq k\) be a fixed integer and suppose inductively that \(D_j\) is a \(\mathcal{D}[(X, Z_j)\)-tail decomposition digraphs for
$j = 0, \ldots, i - 1$: we now construct a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph obtained from $D_{i-1}$ by adding suitable vertices and edges. For, let us consider $(X, Z_i)$ in $\mathcal{S}$. If $(X, Z_i) \in \mathcal{D}$, then we can add to $D_{i-1}$ a vertex $Z_i$ and an edge $(X, Z_i)$. Thus we obtain a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph $D_i$.

On the other hand, suppose that $(X, Z_i) \in \mathcal{D}^+ \setminus \hat{\mathcal{D}}$. This implies that one of the conditions $(C1)$, $(C2)$ holds. Suppose that there exists $T \in \mathcal{P}(\Omega)$ such that the pair $(X, Z_i)$ have been obtained from $(X, Z_i \cup T) = (X, Z_{i-1})$. In this case $D_{i-1}$ is already a $\mathcal{D}|(X, Z_i)$-tail decomposition digraph. Then, we can set $D_i = D_{i-1}$.

On the contrary, suppose that there exist $T, U, W \in \mathcal{P}(\Omega)$ such that $Z_{i-2} = T \cup X_{i-1}$, $Z_{i-1} = U \cup W$ and $Z_k = Z_{k-2} \cup U$. In this case, $D_{i-2}$ is a subgraph of $D_{i-1}$ and in $D_{i-2}$ there are subsets of $\Omega$ whose union is $Z_{i-2}$. Then, we form $D_i$ by adding a vertex labeled $U$ to $D_{i-1}$ and arcs connecting nodes of $D_{i-2}$ labeled by some subsets of $Z_{i-2}$ with $U$. This ends our inductive construction, so a $\mathcal{D}|(X, Y)$-tail decomposition digraph has been formed.

$(iii) \Rightarrow (ii)$: Let us assume that a $\mathcal{D}|(X, Y)$-tail decomposition digraph is given. Let $D$ be a $\mathcal{D}|(X, Y)$-tail decomposition digraph and $D_0, D_1, \ldots, D_k = D$ be a sequence of $\mathcal{D}$-tail decomposition digraphs such that $D_i$ is obtained by $D_{i-1}$ by adding nodes and edges compatibly with Definition 5.4.1. For each $i = 0, \ldots, k$, we assume that $Z_i$ is the union of subsets of $\Omega$ labelling some nodes in $D_i$ and, moreover, suppose that $Z_k \supseteq Y$. We will construct a $\mathcal{D}|(X, Y)$-tail resolution $\mathcal{S}$ from $D$ containing $(X, Z_0) = (X, X), (X, Z_1), \ldots, (X, Z_k)$ as a subsequence. Firstly, set $Z_0 := X$ and $D_0$ be the digraph whose vertex set is exactly $X$ and $\text{Arc}(D_0) = \emptyset$. Let us set $S_0 := \{(X, X)\}$. Let now $i \geq 1$ and suppose that there exists $\mathcal{D}|(X, Z_j)$-tail resolution $S_j$ containing $(X, Z_0) = (X, X), (X, Z_1), \ldots, (X, Z_j)$ as a subsequence, for all $j \leq i - 1$. According to Definition 5.4.1, we obtain $D_i$ from $D_{i-1}$ through a pair $(T, U \cup W) \in \mathcal{D}$, where $T, W \in \mathcal{P}(\Omega)$ and $T \subseteq Z_{i-1}$, by adding a vertex labeled by $U$ and edges of the form $(S_l, U)$ for $l = 1, \ldots, m$, where $m$ is a suitable integer and $\bigcup_{l=1}^{m} S_l \subseteq T$. By considering the pairs $(X, Z_{i-1})$ and $(T, U \cup W)$, we obtain $(X, Z_i)$, where $Z_i = Z_{i-1} \cup U$, by $(C2)$. Therefore, we add both the pairs $(T, U \cup W)$ and $(X, Z_i)$ to $S_{i-1}$.

In this way we construct a $\mathcal{D}|(X, Z_k)$-tail resolution, where $Y \subseteq Z_k$. It could happen that $Y \not\subseteq Z_k$. In this case, since by $(C1)$ we obtain the pair $(X, Y)$ from $(X, Z_k)$, we can add $(X, Y)$ to $S_k$. Denote by $\mathcal{S}$ the sequence so constructed. Thus, we obtain a $\mathcal{D}|(X, Y)$-tail resolution $\mathcal{S}$ from $D$ containing $(X, Z_0) = (X, X), (X, Z_1), \ldots, (X, Z_k)$ as a subsequence. Finally, let us observe that we used only conditions $(C1)$ and $(C2)$ to construct the above $\mathcal{D}|(X, Y)$-tail resolution. Hence $(X, Y) \in \mathcal{D}^+$ and the claim has been proved.

By Theorem 5.4.4 we deduce the following results.

**Corollary 5.4.5.** Let $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{P}(\Omega)^2$. Then, $\mathcal{D} \simeq \mathcal{D}'$ if and only if $\mathcal{D} \sim_{\Omega} \mathcal{D}'$ and $\mathcal{D}' \sim_{\Omega} \mathcal{D}$.

**Proof.** Let us observe that by Theorem 5.4.4, condition $\mathcal{D} \sim_{\Omega} \mathcal{D}'$ becomes $\mathcal{D} \subseteq \mathcal{D}'$ so, by Proposition 5.2.6. Hence, we easily infer the claim.

**Corollary 5.4.6.** Let $\mathcal{D} \subseteq \mathcal{S}(\mathcal{P})$. Then, $\mathcal{D} \sim_{\Omega} \mathcal{S}(\mathcal{P})$ if and only if $\mathcal{D}^+ = \mathcal{S}(\mathcal{P})$.

**Proof.** Suppose that $\mathcal{D} \sim_{\Omega} \mathcal{S}(\mathcal{P})$. By Theorem 5.4.4 it follows that $\mathcal{S}(\mathcal{P}) \subseteq \mathcal{D}^+$ and, by Proposition 5.2.6, we also have the reverse inclusion. Hence, $\mathcal{D}^+ = \mathcal{S}(\mathcal{P})$. 

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Finally, it is now easy to prove that $D \cup (D \cup U)$ is already a $D$-tail resolution and its implicit justification. On the other hand, there also more structural reasons that concern an abstract theory of the binary relations between specific types of families $D \subseteq \mathcal{P}(\Omega)$.

By means of Theorem 5.4.4 we are also able to establish a deeper relation between tail resolutions and tail decomposition digraphs.

**Proposition 5.4.7.** Let $(X, Y) \in \mathcal{P}(\Omega)^2$. If there exists a $D|(X, Y)$-tail resolution $S$, then there exists a $D|(X, Y)$-tail decomposition digraph $D$ such that $\cup_{D}(D) = \mathcal{R}_{\mathcal{D}}^{\delta}(X, Y)$.

**Proof.** Let $S$ be a $D|(X, Y)$-tail resolution and let $(X, Z_0), (X, Z_1), \ldots, (X, Z_k)$ be the ordered subsets pairs occurring in $S$, with $Z_0 = X$ and $Z_k = Y$. Let us construct a collection of $D$-tail decomposition digraphs $D_0, D_1, \ldots, D_k$ such that, for each $i$, $D_i$ is a $D|(X, Z_i)$-tail decomposition digraph. When $i = 0$, the graph with just a node $X$ is clearly a $D|(X, Z_0)$-tail decomposition digraph. Let us suppose, by using induction, that for any $k$ in $0, 1, \ldots, i-1$, $D_k$ is a $D|(X, Z_k)$-tail decomposition digraph and let us construct a $D|(X, Z_k)$-tail decomposition digraph by adding suitable vertices and edges. For this let us consider $(X, Z_i) \in S$. If $(X, Z_i) \in D$, then clearly $D_{i-1}$ is still a $D|(X, Z_i)$-tail decomposition digraph. Let us suppose now that $(X, Z_i)$ is not in $D$ and that there exist $T, W, Z \in \mathcal{P}(\Omega)$ such that $T \subset Z_j$ for some $j < i$, $Z_i = Z_j \cup W$ and $(T, W \cup Z) \in D$, so, by Theorem 5.4.4 and by Theorem 5.3.4, it follows that $(X, Z_j), (T, W \cup Z) \sim_{\Omega}(X, Z_j \cup W) = (X, Z_i)$. In this case $D_j$ is a subgraph of $D_{i-1}$ and in $D_j$ there are vertices of $\Omega$ whose union is $Z_j$. Then, by adding a vertex $W$ to $D_{i-1}$ and suitable edges we obtain a $D|(X, Z_i)$-tail decomposition digraph $D_i$. Finally let us suppose that $(X, Z_i)$ is not in $D$ and there exists $j < i$ such that $Z_j \subseteq Z_i$, so $(X, Z_j) \sim_{\Omega}(X, Z_i)$. In this case $D_j$ is already a $D|(X, Z_j)$-tail decomposition digraph, so is $D_{i-1}$. Then, we can set $D_i = D_{i-1}$. Finally, it is now easy to prove that $D := D_k$ is a $D|(X, Z_k)$-tail decomposition digraph and that $\cup_{D}(D) = \mathcal{R}_{\mathcal{D}}^{\delta}(X, Y)$.

We conclude this section with the following result.

**Proposition 5.4.8.** Let $D \subseteq \mathcal{S}(\mathcal{P})$, $(X, Y) \in \mathcal{P}(\Omega)^2$ and $S$ be a $D|(X, Y)$-tail resolution. If $(U, V) \in \mathcal{R}_{\mathcal{D}}^{\delta}(X, Y)$, then we can obtain a $D|(X, U)$-tail resolution $S'$ by adding eventually $(X, U)$ to a subsequence of $S$.

**Proof.** Let $(U, V) \in \mathcal{R}_{\mathcal{D}}^{\delta}(X, Y)$. If $U = X$, then $(X, U) = (X, X)$ and there’s nothing to prove. If $U \neq X$, then $(U, V) = (X_i, Y_i)$ and, by definition of $D|(X, Y)$-tail resolution $X_{i-1} = X$ and $Y_{i-1} = T \cup X_i = T \cup U$. So by adding $(X, U)$ to the subsequence of $S$, $((X_0, Y_0), (X_1, Y_1), \ldots, (X_{i-1}, Y_{i-1}))$, we obtain a $D|(X, U)$-tail resolution $S'$.

### 5.5 Generators

From the above sections, we observe that the investigation of the mathematical foundations concerning an abstract theory of the binary relations between specific types of families $D \subseteq \mathcal{P}(\Omega)^2$ has an implicit justification. On the other hand, there also more structural reasons that
induce to study these types of binary relations in $\mathcal{P}(\mathcal{P}(\Omega)^2)$. In this perspective we introduce the set map $DP: \mathcal{P}(\mathcal{P}(\Omega)^2) \rightarrow \mathcal{P}(\mathcal{P}(\Omega))$ defined by

$$DP(\mathcal{D}) := \{Z \in \mathcal{P}(\Omega) : X \subseteq Z \implies Y \subseteq Z \forall (X, Y) \in \mathcal{D}\}. \quad (5.15)$$

Let us notice that $DP(\mathcal{D}) \in CLSY(\Omega)$ for any $\mathcal{D} \in \mathcal{P}(\Omega)^2$. Furthermore, if $\mathcal{D} \subseteq \mathcal{D}'$, it results that

$$DP(\mathcal{D}) \supseteq DP(\mathcal{D}'). \quad (5.16)$$

We now show that the map $DP$ is invariant in the passage from a family $\mathcal{D}$ to its union additive envelope.

**Theorem 5.5.1.** $DP(\mathcal{D}^+) = DP(\mathcal{D})$.

**Proof.** By (5.16), we have that $DP(\mathcal{D}^+) \subseteq DP(\mathcal{D})$. Conversely, let $Z \in DP(\mathcal{D})$ and let $(X, Y) \in \mathcal{D}^+$ such that $X \subseteq Z$. We have to prove that $Y \subseteq Z$. Clearly, if $(X, Y) \in \mathcal{D}^+_0$, the claim is obvious.

Assume now that $(X, Y) \in \mathcal{D}^+_k$ for some $k \geq 1$. By Theorem 5.3.1, the pair $(X, Y)$ can be obtained by applying (A1) or (A2) to some pairs belonging to $\mathcal{D}^+_{k-1}$. Thus, we must show our claim inductively. To this regard, let $(X, Y) \in \mathcal{D}^+_0$ and assume that there exists $Y' \in \mathcal{P}(\Omega)$ such that $(X, Y'), (Y', Y) \in \mathcal{D}^+_0$. By our assumptions on $Z$, it follows that $Y' \subseteq Z$ and, then, that $Y \subseteq Z$. On the other hand, if there exist $\{Y_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ for any $i \in I$ and $Y = \bigcup_{i \in I} Y_i$, then $Y_i \subseteq Z$ for any $i \in I$, i.e. $Y = \bigcup_{i \in I} Y_i \subseteq Z$. This proves that $DP(\mathcal{D}) = DP(\mathcal{D}^+_1)$. Assume now that $DP(\mathcal{D}^+_{j-1}) = DP(\mathcal{D}^+_j)$ for any $j \leq k$. Let us prove that $DP(\mathcal{D}^+_k) \subseteq DP(\mathcal{D}^+_{k+1})$ and $Z \in DP(\mathcal{D}^+_k)$ be such that $X \subseteq Z$. If there exists $Y' \in \mathcal{P}(\Omega)$ such that $(X, Y'), (Y', Y) \in \mathcal{D}^+_k$, then $Y' \subseteq Z$ and, then, $Y \subseteq Z$. On the other hand, suppose that there exist $\{Y_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ for any $i \in I$ and $Y = \bigcup_{i \in I} Y_i$. Then, $Y_i \subseteq Z$ for any $i \in I$, i.e. $Y = \bigcup_{i \in I} Y_i \subseteq Z$. This proves that $DP(\mathcal{D}^+_k) = DP(\mathcal{D}^+_{k+1})$ and concludes the proof.

We consider now the family given by

$$\Omega(\Psi) := \{(X, y) : X \in \text{MINP}(\Psi), y \in M_\Psi(X) \setminus X\}. \quad (5.17)$$

Then, through the map $DP$ we can alternative characterize $MAXP(\Psi)$ in terms of the families $\mathcal{G}(\Psi)$ and $\Omega(\Psi)$.

**Theorem 5.5.2.** $DP(\mathcal{G}(\Psi)) = DP(\Omega(\Psi)) = MAXP(\Psi)$.

**Proof.** We firstly prove that $DP(\Omega(\Psi)) = MAXP(\Psi)$. Let $Z \in MAXP(\Psi)$ and let $(X, y) \in \Omega(\Psi)$ with $X \subseteq Z$. Then, or $X \in [Z]_{\approx_\Psi}$ and thus, $y \in Z$ or $M_\Psi(X) \not\subseteq Z$ and, also in this case, $y \in Z$. This proves that $Z \in MAXP(\Psi)$.

On the other hand, let $Z \in DP(\Omega(\Psi))$. Then, there exists $X \in \text{MINP}(\Psi)$ such that $X \approx_\Psi Z$ and $X \subseteq Z$. Hence, by our assumption on $Z$, it results that $M_\Psi(X) \subseteq Z \subseteq M_\Psi(X)$, i.e. $Z = M_\Psi(X)$. Thus, $Z \in MAXP(\Psi)$ and the claim has been shown.
Now, we show that \( DP(\mathcal{G}(\mathfrak{P})) = MAXP(\mathfrak{P}) \). Let \( Z \in MAXP(\mathfrak{P}) \) and \((X,Y) \in \mathcal{G}(\mathfrak{P}) \) with \( X \subseteq Z \). Then, we have that \( Z \supseteq M_{\mathfrak{P}}(X) \supseteq M_{\mathfrak{P}}(Y) \supseteq Y \), hence \( Z \in DP(\mathcal{G}(\mathfrak{P})) \).

On the other hand, let \( Z \in DP(\mathcal{G}(\mathfrak{P})) \). Then, \( Z \subseteq M_{\mathfrak{P}}(Z) \). Moreover, the pair \((Z,M_{\mathfrak{P}}(Z))\) clearly belongs to \( \mathcal{G}(\mathfrak{P}) \). Thus, we conclude that \( M_{\mathfrak{P}}(Z) \subseteq Z \), i.e., \( Z = M_{\mathfrak{P}}(Z) \) and so \( Z \in MAXP(\mathfrak{P}) \).

Based on the results given in Theorems 5.5.1 and 5.5.2, it is natural to investigate some particular types of families \( \mathcal{D} \) that completely characterize all the semi-symmetries of a given information table. The role that these families have in our context is analogous to that of a generator system in an algebraic structure.

**Definition 5.5.3.** We say that:

- \( \mathcal{D} \) *generates the semi-symmetries* of \( \mathfrak{P} \), denoted by \( \mathcal{D} \models \mathfrak{P} \), if \( \mathcal{D} \subseteq \mathcal{G}(\mathfrak{P}) \) and \( \mathcal{D}^{+} = \mathcal{G}(\mathfrak{P}) \).

- \( \mathcal{D} \) *minimally generates the semi-symmetries* of \( \mathfrak{P} \), denoted by \( \mathcal{D} \models_{mgs} \mathfrak{P} \), if \( \mathcal{D} \models \mathfrak{P} \) and for any \( \mathcal{D}' \subseteq \mathcal{D} \) we have that \( (\mathcal{D}')^{+} \subseteq \mathcal{G}(\mathfrak{P}) \);

- \( \mathcal{D} \) is a *essentially generates of the semi-symmetries* of \( \mathfrak{P} \), denoted by \( \mathcal{D} \models_{egs} \mathfrak{P} \), if \( \mathcal{D} \models \mathfrak{P} \) and \( |\mathcal{D}| = \min\{|\mathcal{D}'| : \mathcal{D}' \models \mathfrak{P}\} \).

By virtue of Theorem 5.5.1 and 5.5.2 we obtain the following result.

**Corollary 5.5.4.** If \( \mathcal{D} \models \mathfrak{P} \), then \( DP(\mathcal{D}) = MAXP(\mathfrak{P}) \).

A partial converse of the previous corollary can be given.

**Proposition 5.5.5.** Let \( \mathcal{D} \subseteq \mathcal{P}(\mathcal{P}(\Omega)^{2}) \) be such that \( DP(\mathcal{D}) = MAXP(\mathfrak{P}) \). Then, \( \mathcal{D} \subseteq \mathcal{G}(\mathfrak{P}) \).

**Proof.** Let \((X,Y) \in \mathcal{D} \). Then, \( M_{\mathfrak{P}}(X) \supseteq X \) implies \( M_{\mathfrak{P}}(X) \supseteq Y \), therefore \( M_{\mathfrak{P}}(Y) \subseteq M_{\mathfrak{P}}(X) \), i.e., by (5.4), \( X \geq_{\mathfrak{P}} Y \). This shows that \((X,Y) \in \mathcal{G}(\mathfrak{P}) \). \( \Box \)

We determine now a specific family of ordered pairs of \( \mathcal{P}(\Omega)^{2} \) that generates the semi-symmetries of \( \mathfrak{P} \). To this regard, we set

\[
\mathcal{S}_{\text{minp}}(\mathfrak{P}) := \{(X,Y) \in MINP(\mathfrak{P}) \times (MINP(\mathfrak{P}) \cup \mathcal{P}(M_{\mathfrak{P}}(\emptyset))) \} \cap \mathcal{G}(\mathfrak{P}) .
\]

We have then the following result.

**Theorem 5.5.6.** \( \mathcal{S}_{\text{minp}}(\mathfrak{P}) \models \mathfrak{P} \).

**Proof.** It is obvious that \( \mathcal{S}_{\text{minp}}^{+}(\mathfrak{P}) \subseteq \mathcal{G}(\mathfrak{P}) \), since \( ^{+} \) is a closure operator and \( \mathcal{G}(\mathfrak{P}) \in UAF(\Omega) \). On the other hand, it results that \( \mathcal{G}(\mathfrak{P}) \subseteq \mathcal{S}_{\text{minp}}^{+}(\mathfrak{P}) \). In fact, let \((X,Y) \in \mathcal{G}(\mathfrak{P}) \). Then, there exists \( Z \in MINP(\mathfrak{P}) \) such that \( Z \subseteq X \) and \( Z \approx_{\mathfrak{P}} X \). In particular \( \Gamma_{\mathfrak{P}}(Z,Y) = U \), so \((Z,Y) \sim_{\Omega} (X,Y) \) by (I2). Similarly, there exists \( W_{1} \in MINP(\mathfrak{P}) \) such that \( W_{1} \subseteq Y \) and \( W_{1} \approx_{\mathfrak{P}} Y \). Hence \( Y = W_{1} \cup W' \), where \( W' = Y \setminus W_{1} \). Clearly, \( \Gamma_{\mathfrak{P}}(Z,W_{1}) = \Gamma_{\mathfrak{P}}(Z,W_{1}') = U \). Furthermore, by (I3), we have that \((Z,W_{1}),(Z,W_{1}') \sim_{\Omega} (Z,Y) \). Let us consider the pair
Let \( W_2 \in MINP(\mathfrak{P}) \) such that \( W_2 \cong_{\mathfrak{P}} W'_2 \) and \( W'_1 = W_2 \cup W'_2 \), where \( W'_2 \cap W_2 = \emptyset \). Again, \( \Gamma_{\mathfrak{P}}(Z,W_2) = \Gamma_{\mathfrak{P}}(Z,W'_2) = U \). By iterating this procedure, we obtain a family of subsets \( W_1, \ldots, W_r \) belonging in \( MINP(\mathfrak{P}) \) and such that \( \Gamma_{\mathfrak{P}}(Z,W_i) = U \) for any \( i = 1, \ldots, r \) and a set \( W_r' \in \mathcal{P}(\mathcal{M}(\emptyset)) \) with \( W'_r \cap W_r = \emptyset \). In particular \( \Gamma_{\mathfrak{P}}(Z,W'_r) = U \). Therefore, \( (Z,W_1), \ldots, (Z,W_r), (Z,W'_r) \sim_{\Omega} (Z,Y) \), hence \( S_{\minp}(\mathfrak{P}) \sim_{\Omega} (X,Y) \), i.e. \( (X,Y) \in \mathcal{S}_{\minp}^+(\mathfrak{P}) \).

Given \( X, Y \in \mathcal{P}(\Omega) \), we now consider the case in which both the pairs \( (X,Y) \) and \( (Y,X) \) belong to \( D^+ \). Formally, whenever \( X, Y \in \mathcal{P}(\Omega) \) are such that \( \{(X,Y),(Y,X)\} \subseteq D^+ \), we write \( X \leftrightsquigarrow_D Y \) and we also set

\[
[X]\!\downarrow:=\{X' \in \mathcal{P}(\Omega) : X \leftrightsquigarrow_D X'\},
\]

\[
F_D(X):=\{(X',Y') \in D : X \leftrightsquigarrow_D X'\},
\]

and

\[
f_D(X):=\{X' \in \mathcal{P}(\Omega) : \exists Y' \in \mathcal{P}(\Omega) \text{ such that } (X',Y') \in F_D(X)\}.
\]

**Proposition 5.5.7.** Let \( D, D' \subseteq \mathcal{P}(\Omega)^2 \) such that \( D \models_{\text{mgs}} \mathfrak{P} \) and \( D' \models_{\text{mgs}} \mathfrak{P} \). Then, for any \( (X,Y) \in D \) there exists \( (X',Y') \in D' \) such that \( X \leftrightsquigarrow_D X' \).

**Proof.** By Corollary 5.4.5, it holds \( D' \sim_{\Omega} D \) and thus \( D' \sim_{\Omega} (X,Y) \). Let \( S \) be a \( D'|\{X,Y\} \)-tail resolution. Then, because of the minimality of \( D \), there exists \( (X',Y') \in \mathcal{R}_D(X,Y) \) that can be deduced by \( D \) using \( (X,Y) \). Thus, \( D \sim_{\Omega} (X',X) \). In a similar way, \( D' \sim_{\Omega} (X,X') \) since we use \( (X,Y') \) to deduce \( (X',X) \). This implies that \( D \sim_{\Omega} (X,X') \). The proposition is thus proved.

We prove now that if \( D \) and \( D' \) essentially generate the semi-symmetries of \( \mathfrak{P} \), then the two sets \( L_D(X) \) and \( L_{D'}(X) \) have the same cardinality, for any \( X \in \mathcal{P}(\Omega) \). To this aim, we need a technical notion.

**Definition 5.5.8.** Let \( (X,Y) \in \mathcal{S}(\mathfrak{P}) \). We say that \( X \) resolves \( Y \) in \( \mathfrak{P} \), denoted by \( X|_{\mathfrak{P}} Y \), if there exist \( D \subseteq \mathcal{P}(\Omega) \) such that \( D \models_{\text{mgs}} \mathfrak{P} \) and a \( D|\{X,Y\} \)-tail decomposition digraph \( D \) such that \( F_D(X) \cap \mathcal{U}_D(D) = \emptyset \).

Now, we establish a preliminary technical result.

**Proposition 5.5.9.** (i) Let \( D \models_{\text{egs}} \mathfrak{P} \), \( X \in \mathcal{P}(\Omega) \) and \( \{(Y,Z),(Y',Z')\} \subseteq D \cap F_D(X) \) such that \( Y|_{\mathfrak{P}} Y' \). Then, \( (Y,Z) = (Y',Z') \).

(ii) Let \( D \models_{\text{mgs}} \mathfrak{P} \), \( (X,Y) \in D \) and \( X' \in [X]|_D \). Then, \( X'|_{\mathfrak{P}} Y' \), for some \( Y' \in f_D(X) \).

**Proof.** (i): Let us assume, by contradiction, that the thesis is false. Let \( D \) be a \( D|\{Y,Y'\} \)-tail decomposition digraph such that \( \mathcal{U}_D(D) \cap F_D(X) = \emptyset \) and let \( D' \subseteq \mathcal{S}(\mathfrak{P}) \) obtained from \( D \) by replacing \( (Y,Z) \) and \( (Y',Z') \) with \( (Y',Z \cup Z') \). By (I4) we obtain \( (Y',Z \cup Z') \sim_{\Omega} (Y',Z') \), so \( D' \sim_{\Omega} (Y',Z') \). Since \( (Y,Z) \neq (Y',Z') \), \( D' \) is also a \( D'|\{Y,Y'\} \)-tail decomposition digraph, by the proof of Theorem 5.4.4, \( D' \sim_{\Omega} (Y,Y') \) and thus \( D' \sim_{\Omega} (Y,Z) \), so finally we obtain \( D \sim_{\Omega} D' \). On the other side it is easy to prove that \( D' \sim_{\Omega} D \), so \( D \) and \( D' \) are equivalent, by contradicting the minimality of \( D \). The claim is thus proved.

(ii): If \( X' \in f_D(X) \), then trivially \( X' \) resolves itself and \( (X',X') \notin D \) by the minimality of \( D \).
Proof. Let us suppose, by contradiction, that \(|X', Y'\) ∈ \(\mathcal{S}(\mathcal{P})\) for any choice of \(Y' \in \mathcal{D}(X)\), there exists a \(\mathcal{D}|(X', Y')\)-tail decomposition digraph for every \(Y' \in \mathcal{D}(X)\). Let \(Y'\) be a subset of \(\Omega\) such that the number of nodes of \(\mathcal{D}\) in a \(\mathcal{D}|(X', Y')\)-tail decomposition digraph is minimum and let \(D\) be such a digraph. Let us prove that \(F_D(X') \cap \mathcal{U}_D(D) = \emptyset\). For, let \((U, V) \in \mathcal{D} \cap \mathcal{U}_D(D)\). Then, it follows easily by Theorem 5.4.4 and Proposition 5.4.7 that \(D\) is a \((X', U)\)-tail decomposition digraph and there exists a pair in \(\mathcal{U}_D(D)\) that can be removed and still leave a \(\mathcal{D}|(X', U)\)-tail decomposition digraph \(H'\), contradicting our assumptions. Then, \(F_D(X') \cap \mathcal{U}_D(D) = \emptyset\).

**Theorem 5.5.10.** Let \(\mathcal{D}\) and \(\mathcal{D}'\) such that \(\mathcal{D} \parallel_{\text{egs}} \mathcal{P}\) and \(\mathcal{D}' \parallel_{\text{egs}} \mathcal{P}\). Then, \(|F_D(X)| = |L_{\mathcal{D}'}(X)|\) for any \(X \in \mathcal{P}(\Omega)\).

**Proof.** Let us suppose, by contradiction, that \(F_D(X) = \{(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\}\) and \(L_{\mathcal{D}'}(X) = \{(X'_1, Y'_1), (X'_2, Y'_2), \ldots, (X'_n, Y'_n)\}\), with \(n < m\). By (i) of Proposition 5.5.9 there exist \(j \in \{1, 2, \ldots, m\}\) such that \(X'_j\) is different from any of the \(X_i\)'s.

By (ii) of Proposition 5.5.9 \(X'_j\) resolves \(X_k\) for some \(k \in \{1, 2, \ldots, n\}\). Without loss in generality we can consider \(j = 1\) and \(k = 1\). Let \(\mathcal{D}''\) obtained from \(\mathcal{D}'\) by replacing \((X'_1, Y'_1)\) with \((X_1, Y'_1)\). Then, \(\mathcal{D}''\) is a minimum generator of the semi-symmetries of \(\mathcal{P}\) such that \(|L_{\mathcal{D}''}(X)| = |L_{\mathcal{D}''}(X)|\).

If \(X_1 = X'_1\) for some \(j > 1\), then, by (I3), \((X_1, Y'_1), (X'_j, Y'_1) \sim_{\Omega} (X_1, Y'_1 \cup Y'_1)\) and conversely, by (I6), it holds both \((X_1, Y'_1 \cup Y'_1) \sim_{\Omega} (X_1, Y'_1)\) and \((X_1, Y'_1 \cup Y'_1) \sim_{\Omega} (Y'_1, Y'_1)\), by contradicting the minimality of \(\mathcal{D}''\). Thus \(X_1 \neq X'_j\) for all \(j > 1\) and \(|F_D(X) \cap l_{\mathcal{D}''}(X)| = |F_D(X) \cap l_{\mathcal{D}''}(X)| + 1\).

As in the beginning of the proof we still have that some \(X'_j\) is different from any of the \(X_i\)'s. The previous procedure can thus be iterated, obtaining a minimum generator \(\mathcal{D}(s)\) of the pairs of \(\mathcal{P}\) such that \(F_D(X) \subset l_{\mathcal{D}(s)}(X)\).

Let now \(X'_j \in l_{\mathcal{D}(s)}(X) \setminus F_D(X)\). Then, by (ii) of Proposition 5.5.9, there exists \(k \in \{1, 2, \ldots, n\}\) such that \(X'_j\) resolves \(X_k\). Let \(\mathcal{F}\) be the minimum generator of the semi-symmetries of \(\mathcal{P}\) obtained from \(\mathcal{D}(s)\) by replacing \((X'_j, Y'_j)\) with \((X_k, Y'_j)\). By (i) of Proposition 5.5.9, \(Y'_k = Y'_j\). So \(m = n\) and the theorem is proved.

### 5.6 Pointed Pair Systems on \(\Omega\)

Equation (5.17) allows us to investigate the general properties of the pairs \((X, x) \in \mathcal{P}(\Omega) \times \Omega\) in what follows, we assume that \(\Omega\) is an arbitrary (even non finite) set. We call any ordered pair \((X, x) \in \mathcal{P}(\Omega) \times \Omega\) a pointed pair on \(\Omega\). Let \(\hat{\Omega} := \mathcal{P}(\Omega) \times \Omega\), \(\hat{\Omega}_\text{tr} := \{(X, x) \in \hat{\Omega} : x \in X\}\) and \(\hat{\Omega}_\text{nt} := \hat{\Omega} \setminus \hat{\Omega}_\text{tr}\). We call the elements of \(\hat{\Omega}_\text{tr}\) and \(\hat{\Omega}_\text{nt}\) respectively the **trivial** and the **non-trivial** pointed pairs. Let \(SS(\hat{\Omega}) := \mathcal{P}(\mathcal{P}(\Omega))\) and \(PPS(\hat{\Omega}) := \mathcal{P}(\hat{\Omega})\).

Let us define a map \(\partial : PPS(\hat{\Omega}) \to SS(\hat{\Omega})\) such that

\[
\partial(\mathcal{D}) := \{ A \in \mathcal{P}(\hat{\Omega}) : (A \setminus \{x\}, x) \notin \mathcal{D} \ \forall x \in A\},
\]  

(5.18)

for any \(\mathcal{D} \in PPS(\hat{\Omega})\). We call the set system \(\partial(\mathcal{D})\) the **boundary** of \(\mathcal{D}\).

We now provide some basic properties of the boundary of a pointed pair family \(\mathcal{D}\).
Proposition 5.6.1. Let \( \{ D_i : i \in I \} \subseteq PPS(\Omega) \). Then, the following conditions hold:

(i) If \( D_i \subseteq D_j \) then \( \partial(D_i) \supseteq \partial(D_j) \);
(ii) If \( D := \bigcap_{i \in I} D_i \), then \( \partial(D) = \bigcup_{i \in I} \partial(D_i) \);
(iii) If \( D := \bigcup_{i \in I} D_i \), then \( \partial(D) = \bigcap_{i \in I} \partial(D_i) \).

Proof. (i): Let \( A \in \partial(D_j) \). Then, for any \( x \in A \) it results that \( A \setminus \{ x \}, x \notin D_j \) so, a fortiori, \( A \setminus \{ x \}, x \notin D_i \), i. e. \( A \in \partial(D_i) \).

(ii): It follows immediately by part (i).

(iii): By part (i), it is straightforward to see that \( \partial(D) \subseteq \bigcap_{i \in I} \partial(D_i) \). On the other hand, let \( A \in \bigcap_{i \in I} \partial(D_i) \). Hence, for any \( x \in A \) and any \( i \in I \) it follows that \( A \setminus \{ x \}, x \notin D_i \), i. e. for any \( x \in A \) we have that \( A \setminus \{ x \}, x \notin \bigcup_{i \in I} D_i = D \) or, equivalently, \( A \in \partial(D) \). \( \square \)

We now introduce two fundamental equivalence relations on \( PPS(\Omega) \).

Definition 5.6.2. Let \( D, D' \in PPS(\Omega) \). We say that \( D \) and \( D' \) are:

- boundary equivalent, denoted by \( D \simeq_{\partial} D' \), if \( \partial(D) = \partial(D') \). We denote by \( [D]_{\simeq_{\partial}} \) the equivalence class of \( D \) in \( PPS(\Omega) \) with respect to the equivalence relation \( \simeq_{\partial} \) and we call it the boundary class of \( D \).
- essentially equivalent, denoted by \( D \equiv D' \), if \( D \cap \hat{\Omega}_{\tr} = D' \cap \hat{\Omega}_{\tr} \). We denote by \( [D]_{\equiv} \) the equivalence class of \( D \) in \( PPS(\Omega) \) with respect to the equivalence relation \( \equiv \) and call it the essential class of \( D \).

The basic relation between the essential equivalence and the boundary equivalence is easily established in next result.

Proposition 5.6.3. Let \( D, D' \in PPS(\Omega) \). Then, \( D \equiv D' \implies D \simeq_{\partial} D' \).

Proof. Just observe that \( A \in \partial(D) \) if and only if \( A \setminus \{ x \}, x \notin D \) for any \( x \in A \) and, by essential equivalence, this is equivalent to say that \( A \setminus \{ x \}, x \notin D' \) for any \( x \in A \), i. e. \( A \in \partial(D') \). This proves that \( \partial(D) = \partial(D') \). \( \square \)

At this point, we introduce the specific families of pointed pair systems on \( \Omega \) that are our main element of study.

Definition 5.6.4. Let \( D \in PPS(\Omega) \). Then, we say that \( D \) is:

- extensive, if whenever \( (X, x) \in D, Y \in \mathcal{P}(\Omega) \) and \( X \subseteq Y \), then \( (Y, x) \in D \). We set \( PPS_{ext}(\Omega) := \{ D \in PPS(\Omega) : D \text{ is extensive} \} \);
- reflexive, if \( \hat{\Omega}_{\tr} \subseteq D \). We set \( PPS_{ref}(\Omega) := \{ D \in PPS(\Omega) : D \text{ is reflexive} \} \);
- finitely hereditary, if for any \( (X, x) \in D \) there exists a finite subset \( F \) of \( X \) such that \( (F, x) \in D \). We set \( PPS_{fh}(\Omega) := \{ D \in PPS(\Omega) : D \text{ is finitely hereditary} \} \);
Proposition 5.6.5. The following conditions hold:

(i) if $D$ is extensive then $\partial(D) \subseteq AC(\Omega)$ and, moreover,

\[ A \in \partial(D) \iff (B \subseteq A, x \in A \setminus B \implies (B, x) \notin D); \]

(ii) if $D$ is finitely hereditary then $\partial(D)$ is a finitary set system on $\Omega$;

(iii) if $D$ is extensive and finitely hereditary then

\[ A \in \partial(D) \iff (F \subseteq_f A, x \in A \setminus F \implies (F, x) \notin D); \]

(iv) if $D$ is extensive and matroidal, $A \in \partial(D)$ and $x \in \Omega \setminus A$, then

\[ (A, x) \in D \iff A \cup \{x\} \notin \partial(D); \]

(v) if for any $A \in \partial(D)$ and any $x \in \Omega \setminus A$, we have that $A \cup \{x\} \notin \partial(D)$ if and only if $(A, x) \in D$, then $D$ is matroidal;

(vi) Let $D, D' \in PPS(\Omega)$ such that $D \subseteq D'$ and $D \cong \partial D'$. If $D$ is extensive and matroidal and $D'$ is absorbing, then $D$ is absorbing and $D \cong D'$.

Proof. (i): Clearly, $\emptyset \in \partial(D)$. Let $X \subseteq Y$ with $Y \in \partial(D)$. Suppose by contradiction that $X \notin \partial(D)$. Then, there exists $x \in X$ such that $(X \setminus \{x\}, x) \in D$. Since $D \in PPS_{ext}(\Omega)$, it results that $(Y \setminus \{x\}, x) \in D$, contradicting our hypothesis on $Y$. This shows that $\partial(D) \subseteq AC(\Omega)$.

Let $A \in \partial(D)$, $B \subseteq A$ and $x \in A \setminus B$. It must be $(B, x) \notin D$, otherwise $(A \setminus \{x\}, x) \in D$, contradicting the fact that $A \in \partial(D)$.

On the contrary, let $A \in P(\Omega)$ and assume that for any $B \subseteq A$ and any $x \in A \setminus B$, it follows that $(B, x) \notin D$. Let us fix $x \in A$ and set $B := A \setminus \{x\}$. Then, $(A \setminus \{x\}, x) \notin D$ and, by the arbitrariness of $x$, we infer the thesis.

(ii): Let $X \in P(\Omega)$ and assume that $F \in \partial(D)$ for any $F \subseteq_f X$. Suppose by contradiction that $X \notin \partial(D)$, i.e. there exists $x \in X$ for which $(X \setminus \{x\}, x) \in D$. In particular, there exists $F' \subseteq_f X \setminus \{x\}$ such that $(F', x) \in D$. Let $G := F' \cup \{x\}$. Then, $G \in \partial(D)$ but this would imply that $(G \setminus \{x\}, x) = (F, x) \notin D$, that is a contradiction. This proves that $X \in \partial(D)$.

(iii): Let $A \in P(\Omega)$ such that $(F, x) \notin D$ for any $F \subseteq_f A$ and any $x \in A \setminus F$. Assume by contradiction that $A \notin \partial(D)$, i.e. $(A \setminus \{x\}, x) \in D$ for some $x \in A$. Since $D \in PPS_{abs}(\Omega)$, there exists $F \subseteq_f A \setminus \{x\}$ such that $(F, x) \in D$, but this contradicts our choice of $A$. Thus $A \in \partial(D)$. To show the converse, just take $F \subseteq_f A$ instead of $B$ in (5.19).
$\textbf{(iv)}$: Let $A \in \partial(\Omega)$, $x \in \Omega \setminus A$ and assume that $(A, x) \in \Omega$. It must necessarily be $A \cup \{x\} \notin \partial(\Omega)$, otherwise $(A, x) \in \partial(\Omega)$ by definition of boundary. Vice versa, assume that $B := A \cup \{x\} \notin \partial(\Omega)$, i. e. there exists $y \in B$ for which $(B \setminus \{y\}, y) \in \Omega$. If $y = x$, then $(A, x) \in \Omega$ and the claim has been proved; on the contrary, assume $y \in A$. By part (i), $\partial(\Omega)$ is an abstract complex, therefore $C := A \setminus \{y\} \in \partial(\Omega)$. Note that $(C \cup \{y\}, y) = (B \setminus \{y\}, y) \in \Omega$ and that $(C, y) = (A \setminus \{y\}, y) \notin \Omega$ since $A \in \partial(\Omega)$ and $y \in A$. Now, since $\Omega \in \PPS_{\text{mat}}(\Omega)$, it results that $(C \cup \{y\}, x) = (A, x) \in \Omega$ and this shows the claim.

$\textbf{(v)}$: Let $A \in \partial(\Omega)$ and $x, y \in \Omega$ such that $(A, x) \notin \Omega$ and $(A \cup \{y\}, x) \in \Omega$. In particular, it must necessarily be $A \cup \{x\} \cup \{y\} \notin \partial(\Omega)$. So, $(A \cup \{x\}, y) \in \Omega$, i. e. $\Omega \notin \PPS_{\text{mat}}(\Omega)$.

$\textbf{(vi)}$: Let $(X, x) \in \Omega$ be a non-trivial pair. Then, $(X, x) \in \Omega'$. Since $\Omega' \in \PPS_{\text{abs}}(\Omega)$, there exists $A \in \partial(\Omega')$ such that $A \subseteq X$ and $(A, x) \in \Omega'$. Moreover, by $\partial(\Omega') = \partial(\Omega')$, it results that $A \in \partial(\Omega)$. Now, let us observe that $A \cup \{x\} \notin \partial(\Omega') = \partial(\Omega)$, otherwise $(A, x) \notin \Omega'$, that contradicts the fact that $\Omega' \in \PPS_{\text{abs}}(\Omega)$. By (5.21), it follows that $(A, x) \in \Omega$, i. e. $\Omega \in \Omega' \in \PPS_{\text{abs}}(\Omega)$.

We now show that $\Omega \equiv \Omega'$. Let $(X, x) \in \Omega'$ be non-trivial. Since $\Omega' \in \PPS_{\text{abs}}(\Omega)$, there exists $A \in \partial(\Omega')$ such that $A \subseteq X$ and $(A, x) \in \Omega'$. In particular, (5.21), $A \cup \{x\} \notin \partial(\Omega')$. Thus, $A \in \partial(\Omega)$ and $A \cup \{x\} \notin \partial(\Omega)$ and this implies that $(A, x) \in \Omega$. We conclude the proof by using the fact that $\Omega \in \PPS_{\text{ext}}(\Omega)$.

In what follows we want to understand in which way the boundary and essential classes of a generic $\Omega \in \PPS(\Omega)$ behave with respect to the various families of specific pointed pair systems introduced in Definition 5.6.4. To this regard, we need some notations. We then set:

$$[\Omega]_{\text{ema}} := [\Omega]_{\text{ma}} \cap \PPS_{\text{ext}}(\Omega) \cap \PPS_{\text{mat}}(\Omega) \cap \PPS_{\text{abs}}(\Omega);$$

$$[\Omega]_{\text{efm}} := [\Omega]_{\text{ma}} \cap \PPS_{\text{ext}}(\Omega) \cap \PPS_{\text{fh}}(\Omega) \cap \PPS_{\text{mat}}(\Omega);$$

$$[\Omega]_{\text{refm}} := [\Omega]_{\text{efm}} \cap \PPS_{\text{ref}}(\Omega);$$

$$[\Omega]_{\text{refma}} := [\Omega]_{\text{refm}} \cap \PPS_{\text{abs}}(\Omega);$$

$$[\Omega]_{\text{efm}} := [\Omega] \cap \PPS_{\text{mat}}(\Omega);$$

$$[\Omega]_{\text{efm}} := [\Omega] \cap \PPS_{\text{mat}}(\Omega) \cap \PPS_{\text{fh}}(\Omega).$$

In the next results, we will prove that both $[\Omega]_{\text{efm}}$ and $[\Omega]_{\text{efm}}$ are complete lattices for each $\Omega \in \PPS(\Omega)$.

$\textbf{Theorem 5.6.6.}$ For any $\Omega \in \PPS(\Omega)$, $[\Omega]_{\text{efm}}$ is a complete lattice such that if $\{\Omega_i : i \in I\} \subseteq [\Omega]_{\text{efm}}$, then $\bigvee_{i \in I} \Omega_i = \bigcup_{i \in I} \Omega_i$ and $\bigwedge_{i \in I} \Omega_i = \bigcap_{i \in I} \Omega_i$ when $I$ is finite.

$\textbf{Proof.}$ Let us fix $\Omega \in \PPS(\Omega)$ and let $\{\Omega_i : i \in I\} \subseteq [\Omega]_{\text{efm}}$. We divide the proof in several steps.

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I We will show that $\bigcup_{i \in I} D_i \in [D]|refm|_{\succeq 0}$ and $\bigvee_{i \in I} D_i = \bigcup_{i \in I} D_i$.

It is obvious that $\bigcup_{i \in I} D_i$ is reflexive. Moreover, by (iii) of Proposition 5.6.1 we have that $\partial(\bigcup_{i \in I} D_i) = \bigcap_{i \in I} \partial(D_i) = \partial(D)$, so $\bigcup_{i \in I} D_i \models \partial(D)$ for any $i \in I$. Let us prove now that $\bigcup_{i \in I} D_i$ is extensive. Assume that $(X, x) \in \bigcup_{i \in I} D_i$ and $X \subseteq Y$. Then, there exists an index $k \in I$ such that $(X, x) \in D_k$, so $(Y, y) \in D_k$ and, in particular, $(Y, x) \in \bigcup_{i \in I} D_i$. This shows that $\bigcup_{i \in I} D_i \in PPS_{ext}(\Omega)$.

Let us prove now that $\bigcup_{i \in I} D_i$ is finitely hereditary. For this, assume that $(X, x) \in \bigcup_{i \in I} D_i$.

Then, there exists $k \in I$ such that $(X, x) \in D_k$, so, there exists $F \subseteq_X X$ such that $(F, x) \in D_k$. A fortiori, $(F, x) \in \bigcup_{i \in I} D_i$. This shows that $\bigcup_{i \in I} D_i \in PPS_{fh}(\Omega)$.

II We prove that

$$\overline{D} := \hat{\Omega}_t \cup \{(X, x) \in \hat{\Omega}_{ntr} : \exists A \subseteq X \text{ such that } A \in \partial(D) \text{ and } A \cup \{x\} \notin \partial(D)\}$$

is a minimum in $[D]|refm|_{\succeq 0}$.

Obviously, $\overline{D}$ is reflexive and extensive. Furthermore, it is contained in each $D'$, with $D' \in [D]|refm|_{\succeq 0}$. In fact, if $(X, x) \in \overline{D}$ is non-trivial, by (5.21) it follows that $(A, x) \in D'$ for each $D' \in [D]|refm|_{\succeq 0}$, so $(X, x) \in D'$ by extensivity.

Let us show that $\overline{D}$ is finitely hereditary. Let $(X, x) \in \overline{D}$ be non-trivial. Then, there exists $A \subseteq_X X$ such that $A \in \partial(D)$ and $A \cup \{x\} \notin \partial(D)$. Hence, $(A \cup \{x\}, x) \in \partial(D)$, and $A \cup \{x\} \notin \partial(D)$. This shows that $(F, x) \in \overline{D}$ and the claim follows.

Let us prove now that $\overline{D} \in PPS_{mat}(\Omega)$. Let $A \in \partial(\overline{D})$ and $x, y \in \Omega$ such that $(A, x) \notin \overline{D}$ and $(A \cup \{y\}, x) \notin \overline{D}$. We have to prove that $(A \cup \{x\}, y) \in \overline{D}$, ore, equivalently, the existence of $B \subseteq A \cup \{x\}$ belonging to $\partial(D)$ and such that $B \cup \{y\} \notin \partial(D)$.

By our assumptions, there exists $C \subseteq A \cup \{y\}$ such that $C \in \partial(D)$ and $C \cup \{x\} \notin \partial(D)$. It is straightforward to see that $y \in C$. Now, let $\sigma = \sigma' \cup \{y\}$, where clearly $C' \subseteq A$. We are saying that $C' \cup \{x\} \notin \partial(D)$. We must prove that $C' \cup \{x\} \in \partial(D)$. Suppose by contradiction that $C' \cup \{x\} \notin \partial(D)$. Hence $C' \cup \{x\} \notin \partial(D)$. Again by (5.21), the previous condition is equivalent to say that $A \cup \{x\} \notin \partial(D)$. This contradicts our choice of $A$. Hence $C' \cup \{x\} \in \partial(D)$ and setting $B := \sigma' \cup \{x\}$, the claim has been proved.
Finally, let us prove now that $\overline{D} \in [D]_{\approx_y}$. Let $A \in \partial(\overline{D})$ and suppose by contradiction that $A \not\subseteq \partial(D)$. Since $\partial(D)$ is finitary, there exists $F \subseteq_f A$ such that $F \not\subseteq \partial(D)$. Thus, $F \in \partial(\overline{D})$, otherwise there would be $y \in F$ such that $(\overline{D} \setminus \{y\}, y) \in \overline{D}$ and, by extensivity of $\overline{D}$, $(A \setminus \{y\}, y) \in \overline{D}$, contradicting our choice of $A$. Hence, we proved that $\partial(\overline{D}) \setminus \partial(D)$ is non-empty. Let $G$ be minimal in $\partial(\overline{D}) \setminus \partial(D)$. Surely, $G \neq \emptyset$ since $\emptyset$ belongs by definition to each boundary. Therefore, we infer the existence of some $x \in G$ for which $G \setminus \{x\} \in \partial(D)$. Since $G = (G \setminus \{x\} \cup \{x\} \not\subseteq \partial(D)$, we have that $(G \setminus \{x\}, x) \in \overline{D}$ and, by extensivity of $\overline{D}$, $(A \setminus \{x\}, x) \in \overline{D}$, contradicting our choice of $A$. Thus $\partial(\overline{D}) \subseteq \partial(D)$.

Contrariwise, let $A \in \partial(D)$. If $A \not\subseteq \partial(\overline{D})$, there exists $B \subseteq A \setminus \{x\}$ such that $B \in \partial(D)$ and $B \cup \{x\} \not\in \partial(D)$. By (5.21), this is equivalent to require that $(B, x) \in \overline{D}'$, with $\overline{D}' \in [D]_{\approx_y}$, so $(A \setminus \{x\}, x) \in \overline{D}'$ for any $\overline{D}' \in [D]_{\approx_y}$, contradicting the fact that $A \in \partial(D)$. This proves that $\partial(D) \subseteq \partial(\overline{D})$ and, hence $\partial(D) = \partial(\overline{D})$.

III Let $\mathcal{L}$ be the family of all lower bounds of $\{D_i : i \in I\}$. By the previous part, $\mathcal{L}$ is non-empty. By the first part of the proof, $\bigcup \mathcal{L} \in PPS_{ref}(\Omega) \cap PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega)$. Moreover, it is obvious that $\bigcup \mathcal{L}$ is maximal in $\mathcal{L}$ and that $\partial(\bigcup \mathcal{L}) = \partial(D_i)$ for any $i \in I$. This shows that $\bigcap_{i \in I} D_i = \bigcup \mathcal{L}$.

IV We now prove that $\bigcap_{i \in I} D_i = \bigcap_{i \in I} D_i$ when $I$ is finite. To be more specific, we will prove that $\bigcap_{i \in I} D_i \in PPS_{ref}(\Omega) \cap PPS_{ext}(\Omega) \cap PPS_{mat}(\Omega)$ also in the infinite case and, in the last part of the proof, that $\bigcap_{i \in I} D_i \in PPS_{fh}(\Omega)$ only for $I$ finite.

Obviously $\bigcap_{i \in I} D_i \in PPS_{ref}(\Omega)$. Let us show that $\bigcap_{i \in I} D_i \in PPS_{ext}(\Omega)$. Assume that $(X, x) \in \bigcap_{i \in I} D_i$ and $X \subseteq Y$. Then, $(X, x) \in D_i$ for any $i \in I$ so $(Y, x) \in D_i$ for any $i \in I$ and, consequently, $(Y, x) \in \bigcap_{i \in I} D_i$.

V In order to show that $\bigcap_{i \in I} D_i \in PPS_{mat}(\Omega)$, we need the following technical result. Let $F \subseteq_f \Omega$. Then, $F \in \partial(D)$ if and only if $F \in \partial(\bigcap_{i \in I} D_i)$.

To this regard, let us observe that, since $\bigcap_{i \in I} D_i \subseteq D_i$ for any $i \in I$, by (i) of Proposition 5.6.1, we deduce that if $F \in \partial(D)$ then $F \in \partial(\bigcap_{i \in I} D_i)$.

On the other hand, we prove the converse by induction on $|F|$. Assume that $F = \{x\} \in \partial(\bigcap_{i \in I} D_i)$. If $\{x\} \not\subseteq \partial(D)$, then $(\emptyset, x) \in D_i$ for any $i \in I$, thus $(\emptyset, x) \in \bigcap_{i \in I} D_i$, so $\{x\} \not\subseteq \partial(\bigcap_{i \in I} D_i)$, contradicting our hypothesis. Suppose now the claim to be true for all finite subsets having cardinality at most $n$ and let $F \in \partial(\bigcap_{i \in I} D_i)$ such that $|F| = n + 1$. Suppose by contradiction that $F \not\subseteq \partial(D)$. By using (5.21), it is immediate to prove the existence of some $y \in F$ for which $(F \setminus \{y\}, y) \in D_i$, for any $i \in I$. Since $\bigcap_{i \in I} D_i \in PPS_{ext}(\Omega)$, by (i) of Proposition 5.6.5, $\partial(\bigcap_{i \in I} D_i)$ is an abstract complex. This
implies that $F \setminus \{y\} \in \partial(\bigcap_{i \in I} D_i)$. By the inductive hypothesis, $F \setminus \{y\} \in \partial(\mathcal{D})$. But, since $(F \setminus \{y\}, y) \in \bigcap_{i \in I} D_i$, we are contradicting the fact that $F \in \partial(\bigcap_{i \in I} D_i)$. This shows that $F \in \partial(\mathcal{D})$.

**VI** We now show that $\bigcap_{i \in I} D_i \in [\mathcal{D}]_{\geq 0}$ and that $\bigcap_{i \in I} D_i \in PPS_{mat}(\Omega)$.

By (ii) of Proposition 5.6.5, $\partial(\mathcal{D})$ is finitary on $\Omega$, while by (i) of Proposition 5.6.1, $\partial(\mathcal{D}) \subseteq \partial(\bigcap_{i \in I} D_i)$. On the contrary, let $A \notin \partial(\mathcal{D})$. Since $\partial(\mathcal{D})$ is finitary on $\Omega$, there exists $F \subseteq A$ such that $F \notin \partial(\mathcal{D})$. In particular, by the previous part, $F \notin \partial(\bigcap_{i \in I} D_i)$.

Therefore, there exists $y \in F$ such that $(F \setminus \{y\}, y) \in \bigcap_{i \in I} D_i$. Since $\bigcap_{i \in I} D_i \in PPS_{ext}(\Omega)$, we conclude that $(A \setminus \{y\}, y) \in \bigcap_{i \in I} D_i$, so $A \notin \partial(\bigcap_{i \in I} D_i)$. This shows that $\partial(\bigcap_{i \in I} D_i) = \partial(\mathcal{D})$.

Finally, we must prove that $\bigcap_{i \in I} D_i \in PPS_{mat}(\Omega)$. Let $A \in \partial(\bigcap_{i \in I} D_i)$ and $x, y \in \Omega$ such that $(A, x) \notin \bigcap_{i \in I} D_i$ and $(A \cup \{x\}, y) \in \bigcap_{i \in I} D_i$. By (5.21), it is straightforward to see that $A \cup \{x\} \in \partial(\mathcal{D})$ and, by the same relation, $(A, x) \notin D_i$ for each $i \in I$. Since $D_i \in PPS_{mat}(\Omega)$ for any $i \in I$, it follows that $(A \cup \{x\}, y) \in D_i$ for any $i \in I$, thus it also belongs to $\bigcap_{i \in I} D_i$.

**VII** To conclude the proof, let us show that $\bigcap_{i \in I} D_i \in PPS_{fh}(\Omega)$ when $I$ is finite. We prove the claim for two pointed pair systems $\mathcal{D}', \mathcal{D}'' \in [\mathcal{D}|efm]_{\geq 0}$. To this regard, let $(X, x) \in \mathcal{D}' \cap \mathcal{D}''$. Then, there exist $F, F' \subseteq X$ such that $(F, x) \in \mathcal{D}$ and $(F', x) \in \mathcal{D}''$. Thus $(F \cup F', x) \in \mathcal{D}' \cap \mathcal{D}''$ and the claim has been showed. Now, extending by induction the previous argument, we conclude the proof.

**Remark 5.6.7.** The same claim of Theorem 5.6.6 holds even if we consider $[\mathcal{D}|efm]_{\geq 0}$. The main change in this case is in part (II), since $\mathcal{D}$ can be defined by removing the trivial pairs, i.e.

$$\mathcal{D} := \{(X, x) \in \mathcal{D}_{str} : \exists A \subseteq X \text{ such that } A \in \partial(\mathcal{D}) \text{ and } A \cup \{x\} \notin \partial(\mathcal{D})\}. \quad (5.23)$$

The other modifications are obvious and subsequent.

**Theorem 5.6.8.** For any $\mathcal{D} \in PPS(\Omega)$, we have that $[\mathcal{D}|efm]_{\geq 0}$ is a complete lattice such that if $\{D_i\} \subseteq [\mathcal{D}|efm]_{\geq 0}$, then $\bigvee_{i \in I} D_i = \bigcup_{i \in I} D_i$.

**Proof.** Let $\{D_i : i \in I\} \subseteq [\mathcal{D}|efm]_{\geq 0}$. It has been shown in Theorem 5.6.6 that $\bigcup_{i \in I} D_i \in PPS_{ext}(\Omega) \cap PPS_{fh}(\Omega) \cap PPS_{mat}(\Omega)$. Furthermore, it is straightforward to verify that $\bigcup_{i \in I} D_i \in [\mathcal{D}]_{\geq 0}$ and, hence, that $\bigvee_{i \in I} D_i = \bigcup_{i \in I} D_i$.

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Let us consider the pointed pair system
\[ \hat{\mathcal{D}} := \{(X,x) \in \hat{\Omega} : \exists A \subseteq X \text{ such that } x \notin A \text{ and } (A,x) \in \mathcal{D}\}. \] (5.24)

Obviously \( \hat{\mathcal{D}} \in \text{PPS}_{\text{ext}}(\Omega) \cap \text{PPS}_{\text{fh}}(\Omega) \). We now claim that \( \hat{\mathcal{D}} \text{PPS}_{\text{mat}}(\Omega) \). To this regard, let \( A \in \partial(\hat{\mathcal{D}}) \) and \( x \in \Omega \setminus A \). We argue that \( A \cup \{ x \} \notin \partial(\hat{\mathcal{D}}) \) if and only if \( (A,x) \in \hat{\mathcal{D}} \). In fact, assume that \( A \cup \{ x \} \notin \partial(\hat{\mathcal{D}}) \). It must necessarily be \( (A,x) \in \hat{\mathcal{D}} \), otherwise \( (A,x) \notin \mathcal{D}_i \) for any \( i \in I \) and, by (5.21), this is equivalent to say that \( A \cup \{ x \} \in \partial(\mathcal{D}_i) \subseteq \partial(\hat{\mathcal{D}}) \), that is a contradiction. Contrariwise, if \( (A,x) \in \hat{\mathcal{D}} \), by definition of boundary, we must necessarily have \( A \cup \{ x \} \notin \partial(\hat{\mathcal{D}}) \). We infer the claim by (v) of Proposition 5.6.5.

Furthermore, \( \hat{\mathcal{D}} \) is a minimum in \( [\mathcal{D}]_{\text{efm}} \). In fact, it is immediate to see that if \( (X,x) \in \hat{\mathcal{D}} \), then \( (X,x) \in \mathcal{D}' \) for any \( \mathcal{D}' \in [\mathcal{D}]_{\text{efm}} \). Finally, if \( (X,x) \in \mathcal{D}' \) is a non-trivial pair, then it obviously belongs to \( \hat{\mathcal{D}} \). This shows that \( \hat{\mathcal{D}} \in [\mathcal{D}]_{\text{efm}} \).

Let now \( \mathcal{L} \) be the family of all lower bounds of \( \{ \mathcal{D}_i : i \in I \} \). By the above argument, \( \mathcal{L} \) is non-empty. Moreover \( \bigcup \mathcal{L} \in [\mathcal{D}]_{\text{efm}} \cap \text{PPS}_{\text{ext}}(\Omega) \cap \text{PPS}_{\text{fh}}(\Omega) \cap \text{PPS}_{\text{mat}}(\Omega) \), belongs to \( \mathcal{D}_i \) for any \( i \in I \) and it is maximal in \( \mathcal{L} \). This shows that \( \bigwedge_{i \in I} \mathcal{D}_i = \bigcup \mathcal{L} \) and, hence, that \([\mathcal{D}]_{\text{efm}}\) is a complete lattice.

In the next result, we will show that whenever we take \( \mathcal{D}' \in \text{PPS}(\Omega) \) such that \( \mathcal{D}_1 \subseteq \mathcal{D}' \subseteq \mathcal{D}_2 \), where \( \mathcal{D}_1, \mathcal{D}_2 \in [\mathcal{D}]_{\text{efm}} \), then \( \mathcal{D}' \) inherits only matroidality and essential equivalence.

**Proposition 5.6.9.** For any \( \mathcal{D}, \mathcal{D}' \in \text{PPS}(\Omega) \) and \( \mathcal{D}_1, \mathcal{D}_2 \in [\mathcal{D}]_{\text{efm}} \) such that \( \mathcal{D}_1 \subseteq \mathcal{D}' \subseteq \mathcal{D}_2 \), we have that \( \mathcal{D}' \in [\mathcal{D}]_{\text{efm}} \).

**Proof.** It is immediate to show that \( \mathcal{D}' \in [\mathcal{D}]_{\text{efm}} \). We now prove that \( \mathcal{D} \in \text{PPS}_{\text{mat}}(\Omega) \). To this regard, let \( A \in \partial(\mathcal{D}') \) and \( x \in \Omega \setminus A \). Let us show that \( A \cup \{ x \} \notin \mathcal{D}' \) if and only if \( (A,x) \notin \partial(\mathcal{D}) \). Assume that \( (A,x) \in \mathcal{D}' \). Then, \( (A,x) \in \mathcal{D}_1 \) so, by (5.21), \( A \cup \{ x \} \notin \partial(\mathcal{D}_1) \). In particular, \( A \cup \{ x \} \notin \partial(\mathcal{D}') \). On the contrary, if \( A \cup \{ x \} \notin \partial(\mathcal{D}') \), then \( A \cup \{ x \} \notin \partial(\mathcal{D}_1) \). This entails that \( (A,x) \notin \mathcal{D}_1 \) and, hence, that \( (A,x) \notin \mathcal{D}' \). We conclude by (vi) of Proposition 5.6.5.

We show now that \([\mathcal{D}]_{\text{ema}} \subset \) is a complete lattice.

**Theorem 5.6.10.** For any \( \mathcal{D} \in \text{PPS}(\Omega) \), we have that \([\mathcal{D}]_{\text{ema}} \subset \) is a complete lattice such that if \( \{ \mathcal{D}_i : i \in I \} \subseteq [\mathcal{D}]_{\text{ema}} \subset \), then \( \bigvee_{i \in I} \mathcal{D}_i = \bigcup_{i \in I} \mathcal{D}_i \) and \( \bigwedge_{i \in I} \mathcal{D}_i = \bigcap_{i \in I} \mathcal{D}_i \).

**Proof.** Let \( \{ \mathcal{D}_i : i \in I \} \subseteq [\mathcal{D}]_{\text{ema}} \subset \). It has been shown in the proof of Theorem 5.6.6 that \( \bigcup_{i \in I} \mathcal{D}_i \in [\mathcal{D}]_{\text{ema}} \cap \text{PPS}_{\text{ext}}(\Omega) \cap \text{PPS}_{\text{mat}}(\Omega) \). We now show that \( \bigcup_{i \in I} \mathcal{D}_i \in \text{PPS}_{\text{abs}}(\Omega) \). Let \( (X,x) \in \bigcup_{i \in I} \mathcal{D}_i \) be a non-trivial pair. Then, there exists \( k \in I \) such that \( (X,x) \in \mathcal{D}_k \), so there exists \( A \in \partial(\mathcal{D}_k) \) with \( A \subseteq X \) and \( (A,x) \in \mathcal{D}_k \). We conclude by using the fact that \( \bigcup_{i \in I} \mathcal{D}_i \in [\mathcal{D}]_{\text{ema}} \). This shows that \( \bigcup_{i \in I} \mathcal{D}_i \in \text{PPS}_{\text{abs}}(\Omega) \). In particular, it is straightforward to deduce that \( \bigvee_{i \in I} \mathcal{D}_i = \bigcup_{i \in I} \mathcal{D}_i \).

Let us prove that \( \bigwedge_{i \in I} \mathcal{D}_i = \bigcap_{i \in I} \mathcal{D}_i \). It has been shown in Theorem 5.6.6 that \( \bigcap_{i \in I} \mathcal{D}_i \in \text{PPS}_{\text{ext}}(\Omega) \).
Now, we have to prove that $\bigcap_{i \in I} D_i \in [D]_{\simeq_\partial} \cap \text{PPS}_{\text{abs}}(\Omega) \cap \text{PPS}_{\text{mat}}(\Omega)$. By (i) of Proposition 5.6.1, we have $\partial(D) \subseteq \partial(\bigcap_{i \in I} D_i)$. Contrariwise, let $A \in \partial(\bigcap_{i \in I} D_i) \setminus \partial(D)$. Then, there exists $y \in A$ such that $(A \setminus \{y\}, y) \in D_j$ for some $j \in I$. Since $D_j \in \text{PPS}_{\text{abs}}(\Omega)$, there exists $B \in \partial(\bigcap_{i \in I} D_i)$ such that $B \subseteq A \setminus \{y\}$ and $(B, y) \in D_j$. By (5.21), it follows that $B \cup \{y\} \notin \partial(D)$ and, again by (5.21) and since $D_i \in [D]_{\simeq_\partial}$ for any $i \in I$, we deduce that $(B, y) \in D_i$ for any $i \in I$, i. e. $(B, y) \in \bigcap_{i \in I} D_i$. Since $\bigcap_{i \in I} D_i \in \text{PPS}_{\text{ext}}(\Omega)$, we have that $(A \cup \{y\}, y) \in \bigcap_{i \in I} D_i$, i. e. $A \notin \partial(\bigcap_{i \in I} D_i)$, contradicting our choice of $A$. This shows that $\partial(\bigcap_{i \in I} D_i) = \partial(D)$.

We now prove that $\bigcap_{i \in I} D_i \in \text{PPS}_{\text{mat}}(\Omega)$. Let $A \in \partial(\bigcap_{i \in I} D_i)$ and $x, y \in \Omega$ such that $(A, x) \notin \bigcap_{i \in I} D_i$ and $(A \cup \{y\}, x) \in \bigcap_{i \in I} D_i$. Thus, $(A, x) \notin D_i$ for each $i \in I$ and, by (5.21), it follows that $A \cup \{x\} \in \partial(D_i)$ for any $i \in I$. Since $D_i \in \text{PPS}_{\text{mat}}(\Omega)$ for any $i \in I$, it results that $(A \cup \{x\}, y) \in \bigcap_{i \in I} D_i$ for any $i \in I$, thus it also belongs to $\bigcap_{i \in I} D_i$.

Finally, we show that $\bigcap_{i \in I} D_i \in \text{PPS}_{\text{abs}}(\Omega)$. To this regard, let $(X, x) \in \bigcap_{i \in I} D_i$ be non-trivial. Suppose by contradiction that for each $A \in \partial(\bigcap_{i \in I} D_i)$, with $A \subseteq X$, it results that $(A, x) \notin \bigcap_{i \in I} D_i$. Since $\bigcap_{i \in I} D_i \in \text{PPS}_{\text{ext}}(\Omega) \cap \text{PPS}_{\text{mat}}(\Omega)$, by (5.21) we have that $A \cup \{x\} \in \partial(\bigcap_{i \in I} D_i)$. Then, $A \cup \{x\} \in \partial(D_i)$, so, again by (5.21), $(A, x) \notin D_j$ for some $j \in I$. In other terms, we are saying that $(X, x) \in D_j \setminus X$ and for each $A \in \partial(D_j)$ with $A \subseteq X$, it results that $(A, x) \notin D_j$, contradicting the fact that $D_j \in \text{PPS}_{\text{abs}}(\Omega)$. It is now straightforward to show that $\bigcap_{i \in I} D_i = \bigcap_{i \in I} D_i$.

In the next result we assert that boundary equivalence, together with extensivity, matroidality and absorbency entails essential equivalence.

We now prove that $[D]|\ref{ma}_{\simeq_\partial}$ is an ideal of $[D]|\ref{ma}_{\simeq_\partial}$.

**Theorem 5.6.11.** For any $D \in \text{PPS}(\Omega)$ we have that $[D]|\ref{ma}_{\simeq_\partial}$ is an ideal of the lattice $[D]|\ref{ma}_{\simeq_\partial}$.

**Proof.** First of all, we have to show that $[D]|\ref{ma}_{\simeq_\partial}$ is non-empty. To this regard, let us consider the pointed pair system $\overline{D}$ defined in (5.22). By (5.21), it is straightforward to see that $\overline{D}$ is absorbing. Hence $[D]|\ref{ma}_{\simeq_\partial}$ is non-empty.

Let $D' \subseteq D$, where $D \in [D]|\ref{ma}_{\simeq_\partial}$ and $D' \in [D]|\ref{ma}_{\simeq_\partial}$. By (vi) of Proposition 5.6.5, it follows that $D' \in [D]|\ref{ma}_{\simeq_\partial}$.

Finally, by Theorem 5.6.6, Corollary 5.6.11 and Theorem 5.6.10, given $D', D'' \in [D]|\ref{ma}_{\simeq_\partial}$, consider $D' \cup D'' \in [D]|\ref{ma}_{\simeq_\partial}$, then $D', D'' \subseteq D' \cup D''$ and this proves the claim. □

To conclude this section, we consider the map $\partial : \text{SS}(\Omega) \rightarrow \text{PPS}(\Omega)$ such that

$$\partial(\mathcal{F}) := \hat{\Omega}_{tr} \cup \{(X, x) \in \hat{\Omega}_{ntr} : X \cup \{x\} \notin \mathcal{F}\},$$

(5.25)

for any $\mathcal{F} \in \text{SS}(\Omega)$. We now show that $\partial$ and $\partial$ are inverse each other.
Proposition 5.6.12. Let $\mathcal{F} \in SS(\Omega)$. Then:

(i) $\mathcal{F} = \partial(\mathcal{F})$.

(ii) Let $X \in \mathcal{P}(\Omega)$ and $x, y \notin X$. Assume that $(X, x) \notin \mathcal{F}$ but $(X \cup \{y\}, x) \in \mathcal{F}$. Then, $(X \cup \{x\}, y) \in \mathcal{F}$.

Proof. (i): Let $X \in \partial(\mathcal{F})$. This is equivalent to say that for each $x \in X$ it must be $(X \setminus \{x\}, x) \notin \mathcal{F}$, i.e. $X \setminus \{x\} \cup \{x\} = X \in \mathcal{F}$.

(ii): By our assumptions, we have that $X \cup \{x\} \in \mathcal{F}$ but $X \cup \{x\} \cup \{y\} \notin \mathcal{F}$. It is now obvious that $(X \cup \{x\}, y) \in \mathcal{F}$ and the claim has been shown.

Summarizing suitably the results of this section, we can state the following result.

Theorem 5.6.13. Let $\mathcal{D} \in PPS(\Omega)$ extensive, finitely hereditary and matroidal. Then, $\partial(\mathcal{D}) \in FAC(\Omega)$. Moreover, if $\mathcal{F} \in FAC(\Omega)$ then the $[\mathcal{F}]_{efm}$ is a complete lattice.

5.7 Pre-Closure Operators and Pointed Pair Systems

In this final section we study the interrelations between pre-closure operators on $\Omega$ and pointed pair systems $\mathcal{D} \in PPS(\Omega)$ that are extensive, finitely hereditary and matroidal.

Let $\mathcal{D} \in PPS(\Omega)$, then we consider the set operator $\sigma_{\mathcal{D}} \in OP(\Omega)$ given by

$$\sigma_{\mathcal{D}}(X) := \{x \in \Omega : x \in X \vee (X, x) \in \mathcal{D}\}, \quad (5.26)$$

Contrariwise to any set operator $\sigma \in OP(\Omega)$ we associate the pointed pair system

$$\mathcal{D}_\sigma := \{(X, x) \in \hat{\Omega} : x \in \sigma(X)\} \quad (5.27)$$

In the next result, we prove that, whenever $\sigma$ is an extensive set operator, then $\sigma = \sigma_{\mathcal{D}_\sigma}$.

Proposition 5.7.1. If $\sigma \in OP(\Omega)$ is extensive, then $\sigma = \sigma_{\mathcal{D}_\sigma}$.

Proof. Let $X \in \mathcal{P}(\Omega)$. Then, by both (5.26) and (5.27), it results that $x \in \sigma_{\mathcal{D}_\sigma}(X)$ if and only if $x \in X$ or $(X, x) \in \mathcal{D}_\sigma$, i.e. if and only if $x \in X$ or $x \in \sigma(X)$. By extensiveness of $\sigma$, we conclude that the previous condition is equivalent to say that $x \in \sigma(X)$. We conclude by the arbitrariness of $X \in \mathcal{P}(\Omega)$.

We have the following result.

Theorem 5.7.2. Let $\mathcal{D} \in PPS(\Omega)$ extensive, finitely hereditary and matroidal. Then, $\sigma_{\mathcal{D}}$ is a WMLS, finitary pre-closure operator on $\Omega$. On the contrary, let $\sigma$ be a WMLS, finitary pre-closure operator on $\Omega$. Then, $\mathcal{D}_\sigma$ is a reflexive, extensive, finitely hereditary and matroidal pointed pair system on $\Omega$ and, moreover, $[\mathcal{D}_\sigma]_{efm}$ is a complete lattice.

Proof. We divide the proof in two parts.

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(I) Let $\mathcal{D} \in PPS(\Omega)$ be extensive, finitely hereditary and matroidal. We associate the $\sigma_{\mathcal{D}}$ given in (5.26). It is easy to show that it is a pre-closure operator. Let us prove that

$$\Theta(\sigma_{\mathcal{D}}) = \partial(\mathcal{D}).$$

(5.28)

Let $X \in \Theta(\sigma_{\mathcal{D}})$ and assume that $X \notin \partial(\mathcal{D})$. Thus, there exists $x \in X$ such that $(X \setminus \{x\}, x) \in \mathcal{D}$. In particular, we have that $X \setminus \{x\} \notin \sigma_{\mathcal{D}}(X \setminus \{x\})$, obviously a contradiction with our choice of $X$.

Contriwise, let $A \notin \Theta(\sigma_{\mathcal{D}})$. This means that there exists $B \subseteq A \subseteq \sigma_{\mathcal{D}}(B)$. In particular, there exists $x \in A \setminus B$ such that $(B, x) \in \mathcal{D}$. Since $\mathcal{D} \in PPS_{ext}(\Omega)$, it follows that $(A \setminus \{x\}, x) \in \mathcal{D}$, i. e. $A \notin \partial(\mathcal{D})$. This proves (5.28).

Let us prove that $\sigma_{\mathcal{D}}(X)$ is finitary, i. e. $\sigma_{\mathcal{D}}(X) = \bigcup \{\sigma_{\mathcal{D}}(F) : F \subseteq_f X\}$ for any $X \in \mathcal{P}(\Omega)$. Let $x \in \sigma_{\mathcal{D}}(X)$. If $x \notin X$, then $x \in \sigma_{\mathcal{D}}(\{x\})$. On the contrary, if $x \in X$, then $(X, x) \in \mathcal{D}$ and, since $\mathcal{D} \in PPS_{fh}(\Omega)$, there exists $F \subseteq_f X$ such that $(F, x) \in \mathcal{D}$. Hence, $x \in \sigma_{\mathcal{D}}(F)$. This proves that $\sigma_{\mathcal{D}}(X) \subseteq \bigcup \{\sigma_{\mathcal{D}}(F) : F \subseteq_f X\}$. The reverse inclusion is obvious. This proves that $\sigma_{\mathcal{D}}$ is finitary.

We have to show that $\sigma_{\mathcal{D}}$ is WMLS. Let $A \in \Theta(\sigma_{\mathcal{D}})$ and $x \in \sigma_{\mathcal{D}}(A) \setminus \sigma_{\mathcal{D}}(A)$. This means that $x \notin A$, $(A, x) \notin \mathcal{D}$ and $(A \cup \{y\}, y) \in \mathcal{D}$. Moreover, $A \notin \partial(\mathcal{D})$ by (5.28). Hence, by the fact that $\mathcal{D} \in PPS_{ext}(\Omega)$, it follows that $(A \cup \{x\}, y) \in \mathcal{D}$, i. e. $y \in \sigma_{\mathcal{D}}(A \cup \{x\})$. This shows that $\sigma_{\mathcal{D}}$ is WMLS.

(II) Let $\sigma \in OP(\Omega)$ be a WMLS, finitary pre-closure operator. We associate with it the pointed pair system $\mathcal{D}_{\sigma}$ defined in (5.27). Obviously, $\mathcal{D}_{\sigma} \in PPS_{re_j}(\Omega)$. Moreover, we have that $\mathcal{D}_{\sigma} \in PPS_{ext}(\Omega)$, in fact if $(X, x) \in \mathcal{D}_{\sigma}$ and $X \subseteq Y$, then $x \in \sigma(Y)$, i. e. $(Y, x) \in \mathcal{D}_{\sigma}$.

It is also straightforward to see that $\mathcal{D}_{\sigma} \in PPS_{fh}(\Omega)$. As a matter of fact, let $(X, x) \in \mathcal{D}_{\sigma}$, then $x \in \sigma(X) = \bigcup \{\sigma(F) : F \subseteq_f X\}$, i. e. there exists $F \subseteq_f X$ such that $x \in \sigma(F)$, i. e. $(F, x) \in \mathcal{D}_{\sigma}$. Let us observe that

$$\Theta(\sigma) = \partial(\mathcal{D}_{\sigma})$$

(5.29)

by the same argument used to show (5.28). We now prove that if $\mathcal{D} \cong \mathcal{D}_{\sigma}$, then $\sigma_{\mathcal{D}} = \sigma$.

Given $(X, x) \in \hat{\Omega}_{str}$, we have that $(X, x) \in \mathcal{D}$ if and only if $(X, x) \in \mathcal{D}_{\sigma}$. This implies that $\sigma_{\mathcal{D}}(X) = \sigma_{\mathcal{D}_{\sigma}}(X) = \sigma(X)$ and, by the arbitrariness of $X \in \mathcal{P}(\Omega)$, that $\sigma_{\mathcal{D}} = \sigma$.

Finally, $\mathcal{D}_{\sigma} \in PPS_{mat}(\Omega)$. In fact, let $A \notin \partial(\mathcal{D}_{\sigma})$ and $x, y \in \Omega$ such that $(A, x) \notin \mathcal{D}_{\sigma}$ and $(A \cup \{y\}, x) \in \mathcal{D}_{\sigma}$. This implies that $x \in \sigma_{\mathcal{D}_{\sigma}}(A \cup \{y\}) \setminus \sigma_{\mathcal{D}_{\sigma}}(A)$, i. e. $x \in \sigma(A \cup \{y\}) \setminus \sigma(A)$.

By (5.29), $A \notin \Theta(\sigma)$. Since $\sigma$ is WMLS, then $y \in \sigma(A \cup \{x\})$, i. e. $(A \cup \{x\}, y) \in \mathcal{D}_{\sigma}$ and the thesis follows.

\[ \square \]

Let $\mathcal{F} \in FAC(\Omega)$. By Theorem 5.6.13, $[\partial(\mathcal{F})efm]_{\cong0}$ is a complete lattice. In particular, for any $\mathcal{D} \in [\partial(\mathcal{F})efm]_{\cong0}$, by Theorem 5.7.2 we can consider the WMLS, finitary pre-closure operator $\sigma_{\mathcal{D}}$. In what follows, we denote by $WFPCO(\mathcal{F}, \Omega)$ the family of all WMLS, finitary pre-closure operators $\sigma_{\mathcal{D}}$, when $\mathcal{D} \in [\partial(\mathcal{F})efm]_{\cong0}$.

In the next result, we show that $(WFPCO(\mathcal{F}, \Omega), \subseteq)$ is a complete lattice and we also determine its minimum element.

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Theorem 5.7.3. If \( \sigma \) be a WMLS, finitary pre-closure operator on \( \Omega \) then \( \partial(D_\sigma) \in FAC(\Omega) \). Moreover, if \( \mathcal{F} \in FAC(\Omega) \) then \( (WFPCO(\mathcal{F}, \Omega), \sqsubseteq) \) is a complete lattice whose minimum is

\[
\sigma_\mathcal{F}(X) := \begin{cases} 
X \cup \bigcup \{ x : X \cup \{ x \} \not\in \mathcal{F} \} & \text{if } X \in \mathcal{F} \\
\bigcup \{ \sigma_\mathcal{F}(A) : A \subseteq X \} & \text{otherwise}
\end{cases}
\]

(5.30)

Proof. Let \( \sigma \) be a WMLS, finitary pre-closure operator on \( \Omega \). By Theorem 5.7.2, \( [\sigma_\mathcal{D}|efm]_\approx \) is a complete lattice. By Proposition 5.6.3, for each \( \mathcal{D}, \mathcal{D}' \in [\sigma_\mathcal{D}|efm]_\approx \), we have that \( \partial(D) = \partial(D') \).

In particular, \( \partial(D_\sigma) \in FAC(\Omega) \) by Theorem 5.6.13.

On the contrary, let \( \mathcal{F} \in FAC(\Omega) \). Let us prove that \( (WFPCO(\mathcal{F}, \Omega), \sqsubseteq) \) forms a complete lattice. To this aim, let \( \{ \sigma_i : i \in I \} \subseteq WFPCO(\mathcal{F}, \Omega) \) and let us consider \( \sigma := \bigvee_{i \in I} \sigma_i \), where the join has been taken in \( (OP, \sqsubseteq) \). We have to prove that \( \sigma \) is a WMLS, finitary pre-closure operator. First of all, let us observe that

\[
\sigma(X) := \left\{ x \in \Omega : x \in X \lor (X, x) \in \bigcup_{i \in I} D_i \right\}
\]

for any \( X \in \mathcal{F}(\Omega) \). In other terms, \( \sigma \) is the set operator associated with \( \bigcup_{i \in I} D_i \in [\partial(\mathcal{F})|efm]_\approx \).

By Theorem 5.7.2, \( \sigma \) is a WMLS, finitary pre-closure operator and, by (i) of Proposition 5.6.12, \( \partial(\bigcup D_i) = \mathcal{F} \), i. e. \( \sigma \in WFPCO(\mathcal{F}, \Omega) \).

On the other hand, let \( \{ \sigma_i : i \in I \} \subseteq WFPCO(\mathcal{F}, \Omega) \) and \( \mathcal{D}^* \) be the meet of the family \( \{ [D_\sigma|efm]_\approx : i \in I \} \) in \( [\partial(\mathcal{F})|efm]_\approx \). Moreover, let us consider the set operator

\[
\sigma_{\mathcal{D}^*} := \left\{ x \in \Omega : x \in X \lor (X, x) \in \mathcal{D}^* \right\}
\]

Since it is the set operator associated with \( \mathcal{D}^* \), we deduce it is a WMLS, finitary pre-closure operator such that \( \partial(\bigcup D_i) = \mathcal{F} \). Furthermore, it is straightforward to show that \( \sigma_{\mathcal{D}^*} \sqsubseteq \sigma_i \) for any \( i \in I \). This proves that \( \sigma_{\mathcal{D}^*} \in WFPCO(\mathcal{F}, \Omega) \), i. e. \( (WFPCO(\mathcal{F}, \Omega), \sqsubseteq) \) is a complete lattice.

Let us consider \( \sigma_{\mathcal{D}^*} \), where \( \mathcal{D}^* \) is the minimum of \( [\partial(\mathcal{F})|efm]_\approx \) defined as in (5.23). Hence, by (5.26) it follows that \( \sigma_{\mathcal{D}^*}(X) = \{ x \in \Omega : x \in X \lor (X, x) \in \mathcal{D}^* \} \). Obviously, \( \sigma_{\mathcal{D}^*} \) is a WMLS, finitary pre-closure operator since it is associated with \( \mathcal{D}^* \in [\partial(\mathcal{F})|efm]_\approx \), so \( \sigma_{\mathcal{D}^*} \in WFPCO(\mathcal{F}, \Omega) \).

Furthermore, we observe that if \( X \in \partial(\mathcal{D}^*) = \mathcal{F} \), then it results that \( \sigma_{\mathcal{D}^*}(X) = X \cup \bigcup \{ x : X \cup \{ x \} \not\in \mathcal{F} \} \); on the other hand, assume \( X \not\in \mathcal{F} \). We will show that \( \sigma_{\mathcal{D}^*}(X) := \bigcup_{A \in \mathcal{J}} \{ \sigma_{\mathcal{D}^*}(A) : A \subseteq X \} \). Let \( x \in \sigma_{\mathcal{D}^*}(X) \). Then, or \( x \in A \) for some \( A \subseteq X \) such that \( A \in \mathcal{F} \) or, on the contrary, \( A \cup \{ x \} \not\in \mathcal{F} \), so \( x \in \bigcup_{A \in \mathcal{J}} \{ \sigma_{\mathcal{D}^*}(A) : A \subseteq X \} \). Vice versa, let \( x \in \bigcup_{A \in \mathcal{J}} \{ \sigma_{\mathcal{D}^*}(A) : A \subseteq X \} \). This means that or \( x \in A \) for some \( A \subseteq X \) or there exists \( A \subseteq X \) such that \( A \in \mathcal{F} \) and \( A \cup \{ x \} \not\in \mathcal{F} \), i. e. \( (X, x) \in \mathcal{D}^* \). Thus, \( x \in \sigma_{\mathcal{D}^*}(X) \). In this way, we showed that \( \sigma_{\mathcal{D}^*} \) coincides with the set operator \( \partial_3 \) defined in (5.30).

Finally, by (iii) of Proposition 1.5.1, it results that \( \partial_3 \sqsubseteq \sigma \) for any \( \sigma \in WFPCO(\mathcal{F}, \Omega) \). This concludes the proof.
Chapter 6

Representation Theorems

In this chapter, we deal with representation theorems. In particular, we will show that an arbitrary closure system \( \mathcal{S} \) on a finite set \( \Omega \) are represented by a pairing \( \mathcal{P} \in \text{PAIR}(\Omega) \) whose \( \text{MAXP}(\mathcal{P}) \) coincides with \( \mathcal{S} \) and, by using a classical result by Birkhoff and Frink [19], that for any finite lattice \( L \) there exist a finite set \( \Omega_L \) and a pairing \( \mathcal{P} \in \text{PAIR}(\Omega_L) \) whose maximum partitioner lattice is exactly \( L \). In other terms, we can consider closure system theory and finite lattice theory as sub-theories of that of pairings. Finally, we introduce the notion of symmetry transmission average, that provides a numerical information between all the nodes of the finite lattice \( L \). Thus, we are able to compare through the set map \( \Gamma \) also two elements of the lattice \( L \) that are non-comparable with respect to the order \( \geq_L \).

6.1 Representation Theorems for Closure Systems and Finite Lattices

We now assume that \( \Omega \) is a finite set and in the next result we provide a representation theorem for arbitrary closure systems on \( \Omega \).

**Theorem 6.1.1.** The map \( \text{MAXP} : \mathcal{P} \in \text{PAIR}(\Omega) \mapsto \text{MAXP}(\mathcal{P}) \in \text{CLSY}(\Omega) \) is surjective.

**Proof.** Let \( \mathcal{S} \in \text{CLSY}(\Omega) \), \( E \in \mathcal{S} \) be the minimum element in \( \mathcal{S} \) and let \( C_1, C_2, \ldots, C_k \) be all maximal chains from \( E \) to the top \( \Omega \), where \( C_i \) is the chain in \( \mathcal{S} \) given by \( A_{i_0} = E \subset A_{i_1} \subset \cdots \subset A_{i_l} = \Omega \), for \( i = 1, 2, \ldots, k \). Furthermore, we set \( m_0 = 0 \), \( m_i := \sum_{s=1}^{l_i} l_s \), and \( m := m_k \).

We now define a pairing
\[
\mathcal{P} = \mathcal{P}(\mathcal{S}) := (U_{\mathcal{S}}, \Omega, F_{\mathcal{S}}, N) \in \text{PAIR}(\Omega)
\]
whose maximum partitioner lattices is exactly \( \mathcal{S} \). First of all, we consider the set \( U_{\mathcal{S}} = \{u_1, u_2, \ldots, u_m\} \). On the other hand, we must define the function \( F_{\mathcal{S}} : U_{\mathcal{S}} \times \Omega \rightarrow \mathbb{N} \). To this regard, we set \( F_{\mathcal{S}}(u_s, a_j) := 1 \) if \( a_j \in E \), for any \( u_s \in U_{\mathcal{S}} \). Let now \( a_j \in \Omega \setminus E \). For each \( i \in \{1, 2, \ldots, k\} \) we define \( t_{ij} \) be the smallest positive integer such that \( a_j \in A_{i_{t_{ij}j}} \setminus A_{i_{t_{ij}-1}} \).
Finally, we set:

\[
F_\mathcal{S}(u_s, a_j) := \begin{cases} 
1 & \text{if } s = 1, \\
F_\mathcal{S}(u_{s-1}, a_j) & \text{if } m_{i-1} + 2 \leq s \leq m_i - t_{ij} + 1, \\
F_\mathcal{S}(u_{s-1}, a_j) + 1 & \text{otherwise.}
\end{cases}
\]  

(6.2)

Let us firstly show that if \( A \in \mathcal{S} \) then \( A \in MAXP(\mathcal{Q}) \). If \( A = \Omega \) there is nothing to prove. Hence, assume \( A \neq \Omega \). Let \( C_i : A_{i_0} = E \subset A_{i_1} \subset \cdots A_{i_t} \subset \Omega \) be a chain from \( E \) to \( \Omega \) and \( t \in \{1, 2, \ldots, t_i\} \) be such that \( A = A_{i_t} \). Set moreover \( s := m_i - t \). Let us prove now that \( u_s \equiv_A u_{s+1} \) but, for each choice of \( a_j \in \Omega \setminus A \), it holds \( F_\mathcal{S}(u_s, a_j) \neq F_\mathcal{S}(u_{s+1}, a_j) \). Let us notice that if \( a_j \in A \), then \( t_{ij} \leq t \), and \( m_{i-1} + 1 \leq s \leq m_i - t_{ij} \). By (6.2), it follows that \( F_\mathcal{S}(u_s, a_j) = F_\mathcal{S}(u_{s+1}, a_j) \), therefore we have shown that \( u_s \equiv_A u_{s+1} \). On the other hand, if \( a_j \notin A \), then \( t_{ij} > t \), so \( s = m_i - t > m_i - t_{ij} \). Again by (6.2), it follows that \( F_\mathcal{S}(u_{s+1}, a_j) = F_\mathcal{S}(u_s, a_j) + 1, \) i.e. \( u_s \neq_A u_{s+1} \). This proves that \( A \in MAXP(\mathcal{Q}) \).

Let now \( A \in MAXP(\mathcal{Q}) \). We will now prove that \( A \in \mathcal{S} \). To this regard, let us consider the minimum element \( B \) in \( \mathcal{S} \) such that \( A \subseteq B \). We now prove that \( A \equiv_A B \). Let us notice that, for any \( B' \in \mathcal{S} \) such that \( B' \subset B \), it results that \( B' \subset A \). In fact, let \( a_k \in B' \) and \( a_k \in A \setminus B' \subseteq B \setminus B' \). Let \( s \) be such that \( u_{s-1} \equiv_A u_s \). Thus \( F_\mathcal{S}(u_s, a_h) = F_\mathcal{S}(u_{s-1}, a_h) \), so \( m_{i-1} + 2 \leq s \leq m_i - t_{ih} + 1 \). But both the conditions \( a_h \in B \setminus B' \) and \( a_k \in B' \) imply that \( t_{ih} > t_{ik} \), so \( m_{i-1} + 2 \leq s \leq m_i - t_{ik} + 1 \) and, then, \( F_\mathcal{S}(u_s, a_k) = F_\mathcal{S}(u_{s-1}, a_k) \). Since \( A \in MAXP(\mathcal{Q}) \), it holds that \( a_k \in A \) and thus that \( B' \subset A \). Without loss in generality, we can assume that \( B' \) is maximal among the proper subsets of \( B \). Let now \( C_i \) be a chain through \( B \). Then, there exists \( t \) such that \( B' := A_{i_t-1} \subset A \subseteq A_{i_t} = B \) and let \( s \in \{1, 2, \ldots, m\} \) such that \( u_{s-1} \equiv_A u_s \). If \( a_k \in B \) and \( a_k \in A \setminus B' \), then \( t_{ik} < t_{ih} \), so, since \( u_{s-1} \equiv_A u_s \), it holds that \( F_\mathcal{S}(u_s, a_k) = F_\mathcal{S}(u_{s-1}, a_k) \), thus \( a_k \in M_{\mathcal{Q}}(A) = A \). This proves that \( A = B \in \mathcal{S} \) and the thesis has been shown. \( \square \)

Example 6.1.2. Let us consider \( \Omega := \{a_1, a_2, a_3, a_4, a_5\} \) and let

\[ \mathcal{S} := \{\Omega, \{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_2, a_3\}, \{a_2, a_5\}, \{a_2\}\} \]

be a closure system on \( \Omega \). The minimum in \( \mathcal{S} \) is \( E = \{a_2\} \). The lattice associated with \( \mathcal{S} \) has the following Hasse diagram

![Hasse diagram](image-url)
There are three chains from $E$ to $\Omega$:

- $C_1 : \{a_2\} \subset \{a_2, a_3\} \subset \{a_1, a_2, a_3\} \subset \Omega$
- $C_2 : \{a_2\} \subset \{a_2, a_3\} \subset \{a_2, a_3, a_4\} \subset \Omega$  \hspace{1cm} (6.3)
- $C_3 : \{a_2\} \subset \{a_2, a_5\} \subset \Omega$

Thus $m_0 = 0, m_1 = 3, m_2 = 6$ and $m_3 = 8$. Moreover $t_{11} = 2, t_{21} = 3, t_{31} = 2, t_{13} = 1, t_{23} = 1, t_{33} = 2, t_{14} = 3, t_{24} = 2, t_{34} = 2, t_{15} = 3, t_{25} = 3$ and $t_{35} = 1$. Then, $U_\mathcal{G} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$, and, by (6.2) and (6.1) we can construct the pairing in Figure 6.1.

Moreover, in Figure 6.2 we represent the Hasse diagram of $G(\mathcal{P}(\mathcal{G}))$ (we use string notation in the figure).

Corollary 6.1.3. Let $\mathcal{G} \in CLS\mathcal{Y}(\Omega)$ and let $\mathcal{P} \in PAIR(\Omega)$ such that $MAXP(\mathcal{P}) = \mathcal{G}$. Then, the lattices $PSYM(\mathcal{P}), M(\mathcal{P})$ and $(\mathcal{G}, \subseteq^*)$ are order isomorphic.

Proof. The result is a direct consequence of Theorem 6.1.1 and part (ii) of Corollary 2.2.2. \hfill $\Box$

Another consequence of Theorem 6.1.1 is that any union additive relation on $\Omega$ satisfies a condition similar to the definition of the relation $\geq_\mathcal{P}$. Therefore, this means that in the finite case, it is not restrictive to consider the union additive relations induced by pairing systems.

Theorem 6.1.4. Let $\geq$ be a union additive relation on $\Omega$. Then, there exists a pairing $\mathcal{P} \in PAIR(\Omega)$ such that $\geq$ coincides with $\geq_\mathcal{P}$.

Proof. Let $\mathcal{D} := \mathcal{D}_{\geq}$ be the union additive family associated with union additive relation $\geq$. By Proposition 5.2.8, we can associate with $\mathcal{D}$ a closure operator $c_\mathcal{D}$ such that $c_\mathcal{D}(X) := \bigcup\{Y \in \mathcal{P}(\Omega) : (X, Y) \in \mathcal{D}\}$ for any $X \in \mathcal{P}(\Omega)$. By Theorem 1.5.7, to the closure operator $c_\mathcal{D}$ corresponds a closure system $\mathcal{G}$. Moreover, by Theorem 6.1.1, there exists a pairing $\mathcal{P} \in PAIR(\Omega)$ such that $MAXP(\mathcal{P}) = \mathcal{G}$. As a direct consequence, it is easy to show that $c_\mathcal{D}(X) = M_\mathcal{G}(X)$ for any $X \in \mathcal{P}(\Omega)$. Indeed, $X \subseteq M_\mathcal{G}(X)$ and since $c_\mathcal{D}(X) \in MAXP(\mathcal{P})$, we conclude that $M_\mathcal{G}(X) \subseteq c_\mathcal{D}(X)$. Vice versa, since $M_\mathcal{G}(X) \in \mathcal{G}$, then $c_\mathcal{D}(X) \subseteq M_\mathcal{G}(X)$.

Now, we have to prove that $X \geq Y$ if and only if $X \geq_\mathcal{P} Y$. Assume that $X \geq Y$. Then,
Figure 6.2: Hasse Diagram of $G(\mathcal{P}(\emptyset))$. 
Y ⊆ cD(X) = Mṇ(Y) ⊆ Mṇ(X) and, by definition of ≥P, it follows that X ≥P Y. On the other hand, if X ≥P Y, it follows that Y ⊆ Mṇ(X) = cD(X), so (X, Y) ∈ D and, thus, X ≥ Y. This concludes the proof.

At this point, based on a classical representation theorem of Birkhoff and Frink (see [19]), we can establish the more important theoretical result of this section, that we can consider a pairing representation theorem for finite lattices.

**Theorem 6.1.5.** Let L = (L, ≤L) be a finite lattice. Then, there exist a finite set Ω, a positive integer n such that |Ω| = n and a pairing Ψ ∈ PAIR(Ω) such that the lattices L, M(Ψ), G(Ψ) and P(Ψ) are order isomorphic. Therefore, if ηΨ : L → MAXP(Ψ) is the map that induces such an isomorphism between L and M(Ψ), we have that

\[ x ≤L x' ⇔ ηΨ(x) ≤ ηΨ(x') ⇔ πΨ(ηΨ(x)) ≤ πΨ(ηΨ(x')) ⇔ γΨ(ηΨ(x), ηΨ(x')) = 1 \]  

(6.4)

for any x, x′ ∈ L.

**Proof.** In [19], Birkhoff and Frink proved that any finite lattice is order isomorphic to the corresponding lattice induced from some closure system. So that there exist a finite set Ω, a closure system S ∈ CLSY(Ω) and a map η : L → S that is an order isomorphism between the dual lattice L* and the lattice (S, ⊆). Then, by Theorem 6.1.1, we can construct a pairing Ψ ∈ PAIR(Ω) such that MAXP(Ψ) = S. Therefore ηΨ := η is an order isomorphism between L and M(Ψ), and (6.4) becomes a direct consequence of (2.17).

The relation established in (6.4) has deep theoretical consequences: it says us that the study of the symmetry transmission measure between subsets of finite sets is equivalent to the study of order relations on finite lattices. Therefore, we now investigate the basic theoretical properties of the symmetry transmission measure between subsets of finite sets and the direct interrelation of this notion with other classical notions of lattice theory.

**Definition 6.1.6.** We call pairing characteristic of the lattice L, denoted by pc(L), the minimum positive integer N for which the thesis of Theorem 6.1.5 is verified.

Let N = pc(L). We set then

\[ PAIR(L) := \{ Ψ ∈ PAIR(Ω) : M(Ψ) ≅ L \}. \]

Now, if Ψ ∈ PAIR(L), with the same notations introduced in the statement of Theorem 6.1.5, we can consider the inverse order isomorphism ηΨ⁻¹ : MAXP(Ψ) → L and the closure operator Mp : P(Ω) → MAXP(Ψ), so that we obtain the surjective map

\[ ξΨ := ηΨ⁻¹ ∘ Mp : P(Ω) → L \]

Let us note that the map ξΨ is not canonically determined. In fact, it depends from the order isomorphism ηΨ, which in turn depends on the not uniquely determined closure system S given in the proof of Theorem 6.1.5. However, by means of the map ξΨ we can formally describe the following equivalences.
Theorem 6.1.7. Let $L = (L, \leq_L)$ be a finite lattice, $N = pc(L)$ and $X, Y \in \mathcal{P}(\Omega)$. Then, the following conditions are equivalent:
(i) for any $\Psi \in \text{PAIR}(L)$ we have $\xi_\Psi(X) \leq_L \xi_\Psi(Y)$;
(ii) for any $\Psi \in \text{PAIR}(L)$ we have $\gamma_\Psi(X, Y) = 1$;
(iii) there exists $\Psi \in \text{PAIR}(L)$ such that $\gamma_\Psi(X, Y) = 1$.

Proof. (i) $\implies$ (ii): Let $x := \xi_\Psi(X) \leq_L y := \xi_\Psi(Y)$. Let $\Psi' \in \text{PAIR}(L)$ and $\eta_\Psi : L \to \text{MAXP}(\Psi')$. By (6.4), $\eta_\Psi(x) \leq^* \eta_\Psi(y)$, i.e.
$$\pi_{\Psi'}(\eta_\Psi(x)) = \pi_{\Psi'}(\eta_\Psi(x')) \implies \gamma_{\Psi'}(\eta_\Psi(x), \eta_\Psi(x')) = 1$$
and the claim has been shown.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (i): By (6.4), we have that $\gamma_\Psi(X, Y) = \gamma_\Psi(M_\Psi(X), M_\Psi(Y)) = 1$ if and only if $\pi_\Psi(X) \leq \pi_\Psi(Y)$ or, equivalently, $M_\Psi(Y) \leq M_\Psi(X)$, that is equivalent to $\eta_\Psi^{-1}(M_\Psi(X)) \leq \eta_\Psi^{-1}(M_\Psi(Y))$, i.e. $\xi_\Psi(X) \leq_L \xi_\Psi(Y)$.

\[\square\]

6.2 Pairing Representation of Finite Complementary Involution Lattices

In this section, we prove a second representation theorem for finite lattices. For, we assume that $M_\Psi(\emptyset) = \emptyset$ and define a set operator $I_\Psi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ that behaves as the topological interior part of a set. To be more specific, we set
$$I_\Psi(A) := (M_\Psi(A^c))^c.$$  \hspace{1cm} (6.5)

Definition 6.2.1. We call $I_\Psi(A)$ the interior of $A$. We say that $A \in \mathcal{P}(\Omega)$ is regularly closed if $A = M_\Psi(I_\Psi(A))$ and denote by $\mathcal{R}(\Psi)$ the family of all regularly closed subsets.

Similarly to what happens to its topological counterpart, the complement of the interior of any subset belongs to $\text{MAXP}(\Psi)$.

Proposition 6.2.2. Let $A \in \mathcal{P}(\Omega)$. Then:
(i) $I_\Psi(A)$ is the greatest subset of $A$ whose complement is a maximum partitioner.
(ii) If $A \subseteq B$ then $I_\Psi(A) \subseteq I_\Psi(B)$.
(iii) $I_\Psi(I_\Psi(A)) = I_\Psi(A)$.
(iv) If $Z \in \mathcal{P}(\Omega)$ is the complement of some maximum partitioner, then $M_\Psi(Z) \in \mathcal{R}(\Psi)$.
(v) If $A$ and $A^c$ both belong to $\text{MAXP}(\Psi)$ then $A$ is regularly closed.

Proof. (i): Let $Z \subseteq A$ such that $Z = \Omega \setminus M_\Psi(B)$ for some $B \in \mathcal{P}(\Omega)$. Then, $Z = \Omega \setminus M_\Psi(B) \subseteq A$ implies that $M_\Psi(B) \supseteq A^c$, i.e. $M_\Psi(B) \supseteq M_\Psi(A^c)$, so $Z \subseteq \Omega \setminus M_\Psi(A^c) = I_\Psi(A)$.

(ii): Straightforward.

(iii): $I_\Psi(I_\Psi(A)) = \Omega \setminus M_\Psi(\Omega \setminus I_\Psi(A)) = \Omega \setminus M_\Psi(\Omega \setminus (\Omega \setminus M_\Psi(A^c))) = \Omega \setminus M_\Psi(A^c) = I_\Psi(A)$.

(iv): Let $Z \in \mathcal{P}(\Omega)$ be the complement of some maximum partitioner. By (i), it readily follows that $Z \subseteq I_\Psi(M_\Psi(Z)) \subseteq M_\Psi(Z)$, therefore $M_\Psi(I_\Psi(M_\Psi(Z))) = M_\Psi(Z)$, i.e. $M_\Psi(Z) \in \mathcal{R}(\Psi)$.

(v): Suppose that $A$ and $A^c$ both belong to $\text{MAXP}(\Psi)$. Then, $I_\Psi(A) = (M_\Psi(A^c))^c = A$, thus $M_\Psi(I_\Psi(A)) = A$, i.e. $A$ is regularly closed. \[\square\]
Let us set \( \mathcal{R}(\mathfrak{P}) := (\mathcal{R}(\mathfrak{P}), \subseteq) \). In next result we will show that \( \mathcal{R}(\mathfrak{P}) \) is a complete lattice.

**Theorem 6.2.3.** \( \mathcal{R}(\mathfrak{P}) \) is a complete lattice with \( \hat{0}_{\mathcal{R}(\mathfrak{P})} = \emptyset \) and \( \hat{1}_{\mathcal{R}(\mathfrak{P})} = \Omega \).

**Proof.** It is evident that \( \emptyset \) and \( \Omega \) are both regularly closed subsets. Let \( \Sigma \) be a subfamily of \( \mathcal{R}(\mathfrak{P}) \) and set

\[
A := M_{\mathfrak{P}} \left( I_{\mathfrak{P}} \left( \bigcap_{B \in \Sigma} B \right) \right). \quad (6.6)
\]

Since \( I_{\mathfrak{P}} \left( \bigcap_{B \in \Sigma} B \right) \) is the complement of a maximum partitioner, by (iv) of Proposition 6.2.2, we deduce that \( A \in \mathcal{R}(\mathfrak{P}) \). We now prove that \( A = \inf \Sigma \). If there exists \( A' \in \mathcal{R}(\mathfrak{P}) \) such that \( A' \subseteq B \) for any \( B \in \Sigma \), then \( A' \subseteq \bigcap_{B \in \Sigma} B \), so \( I_{\mathfrak{P}}(A') \subseteq I_{\mathfrak{P}} \left( \bigcap_{B \in \Sigma} B \right) \) by (ii) of Proposition 6.2.2. Applying \( M_{\mathfrak{P}} \) to both sides of the previous relation, we obtain \( A' \subseteq A \). This implies that \( A \) is the g.l.b. of \( \Sigma \) and our claim has been proved.

On the other hand, set

\[
C := M_{\mathfrak{P}} \left( \bigcup_{B \in \Sigma} I_{\mathfrak{P}}(B) \right). \quad (6.7)
\]

By (iv) of Proposition 6.2.2, it is clear that \( C \in \mathcal{R}(\mathfrak{P}) \), since \( \bigcup_{B \in \Sigma} I_{\mathfrak{P}}(B) = \Omega \setminus \bigcap_{B \in \Sigma} M_{\mathfrak{P}}(B^c) \) is the complement of a maximum partitioner. We now show that \( C = \sup \Sigma \). Let \( C' \in \mathcal{R}(\mathfrak{P}) \) such that \( B \subseteq C' \) for any \( B \in \Sigma \), then, by (ii) of Proposition 6.2.2, it follows that \( \bigcup_{B \in \Sigma} I_{\mathfrak{P}}(B) \subseteq I_{\mathfrak{P}}(C') \). Applying \( M_{\mathfrak{P}} \) to both sides of the previous relation, we obtain \( C \subseteq C' \). This implies that \( A \) is the l.u.b. of \( \Sigma \) and our claim has been shown. \( \square \)

Let us define a map \( \psi : \mathcal{R}(\mathfrak{P}) \to MAXP(\mathfrak{P}) \) by setting \( \psi(A) := M_{\mathfrak{P}}(A^c) \).

**Theorem 6.2.4.** The map \( \psi \) satisfies the following properties:

(i) \( \psi(\mathcal{R}(\mathfrak{P})) \subseteq \mathcal{R}(\mathfrak{P}) \);

(ii) If \( A \subseteq B \), then \( \psi(A) \geq_{\mathfrak{P}} \psi(B) \);

(iii) \( \psi \) is an involution;

(iv) \( \psi(A) \land A = \emptyset \) and \( \psi(A) \lor A = \Omega \).

**Proof.** (i): Let \( A \in \mathcal{R}(\mathfrak{P}) \). Since \( A \in MAXP(\mathfrak{P}) \), then, by (iv) of Proposition 6.2.2, \( M_{\mathfrak{P}}(A^c) \in \mathcal{R}(\mathfrak{P}) \).

(ii): \( \psi \) is clearly order-reversing, in fact if \( A \subseteq B \), then \( \Omega \setminus B \subseteq \Omega \setminus A \), i.e. \( M_{\mathfrak{P}}(\Omega \setminus B) \subseteq M_{\mathfrak{P}}(\Omega \setminus A) \) or, in terms of union additive relation, \( M_{\mathfrak{P}}(\Omega \setminus A) \geq_{\mathfrak{P}} M_{\mathfrak{P}}(\Omega \setminus B) \), i.e. \( \psi(A) \geq_{\mathfrak{P}} \psi(B) \).

(iii) Let \( A \in \mathcal{R}(\mathfrak{P}) \). Then

\[
\psi(\psi(A)) = \psi(M_{\mathfrak{P}}(A^c)) = M_{\mathfrak{P}}(\Omega \setminus M_{\mathfrak{P}}(A^c)) = M_{\mathfrak{P}}(I_{\mathfrak{P}}(A)) = A.
\]
Theorem 6.2.5. Let pairing on a certain finite set $\Omega$. Thus, $\psi$ is a finite set.

We divide the proof in three steps:

Proof. We now prove that the unique subset contained in $A$ is the empty set. As a matter of fact, let $Z \subseteq A$ and, in particular, by (i) of Proposition 6.2.2, $Z \subseteq I_{\mathcal{P}}(A) = \Omega \setminus M_{\mathcal{P}}(A^c)$. But it happens if and only if $Z = \emptyset$. Thus, $I_{\mathcal{P}}(M_{\mathcal{P}}(A^c) \cap A)$ and, since $M_{\mathcal{P}}(\emptyset) = \emptyset$, we conclude that $\psi(A) \cap A = M_{\mathcal{P}}(\emptyset) = \emptyset$.

On the other hand, let us now compute $\psi(A) \lor A$. By (6.7), it results that

$$\psi(A) \lor A = M_{\mathcal{P}}(I_{\mathcal{P}}(A) \cup I_{\mathcal{P}}(\psi(A))) = M_{\mathcal{P}}(\Omega \setminus M_{\mathcal{P}}(A^c)) \cup ((\Omega \setminus M_{\mathcal{P}}(\psi(A)^c)))$$

but it follows that

$$M_{\mathcal{P}}(\Omega \setminus M_{\mathcal{P}}(A^c)) \cup M_{\mathcal{P}}(A^c) = \Omega.$$

by using (b) of Corollary 2.2.2 and the definition of $\psi$. This shows that $\psi(A) \lor A = \Omega$ and the proof concludes here. $\square$

In next result, we prove that for any finite complementary involution lattice there exists a pairing on a certain finite set $\Omega_{L,\psi}$ whose regularly closed family is isomorphic to $L$ itself.

**Theorem 6.2.5.** Let $(L, \psi)$ be a finite complementary involution lattice. Then, there exist a finite set $\Omega_{L,\psi}$ and a pairing $\mathcal{P} \in PAIR(\Omega_{L,\psi})$ such that $\mathcal{R}(\mathcal{P})$ is order-isomorphic to $(L, \leq_L)$.

**Proof.** We divide the proof in three steps:

I Let us show that given an order-preserving $\psi$-complementary map $f : L \to \mathcal{F}$, where $\mathcal{F}$ is a complement-closed family on a certain finite set $\Omega_n$, there exist a finite set $\Omega$, a complement-closed family $\mathcal{G}$ on it and, fixed two elements $\alpha, \beta \in L$, there exists an order-preserving $\psi$-complementary map $\hat{g} : L \to \mathcal{G}$-preserving such that:

1. $\Omega := \Omega_n$ and $\mathcal{F} = \mathcal{G}$ or there exists $\xi \notin \Omega_n$ such that $\Omega := \Omega_{n+1} = \Omega_n \cup \{\xi\}$ or there exist some $\eta, \xi \notin \Omega_n$ such that $\Omega := \Omega_{n+2} = \Omega_n \cup \{\eta, \xi\}$;
2. for any $\gamma \in L$, $\hat{g}({\gamma}) = f({\gamma})$ or $\hat{g}({\gamma}) = f({\gamma}) \cup \{\xi\}$ or $\hat{g}({\gamma}) = f({\gamma}) \cup \{\eta, \xi\}$.

Clearly, if $f$ is $\{\alpha, \beta\}$-preserving, there is nothing to prove since we can set $\hat{g} := f$, $\mathcal{G} := \mathcal{F}$ and $\Omega := \Omega_n$. Hence, we must prove the claim when $f$ is not $\{\alpha, \beta\}$-preserving. For, we have to investigate two cases: in the first case, let us assume that $\alpha \leq_L \beta$ (or that they are non-comparable) and that $f(\alpha) = f(\beta)$. Let $\xi \notin \Omega_n$ and $\Omega' := \Omega_n \cup \{\xi\}$. We now define a map $h_1 : L \to h_1(L) \subseteq \Omega'$ where, for any $\gamma \in L$,

$$h_1(\gamma) := \begin{cases} f(\gamma) \cup \{\xi\} & \text{if } \beta \leq_L \gamma \\ f(\gamma) & \text{otherwise.} \end{cases}$$

First of all, we observe that $h_1$ is an order-preserving map. Moreover it holds that, whenever $f$ is $\{\gamma, \delta\}$-preserving for some $\gamma, \delta \in L$, then also $h_1$ satisfies the same property.
In fact, if $\gamma$ and $\delta$ are non-comparable, then one of the following situations occur: if $\beta \leq_L \gamma, \delta$, then $h_1(\gamma) = f(\gamma) \cup \{\xi\}$ and $h_1(\delta) = f(\delta) \cup \{\xi\}$ and since $f(\gamma)$ and $f(\delta)$ are non-comparable, it follows that $h_1(\gamma)$ and $h_1(\delta)$ are also non-comparable; if $\gamma, \delta <_L \beta$, then $h_1(\gamma) = f(\gamma)$ and $h_1(\delta) = f(\delta)$, so $h_1(\gamma)$ and $h_1(\delta)$ are also non-comparable; if $\gamma <_L \beta$ and $\beta$ and $\delta$ are non-comparable, then $h_1(\gamma) = f(\gamma)$ and $h_1(\delta) = f(\delta)$, so $h_1(\gamma)$ and $h_1(\delta)$ are also non-comparable and, finally, if $\beta \leq_L \gamma$ and $\beta$ and $\delta$ are non-comparable, then $h_1(\gamma) = f(\gamma) \cup \{\xi\}$ and $h_1(\delta) = f(\delta)$. Since $f(\gamma)$ and $f(\delta)$ are non-comparable, also $h_1(\gamma)$ and $h_1(\delta)$ are.

The other cases to be analyzed are similar. This proves that $h_1$ is $\{\gamma, \delta\}$-preserving. Moreover, it results that

$$h_1(\gamma) \cap h_1(\psi(\gamma)) = \emptyset \quad (6.8)$$

for any $\gamma \in L$. Let us suppose that $\beta \leq_L \gamma$. This implies that $\psi(\gamma)$ and $\beta$ are non-comparable. Therefore $h_1(\gamma) = f(\gamma) \cup \{\xi\}$ and $h_1(\psi(\gamma)) = f(\psi(\gamma)) = \Omega_\gamma \setminus f(\gamma)$ and (6.8) holds. The other possible cases are similar.

Now, let us consider all possible pairs $\{\gamma, \psi(\gamma)\}$, for each $\gamma \in L$. We order these pairs in the following way

$$\{\gamma_0, \psi(\gamma_0)\}, \{\gamma_1, \psi(\gamma_1)\}, \ldots, \{\gamma_m, \psi(\gamma_m)\},$$

so that $\gamma_i \not\leq_L \alpha$ for each $i = 0, \ldots, m$.

Let us set $g_0 := h_1$. Suppose that, for each $i = 1, \ldots, m - 1$, there exist a complement-closed family $\mathcal{G}_i$ on $\Omega'$ and a mapping $\hat{g}_i : L \to \mathcal{G}_i$ such that:

(i) $\hat{g}_i(\gamma) = h_1(\gamma)$ or $\hat{g}_i(\gamma) = h_1(\gamma) \cup \{\xi\}$ for any $\gamma \in L$;

(ii) if $f$ is $\{\gamma, \delta\}$-preserving, then also $\hat{g}_i$ is;

(iii) if $i \leq j$, then $\hat{g}_i(\gamma) \subseteq \hat{g}_j(\gamma)$ for any $\gamma \in L$;

(iv) $\hat{g}_i(\gamma) \cap \hat{g}_i(\psi(\gamma)) = \emptyset$;

(v) if $\gamma \leq_L \delta$ and $\xi \in \hat{g}_i(\gamma)$, then $\xi \in \hat{g}_i(\delta)$;

(vi) $\hat{g}_i(\psi(\gamma_i)) = \Omega' \setminus \hat{g}_i(\gamma_i)$, i.e. $\hat{g}_i$ is an order-preserving $\psi$-complementary map.

We set, for any $\gamma \in L$ and $1 \leq k \leq m - 1$,

$$\mathcal{G}_k(\gamma) := \bigcup_{i=0}^{k-1} \hat{g}_i(\gamma).$$

If $\mathcal{G}_i(\psi(\gamma_q)) \cup \mathcal{G}_i(\gamma_q) = \Omega_\gamma$ for some $\gamma_q \in L$ and $i = 0, \ldots, k - 1$, then we set

$$g_i(\gamma) := \begin{cases} \mathcal{G}_i(\gamma) \cup \{\xi\} & \text{if } \gamma_q \leq_L \gamma \\ \mathcal{G}_i(\gamma) & \text{otherwise.} \end{cases}$$

whereas, if $\mathcal{G}_i(\psi(\gamma)) = \Omega \setminus \mathcal{G}_i(\gamma)$ for any $\gamma \in L$, we set $g_i := \mathcal{G}_i$. Clearly, if $\mathcal{G}_i$ satisfies (i) – (vi), then also $g_i$ does. Moreover, in both cases, we set

$$\mathcal{G}_i := \{g_i(\gamma) : \gamma \in L\}.$$
We now prove that there exist a finite set $\Omega$. Suppose that for any $0 < L \in \mathbb{N}$, construct $\hat{g}$. In the second case, we assume $\alpha < \xi$. Clearly, since $g$ implies that $g = g_0$, we obtain $(g) = f(\gamma)$ and this shows that $\hat{g}(\gamma) = f(\gamma) \cup \{\xi\}$. Otherwise, if $\alpha$ and $\beta$ are non-comparable, let $\eta \notin \Omega$ and set $\Omega := \Omega \cup \{\eta\}$. Moreover, for any $\gamma \in \Omega$, we set

$$h_2(\gamma) := \begin{cases} g_m(\gamma) \cup \{\eta\} & \text{if } \alpha \leq_L \gamma \\ g_m(\gamma) & \text{otherwise.} \end{cases}$$

By reasoning as above, we can construct $g_m'$ through $g_m$ and $h_2$ using $\eta$ instead of $\xi$.

Clearly, since $\xi \in g_m'(\beta \setminus g_m'(\alpha))$, we deduce that $g_m'$ is $\{\alpha, \beta\}$-preserving. Therefore, we can set $\hat{g} := g_m'$ and $\hat{g} := \hat{g}(\gamma) : \gamma \in \Omega$. This concludes the proof in the case $\alpha < \beta$ (or they are non-comparable) and $f(\alpha) = f(\beta)$.

In the second case, we assume $\alpha$ and $\beta$ non-comparable and $f(\alpha) \subseteq f(\beta)$. We can construct $\hat{g}$ by starting from $f$ in the same way we obtained $g_m'$ from $g_m$, by replacing $\eta$ with $\xi$ and it holds $\hat{g}(\gamma) = f(\gamma) \cup \{\eta, \xi\}$.

II We now prove that there exist a finite set $\Omega_{L, \psi}$ and an isomorphism $f : L \rightarrow \mathcal{F}$, where $\mathcal{F}$ is a complement-closed family on $\Omega_{L, \psi}$. Firstly, let $g : L \rightarrow \emptyset$. Let us consider the family of all 2-subsets of $L$, i.e. $\{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$. We set $f_0 := \hat{g}$, $\mathcal{F}_0 := \mathcal{G}$ and $G_0 := \Omega$, where $\hat{g}$, $\mathcal{G}$, $\Omega$ have been constructed in the previous step, $\hat{g}$ is $\{\alpha_0, \beta_0\}$-preserving and $\mathcal{F}_0$ is a complement-closed family on $G_0$.

Suppose that for any $0 \leq i \leq m - 1$ there exist $f_i$, $\mathcal{F}_i$ and $G_i$ such that:

(A) $f_i : L \rightarrow \mathcal{F}_i$ is an order-preserving complementarity $\{x_i, y_i\}$-preserving and $\mathcal{F}_i$ is a complement-closed family on $G_i$;

(B) If $i < j < m$ and $f_i$ is $\{\alpha, \beta\}$-preserving, then $f_j$ does. Moreover, for any $\alpha \in L$, it results that $f_j(\alpha) \cap G_i = f_i(\alpha)$.
Let us set:
\[
f_m(\alpha) := \bigcup_{i=0}^{m-1} f_i(\alpha)
\]
for any \(\alpha \in \mathbb{L}\),
\[
\mathcal{F}_m := \{f_m(\alpha) : \alpha \in \mathbb{L}\}
\]
and
\[
G_m := \bigcup \mathcal{F}_m.
\]
Let us prove that \(f_m\) satisfies both (A) and (B). It is obvious that \(f_m\) is order-preserving since if \(\alpha \leq \beta\), then \(f_i(\alpha) \subseteq f_i(\beta)\) for each \(i < m\) and, by the definition of \(f_m\), a fortiori, it follows that \(f_m(\alpha) \subseteq f_m(\beta)\). We now prove that \(f_m\) is an order-preserving \(\psi\)-complementary map. Suppose that \(f_m(\psi(\alpha)) \cap f_m(\alpha) \neq \emptyset\) for some \(\alpha \in \mathbb{L}\). This implies that there exist two indices \(i < j < m\) such that \(b \in f_i(\psi(\alpha)) \cap f_j(\alpha)\) but, by (2), it follows that \(f_i(\psi(\alpha)) \subseteq f_j(\psi(\alpha))\), i.e., \(b \in f_j(\psi(\alpha)) \cap f_j(\alpha)\), that is a contradiction. Thus, \(f_m(\psi(\alpha)) \subseteq G_m \setminus f_m(\alpha)\). Indeed, the previous relation is an equality. In fact, let \(b \in G_m\), then there exists \(\beta \in \mathbb{L}\) and an index \(i < m\) such that \(b \in f_i(\beta)\). In particular, \(b \in G_i = f_i(\alpha) \cup f_i(\psi(\alpha))\). Therefore it necessarily must be \(f_m(\psi(\alpha)) = G_m \setminus f_m(\alpha)\). So, \(f_m\) satisfies (A).

Suppose that \(f_i\) is \(\{\alpha, \beta\}\)-preserving. Let us show that \(f_m\) is also \(\{\alpha, \beta\}\)-preserving. In fact, suppose without loss of generality that \(\alpha \leq \beta\) and let \(b \in f_i(\beta) \setminus f_i(\alpha)\). Then, clearly \(b \in f_m(\beta)\). If we assume by contradiction that \(b \in f_m(\alpha)\), there would be an index \(j > i\) such that \(b \in f_j(\alpha)\), so \(b \in f_j(\beta) \cap G_i = f_i(\alpha)\), that is a contradiction. Hence \(b \in f_m(\beta) \setminus f_m(\alpha)\). The converse is obviously true. Therefore \(f_m\) is \(\{\alpha, \beta\}\)-preserving.

Furthermore, it is immediate that \(f_m(\alpha) \cap G_i = f_i(\alpha)\) for any \(i = 0, \ldots, m - 1\). So \(f_m\) satisfies (B). It is also obvious that \(\mathcal{F}_m\) is a complement-closed family on \(G_m\).

Therefore we have constructed an order-preserving \(\psi\)-complementary map \(f_m : \mathbb{L} \rightarrow \mathcal{F}_m\). By starting from \(f_m\), \(\mathcal{F}_m\) and \(G_m\), as in the first step of the proof, we can construct a finite set \(\Omega_{\mathbb{L}, \psi}\), a complement-closed family \(\mathcal{F}\) on it and an order-preserving \(\psi\)-complementary map \(f : \mathbb{L} \rightarrow \mathcal{F}\). Obviously, \(f\) is an isomorphism since it is \(\{\alpha, \beta\}\)-preserving for each 2-subset \(\{\alpha, \beta\}\) of \(\mathbb{L}\). Thus, since \((\mathcal{F}, \subseteq)\) is a poset, the isomorphism \(f : \mathbb{L} \rightarrow \mathcal{F}\) induces on \((\mathcal{F}, \subseteq)\) a natural lattice structure.

III The last step of the proof consists of proving the existence of pairing \(\mathfrak{P}\) on \(\Omega_{\mathbb{L}, \psi}\) such that \(\mathcal{F} = \mathcal{R}(\mathfrak{P})\).

Let us consider the map \(\phi : \mathcal{P}(\Omega_{\mathbb{L}, \psi}) \rightarrow \mathcal{P}(\Omega_{\mathbb{L}, \psi})\) defined as follows:
\[
\phi(X) = \bigcap\{Y \in \mathcal{F} : X \subseteq Y\}.
\]
(6.9)

It is immediate to see that \(X \subseteq \phi(X)\) and that \(\phi(X) \subseteq \phi(Y)\) whenever \(X \subseteq Y\). Moreover, it also results that \(\phi\) is idempotent. In fact, by (6.9), if \(Z \in \mathcal{F}\) contains \(X\), then it also contains \(\phi(X)\). The converse is obvious. This implies that \(\phi\) is a closure operator. Furthermore, it is evident that \(\phi(\emptyset) = \emptyset\). Thus, by Theorem 6.1.1, there exists \(\mathfrak{P} \in \text{PAIR}(\Omega)\) such that \(\text{MAXP}(\mathfrak{P})\) coincides with the family of all the fixed subsets of \(\phi\). This means that in next calculations we are allowed to use \(M_{\mathfrak{P}}\) instead of \(\phi\).
We must prove that $F = R(\mathfrak{P})$. For, let us observe that if $Z \in F$, then $M_{\mathfrak{P}}(Z) = Z$. So, $F \subseteq MAXP(\mathfrak{P})$. In particular, we have that $I_{\mathfrak{P}}(Z) = Z \in F$, so $Z = M_{\mathfrak{P}}(I_{\mathfrak{P}}(Z))$, i.e. $Z \in R(\mathfrak{P})$.

On the other hand, let $Z \in R(\mathfrak{P})$. Since $F$ is a complete lattice, it follows that

$$Z^* := \inf\{Y \in F : Z \subseteq Y\} \in F.$$ 

Furthermore, we have that $Z^* \subseteq \bigcap\{Y \in F : Z \subseteq Y\} = M_{\mathfrak{P}}(Z) = Z$. Let $A \subseteq Z$ such that $A^c \in MAXP(\mathfrak{P})$. Let us show that

$$A = \bigcup\{Y \in F : Y \subseteq A\}. \quad (6.10)$$

In fact,

$$A^c = M_{\mathfrak{P}}(A^c) = \bigcap\{Y \in F : A^c \subseteq Y\} = \bigcup\{Z \in F : Z \subseteq A\}.$$

as claimed.

Moreover, let us note that $Z^*$ contains each $Y$ in $(6.10)$. This implies that $A \subseteq Z^*$. Now, since $Z^* \in F$, then $(Z^*)^c \in MAXP(\mathfrak{P})$. By $(i)$ of Proposition 6.2.2, we conclude that $I_{\mathfrak{P}}(Z) = Z^*$. But since $F \subseteq MAXP(\mathfrak{P})$, we deduce that $Z = M_{\mathfrak{P}}(I_{\mathfrak{P}}(Z)) = I_{\mathfrak{P}}(Z) \in F$. Hence $F = R(\mathfrak{P})$. This concludes the proof. 

\[\square\]

### 6.3 Symmetry Transmission Table

Let $L$ be a finite lattice, $N = pc(L)$ and $\mathfrak{P} \in PAIR(L)$. By means of $(6.4)$ of Theorem 6.1.5, we can note that the function $\gamma_{\mathfrak{P}}$ is a refinement of the partial order $\leq_L$, since it provides a numerical information also between two non-comparable nodes of the lattice $L$. In fact, for any two elements $x, y \in L$, we have that $x \leq_L y$ if and only if $\gamma_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x), \eta_{\mathfrak{P}}(y)) = 1$ and, moreover, we can always compute $\gamma_{\mathfrak{P}}(\eta_{\mathfrak{P}}(x), \eta_{\mathfrak{P}}(y))$ even if $x$ and $y$ are non-comparable each other. Therefore, it is natural to investigate all possible values of $\gamma_{\mathfrak{P}}(X,Y)$ when $(X,Y)$ runs over $\mathcal{P}(\Omega_N) \times \mathcal{P}(\Omega_N)$.

To this aim, we introduce the **symmetry transmission table** of a pairing $\mathfrak{P} \in PAIR(\Omega_N)$ within which we collect all values $\gamma_{\mathfrak{P}}(X,Y)$ when $(X,Y) \in \mathcal{P}(\Omega_N) \times \mathcal{P}(\Omega_N)$.

**Definition 6.3.1.** Let $\mathfrak{P} \in PAIR(\Omega_N)$. We call **symmetry transmission table** of $\mathfrak{P}$, denoted by $T_{dep}(\mathfrak{P})$, the $2^N \times 2^N$ table having as rows and columns all elements of $\mathcal{P}(\Omega_N)$ and in the place corresponding to row $X$ and column $Y$ the value $\gamma_{\mathfrak{P}}(X,Y)$.

The symmetry transmission table is a source of very useful information when one studies the properties of $\mathfrak{P}$. In fact, a complete knowledge of the symmetry transmission table enables us to build the symmetry partition lattice of $\mathfrak{P}$. Hence, we can think the table $T_{dep}(\mathfrak{P})$ as a type of numerical completion of the symmetry partition lattice of $\mathfrak{P}$.

Since the symmetry transmission table provides numerical informations for any pair $(X,Y) \in \mathcal{P}(\Omega_N) \times \mathcal{P}(\Omega_N)$, it is natural to introduce a global symmetry transmission measure as well as it is usually done in many sectors of discrete mathematics.
To this regard, if \( A \subseteq \Omega_N \), we set

\[
\lambda_r(A) := \frac{1}{2^n} \sum_{X \subseteq \Omega_N} \gamma_\Psi(A, X)
\]

and

\[
\lambda_c(A) := \frac{1}{2^n} \sum_{X \subseteq \Omega_N} \gamma_\Psi(X, A).
\]

**Definition 6.3.2.** We call symmetry transmission average of \( \Psi \) the number

\[
\lambda(\Psi) := \frac{1}{2^n} \sum_{A \subseteq \Omega_N} \lambda_r(A) = \frac{1}{2^n} \sum_{A \subseteq \Omega_N} \lambda_c(A).
\]

We can interpret the number \( \lambda(\Psi) \) as the average capacity to transmit (or to receive) symmetry of a generic subset of \( \Omega_N \).

### 6.4 Symmetry Transmission Tables on Some Examples

In this section, we compute the symmetry transmission average for some basic digraph family, also providing an asymptotic estimate. On the other hand, we consider two cases of pairings on infinite sets \( \Omega \) and we compute the map \( \Gamma_\Psi \) in this cases.

#### 6.4.1 The Directed \( n \)-Cycle

We determine the symmetry transmission table for the pairing \( \Psi[C_n^r] \).

**Proposition 6.4.1.** Let \( A \) and \( B \) be two distinct vertex subsets of \( V = V(C_n^r) = \{v_1, \ldots, v_n\} \). Then

\[
\Gamma_\Psi(A, B) = \begin{cases} 
V & \text{if } B = \emptyset \text{ or } A = V \text{ or } |A| = n-1 \text{ or } B \subseteq A \\
0 & \text{if } A = \emptyset \text{ and } B \neq \emptyset \\
A_{-1} & \text{otherwise}
\end{cases} \tag{6.11}
\]

and

\[
\gamma_\Psi(A, B) = \begin{cases} 
1 & \text{if } B = \emptyset \text{ or } A = V \text{ or } |A| = n-1 \text{ or } B \subseteq A \\
0 & \text{if } A = \emptyset \text{ and } B \neq \emptyset \\
\frac{|A|}{n} & \text{otherwise}
\end{cases} \tag{6.12}
\]

**Proof.** By \((ii)\) of Proposition 2.3.9, we have that if \( B = \emptyset \), then \( \pi_{C_n^r}(B) = V \). Therefore if \( A \) is any subset of \( V \) and \( v \in V \) is any vertex, it follows that \( [v]_A \subseteq [v]_B = V \). Similarly, if \( A = V \) or if \( |A| = n-1 \), we have \( \pi_{C_n^r}(A) = v_1|\ldots|v_n \). In other words, in these situations \( \Gamma_\Psi(A, B) = V \). Furthermore, by (2.13), if \( B \subseteq A \), we deduce that \( \Gamma_\Psi(A, B) = V \). Again by \((ii)\) of Proposition 2.3.9, it is clear that if \( A = \emptyset \) and \( B \neq \emptyset \), we have \( [v]_A = V \) for every \( v \in V \), but \( [v]_B \nsubseteq V \). So, \( \Gamma_\Psi(A, B) = \emptyset \). Let \( |A| = k \) and \( |B| = l \). By \((ii)\) of Proposition 2.3.9, we have \( \pi_{C_n^r}(A) = v_{i_1-1}\ldots|v_{i_{k-1}}|A_{-1} \) and \( \pi_{C_n^r}(B) = v_{j_1-1}\ldots|v_{j_{l-1}}|B_{-1} \). This means that \( [v]_A \subseteq [v]_B \) only for \( x = v_{i_1-1}, \ldots, v_{i_{k-1}} \), therefore \( \Gamma_\Psi(A, B) \supseteq \{v_{i_1-1}, \ldots, v_{i_{k-1}}\} \). Let \( v \in \Gamma_\Psi(A, B) \setminus A_{-1} \). Then, \( [v]_A = A_{-1} \). We have two cases: \( v \in B \) or \( v \notin B \). In the first
case, \([v]_B\) has a single element, so \([v]_A\) can’t be contained in \([v]_B\) unless \(A^c - 1\) consists of a single element, but this happens if \(|A| = n - 1\) and we have excluded this situation. In the second case, we have \([v]_A \subseteq [v]_B\) if and only if \(A^c - 1 \subseteq B^{-1}\), i.e. if and only if \(B^{-1} \subseteq A^{-1}\). This means that \(B \subseteq A\), but this situation has been excluded. Hence the proof of (6.11) is complete. Finally, by definition we have \(\gamma_P(A, B) = \frac{\|P(A, B)\|}{|V|}\), therefore (6.12) follows directly from (6.11). \(\square\)

We compute the previous average numbers in the particular case of the pairing \(P[C_n]\) and we also provide an asymptotic estimate for the symmetry transmission average.

**Theorem 6.4.2.** (i): Let \(A \subseteq V(C_n^r)\). Then

\[
\lambda_r(A) = \begin{cases} 
\frac{1}{2^r} & \text{if } A = \emptyset \\
\frac{1}{2^r - \pi} + \frac{k}{n2^{n-r}}(2^{n-k} - 1) & \text{if } k = |A| < n - 1 \\
1 & \text{if } |A| \geq n - 1 
\end{cases}
\]

(ii): Let \(B \subseteq V = V(C_n^r)\). Then, we have:

\[
\lambda_c(B) = \begin{cases} 
1 & \text{if } B = \emptyset \\
\frac{3n-1}{4n} + \frac{1}{n2^r} & \text{if } |B| = 1 \\
\frac{1}{2^r} + \frac{k}{2^r} + \frac{1}{2^r} \left[ \sum_{l=1}^{k-1} \binom{n}{l} \frac{l}{n} + \sum_{l=k}^{n-2} \left( \binom{n}{l} - \binom{n-k}{l-1} \right) \frac{l}{n} \right] & \text{if } 1 < k = |B| < n \\
\frac{1}{2^r} + \frac{1}{2} & \text{if } |B| = n 
\end{cases}
\]

(iii): \(\lambda(C_n^r) = \frac{1}{4^r} \left\{ \sum_{k=0}^{n-2} \binom{n}{k} \left[ 2^k + k \frac{2^{n-2k}}{n} \right] + 2^n(n+1) \right\} \) and \(\lim_{n \to \infty} \lambda(C_n^r) = \frac{1}{2}\).

**Proof.** (i): If \(A = \emptyset\), by (6.12), in the first row of symmetry transmission table of \(C_n^r\) we have all the entries 0 except the first, which is 1, so \(\lambda_r(\emptyset) = \frac{1}{2^r}\). Otherwise, if \(|A| = r\), where \(r = 1, \ldots, n - 2\), we have exactly \(2^r\) entries equal to 1, corresponding to all its possible subsets, and the other are equal to \(\frac{k}{n}\). Finally, when \(|A| = n - 1\) or \(A = V\), the entries of the corresponding rows in the symmetry transmission table are all 1.

(ii): We use the numbers \(\gamma_P(A, B)\) determined in (6.12) of Proposition 6.4.1. The first column, corresponding to \(B = \emptyset\), has all entries 1, therefore, \(\lambda_c(\emptyset) = 1\). Let \(B\) be a singleton. Then, if \(B \subseteq A\), the corresponding entries of the symmetry transmission table are 1; this happens \(2^{n-1}\) times. Moreover, it is true also if \(|A| = n - 1\). Now, we compute the other values. There are exactly \(\binom{n-1}{k}\) \(k\)-subsets of \(V\) not containing \(B\) and the corresponding value of the symmetry transmission table is \(\frac{k}{n}\). Therefore we have

\[
\lambda_c(\{v_i\}) = 2^{n-1} + 1 + \sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k}{n}.
\]
transmission table is 

\[ \lambda_c(v_i) = \frac{3n - 1}{4n} + \frac{1}{n2^n}. \]

Let \( B \) be a generic subset with \( 1 < |B| < n \) elements. First of all, we observe that there are exactly \( 2^{n-k} \) subsets of \( V \) containing it and there are \( \binom{n}{n-1} - \binom{n-k}{n-k-1} = k \) subsets of cardinality \( n - 1 \) not containing \( B \), to which correspond the value 1. Furthermore there are no \( l \)-subsets containing \( B \) for \( l = 1, \ldots, k - 1 \) and, for each of them, the corresponding value in the symmetry transmission table is \( \frac{l}{n} \), i.e. we can sum the quantity \( \sum_{l=1}^{k-1} \binom{n}{l} \frac{l}{n} \). Now, suppose that \( k \leq l \leq n - 2 \).

There are exactly \( \binom{n}{k} \) subsets of cardinality \( l \) not containing \( B \) and in correspondence of them, the value taken in the symmetry transmission table is \( \frac{l}{n} \). Therefore we have:

\[
\lambda_c(B) = \frac{1}{2^k} + \frac{k}{2^n} + \frac{1}{2^n} \sum_{l=1}^{k-1} \left( \binom{n}{l} \frac{l}{n} \right) + \frac{1}{2^n} \sum_{l=k}^{n-k} \left( \binom{n}{l} - \left( \binom{n-k}{l-k} \right) \frac{l}{n} \right).
\]

Finally, when \(|B| = n\), we have \( n + 1 \) entries equal to 1. Furthermore, there are no \( l \)-subsets containing \( B \) for \( l = 1, \ldots, n - 2 \) and, for each of them, the corresponding value in the symmetry transmission table is \( \frac{1}{n} \). So \( \lambda_c(V) = \frac{n+1}{2^n} + \frac{1}{2^n} \sum_{l=1}^{n-k} \binom{n}{l} \frac{l}{n} \). Adding and subtracting the quantities \( n(n-1) + n \), using the fact that \( \sum_{k=0}^{n} \binom{n}{k}k = n2^{n-1} \), with algebraic manipulations we obtain

\[
\lambda_c(V) = \frac{1}{2^n} + \frac{1}{2}.
\]

(iii): Setting \( k = |A| \), we have \( 2^{2n} \) elements in the table, while, by (i), the sum of elements in the same row is \( 2^k + \frac{k2^k}{2^n}(2^{n-k} - 1) \) for \( k = 1, \ldots, n - 2 \). We observe that if \( k = 0 \), then the previous formula gives the result 1, in agreement with part (i). Therefore we have

\[
\lambda(C_n) = \frac{1}{4^n} \left\{ \sum_{k=0}^{n-2} \binom{n}{k} \left[ 2^k + \frac{2^{n-k} - 2^k}{n} \right] + 2^n(n+1) \right\}.
\]

Now, let us note that \( \lim_{n \to \infty} \frac{2^{n(n+1)}}{4^n} = 0 \) and \( \lim_{n \to \infty} \frac{1}{4^n} \sum_{k=0}^{n-2} \binom{n}{k}2^k = 0 \), since \( \sum_{k=0}^{n-2} \binom{n}{k}2^k = \sum_{k=0}^{n-2} \binom{n}{k}2^k - n2^{n-1} - 2^n = 3^n - n2^{n-1} - 2^n \). Moreover

\[
\sum_{k=0}^{n-2} \binom{n}{k} \frac{k(2^n - 2^k)}{n} \leq \sum_{k=0}^{n-2} \binom{n}{k} \frac{k2^n}{n} = \frac{2^n}{n} \sum_{k=0}^{n} \binom{n}{k} - n(n-1) - n = 2^{n-1} - n2^n.
\]

Hence

\[
\lim_{n \to \infty} \lambda(C_n) \leq \lim_{n \to \infty} \frac{2^{2n-1} - n2^n}{4^n} = \frac{1}{2}.
\]

Similarly, since \( \sum_{k=0}^{n} \binom{n}{k}k = n2^{n-1} \), by (6.13) we obtain

\[
\lambda(C_n) = \frac{1}{4^n} \left\{ \sum_{k=0}^{n-2} \binom{n}{k} \left[ \frac{2^n}{n} + 2^k \left( 1 - \frac{k}{n} \right) \right] + 2^n(n+1) \right\} \geq \frac{1}{4^n} [2^{2n-1} + 2^n(1 + n)].
\]
Therefore
\[ \lim_{n \to \infty} \lambda(C_n^*) \geq \lim_{n \to \infty} \frac{1}{4n}[2^{2n-1} + 2^n(1 + n)] = \frac{1}{2} \]
and the thesis follows.

\subsection*{6.4.2 The directed \( n \)-path}

Now we focus our attention to the case of the directed \( n \)-path \( \vec{P}_n \).

**Proposition 6.4.3.** Let \( A \) and \( B \) be two distinct vertex subsets of \( V = V(\vec{P}_n) = \{v_1, \ldots, v_n\} \). Then
\[
\Gamma_p(A, B) = \begin{cases} 
V & \text{if } B = \emptyset \lor B = \{v_1\} \lor A = V \lor A = \{v_2, \ldots, v_n\} \lor B \subseteq A \\
\emptyset & \text{if } (A = \emptyset \lor A = \{v_1\}) \land (B \neq \emptyset \land B \neq \{v_1\}) \\
A_{-1} & \text{otherwise}
\end{cases} \tag{6.14}
\]
Moreover
\[
\gamma_p(A, B) = \begin{cases} 
1 & \text{if } B = \emptyset \lor B = \{v_1\} \lor A = V \lor A = \{v_2, \ldots, v_n\} \lor B \subseteq A \\
0 & \text{if } (A = \emptyset \lor A = \{v_1\}) \land (B \neq \emptyset \land B \neq \{v_1\}) \\
\frac{k}{n} & \text{if } |A| = k \land v_1 \notin A \\
\frac{k-1}{n} & \text{if } |A| = k \land v_1 \in A
\end{cases} \tag{6.15}
\]

**Proof.** By Proposition 2.3.9, we have that if \( B = \emptyset \) or \( B = \{v_1\} \), then \( \pi_{\vec{P}_n}(B) = V \). Therefore, if \( A \) is any subset of \( V \) and \( v \in V \) is any vertex, it follows that \([v]_A \subseteq [v]_B = V \). Similarly, if \( A = V \) or if \( A = \{v_2, \ldots, v_n\} \), we have \( \pi_{\vec{P}_n}(A) = v_1 \ldots |v_n| \). Moreover, by (2.13), if \( B \subseteq A \), then \( \pi_{\vec{P}_n}(A) \leq \pi_{\vec{P}_n}(B) \). In other words, in these cases \( \Gamma_{\tilde{P}_n}(A, B) = V \). A fortiori, even if \( A = \emptyset \) or \( A = \{v_1\} \), then \( \Gamma_{\tilde{P}_n}(A, B) = V \). This is false when \( B \neq \emptyset \), \( \{v_1\} \), in fact in this case we would have \([v]_A = V \) contained in one of its subsets, that is an absurd. Let \(|A| = k \) and \(|B| = l \) vertex subsets different from the previous. We have respectively \( \pi_{\vec{P}_n}(A) = v_{i_1-1} \ldots v_{i_k-1} A_{-1} \) and \( \pi_B = v_{j_1-1} \ldots v_{j_l-1} B_{-1} \). It is clear that the classes of vertices which form single blocks with respective to \( A \) are contained in their classes with respect to \( B \), so \( A_{-1} \subseteq \Gamma_{\tilde{P}_n}(A, B) \). Let \( v \in \Gamma_{\tilde{P}_n}(A, B) \setminus A_{-1} \). Then, \([v]_A = A_{-1} \). We have two cases: \( v \in B \) or \( v \notin B \). In the first case, \([v]_B \) has a single element, so \([v]_A \) can’t be contained in \([v]_B \) unless \( A_{-1} \) consists of a single element, but this happens if \( A = \{v_2, \ldots, v_n\} \) and we have excluded this situation. In the second case, we have \([v]_A \subseteq [v]_B \) if and only if \( A_{-1} \subseteq B_{-1} \), i.e. if and only if \( B_{-1} \subseteq A_{-1} \). This means that \( B \subseteq A \), but this situation has been excluded. Thus \( \Gamma_{\tilde{P}_n}(A, B) = \{v_{i_1-1} \ldots, v_{i_k-1}\} \). This complete the proof of (6.14). Finally, by definition, we have \( \gamma_{\tilde{P}_n}(A, B) = \frac{|\Gamma_{\tilde{P}_n}(A, B)|}{|V|} \), therefore (6.15) follows directly from (6.14). \( \square \)
Lemma 6.4.4. Let \( A \subseteq V = V(\tilde{P}_n) \). Then, we have

\[
\lambda_r(A) = \begin{cases} 
\frac{1}{2n-1} & \text{if } A = \emptyset \lor A = \{v_1\} \\
1 & \text{if } \{v_2, \ldots, v_n\} \subseteq A \\
\frac{1}{n} + \frac{3}{2n} (1 - \frac{1}{n}) & \text{if } |A| = 1 \\
\frac{1}{2n-1} (1 + \frac{k-1}{n} (2^{n-k} - 1)) & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \\
\frac{1+2^k + \frac{k}{n} (2^{n-2^k} - 1)}{2^n} & \text{if } |A| = k \neq n-1 \land v_1 \notin A 
\end{cases}
\]

Proof. By (6.15) of Lemma 6.4.3, in the rows corresponding to \( A = \emptyset \) and \( A = \{v_1\} \) we have only two entries \( 1 \) and the others are all \( 0 \), so \( \lambda_r(\emptyset) = \lambda_r(\{v_1\}) = \frac{1}{2n-1} \). When \( A = V \), we have that \( \gamma_r^k(B) = 1 \) for every \( B \subseteq V = V(\tilde{P}_n) \), therefore \( \lambda_r(V) = 1 \). If \( A \) is singleton \( \{v_i\} \neq \{v_1\} \), by (6.15) only in three entries we have \( 1 \), in the other entries we have \( \frac{1}{n} \). Therefore we have \( \lambda_r(A) = \frac{1}{2n} \left( \frac{2n}{n} - \frac{3}{n} + 3 \right) \), i.e. \( \frac{1}{n} + \frac{3}{2n} (1 - \frac{1}{n}) \). Let \( A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \). By (6.15), the entries are \( 1 \) when the sets in the columns of the symmetry transmission table are contained in \( A \), therefore there are exactly \( \left( \frac{k}{n} \right) + \cdots + \left( \frac{k}{n} \right) = 2^k \) entries equal to \( 1 \), i.e. the number of all vertex subsets of a set of cardinality \( k \), while the other entries are equal to \( \frac{k-1}{n} \). If \( A \) doesn’t contain the vertex \( v_1 \) and has cardinality \( |A| = k \neq n-1 \), there are \( 2^k + 1 \) entries equal to \( 1 \), i.e. those corresponding to all its subsets and one corresponding to the second column. By (6.15), we deduce that the remaining entries have value \( \frac{k}{n} \). \( \square \)

Theorem 6.4.5. The quantity \( \lambda(\tilde{P}_n) \) is equal to:

\[
\frac{1}{4^n} \left\{ 4 + 2^{n+1} + (n-2) \left( 3 + \frac{2n-3}{n} \right) + \sum_{k=2}^{n-1} \binom{n-1}{k-1} 2^k \left[ 1 + \frac{k-1}{n} (2^{n-k} - 1) \right] \right\} + \\
\frac{1}{4^n} \left\{ \sum_{k=2}^{n-2} \binom{n-1}{k} \left[ 1 + 2^k + \frac{k}{n} (2^{n-2^k} - 1) \right] \right\}.
\]

and \( \lim_{n \to \infty} \lambda(\tilde{P}_n) = \frac{1}{2} \).

Proof. We must add the numbers \( \lambda_r(A) \) determined in Lemma 6.4.4, when \( A \) varies among the subsets of \( V(\tilde{P}_n) \). So, if \( A = \emptyset \) or \( A = \{v_1\} \), we have to add 2, while if \( A = \{v_2, \ldots, v_n\} \) or \( A = v \) we have to add \( 2^n \). If \( A \) is a singleton \( v_i \), then we have exactly three entries equal to \( 1 \), while the others take the value \( \frac{1}{n} \). Since there are \( n-2 \) of these singletons, we can add the quantity \( (n-2) \left( 3 + \frac{2n-3}{n} \right) \). Since, for \( k = 2, \ldots, n-1 \) the number of sets of cardinality \( k \) containing the vertex \( v_1 \) is \( \binom{n-1}{k-1} \), we have to add the quantity \( \sum_{k=2}^{n-1} \binom{n-1}{k-1} 2^k \left[ 1 + \frac{k-1}{n} (2^{n-k} - 1) \right] \). Moreover, since for \( k = 2, \ldots, n-2 \) the number of sets of cardinality \( k \) not containing the vertex
we also have to add the quantity \( \sum_{k=2}^{n-2} \binom{n-1}{k} 2^k [1 + 2^k + \frac{k}{n}(2^n - 2^k - 1)] \). Thus, we conclude

\[
\lambda(\vec{P}_n) = \frac{1}{4^n} \left\{ 4 + 2^{n+1} + (n-2) \left( 3 + \frac{2^n - 3}{n} \right) + \sum_{k=2}^{n-1} \binom{n-1}{k} 2^k \left[ 1 + \frac{k-1}{n}(2^{n-k} - 1) \right] \right\} + \frac{1}{4^n} \left\{ \sum_{k=2}^{n-2} \binom{n-1}{k} \left[ 1 + 2^k + \frac{k}{n}(2^n - 2^k - 1) \right] \right\}.
\]

We set \( R = \sum_{k=2}^{n-1} \binom{n-1}{k-1} 2^k [1 + \frac{k-1}{n}(2^{n-k} - 1)] \) and \( S = \sum_{k=2}^{n-2} \binom{n-1}{k} \sum_{k=0}^{n} k n^{k-1}, \) we also have to add the quantity \( \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1} \), with algebraic manipulations we obtain

\[
R + S \leq 3^n + 2^{2n-1} - 2 - 3 \cdot 2^n - 3n - \frac{n-1}{n} 2^n.
\]

Then, we have

\[
\lambda(\vec{P}_n) = \frac{1}{4^n} \left[ 4 + 2^{n+1} + (n-2) \left( 3 + \frac{2^n - 3}{n} \right) + R + S \right] \leq \frac{1}{4^n} \left[ 4 + 2^{n+1} + (n-2) \left( 3 + \frac{2^n - 3}{n} \right) + 3^n + 2^{2n-1} - 2 - 3 \cdot 2^n - 3n - \frac{n-1}{n} 2^n \right],
\]

hence

\[
\lim_{n \to \infty} \lambda(\vec{P}_n) \leq \frac{1}{2}.
\]

In a similar way, again by means of both the Stifel formula and the identities \( \sum_{k=0}^{n} \binom{n}{k} k = n 2^{n-1} \),
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k = 2n \cdot 3^{n-1}, \quad \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}, \quad \sum_{k=0}^{n-1} \binom{n-1}{k} 2^k = 3^{n-1}, \quad \sum_{k=0}^{n-1} \binom{n-1}{k} k = (n-1) 2^{n-2},
\]
we get

\[
R + S \geq \frac{1}{n} \left[ 2^{n+1} - 2^n + 2^{n-1}(n-1)^2 + 2n - 2 - 2(n-1)(2^{n-3} + 1)] + \left( 1 - \frac{1}{n} \right) 2^{2n-1} - (2 \cdot 3^{n-1} + 2^{n-1}) \left( 1 - \frac{1}{n} \right) \right]
\]

(6.16)

Adding the term \( Q = 4 + 2^{n+1} + (n-2) \left( 3 + \frac{2^n - 3}{n} \right) \), dividing by \( 4^n \) and passing to the limit as \( n \to \infty \), we obtain

\[
\lim_{n \to \infty} \lambda(\vec{P}_n) \geq \frac{1}{2}.
\]

Hence \( \lim_{n \to \infty} \lambda(\vec{P}_n) = \frac{1}{2} \). \qed

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6.4.3 The Case of \((V, \varphi, K)\)-VBP

Let us analyze the case of \((V, \varphi, K)\)-VBP and that of pairings associated with group actions on \(\Omega\).

**Proposition 6.4.6.** Let \(V\) be a \((V, \varphi, K)\)-VBP and \(A, B \subseteq V\). Then

\[
\Gamma_V(A, B) = \begin{cases} 
V & \text{if } B \subseteq A \\
\emptyset & \text{otherwise}
\end{cases}
\]

**Proof.** Since \(\pi_p(A) = \{u + A^\perp : u \in V\}\), we deduce that \(u + A^\perp \subseteq u + B^\perp\) if and only if \(A^\perp \subseteq B^\perp\) or, equivalently, if \(B \subseteq A\). Hence, \(\Gamma_V(A, B) = V\) if and only if \(B \subseteq A\), otherwise \(\Gamma_V(A, B) = \emptyset\). \(\square\)

6.4.4 The Case of Pairings Arising from Group Actions on \(\Omega\)

Let us analyze the case of pairings associated with group actions on \(\Omega\).

**Proposition 6.4.7.** Let \((G, \psi)\) be the pairing associated with the action \(\psi\) of the group \(G\) on \(\Omega\) and \(A, B \subseteq \Omega\). Then

\[
\Gamma_{G, \psi}(A, B) = \begin{cases} 
G & \text{if } \text{Stab}_{G, \psi}(A) \leq \text{Stab}_{G, \psi}(B) \\
\emptyset & \text{otherwise}
\end{cases}
\]

**Proof.** Since \(\pi_p(A) = \{g \text{ Stab}_{G, \psi}(A) : g \in G\}\), we deduce that \(g \text{ Stab}_{G, \psi}(A) \subseteq g \text{ Stab}_{G, \psi}(B)\) if and only if \(\text{Stab}_{G, \psi}(A) \leq \text{Stab}_{G, \psi}(B)\). Hence, for any \(g' \in G\), it results that \(g' \text{ Stab}_{G, \psi}(A) \subseteq g' \text{ Stab}_{G, \psi}(B)\), so \(\Gamma_{G, \psi}(A, B) = G\). Otherwise, we must have \(\Gamma_{G, \psi}(A, B) = \emptyset\). \(\square\)
Chapter 7

Matroidal Properties of $MINP(\mathfrak{P})$

The result provided in Theorem 6.1.1 can be linked in a deeper way with the classical notion of matroid. As a matter of fact, Theorem 1.6.5 gives a necessary and sufficient condition so that a rank-symmetry operator of a matroid $M$ on a finite set $\Omega$ coincides with a specific closure operator on $\Omega$.

This means that we can pass from the rank-symmetry operator $\sigma$ of a given matroid on $\Omega$ to the closure operator associated with some pairing $\mathfrak{P}$ on $\Omega$ itself by means of Theorem 6.1.1. To be more specific, in [20] it has been showed that $MINP(\mathfrak{P}) = \{A \in \mathcal{P}(\Omega) : a \not\in M_{\mathfrak{P}}(A \setminus \{a\}) \forall a \in A\}$. (7.1)

Then, the identity provided in (7.1) tells us that $MINP(\mathfrak{P})$ behaves as the family of independent sets of a potential matroid associated with the closure operator $M_{\mathfrak{P}}$. In other terms, $MINP(\mathfrak{P})$ is related to $MAXP(\mathfrak{P})$ in a similar way in which the independent set family of a matroid is related to its closed set family.

Therefore, the representation Theorem 6.1.1 leads us to conclude that we do not lose in generality whenever we study closure systems or abstract simplicial complexes through pairings and, furthermore, we also obtain explicit constructive criteria in order to study matroids as particular cases of abstract simplicial complexes and finite lattice theory as a subtheory of closure system theory.

In this chapter, we assume that $\Omega$ is a finite fixed set. We will establish new links between matroid theory and the maximum partitioners, the minimal ones and the symmetry bases.

To this regard, our first basic result is a new representation theorem, where we prove that any matroid $M$ on $\Omega$ can be represented through a pairing whose $MINP(\mathfrak{P})$ coincides with $M$.

**Theorem 7.0.8.** Let $M$ be the independent set family of a matroid on $\Omega$. Then, there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $MINP(\mathfrak{P}) = M$.

**Proof.** Let $\sigma$ be the closure operator associated with $M$ (see [182]). Then, it is well known that $M = \{A \in \mathcal{P}(\Omega) : a \not\in \sigma(A \setminus \{a\}) \forall a \in A\}$. (7.2)

By Theorem 6.1.1 there exists a pairing $\mathfrak{P} \in PAIR(\Omega)$ such that $M_{\mathfrak{P}} = \sigma$. Then, the thesis follows by (7.2) and by $(iv)$ of Theorem 2.6.1. \qed

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We now study the two maps \( r_{\mathcal{P}} : = r_{MINP(\mathcal{P})} \) and \( Dr_{\mathcal{P}} : = Dr_{MINP(\mathcal{P})} \). To this regard, we firstly note that
\[
 r_{\mathcal{P}}(A) = \max\{|X| : X \in MINP_\mathcal{P}(A)\} = \max\{|X| : X \in \max(MINP_\mathcal{P}(A))\} \tag{7.3}
\]
and
\[
 Dr_{\mathcal{P}}(A) := \{x \in \Omega : r_{\mathcal{P}}(A \cup \{x\}) = r_{\mathcal{P}}(A)\}. \tag{7.4}
\]

**Example 7.0.9.** In reference to the pairing \( \mathcal{P} \) given in Example 3.4.5, it results that:
\[
 r_{\mathcal{P}}(\emptyset) = 0; \quad r_{\mathcal{P}}(\{r\}) = r_{\mathcal{P}}(\{c\}) = r_{\mathcal{P}}(\{m\}) = r_{\mathcal{P}}(\{b\}) = r_{\mathcal{P}}(\{m, c\}) = 1;
\]
\[
 r_{\mathcal{P}}(\{r, m\}) = r_{\mathcal{P}}(\{r, c\}) = r_{\mathcal{P}}(\{r, b\}) = r_{\mathcal{P}}(\{b, c\}) = r_{\mathcal{P}}(\{m, b\}) = r_{\mathcal{P}}(\{r, m, c\}) =
\]
\[
 = r_{\mathcal{P}}(\{r, m, b\}) = r_{\mathcal{P}}(\{m, c, b\}) = r_{\mathcal{P}}(\{r, c, b\}) = r_{\mathcal{P}}(\{r, m, c, b\}) = 2
\]
Therefore, we deduce that
\[
 Dr_{\mathcal{P}}(\emptyset) = \emptyset; \quad Dr_{\mathcal{P}}(\{r\}) = \{r\}; \quad Dr_{\mathcal{P}}(\{b\}) = \{b\};
\]
\[
 Dr_{\mathcal{P}}(\{c\}) = Dr_{\mathcal{P}}(\{m\}) = Dr_{\mathcal{P}}(\{m, c\}) = \{m, c\};
\]
\[
 Dr_{\mathcal{P}}(\{r, m\}) = Dr_{\mathcal{P}}(\{r, c\}) = Dr_{\mathcal{P}}(\{r, b\}) = Dr_{\mathcal{P}}(\{m, b\}) = Dr_{\mathcal{P}}(\{b, c\}) =
\]
\[
 = Dr_{\mathcal{P}}(\{r, m, c\}) = Dr_{\mathcal{P}}(\{r, c, b\}) = Dr_{\mathcal{P}}(\{m, c, b\}) = Dr_{\mathcal{P}}(\{r, m, b\}) =
\]
\[
 = Dr_{\mathcal{P}}(\{r, m, c, b\}) = \{r, m, c, b\}.
\]

**Proposition 7.0.10.** The function \( r_{\mathcal{P}} \) satisfies the following properties:
(i) \( r_{\mathcal{P}}(\emptyset) = 0 \);
(ii) \( \text{If } A \subseteq B, \text{ then } r_{\mathcal{P}}(A) \leq r_{\mathcal{P}}(B) \);
(iii) \( 0 \leq r_{\mathcal{P}}(A) \leq |A| \text{ for any } A \in \mathcal{P}(\Omega) \);
(iv) \( r_{\mathcal{P}}(A) \leq r_{\mathcal{P}}(A \cup \{x\}) \leq r_{\mathcal{P}}(A) + 1 \).

**Proof.** All these properties are direct consequences of (7.3). \( \square \)

In [20], it has been proved the next characterization for set system \( MINP(\mathcal{P}) \) in terms of \( r_{\mathcal{P}} \):
\[
 MINP(\mathcal{P}) = \{A \in \mathcal{P}(\Omega) : r_{\mathcal{P}}(A) = |A|\}. \tag{7.5}
\]

**Remark 7.0.11.** Let us observe that the map \( r_{\mathcal{P}} \) behaves as the rank function of a matroid whose independent set coincides with \( MINP(\mathcal{P}) \) and whose bases are the elements of \( \max(MINP(\mathcal{P})) \). However, in general, \( MINP(\mathcal{P}) \) is not a matroid, even if it has many matroidal-like properties.

We can establish now two basic links between some properties of the symmetry bases and the matroidal properties of the set system \( MINP(\mathcal{P}) \).
**Theorem 7.0.14.** If $\text{MINP}(\mathfrak{P})$ is a matroid, then $\text{BAS}_{\mathfrak{P}}(A)$ has uniform cardinality for any $A \in \mathcal{P}(\Omega)$.

(ii) If $\mathfrak{M}$ is a MLS closure operator on $\Omega$ then $\text{MINP}(\mathfrak{P})$ is a matroid.

**Proof.** (i): It follows immediately by Theorem 1.6.4 and by (iii) of Theorem 3.2.2.

(ii): Since $\mathfrak{M}$ is a MLS closure operator on $\Omega$, the family

$$\mathcal{F} := \{ A \in \mathcal{P}(\Omega) : a \notin \mathfrak{M}(A \setminus \{a\}) \ \forall a \in A \}$$

coincides with the independent set family of a matroid on $\Omega$ (see [182]). By virtue of (7.1), we deduce that $\mathcal{F} = \text{MINP}(\mathfrak{P})$, hence the claim has been showed.

We now ask which relation there is between the set operators $Drk_{\mathfrak{P}}$ and $\mathfrak{M}$. In general, the identity $Drk_{\mathfrak{P}}(A) = \mathfrak{M}(A)$ does not hold, as we can see in the next example.

**Example 7.0.13.** Let $\mathfrak{P}$ be the pairing given in Example 3.2.3 and let $A = \{1, 2\}$. It is immediate to see that $rk_{\mathfrak{P}}(\{1, 2\}) = rk_{\mathfrak{P}}(\{1, 2, 5\}) = 2$ and that $5 \notin \mathfrak{M}_{\mathfrak{P}}(\{1, 2\}) = \{1, 2\}$. Hence $\mathfrak{M}_{\mathfrak{P}}(A) \neq Drk_{\mathfrak{P}}(A)$. Furthermore, let us observe that $\text{BAS}(\mathfrak{P}) = \{\{2, 5\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 2, 3\}, \{2, 3, 4\}\}$. In fact, we observe by Figure 3.2 that $\text{BAS}(\mathfrak{P}) = \{\{2, 5\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 2, 3\}, \{2, 3, 4\}\}$, whilst $\max(\text{MINP}(\mathfrak{P})) = \{\{2, 5\}, \{4, 5\}\}$. Therefore, $\text{MINP}(\mathfrak{P})$ is not a matroid: let $A = \{2, 5\}$ and $B = \{1, 2, 4\}$. Then, for any $b \in B \setminus A$, it follows that $A \cup \{b\} \notin \text{MINP}(\mathfrak{P})$. As a matter of fact, neither $\{1, 2, 5\}$ nor $\{2, 4, 5\}$ belong to $\text{MINP}(\mathfrak{P})$, as Figure 3.2 shows.

However, we show now that $\text{MINP}(\mathfrak{P})$ is a matroid on $\Omega$ when the set operators $Drk_{\mathfrak{P}}$ and $\mathfrak{M}$ coincide.

**Theorem 7.0.14.** If $Drk_{\mathfrak{P}}(A) = \mathfrak{M}(A)$ for any $A \in \mathcal{P}(\Omega)$, then $\text{MINP}(\mathfrak{P}) \in \text{MATR}(\Omega)$.

**Proof.** Let us prove that $rk_{\mathfrak{P}}$ satisfies (Rk1), (Rk2) and (Rk3). By (i) and (ii) of Proposition 7.0.10, we have that $rk_{\mathfrak{P}}$ satisfies (Rk1) and (Rk2). Let us assume that $rk_{\mathfrak{P}}(A \cup \{a\}) = rk_{\mathfrak{P}}(A \cup \{b\}) = rk_{\mathfrak{P}}(A)$. This means that $a, b \in Drk_{\mathfrak{P}}(A) = \mathfrak{M}(A)$. Hence, $A \cup \{a\} \approx_{\mathfrak{P}} A$. Moreover, it clearly results that $\mathfrak{M}(A \cup \{a\}) = Drk_{\mathfrak{P}}(A \cup \{a\}) = Drk_{\mathfrak{P}}(A) = \mathfrak{M}(A)$ and $\mathfrak{M}(A \cup \{b\}) = Drk_{\mathfrak{P}}(A \cup \{b\}) = Drk_{\mathfrak{P}}(A) = \mathfrak{M}(A)$. Thus, $b \in \mathfrak{M}(A \cup \{a\}) = Drk_{\mathfrak{P}}(A \cup \{a\})$, therefore we conclude that $rk_{\mathfrak{P}}(A \cup \{a\} \cup \{b\}) = rk_{\mathfrak{P}}(A \cup \{a\}) = rk_{\mathfrak{P}}(A)$. This shows that $rk_{\mathfrak{P}}$ satisfies (Rk3) and that $rk_{\mathfrak{P}}$ is the rank function of a matroid on $\Omega$. By Theorem 1.6.2 and (7.5), we deduce that $\text{MINP}(\mathfrak{P})$ is a matroid.

The reverse theorem does not hold, as we can see in the next example.

**Example 7.0.15.** Let us consider the pairing $\mathfrak{P}$ of Example 3.4.5. It is easy to see that $\text{MINP}(\mathfrak{P})$ is a matroid. In fact, for the empty set, Property (M2) holds trivially. Moreover, if we take $A, B$ such that $|A| = 1$ and $|B| = 2$, two cases occur: if $A \subsetneq B$, (M2) is trivially verified, otherwise, we clearly have that $A \cap B = \emptyset$. Let us consider, for example $A = \{r\}$ and $B = \{b, c\}$. Since both $\{r, c\}$ and $\{r, b\}$ belong to $\text{MINP}(\mathfrak{P})$, we conclude that (M2) is satisfied for these two particular sets. A similar argument holds for any pair of subsets as $A$ and $B$. Therefore $\text{MINP}(\mathfrak{P})$ is a matroid. Nevertheless, by Example 7.0.9, we have that $Drk_{\mathfrak{P}}(\{r, b\}) = \{r, m, c, b\} \notin \mathfrak{M}_{\mathfrak{P}}(\{r, b\}) = \{r, b\}$. 161
Nevertheless, the following partial reverse of Theorem 7.0.14 can be provided.

**Theorem 7.0.16.** If $\text{MINP}(\mathfrak{P}) \in \text{MATR}(\Omega)$, then $M_{\mathfrak{P}}(A) \subseteq \text{Drk}_{\mathfrak{P}}(A)$ for any $A \in \mathfrak{P}(\Omega)$.

**Proof.** Let $A \in \mathfrak{P}(\Omega)$ and $a \notin \text{Drk}_{\mathfrak{P}}(A)$. Then, $r_{\mathfrak{P}}(A \cup \{a\}) = r_{\mathfrak{P}}(A) + 1$. In particular, by Theorem 1.6.4, it results that $||\text{BAS}_{\mathfrak{P}}(A)|| = r_{\mathfrak{P}}(A)$ and $||\text{BAS}_{\mathfrak{P}}(A \cup \{a\})|| = r_{\mathfrak{P}}(A) + 1$. If $a$ were in $M_{\mathfrak{P}}(A)$, we would have that $\text{BAS}_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ has not uniform cardinality. But, $\text{BAS}_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \subseteq \max(\text{MINP}_{\mathfrak{P}}(M_{\mathfrak{P}}(A)))$ so, by Theorem 1.6.4, $\text{BAS}_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ must have uniform cardinality too. Therefore, $a \notin M_{\mathfrak{P}}(A)$.

In the last part of this chapter, we first recall the notion of maxp-symmetry base uniform pairing introduced in [20] and we show the link between such pairings and symmetry bases, minimal partitioners and the coincidence of the two set operators $\text{Drk}_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$.

**Definition 7.0.17.** We say that $\mathfrak{P} \in \text{PAIR}(\Omega)$ is maxp-symmetry base uniform if the following two conditions are satisfied:

(i) for any $A \in \mathfrak{P}(\Omega)$, $\text{BAS}_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ has uniform cardinality;

(ii) if $A, B \in \mathfrak{P}(\Omega)$, $A \not\subseteq B$ and $A \not\approx_{\mathfrak{P}} B$, then $||\text{BAS}_{\mathfrak{P}}(M_{\mathfrak{P}}(A))|| < ||\text{BAS}_{\mathfrak{P}}(M_{\mathfrak{P}}(B))||$.

We denote by $\text{MSBUP}(\Omega)$ the set of all maxp-symmetry base uniform pairings.

In the next fundamental result, we characterize the maxp-symmetry base uniform pairings in terms of equality of the two operators $\text{Drk}_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$ and of the two set systems $\text{BAS}_{\mathfrak{P}}(A)$ and $\max(\text{MINP}_{\mathfrak{P}}(A))$ for any $A \in \mathfrak{P}(\Omega)$.

**Theorem 7.0.18.** The following conditions are equivalent:

(i) $\mathfrak{P} \in \text{MSBUP}(\Omega)$;

(ii) $\max(\text{MINP}_{\mathfrak{P}}(A)) = \text{BAS}_{\mathfrak{P}}(A)$ for any $A \in \mathfrak{P}(\Omega)$;

(iii) $\text{Drk}_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A)$ for any $A \in \mathfrak{P}(\Omega)$.

**Proof.** (i) $\implies$ (ii): By (iii) of Theorem 3.2.2, we have that $\text{BAS}_{\mathfrak{P}}(A) \subseteq \max(\text{MINP}_{\mathfrak{P}}(A))$ for any $A \in \mathfrak{P}(\Omega)$. Let now $B \in \max(\text{MINP}_{\mathfrak{P}}(A)) \setminus \text{BAS}_{\mathfrak{P}}(A)$. Therefore $\pi_{\mathfrak{P}}(B) \neq \pi_{\mathfrak{P}}(A)$, so there exists $a \in A \setminus M_{\mathfrak{P}}(B)$ such that $B \cup \{a\} \not\approx_{\mathfrak{P}} B$. Since $B \in \max(\text{MINP}_{\mathfrak{P}}(A))$, it follows that $\text{BAS}_{\mathfrak{P}}(B) = \{B\}$. Let us prove that $a \in C$ for any $C \in \text{BAS}_{\mathfrak{P}}(B \cup \{a\})$. Suppose the claim false. Then, $C \subseteq B \subseteq B \cup \{a\}$, but this implies $\pi_{\mathfrak{P}}(B \cup \{a\}) \leq \pi_{\mathfrak{P}}(B) \leq \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(B \cup \{a\})$, i.e. $\pi_{\mathfrak{P}}(B \cup \{a\}) = \pi_{\mathfrak{P}}(B)$, contradiction. Thus, $a \in C$ for any $C \in \text{BAS}_{\mathfrak{P}}(B \cup \{a\})$. Since $\mathfrak{P}$ is a maxp-symmetry base uniform pairing, $||\text{BAS}_{\mathfrak{P}}(B \cup \{a\})|| > ||\text{BAS}_{\mathfrak{P}}(B)||$. Moreover, if $C \in \text{MINP}_{\mathfrak{P}}(B)$, then $C \in \text{MINP}_{\mathfrak{P}}(B \cup \{a\})$ hence, in particular, $B \in \text{MINP}_{\mathfrak{P}}(B \cup \{a\})$. Therefore, $\text{BAS}_{\mathfrak{P}}(B \cup \{a\}) = B \cup \{a\}$. In other terms, we have found a subset containing $B$ and belonging to $\max(\text{MINP}_{\mathfrak{P}}(A))$, i.e. $B \not\in \max(\text{MINP}_{\mathfrak{P}}(A))$, contradiction. This shows the thesis.

(ii) $\implies$ (iii): Let $a \notin M_{\mathfrak{P}}(A)$. Hence $A \cup \{a\} \not\approx_{\mathfrak{P}} A$. Suppose by contradiction that $a \in \text{Drk}_{\mathfrak{P}}(A)$, that is $r_{\mathfrak{P}}(A) = r_{\mathfrak{P}}(A \cup \{a\})$. Let $B \in \text{MINP}_{\mathfrak{P}}(A)$ such that $|B| = r_{\mathfrak{P}}(A)$. It must necessarily be $B \in \max(\text{MINP}_{\mathfrak{P}}(A \cup \{a\})) = \text{BAS}_{\mathfrak{P}}(A \cup \{a\})$, so $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A \cup \{a\})$, contradicting the fact that $a \notin M_{\mathfrak{P}}(A)$. Thus, $a \notin \text{Drk}_{\mathfrak{P}}(A)$.  

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By Theorem 7.0.18, it results that

\[ \text{Proof.} \]

fortiori, \( M \) by Theorem 7.0.14, it follows that \( \text{MINP} \) is a symmetry base of \( C \).

Let \( \text{MINP} \) be the \( \text{MINP} \) of Theorem 7.0.18 and 7.0.14.

Corollary 7.0.20.

Let \( \text{BAS} \) is a MLS closure operator, but \( \text{MINP} \) is a maxp-symmetry base uniform pairing.

Corollary 7.0.22. Let \( \mathcal{P} \in \text{MSBUP}(\Omega) \). Then, \( \text{MINP}(\mathcal{P}) \) is a matroid whose basis family is \( \text{BAS}(\mathcal{P}) \).

\[ \text{Proof.} \]

It follows immediately by Theorems 7.0.18 and 7.0.14.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{u}_1 & 1 & 0 & 0 & 1 & 0 \\
\text{u}_2 & 0 & 1 & 0 & 0 & 1 \\
\text{u}_3 & 0 & 0 & 0 & 0 & 1 \\
\text{u}_4 & 1 & 0 & 0 & 1 & 0 \\
\text{u}_5 & 1 & 0 & 0 & 0 & 1 \\
\hline
\end{tabular}
\caption{Functional Table \( \mathcal{P} \).}
\end{table}

Vice versa, let \( a \in M_{\mathcal{P}}(A) \), then \( A \cup \{a\} \approx_{\mathcal{P}} A \). Let us consider \( B \in BAS_{\mathcal{P}}(A \cup \{a\}) \). We have two possibilities: if also \( B \in BAS_{\mathcal{P}}(A) \), then \( rk_{\mathcal{P}}(A) = rk_{\mathcal{P}}(A \cup \{a\}) \) by the definition of \( rk_{\mathcal{P}} \); otherwise \( a \in B \). In the latter case, we have \( B \neq C \cup \{a\} \) for each symmetry base \( C \) of \( A \), otherwise \( \pi_{\mathcal{P}}(B \setminus \{a\}) = \pi_{\mathcal{P}}(C) = \pi_{\mathcal{P}}(A) \) but, by (ii) of Theorem 2.6.1, it must result \( \pi_{\mathcal{P}}(B \setminus \{a\}) \neq \pi_{\mathcal{P}}(B) = \pi_{\mathcal{P}}(A) \). It must necessarily result \( B = B' \cup \{a\} \) for some \( B' \subsetneq C \), where \( C \) is a symmetry base of \( A \). This implies that \( rk_{\mathcal{P}}(A \cup \{a\}) \leq rk_{\mathcal{P}}(A) \). Thus, by monotonicity of \( rk_{\mathcal{P}} \), we conclude that \( rk_{\mathcal{P}}(A) = rk_{\mathcal{P}}(A \cup \{a\}) \), i.e. \( a \in Drk_{\mathcal{P}}(A) \).

(iii) \( \implies \) (i): By Theorem 7.0.14, we deduce that \( \text{MINP}(\mathcal{P}) \) is a matroid. This implies that \( ||BAS_{\mathcal{P}}(A)|| \) has uniform cardinality for any \( A \in \mathcal{P}(\Omega) \), so, in particular, \( ||BAS_{\mathcal{P}}(M_{\mathcal{P}}(A))|| \) has uniform cardinality for any \( A \in \mathcal{P}(\Omega) \). Moreover, if \( A, B \in \mathcal{P}(\Omega) \), \( A \subsetneq B \) and \( A \neq \mathcal{P} \) \( B \), we conclude by (ii) of Proposition 7.0.10 that \( rk_{\mathcal{P}}(A) < rk_{\mathcal{P}}(B) \), so \( ||BAS_{\mathcal{P}}(M_{\mathcal{P}}(A))|| < ||BAS_{\mathcal{P}}(M_{\mathcal{P}}(B))|| \) and the thesis follows.

In the following example, we provide an example of maxp-symmetry base uniform pairing.

\textbf{Example 7.0.19.} Let us consider the pairing \( \mathcal{P} \) represented in Figure 7.1.

The indistinguishability lattice of \( \mathcal{P}(\mathcal{P}) \) has been drawn in Figure 7.2.

It is easy to verify that \( Drk_{\mathcal{P}}(A) = M_{\mathcal{P}}(A) \) for any \( A \in \mathcal{P}(\Omega) \). Thus, we conclude that \( \text{MINP}(\mathcal{P}) \) is a matroid. As it can be easily seen by Figure 7.2, \( \mathcal{P} \) is maxp-symmetry base uniform, whence it follows that \( BAS_{\mathcal{P}}(A) = \max(\text{MINP}(\mathcal{P})) \) for any \( A \in \mathcal{P}(\Omega) \).

As immediate consequences of the previous theorem, we obtain the following results.

\textbf{Corollary 7.0.20.} Let \( \mathcal{P} \in \text{MSBUP}(\Omega) \). Then, \( M_{\mathcal{P}} \) is a MLS closure operator.

\textbf{Proof.} By Theorem 7.0.18, it results that \( Drk_{\mathcal{P}}(A) = M_{\mathcal{P}}(A) \) for any \( A \in \mathcal{P}(\Omega) \). Furthermore, by Theorem 7.0.14, it follows that \( \text{MINP}(\mathcal{P}) \), therefore \( Drk_{\mathcal{P}} \) is a MLS closure operator. A fortiori, \( M_{\mathcal{P}} \) is a MLS closure operator.

Let us note that the converse of Corollary 7.0.20 does not hold as we see in next example.

\textbf{Example 7.0.21.} In reference to the pairing \( \mathcal{P} \) given in Example 3.4.5, it results that \( M_{\mathcal{P}} \) is a MLS closure operator, but \( \mathcal{P} \) is not a maxp-symmetry base uniform pairing.

\textbf{Corollary 7.0.22.} Let \( \mathcal{P} \in \text{MSBUP}(\Omega) \). Then, \( \text{MINP}(\mathcal{P}) \) is a matroid whose basis family is \( \text{BAS}(\mathcal{P}) \).

\textbf{Proof.} It follows immediately by Theorems 7.0.18 and 7.0.14.

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Figure 7.2: The indistinguishability lattice $I(\Psi)$.
Chapter 8

Symmetry Approximations and Indistinguishability Linear Systems

In this chapter, we introduce the notions of lower and upper symmetry approximations and determine in specific cases, namely graphs and digraphs, the corresponding sets. Furthermore, we link these approximations to a class of pairings that gives rise to an analogy with linear systems and that we call indistinguishability linear systems.

8.1 Symmetry Approximations

We provide now the basic definitions concerning symmetry approximation.

**Definition 8.1.1.** Let \( \mathfrak{P} = (U, \Lambda, F) \in \text{PAIR}(\Omega) \), \( A \subseteq \Omega \) and \( X \subseteq U \).

- The **A-lower symmetry approximation** of \( X \) is the following subset of \( U \):
  \[
  l_A(X) := \{x \in U : [x]_A \subseteq X\}.
  \]

- The **A-upper symmetry approximation** of \( X \) is the following subset of \( U \):
  \[
  u_A(X) := \{x \in U : [x]_A \cap X \neq \emptyset\}.
  \]

- The subset \( X \) is called **A-symmetry exact** if and only if \( l_A(X) = u_A(X) \) and **A-rough** otherwise.

- The subset
  \[
  BN_A(X) := u_A(X) \setminus l_A(X)
  \]
  is called the **A-boundary region** of \( X \).

- If \( X \neq \emptyset \), the **A-symmetry approximation accuracy** of \( X \) is the quantity
  \[
  \alpha_A(X) := \frac{|l_A(X)|}{|u_A(X)|}.
  \]

**Remark 8.1.2.** It is immediate to verify that both the following relations hold:

(i) \( l_A(X) = \bigcup\{C \in \pi_{\mathfrak{P}}(A) : C \subseteq X\} \);

(ii) \( u_A(X) = \bigcup\{C \in \pi_{\mathfrak{P}}(A) : C \cap X \neq \emptyset\} \).
8.2 Symmetry Approximation in Some Examples

In this section we compute the \( A \)-upper and the \( A \)-lower symmetry approximations for some specific cases of pairings.

8.2.1 The \( A \)-upper and the \( A \)-lower Approximations for \( C_n \)

In this subsection, we compute the \( A^{−}\)-upper and the \( A^{-}\)-lower approximation functions for the \( n \)-cycle \( C_n \). However, we interpret in geometric terms what the \( A^{−}\)-lower and \( A^{-}\)-upper approximation represent in graph context.

**Proposition 8.2.1.** Let \( G = (V(G), E(G)) \) be a simple undirected graph and let \( A \) and \( Y \) be two subsets of \( V(G) \). Then:

(i) \( u_A(Y) = \{v \in V(G) : \exists u \in Y : N_G(u) \cap A = N_G(v) \cap A\} \).

Therefore, \( v \in u_A(Y) \) if and only if \( v \) is an \( A \)-symmetric vertex of some \( u \in Y \).

(ii) \( l_A(Y) = \{v \in V(G) : (u \in V(G) \land N_G(u) \cap A = N_G(v) \cap A) \Rightarrow u \in Y\} \).

Therefore, \( v \in l_A(Y) \) if and only if all \( A \)-symmetric vertices of \( v \) are in \( Y \).

**Proof.** It is an immediate consequence of (2.2) and of the definitions of lower and upper symmetry approximations. 

Hence the lower symmetry approximation of a vertex set \( Y \) represents a subset of \( Y \) such that there are no elements outside \( Y \) with the same connections of any vertex in \( l_A(Y) \) (relatively to \( A \)). The upper symmetry approximation of \( Y \) is the set of vertices with the same connections (with respect to \( A \)) of at least one element in \( Y \). By the previous proposition it is natural to call \( l_A(Y) \) the \( A \)-symmetry kernel of \( Y \) and \( u_A(Y) \) the \( A \)-symmetry closure of \( Y \).

In order to determine the general form of the \( A \)-upper approximation function of \( C_n \), we must examine all possible relations between the vertex subset \( Y \) and the three subsets \( B_{C_n}(A) \), \( K_{C_n}(A) \) and \( S_{C_n}(A) \). Next, we also show that any possible choice of the vertex subsets \( A \) and \( Y \) is included in the cases we examined.

If \( A \) and \( Y \) are two vertex subsets of \( C_n \) we set

\[
Q_A(Y) := \bigcup \{N_{C_n}(v) : v \in C_{C_n}(A) \land N_{C_n}(v) \cap Y \neq \emptyset\}.
\]

**Theorem 8.2.2.** Let \( A \) and \( Y \) be two vertex subsets of \( C_n \). The map \( u : (A, Y) \in \mathcal{P}(V(C_n)) \times \mathcal{P}(V(C_n)) \mapsto u_A(Y) \in \mathcal{P}(V(C_n)) \) is completely described from the cases listed in the following table:

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Proof. 1): Let $Y = V(C_n)$ and $A$ any vertex subset of $V(C_n)$. Obviously $[v]_A \cap Y \neq \emptyset$ for every vertex $v$, so $u_A(V(C_n)) = V(C_n)$.

2): Let $Y = \emptyset$ and $A$ any vertex subset of $V(C_n)$. Obviously $[v]_A \cap \emptyset = \emptyset$ for every vertex $v$, so $u_A(\emptyset) = \emptyset$.

3): Let $A$ and $Y$ be two vertex subsets such that $Y \neq \emptyset \cap Y \subseteq B_{C_n}(A)$. This means that the indiscernibility block intersecting $Y$ is exactly $B_{C_n}(A)$, therefore $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$.

4): Let $A$ and $Y$ be two vertex subsets such that $Y \neq \emptyset \cap Y \subseteq K_{C_n}(A)$. Recalling that $K_{C_n}(A) = N_{C_n}(C_n(A))$, we deduce that $Y$ intersects only the neighbourhoods of some points $v_1, \ldots, v_m \in C_n(A)$, therefore $Q_A(Y) \neq \emptyset$ and $u_A(Y) = Q_A(Y)$.

5): Let $A$ and $Y$ be two vertex subsets such that $Y \neq \emptyset \cap Y \subseteq S_{C_n}(A)$. Since the elements of $S_{C_n}(A)$ form single blocks in the $A$–indiscernibility partition, we have that $Y \cap [v]_A \neq \emptyset$ if and only if $v \in S_{C_n}(A) \cap Y = Y$. Hence $u_A(Y) = Y$.

6): Let $A$ and $Y$ be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \cap Y \cap K_{C_n}(A) \neq \emptyset \cap Y \cap S_{C_n}(A) = \emptyset$. In other words, $Y$ is transversal only to $B_{C_n}(A)$ and $K_{C_n}(A)$; therefore we have that $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$ or $\exists w \in C_n(A) : v \in N_{C_n}(w) \cap N_{C_n}(w) \subseteq Q_A(Y)$. Thus $u_A(Y) = B_{C_n}(A) \cap Q_A(Y)$.

7): Let $A$ and $Y$ be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \cap Y \cap S_{C_n}(A) \neq \emptyset \cap Y \cap K_{C_n}(A) = \emptyset$. In this case, $Y$ is transversal only to $B_{C_n}(A)$ and $S_{C_n}(A)$, hence $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$ or $v \in Y \cap S_{C_n}(A)$. Thus $u_A(Y) = B_{C_n}(A) \cup (Y \cap S_{C_n}(A)) = Y \cup B_{C_n}(A)$.

8): Let $A$ and $Y$ be two vertex subsets such that $Y \cap K_{C_n}(A) \neq \emptyset \cap Y \cap S_{C_n}(A) \neq \emptyset \cap Y \cap B_{C_n}(A) = \emptyset$. Then, $[v]_A \cap Y \neq \emptyset$ if and only if $v \in Y \cap S_{C_n}(A)$ or $\exists w \in C_n(A) : v \in N_{C_n}(w) \cap N_{C_n}(w) \subseteq Q_A(Y)$, since $Y$ is transversal to both $K_{C_n}(A)$ and $S_{C_n}(A)$. So, we conclude that $u_A(Y) = Q_A(Y) \cup (Y \cap S_{C_n}(A)) = Q_A(Y) \cup Y$.

9): Let $A$ and $Y$ be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset \cap Y \cap K_{C_n}(A) \neq \emptyset \cap Y \cap S_{C_n}(A) \neq \emptyset$. It means that $Y$ is transversal to the three sets. Therefore we have $[v]_A \cap Y \neq \emptyset$ if and only if $v \in B_{C_n}(A)$ or $v \in Y \cap S_{C_n}(A)$ or $\exists w \in C_n(A) : v \in N_{C_n}(w) \cap N_{C_n}(w) \subseteq Q_A(Y)$. Therefore, $u_A(Y) = B_{C_n}(A) \cup Q_A(Y) \cup (Y \cap S_{C_n}(A)) = Y \cup B_{C_n}(A) \cup Q_A(Y)$ and we are done.

At this point, we prove that the previous cases are all disjoint each other and they are all possible cases that can occur. Let $Y$ be a proper vertex subset of $V = V(C_n)$, then since $V(C_n) = B_{C_n}(A) \cup K_{C_n}(A) \cup S_{C_n}(A)$, we deduce that $Y$ can be a subset of one of these three sets, as we have said writing down the conditions 3), 4) and 5), or it can be transversal to two of them, without containing none of them and without intersecting the third, as we have said.
writing down the conditions 6), 7) and 8). Finally, \( Y \) can be transversal to every set, without containing none of them, as written in the last condition. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. In this way we have shown the theorem.

Let us compute now the \( A \)-lower approximation function for the \( n \)-cycle \( C_n \). Also to compute the \( A \)-lower approximation function of \( C_n \) we will use the previous technique, namely to study all possible relations between the vertex subset \( Y \) and the three subsets \( B_{C_n}(A) \), \( K_{C_n}(A) \) and \( S_{C_n}(A) \) determining the \( A \)-symmetry partition.

If \( A \) and \( Y \) are two vertex subsets of \( C_n \) we set

\[
T_A(Y) := \bigcup_{v \in C_{C_n}(A)} \{ N_{C_n}(v) : N_{C_n}(v) \subseteq Y \}.
\]

**Theorem 8.2.3.** Let \( A \) and \( Y \) be two vertex subsets of \( C_n \). The map \( I : (A,Y) \in \mathcal{P}(V(C_n)) \times \mathcal{P}(V(C_n)) \mapsto I_A(Y) \in \mathcal{P}(V(C_n)) \) is completely described from the cases listed in the following table:

<table>
<thead>
<tr>
<th>CONDITIONS</th>
<th>( I_A(Y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = V(C_n) )</td>
<td>( V(C_n) )</td>
</tr>
<tr>
<td>( Y \not\subseteq B_{C_n}(A) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( Y \supseteq B_{C_n}(A) \land T_A(Y) = \emptyset )</td>
<td>( B_{C_n}(A) \cup (Y \cap S_C(A)) )</td>
</tr>
<tr>
<td>( Y \supseteq B_{C_n}(A) \land T_A(Y) \neq \emptyset )</td>
<td>( B_{C_n}(A) \cup (Y \cap S_C(A)) \cup T_A(Y) )</td>
</tr>
<tr>
<td>( Y \cap B_{C_n}(A) \neq \emptyset \land B_{C_n}(A) \not\subseteq Y \land T_A(Y) = \emptyset )</td>
<td>( Y \cap S_C(A) )</td>
</tr>
<tr>
<td>( Y \cap B_{C_n}(A) \neq \emptyset \land B_{C_n}(A) \not\subseteq Y \land T_A(Y) \neq \emptyset )</td>
<td>( (Y \cap S_C(A)) \cup T_A(Y) )</td>
</tr>
</tbody>
</table>

**Proof.**

1): Let \( Y = V(C_n) \) and \( A \) be any vertex subset of \( V(C_n) \). Obviously \([v]_A \subseteq V(C_n)\) for every vertex \( v \), so \( I_A(V(C_n)) = V(C_n) \).

2): Let \( Y = \emptyset \) and \( A \) be any vertex subset of \( V(C_n) \). Obviously no indiscernibility class is contained in the empty set, so \( I_A(\emptyset) = \emptyset \).

3): Let \( A \) and \( Y \) be two vertex subsets such that \( Y \not\subseteq B_{C_n}(A) \). Since \( B_{C_n}(A) \) forms a single block and since \( Y \) is disjoint from \( K_{C_n}(A) \) and \( S_{C_n}(A) \), we deduce that there is no vertex whose indiscernibility class is contained in \( Y \), thus \( I_A(Y) = \emptyset \).

4): Let \( A \) and \( Y \) be two vertex subsets such that \( Y \supseteq B_{C_n}(A) \) and \( T_A(Y) = \emptyset \). We observe that \( Y \) may or not intersect the vertex subset \( K_{C_n}(A) \) and, in the first case, in such a way that there not exists any vertex \( w \in C_{C_n}(A) \) whose neighbourhood is contained in \( Y \). Furthermore, \( Y \) may or not intersect \( S_{C_n}(A) \). Thus we conclude that \([v]_A \subseteq Y \) if and only if \( v \in B_{C_n}(A) \) or, possibly, \( v \in Y \cap S_C(A) \), i.e. \( I_A(Y) = B_{C_n}(A) \cup (Y \cap S_C(A)) \).

5): Let \( A \) and \( Y \) be two vertex subsets such that \( Y \supseteq B_{C_n}(A) \) and \( T_A(Y) \neq \emptyset \). We observe that \( Y \) may or not intersect the vertex subset \( S_{C_n}(A) \). Moreover, there exists at least one vertex \( w \in C_{C_n}(A) \) whose neighbourhood is contained in \( Y \). Hence we conclude that \([v]_A \subseteq Y \) if and only if \( v \in B_{C_n}(A) \) or \( v \in N_{C_n}(w) \subseteq T_A(Y) \) for some \( w \in C_{C_n}(A) \) or, possibly, \( v \in Y \cap S_C(A) \), i.e. \( I_A(Y) = B_{C_n}(A) \cup N_{C_n}(w) \cup (Y \cap S_C(A)) \).

6): Let \( A \) and \( Y \) be two vertex subsets such that \( Y \cap B_{C_n}(A) \neq \emptyset \land B_{C_n}(A) \not\subseteq Y \land T_A(Y) = \emptyset \). Since \( B_{C_n}(A) \) forms a single block, we conclude that \( B_{C_n}(A) \not\subseteq I_A(Y) \). We also observe that \( Y \)
may or not intersect the vertex subset $S_{C_n}(A)$. Moreover, there is no vertex $w \in C_{C_n}(A)$ such that $v \in N_{C_n}(w)$ and $N_{C_n}(w) \subseteq Y$. This means that $[v]_A \subseteq Y$ if and only if $v \in Y \cap S_{C_n}(A)$, i.e. $I_A(Y) = Y \cap S_{C_n}(A)$.

7): Let $A$ and $Y$ be two vertex subsets such that $Y \cap B_{C_n}(A) \neq \emptyset$ and $B_{C_n}(A) \nsubseteq Y \cap T_A(Y) \neq \emptyset$. Since $B_{C_n}(A)$ forms a single block, we conclude that $B_{C_n}(A) \nsubseteq I_A(Y)$. We also observe that $Y$ may or not intersect the vertex subset $S_{C_n}(A)$. Furthermore, there exists at least a vertex $w \in C_{C_n}(A)$ such that $v \in N_{C_n}(w)$ and $N_{C_n}(w) \subseteq Y$. In other words, we are saying that $[v]_A \subseteq Y$ if and only if $v \in Y \cap S_{C_n}(A)$ or $\exists w \in C_{C_n}(A) : v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq T_A(Y)$, i.e. $I_A(Y) = (Y \cap S_{C_n}(A)) \cup T_A(Y)$.

At this point, we show that we have studied all the occurring cases. Let $Y$ be a proper vertex subset of $V = V(C_n)$. In case 3) we have that $Y \nsubseteq B_{C_n}(A)$ while in cases 4) and 5) we are requiring that $Y$ contains $B_{C_n}(A)$ and it may (or not) be transversal to both $K_{C_n}(A)$ and $S_{C_n}(A)$. Finally, $Y$ may only transversal to $B_{C_n}(A)$, without containing it, as we have seen in the last two cases. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. Hence, the theorem is proved.

To conclude this section, we recall that $A$-symmetry exactness has been investigated in [52].

### 8.2.2 Symmetry Approximation in Some Basic Digraphs

Let $D = (V(D), Arc(D))$ be a digraph and let $A, Y$ be two vertex subsets of $D$. We denote by $I^*_A(Y) := \{v \in V(D) : [v]_A \subseteq Y\}$ the $A$-lower approximation of $Y$ and by $u^*_A(Y) := \{v \in V(D) : [v]_A \cap Y \neq \emptyset\}$ the $A$-upper approximation of $Y$.

In the next results we completely determine $A$-lower approximation, $A$-upper approximation and $A$-symmetry exactness for the four basic digraph families $K_{p,q}$, $F$, $C_n$, and $T_n$.

We begin with the case of the complete bipartite digraph $K_{p,q}$.

**Proposition 8.2.4.** Let $A$ and $Y$ be two subsets of $V(K_{p,q}) = \{x_1, \ldots, x_p, y_1, \ldots, y_q\}$, where $Y \neq \emptyset$ and $Y \neq V$. Then:

(i) the $A$-lower approximation of $Y$ is

$$ I^*_A(Y) = \begin{cases} Y & \text{if } A = V \\ \emptyset & \text{if } A = \emptyset \\ A \subseteq B_1 \lor (Y \cap B_1 \neq \emptyset \land Y \cap B_2 \neq \emptyset \land B_1 \nsubseteq Y \land B_2 \nsubseteq Y) & \text{if } A \subseteq B_1 \lor (Y \cap B_1 \neq \emptyset \land Y \cap B_2 \neq \emptyset \land B_1 \nsubseteq Y \land B_2 \nsubseteq Y) \\ B_1 & \text{if } B_1 \nsubseteq Y \\ B_2 & \text{if } B_2 \nsubseteq Y \end{cases} $$

(ii) The $A$-upper approximation of $Y$ is

$$ u^*_A(Y) = \begin{cases} B_1 & \text{if } Y \subseteq B_1 \land A \nsubseteq B_1 \\ B_2 & \text{if } Y \subseteq B_2 \land A \nsubseteq B_1 \\ V & \text{otherwise} \end{cases} $$

(iii) If $A = V$, then $Y$ any vertex subset is $A$-symmetry exact. In the other cases, $Y$ is $A$-symmetry exact if and only if $Y = B_1$ or $Y = B_2$.  

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Proof. (i): The statement in the case \( A = V \) is obvious. Moreover, let \( A = \emptyset \lor A \subseteq B_1 \). Then \([v]_A = V\), whence \( U_A^+(Y) = \emptyset \). Let \( A \) be a non-empty vertex subset not contained in \( B_1 \). By Proposition 2.3.9, \( \pi_{\mathcal{K}_{p,q}}(A) = B_1|B_2 \), therefore if \( Y \) is any vertex subset containing \( B_1 \) or \( B_2 \), it is clear that, respectively, \( U_A^+(Y) = B_1 \) or \( U_A^+(Y) = B_2 \); otherwise, if \( Y \) is transversal to both \( B_1 \) and \( B_2 \) without containing none of them, then \([v]_A \not\subseteq Y\), hence \( U_A^+(Y) = \emptyset \).

(ii): By Proposition 2.3.9, if we suppose that \( A = \emptyset \) or \( A \subseteq B_1 \), we conclude easily that \( U_A^+(Y) = V \). Let \( A \) be a non-empty vertex subset such that \( A \not\subseteq B_1 \) and let \( Y \) be another vertex subset. We have three cases: if \( Y \subseteq B_1 \), clearly it results \( U_A^+(Y) = B_1 \), if \( Y \subseteq B_2 \), clearly it results \( U_A^+(Y) = B_2 \), otherwise \( Y \) is transversal to both \( B_1 \) and \( B_2 \), so it intersects both the symmetry blocks of \( \pi_{\mathcal{K}_{p,q}}(A) \). This means that, in this case, \( U_A^+(Y) = V \).

(iii): It follows immediately by the previous parts. \( \square \)

We can now examine the digraph \( \vec{P}_n \).

**Proposition 8.2.5.** Let \( A \) and \( Y \) be two subsets of \( V(\vec{P}_n) = \{v_1, \ldots, v_n\} \), where \( Y \neq \emptyset \) and \( Y \neq V \). Then:

(i) the \( A \)-lower approximation of \( Y \) is

\[
U_A^-(Y) = \begin{cases} 
Y & \text{if } A = V \lor A = \{v_2, \ldots, v_n\} \lor (|A| = k \land Y \supseteq A_{c_1}) \\
\emptyset & \text{if } A = \emptyset \lor A = \{v_1\} \\
Y \cap A_{-1} & \text{otherwise.}
\end{cases}
\]

(ii) The \( A \)-upper approximation of \( Y \) is

\[
U_A^+(Y) = \begin{cases} 
V & \text{if } A = \emptyset \lor A = \{v_1\} \\
Y \cup A_{c_1} & \text{if } |A| = k \land Y \cap A_{c_1} \neq \emptyset \\
Y & \text{otherwise.}
\end{cases}
\]

(iii) If \( A = V \) or \( A = \{v_2, \ldots, v_n\} \) then \( Y \) is always \( A \)-symmetry exact. In the other cases, \( Y \) is \( A \)-symmetry exact if and only if \( A \neq \emptyset \land A \neq \{v_1\} \land (|A| = k \land (Y \supseteq A_{c_1} \lor Y \subseteq A_{-1})) \).

**Proof.** (i) By Proposition 2.3.9, if \( A = V \) or \( A = \{v_2, \ldots, v_n\} \), then \( \pi_{\mathcal{K}_{p,q}}(A) = v_1|v_n \) and therefore \( U_A^+(Y) = Y \) for any choice of the vertex subset \( Y \). Again, by Proposition 2.3.9, it is obvious that if \( A = \emptyset \) or \( A = \{v_1\} \), then \( U_A^+(Y) = \emptyset \), since the only subset \( Y \) containing \( V \) is \( V \) itself, but by hypothesis \( Y \neq V \). Let \( |A| = k \); then, by Proposition 2.3.9, \( \pi_{\mathcal{K}_{p,q}}(A) = v_{i_1-1}|v_{i_k-1}A_{c_1} \lor \pi_{\mathcal{K}_{p,q}}(A) = v_{i_2-1}|v_{i_k-1}A_{c_1} \) depending on whether \( v_1 \notin A \) or \( v_1 \in A \). In both cases, if \( Y \supseteq A_{c_1} \), then \([v]_A \subseteq Y \) if and only if \( v \in A_{c_1} \) or \( v \in Y \cap A_{-1} \) and \( U_A^+(Y) = A_{c_1} \lor \cap Y \cap A_{-1} \) = \( Y \). If \(|A| = k \) and \( Y \supseteq A_{-1} \), then \([v]_A \subseteq Y \) if and only if \( v \in Y \cap A_{-1} \), so \( U_A^+(Y) = Y \cap A_{-1} \).

(ii) By Proposition 2.3.9, if \( A = \emptyset \) or \( A = \{v_1\} \), then \( \pi_{\mathcal{K}_{p,q}}(A) = V \) and therefore all the vertices of the digraph are contained in a single block that obviously intersects \( Y \). Let \( A = V \), then by Proposition 2.3.9, \( \pi_{\mathcal{K}_{p,q}}(A) = v_1|v_n \). If \(|A| = k \) and \( Y \cap A_{c_1} \neq \emptyset \), then \([v]_A \cap Y \neq \emptyset \) if and only if \( v \in A_{c_1} \) or \( v \in Y \cap A_{c_1} \) and thus in this case \( U_A^+(Y) = Y \lor A_{c_1} \). If \(|A| = k \) and \( Y \subseteq A_{-1} \), then \([v]_A \cap Y \neq \emptyset \) if and only if \( v \in Y \cap A_{-1} \), so \( U_A^+(Y) = Y \). If \( A = V \) or \( A = \{v_2, \ldots, v_n\} \), then by using Proposition 2.3.9, it is easy to see that \([v]_A \cap Y \neq \emptyset \) if and
only if \( v \in Y \). This completes the proof of (ii).

(iii) By the previous parts, it is clear that if \( A = V \) or if \( A = \{v_2, \ldots, v_n\} \) then \( Y \) is always \( A \)-symmetry exact. Moreover if \( A \neq \emptyset \) and \( A \neq \{v_1\} \) and \( (Y \supseteq A_1 \cup Y \subseteq A_{-1}) \) then \( Y \) is \( A \)-symmetry exact. On the other hand, it is obvious that if \( A = \emptyset \) or \( A = \{v_1\} \) then no proper vertex subset \( Y \) is \( A \)-symmetry exact. Now, let \( |A| = k \), where \( A \neq \{v_2, \ldots, v_n\} \). A vertex subset \( Y \) is \( A \)-symmetry exact if and only if \( \prod_A(Y) = Y = u_A^+(Y) \). By part (i), \( \prod_A(Y) = Y \) if and only if \( Y \supseteq A_1 \cup Y \subseteq A_{-1} \). Finally, by part (ii), \( Y = u_A^+(Y) \) if and only if \( Y \cap A_1 \neq \emptyset \) and \( Y \cup A_{-1} = Y \) (equivalent to say that \( Y \supseteq A_1 \) or \( Y \cap A_{-1} = \emptyset \) (equivalent to say that \( Y \subseteq A_{-1} \)). The proposition is thus proved. \( \square \)

We determine now the rough approximation functions and the corresponding exactness for the \( n \)-directed cycle.

**Proposition 8.2.6.** Let \( A \) and \( Y \) be two subsets of \( V(C_n^7) = \{v_1, \ldots, v_n\} \), where \( Y \neq \emptyset \) and \( Y \neq V \). Then:

(i) the \( A \)-lower approximation of \( Y \) is

\[
U_A^+(Y) = \begin{cases} 
Y & \text{if } (A = V) \lor (1 \leq |A| \leq n - 2 \land Y \supseteq A_{-1}) \lor |A| = n - 1 \\
\emptyset & \text{if } A = \emptyset \\
A_{-1} \cap Y & \text{otherwise}
\end{cases}
\]

(ii) The \( A \)-upper approximation of \( Y \) is

\[
u_A^+(Y) = \begin{cases} 
V & \text{if } A = \emptyset \\
Y \cup A_{-1} & \text{if } 1 \leq |A| \leq n - 2 \land Y \cap A_{-1} \neq \emptyset \\
Y & \text{otherwise}
\end{cases}
\]

(iii) If \( A = V \) or \( |A| = n - 1 \), then \( Y \) is always \( A \)-symmetry exact. In the other cases, \( Y \) is \( A \)-symmetry exact if and only if \( 1 \leq |A| \leq n - 2 \land (Y \supseteq A_1 \lor Y \subseteq A_{-1}) \).

**Proof.** (i) : By Proposition 2.3.9, if \( A = \emptyset \) then \( \prod_A(Y) = \emptyset \), since the only subset \( Y \) containing \( V \) is \( V \) itself, but by hypothesis \( Y \neq V \). Suppose that \( |A| \geq n - 1 \), then by Proposition 2.3.9, it results that \( \prod_{C_n}(A) = \{v_1, \ldots, v_n\} \), therefore it is clear that \( \prod_A(Y) = Y \) for any choice of the vertex subset \( Y \). Let \( 1 \leq |A| = k \leq n - 2 \); by Proposition 2.3.9 we deduce that if \( Y \supseteq A_{-1} \), then \([v]_A \subseteq Y\) if and only if \( v \in A_{-1} \) or \( v \in Y \cap A_{-1} \) and \( \prod_A(Y) = A_{-1} \cup (Y \cap A_{-1}) \). If \( 1 \leq |A| \leq n - 2 \land Y \supseteq A_{-1} \), then \([v]_A \subseteq Y\) if and only if \( v \in A_{-1} \) or \( v \in Y \cap A_{-1} \) and \( \prod_A(Y) = A_{-1} \cup (Y \cap A_{-1}) \). If \( 1 \leq |A| \leq n - 2 \land Y \supseteq A_{-1} \), then \([v]_A \subseteq Y\) if and only if \( v \in A_{-1} \) or \( v \in Y \cap A_{-1} \) and \( \prod_A(Y) = A_{-1} \cup (Y \cap A_{-1}) \). Finally, if \( 1 \leq |A| \leq n - 2 \land Y \subseteq A_{-1} \), then \([v]_A \subseteq Y\) if and only if \( v \in Y \cap A_{-1} = Y \), hence we conclude that \( \prod_A(Y) = Y \). We have thus shown (ii).

(iii) : By the previous parts, it is clear that if \( |A| \geq n - 1 \) then \( Y \) is always \( A \)-symmetry exact. it is also clear that if \( 1 \leq |A| \leq n - 2 \land (Y \supseteq A_1 \lor Y \subseteq A_{-1}) \) then \( Y \) is \( A \)-symmetry exact. Moreover, it is obvious that if \( A = \emptyset \) then no proper vertex subset \( Y \) is \( A \)-symmetry exact. We have thus shown (ii).

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exact. Now, let $1 \leq |A| \leq n - 2$. A vertex subset $Y$ is $A$-symmetry exact if and only if $V(Y) = Y = u_A(Y)$. By part (i), $V(Y) = Y$ if and only if $Y \supseteq A_c$ or $Y \cap A_c = \emptyset$, i.e. if and only if $Y \subseteq A_c$. Furthermore, by part (ii), $Y = u_A(Y)$ if and only if $Y \cap A_c \neq \emptyset$ and $Y \cup A_c = Y$ or $Y \cap A_c = \emptyset$, i.e. if and only if $Y \supseteq A_c$ or $Y \subseteq A_c$. This completes the proof.

Finally, we focus our attention on the $n$-transitive tournament $\vec{T}_n$. Let $A = \{v_1, \ldots, v_k\} \subseteq V(\vec{T}_n)$. We set then, for each choice of $j \in \{0, 1, \ldots, k\}$:

$$C_A^j := \begin{cases} \{v_1, \ldots, v_{i_1-1}\} & \text{if } i_1 \neq 1 \land j = 0, \\ \{v_{i_j}, \ldots, v_{j+1-1}\} & \text{if } 1 \leq j < k, \\ \{v_{i_k}, \ldots, v_n\} & \text{if } j = k. \end{cases}$$

These sets are the blocks of the partition $\pi_{\vec{T}_n}(A)$ (see Proposition 2.3.9). Thus clearly it holds:

$$V(Y) = \begin{cases} \emptyset & \text{if } A = V \lor A = \{v_2, \ldots, v_n\} \\ \bigcup_{v_j \in A} \{C_A^j : C_A^j \subseteq Y\} & \text{otherwise.} \end{cases}$$

and

$$V(Y) = \begin{cases} \{v_1\} & \text{if } A = \emptyset \lor A = \{v_1\} \\ \bigcup_{v_j \in A} \{C_A^j : C_A^j \cap Y \neq \emptyset\} & \text{otherwise.} \end{cases}$$

### 8.3 Exactness as a Type of Dependency

Exactness with respect to a fixed subset can be characterized through symmetry approximation accuracy, in fact, an element subset $X$ is $A$-symmetry exact if and only if $\alpha_A(X) = 1$. In the sequel, we introduce the notion of indistinguishability linear system and then we see in which way this notion is linked to symmetry approximation accuracy.

**Definition 8.3.1.** We call indistinguishability linear system a structure $\mathcal{S} = (U_S, C_S, D_S, F_S, \Lambda_S)$, where $U_S, C_S, \Lambda_S$ are non-empty sets, $D_S$ is empty or consists of a single element $d_S$ such that $d_S \notin C_S$. $F_S$ is a map having domain $U_S \times (C_S \cup D_S)$ and codomain $\Lambda_S$. We also call:

- the elements of $U_S$ the equations of $\mathcal{S}$;
- the elements of $C_S$ the variables of $\mathcal{S}$;
- the element $d_S$ the constant of $\mathcal{S}$;
- the set $D_S$ the constant set of $\mathcal{S}$;
- the elements of $\Lambda_S$ the coefficients of $\mathcal{S}$;
- the map $F_S$ the coefficient map of $\mathcal{S}$.
We say that $S$ is a homogeneous indistinguishability linear system when $D_S = \emptyset$. If the sets $U_S$ and $\Omega_S$ are both finite we say that $S$ is a finite indistinguishability linear system. In what follows we will set $\Omega_S := C_S \cup D_S$.

In what follows we denote by $S = \langle U_S, \Omega_S, C_S, D_S, F_S, \Lambda_S \rangle$ a given indistinguishability linear system.

If $W \in \mathcal{P}(U_S)$ and $A \in \mathcal{P}(C_S)$ we denote by $S_{W,A}$ the indistinguishability linear system having equation set $W$, variable set $A$, constant set $D_S$, coefficient map the restriction of $F_S$ to the subset $W \times (A \cup D_S) \subseteq U_S \times \Omega_S$, coefficient set $\Lambda_S$. In particular, we write $S_A$ instead of $S_{U_S,A}$ and $S_W$ instead of $S_{W,C_S}$.

**Remark 8.3.2.** Let us observe that indistinguishability linear systems are a particular case of pairings. Hence, the notion of $A$-symmetry partition can be given for them, recalling that $\Omega = C_S \cup D_S$. At an interpretative level, given a linear system, we fix a subset of variables (and possibly also the constant term) and we consider all the equations restricted to the previous set of variables. At this point, there may be some identical equations that we identify. By means of this equivalence relation we reinterpret many concepts of local and global compatibility relative to the indistinguishability linear system $S$. In what follows, we denote by $\pi_S(A)$ the set partition on $U_S$ induced by $\equiv_A$.

On the other hand, if we restrict our attention only to variable subsets, we can introduce the indistinguishability relation $\approx_S$ and the notion of maximum partitioner (as in the pairing context) in the case of indistinguishability linear systems and, hence, the other structures related to $\approx_S$. Therefore, we can extend to indistinguishability linear systems all the results we proved in the previous chapters.

Let $S$ be a finite indistinguishability linear system having Boolean values 0 and 1 in the column corresponding to the constant $d_j$, for some $1 \leq j \leq p$. Then, there is only a subset $X \subseteq U_S$ such that $F_S(u, d_j) = 1$ if $u \in X$ and $F_S(u, d_j) = 0$ if $u \in U_S \setminus X$.

**Definition 8.3.3.** Let $S$ and $X$ as above. We say that:

- $S$ is a decision table with $j$-th decision $X$, and we set $d_j := d^X$;
- $X$ is a decision of $S$ if $S$ is a decision table with $j$-th decision $X$, for some $1 \leq j \leq p$.

If $S$ is a simple decision table that such $d_S = d^X$, for some $X \subseteq U_S$, we say that $S$ is a simple $X$-decision table.

We have then the following result.

**Proposition 8.3.4.** Let $X \subseteq U_S$ and $A \subseteq C_S$. Then,

$$\gamma_S(A, \{d^X\}) = 1 - \frac{|BN_A(X)|}{|U_S|} \quad (8.1)$$

Hence, in particular, $X$ is $A$-symmetry exact if and only if $\gamma(A, \{d^X\}) = 1$.  

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Proof. We have
\[ \Gamma_S(A, \{d^X\}) = I_A(X) \cup I_A(X^c) = I_A(X) \cup (U_S \setminus \mathbf{u}_A(X)), \]
therefore
\[ \gamma_S(A, \{d^X\}) = \left| \Gamma_S(A, \{d^X\}) \right| = \frac{|I_A(X)| + |U_S| - |\mathbf{u}_A(X)|}{|U_S|} = \frac{|U_S| - |\mathbf{u}_A(X) \setminus I_A(X)|}{|U_S|}, \]
and the thesis follows. \(\square\)

Remark 8.3.5. The result established in Proposition 8.3.4 is simple but important, since it connects the A-symmetry exactness of the subset \(X\) to the value of the symmetry transmission measure \(\gamma(A, \{d^X\})\). In other terms, we can see that \(X\) is \(A\)-symmetry exact if and only if the single element \(d^X\) totally depends by the subset \(A\) in the indistinguishability linear system \(S\).

Definition 8.3.6. If \(X \subseteq U_S\) and \(A \subseteq C_S\), we set
\[ \beta_S^A(X) := \gamma_S(A, \{d^X\}) \]
We call \(\beta_A(X)\) the \((A,d^X)\)-approximation symmetry measure of \(X\).

The identity established in (8.1) can also be reformulated in the following two equivalent ways:
\[ \beta_A(X) = 1 - (1 - \alpha_A(X)) \frac{|\mathbf{u}_A(X)|}{|U_S|} \] (8.2)
and
\[ \alpha_A(X) = 1 - (1 - \beta_A(X)) \frac{|U_S|}{|\mathbf{u}_A(X)|} \] (8.3)

By (8.2) we immediately deduce the following result

Proposition 8.3.7. If \(\alpha_A(X) = \beta_A(X)\) then \(\alpha_A(X) = \beta_A(X) = 1\) or \(\mathbf{u}_A(X) = U_S\).

Here we simply recall the following differences between \(\alpha_A\) and \(\beta_A\):

\((\alpha_1)\) \(\alpha_A(X) = 0 \iff I_A(X) = \emptyset\),

\((\alpha_2)\) for a fixed \(I_A(X) \neq \emptyset\), \(\alpha_A(X)\) strictly monotonically decreases with \(|\mathbf{u}_A(X)|\),

and

\((\beta_1)\) \(\beta_A(X) = 0 \iff (I_A(X) = \emptyset \land \mathbf{u}_A(X) = U_S)\),

\((\beta_2)\) for a fixed \(I_A(X)\), \(\beta_A(X)\) strictly monotonically decrease with \(|\mathbf{u}_A(X)|\).
8.4 Compatibility for Indistinguishability Linear Systems

Let $X \in \mathcal{P}(U_S)$ and $Y \in \mathcal{P}(C_S)$. We set

$$\Theta_S(X,Y) := \{v \in X : [v]_Y \cap X \subseteq [v]_{D_S} \cap X\} \quad (8.4)$$

and

$$\theta_S(X,Y) := \frac{|\Theta_S(X,Y)|}{|X|} \quad (8.5)$$

if $S$ is finite.

Moreover, if $Y, Z \in \mathcal{P}(\Omega_S)$ we also set

$$\Gamma_S(Y,Z) := \{v \in U_S : [v]_Y \subseteq [v]_Z\}$$

and

$$\gamma_S(Y,Z) := \frac{|\Gamma_S(Y,Z)|}{|U_S|}$$

if $S$ is finite.

For $A \in \mathcal{P}(C_S)$ and $u \in U_S$ we denote by $F^u_{S,A}$ the map from $A$ to $\Lambda_S$ defined by $F^u_{S,A}(a) := F_S(u,a)$ for any $a \in A$. Moreover, we denote by $\eta^w_{S,A}$ the map from $U_S$ to $\Lambda^A_S$ defined by $\eta^w_{S,A}(u) := F^w_{S,A}$ for any $u \in U_S$. We will denote by $\eta^w_{S,A}$ the restriction of $\eta^w_{S,A}$ to $W \in \mathcal{P}(U_S)$. By means of these notations we establish (also in the non finite case) an equivalence between the notion of compatibility ($ILS1$), discussed in the introductory section, and one of its formulations expressed in terms of symmetry classes.

**Proposition 8.4.1.** Let $W \in \mathcal{P}(U_S)$ and $A \in \mathcal{P}(C_S)$. Then, the following conditions are equivalent.

(1) There exists a map $f : \Lambda^A_S \to \Lambda_S$ such that $F_S(u,d_S) = f(\eta^w_{S,A}(u))$ for any $u \in W$.

(2) $\Theta_S(W,A) = W$.

**Proof.** Suppose that (1) holds and let $u \in W$ and $u' \in W$ such that $u' \equiv_A u$. Thus $F_S(u,a) = F_S(u',a)$ for any $a \in A$. In other terms it results that $F^w_{S,A} = F^w_{S,A}$, so $\eta^w_{S,A}(u) = \eta^w_{S,A}(u')$. By hypothesis, we conclude that $F_S(u,d_S) = F_S(u',d_S)$, therefore $u' \in [u]_{D_S} \cap W$. This means that $\Theta_S(W,A) = W$.

On the other hand, suppose that (2) holds. Let us consider an application $f : \Lambda^A_S \to \Lambda_S$ such that $F_S(u,d_S) := (f \circ \eta^w_{S,A})(u)$ for any $u \in W$. Let $u,u' \in W$ such that $u \equiv_A u'$. Hence, by our assumption, $F_S(u,d_S) = F_S(u',d_S)$ and, in particular, $f(\eta^w_{S,A}(u)) = f(\eta^w_{S,A}(u'))$, so $f$ is well defined and the claim has been shown. □

**Remark 8.4.2.** (i) When $A = \{a_1,\ldots,a_p\}$ is a finite set of $C_S$ the condition (ILS1) is equivalent to the following:

(1') There exists a map $f : \Lambda^A_S \to \Lambda_S$ such that $F_S(u,d_S) = f(F_S(u,a_1),\ldots,F_S(u,a_p))$ for any $u \in W$.

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(ii) When $S$ is a finite indistinguishability linear system the condition \((ILS2)\) is equivalent to the condition $\theta_S(W, A) = 1$.

**Definition 8.4.3.** If the condition \((ILS1)\), or equivalently \((ILS2)\), is satisfied we say that $S$ is \((W, A)\)-**compatible**, otherwise we say that $S$ is \((W, A)\)-**incompatible**. In particular, we say that $S$ is \(W\)-**equation compatible** if $S$ is \((W, C_S)\)-compatible and \(A\)-**variable compatible** if it is \((U_S, A)\)-compatible. Finally, we say that $S$ is **compatible** if it is \((U_S, C_S)\)-compatible.

**Remark 8.4.4.** (i) Let us note that when $S$ is a homogeneous indistinguishability linear system the condition \((ILS2)\) is trivially satisfied for any $W \in \mathcal{P}(U_S)$ and $A \in \mathcal{P}(C_S)$. On the other hand, as we will see in next sections, several constructions depend exclusively by the variable set $C_S$, therefore, maintaining the analogy with classical linear systems, we can assert that these constructions depend by the homogeneous indistinguishability linear system associated with $S$. Therefore, the investigation of homogeneous indistinguishability linear systems is also important in the analysis of compatibility; in fact we restrict our attention to the variable set $C_S$ and then "trasle" to the non-homogeneous case our results.

(ii) When we have a classical compatible linear system, if we keep all variables fixed and take a subset of equations, the compatibility of the system is preserved. This is true also in our context, in fact it is immediate to verify that $S$ is compatible if and only if $S$ if $W$-equation compatible for any $W \in \mathcal{P}(U_S)$.

Based on the terminology introduced in Definition 8.4.3, we note that $\Theta_S(X, Y)$ is the set of all equations of $X$ that form a compatible system relatively to the variables of $Y$. Therefore we call $\Theta_S(X, Y)$ the \((X, Y)\)-**compatibility region** and $\theta_S(X, Y)$ the \((X, Y)\)-**compatibility measure** (when $S$ is finite).

Analogously, $\Gamma_S(Y, Z)$ is the set of all equations whose symmetry with respect to the variables in $Y$ implies also their symmetry with respect to the variables in $Z$.

### 8.5 Compatibility Operators

In this section we investigate the notion of local compatibility of $S$ in terms of compatibility operators. In fact, we can introduce two operators $\phi_A : \mathcal{P}(U_S) \to \mathcal{P}(U_S)$ and $\varphi_W : \mathcal{P}(C_S) \to \mathcal{P}(U_S)$ by setting respectively

$$\phi_A(X) = \Theta_S(X, A)$$

and

$$\varphi_W(Y) := \Theta_S(W, Y).$$

We call $\phi_A$ the $A$-**equation compatibility operator** of $S$ and $\varphi_W$ the $W$-**variable compatibility operator** of $S$. Moreover, let us set $\hat{\phi}_A(X) := \theta_S(X, A)$ and $\hat{\varphi}_W(Y) := \theta_S(W, Y)$.

The compatibility operators are directly related to the local compatibility of $S$ by means of the next result.

**Proposition 8.5.1.** (i) $S_{W, A}$ is compatible if and only if $\phi_A(W) = W$.

(ii) If $\varphi_W(A) = \emptyset$ then $S_{W, A}$ is incompatible.
Proof. (i): \( S_{W,A} \) is compatible if and only if for any pair of points \( u, u' \in W \) such that \( u \equiv_A u' \) we have that \( d_S(u) = d_S(u') \). But this is equivalent to require that \( [u]_A \cap W \subseteq [u]_{D_S} \cap W \) for any \( u \in W \), i.e. \( \Theta_S(W,A) = W \) or, equivalently, \( \phi_A(W) = W \).

(ii): By our assumptions, \( \varphi_W(A) = \emptyset \), i.e. \( \Theta_S(W,A) = \emptyset \). This means that

\[
[u]_A \cap W \not\subseteq [u]_{D_S} \cap W
\]

for any \( u \in W \). Thus, for any \( u \in W \) there exists \( u' \in W \) such that \( u \equiv_A u' \) and \( u \neq_{d_S} u' \). Hence \( d_S(u) \neq d_S(u') \), therefore \( S_{W,A} \) is incompatible.

We state now some basic properties of the \( A \)-equation compatibility operator.

**Proposition 8.5.2.** The \( A \)-equation compatibility operator \( \phi_A \) satisfies the following properties:

(i) \( \phi_A(W) \subseteq W \) for any \( W \in \mathcal{P}(U_S) \);

(ii) \( \phi_A^2(W) = \phi_A(W) \) for any \( W \in \mathcal{P}(U_S) \);

(iii) if \( W' \subseteq W \) then \( \phi_A(W) \cap W' \subseteq \phi_A(W') \);

(iv) \( \phi_A(W) = W \) if and only if \( \phi_A(W'') \subseteq \phi_A(W') \) for any \( W', W'' \in \mathcal{P}(W) \) such that \( W'' \subseteq W' \).

Proof. (i): It follows directly by the definition of \( \phi_A(W) \).

(ii): By (i) it’s clear that \( \phi_A^2(W) \subseteq \phi_A(W) \). On the other hand, let \( v \in \phi_A(W) = \Theta_S(W,A) \).

Hence \( [v]_A \cap W \subseteq [v]_{D_S} \cap W \). By (i), we have that \( \Theta_S(W,A) \subseteq W \); therefore, if \( v \in \phi_A(W) \), then \( [v]_A \cap W \subseteq [v]_{D_S} \cap W \), thus

\[
[v]_A \cap \phi_A(W) = [v]_A \cap W \cap \phi_A(W) \subseteq [v]_{D_S} \cap W \cap \phi_A(W) = [v]_{D_S} \cap \phi_A(W),
\]

so \( v \in \Theta_S(\phi_A(W), A) = \phi_A^2(W) \). Hence \( \phi_A(W) \subseteq \phi_A^2(W) \) and (ii) follows.

(iii): Let \( v \in \phi_A(W) \cap W' = \Theta_S(W,A) \cap W' \). Then \( [v]_A \cap W \subseteq [v]_{D_S} \cap W \), and this relation is preserved if we both intersect members with \( W' \), thus \( [v]_A \cap W' \subseteq [v]_{D_S} \cap W' \), i.e. \( v \in \Theta_S(W', A) = \phi_A(W') \).

(iv): Suppose that \( \Theta_S(W,A) = W \) and let \( W'' \subseteq W' \subseteq W \). Firstly we observe that \( [v]_A \cap W \subseteq [v]_{D_S} \cap W, \) for any \( v \in W, \) hence, obviously, \( [v]_A \cap W'' \subseteq [v]_{D_S} \cap W' \) for any \( W' \subseteq W \). A fortiori, if \( v \in \phi_A(W'') \), then it results that \( v \in \phi_A(W') \) and the claim follows.

Conversely, since \( \phi_A(\{v\}) = \{v\} \) for any \( v \in W \), we deduce that \( \phi_A(W) = \Theta_S(W,A) = W \).

\( \square \)

**Corollary 8.5.3.** \( \phi_A \) is a kernel operator if and only if \( \phi_A(U_S) = U_S \).

Proof. The thesis follows immediately by (i), (ii), (iv) of Proposition 8.5.2.

\( \square \)

In the next result we establish the first basic properties of the \( W \)-variable compatibility operator.

**Proposition 8.5.4.** The \( W \)-variable compatibility operator \( \varphi_W \) satisfies the following properties:
(i) If \( A' \in \mathcal{P}(C_{\mathcal{S}}) \) and \( A \subseteq A' \) then \( \varphi_W(A) \subseteq \varphi_W(A') \).

(ii) Let \( A, B \in \text{MAXP}(\mathcal{S}) \). Then,
\[
\varphi_W(A \lor B) \subseteq \varphi_W(A) \cap \varphi_W(B) \subseteq \varphi_W(A) \cup \varphi_W(B) \subseteq \varphi_W(A \land B). \tag{8.8}
\]
Moreover, if \( \varphi_W(A \cap B) = \varphi_W(A) \cap \varphi_W(B) \) it results that
\[
\hat{\varphi}_W(A \land B) + \hat{\varphi}_W(A \lor B) \geq \hat{\varphi}_W(A) + \hat{\varphi}_W(B). \tag{8.9}
\]

(iii) If \( A, B \in \mathcal{P}(C_{\mathcal{S}}) \) and \( A \approx \mathcal{S} B \) then \( \varphi_W(A) = \varphi_W(B) \).

(iv) \( \text{Im}_{\mathcal{S}}(\varphi_W) = \{ \varphi_W(A) : A \in \text{MAXP}(\mathcal{S}) \} \).

**Proof.** (i): Let \( u \in \varphi_W(A) = \Theta_{\mathcal{S}}(W, A) \). Since \( A \subseteq A' \), it results that \([u]_{A'} \subseteq [u]_A\), therefore
\[
[u]_{A'} \cap W \subseteq [u]_A \cap W, \tag{8.10}
\]
But the condition \( u \in \Theta_{\mathcal{S}}(W, A) \) is equivalent to require that \([u]_A \cap W \subseteq [u]_D \cap W \). By (8.10), we conclude that \([u]_{A'} \cap W \subseteq [u]_{D_s} \cap W \), i.e. \( u \in \varphi_W(A) = \Theta_{\mathcal{S}}(W, A') \).

(ii): We have that \( A \lor B = A \cap B \) and \( A \land B = M_{\mathcal{S}}(A \cup B) \). Therefore, by (i), we deduce that
\[
\varphi_W(A \lor B) = \hat{\varphi}_{M_{\mathcal{S}}(A \lor B)}(W) \subseteq \hat{\varphi}_{A}(W) \cap \hat{\varphi}_{B}(W) = \varphi_W(A) \cap \varphi_W(B)
\]
and
\[
\varphi_W(A \land B) = \hat{\varphi}_{M_{\mathcal{S}}(A \land B)}(W) \supseteq \hat{\varphi}_{A}(W) \cup \hat{\varphi}_{B}(W) = \varphi_W(A) \cup \varphi_W(B).
\]
This proves (8.8).

Let now \( A, B \in \text{MAXP}(\mathcal{S}) \) satisfying the condition \( \varphi_W(A \cap B) = \varphi_W(A) \cap \varphi_W(B) \). We prove the following equivalent version of (8.9):
\[
\hat{\theta}_{A \land B}(W) + \hat{\theta}_{A \lor B}(W) \geq \hat{\theta}_{A}(W) + \hat{\theta}_{B}(W). \tag{8.11}
\]
Since
\[
|\Theta_{\mathcal{S}}(W, A) \cup \Theta_{\mathcal{S}}(W, B)| = |\Theta_{\mathcal{S}}(W, A)| + |\Theta_{\mathcal{S}}(W, B)| - |\Theta_{\mathcal{S}}(W, A) \cap \Theta_{\mathcal{S}}(W, B)|
\]
and \( A \lor B = A \land B \), from our hypothesis on \( A \) and \( B \) it follows that
\[
|\Theta_{\mathcal{S}}(W, A) \cup \Theta_{\mathcal{S}}(W, B)| + |\Theta_{\mathcal{S}}(W, A \lor B)| = |\Theta_{\mathcal{S}}(W, A)| + |\Theta_{\mathcal{S}}(W, B)| \tag{8.12}
\]
On the other hand, we have that \( A \land B = M_{\mathcal{S}}(A \cup B) \), therefore, since \( A, B \subseteq M_{\mathcal{S}}(A \cup B) \) it follows that
\[
\Theta_{\mathcal{S}}(W, A) \cup \Theta_{\mathcal{S}}(W, B) \subseteq \Theta_{\mathcal{S}}(W, M_{\mathcal{S}}(A \cup B)) = \Theta_{\mathcal{S}}(W, A \land B). \tag{8.13}
\]
Then, by (8.12) and (8.13) we have that
\[
|\Theta_{\mathcal{S}}(W, A \land B)| + |\Theta_{\mathcal{S}}(W, A \lor B)| \geq |\Theta_{\mathcal{S}}(W, A)| + |\Theta_{\mathcal{S}}(W, B)|,
\]
that is equivalent to (8.11).

(iii): Clearly, \( A \approx \mathcal{S} \) if and only if \( \pi_{\mathcal{S}}(A) = \pi_{\mathcal{S}}(B) \). Thus it is clear that \( \Theta_{\mathcal{S}}(W, A) = \Theta_{\mathcal{S}}(W, B) \), i.e. \( \varphi_W(A) = \hat{\varphi}_{A}(W) = \hat{\varphi}_{B}(W) = \varphi_W(B) \).

(iv): It follows directly by (iii), since \( \varphi_W(A) = \varphi_W(M_{\mathcal{S}}(A)) \) for any \( A \in \mathcal{P}(\mathcal{S}) \). 
\[\square\]
Remark 8.5.5. By (iii) and (iv) of Proposition 8.5.4, we can identify the $W$-variable compatibility operator $\varphi_W$ with its restriction on $\text{MAXP}(S)$. Therefore in what follows we will assume that

$$\varphi_W : \text{MAXP}(S) \to \mathcal{P}(U_S).$$

We have so far undertaken a local study of compatibility. We now want to focus our attention to the global counterpart of the theory.

For any $W \in \mathcal{P}(U_S)$ we set

$$\Phi_S(W) := \{(A, \varphi_W(A)) : A \in \text{MAXP}(S)\}, \quad \mathcal{P}_S(W) := (\Phi_S(W), \sqsubseteq)$$

and

$$\hat{\Phi}_S(W) := \{(A, \theta_S(W, A)) : A \in \text{MAXP}(S)\}, \quad \hat{\mathcal{P}}_S(W) := (\hat{\Phi}_S(W), \sqsubseteq),$$

if $S$ is finite, where

$$(A, \varphi_W(A)) \sqsubseteq (A', \varphi_W(A')) : \iff (A, \theta_S(W, A)) \sqsubseteq (A', \theta_S(W, A')) : \iff A \subseteq^* A',$$

so that $\mathcal{P}_S(W), \hat{\mathcal{P}}_S(W)$ and $M_S(S)$ are three order isomorphic lattices.

We can interpret, respectively, the lattices $\mathcal{P}_S(W)$ and $\hat{\mathcal{P}}_S(W)$ as a type of $W$-projection and $W$-numerical projection of the maximum partitioner lattice $M_S(S)$ on the corresponding equation subset $W$.

Definition 8.5.6. We call $\mathcal{P}_S(W)$ the $W$-projection lattice and $\hat{\mathcal{P}}_S(W)$ the $W$-numerical projection lattice of $S$.

At this point we can provide an interpretation of $W$-equation compatibility and $A$-variable compatibility in terms of the previous set operator language.

Proposition 8.5.7. (i) $S$ is $W$-equation compatible $\iff (C_S, W) \in \Phi_S(W) \iff \varphi_W(C_S) = W$.

(ii) $S$ is $A$-variable compatible $\iff \phi_A$ is a kernel operator.

Proof. (i): By (iv) of Proposition 8.5.4, it results that $(A, W) \in \Phi_S(W)$ if and only if there exists $A \in \text{MAXP}(S)$ such that $\varphi_W(A) = \Theta_S(W, A) = W$. By (i) of the same Proposition, we deduce that $\varphi_W(C_S) = W$. By Proposition 8.5.1 it follows immediately that $S_W$ is compatible if and only if $\Theta_S(C_S) = W$.

(ii): Straightforward.

We can relate from a global perspective the local incompatibilities of an indistinguishability linear system $S$ and the $W$-variable compatibility operator. In order to analyze this interrelation, let $\pi_S(C_S) = \{B_i : i \in I\}$ and $\pi_S(d_S) = \{C_j : j \in J\}$. For any index $i \in I$ we set then

$$(SB)_i := \{B' : B_i \not\subseteq C_j \forall j \in J\}$$

and

$$N_S := \bigcup_{i \in I} (SB)_i.$$

We have then the following result.
Proposition 8.5.8. Let $\emptyset \neq W \subseteq \mathbb{U}_S$. Then, $\Phi_S(W) = \{(A, \emptyset), A \in \text{MAXP}(S)\}$ if and only if $W = \bigcup_{k \in K} B'_k$, for some sub-family $\{B'_k : k \in K\} \subseteq N_S$.

Proof. By (i) of Proposition 8.5.4 the condition $\{(A, \emptyset), A \in \text{MAXP}(S)\}$ is equivalent to $\varphi_W(C_S) = \emptyset$. Let us note that $\varphi_W(C_S) = \emptyset$ if and only if for any $u \in W$ it results that $[u]_{C_S} \cap W \not\subseteq [u]_{D_S} \cap W$. This is equivalent to require that, for any $u \in W$, there exists at least a equation $u' \in W$ such that $u \equiv_{U_S} u'$ and $u \not\equiv_{D_S} u'$. In other terms, we are saying that there exist some $C_S$-symmetry sub-blocks $B'_k \in N_S$, for some index set $k \in K$, such that $\bigcup_{k \in K} B'_k = W$. Hence the thesis follows.

We set now

$$DEP_S(A) := \{W \in \mathcal{P}(U_S) : \phi_A(W) = W\},$$  \hspace{1cm} (8.14)

that is equivalent to the following

$$DEP_S(A) = \{W \in \mathcal{P}(U_S) : \forall v \in W, [v]_A \cap W \subseteq [v]_{D_S} \cap W\}. \hspace{1cm} (8.15)$$

We observe that if we consider the operator $DEP_S : \mathcal{P}(C_S) \rightarrow SS(U_S)$, this is constant over each global symmetry class. Therefore it is sufficient to consider its restriction $DEP_S : \text{MAXP}(S) \rightarrow SS(U_S)$.

We now prove three basic properties holding for the above operator $DEP_S$.

Proposition 8.5.9. The following properties hold.

(i) $\emptyset \in DEP_S(A)$ and $\{v\} \in DEP_S(A)$ for any $v \in U_S$;
(ii) $\phi_A(W) = \varphi_W(A) \in DEP_S(A)$ for any $W \in \mathcal{P}(U_S)$;
(iii) If $W \in DEP_S(A)$ and $W' \subseteq W$ then $W' \in DEP_S(A)$.

Proof. (i): It is immediate to see that (8.15) holds for $W = \emptyset$. Moreover, let $W = \{v\}$. Then

$$[v]_A \cap \{v\} = \{v\} = [v]_{D_S} \cap \{v\}$$

and the claim is proved.

(ii): It is a direct consequence of (ii) of Proposition 8.5.2 and (8.14).

(iii): Let $W \in DEP_S(A)$ and $W' \subseteq W$. Then, by (8.15), we have

$$[v]_A \cap W \subseteq [v]_{D_S} \cap W, \forall v \in W.$$  \hspace{1cm} (8.16)

In particular, it results that

$$[v]_A \cap W \cap W' = [v]_A \cap W' \subseteq [v]_{D_S} \cap W \cap W' = [v]_{D_S} \cap W' \forall v \in W'. \hspace{1cm} (8.16)$$

So (iii) follows.
In next result we state some important properties of the operator $DEP_S$. We will use the notation

$$MDEP_S(A) := \max(DEP_S(A)).$$

**Theorem 8.5.10.** The following conditions hold.

(i) If $v \in W \setminus \varphi_W(A)$, then $[v]_A \cap W \subseteq W \setminus \varphi_W(A)$.

(ii) We have that

$$\varphi_U(A) = \bigcap_{Z \in MDEP_S(A)} Z.$$  \hspace{1cm} (8.17)

(iii) $\varphi_U(A) = U_S \iff DEP_S(A) = \mathcal{P}(U_S)$.

(iv) If $\varphi_U(A) \neq U_S$ and $W \in DEP_S(A)$, then $\emptyset \neq \varphi_U(A)^c \not\subseteq W$.

(v) If $W \in DEP_S(A)$ then $[v]_A \not\subseteq W$ for any $v \in \varphi_U(A)^c$.

(vi) Suppose that $[v]_A \cap W = \emptyset$ for any $v \in \varphi_U(A)^c$, whence $W \in DEP_S(A)$.

(vii) Let $A \subseteq B$. Then $DEP_S(A) \subseteq DEP_S(B)$.

*Proof.* (i): Let $v \in W \setminus \varphi_W(A)$ and let $u' \in [v]_A \cap W$. Suppose by contradiction that $u' \in \varphi_W(A)$, i.e. $[u']_A \cap W \subseteq [u']_D \cap W$. Then, we have $v \in [v]_A \cap W = [u']_A \cap W \subseteq [u']_D \cap W$, hence $[v]_D \cap W = [u']_D \cap W$, therefore $[v]_A \cap W \subseteq [v]_D \cap W$, i.e. $v \in \varphi_W(A)$, contradiction. Then, $u' \in W \setminus \varphi_W(A)$ and the claim follows.

(ii): Let $v \in \varphi_U(A) = \Gamma_S(A, \{d_S\})$. Then $[v]_A \subseteq [v]_D$. Let us prove now that, if $W \in DEP_S(A)$, then $W \cup \{v\} \in DEP_S(A)$. This implies that, for all $Z \in MDEP_S(A)$, $v \in Z$. If $u \in W$, then, by definition of $DEP_S$, $[u]_A \cap W \subseteq [u]_D \cap W$. If $w \not\equiv A u$, then $[w]_A \cap (W \cup \{w\}) = [w]_A \cap W \subseteq [u]_D \cap (W \cup \{w\})$. If $w \equiv A u$, then $w \equiv d u$, so $[w]_A \cap (W \cup \{w\}) \subseteq [u]_D \cap (W \cup \{w\})$. Moreover $[u]_A \subseteq [u]_D$ implies $[u]_A \cap (W \cup \{w\}) \subseteq [u]_D \cap (W \cup \{w\})$.

Conversely, let us suppose that $v \in Z$, $\forall Z \in MDEP_S(A)$. Then, by Proposition 8.5.9, part (ii), we obtain:

$$\bigcup_{Z \in MDEP_S(A)} Z = U_S,$$

thus

$$[v]_A = \bigcup_{Z \in MDEP_S(A)} (Z \cup [v]_A) \subseteq \bigcup_{Z \in MDEP_S(A)} (Z \cup [v]_D) = [v]_D,$$

so $v \in \Gamma_S(A, \{d_S\}) = \varphi_U(A)$.

(iii): Since $v \in \varphi_U(A) = \Gamma_S(A, \{d_S\}) = U_S$, then $[v]_A \subseteq [v]_D$, so it is obvious that $[v]_A \cap W \subseteq [v]_D \cap W$ for any $W \in \mathcal{P}(U_S)$, i.e. $DEP_S(A) = \mathcal{P}(U_S)$.

Conversely, suppose that $DEP_S(A) = \mathcal{P}(U_S)$. Hence, $U_S \in DEP_S(A)$, so, by (8.15), we have

$$[v]_A \cap U_S = [v]_A \subseteq [v]_D \cap U_S = [v]_D$$

for any $v \in U_S$. Thus $v \in \Gamma_S(A, \{d_S\})$ for any $v \in U_S$.

(iv): Let $X \in DEP_S(A)$. Since $\varphi_U(A) \not\subseteq U_S$, we clearly have that $\varphi_U(A)^c \not\subseteq \emptyset$. Now, suppose
by contradiction that \( \varphi_{U_h}(A)^c \subseteq X \) and let \( u \in \varphi_{U_h}(A)^c \). By (i), we have that if \( u \in \varphi_{U_h}(A)^c \) and \( u' \equiv_A u \) then \( u' \in \varphi_{U_h}(A)^c \). An immediate consequence is that

\[
[v]_A \cap W = [v]_A.
\]  

Moreover, since \( u \in \varphi_{U_h}(A)^c \), it results that \( [v]_A \not\subseteq [v]_{D_h} \), hence, a fortiori, \( [v]_A \not\subseteq [v]_{D_h} \setminus W \) and the thesis follows.

(v): Let \( W \in DEP_h(A) \) and suppose by contradiction that there exists \( u \in \varphi_{U_h}(A)^c \) such that \( [v]_A \subseteq W \). By (i) we have that if \( u' \equiv_A u \), then \( u' \in \varphi_{U_h}(A)^c \). Therefore \( u' \in W \) for any \( u' \in [v]_A \). In particular, we have

\[
[v]_A \cap W = [v]_A.
\]  

Moreover, since \( u \in \varphi_{U_h}(A)(D_h)^c \), we have that \( [v]_A \not\subseteq [v]_{D_h} \), therefore \( [v]_A \not\subseteq [v]_{D_h} \setminus W \) and the thesis follows.

(vi): Suppose that \( [v]_A \cap W = \emptyset \) for any \( v \in \varphi_{U_h}(A)^c \). Then \( v \notin W \) for any \( v \in \varphi_{U_h}(A)^c \), i.e. \( \varphi_{U_h}(A)^c \subseteq W^c \) or, equivalently, \( W \subseteq \varphi_{U_h}(A) \). Therefore, for any \( v \in W \) it results \( [v]_A \subseteq [v]_{D_h} \), so that

\[
[v]_A \cap W \subseteq [v]_{D_h} \cap W.
\]

Hence, by (8.15), we have that \( W \in DEP_h(A) \).

(vii): Let \( A \subseteq B \). We want to show that any equation of \( DEP_h(A) \) belongs also to \( DEP_B(S) \). Let \( W \in DEP_h(A) \), thus \( [v]_A \cap W \subseteq [v]_{D_h} \cap W \) for any \( v \in W \). Moreover, by (2.13) it follows that \( \pi_S(B) \leq \pi_S(A) \) or, equivalently, \( [v]_B \subseteq [v]_A \). In particular, it results that \( [v]_B \cap W \subseteq [v]_A \cap W \), hence we obtain

\[
[v]_B \cap W \subseteq [v]_A \cap W \subseteq [v]_{D_h} \cap W,
\]
i.e. \( W \in DEP_B(S) \).

\[\square\]

### 8.6 Reduction of Indistinguishability Linear Systems

In this section we investigate the question concerning the reduction of an indistinguishability linear system \( S \). In other terms, we fix an equation subset \( W \) and a variable subset \( A \). The reduction technique consists of the choice of a criterion in order to reduce the complexity of a linear system. Clearly, this criterion refers to the choice of an operator or function associated with the property we want to preserve. In what follows, we provide two different reduction criteria. In fact, we can reduce the linear system by taking the smallest variable sets providing the same set of constants relative to all the equations of a given equation subset \( W \). On the other hand, the second kind of reduction we consider here consists of taking the smallest variable set \( B \) of \( A \) such that any equation compatible with respect to \( A \) is also compatible with respect to \( B \).

We set now

\[
\Delta_{W,A}^{d_s}(u, u') := \begin{cases} 
\Delta_{W,A}(u, u') & \text{if } u \not\equiv_{d_s} u', \\
0 & \text{otherwise,}
\end{cases}
\]  

for all \( u, u' \in W \). In particular, we set \( \Delta_s := \Delta_{U_h, C_s}^{d_s} \).
We call $\Delta_{W,A}^{d_{S}}(u,u')$ the \textit{(}$d_{S},W,A$\textit{)-discernibility neighborhood} of $u$ and $u'$ in $S$ and \textit{(}$d_{S},W,A$\textit{)-discernibility set system} of $S$ the system $\mathcal{D}(W,A,d_{S}):=(A,DIS_{S}(d_{S},W,A))$, where

$$DIS_{S}(d_{S},W,A):=\{\Delta_{W,A}^{d_{S}}(u,u') : u,u' \in W \text{ and } \Delta_{W,A}^{d_{S}}(u,u') \neq \emptyset\}.$$ 

In particular, we set $DIS(d_{S},S):=DIS_{S}(d_{S},U_{S},C_{S})$ and $\mathcal{D}(d_{S},S):=(U_{S},DIS(d_{S},S))$, moreover we call $\Delta_{S}^{d_{S}}(u,u')$ the $d_{S}$-discernibility neighborhood of $u$ and $u'$ and $\mathcal{D}(d_{S},S)$ the $d_{S}$-discernibility set system of $S$. If $W$ and $A$ are both finite sets with $W = \{u_{1},\ldots,u_{m}\}$, we denote by $DT_{S}[d_{S},W,A]$ the $m \times m$ table having the equations $u_{1},\ldots,u_{m}$ on both its rows and columns and the subsets $\Delta_{W,A}^{d_{S}}(u_{i},u_{j})$ in the $(u_{i},u_{j})$-entry and we call $DT_{S}[d_{S},W,A]$ the \textit{(}$W,A,d_{S}$\textit{)-discernibility table} of $S$. In particular, we set $DT_{S}[d_{S}]:=DT_{S}[U_{S},C_{S},d_{S}]$ and we call it the \textit{d_{S}-discernibility table} of $S$.

In what follows we fix a variable set $A$ and an equation $u$. We associate with $u$ the set of all constants relative to all the equations that are $A$-symmetric to $u$.

We introduce the following function $\xi_{A} : W \rightarrow \mathcal{P}(\Lambda_{S})$:

$$\xi_{A}(u) = \{F_{S}(u',d_{S}) : u' \in [u]_{A}\}. \quad (8.21)$$

\textbf{Remark 8.6.1.} (i): If $A$ and $B$ are two variable subsets such that $A \subseteq B$, then $\xi_{A}(w) \subseteq \xi_{B}(w)$ for any $w \in W$.

(ii): If $S_{W,A}$ is compatible, then $\xi_{A}(w) = \{F_{S}(w,d_{S})\}$.

We now introduce the \textit{(}$\xi,W,A$\textit{)-essentials}.

\textbf{Definition 8.6.2.} We say that a subset $B \subseteq A$ is a \textit{(}$\xi,W,A$\textit{)-essential} if:

- there exists $w \in W$ such that $\xi_{A \setminus B}(w) \neq \xi_{A}(w)$,

- for any $B' \subsetneq B$ and for any $w \in W$, we have $\xi_{A \setminus B'}(w) = \xi_{A}(w)$.

We denote by $ESS_{S}(\xi,W,A)$ the family of all \textit{(}$\xi,W,A$\textit{)-essentials}. We denote by $ESS(\xi,S):=ESS_{S}(\xi,U_{S},C_{S})$ the family of all \textit{\xi-essentials} of $S$.

We introduce the \textit{(}$\xi,W,A$\textit{)-reducts}.

\textbf{Definition 8.6.3.} We say that a subset $B \subseteq A$ is a \textit{(}$\xi,W,A$\textit{)-reduct} if:

- $\xi_{A}(w) = \xi_{B}(w)$ for any $w \in W$,

- for any $B' \subsetneq B$ there exists $w_{B'} \in W$ such that $\xi_{B'}(w_{B'}) \neq \xi_{A}(w_{B'})$.

We denote by $RED_{S}(\xi,W,A)$ the family of all \textit{(}$\xi,W,A$\textit{)-reducts}. We denote by $RED(\xi,S):=RED_{S}(\xi,U_{S},C_{S})$ the family of all \textit{\xi-reducts} of $S$.

In next result we relate the \textit{(}$\xi,W,A$\textit{)-reducts} to the compatibility of $S_{W,A}$.

\textbf{Theorem 8.6.4.} Let $S_{W,A}$ be a compatible indistinguishability linear system. Then $RED_{S}(\xi,W,A)$ coincides with the family of variable sets $B$ such that:
(i) $\Theta_S(W,B) = W$;

(ii) $\Theta_S(W,B') \neq W$ for any $B' \subsetneq B$.

Proof. Let $B \in RED_S(\xi, W, A)$, then $\xi_B(w) = \xi_A(w) = \{F_S(w, d_S)\}$ for any $w \in W$. We must prove that $[w]_B \cap W \subseteq [w]_{D_S} \cap W$. Let $w' \in [w]_B \cap W$ and set $t = F_S(w', d_S)$. Then, $t \in \xi_B(w) = \xi_A(w)$, so that $t = F_S(w, d_S)$ or, equivalently, $w' \in [w]_{D_S} \cap W$. So $\Theta_S(W,B) = W$. Let us consider $B' \subsetneq B$. Then, $B' \notin RED_S(\xi, W, A)$, hence there exists $w \in W$ such that $\xi_B(w) \neq \xi_A(w)$, so $\xi_A(w) \subsetneq \xi_B(w)$. This means that there exists $w' \in W$ such that $w' \neq A w, w' \equiv_{B'} w$ and $t := F_S(w', d_S) \neq F_S(w, d_S)$. Hence, $[w]_{B'} \cap W \not\subseteq [w]_{D_S} \cap W$, i.e. $w \in W \setminus \Theta_S(W,B')$.

On the other hand, let $B \in \mathcal{P}(S)$ satisfying (i) and (ii). By Proposition 8.5.7, $S_{W,B}$ is compatible, hence $\xi_B(w) = \xi_A(w) = \{F_S(w, d_S)\}$. Now, let $B' \subsetneq B$. Then $\Theta(W,B') \subsetneq W$, so there exists $w \in W$ such that $[w]_{B'} \cap W \not\subseteq [w]_{D_S} \cap W$. This means that there exists $w' \in W$ such that $w' \equiv_{B'} w$ but $F_S(w, d_S) \neq F_S(w', d_S)$, so $\xi_A(w) \neq \xi_B(w)$. We conclude that $B \in RED_S(\xi, W, A)$.

We now provide a geometric property for the set system $ESS_S(\xi, W, A)$.

Theorem 8.6.5. Let $S_{W,A}$ be a compatible indistinguishability linear system. Then,

$$ESS_S(\xi, W, A) = \min(DIS_S(d_S, W, A)).$$

Proof. Let $B \in ESS_S(\xi, W, A)$. Then, there exists $u \in W$ such that $\xi_A(u) \neq \xi_A \setminus B(u)$. By compatibility of $S_{W,A}$, $\xi_A(u)$ consists of the single element $F_S(u, d)$. Since $\pi_S(A) \leq \pi_S(A \setminus B)$, there exists $u' \in W$ such that:

(a) $u \equiv_{A \setminus B} u'$;

(b) $u \equiv_A u'$;

(c) $F_S(u, d) \neq F_S(u', d)$.

Hence $\Delta_{W,A}^{d_S}(u, u') \subseteq B$. Now, let $b \in B$ and set $C = B \setminus \{b\}$. By (ii) of Definition 8.6.2, it results $\xi_{A \setminus C}(u) = \xi_A(u)$ for any $u \in W$. Let $u, u'$ as before. It must results

$$u \neq A \setminus C u',$$

otherwise, we would have $F_S(w, d), F_S(u, d) \in \xi_{A \setminus C}(u)$, so $\xi_{A \setminus C}(u) \neq \xi_A(u)$, that is an absurd.

In other terms, we have shown that $B \in \min(DIS_S(d_S, W, A))$.

Conversely, let $B \in \min(DIS_S(d_S, W, A))$. Hence there are two equations $u, u' \in W$ such that $B = \Delta_{W,A}^{d_S}(u, u')$. By (8.20), we have $u \neq d_S u'$ and $u \neq A u'$. It follows immediately that $u \equiv_{A \setminus B} u'$, therefore $\xi_A(u) \neq \xi_{A \setminus B}(u)$. Let $C \subsetneq B$. We firstly observe that by minimality of $B$, we must have $\Delta_{W,A}^{d_S}(u, u') \subseteq C$ for any $u, u' \in W$. We show that $\pi_S(A \setminus C) = \pi_S(A)$. In fact, if $u \equiv_{A \setminus C} u'$ and $u \neq A u'$, we would have $\Delta_{W,A}^{d_S}(u, u') \subseteq C$, that is an absurd. Hence $u \equiv A u' \iff u \equiv_{A \setminus C} u'$, so $\pi_S(A \setminus C) = \pi_S(A)$. So $B$ satisfies also condition (ii) of Definition 8.6.2 and the theorem is proved.

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Theorem 8.6.6. Let $S_{W,A}$ be a compatible indistinguishability linear system. Then:
(i) $\xi_B(u) = \xi_A(u)$ for any $u \in W$ if and only if $B$ is a transversal of $DIS_S(d_S,W,A)$.
(ii) $RED_S(\xi,W,A) = Tr(DIS_S(d_S,W,A))$.

Proof. Assume that $\xi_A(u) = \xi_B(u)$ for any $u \in W$. We show that $B$ is a transversal of $DIS_S(d_S,W,A)$. Let $E \in DIS_S(d_S,W,A)$. Then, there exists two distinct equations $u, u' \in W$ such that $u \neq_S u'$ and $E = \{a \in A : (F_S(u,a) \neq F_S(u',a))\}$. Since $S_{W,A}$ is compatible, we deduce that $u \not\eq_A u'$ and so $E \neq \emptyset$. If $u \equiv_B u'$, by hypothesis we would have $\xi_A(u) \neq \xi_B(u)$, that is an absurd. So, we have thus found a variable $a' \in B$ such that $F_S(u,a') \neq F_S(u',a')$, i.e. $a' \in B \cap E$. Therefore $B$ is a transversal of $DIS_S(d_S,W,A)$.

Conversely, let $B$ be a transversal of $DIS_S(d_S,W,A)$. Moreover, let $u, u' \in W$ such that $u \equiv_B u'$. Suppose by contradiction that $F_S(u,a) \neq F_S(u',a')$, $u \equiv_B u'$ and, by consistency, $u \neq_B u'$. Therefore $\Delta^S_{W,A}(u,u') \neq \emptyset$ but $\Delta^S_{W,A}(u,u') \cap B = \emptyset$, that is an absurd. Hence, $\xi_B(u) = \xi_A(u)$ for any $u \in W$.

(ii) : Let $B \in RED_S(\xi,W,A)$. Hence $\xi_A(u) = \xi_B(u)$ for any $u \in W$, and by (i) this implies that $B$ is a transversal of $DIS_S(d_S,W,A)$.

In what follows we provide another criterion of reduction relied on the operator $\varphi_W$.

Definition 8.6.7. We say that a subset $B \subseteq A$ is a $(\varphi,W,A)$-essential if:

- $\varphi_W(A \setminus B) \subseteq \varphi_W(A)$,
- for any $B' \subseteq B$ we have $\varphi_W(A \setminus B') = \varphi_W(A)$,

that are equivalent to the following conditions,

- $\hat{\varphi}_W(A \setminus B) < \hat{\varphi}_W(A)$;
- for any $B' \subseteq B$ we have $\hat{\varphi}_W(A \setminus B') = \hat{\varphi}_W(A)$.

We denote by $ESS_\varphi(\varphi,W,A)$ the family of all $(\varphi,W,A)$-essentials and we denote by $ESS(\varphi,\varphi,S) := ESS_\varphi(\varphi,U_\varphi,C_\varphi)$ the family of all $\varphi$-essentials of $S$.

We now introduce the notion of $(\varphi,W,A)$-reducts.

Definition 8.6.8. We say that a subset $B \subseteq A$ is a $(\varphi,W,A)$-reduct if:

- $\varphi_W(A) = \varphi_W(B)$,
- for any $B' \subseteq B$ we have $\varphi_W(B') \subseteq \varphi_W(A)$,

that are equivalent to the following conditions,
\[ S_2 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad d_{S_2} \\
\begin{array}{cccccc}
u_1 & 1 & 0 & 0 & 0 & 0 & 1 \\
u_2 & 0 & 1 & 1 & 0 & 0 & 2 \\
u_3 & 0 & 1 & 1 & 0 & 0 & 3 \\
u_4 & 1 & 0 & 0 & 0 & 0 & 4 \\
u_5 & 0 & 1 & 1 & 1 & 1 & 5 \\
\end{array} \]

Figure 8.1: Indistinguishability linear system \( S_2 \).

- \( \hat{\varphi}_W(A) = \hat{\varphi}_W(B) \).
- for any \( B' \subseteq B \) we have \( \hat{\varphi}_W(B') < \hat{\varphi}_W(A) \).

We denote by \( RED_8(\varphi, W, A) \) the family of all the \( (\varphi, W, A) \)-reducts. We denote by \( RED(S) := RED_8(\varphi, U_5, C_5) \) the family of all the \( \varphi \)-reducts of \( S \).

In the next example we see that the \( ESS_8(\varphi, W, A) \) does not coincide with the minimals of \( DIS_8(d_5, W, A) \) and that \( RED_8(\varphi, W, A) \) is not the family of the minimal transversals of \( DIS(d_5, S) \).

**Example 8.6.9.** Let us consider the indistinguishability linear system \( S_2 \) represented in Figure 8.1.

Then,

\[ \pi_{S_2}(d_{S_2}) = u_1|u_2|u_3|u_4|u_5 \]

and

\[ \pi_{S_2}(C_{S_2}) = u_1u_4|u_2u_3u_5. \]

Furthermore, it is easy to see that

\[ DIS(d_{S_2}, S_2) = \{\{a_1, a_2, a_3\}, \{a_4, a_5\}, \{a_1, a_2, a_3, a_4, a_5\}\}, \]

and

\[ \min(DIS(d_{S_2}, S_2)) = \{\{a_1, a_2, a_3\}, \{a_4, a_5\}\}. \]

By Definition 8.6.7, we have that

\[ ESS(\varphi, S_2) = \{a_4, a_5\}. \]

Thus, we conclude that, in general, \( ESS(\varphi, S_2) \) does not coincide with the family of all minimal hyper-edges of \( DIS(d_{S_2}, S_2) \). Finally, by Definition 8.6.8, we see that

\[ RED(\varphi, S_2) = \{\{a_4\}, \{a_5\}\}, \]

so the \( \varphi \)-reducts family is not the transversal of \( DIS(d_{S_2}, S_2) \).

The next proposition asserts that we can focus our attention only on the maximum partitioners in order to study the family of all \( \varphi \)-reducts of an global symmetry class.
Proposition 8.6.10. Let $A \approx S B$ and $A \subseteq B$, then

$$RED_S(\varphi, W, A) \subseteq RED_S(\varphi, W, B).$$  \hfill (8.22)

In particular,

$$RED_S(\varphi, W, A) \subseteq RED_S(\varphi, W, M_S(A)).$$  \hfill (8.23)

Proof. We firstly observe that since $A \approx S B$, then $\phi_A(W) = \phi_B(W)$. By Definition 8.6.8 it follows that, if $C \in RED_S(\varphi, W, A)$, then $\phi_B(W) = \phi_C(W)$ and $\phi_C'(W) \subseteq \phi_B(W)$ for any $C' \not\subseteq C$. Thus, $C \in RED_S(\varphi, W, B)$. $\square$

In next result we show the interrelation between $(\varphi, W, A)$-reducts and the $(\varphi, W, A)$-essentials.

Theorem 8.6.11.

$$RED_S(\varphi, W, A) = Tr(ESS_S(\varphi, W, A))$$

Proof. Let $B \subseteq A$ such that $\varphi_W(A) = \varphi_W(B)$. This means that

$$\Theta_S(W, A) = \Theta_S(W, B)$$  \hfill (8.24)

i.e.

$$[u]_A \cap W \subseteq [u]_{D_S} \cap W \iff [u]_B \cap W \subseteq [u]_{D_S} \cap W.$$  \hfill (8.25)

We claim that $B$ is a transversal of $ESS_S(\varphi, W, A)$. For, let $E \in ESS_S(\varphi, W, A)$. Hence

$$\Theta_S(W, A \setminus E) \not\subseteq \Theta_S(W, A).$$  \hfill (8.26)

In other terms, there exists an equation $u \in W$ such that

$$[u]_A \cap W \subseteq [u]_{D_S} \cap W$$  \hfill (8.27)

and an equation $u' \in W$ such that

(a) $u' \equiv_{A \setminus E} u$;

(b) $u' \not\equiv_{D_S} u$.

This implies by (8.27) that $u \neq_A u'$ and, in particular, by (8.25), it follows that

$$u' \neq_B u.$$  \hfill (8.28)

Thus, there exists $b \in B$ such that $F_S(u, b) \neq F_S(u', b)$. Therefore, we have that $b \in \Delta^D_{W, A}(u, u')$. By (a) and (b) it is easy to see that $\Delta^D_{W, A}(u, u') \subseteq E$, in fact for any $a \in A \setminus E$ it results that

$$F_S(u, a) = F_S(u', a),$$

hence $A \setminus E \subseteq A \setminus \Delta^D_{W, A}(u, u')$, i.e. $\Delta^D_{W, A}(u, u') \subseteq E$. So $b \in E$ and $B$ is a transversal of $E$ and, since the choice of $E \in ESS_S(\varphi, W, A)$ is arbitrary, $B$ is a transversal of $ESS_S(\varphi, W, A)$.
On the other hand, let \( B \) be a transversal of \( \text{ESS}_S(\varphi, W, A) \) and fix \( E \in \text{ESS}_S(\varphi, W, A) \). By Definition 8.6.7, there exists \( u \in \Theta_S(W, A) \) such that \( u \not\in \Theta_S(W, A \setminus E) \). Hence

\[
[u]_A \cap W \subseteq [u]_{D_S} \cap W
\]

and there exists \( u' \in W \) satisfying \((a)\) and \((b)\). In other terms, it results that \( u' \in [u]_A \cap W \cap [u]_{A \setminus E} \). Let \( b \in B \cap E \). We have that \( F_S(u, b) \neq F_S(u', b) \), otherwise

\[
F_S(u, a) = F_S(u', a)
\]  

(8.29)

for any \( a \in (A \setminus E) \cup \{ b \} \). If we set \( E' = E \setminus \{ b \} \), we are saying that \((8.29)\) holds for any \( a \in A \setminus E' \), thus \( u \equiv_{A \setminus E'} u' \), so \( u \not\in \Theta_S(W, A \setminus E') \), contradicting Definition 8.6.7. Therefore, \( F_S(u, b) \neq F_S(u', b) \), i.e. \( u \neq_B u' \) or, equivalently, \( u' \in [u]_B \cap W \cap [u]_{A \setminus E} \). Since the previous argument can be extended to any \( u' \in [u]_A \cap W \cap [u]_{A \setminus E} \), we conclude that

\[
[u]_A \cap W \cap [u]_{A \setminus E} \subseteq [u]_B \cap W \cap [u]_{A \setminus E},
\]

i.e.

\[
[u]_B \cap W \cap [u]_{A \setminus E} \subseteq [u]_A \cap W \cap [u]_{A \setminus E}
\]

or, equivalently,

\[
[u]_B \cap W \subseteq [u]_A \cap W.
\]

Since \( u \in \Theta_S(W, A) \), we deduce that \( u \in \Theta_S(W, B) \) or, equivalently, that \( \Theta_S(W, A) \subseteq \Theta_S(W, B) \). The reverse inclusion holds by Proposition 8.5.4. So, \( \varphi_W(B) = \varphi_W(A) \). We have thus shown that \( B \subseteq A \) is a transversal of \( \text{ESS}_S(\varphi, W, A) \) if and only if \( \varphi_W(B) = \varphi_W(A) \). We can now conclude the proof. Let \( c \in B \) and \( B' = B \setminus \{ c \} \). By \((ii)\) of Definition 8.6.8, we have that

\[
\Theta_S(W, B') \not\subseteq \Theta_S(W, A)
\]

therefore, by the above argument, \( B' \) is not a transversal of \( \text{ESS}_S(\varphi, W, A) \). Thus

\[
\text{RED}_S(\varphi, W, A) \subseteq \text{Tr}(\text{ESS}_S(\varphi, W, A)).
\]

Conversely, if \( B \in \text{Tr}(\text{ESS}_S(\varphi, W, A)) \), by the above argument it results that

\[
\Theta_S(W, B) = \Theta_S(W, A)
\]

and by minimality, \((ii)\) of Definition 8.6.8 follows immediately. This shows the claim. \( \square \)

In what follows, we want to define a new class of indistinguishability linear systems such that the \((\varphi, W, A)\)-essentials are exactly the minimal elements of \( \text{DIS}_S(d_S, W, A) \) and the \((\varphi, W, A)\)-reducts are the minimal transversal of the same set system. In order to do this, we introduce the following terminology.

**Definition 8.6.12.** We say that \( S \) is \((\varphi, W, A)\)-disjunctive if for any pair of equations \( u, v \in W \) the following condition holds:

\[
\Delta_{W, A}^{d_S}(u, v) \neq \emptyset \implies u \in \varphi_W(A) \lor v \in \varphi_W(A).
\]  

(8.30)

In particular, if \( W = U_S \) and \( A = C_S \), we say that \( S \) is \( \varphi \)-disjunctive.
Figure 8.2: Indistinguishability linear system $S_3$.

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$d_3$</th>
</tr>
</thead>
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<tr>
<td>$u_1$</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$u_2$</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$u_3$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u_4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$u_5$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

In the next example we show that a $\varphi$-disjunctive indistinguishability linear system fails to be a $(\varphi, W, A)$-disjunctive indistinguishability linear system for any $A \subseteq C_S$ and any $W \subseteq U_S$.

**Example 8.6.13.** Let us consider the indistinguishability linear system $S_3$ given in Figure 8.2.

It is immediate to see that $S_3$ is a $\varphi$-disjunctive indistinguishability linear system. On the other hand, let $W = \{u_1, u_2, u_3, u_4\}$ and $A = \{a_1\}$. It is easy to verify that

$$[u]_A \cap W = \begin{cases} \{u_1, u_4\} & \text{if } u = u_1 \lor u = u_4 \lor u = u_5 \\ \{u_2, u_3\} & \text{if } u = u_2 \lor u = u_3 \end{cases}$$

and

$$[u]_{d_3} \cap W = \begin{cases} \{u_1\} & \text{if } u = u_1 \\ \{u_2\} & \text{if } u = u_2 \\ \{u_3\} & \text{if } u = u_3 \lor u = u_5 \\ \{u_4\} & \text{if } u = u_4 \end{cases}$$

Thus

$$\Theta_S(W, A) = \emptyset.$$ 

This shows that $S_3$ is not a $(\varphi, W, A)$-disjunctive indistinguishability linear system.

As an immediate result, we prove that all the compatible indistinguishability linear systems are also $\varphi$-disjunctive.

**Proposition 8.6.14.** If $S$ is compatible, then it is also a $\varphi$-disjunctive indistinguishability linear system.

**Proof.** By Proposition 8.4.1, compatibility of $S$ is equivalent to the condition $\Theta_S(U_S, C_S) = U_S$. \qed

**Remark 8.6.15.** Let us note that there exist incompatible indistinguishability linear systems that are $\varphi$-disjunctive, as we can see taking the indistinguishability linear system $S_3$ of Figure 8.2.

In the next result we show that for any $(\varphi, W, A)$-disjunctive indistinguishability linear system the essentials $ESS_S(\varphi, W, A)$ are exactly the minimal elements of $DIS_S(d_S, W, A)$.
Let then
\[ \theta(i) \]

**Theorem 8.6.16.** Let \( S \) be a \((\varphi,W,A)\)-disjunctive indistinguishability linear system. Then,
\[
ESS_{S}(\varphi,W,A) = \min(DIS_{S}(d_{S},W,A)).
\] (8.31)

**Proof.** We will prove that a variable subset \( B \subseteq A \) belongs to \( ESS_{S}(\varphi,W,A) \) if and only if it satisfies the following two conditions:

(a) \( B \supseteq \Delta_{W,A}^{d_{S}}(u,v) \neq \emptyset \), for some \( u,v \in W \);

(b) if there exists \( u,v \in W \) such that \( \emptyset \neq \Delta_{W,A}^{d_{S}}(u,v) \subseteq B \), then \( \Delta_{W,A}^{d_{S}}(u,v) = B \).

Let \( ESS_{S}(\varphi,W,A) \). By Definition 8.6.7 and by Proposition 8.5.4, this means that there exists \( u \in W \) such that \([u]_{A} \subseteq [u]_{D_{S}} \subseteq [u]_{A,B} \subseteq [u]_{D_{S}} \subseteq W \). This means that there exists \( v \in W \) such that \( u \equiv_{A,B} v \) and \( u \not\equiv_{D_{S}} v \), so, \( F_{S}(u,v) = F_{S}(v,a) \) for any \( a \in A \setminus B \). By (8.20), we deduce that \( A \setminus B \subseteq A \setminus \Delta_{W,A}^{d_{S}}(u,v) \), so \( B \) satisfies condition (a). Now, suppose that there exists \( u,v \in W \) such that \( \emptyset \neq \Delta_{W,A}^{d_{S}}(u,v) \subseteq B \). Let \( b \not\equiv_{D_{S}} \Delta_{W,A}^{d_{S}}(u,v) \); hence \( F_{S}(u,b) = F_{S}(v,b) \). Let \( B' := B \setminus \{b\} \). Since \( F_{S}(u,a) = F_{S}(v,a) \) for all \( a \in A \setminus B \) and \( F_{S}(u,b) = F_{S}(v,b) \), we have \( F_{S}(u,a) = F_{S}(v,a) \) for all \( a \in (A \setminus B) \setminus \{w\} \) = \( A \setminus B' \). In other terms, we have \( u \equiv_{A,B'} v \).

Then, since \( u \not\equiv_{D_{S}} v \), we deduce that \( u \not\equiv_{\Theta_{S}(W,A \setminus B')} \). But, we have that \( u \in \Theta_{S}(W,A) \), so \( b \not\in B \), by Definition 8.6.7. Hence part (b) holds.

Assume now that \( B \subseteq A \) satisfies (a) and (b). Since \( \Delta_{W,A}^{d_{S}}(u,v) \neq \emptyset \), it holds that \( u \not\equiv_{A} v \) and \( u \not\equiv_{D_{S}} v \). We may assume that \( u \in \Theta_{S}(W,A) \). Moreover \( B \supseteq \Delta_{W,A}^{d_{S}}(u,v) \) implies that \( u \equiv_{A,B} v \) and \( u \not\equiv_{D_{S}} v \), thus \( u \not\in \Theta_{S}(W,A \setminus B) \), so (i) of Definition 8.6.7 holds.

Let now \( B' \subsetneq B \). Condition (b) implies that, for all \( u,v \in W \) such that \( \emptyset \neq \Delta_{W,A}^{d_{S}}(u,v) \), it holds that \( \Delta_{W,A}^{d_{S}}(u,v) \not\subseteq B' \). This is equivalent to say that \( \pi_{S}(A \setminus B') = \pi_{S}(A) \). In fact, it is obvious that \( u \equiv_{A} u' \) implies \( u \equiv_{A,B'} u' \); moreover, suppose that \( u \equiv_{A,B'} u' \) and assume by contradiction that \( u \not\equiv_{A} u' \). Then, we have \( \Delta_{W,A}^{d_{S}}(u,u') \subseteq B' \), that is an absurd. So (ii) of Definition 8.6.7 holds too and \( B \in ESS_{S}(\varphi,W,A) \).

The next result shows that the reducts of \( S \) are exactly the minimal transversals of the essentials family \( ESS(S) \) for any \( \varphi \)-disjunctive indistinguishability linear system \( S \).

**Theorem 8.6.17.** Let \( S \) be a \((\varphi,W,A)\)-disjunctive indistinguishability linear system and let \( B \subseteq A \). Then:
(i) \( \theta_{S}(W,B) = \theta_{S}(W,A) \) if and only if \( B \) is a transversal of \( DIS_{S}(d_{S},W,A) \).
(ii) \( RED_{S}(\varphi,W,A) = Tr(DIS_{S}(d_{S},W,A)) = Tr(ESS_{S}(\varphi,W,A)) \).

**Proof.** (i) Assume that \( \theta(W,B) = \theta(W,A) \). We show that \( B \) is a transversal of \( DIS_{S}(d_{S},W,A) \).

Let then \( E \in DIS_{S}(d_{S},W,A) \). Then, there exists two distinct equations \( u,u' \in W \) such that \( u \not\equiv_{D_{S}} u' \) and \( E = \{a \in A : (F_{S}(u,a) \not\equiv_{S}(u',a')\} \). We deduce that \( u \not\equiv_{A} u' \) and we can assume that \( u \in \Theta(W,A) \). By hypothesis, \( u \in \Theta(W,B) \), hence \( u \not\equiv_{B} u' \). Thus, there exists \( b \in B \) such that \( F_{S}(u,b) \not\equiv_{S}(u',b) \), i.e. \( b \in B \cap E \). Therefore \( B \) is a transversal of \( DIS_{S}(d_{S},W,A) \).

On the other hand, we suppose now that \( B \) is a transversal of \( DIS_{S}(d_{S},W,A) \). Hence, there exist \( u,u' \in W \) such that \( \Delta_{W,A}^{d_{S}}(u,u') \neq \emptyset \). We clearly have that \( u \not\equiv_{A} u' \). We set now \( E := \Delta_{W,A}^{d_{S}}(u,u') \). Let \( b \in B \cap E \), then \( F_{S}(u,b) \not\equiv_{S}(u',b) \), so \( u \not\equiv_{B} u' \). We are saying that for
any $u, u' \in W$ such that $u \not\equiv_{dS} u'$ and $u \not\equiv_A u'$, then $u \not\equiv_B u'$. In other terms, we have shown that

$$[u]^c_A \cap W \subseteq [u]^c_B \cap W$$

or, equivalently, that

$$[u]_B \cap W \subseteq [u]_A \cap W.$$

Since $u \in \Theta_S(W, A)$, we deduce that $u \in \Theta_S(W, B)$, i.e. $\Theta_S(W, A) \subseteq \Theta_S(W, B)$. The reverse inclusion follows by Proposition 8.5.4. So $\theta_S(W, B) = \theta_S(W, A)$.

$(ii)$: Let $B \in RED_S(\varphi, W, A)$. Hence $\theta_S(W, B) = \theta(W, A)$, and by $(i)$ this implies that $B$ is a transversal of $DIS_S(d_S, W, A)$. Now, let $b \in B$ and set $B' = B \setminus \{b\}$. By Definition 8.6.8 we have that $\theta_S(W, B') < \theta_S(W, A)$, therefore by $(i)$ it follows that $B'$ is not a transversal of $DIS_S(d_S, W, A)$. Hence $B$ is a minimal transversal of $DIS_S(d_S, W, A)$. On the other hand, let $B$ be a minimal transversal of $DIS_S(d_S, W, A)$, then by $(i)$ it follows that $\theta_S(W, B) = \theta_S(W, A)$. Now, if $b \in B$, the subset $B' := B \setminus \{b\}$ is not a transversal of $DIS_S(d_S, W, A)$ by virtue of the minimality of $B$, therefore, again by $(i)$ we obtain $\theta_S(W, B') < \theta_S(W, A)$. Hence $B \in RED_S(\varphi, W, A)$. Thus, $RED_S(\varphi, W, A) = Tr(DIS_S(d_S, W, A))$. By Theorem 8.6.16, it results $Tr(DIS_S(d_S, W, A)) = Tr(ESS_S(\varphi, W, A))$, hence $(ii)$ has been proved. \qed

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