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Summary. Kapteyn series are a very useful mathematical toolbox occurring in a wide field of applications even though their mathematical framework is not trivial. Indeed, these series considered as toolbox, are sometimes difficult to use because of their numerical convergence problems that are the motive of slow computation times. This is the reason why their use in practical contexts can be tough.
Taking this into consideration, in this thesis a new approach to manage Kapteyn series has been proposed. In fact, an integral representation in connection to them has been considered. This integral approach allows to obtain more or less slow computational times with respect to the computation of the Kapteyn series depending on the particular integrand structure of the integral form of the considered application.
This thesis is organized as follows. The first two chapters are purely mathematical and report the history, the state of the art, and the main properties of Bessel functions and of the Kapteyn series respectively.
In the third one the definition and the properties of uniform convergent series has been shown and then the proposed approach is illustrated. This lets us to obtain an integral representation of a considered class of Kapteyn series.
From the fourth chapter onwards, have been pointed out applicative scenarios in which the proposed approach could be useful. The fourth chapter is completely dedicated to the Kepler problem. The history of this important problem has been briefly reported and then the solution in integral form obtained by the proposed approach has been explained. In the fifth and the six-th chapters have been discussed applications in the power electronics and the control theory fields. Indeed, the PWM modulation has been analyzed in the use of the DC-AC conversion and in the feedback control systems respectively. This analysis is based on the isomorphism between Kepler's problem and the problem of determinating the switching instants of the PWM output.
Lastly, in the seven-th chapter the proposed approach is used both to obtain the integral forms of the Kapteyn series that occur in the modern theory of electromagnetic radiation and in the models of the energy spectrum of fronts, and to generalize a Meissel's result.

1

Bessel functions

Bessel functions belong to a particular class of mathematical functions, known as special functions [1, 2].
'Special functions' are particular functions of real or complex variables characterized by particular properties that let them be useful in many applications. This is the reason why, an in-depth study of them and their connections with other functions, is essential.

1.1 Historical background

Bessel functions are a particular solution $y(x)$ of the following differential equation, known as 'Bessel's equation for functions of order ν'[3]

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \tag{1.1}$$

where $\nu \in \mathbb{C}$ and $x \in \mathbb{C}$ are respectively the order and the argument of the Bessel functions.
The name of these functions derives from the german mathematician Friederich Bessel who in 1826 thoroughly analyzed them. If as said, in one hand Bessel defined and characterized these functions, in the other hand other mathematicians already dealt with these functions, which were not yet known as Bessel functions. So the paper of Bessel is not the first step of the historical development of the Bessel functions.
The history of these functions began in 1732, when Daniel Bernoulli [4], studying necessary conditions for an heavy chain of uniform density suspended by an extreme point A to have infinitesimal periodical oscillations, calculated that the ratio between the displacement of a generic point with distance x from the free extreme point B and the length l of the chain can be expressed by a series, today known as a Bessel function of order zero and argument $2\sqrt{\frac{x}{n}}$.
In [5], Bernoulli himself rewrote the relation between the displacement of the

point and the length of the chain as a differential equation, which considering a substitution of a variable, is the Bessel equation of order zero.
Eulero came to the same equation [6, 7] studying the same subject with a different approach.
In 1770, Lagrange [8] studying the elliptical motion of a planet around the sun and taking into account the following relation between the 'radius vector' r, the 'mean anomaly' M and the 'eccentric anomaly' E

$$M = E - \epsilon \sin E, \qquad r = a(1 - \cos E) \tag{1.2}$$

demonstrated that it was possible to formulate the following expansion

$$E = M + \sum_{n=1}^{\infty} A_n \sin nM, \qquad \frac{r}{a} = 1 + \frac{1}{2}\epsilon^2 + \sum_{n=1}^{\infty} B_n \cos nM \tag{1.3}$$

where a and ϵ are respectively the 'semi-major axis' and the 'eccentricity' of the orbit and where the expressions of A_n and B_n are

$$\begin{aligned} A_n &= 2 \sum_{m=0}^{\infty} \frac{(-1)^m n^{n+2m-1} \epsilon^{n+2m}}{2^{n+2m} m!(n+m)!}, \\ B_n &= -2 \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m) n^{n+2m-2} \epsilon^{n+2m}}{2^{n+2m} m!(n+m)!}. \end{aligned} \tag{1.4}$$

The italian astronomer Carlini studied the possibility to approximate the expression of A_n and B_n when n is sufficiently large and $0 < \epsilon < 1$ [9].
However, it was Poisson, who in 1836[10] found the following relation

$$B_n = -\frac{\epsilon}{n} \frac{dA_n}{d\epsilon} \tag{1.5}$$

that allows to establish the eq. (1.4) as follows

$$A_n = 2J_n(n\epsilon)/n, \qquad B_n = -2(\epsilon/n) J_n^{'}(n\epsilon), \tag{1.6}$$

where $J_n(\cdot)$ is the Bessel function of order n and argument $(\cdot)$.
Up to now, there are still a lot of open questions about these functions, and in particular as regards the computation and the properties of the roots of Bessel functions and their derivatives [11, 12] or as regards the evaluation of ratios of those functions [13].
Anyway, the more interesting subject probably concerns their numerical evaluation, that is considered to be onerous especially when the order is large. It is still a problem continuing to be discussed, e.g. in [14] a new algorithm that handles this problem has been proposed.

1.2 Applications

Bessel functions have a role of essential importance in different fields: for example they are useful in the study of the propagation of electromagnetic waves [15, 16] as well as in the analysis of the vibration of a stretched membrane [17]. These functions are also closely connected to the theory of heat conduction[18].
Bessel functions are also useful in the study of waves in a fluid. In this context the study of the generation of the tsunami can be found[19].
Those functions appear also in the theory of the estimation of terrestrial ice ages [20] and in the analytic theory of the divisors of the numbers. [3, 21].

1.3 Bessel functions of the first kind

In this section it is illustrated how it is possible to come to the following representation of Bessel functions $J_\nu(x)$ in terms of infinite series

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)}. \tag{1.7}$$

where $\Gamma(\nu)$ represents the gamma function [1], defined as

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt \qquad con \quad \nu \in \mathbb{C}, \tag{1.8}$$

and that is an analytical extension of the factorial function in the complex field.
As said before, Bessel functions provide a particular solution of the differential equation (1.1). The form of the solution is

$$y(x) = \Phi(x,r) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad , a_0 \neq 0, \tag{1.9}$$

and by the derivative of this solution with respect to x it is possible to word the following expression

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \tag{1.10}$$

from which, through another operation of derivative, follows

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} \tag{1.11}$$

Substituting eqs. (1.9), (1.10) and (1.11) in (1.1), it is possible to frame:

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}+\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}+ \\ +\sum_{m=0}^{\infty}a_m x^{m+r+2}-\sum_{m=0}^{\infty}\nu^2 a_m x^{m+r}=0. \tag{1.12}$$

Eq. (1.12) can be rewritten more compactly as:

$$\sum_{m=0}^{\infty}\alpha_m x^{m+r}=0, \tag{1.13}$$

where it is possible to notice that the equation has a solution if each coefficient α_m con $m=0,1,\dots$ is equal to zero, that is

$$\begin{gathered}\alpha_0=r(r-1)a_0+ra_0-\nu^2a_0=0\\ \alpha_1=(r+1)ra_1+(r+1)a_1-\nu^2a_1=0\\ \vdots\\ \alpha_m=(m+r)(m+r-1)a_m+(m+r)a_m+a_{m-2}-\nu^2a_m=0 \qquad (m=2,3,...)\end{gathered} \tag{1.14}$$

In order to annihilate the first coefficient α_0 it is necessary that

$$(r+\nu)(r-\nu)=0 \tag{1.15}$$

Then it is possible one of the two chooses:

$$r=v \qquad r=-v. \tag{1.16}$$

Choosing $r=v$ and annihilating the coefficient α_1 it follows that $a_1=0$. Then, so that each α_m $(m=2,3,...)$ is equal to zero it is necessary that:

$$(m+2\nu)ma_m+a_{m-2}=0 \qquad \Rightarrow a_m=-\frac{1}{(m+2\nu)m}a_{m-2}. \tag{1.17}$$

From eq. (1.17) it follows that

$$\begin{aligned}a_{2m+1}&=0 & m=1,2,...\\ a_{2m}&=-\tfrac{1}{4m(\nu+m)}a_{2m-2} & m=1,2,...\end{aligned} \tag{1.18}$$

and then

$$a_{2m}=-\frac{(-1)^m}{4^m m!(\nu+1)(\nu+2)\dots(\nu+m)}a_0 \qquad m=1,2,\dots \tag{1.19}$$

The value of the coefficient a_0 is arbitrary and Bessel functions are the subset of the solution set of the Bessel equation, marked with the notation $y(x)=J_n(x)$, that have been obtained, choosing

$$a_0 = \frac{1}{2^n n!}. \tag{1.20}$$

Under the hypothesis:

$$\nu = n \qquad e \qquad n \in \mathbb{Z}, \tag{1.21}$$

Bessel functions of the first kind of order n are represented as

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \tag{1.22}$$

The Hypothesis in eq. (1.21) that n is an integer number is necessary because in eq. (1.22) appears the factorial function that is defined in the set of non negative integer numbers. This hypothesis can be removed considering the gamma function, above defined, that lets eq. (1.22) be rewritten as eq. (1.7).

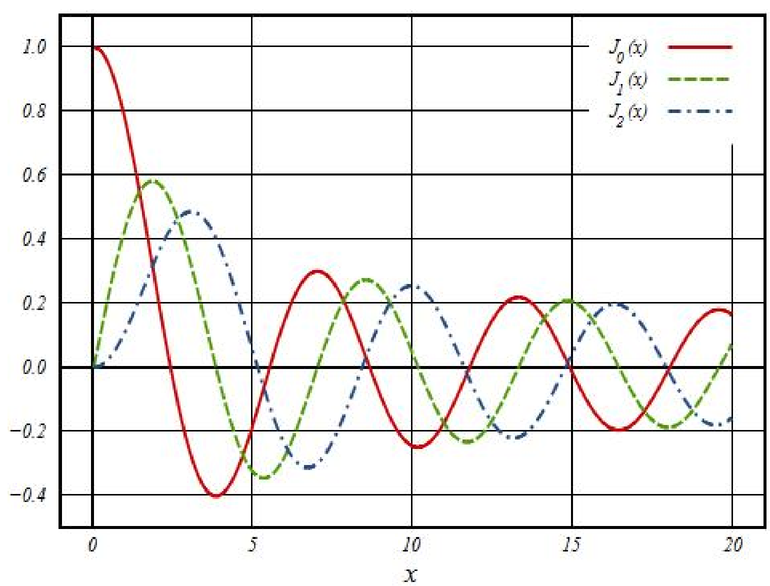

Fig. 1.1. Bessel functions of the first kind of order $n = 0, 1, 2$

In fig. 1.1 are reported Bessel functions of the first kind of order $n = 0, 1, 2$. As the graphic shows, all Bessel functions for $x = 0$ are equal to zero, unless the Bessel functions of order zero that have value equal to 1, and have oscillations around the value zero of decreasing amplitude.

1.4 Integral representations of Bessel functions

Bessel functions of the first kind, for integer values of n, can be also represented in the following integral form that was introduced by Bessel himself [22] as definition of $J_n(z)$

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - z\sin\theta)d\theta. \tag{1.23}$$

Clearly, this kind of representation allows to apply the properties of integrals and it is by those properties that derive a lot of properties of Bessel functions. In literature there are several integral representations of those functions [3] and the same eq. (1.23) can be expressed in the following equivalent forms:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)d\theta, \tag{1.24}$$

and

$$J_n(z) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} \cos(n\theta - z\sin\theta)d\theta, \tag{1.25}$$

where α is an arbitrarily chosen angle.
Moreover, there are also the two following representations introduced by Jacobi[23], that holds respectively for odd and even values of n:

$$J_n(z) = \frac{1}{2\pi}(-1)^{\frac{1}{2}(n-1)}\int_0^{\frac{1}{2}\pi} \cos n\eta \sin(z\cos\eta)d\eta \qquad (n \quad odd), \tag{1.26}$$

and

$$J_n(z) = \frac{1}{2\pi}(-1)^{\frac{1}{2}n}\int_0^{\frac{1}{2}\pi} \cos n\eta \cos(z\cos\eta)d\eta \qquad (n \quad even). \tag{1.27}$$

To represent Bessel functions in the case of complex order there is an alternative integral form given by Schläfli[24, 25, 26]:

$$J_\nu(z) = \frac{1}{\pi}\int_0^{\pi} \cos(\nu\theta - z\sin\theta)d\theta - \frac{\sin\nu\pi}{\pi}\int_0^{\infty} e^{-\nu t - z\sinh t}dt, \tag{1.28}$$

that holds for $|argz| < \frac{1}{2}\pi$, this form is known as generalization of Schläfli of the Bessel integral.
An other integral representation for n integer, given by Poisson [27] is the following one:

$$J_n(x) = \frac{(\frac{1}{2}z)^n}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(z\cos\theta)\sin^{2n}\theta d\theta, \tag{1.29}$$

that after standard manipulations can be expressed as

$$J_n(x) = \frac{(\frac{1}{2}z)^n}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{iz\cos\theta}\sin^{2n}\theta d\theta, \tag{1.30}$$

from which it follows

$$J_n(x) = \frac{(\frac{1}{2}z)^n}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{izt}(1-t^2)^{n-\frac{1}{2}} dt. \tag{1.31}$$

From the integral forms given by Poisson, Lommel [28] found the following expression of the $J_\nu(z)$ that holds for each order $\nu \in \mathbb{C}$

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 t^{\nu-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\cos\{z(1-t)^{\frac{1}{2}}\}dt. \tag{1.32}$$

Taking into account eq. (1.32) by simple manipulations it is possible to write the following equivalent forms:

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cos(zt)dt, \tag{1.33}$$

and

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}\cos(zt)dt, \tag{1.34}$$

and

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}e^{izt}dt, \tag{1.35}$$

and

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{1}{2}\pi} \cos(z\cos\theta)\sin^{2\nu}\theta d\theta, \tag{1.36}$$

and

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{iz\cos\theta}\sin^{2\nu}\theta d\theta. \tag{1.37}$$

Recently the following integral representation [29] of Bessel functions has been introduced:

$$J_\nu(x) = \frac{2^{\nu-2}\Gamma(\nu+\frac{5}{2})}{\pi^{\frac{3}{2}i}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{B(\frac{\nu+s}{2},\frac{\nu-s+5}{2})\int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{t^{(\nu-s+5)/2}}dt}{\Gamma(\nu-s+2)\zeta(\nu-s+2)x^s} ds, \quad (1.38)$$

where $\zeta(\cdot)$ and $B(\cdot,\cdot)$ are the zeta function of Riemann and the beta function (also known as eulerian integral of the first kind) respectively:

1.5 Main properties

Bessel functions have several useful properties [3].
For integer values of n it holds:

$$J_{-n}(x) = (-1)^n J_n(x) \quad (1.39)$$

and

$$J_n(-x) = (-1)^n J_n(x). \quad (1.40)$$

There are also the following recurrence formulas

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad (1.41)$$

and

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x), \quad (1.42)$$

which connect three contiguous Bessel functions.
Adding and subtracting eq. (1.41) and eq. (1.42) the following properties can be written

$$zJ_n'(z) + nJ_n(z) = zJ_{n-1}(z) \quad (1.43)$$

and

$$zJ_n'(z) - nJ_n(z) = -zJ_{n+1}(z). \quad (1.44)$$

Considering in these expressions ν instead of n, it is clear that they hold the equivalent properties generalized in the complex field. The feasibility of this generalization has been proved by Lommel[28].
The square of Bessel functions can be expressed in integral form [3] as

$$J_n^2(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} J_{2n}[2x\cos(\phi)]d\phi. \quad (1.45)$$

1.6 Connections with other functions

In this section some relationships between Bessel function and other special functions have been considered.
Bessel functions can be expressed throughout hypergeometric series [2]as

$$J_\alpha(x) = \frac{(x/2)^\alpha}{\Gamma(\alpha+1)}\, {}_0F_1(\alpha+1; -x^2/4) \tag{1.46}$$

where the hypergeometric series is defined as

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!} \tag{1.47}$$

where

$$(a)_n = a(a+1)(a+2)...(a+n-1),\ (a)_0 = 1 \tag{1.48}$$

is the Pochhammer symbol[126].
Bessel functions are also connected to the generalized Laguerre polynomials [127] defined as

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}. \tag{1.49}$$

In fact it holds the following eq.

$$\frac{J_\alpha(x)}{\left(\frac{x}{2}\right)^\alpha} = \frac{e^{-t}}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}\left(\frac{x^2}{4t}\right)}{\binom{k+\alpha}{k}} \frac{t^k}{k!}. \tag{1.50}$$

Note that the hypergeometric series as well as the generalized Laguerre polynomial have relations with other special functions. So, from those connections ensue other connections between Bessel functions and other special functions. For example, taking into account the following expression

$$Li_2(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2} = {}_3F_2(1,1,1;2,2;x)x, \tag{1.51}$$

that links the hypergeometric series and the dilogarithm $Li_2(x)$, that is the particular case for $s=2$ of the Jonquiére function $Li_s(x)$ [58] (also known as polylogarithm) defined as

$$Li_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}, \tag{1.52}$$

by eq. (1.46) it is possible to connect Bessel functions with the dilogarithm.

1.7 Bessel functions of the second kind and Hankel functions

Bessel equation (1.1) is a differential equation of the second order. For this reason, it has a general solution that is a linear combination of two particular linearly independent solutions. An other particular solution is given by $J_{-\nu}(x)$. But it holds eq.(1.39) and then $J_n(x)$ and $J_{-n}(x)$ are linearly dependent functions. However, an other particular solution that is linearly independent from $J_n(x)$ and which is useful to have a general solution is given by Bessel functions of the second kind $Y_\nu(x)$ defined as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{1.53}$$

Another couple of linearly independent solutions of the Bessel equation is given considering the Hankel functions (also known as Bessel function of the third kind) defined as

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x). \end{aligned} \tag{1.54}$$

2

Kapteyn series

2.1 Introduction and historical background

Each series of the kind

$$\sum_{n=0}^{\infty} \alpha_n J_{\nu+n}\{(\nu+n)z\}, \tag{2.1}$$

in which $J_m(x)$ is the Bessel function of the first kind and where α_n, ν e z $\in C$ and coefficients α_n e ν are independent on z, is called a Kapteyn series. They were named after the mathematician Willem Kapteyn, who first [30] systematically investigated them and examined the possibility of expanding an arbitrary analytic function into a Kapteyn series.
Such series are characterized by their own complexity that arise from a background of poor theories. Indeed the properties of these series not only use a not trivial mathematics, but also they are in a very exiguous number [3]. Although, as known these series converge, for very few of them there are explicit closed forms and moreover it is well known that such series suffer from problems of numerical convergence. This happens because sometimes it is necessary to evaluate a hight number of terms in order to have acceptable approximation errors but also because the evaluation of the single term can be slow involving a Bessel function that, as said in chapter 1, can produce computational problems.
Many authors have proposed methods to improve convergence speed as in [31] where Wynn's epsilon method [32] is used to accelerate the convergence of these series.
To sum up, on the one hand to face problems in which Kapteyn series are involved, can be an onerous matter and on the other hand they have an extremely practical importance in many applications in several different fields [3].
An interesting approach seems to be the one which represents Kapteyn series in integral forms. In [31, 33, 34, 35] integral representations are given for

Kapteyn series of specific problems, whereas, in [36] the problem of finding an integral representation is treated over a more general class of Kapteyn series. The history of these series is related to the problem of the elliptic motion in the mechanics of the bodies, in particular it is related to Kepler's problem. The solution of this problem, introduced by Lagrange [8] was manipulated after an half century by Bessel, obtaining a solution in terms of Kapteyn series.

Let

$$M = E - \epsilon \sin(E), \tag{2.2}$$

the second Kepler's equation [37], in which

- $\epsilon \in (0,1]$ represents the eccentricity of the ellipse;
- E is the eccentric anomaly, defined as the angle between the major axis of the ellipse and the line that links the center of the ellipse and the planet;
- $M \in (0,\pi)$ is the mean anomaly that is the product between the mean speed of rotation of the planet and the time of the last pass at the perihelion.

The Kepler problem consists in solving eq. (2.2) for E. Its solution in terms of Kapteyn series is

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\epsilon) \sin(nM). \tag{2.3}$$

Related to Kepler's problem, we find also the following Kaptyn series:

$$\cos(E) = -\frac{1}{2}\epsilon + \sum_{n=1}^{\infty} \frac{2}{n} J'_n(n\epsilon) \cos(nM), \tag{2.4}$$

and

$$\cos(mE) = m \sum_{n=1}^{\infty} \frac{1}{n} \{J_{n-m}(n\epsilon) - J_{n+m}(n\epsilon)\} \cos(nM) \quad m = 2, 3, .., \tag{2.5}$$

and

$$\sin(mE) = m \sum_{n=1}^{\infty} \frac{1}{n} \{J_{n-m}(n\epsilon) + J_{n+m}(n\epsilon)\} \sin(nM) \quad m = 1, 2, ... \tag{2.6}$$

Another series derived by Kepler equation and discovered by Herz[53] is the following one:

$$\frac{a}{r} = 1 + 2 \sum_{n=1}^{\infty} J_n(n\epsilon) \cos(nM), \tag{2.7}$$

where a is the major semi-axis of the ellipse and r is the radius vector, e.g. the distance of the point of the ellipse where is set the moving planet from the focus of the ellipse representing the sun.

2.2 Applications

Apart from the astronomical field, Kapteyn series are a toolbox frequently used in many applications of several disciplines, as for example in the modern theory of electromagnetic radiation [38, 39, 40, 31].
They are also related to Markov chains and then they have applications in the queueing theory [41, 42].
Kapteyn series occur also studying atmospheric phenomenas and more precisely in models of the energy spectrum of fronts [43, 44, 45, 46].
In dynamics of fluids Kapteyn series have a key role in the theory of the so called 'Concentrated vortices' that is space-localized zone with non-zero vorticity surrounded by potential flow [47]. More precisely, Kapteyn series appear in modeling the behaviour of Kelvin waves that are generated in concentrated vortices and that are cause of perturbations.
Kapteyn series play also an important role in problems of instability in plasmas [48].
Then they appear in models of spectra in optical regime of quantum modulated by magnetic fields at frequencies of order of teraherz [49, 50] and in any problems of conduction in semiconductors with a superlattice.
Another interesting applicative scenario concerns the spectral analysis in DC-AC conversion [51], where throughout an isomorphism with Kepler's problem, analytic expressions of switching instants of the inverter can be written in terms of Kapteyn series. Furthermore, in the same context these series are used for the analysis of the so called 'ripple stability' in PWM feedback control system [52].

2.3 Special Kapteyn series.

Kapteyn series, whose explicit closed form is known, are really in an exiguous number. In [53] Herz systematically studied some Kapteyn series giving for them analytic closed form. Among his results, there are the following ones:

$$\frac{1}{2}+\frac{1}{4}\epsilon=\sum_{n=1}^{\infty}\frac{J_n'(n\epsilon)}{n}, \quad \frac{1}{2}-\frac{1}{4}\epsilon=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{J_n'(n\epsilon)}{n}, \tag{2.8}$$

and

$$\frac{\frac{1}{2}\epsilon}{1-\epsilon} = \sum_{n=1}^{\infty} J_n(n\epsilon), \quad \frac{\frac{1}{2}\epsilon}{1+\epsilon} = \sum_{n=1}^{\infty}(-1)^{n-1} J_n(n\epsilon), \tag{2.9}$$

and

$$\frac{\frac{1}{2}}{(1-\epsilon)^2} = \sum_{n=1}^{\infty} nJ'_n(n\epsilon), \quad \frac{\frac{1}{2}}{(1+\epsilon)^2} = \sum_{n=1}^{\infty}(-1)^{n-1} nJ'_n(n\epsilon). \tag{2.10}$$

Other researchers, apart from Herz, were interested in Kapteyn series and in the problem of finding analytic forms too. Among them, there is Meissel [54, 55] that for $m = 1, 2, 3, 4, 5$ found the closure of the following series

$$\sum_{n=1}^{\infty} \frac{J_{2n}(2n\epsilon)}{n^{2m}}, \tag{2.11}$$

and

$$\sum_{n=1}^{\infty} \frac{J_{2n-1}(2n-1)\epsilon}{(2n-1)^{2m}}. \tag{2.12}$$

Eq. (2.11) can be expressed in a polynomial form of degree $2m$. Here there are the values obtained by Meissel for $m = 1, 2, 3$

$$\begin{array}{lc} m=1 & \frac{\epsilon^2}{2} \\ m=2 & \frac{\epsilon^2}{2} - \frac{\epsilon^4}{8} \\ m=3 & \frac{\epsilon^2}{2} - \frac{5\epsilon^4}{32} + \frac{\epsilon^6}{72} \end{array} \tag{2.13}$$

And eq. (2.12) can be expressed in a polynomial form of degree $2m-1$, for $m = 1, 2, 3$ there are the following values:

$$\begin{array}{lc} m=1 & \frac{\epsilon}{2} \\ m=2 & \frac{\epsilon}{2} - \frac{\epsilon^3}{18} \\ m=3 & \frac{\epsilon}{2} - \frac{5\epsilon^3}{81} + \frac{\epsilon^5}{450}. \end{array} \tag{2.14}$$

The following one is a result of Schott [38]:

$$\sum_{n=1}^{\infty} n^2 J_{2n}(2n\epsilon) = \frac{\epsilon^2(1+\epsilon^2)}{2(1-\epsilon^2)^4}. \tag{2.15}$$

Kapteyn [30] demonstrated that the following eq.

$$1 + 2\sum_{n=1}^{\infty} J_n(nz) = \frac{1}{1-z} \tag{2.16}$$

holds for z complex such that

$$\left| \frac{z \exp(\sqrt{1-z^2})}{1+\sqrt{1-z^2}} \right| < 1. \tag{2.17}$$

or equivalently $-1 < z < 1$, if z is real.

2.4 Integral representations of any Kapteyn series involving specific problems

In literature there are many integral representations of Kapteyn series related to specific contexts .
Meissel [54] gave the following representation

$$2\sum_{n=1}^{\infty} \frac{J_{2n}(2n\epsilon)}{n^2+\xi^2} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left\{ \frac{\pi \cosh 2\xi\theta \cosh(2\epsilon\xi\cos\theta)}{\xi \sinh \pi\xi} - \frac{1}{\xi^2} \right\} d\theta \tag{2.18}$$
$$\xi \quad \in \mathbb{R} \qquad 0 < \epsilon \leq 1.$$

A significant work has been done on the Kepler problem.
In [33] eq. (2.2) is rewritten in integral form as

$$E = M + c \tag{2.19}$$

where

$$\begin{aligned} c = \pm\frac{1}{\pi} \Bigg\{ & \int_{-\epsilon}^{\epsilon} \Bigg[\frac{\rho}{f^2+\rho^2} \Bigg(xf' - f - \frac{f''}{2f'^2} f^2 + \frac{f'''f' - 3f''^2}{6f'^4} f^3 + \\ & - \frac{f'^2 - 10f'f''f''' + 15f''^3}{24f'^6} f^4 \Bigg) + \\ & + \frac{f'^3 f''''' - 15f'^2 f'' f'''' + 105 f' f''^2 f''' - 10 f'^2 f'''^2 - 105 f''^4}{24 f'^8} f^4 \arctan\left(\frac{\rho}{f}\right) \Bigg] dx + \\ & + \Bigg[\Bigg(x - \frac{f}{f'} - \frac{f''}{2f'^3} f^2 + \frac{f'''f' - 3f''^2}{6f'^5} f^3 + \\ & - \frac{f'^2 f'''' - 10 f' f'' f''' - 15 f''^3}{24 f'^7} f^4 \Bigg) \arctan\left(\frac{\rho}{f}\right) \Bigg]_{-\epsilon}^{\epsilon} \Bigg\} \end{aligned} \tag{2.20}$$

where we can consider the operator $[f(x)]_a^b = f(a) - f(b)$, where ρ is an arbitrary positive constant, and where $f = x - \epsilon\sin(x+M)$, $f' = 1 - \epsilon\cos(x+M)$, $f'' = -f'''' = \epsilon\sin(x+M)$ and $f''' = -f''''' = \epsilon\cos(x+M)$.
In [35] eq. (2.2) if $\frac{\pi}{2} - \epsilon < M < \pi$ or equivalently $\frac{\pi}{2} < E < \pi$ are given the following integral representations, the first one is

$$E = M + (\pi - M)\exp\left\{ \frac{1}{\pi} \int_0^{\infty} \arctan\left[\frac{2x\left(\epsilon\cosh x + \frac{\pi}{2}\right)\tanh x}{(\pi-M)^2 + x^2 - \left(\epsilon\cosh x + \frac{\pi}{2}\right)^2} \right] dx \right\} \tag{2.21}$$

that is a simplification of the solution in[34] and the other one is

$$E = M + \frac{V\left(\frac{3}{2}\pi - M, \epsilon\right) - V\left(M - \frac{1}{2}\pi, \epsilon\right)}{\left[L\left(\frac{3}{2}\pi - M, \epsilon\right) + L\left(M - \frac{1}{2}\pi, \epsilon\right)\right]^3}, \tag{2.22}$$

where

$$V(\alpha, \epsilon) = -\frac{\delta}{\delta\alpha} L(\alpha, \epsilon) = \frac{2}{\pi} \int_0^\infty \frac{(\epsilon \cosh x + \alpha)^2 - x^2}{[(\epsilon \cosh x + \alpha)^2 + x^2]^2} dx. \tag{2.23}$$

In the context of the modern theory of electromagnetic radiation [31] are provided integral representations of the following Kapteyn series

$$F^+ = \sum_{n=1}^\infty \frac{J_n^2(nb)}{n} \qquad , \tag{2.24}$$

and

$$F^- = \sum_{n=1}^\infty (-1)^n \frac{J_n^2(nb)}{n}, \tag{2.25}$$

that are respectively

$$F^+ = -\frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} d\theta \ln \left[\frac{\sin^2(\theta - b\cos\phi\sin\theta)\sin^2(\theta + b\cos\phi\sin\theta)}{\sin^4\theta} \right] \tag{2.26}$$

and

$$F^- = -\frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} d\theta \ln \left[\frac{\sin^2(\theta - b\cos\phi\cos\theta)\sin^2(\theta + b\cos\phi\cos\theta)}{\sin^4\theta} \right]. \tag{2.27}$$

In the next chapter a more systematic approach that allows to face the problem of finding an integral representation of a class of Kapteyn series, is illustrated. Considering them as particular cases, we are going to examine the Kapteyn series of the solution of the Kepler problem and those in eqs. (2.24) and (2.25).

3

Integral representation of a class of Kapteyn series

In this chapter an approach that allows to obtain an integral representation of a class of Kapteyn series is presented. This approach is based on the property of uniform convergence of the considered series and on the rapresentation of the Bessel function by an integral form. This class of series permits not only to consider Kapteyn series occurring in several useful application but also the solution of the Kepler problem.

3.1 Uniform convergent series

In this section we are going to discuss the definition and some concepts of uniform convergent series that will be useful later on (see [56] for a more comprehensive treatment).
Let consider the series

$$\sum_{n=1}^{\infty} u_n(z) \tag{3.1}$$

and let

$$S_n(z) = \sum_{i=1}^{n} u_i(z); \tag{3.2}$$

given $\epsilon > 0$, the series converges if n exists such that for each $p > 0$

$$|S_{n+p}(z) - S_n(z)| < \epsilon; \tag{3.3}$$

and if in addition the value of n is independent from z the series is uniformly convergent [56].

Among the criteria of uniform convergence there is the Hardy's test: the series

$$\sum_{n=1}^{\infty} a_n(z) f_n(z) \tag{3.4}$$

converges uniformly if $\left|\sum_{n=1}^{p} a_n(z)\right| \leq k$ where $a_n(z)$ is real, k is limited and independent from p and z, and if $f_n(z) \geq f_{n+1}(z)$ and $f_n(z) \to 0$ uniformly if $n \to \infty$.
A property of uniformly convergent series regards the possibility of changing the order of the operators of integration and summation. To be more precise, if the series in eq. (3.1) converges uniformly in the interval (a, b) and let $x \in (a, b)$, the following relationship holds:

$$\sum_{n=1}^{\infty} \int_{a}^{x} u_n(z) dz = \int_{a}^{x} \sum_{n=1}^{\infty} u_n(z) dz. \tag{3.5}$$

3.2 Considered class of Kapteyn series

The considered class of Kapteyn series is the following one

$$\gamma(a, b, c, d, g, s, \epsilon, \nu, \phi) = \sum_{n=1}^{\infty} \frac{a^n}{(dn + b)^s} J_{gn+\nu} \left[(gn + \nu)\, \epsilon\right] \sin\left(2\pi nc + \phi\right). \tag{3.6}$$

As regards the Bessel function of the first kind $J_n(z)$ it is considered the integral representation in eq. (1.24) that allows to rearrange the series in (3.6) as

$$\gamma = \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{0}^{\pi} \frac{a^n}{(dn + b)^s} \cos\left[(gn + \nu)\left(\theta - \epsilon \sin(\theta)\right)\right] \sin\left(2\pi nc + \phi\right) d\theta \tag{3.7}$$

where the dependence of γ on parameters is omitted for ease of notation.
Eq. (3.7) can be rewritten as

$$\gamma = \frac{1}{\pi} \int_{0}^{\pi} \sum_{n=1}^{\infty} \frac{a^n}{(dn + b)^s} \cos\left[(gn + \nu)\left(\theta - \epsilon \sin(\theta)\right)\right] \sin\left(2\pi nc + \phi\right) d\theta \tag{3.8}$$

since the series on the right is uniformly convergent.

Lemma 3.1. *The series*

$$\tilde{\gamma} = \sum_{n=1}^{\infty} \frac{a^n}{(n + b)^s} \cos\left[(gn + \nu)\left(\theta - \epsilon \sin(\theta)\right)\right] \sin\left(2\pi nc + \phi\right) \tag{3.9}$$

is uniformly convergent for s integer, $s \geq 1$, $|a| \leq 1$, $b \geq 0$ and

$$2\delta \leq 2\pi c - g\theta + \epsilon g \sin(\theta) \leq 2\pi - 2\delta, \tag{3.10}$$

$$2\delta \leq 2\pi c + g\theta - \epsilon g \sin(\theta) \leq 2\pi - 2\delta. \tag{3.11}$$

where δ is an arbitrary small positive constant.

Proof. The proof follows by using the Hardy's test. Eq. (3.9) can be rewritten by algebraic manipulations as

$$\tilde{\gamma} = \sigma_1 + \sigma_2 \tag{3.12}$$

with

$$\sigma_i = \sum_{n=1}^{\infty} \frac{a^n}{(n+b)^s} \sin\left[n\alpha_i + \beta_i\right] \qquad i = 1, 2 \tag{3.13}$$

where

$$\alpha_1 = 2c\pi - g\theta + \epsilon g \sin(\theta) \qquad \beta_1 = \phi - \theta\nu + \epsilon\nu \sin(\theta), \tag{3.14}$$

and

$$\alpha_2 = 2c\pi + g\theta - \epsilon g \sin(\theta) \qquad \beta_2 = \phi + \theta\nu - \epsilon\nu \sin(\theta). \tag{3.15}$$

By considering $f_n(z) = \frac{a^n}{(n+b)^s}$, the conditions on $f_n(z)$ function hold if $|a| \leq 1$ and $b \geq 0$ with s integer, $s \geq 1$. Taking into account the following equality [121]

$$\sum_{\xi=1}^{p} \sin(x + \xi y) = \sin\left(x + \frac{p+1}{2} y\right) \sin\left(\frac{py}{2}\right) cosec\left(\frac{y}{2}\right). \tag{3.16}$$

it is straightforward to obtain

$$\left|\sum_{n=1}^{p} \sin\left[n\alpha_i + \beta_i\right]\right| \leq \left|cosec\left(\frac{\alpha_i}{2}\right)\right| \leq \left|cosec\left(\delta\right)\right|, \qquad i = 1, 2 \tag{3.17}$$

from which the proof follows.

The following theorem gives an integral representation for the series in eq. (3.8).

Theorem 3.2. *Let $\gamma_G(\theta)$ be the function defined as*

$$\begin{aligned}
\gamma_G(\theta) = \jmath \frac{a}{4d^s} \Big\{ & -e^{\jmath[\alpha_1(\theta)+\beta_1(\theta)]} \Phi\left[ae^{\jmath\alpha_1(\theta)}, s, \frac{b}{d}+1\right] + \\
& + e^{-\jmath[\alpha_1(\theta)+\beta_1(\theta)]} \Phi\left[ae^{-\jmath\alpha_1(\theta)}, s, \frac{b}{d}+1\right] + \\
& - e^{\jmath[\alpha_2(\theta)+\beta_2(\theta)]} \Phi\left[ae^{\jmath\alpha_2(\theta)}, s, \frac{b}{d}+1\right] + \\
& + e^{-\jmath[\alpha_2(\theta)+\beta_2(\theta)]} \Phi\left[ae^{-\jmath\alpha_2(\theta)}, s, \frac{b}{d}+1\right] \Big\}
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \alpha_1(\theta) &= 2c\pi - g\theta + \epsilon g \sin(\theta), \\ \alpha_2(\theta) &= 2c\pi + g\theta - \epsilon g \sin(\theta), \\ \beta_1(\theta) &= \phi - \theta\nu + \epsilon\nu \sin(\theta), \\ \beta_2(\theta) &= \phi + \theta\nu - \epsilon\nu \sin(\theta), \end{aligned} \tag{3.19}$$

and $\Phi[\cdot,\cdot,\cdot]$ *is the Lerch transcendent zeta function [57] defined as*

$$\Phi(z,t,h) = \sum_{n=0}^{\infty} \frac{z^n}{(n+h)^t} \tag{3.20}$$

and $\jmath$ *is the imaginary unit.*
The series in eq. (3.8) *is equivalent to the following integral form*

$$\gamma = \frac{1}{\pi} \int_0^{\pi} \gamma_G(\theta) d\theta \tag{3.21}$$

for $Re\left[\frac{b}{d}+1\right] > 0$ *and one of the following conditions satisfied:*

- $|a| \leq 1,\ ae^{\jmath\alpha_i(\theta)}\big|_{i=1,2} \neq 1,\ Re[s] > 0,$
- $ae^{\jmath\alpha_1(\theta)} = 1,\ Re[s] > 1$ *or* $ae^{\jmath\alpha_2(\theta)} = 1,\ Re[s] > 1.$

Proof. By applying the Euler's formula and by standard algebraic manipulations eq. (3.8) can be rewritten as

$$\gamma = \frac{1}{4\pi d^s}\jmath \int_0^{\pi} \sum_{n=1}^{\infty} \frac{a^n}{\left(n+\frac{b}{d}\right)^s} \Big\{ e^{-\jmath[n\alpha_1(\theta)+\beta_1(\theta)]} - e^{\jmath[n\alpha_1(\theta)+\beta_1(\theta)]} + \\ + e^{-\jmath[n\alpha_2(\theta)+\beta_2(\theta)]} - e^{\jmath[n\alpha_2(\theta)+\beta_2(\theta)]} \Big\} d\theta. \tag{3.22}$$

Taking into account eq. (3.22), the proof follows by the definition of the transcendental Lerch function in (3.20) and by the following relationship

$$\sum_{n=1}^{\infty} \frac{z^n}{(n+h)^t} = z\Phi(z,t,h+1). \tag{3.23}$$

Hypotheses on parameters s, b, d, a and ϵ are related to the following conditions that assure $\Phi(z,t,h)$ to be analytic, i.e: $Re[h] > 0$ and either $|z| \leq 1$, $z \neq 1$, $Re[t] > 0$ or $z = 1$, $Re[t] > 1$ [57].

Remark 3.3. Note that if $s = 1$ and there is a value of $\theta \in (0,\pi)$ such as $ae^{\jmath\alpha_1(\theta)} = 1$ or $ae^{\jmath\alpha_2(\theta)} = 1$ then the function $\gamma_G(\theta)$ has a singular point in θ.

3.3 Relationship between the transcendental Lerch function $\Phi(z, t, h)$ and other functions

The following properties will allow to simplify the obtained integral representation in particular cases of applications that are going to be considered in the next chapters. In eq. (3.25) we can notice the connection between the function $\Phi(z, t, h)$ and the Jonquiére function $Li_s(x)$ [58], commonly called the polylogarithm function, defined in eq. (1.52)

$$Li_s(z) = z\Phi(z, s, 1). \tag{3.24}$$

Furthermore, there is a relationship with the Bernoulli polynomials $B_s(z)$ [57] defined as

$$B_n(x) = \sum_{m=0}^{n} \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (x+k)^n. \tag{3.25}$$

through the following eq.

$$Li_s(e^{2\pi \jmath z}) + (-1)^s Li_s(e^{-2\pi \jmath z}) = -\frac{(2\pi \jmath)^s}{s!} B_s(z) \tag{3.26}$$

that holds for $s = 0, 1, ...$ and for $0 \leq Re[z] < 1$ if $Im[z] \geq 0$ or $0 < Re[z] \leq 1$ if $Im[z] < 0$.

4

Kepler's problem

In this chapter the approach described in the third chapter is used to obtain an integral representation of the solution of the Kepler problem and of the Kapteyn series related to the problem itself. The properties of the integrand are also analyzed and these properties allow to solve the problem via a bisection algorithm.

4.1 Introduction

Not so many time ago, it has been marked the 400th anniversary of the publication 'Astronomia Nova' [37] by Keplero. As mentioned in the second chapter the Kepler problem consists in solving eq. (2.2) in respect of the eccentric anomaly E given the mean anomaly M and the eccentricity of the elliptic orbit ϵ.
Kepler described the problem with these words [37]
At data anomalia media, nulla Geometrica methodus est, perveniendi ad coaequatam, videlicet ad anomaliam eccentri. Nam anomalia media est composita ex duabus areae partibus, sectore et triangulo: quorum ille quidem numeratur ab arcu eccentri; hoc, ab eius arcus sinu, in valorem trianguli maximi multiplicato resectis ultimis. At proportiones inter arcus et eorum sinus infinitae sunt numero. Itaque summa utriusque proposita, dici non potest, quantus sit arcus, quantus eius sinus, respondens huic summae; nisi prius exploremus, dato arcu, quanta evadat area: hoc est, nisi tabulas contruxeris, et ex iis postea operaris. Haec est mea sententia. Quae quo minus habere videbitur Geometricae pulchritudinis, hoc magis adhortor Geometras, uti mihi solvant hoc problema: Data area partis semicirculi, datoque puncto diametri, invenire arcum, angulum ad illud punctum, cuius anguli cruribus, et quo arcu, data area comprehenditur. Vel: Aream semicirculi ex quocunque puncto diametri in data ratione secare. Mihi sufficit credere, solvi a priori non posse, propter arcus et sinus ετερογένειαν. Erranti mihi, quicunque viam monstraverit, is erit mihi magnus Apollonius.

This means: "But given the mean anomaly there is not any geometric method to determine the eccentric one. In fact the mean anomaly is composed by two parts of area, a sector and a triangle: the former is measured from the eccentric arc; the latter by multiplying the sine of that arc and the area of the maximum triangle, and subtracting this to the sector. Infinite numbers are the proportions possible between sines and arcs. For this reason, given their sum, it is not possible to establish the value of the arc and that of the sine corresponding to the sum itself, unless we have already examined the increase of the area: in other words, unless you have made tables so to work with them. This is my opinion, and although it does not seem a geometric problem particularly attractive, now more than ever, I interrogate the geometrians so that they can give me the solution of this problem: Given the area of a part of the half circle and given a point of the diameter, you should find the arc and the angle having a vertex in that point so that the arc and the sides of the angle include the given area. To be more precise: you should divide the area of the half circle in a assigned ratio from a point of the diameter. According to me, it is sufficient to believe a priori that is not possible to solve the problem, because the arc and the sine are heterogeneous. I am wandering aimlessly, but if somebody shows me the way, he will be for me a New Apolonio.".

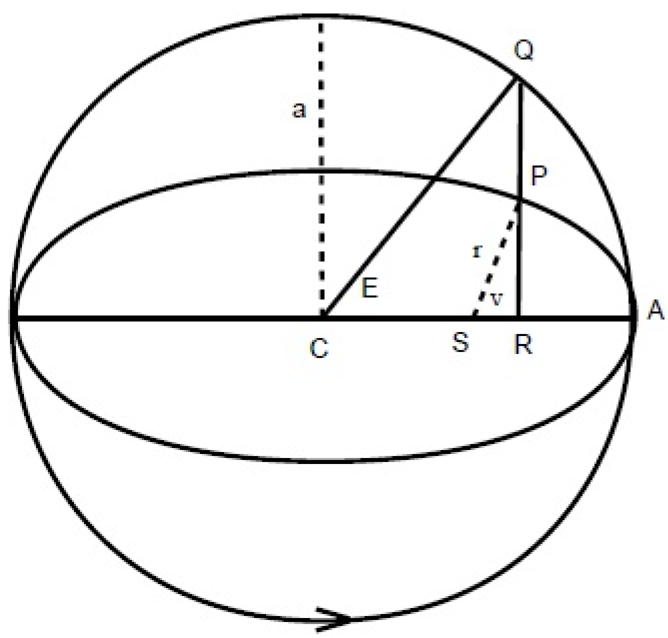

Fig. 4.1. Graphical representation of orbital elements.

The basic physical meaning of the Kepler equation is explained in fig. 4.1, in which an ellipse with eccentricity ϵ is shown, that is, the orbit of a body moving around the stationary gravitating center placed in the focus of the ellipse S. The center and the pericenter of the ellipse are marked with letters C and A, respectively. Moreover, a circle is constructed with its center coinciding in point C and radius equal to the major semiaxis of the ellipse. At the same time, let the position of the rotating body be determined by point P. From P, a perpendicular is marked to the major axis of the ellipse and foot of the perpendicular is marked with letter R. This perpendicular is extended until it intersects with the circle at point Q. Then, the angle $\angle ACQ$ is just the eccentric anomaly E. Suppose the planet P, having passed through perihelion A, is at position P after elapsed time t, it is possible to express the polar coordinates of P, (r, v), relative to the sun in terms of t, where r and v are the vector radius and the true anomaly respectively. Keplers equation relates E to time t by means of a quantity

$$M = \frac{2\pi t}{T}, \tag{4.1}$$

where T is the time required for the planet to complete one trip in its orbit around the sun. The quantity M represents the average angular speed of the radius vector SP. Classical formulas to find r and v are

$$r = a\left(1 - \epsilon\cos(E)\right), \tag{4.2}$$

and

$$\tan\left(\frac{v}{2}\right) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\left(\frac{E}{2}\right). \tag{4.3}$$

Therefore, if t and M are known, it is possible to solve the Kepler equation for E and to determine position (r, v) at time t using eq. (4.2) and (4.3).
The search of a solution of this equations has been a long time open problem, in [125] it is well reported the history of this research. It is worth noting that many proposed solutions, starting from the early solution given by Lagrange in 1770 (that Laplace demonstrated be correct only for $0 < \epsilon < 0.6627$)

$$E = M + \sum_{n=1}^{\infty} a_n(M) e^n \tag{4.4}$$

where

$$a_n(M) = \frac{1}{n!}\frac{d^{n-1}}{dM^{n-1}}\left[sin^n(M)\right], \tag{4.5}$$

are an application to Kepler's equation of more general techniques.
As said in the second chapter, a solution is given in terms of Kapteyn series (2.2). This solution was given by Kapteyn himself in his memoir in 1893 [30] where he studied the possibility to express function by series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(f) J_n(nx), \tag{4.6}$$

Taking into account $g(M) = E$ and expressing the quantity $g(M) - M$ in Fourier series

$$g(M) - M = \epsilon \sin(E) = \sum_{n=1}^{\infty} A_n \sin(nM), \tag{4.7}$$

where

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} [g(M) - M] \sin(nM) dM \\
&= -\frac{2}{n\pi}\int_0^{\pi} [g(M) - M]\, d\left(\cos(nM)\right) \\
&= -\frac{2}{n\pi}\left\{[g(M) - M]\cos(nM)\right\}_0^{\pi} + \\
&\quad + \frac{2}{n\pi}\int_0^{\pi} \cos(nM)\left[g^{'}(M) - 1\right] dM \\
&= \frac{2}{n\pi}\int_0^{\pi} \cos(nM) g^{'}(M) dM \\
&= \frac{2}{n\pi}\int_0^{\pi} \cos(nM) d(g(M)).
\end{aligned} \tag{4.8}$$

being $M = E - \epsilon \sin(E) = g(M) - \epsilon \sin(g(M))$, it follows

$$\begin{aligned}
A_n &= \frac{2}{n\pi}\int_0^{2\pi} \cos\left[ng(M) - n\epsilon \sin(g(M))\right] d(g(M)) \\
&= \frac{2}{n}\left\{\frac{1}{\pi}\int_0^{\pi} \cos\left(nE - n\epsilon \sin(E)\right)\right\} \qquad (4.9) \\
&= \frac{2}{n} J_n(n\epsilon). \qquad (4.10)
\end{aligned}$$

that allows to obtain the solution in terms of Kapteyn series in eq. (2.2). Moreover, in literature there are several numerical methods attempting to solve the Kepler equation, and they can be divided into three categories: iterative methods [59, 60, 61, 62], discretization methods [63, 64] and approximation-based methods [65, 66, 67, 68, 69, 70, 71]. This equation has been also managed by an analytic point of view, not only by the integral representations analyzed in the previous chapter but also through the Lambert hyperfunction [72].
Some properties of Kepler's equation

- $M(E) = E - \epsilon \sin(E)$ $(0 < \epsilon < 1)$ is a monotonic function; the Kepler equation admits a unique solution;

- $E - M$ is a 2π periodic function in E and then in M;
- E(M) is an odd function and for this it is possible to consider only the case $0 < M < \pi$ [35], $M = 0$ and $M = \pi$ are both trivial cases and give $E = 0$ e $E = \pi$ respectively.

4.2 Proposed solution

The following series

$$\sum_{n=1}^{\infty} \frac{2}{n} J_n(n\epsilon)\sin(nM) \tag{4.11}$$

that is involved in the solution of the Kepler problem in terms of Kapteyn series in eq. (2.2) is the particular case of the class of Kapteyn series considered in the third chapter in eq. (2.2). More precisely it is the case $\gamma(1, 0, \frac{M}{2\pi}, 1, 1, 1, \epsilon, 0, 0)$. Therefore, by using the integral representation in eq. (3.21), the definition of the polylogarithm in eq. (1.52) and his connection with the transcendental Lerch function in eq. (3.24), it is possible to obtain the following integral representation of the solution of the Kepler problem

$$E = M + \frac{\jmath}{2\pi}\int_0^{\pi} \Big\{ \ln[1 - e^{\jmath[\theta - \epsilon\sin(\theta) + M]}] - \ln[1 - e^{-\jmath[\theta - \epsilon\sin(\theta) + M]}] + \\ + \ln[1 - e^{-\jmath[\theta - \epsilon\sin(\theta) - M]}] - \ln[1 - e^{\jmath[\theta - \epsilon\sin(\theta) - M]}] \Big\}\, d\theta \tag{4.12}$$

where the relationship

$$Li_1(x) = -\ln(1 - x) \tag{4.13}$$

has been considered where $\ln(x)$ represents the natural logarithm of x.
Taking into account the fact that the logarithm of a complex number can be expressed as

$$\ln(z) = \ln|z| + \jmath\arg(z) \tag{4.14}$$

eq. (4.12) can be rewritten as

$$E = M + \frac{1}{\pi}\int_0^{\pi} \left\{ \arctan\left[\tan\left(\frac{\pi - M + \theta - \epsilon\sin(\theta)}{2}\right)\right] + \\ + \arctan\left[\tan\left(\frac{\pi - M - \theta + \epsilon\sin(\theta)}{2}\right)\right]\right\} d\theta. \tag{4.15}$$

Noting that

$$\frac{\pi - M - \theta + \epsilon\sin(\theta)}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{4.16}$$

eq. (4.15) becomes

$$E = \frac{\epsilon}{\pi} + \frac{1}{4}(2M + \pi) + \frac{1}{\pi} \int_0^\pi \arctan\left[\tan\left(\frac{\pi - M + \theta - \epsilon \sin(\theta)}{2}\right)\right] d\theta. \quad (4.17)$$

Further simplifications of this expression are not possible, because

$$\frac{\pi - M + \theta - \epsilon \sin(\theta)}{2} \in (0, \pi) \quad (4.18)$$

and so it is not possible to use the well known relationship [121] between the inverse and the direct trigonometric function

$$\arctan[\tan(x)] = x - n\pi, \quad nx - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}, \quad (4.19)$$

because the range of x change with θ.

4.3 Properties of the integrand

In this section the properties of the integrand of the solution of Kepler problem in eq. (4.15)

$$\begin{aligned} f(M, \epsilon, \theta) = \arctan\{\tan[\alpha_1(M, \epsilon, \theta)]\} + \\ + \arctan\{\tan[\alpha_2(M, \epsilon, \theta)]\}, \quad \theta \in [0, \pi] \end{aligned} \quad (4.20)$$

are analyzed, where:

$$\alpha_1(M, \epsilon, \theta) = \frac{1}{2}[\pi - M + \theta - \epsilon \sin(\theta)], \quad (4.21)$$

and

$$\alpha_2(M, \epsilon, \theta) = \frac{1}{2}[\pi - M - \theta + \epsilon \sin(\theta)]. \quad (4.22)$$

Given $M \in (0, \pi)$, $\epsilon \in (0, 1]$ and $\theta \in [0, \pi]$, it is easy to show that

$$\alpha_1(M, \epsilon, \theta) \in (0, \pi), \quad (4.23)$$

and

$$\alpha_2(M, \epsilon, \theta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (4.24)$$

Moreover:

$$\frac{df(M, \epsilon, \theta)}{d\theta} = 0, \quad \forall\theta, \quad (4.25)$$

and

$$M \in (0, \pi) \Rightarrow \begin{cases} f(M, \epsilon, 0) = \pi - M > 0, \\ f(M, \epsilon, \pi) = -M < 0. \end{cases} \quad (4.26)$$

Considering eq. (4.25) and (4.26) it is possible to state that $f(M, \epsilon, \pi)$ is a piecewise constant function having at least a discontinuity for $\theta \in [0, \pi]$. Furthermore, taking into account that arctan[tan(x)] has no discontinuity in $(-\pi/2, \pi/2)$ and admits a unique discontinuity in $x = \pi/2$ in the range $(0, \pi)$, the function $f(M, \epsilon, \pi)$ has a unique point of discontinuity in $\theta = \bar{\theta}$ such that

$$\alpha_1\left(M, \epsilon, \bar{\theta}\right) - \frac{\pi}{2} = 0. \tag{4.27}$$

From eq. (4.21), the condition in eq. (4.27) becomes:

$$M = \bar{\theta} - \epsilon \sin\left(\bar{\theta}\right). \tag{4.28}$$

Therefore, considering the generic triplet (M, ϵ, E) that satisfies Kepler's equation and eq. (4.28) it is possible to write the following equation:

$$E - \bar{\theta} = \epsilon\left[\sin(E) - \sin\left(\bar{\theta}\right)\right], \tag{4.29}$$

from which, it follows that the unique point of discontinuity of the function $f(M, \epsilon, \theta)$ coincides with the solution of the Kepler equation $\bar{\theta} = E$. Note that it is possible to reach the same consideration through Remark 3.3 in the previous chapter.
In fig. 4.2 the trend of the integrand respect to θ has been shown.

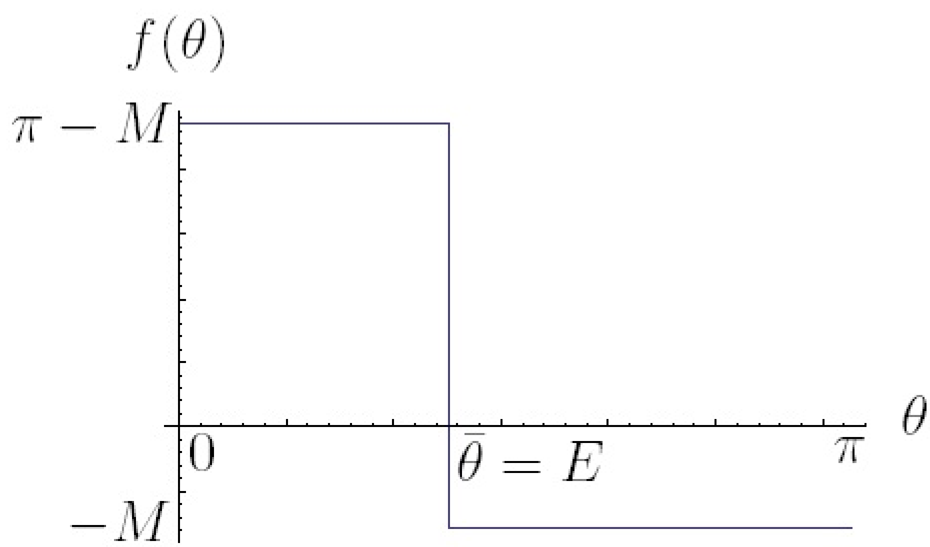

Fig. 4.2. Integrand in eq. (4.15)

The properties of the integrand permit to obtain a numeric solution through a bisection algorithm.

4.4 Integral representations of Kapteyn series related to Kepler's problem

Following the same approach used in the third chapter it is possible to obtain the integral representations of the Kapteyn series related to the Kepler equation, that are:

$$\cos(E) = -\frac{1}{2}\epsilon + \frac{1}{\pi}\int_0^\pi \sin(\theta)\Big\{\arctan\Big[\tan\Big(\frac{\pi - M - \theta + \epsilon\sin(\theta)}{2}\Big)\Big] + \\ - \arctan\Big[\tan\Big(\frac{\pi - M + \theta - \epsilon\sin(\theta)}{2}\Big)\Big]\Big\}d\theta \quad (4.30)$$

and

$$\cos(mE) = -\frac{m}{\pi}\int_0^\pi \sin(m\theta)\Big\{\arctan\Big[\tan\Big(\frac{\pi - M + \theta - \epsilon\sin(\theta)}{2}\Big)\Big] + \\ - \arctan\Big[\tan\Big(\frac{\pi - M - \theta + \epsilon\sin(\theta)}{2}\Big)\Big]\Big\}d\theta \quad (4.31)$$

and

$$\sin(mE) = \frac{m}{\pi}\int_0^\pi \cos(m\theta)\Big\{\arctan\Big[\tan\Big(\frac{\pi - M + \theta - \epsilon\sin(\theta)}{2}\Big)\Big] + \\ + \arctan\Big[\tan\Big(\frac{\pi - M - \theta + \epsilon\sin(\theta)}{2}\Big)\Big]\Big\}d\theta. \quad (4.32)$$

4.5 Numerical aspects

In this section, by using the software Mathematica 6 [131] two numerical experiments on the Kepler problem have been conducted.

4.5.1 Numerical example 1

Eq. (4.30) and (2.4) are equivalent but they are very different from a computational point of view.
In this example it has been chosen a "true" triplet $E_v = \frac{5\pi}{13}$, $\epsilon_v = 1$ e $M_v = \frac{5\pi}{13} - \sin\left(\frac{5\pi}{13}\right)$ and an estimation of quantity $\cos(E)_s$ has been evaluated considering both the Kapteyn series truncated at N terms and the integral representation (evaluated with machine precision and 32 digits precision). For both of them the relative error $\frac{cos(E)_s}{cos(E_v)} - 1$ has been considered.
In the following tables the results of the experiments have been reported.

N	Relative error
100	2.9000 − 3
200	1.2000 − 3
300	2.0000 − 4
400	2.0000 − 4
500	3.0000 − 4
600	1.0000 − 4
700	3.1565 − 5
800	1.1228 − 4
900	8.7421 − 5
1000	8.5207 − 6
2000	1.9702 − 6
3000	6.0981 − 7
4000	1.2966 − 7
5000	8.2830 − 8

Table 4.1. Obtained results by eq. (2.4).

Precision	Relative errors
Machine precision	3.2118 − 5
32 digits	1.4215 − 22

Table 4.2. Obtained results by eq. (4.30).

4.5.2 Numerical example 2

In Table 4.3, for each specified subinterval of M and ϵ, have been reported the maximum and the mean values on 10^4 random trials of the absolute errors between the estimated value of E, computed by 54 steps of bisection, and the true one. The value of 54 steps of bisection has been chosen, according to [**?**], in order to obtain a tolerance of 2^{-53} not considering the effect of rounding errors.
It is worth noting that the max errors occur in the boundary values of parameters. Indeed, as far as the maximum error is concerned, it is achieved in the sub-region $M \in \left(\frac{9\pi}{5}, 2\pi\right) \wedge \epsilon \in (0.9, 1)$,while the maximum values of the mean errors are achieved in the sub-regions $\mathbf{I}_1 = M \in \left(\frac{9\pi}{5}, 2\pi\right) \wedge \epsilon \in (0.9, 1)$ and $\mathbf{I}_2 = M \in \left(0, \frac{\pi}{5}\right) \wedge \epsilon \in (0.9, 1)$.

err_{max} err_{mean}	$M \in (0, \frac{\pi}{5})$	$M \in (\frac{\pi}{5}, \frac{2\pi}{5})$	$M \in (\frac{2\pi}{5}, \frac{3\pi}{5})$	$M \in (\frac{3\pi}{5}, \frac{4\pi}{5})$	$M \in (\frac{4\pi}{5}, \pi)$	$M \in (\pi, \frac{6\pi}{5})$	$M \in (\frac{6\pi}{5}, \frac{7\pi}{5})$	$M \in (\frac{7\pi}{5}, \frac{8\pi}{5})$	$M \in (\frac{8\pi}{5}, \frac{9\pi}{5})$	$M \in (\frac{9\pi}{5}, 2\pi)$
	4.44089 − 16	6.66134 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	1.77636 − 15	1.77636 − 15	1.77636 − 15
$\epsilon \in (0, 0.1)$	2.59302 − 16	3.11229 − 16	3.28715 − 16	2.40630 − 16	2.22444 − 16	2.23510 − 16	3.78853 − 16	4.48797 − 16	4.76064 − 16	4.79350 − 16
	4.44089 − 16	6.66134 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	1.77636 − 15	1.77636 − 15	1.77636 − 15
$\epsilon \in (0.1, 0.2)$	2.82916 − 16	3.23930 − 16	3.24918 − 16	2.40075 − 16	2.19380 − 16	2.19780 − 16	3.93552 − 16	4.56168 − 16	5.16742 − 16	5.70832 − 16
	4.99600 − 16	6.66134 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	1.77636 − 15	1.77636 − 15	1.77636 − 15
$\epsilon \in (0.2, 0.3)$	3.15755 − 16	3.35354 − 16	3.22387 − 16	2.35478 − 16	2.19247 − 16	2.19247 − 16	4.11493 − 16	4.62208 − 16	5.71987 − 16	6.46594 − 16
	5.55112 − 16	6.66134 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	1.77636 − 15	1.77636 − 15	1.77636 − 15
$\epsilon \in (0.3, 0.4)$	3.60329 − 16	3.58313 − 16	3.15303 − 16	2.34879 − 16	2.21512 − 16	2.22045 − 16	4.19531 − 16	4.71534 − 16	6.25988 − 16	7.27152 − 16
	6.66134 − 16	6.66134 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	8.88178 − 16	1.77636 − 15	1.77636 − 15	1.77636 − 15
$\epsilon \in (0.4, 0.5)$	4.12233 − 16	3.79097 − 16	3.13305 − 16	2.34079 − 16	2.22000 − 16	2.28839 − 16	4.11093 − 16	4.83880 − 16	6.75282 − 16	8.17657 − 16
	7.77156 − 16	6.66134 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	8.88178 − 16	1.77636 − 15	2.66454 − 15	2.66454 − 15
$\epsilon \in (0.5, 0.6)$	4.87672 − 16	3.96638 − 16	2.99227 − 16	2.29261 − 16	2.20624 − 16	2.35101 − 16	4.06208 − 16	5.05906 − 16	7.30083 − 16	9.47953 − 16
	8.88178 − 16	7.77156 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	8.88178 − 16	1.77636 − 15	2.66454 − 15	3.55271 − 15
$\epsilon \in (0.6, 0.7)$	6.11167 − 16	4.33387 − 16	3.05311 − 16	2.25819 − 16	2.17915 − 16	2.37721 − 16	4.07896 − 16	5.02798 − 16	7.93410 − 16	1.18785 − 15
	1.27676 − 15	9.99201 − 16	6.66134 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	8.88178 − 16	1.77636 − 15	2.66454 − 15	4.44089 − 15
$\epsilon \in (0.7, 0.8)$	8.23703 − 16	5.00489 − 16	3.09841 − 16	2.27551 − 16	2.20579 − 16	2.35456 − 16	3.95239 − 16	5.15055 − 16	8.72369 − 16	1.55440 − 15
	2.35922 − 15	1.33227 − 15	6.66134 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	8.88178 − 16	1.77636 − 15	3.55271 − 15	8.88178 − 15
$\epsilon \in (0.8, 0.9)$	1.27216 − 15	5.73142 − 16	3.07709 − 16	2.22866 − 16	2.23288 − 16	2.37321 − 16	3.92930 − 16	5.18696 − 16	9.97336 − 16	2.34950 − 15
	9.00063 − 13	1.77636 − 15	6.66134 − 16	4.44089 − 16	4.44089 − 16	8.88178 − 16	8.88178 − 16	1.77636 − 15	5.32907 − 15	1.03917 − 12
$\epsilon \in (0.9, 1)$	4.09682 − 15	6.86950 − 16	3.05844 − 16	2.23466 − 16	2.21467 − 16	2.31593 − 16	3.84226 − 16	5.16920 − 16	1.15410 − 15	7.88161 − 15

Table 4.3. Maximum and mean absolute errors of E

5

Spectral analysis of a class of DC-AC PWM inverters.

In this chapter an application of the Kapteyn series on the PWM modulation technique for single-phase bipolar switching inverters with a modulating signal that is a sinusoid of period multiple of the period of the triangular carrier, has been shown [51].
This is the most common PMW inverter, due to the fact that it improves the harmonic contents of the output waveform [88].
Particularly, it is shown how it is possible to analyze the outcoming signal spectrum of this technique through an isomorphism between the equation concerning the PWM switching instants and the Kepler equation.
The solution of the Kepler problem allows to obtain the switching instants, not only in form of Kapteyn series but also in integral form and closed expressions of THD (total harmonic distortion) and WTHD (weighted total harmonic distortion).

5.1 Introduction

The PWM (pulse-width modulation) has been and continues to be object of research, because it is a useful tool in a large application field [81], such as:

- the conversion of digital signals into analogical ones [82];
- the transmission of analogical signals through optical fibres [83];
- fluids control [84];
- the design of efficient power amplifiers [85];
- the robust closed-loop control of isometric muscle force [86];
- the converter switching instants control [87];
- the active filter planning [101];
- the nipple reduction of the incoming current in the inverter [102].

Although many works presented and recognized by the scientific community, have contributed to improve the classic approach performances, at the present

the aim is to reduce the harmonic distortion and to increase of the output magnitudes for a given switching frequency [88, 89, 90, 91, 94, 95, 96].
It is knowledge that the inverter performances, whichever switching strategy is, are linked to the harmonic contents of the outcoming signal. For this reason, it is important to conduct an in-depth study on the contents of this wave form and on its variation in reference to the variation of the operative parameters, such as the modulation index, or better the ratio between the amplitude of the modulating signal and of the carrier.
Although it is possible to run simulations which permit to evaluate the harmonic contents of the outcoming wave form, a theoretic solution is required. In [97] a new approach, which provides an analytic expression for the signal of PWM spectrum with natural or uniform sampling, in the case of single-edge or double-edge modulations, has been introduced. In the particular case of sinusoidal modulation, the results of this approach are the same to those obtained in the past, using the double Fourier series method [88].
The spectral analysis based on double Fourier series method, commonly used in the telecommunication system [100], is however mathematically complex [51], and therefore an alternative method may be important and it can provide further information on the harmonic contents of the outcoming signal.
In an ideal device, the output signal is perfectly sinusoidal, while actually we have a periodic signal with not null harmonics of superior orders.
To quantify the distortion caused by the unsought harmonics, it is necessary to introduce some performance indices, which measure the harmonic distortions. The total harmonic distortion (THD) is an important element, used for this aim. In literature it is possible to find several definition to it. According to the standard one [93] the harmonic contents of the signal is related to the fundamental, as the following expression shows:

$$THD = \sqrt{\frac{\sum_{j=2}^{\infty}(a_j^2 + b_j^2)}{a_1^2 + b_1^2}}, \tag{5.1}$$

where a_j e b_j, $j = 1, 2, ...$ are the coefficients of the Fourier series of the given signal.
In order to calculate this index, it is necessary to determine the spectrum of the given signal, but it is not so easy because of the nonlinearity involved in the system [97]. Furthermore, the TDH definition in eq (5.1) also means the sum of an infinite number of terms, but in the practical applications only a number of finished terms can be calculated. This one is commonly $j \leq 25$, however with an error of approximation [98]. In [99] it is shown how the number of the considered harmonics can influence the THD result.
In [88] it is underlined how THD cannot sometimes be a good index in considering a strategy better than the other one. Indeed it is shown that this index depends only on the RMS and on its fundamental and therefore it cannot distinguish two wave forms having the same fundamentals and RMS.

However THD is still an useful index for sizing filter requirements at the input side converter, since voltage distortion is an important criterion for specifying a clean voltage waveform [88].
An other important element is the index of the weight harmonic distortion (WTHD) [88], defined as

$$WTHD = \sqrt{\frac{\sum_{j=2}^{\infty} \frac{(a_j^2 + b_j^2)}{j^2}}{(a_1^2 + b_1^2)}}, \tag{5.2}$$

where the lower frequencies are weighted more heavily.

5.2 DC-AC Conversion

In the classic DC-AC conversion problem, the perfect case would be that in which we would obtain a sinusoidal signal with the desired frequency and amplitude. In the logical inverter DC-AC scheme in fig. 5.1, in order to obtain the desired signal, a sinusoidal control signal $m(t)$ at the desired frequency is compared with a carrier waveform $c(t)$. In the inverter, the switches are controlled by referring to the comparison of $m(t)$ and $c(t)$ and an ONOFF logic is implemented in the switches controller block.

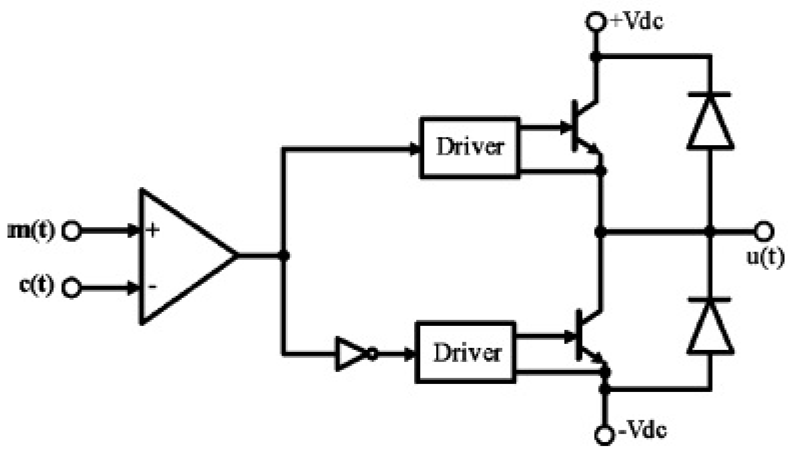

Fig. 5.1. Logical inverter DC-AC scheme

As already said, let us consider a triangular carrier with amplitude E_c and period T_c, defined as

$$c(t) = \begin{cases} E_c\left(\frac{4}{T_c}t - 4q - 1\right),\ qT_c \leq t \leq qT_c + \frac{T_c}{2}, \\ -E_c\left(\frac{4}{T_c}t - 4q - 3\right),\ qT_c + \frac{T_c}{2} < t < (q+1)T_c, \end{cases} \tag{5.3}$$

and a sinusoidal modulating signal with amplitude E_m,angular frequency $\omega_r = \frac{2\pi}{kT_c}$ ($k > 1$ integer) and phase ϕ, defined as

$$m(t) = E_m \sin\left(\frac{2\pi}{kT_c}t + \phi\right). \tag{5.4}$$

The inverter outcoming signal $u(t)$ is

$$u(t) = \begin{cases} V_{dc},\ m(t) \geq c(t), \\ -V_{dc},\ m(t) < c(t), \end{cases} \tag{5.5}$$

where $2V_{dc}$ is the value of the dc supply voltage. Eq. (5.5) can be rewritten as

$$u(t) = \begin{cases} V_{dc},\ qT_c \leq t \leq \tau_{1q} \text{ or } \tau_{2q} \leq t < (q+1)T_c, \\ -V_{dc},\ \tau_{1q} < t < \tau_{2q}. \end{cases} \tag{5.6}$$

with $q = 0, 1, ..., k-1$.
τ_{1q} e τ_{2q} are the switching instants uniquely determined by a logical comparison between the sinusoidal modulating signal $m(t)$ and the triangular carrier $c(t)$ as shown in fig. 5.2.
Taking into account a natural sampling logic, the switching instants τ_{1q} e τ_{2q} are determined by the following equations:

$$E_m \sin\left(\frac{2\pi}{kT_c}\tau_{1q} + \phi\right) = E_c\left(\frac{4}{T_c}\tau_{1q} - 4q - 1\right) \tag{5.7}$$

and

$$E_m \sin\left(\frac{2\pi}{kT_c}\tau_{2q} + \phi\right) = -E_c\left(\frac{4}{T_c}\tau_{2q} - 4q - 3\right). \tag{5.8}$$

With the classic approach based on double Fourier series it is not possible to determine in an analytic way the switching instants [104]. On the contrary, thanks to an interesting relation with the Kepler problem, it is possible to express the switching instants through Kapeyn series.
Defining the following quantities

$$\epsilon_i = (-1)^{i+1}\frac{\pi}{2k}\frac{E_m}{E_c}, \quad i = 1, 2, \tag{5.9}$$

and

$$E_{iq} = \frac{2\pi}{kT_c}\tau_{iq} + \phi, \quad i = 1, 2, \tag{5.10}$$

and

$$M_{iq} = \frac{\pi}{2k}(4q + 2i - 1) + \phi, \quad i = 1, 2 \tag{5.11}$$

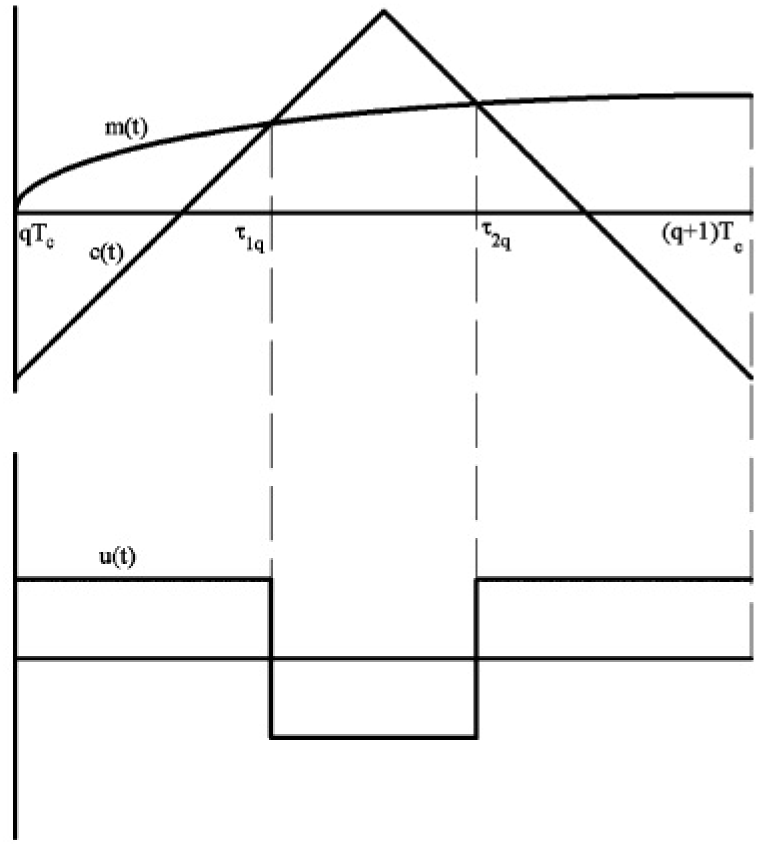

Fig. 5.2. $m(t)$, $c(t)$ and $u(t)$ in the q-th period of the carrier.

it is possible to rewrite the equations (5.7) and (5.8) as follows

$$E_{iq} - \epsilon_i \sin(E_{iq}) = M_{iq}, \quad i = 1, 2, ... \tag{5.12}$$

For this reason, the isomorphism with the Kepler problem, described in the second chapter can be useful to determine the switching instants τ_{1q} in terms of Kapteyn series

$$\begin{aligned} \tau_{1q} = & \tfrac{T_c}{4}(4q+1)+ \\ & + \tfrac{kT_c}{\pi} \sum_{n=1}^{\infty} \tfrac{1}{n} J_n \left(\tfrac{n\pi E_m}{2kE_c} \right) \sin\left[\tfrac{n\pi}{2k}(4q+1) + n\phi \right], \end{aligned} \tag{5.13}$$

and similarly, using the Bessel function property in eq.(1.40), τ_{2q}

$$\begin{aligned} \tau_{2q} = & \tfrac{T_c}{4}(4q+3)+ \\ & + \tfrac{kT_c}{\pi} \sum_{n=1}^{\infty} \tfrac{(-1)^n}{n} J_n \left(\tfrac{n\pi E_m}{2kE_c} \right) \sin\left[\tfrac{n\pi}{2k}(4q+3) + n\phi \right]. \end{aligned} \tag{5.14}$$

In the following lemmas and theorems, some properties concerning the switching instants have been provided.

Lemma 5.1.

$$l_{11} = \sum_{q=0}^{k-1} \sin\left[\frac{n\pi}{2k}(4q+1) + n\phi\right] = \begin{cases} k\sin\left[m\left(k\phi + \frac{\pi}{2}\right)\right], & n = mk,\ \ m, n \in \mathbb{Z}, \\ 0, & \textit{elsewhere} \end{cases} \tag{5.15}$$

$$l_{21} = \sum_{q=0}^{k-1} \sin\left[\frac{n\pi}{2k}(4q+3) + n\phi\right] = \begin{cases} (-1)^m k\sin\left[m\left(k\phi + \frac{\pi}{2}\right)\right], & n = mk,\ \ m, n \in \mathbb{Z}, \\ 0, & \textit{elsewhere}, \end{cases} \tag{5.16}$$

Proof. Eq. (5.15) can be rewritten as:

$$l_{11} = \cos\left[n\left(\frac{\pi}{2k} + \phi\right)\right]\sum_{q=0}^{k-1} \sin\left(n\frac{2\pi}{k}q\right) + + \sin\left[n\left(\frac{\pi}{2k} + \phi\right)\right]\sum_{q=0}^{k-1} \cos\left(n\frac{2\pi}{k}q\right). \tag{5.17}$$

Taking into consideration that [121]:

$$\sum_{q=0}^{k-1} \sin\left(n\frac{2\pi}{k}q\right) = \sin\left(\frac{k-1}{k}n\pi\right)\frac{\sin(n\pi)}{\sin\left(\frac{n\pi}{k}\right)} \tag{5.18}$$

and

$$\sum_{q=0}^{k-1} \cos\left(n\frac{2\pi}{k}q\right) = \cos\left(\frac{k-1}{k}n\pi\right)\frac{\sin(n\pi)}{\sin\left(\frac{n\pi}{k}\right)} \tag{5.19}$$

it follows that n must be equal to mk with $m \in \mathbb{N}$, then

$$\sum_{q=0}^{k-1} \sin\left(n\frac{2\pi}{k}q\right) = 0, \tag{5.20}$$

e

$$\sum_{q=0}^{k-1} \cos\left(n\frac{2\pi}{k}q\right) = k, \tag{5.21}$$

from which the proof follows.
With similar technicalities, the proof of l_{21} can be approached.

Lemma 5.2.

$$l_{12} = \sum_{q=0}^{k-1} \left\{ \sin \left[\frac{2\pi}{k}(q+1)j \right] - \sin \left[\frac{2\pi}{k} qj \right] \right\} = 0, \quad j \in \mathbb{Z}, \tag{5.22}$$

$$l_{22} = \sum_{q=0}^{k-1} \left\{ \cos \left[\frac{2\pi}{k} qj \right] - \cos \left[\frac{2\pi}{k}(q+1)j \right] \right\} = 0, \quad j \in \mathbb{Z}. \tag{5.23}$$

Proof. Through the prostaferesis formula

$$l_{21} = 2 \sin \left(\frac{j\pi}{k} \right) \sum_{q=0}^{k-1} \cos \left[\frac{(2q+1)j\pi}{k} \right]. \tag{5.24}$$

Taking into account [121]

$$\sum_{q=0}^{k-1} \cos \left[\frac{(2q+1)j\pi}{k} \right] = \cos(j\pi) \frac{\sin(j\pi)}{\sin\left(\frac{j\pi}{k}\right)}, \tag{5.25}$$

the proof follows and similarly for l_{22}.

Theorem 5.3.

$$\sum_{q=0}^{k-1} (\tau_{2q} - \tau_{1q}) = \frac{1}{2} k T_c, \quad k \textit{ odd}. \tag{5.26}$$

Proof. Considering Lemma 5.1, the sum on $q = 0, ..k-1$ of τ_{1q} and τ_{2q} can be rewritten as

$$\begin{aligned} \sum_{q=0}^{k-1} \tau_{1q} &= \tfrac{T_c}{4} k(2k-1) + \\ &+ \tfrac{kT_c}{\pi} \sum_{m=1}^{\infty} \tfrac{1}{m} J_{km} \left(\tfrac{m\pi E_m}{2E_c} \right) \sin \left[m \left(k\phi + \tfrac{\pi}{2} \right) \right] \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} \sum_{q=0}^{k-1} \tau_{2q} &= \tfrac{T_c}{4} k(2k+1) + \\ &+ \tfrac{kT_c}{\pi} \sum_{m=1}^{\infty} \tfrac{(-1)^{m(k+1)}}{m} J_{km} \left(\tfrac{m\pi E_m}{2E_c} \right) \sin \left[m \left(k\phi + \tfrac{\pi}{2} \right) \right]. \end{aligned} \tag{5.28}$$

From which

$$\begin{aligned} \sum_{q=0}^{k-1} (\tau_{2q} - \tau_{1q}) &= \tfrac{1}{2} k T_c + \\ &+ \tfrac{kT_c}{\pi} \sum_{m=1}^{\infty} \tfrac{J_{km}\left(\frac{m\pi E_m}{2E_c} \right)}{m} \sin \left[m \left(k\phi + \tfrac{\pi}{2} \right) \right] \left[(-1)^{m(k+1)} - 1 \right], \end{aligned} \tag{5.29}$$

and for k odd the term $\left[(-1)^{m(k+1)} - 1\right]$ é is null and for this reason the theorem is demonstrated.

This approach may be used in all class of PWM inverters in which the switching instants are determined by an intersection between the sinusoidal modulating signal and a triangular or sawtooth carrier.

5.3 Spectrum of the output signal by Fourier series and explicit expression of the THD

The output signal $u(t)$ can be expressed through a double Fourier series [88] as follows:

$$\begin{aligned} u(t) =& \frac{V_{dc}E_m}{E_c} \sin\left(\frac{2\pi}{kT_c}t + \phi + \frac{\pi}{2k}\right) + \\ &+ \frac{4V_{dc}}{\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left\{ \sum_{n=-\infty}^{+\infty} Z\left(n, r, \frac{E_m}{E_c}\right) \times \right. \\ &\left. \times \sin\left[\left(\frac{2\pi}{T_c}r + \frac{2\pi}{kT_c}n\right)t + q\left(\phi + \frac{\pi}{2k}\right)\right]\right\} \end{aligned} \tag{5.30}$$

where

$$Z\left(n, r, \frac{E_m}{E_c}\right) = \begin{cases} 0, & r+n \text{ even} \\ J_n\left(\frac{r\pi E_m}{2E_c}\right), & r+n \text{ odd}, \end{cases} \tag{5.31}$$

Eq. (5.30) is made up of two terms: the first one is the fundamental of amplitude directly proportional to the modulation index E_m/E_c; the second one is made up of the harmonics, which are odd multiple of the carrier frequency (for $n = 0$) and of the harmonics in the lateral band for every multiple of carrier frequency for the other values of n.
The amplitude of the i-th odd harmonic A_i, can be expressed as [88]:

$$\begin{aligned} A_i =& \left| \frac{V_{dc}E_m}{E_c}\delta_{i,1} + \frac{4V_{dc}}{\pi} \sum_{p=1}^{\infty} \frac{J_{pk+i}\left(\frac{p\pi E_m}{2E_c}\right) e^{jp\bar{\phi}}}{p} + \right. \\ &\left. - \frac{4V_{dc}}{\pi} \sum_{p=1}^{\infty} \frac{J_{pk-i}\left(\frac{p\pi E_m}{2E_c}\right) e^{-jp\bar{\phi}}}{p} \right|, \end{aligned} \tag{5.32}$$

where $\bar{\phi} = k\phi + \frac{\pi}{2}$ And $\delta_{x,y}$ is the Kronecker delta function, defined as:

$$\delta_{x,y} := \begin{cases} 1 & x = y \\ 0 & \text{altrimenti} \end{cases} . \tag{5.33}$$

An alternative way to express the harmonic contents of the signal can be that of considering this isomorphism with the Kepler problem; in this case $u(t)$ signal can be expressed in Fourier series as:

$$u(t) = a_0 + \sum_{j=1}^{\infty} \left[a_j \cos\left(\frac{2\pi}{kT_c} jt\right) + b_j \sin\left(\frac{2\pi}{kT_c} jt\right)\right], \tag{5.34}$$

where

$$a_0 = \frac{2V_{dc}}{kT_c} \sum_{q=0}^{k-1} \left(\tau_{1q} - \tau_{2q} + \frac{T_c}{2}\right), \tag{5.35}$$

and

$$\begin{aligned} a_j = \frac{V_{dc}}{j\pi} \sum_{q=0}^{k-1} \Bigg\{ & 2\sin\left[\frac{2\pi}{kT_c} j\tau_{1q}\right] - 2\sin\left[\frac{2\pi}{kT_c} j\tau_{2q}\right] + \\ & + \sin\left[\frac{2\pi}{k}(q+1)j\right] - \sin\left[\frac{2\pi}{k} qj\right]\Bigg\}, \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} b_j = \frac{V_{dc}}{j\pi} \sum_{q=0}^{k-1} \Bigg\{ & 2\cos\left[\frac{2\pi}{kT_c} j\tau_{2q}\right] - 2\cos\left[\frac{2\pi}{kT_c} j\tau_{1q}\right] + \\ & + \cos\left[\frac{2\pi}{k} qj\right] - \cos\left[\frac{2\pi}{k}(q+1)j\right]\Bigg\}. \end{aligned} \tag{5.37}$$

Remark 5.4. Considering theorem 5.3, it is evident that a_0 is equal to zero for k odd. So, as known, in DC-AC convertion, the ratio between the modulating period and that of the carrier must be odd so that to obtain a null mean value.

Taking into consideration lemma 5.2, eqs. (5.36) and (5.37) can be rewritten as

$$a_j = \frac{2V_{dc}}{j\pi} \sum_{q=0}^{k-1} \left\{ \sin\left[\frac{2\pi}{kT_c} j\tau_{1q}\right] - \sin\left[\frac{2\pi}{kT_c} j\tau_{2q}\right]\right\}, \tag{5.38}$$

and

$$b_j = \frac{2V_{dc}}{j\pi} \sum_{q=0}^{k-1} \left\{ \cos\left[\frac{2\pi}{kT_c} j\tau_{2q}\right] - \cos\left[\frac{2\pi}{kT_c} j\tau_{1q}\right]\right\}. \tag{5.39}$$

Adding up and subtracting $j\phi$ in the trigonometrical function in eqs. (5.44) and (5.47) it follows that

$$a_j = \frac{2V_{dc}}{j\pi}\left\{\cos(j\phi)\sum_{q=0}^{k-1}\left[\sin(jE_{1q}) - \sin(jE_{2q})\right] + \right.$$
$$\left. - \sin(j\phi)\sum_{q=0}^{k-1}\left[\cos(jE_{1q}) - \cos(jE_{2q})\right]\right\}, \tag{5.40}$$

and

$$b_j = \frac{2V_{dc}}{j\pi}\left\{\cos(j\phi)\sum_{q=0}^{k-1}\left[\cos(jE_{1q}) - \cos(jE_{2q})\right] + \right.$$
$$\left. + \sin(j\phi)\sum_{q=0}^{k-1}\left[\sin(jE_{1q}) - \sin(jE_{2q})\right]\right\}. \tag{5.41}$$

Considering the Kapteyn series connected to Kepler problem eqs. (2.4), (2.5) and (2.6) it is possible to find analytic expressions of a_j and b_j. Because of different formulas for $\cos(E)$ (Eq. (2.4)) and for $\cos(mE)$ (Eq. (2.5)) it is necessary to consider harmonics of index $j \geq 2$ separately to the fundamental. Substituting eqs. (2.5) and (2.6) in eq. (5.44) and taking into consideration eqs. (5.15) and (5.16) and the following ones (whose demonstration are similar to that of lemma 5.1)

$$\sum_{q=0}^{k-1}\cos\left[\tfrac{n\pi}{2k}(4q+1) + n\phi\right] = \begin{cases} k\cos\left[m\left(k\phi + \frac{\pi}{2}\right)\right], & n = mk,\ m,n \in \mathbb{Z}, \\ 0, & \text{altrimenti} \end{cases} \tag{5.42}$$

and

$$\sum_{q=0}^{k-1}\cos\left[\tfrac{n\pi}{2k}(4q+3) + n\phi\right] = \begin{cases} (-1)^m k\cos\left[m\left(k\phi + \frac{\pi}{2}\right)\right], & n = mk,\ m,n \in \mathbb{Z}, \\ 0, & \text{altrimenti,} \end{cases} \tag{5.43}$$

it follows that

$$\begin{aligned} a_j = {} & \tfrac{2V_{dc}}{\pi}\cos(j\phi)\sum_{n=1}^{\infty}\tfrac{\alpha_{nj}\sin\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1-(-1)^{n(k+1)+j}\right] + \\ & -\tfrac{2V_{dc}}{\pi}\sin(j\phi)\sum_{n=1}^{\infty}\tfrac{\beta_{nj} cos\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1-(-1)^{n(k+1)+j}\right], \\ & j = 2,3,... \end{aligned} \tag{5.44}$$

where

$$\alpha_{nj} = J_{kn-j}\left(\frac{n\pi E_m}{2E_c}\right) + J_{kn+j}\left(\frac{n\pi E_m}{2E_c}\right) \tag{5.45}$$

and

$$\beta_{nj} = J_{kn-j}\left(\frac{n\pi E_m}{2E_c}\right) - J_{kn+j}\left(\frac{n\pi E_m}{2E_c}\right) \tag{5.46}$$

Similarly it is possible to write the expression of b_j:

$$\begin{aligned} b_j = &-\tfrac{2V_{dc}}{\pi}\cos(j\phi)\sum_{n=1}^{\infty}\tfrac{\beta_{nj}\cos\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1-(-1)^{n(k+1)+j}\right]+ \\ &-\tfrac{2V_{dc}}{\pi}\sin(j\phi)\sum_{n=1}^{\infty}\tfrac{\alpha_{nj}\sin\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1-(-1)^{n(k+1)+j}\right], \\ &j = 2, 3, ... \end{aligned} \tag{5.47}$$

It can be noted that for k odd, the even harmonics are null. Through the same approach and using Bessel function properties, described by eqs. (1.41) and (1.42), the following analytic expressions of the coefficient concerning the fundamental a_1 and b_1 are given:

$$\begin{aligned} a_1 = &\tfrac{V_{dc}E_m}{E_c}\sin(\phi) + \tfrac{8k}{\pi^2}\tfrac{V_{dc}E_c}{E_m}\cos(\phi)\times \\ &\times\sum_{n=1}^{\infty}\tfrac{J_{kn}\left(\frac{n\pi E_m}{2E_c}\right)\sin\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1+(-1)^{n(k+1)}\right]+ \\ &-\tfrac{2V_{dc}}{\pi}\sin(\phi)\sum_{n=1}^{\infty}\tfrac{\beta_{n1}\cos\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1+(-1)^{n(k+1)}\right], \end{aligned} \tag{5.48}$$

and

$$\begin{aligned} b_1 = &\tfrac{V_{dc}E_m}{E_c}\cos(\phi) - \tfrac{2V_{dc}}{\pi}\cos(\phi)\times \\ &\times\sum_{n=1}^{\infty}\tfrac{\beta_{n1}\cos\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1+(-1)^{n(k+1)}\right]+ \\ &-\tfrac{8k}{\pi^2}\tfrac{V_{dc}E_c}{E_m}\sin(\phi)\times \\ &\times\sum_{n=1}^{\infty}\tfrac{J_{kn}\left(\frac{n\pi E_m}{2E_c}\right)\sin\left[n\left(k\phi+\frac{\pi}{2}\right)\right]}{n}\left[1+(-1)^{n(k+1)}\right]. \end{aligned} \tag{5.49}$$

The following theorem provides a useful result for an analytic calculation of THD and WTHD

Theorem 5.5. *The sum of the harmonics to the square, weighted by the p-th power of the harmonic index, $\Gamma(p)$, with $p \in \mathbb{N}^+$, can be expressed in term of Bernoulli polynomial $B_n(x)$ as:*

$$\Gamma(p) = \sum_{j=1}^{\infty} \frac{a_j^2+b_j^2}{j^{2p}} = (-1)^p \frac{2^{2p+3}\pi^{2p}V_{dc}^2}{(2p+2)!} \times$$
$$\times \left[\sum_{q=0}^{k-1} \sum_{s=0}^{q-1} \left(B_{2p+2}\left(\frac{\tau_{1q}-\tau_{1s}}{kT_c}\right) - 2B_{2p+2}\left(\frac{\tau_{1q}-\tau_{2s}}{kT_c}\right) + B_{2p+2}\left(\frac{\tau_{2q}-\tau_{2s}}{kT_c}\right) \right) + \right.$$
$$\left. + \sum_{q=0}^{k-1} \sum_{s=q}^{k-1} \left(B_{2p+2}\left(\frac{\tau_{1s}-\tau_{1q}}{kT_c}\right) - 2B_{2p+2}\left(\frac{\tau_{2s}-\tau_{1q}}{kT_c}\right) + B_{2p+2}\left(\frac{\tau_{2s}-\tau_{2q}}{kT_c}\right) \right) \right],$$
$$p \in \mathbb{N}^+. \tag{5.50}$$

Proof. From eqs. (5.44) and (5.47) it is possible to write

$$\Gamma(p) = \frac{4V_{dc}^2}{\pi^2} \sum_{q=0}^{k-1} \sum_{s=0}^{k-1} \sum_{j=1}^{\infty} \left\{ \frac{\cos\left[2\pi j \frac{\tau_{1q}-\tau_{1s}}{kT_c}\right]}{j^{2p+2}} \right.$$
$$\left. - 2\frac{\cos\left[2\pi j \frac{\tau_{1q}-\tau_{2s}}{kT_c}\right]}{j^{2p+2}} + \frac{\cos\left[2\pi j \frac{\tau_{2q}-\tau_{2s}}{kT_c}\right]}{j^{2p+2}} \right\}. \tag{5.51}$$

From representation in Fourier series of Bernoulli polynomial of even order $B_{2n}(x)$ [2]:

$$B_{2n}(x) = \frac{(-1)^{n+1}2(2n)!}{(2\pi)^{2n}} \sum_{j=1}^{\infty} \frac{\cos(2j\pi x)}{j^{2n}}, \quad 0 \le x \le 1, \tag{5.52}$$

and considering the interval of the variable x, the theorem is demonstrated.

The following corollary of the theorem permits to calculate the THD.

Corollary 5.6. *If k is odd and $p = 0$, it follows that*

$$\sum_{j=1}^{\infty} (a_j^2 + b_j^2) = 2V_{dc}^2. \tag{5.53}$$

Proof. For $p = 0$ from theorem 5.5 it follows that

$$\sum_{j=1}^{\infty} (a_j^2 + b_j^2) = 4V_{dc}^2 \times$$
$$\times \left[\sum_{q=0}^{k-1} \sum_{s=0}^{q-1} \left(B_2\left(\frac{\tau_{1q}-\tau_{1s}}{kT_c}\right) - 2B_2\left(\frac{\tau_{1q}-\tau_{2s}}{kT_c}\right) + B_2\left(\frac{\tau_{2q}-\tau_{2s}}{kT_c}\right) \right) + \right. \tag{5.54}$$
$$\left. + \sum_{q=0}^{k-1} \sum_{s=q}^{k-1} \left(B_2\left(\frac{\tau_{1s}-\tau_{1q}}{kT_c}\right) - 2B_2\left(\frac{\tau_{2s}-\tau_{1q}}{kT_c}\right) + B_2\left(\frac{\tau_{2s}-\tau_{2q}}{kT_c}\right) \right) \right].$$

Substituting the expression of $B_2(y)$ it follows that

$$\sum_{j=1}^{\infty}(a_j^2+b_j^2) = \frac{8V_{dc}^2}{k^2T_c^2}\sum_{q=0}^{k-1}(\tau_{2q}-\tau_{1q})\left[kT_c-\sum_{q=0}^{k-1}(\tau_{2q}-\tau_{1q})\right], \tag{5.55}$$

And considering the theorem 5.3, the proof follows.

From the corollary it is in fact possible to obtain the known TDH expression [88]:

$$THD = \sqrt{\frac{2V_{dc}^2-(a_1^2+b_1^2)}{a_1^2+b_1^2}}, \tag{5.56}$$

and its approximation. Indeed, considering that the amplitude of the fundamental is well approximated as [92]

$$\sqrt{a_1^2+b_1^2} \approx \frac{V_{dc}E_m}{E_c}, \tag{5.57}$$

an easy formula which permits to calculate the TDH is

$$THD \approx \sqrt{\frac{2E_c^2-E_m^2}{E_m^2}}. \tag{5.58}$$

In the same way, from theorem 5.5, WTHD can be expressed in a closed form as

$$WTHD = \sqrt{\frac{\Gamma(1)-(a_1^2+b_1^2)}{a_1^2+b_1^2}}. \tag{5.59}$$

5.4 Integral representation of the switching instants and of harmonics coefficients

From eqs. (5.13) and (5.14) through the solution in integral form of Kepler's problem, already shown in equation (4.15), it is possible to derive the following expressions connected with the switching instants.

$$\begin{aligned}\tau_{iq} =& \frac{kT_c}{2\pi}(M_{iq}-\phi)+\\ &+\frac{kT_c}{2\pi^2}\int_0^{\pi}\arctan\left[\tan\left(\frac{\pi-M_{iq}+\theta-\epsilon_i\sin(\theta)}{2}\right)\right]+\\ &+\arctan\left[\tan\left(\frac{\pi-M_{iq}-\theta+\epsilon_i\sin(\theta)}{2}\right)\right]d\theta,\\ &i=1,2.\end{aligned} \tag{5.60}$$

Moreover, considering k odd, in order to obtain a null mean value, the result is, as said before, that the even harmonic coefficients are null and the following ones are the expressions of the odd harmonic coefficients, taking into consideration for ease of notation the phase $\phi = 0$

$$a_1 = \frac{16kV_{dc}E_c}{E_m\pi^2}\sum_{n=1}^{\infty}\frac{J_{kn}\left(\frac{n\pi E_m}{2E_c}\right)\sin\left(n\frac{\pi}{2}\right)}{n} \tag{5.61}$$

and

$$b_1 = \frac{V_{dc}E_m}{E_c} - \frac{4V_{dc}}{\pi}\sum_{n=1}^{\infty}\frac{\left\{J_{kn-1}\left(\frac{n\pi E_m}{2E_c}\right) - J_{kn+1}\left(\frac{n\pi E_m}{2E_c}\right)\right\}\cos\left(n\frac{\pi}{2}\right)}{n}. \tag{5.62}$$

and

$$a_j = \frac{4V_{dc}}{\pi}\sum_{n=1}^{\infty}\frac{4V_{dc}}{\pi}\sum_{n=1}^{\infty}\frac{\left\{J_{kn-j}\left(\frac{n\pi E_m}{2E_c}\right)+J_{kn+j}\left(\frac{n\pi E_m}{2E_c}\right)\right\}\sin\left(n\frac{\pi}{2}\right)}{n} \tag{5.63}$$
$$j = 3, 5, ...$$

and

$$b_j = \frac{4V_{dc}}{\pi}\sum_{n=1}^{\infty}\frac{4V_{dc}}{\pi}\sum_{n=1}^{\infty}\frac{\left\{J_{kn-j}\left(\frac{n\pi E_m}{2E_c}\right)-J_{kn+j}\left(\frac{n\pi E_m}{2E_c}\right)\right\}\cos\left(n\frac{\pi}{2}\right)}{n}. \tag{5.64}$$
$$j = 3, 5, ...$$

From these ones, following the same approach of the third chapter, these integral representations can be obtained

$$a_1 = \frac{8kV_{dc}E_c}{\pi^3 E_m}\int_0^{\pi}\left\{\arctan\left[\frac{\cos\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}{1-\sin\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}\right]\right.$$
$$\left. + \arctan\left[\frac{\cos\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}{1+\sin\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}\right]\right\}d\theta \tag{5.65}$$

and

$$b_1 = \frac{V_{dc}E_m}{E_c} + \frac{4V_{dc}}{\pi^2}\int_0^{\pi}\left\{\arctan\left[\frac{\cos\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}{1-\sin\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}\right]\right.$$
$$\left. - \arctan\left[\frac{\cos\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}{1+\sin\left(k\theta - \frac{E_m\pi\sin\theta}{2E_c}\right)}\right]\right\}\sin\theta d\theta \tag{5.66}$$

and

$$
\begin{aligned}
a_j &= \frac{4V_{dc}}{\pi^2}\int_0^{\pi}\Big\{\arctan\Big[\frac{\cos\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}{1-\sin\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}\Big]\\
&+ \arctan\Big[\frac{\cos\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}{1+\sin\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}\Big]\Big\}\cos j\theta d\theta\\
j &= 3,5,...
\end{aligned}
\tag{5.67}
$$

and

$$
\begin{aligned}
b_j = \frac{4V_{dc}}{\pi^2}\int_0^{\pi}\Big\{&\arctan\Big[\frac{\cos\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}{1-\sin\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}\Big]\\
&-\arctan\Big[\frac{\cos\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}{1+\sin\big(k\theta-\frac{E_m\pi\sin\theta}{2E_c}\big)}\Big]\Big\}\sin j\theta d\theta\\
j = 3,5,...
\end{aligned}
\tag{5.68}
$$

5.5 Approximation for the calculation of the harmonics

With reference to eqs. (5.63) and (5.64), the j-th harmonics is the quantity

$$h_j = \sqrt{a_j^2 + b_j^2}, \tag{5.69}$$

For$k$ odd and phase $\phi = 0$, an approximation of the odd harmonics has been conjectured in the following expression

$$h_j \simeq \frac{4V_{dc}}{\pi\Big(\delta_{0[\frac{j}{k}]} + \big[\frac{j}{k}\big]\Big)}\Big|J_{k[\frac{j}{k}]-j}\Big(\frac{\pi Em\,\big[\frac{j}{k}\big]}{2E_c}\Big)\Big|, \tag{5.70}$$

where the notation $[\cdot]$ is defined as the Round function, which approximates its argument to the nearest integer.

5.6 Numerical considerations

In this section some numerical considerations have been presented: firstly, the spectrum of the PWM output has been calculated through Kapteyn series, the integral representations and approximate forms; secondly the advantages of calculating WTHD through the formulas (5.59) have been highlighted.

5.6.1 Spectrum calculation

In the following illustrations, the spectrum of the output signals (considering the following values of parameters $k = 21$, $E_m/E_c = 9/10$, $V_{dc} = 1$ and $\phi = 0$) has been shown, calculated respectively through: Kapteyn series in fig.5.3 truncated at 100 terms, the integral representations calculated at machine precision in fig.5.4 and the approximated expression in fig. 5.5.

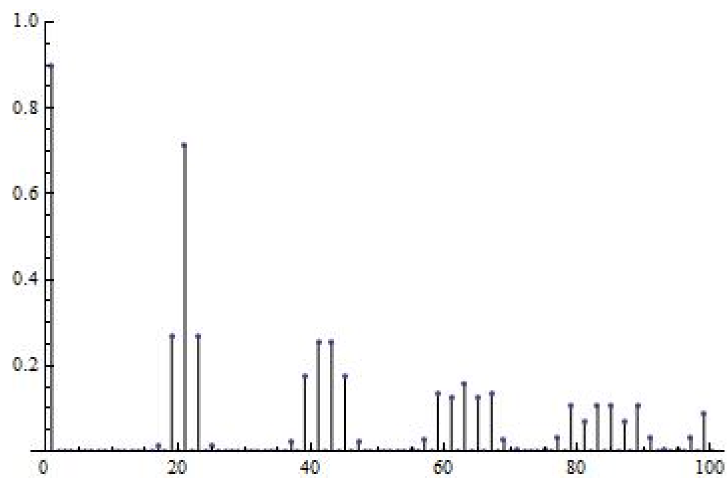

Fig. 5.3. Spectrum calculated by Kapteyn series with $k = 21$, $E_m/E_c = 9/10$, $V_{dc} = 1$ and $\phi = 0$

It is possible to note that the three spectra are identical.

5.6.2 Considerations on WTHD

There are discussions on the WTHD calculation through the formula (5.59) and equations (5.13) and (5.14) , which allow to calculate the switching instants in terms of Kapteyn series, which in this case, rapidly converge for given values of the parameters, as represented in the following experiments conducted summing up n terms ($n = 10, 11, ..., 15$), considering the same values of the instanced parameters.
Defining $\tau_{1q}(n)$ and $\tau_{2q}(n)$ the Kapteyn series in equations (5.13) and (5.14) truncated at n terms, and

$$e_1(n) = \max_{q=0,1,...,k-1} \left| \frac{\tau_{1q}(n)}{\tau_{1q}(n-1)} - 1 \right|, \quad n = 10, 11, ..., 15, \tag{5.71}$$

and

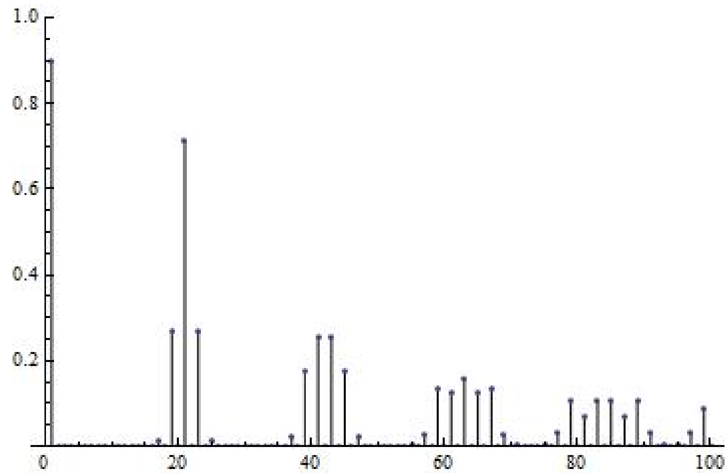

Fig. 5.4. Spectrum calculated by integral representations with $k = 21$, $E_m/E_c = 9/10$, $V_{dc} = 1$ and $\phi = 0$

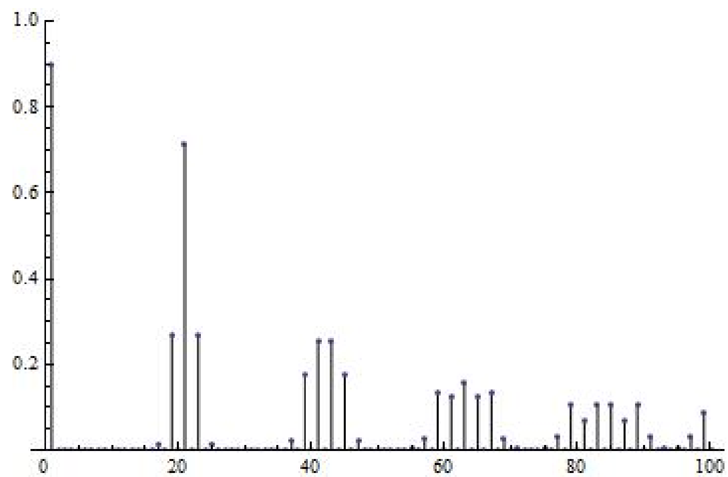

Fig. 5.5. Spectrum calculated by approximated formulas with $k = 21$, $E_m/E_c = 9/10$, $V_{dc} = 1$ and $\phi = 0$

$$e_2(n) = \max_{q=0,1,\ldots,k-1} \left| \frac{\tau_{2q}(n)}{\tau_{2q}(n-1)} - 1 \right|, \quad n = 10, 11, ..., 15, \tag{5.72}$$

The absolute value of the incremental errors, that can be made considering one more term with respect to that considered in the sum, in table 6.1, it has been shown that with $n \approx 15$, errors are lower than 10^{16}, and for this reason 15 terms in the expressions (5.13) and (5.14) are sufficient to obtain the switching instants with a reasonable accuracy.

n	10	11	12	13	14	15
$e_1(n)$	$8.64 \cdot 10^{-12}$	$7.38 \cdot 10^{-13}$	$6.32 \cdot 10^{-14}$	$5.41 \cdot 10^{-15}$	$4.64 \cdot 10^{-16}$	$3.98 \cdot 10^{-17}$
$e_2(n)$	$3.78 \cdot 10^{-12}$	$2.39 \cdot 10^{-13}$	$1.34 \cdot 10^{-14}$	$6.47 \cdot 10^{-16}$	$7.57 \cdot 10^{-17}$	$7.21 \cdot 10^{-18}$

Table 5.1. Incremental relative errors $e_1(n),\ e_2(n),\ n = 2, 3, ..., 15$ with $k = 21$ and $E_m/E_c = 9/10$.

After having calculated the switching instants, equation (5.50) provides the sum weighted on all the harmonics without any simplification and a_1 and b_1, which appear in equation (5.59), can be calculated in terms of τ_{1q} and τ_{2q}, from equations (5.36) e (5.37) respectively. Vice versa the classic way to compute the WTHD uses the weighted summation of harmonics amplitudes, thus presupposes a first approximation on a number of terms necessary to compute the harmonics amplitude and a second one on a number of harmonics to be summed up.
Consider WTHD_{FF} as the value of the calculated WTHD by eq. (5.59) with an approximation of τ_{1q} e τ_{2q} with 15 terms, and consider $\text{WTHD}_{CL}(n_h)$ as the value for the WTHD calculated, summing up the first odd n_h harmonics (the even ones are all null), calculated using the expressions (5.32) with 100 terms. In fig.5.6 it has been shown in logarithmic scale, the relative error.

$$w(n_h) = \left| \frac{\text{WTHD}_{CL}(n_h)}{\text{WTHD}_{FF}} - 1 \right|, \quad n_h = 1, 2, ..., 500.$$

It can be noted that the calculation of WTHD through$\text{WTHD}_{CL}(n_h)$ needs approximately 500 harmonics, in order to obtain a relative error of 10^{-6}. The proposed approach, differently from the classic one, which calculates at first the switching instants and then the function $\Gamma(1)$ without further approximations, seems to be more accurate and more useful as regards computational points of view.

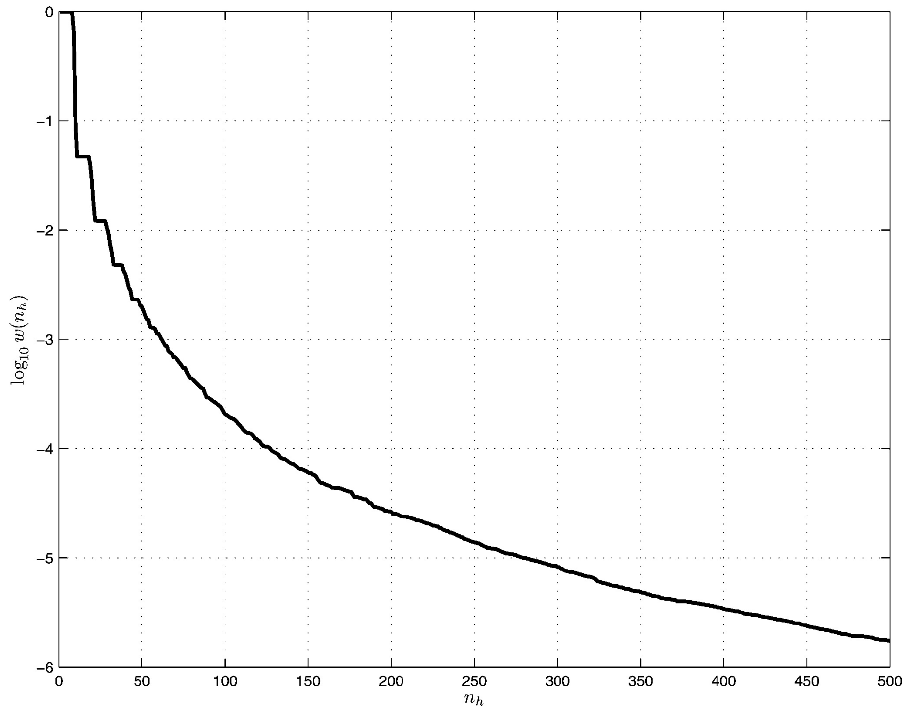

Fig. 5.6. Comparison between $WTHD_{FF}$ and $WHTD_{CL}(n_h)$

6

Cycle limits in a class of PWM feedback control system

In this chapter has been analyzed the ripple-stability of a feedback control system with a natural sampling PWM modulator as power amplifier.
The analysis is based on the describing dual-input function and on the isomorphismus between the problem of calculating PWM switching instants and the Kepler problem, on an easy stability criterion and on a sufficient condition for the absence of solutions for the balance equation by the descriptive function and therefore the probable absence of a cycle limit.
The obtained stability criterion has been related to the stability of the feedback PWM control system with first or second order apparatus: in the first case it is formally demonstrated that the proposed criterion provides the local stability of an equilibrium point while in the second case a simulation based on Monte Carlo methods confirms that the choose of the parameters together with the proposed criterion avoid cycle limits and assures the local stability of the system.

6.1 Introduction

The PWM logic, illustrated in the previous chapter connected to the conversion DC-AC, can be useful in non linear on-off control, in which the control law is active only for a time portion, proportional to the width of the error; to be more precise it is quite the same illustrated in the previous chapter: the PWM output signal in this case is the control signal which is determined by the comparison between the modular signal (in this case it represents the error signal) and a periodic carrier.
The main advantage of this control technique is bound to the fact that they are devices with small size and weight and lower energetic dissipation [105]. Moreover, the analysis of the feedback PWM control system can be really hard, due to the fact that it is a non linear system.
In literature, the describing function method has been largely used in order to determine cycle-limits and to study the dynamics of non linear systems

[108, 109, 110, 111, 112, 113], because as known, it is really important to determine conditions which can allow to anticipate and avoid the ripple stability and therefore the presence of cycle-limits.

In literature there are several approaches to determine cycle limits in the PWM control system [114, 115, 116, 117].

In [108] it has been demonstrated the use of the theory of the describing function as regards feedback PWM control system with null reference.

The logic of the existing approaches is quite the same: the development of the cycle limit modus, which defines the non linear output form, has been made as hypothesis. The output signal of the linear plant has been placed as input of the nonlinear block, changing its own sign. Then it is possible to obtain the describing function of the non linear block (in this case the PWM block), which relates the parameters of the fundamental component of the output with the amplitude of its input.

The system is therefore analyzed as a linear block, connected to a non linear one represented by its describing function depending on the amplitude. The obtained equations of the supposed cycle limits, for amplitude and frequency, can be solved through a graphic procedure, and to be more precise through the intersection in the Nyquist plane of the frequency-response of the linear plant and the critic locus (the contrary of the descriptive function) of the non linear block.

Firstly, the disadvantages of this technique are that the form of the signal input to the non linear part must be known a priori, secondly that there is the possibility to obtain multiple solutions thanks to the describing function structure, and lastly the plant to be controlled must be low-pass so that it is possible to consider only the fundamental component of the output.

In realty, in the case we take into consideration, the method of the descriptive function has been applied in a more general version; to be more precise considering the method of the descriptive dual input function, in which the input is made up of a sinusoidal element and a bias.

In this case the hypothesis that the reference is null could be removed, the same as regards the characteristic of the non linear part be symmetric with respect to the origin, which characterizes the descriptive function.

The describing functions to be detected, are two: the first one represents the output of the non linear block mean value related to the input mean value; the second one is the link between the parameter of the sinusoidal input and the parameters of the fundamental component of the output.

The idea of this approach appears to be the same of the previous chapter: taking into account a sinusoid plus a constant as the modulator input, the input switching instants of input of the linear part can be determined using an isomorphismus with the Kepler problem, and it is possible to obtain Fourier series coefficients of the input signal to the linear part and therefore an explicit expression of the describing function.

6.2 System model

The system to be analyzed is a PWM feedback control system on an linear plant, time invariant , with low-pass filter characteristics, shown in fig. 6.1. The PWM output $u(t)$ is a constant piecewise function, defined on two subintervals, $[kT, kT+\tau_k)$ and $(kT+\tau_k, kT+T)$, where $\tau_k \in [0,T]$, is the switching instant within the period of sampling T.

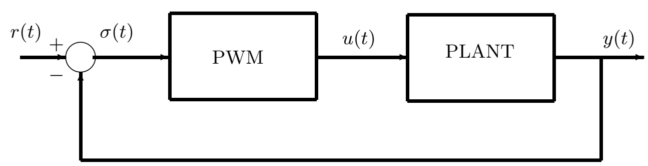

Fig. 6.1. PWM feedback control

As illustrated in fig. 6.2, the modulator output can assume values such as M and $-M$ as regards the first subinterval (in base of the error signal $\sigma(t)$, input to the modulator, valued in kT) and it is null in the second subinterval.

$$u(t) = \begin{cases} M sgn[\sigma(kT)], & kT \leq t \leq kT + \tau_k, \\ 0, & kT + \tau_k < t < kT + T. \end{cases} \tag{6.1}$$

The value of τ_k is determined by the intersection of the error signal $\sigma(t)$, and the sawtooth carrier $c(t)$, of period T having the following expression:

$$c(t) = sgn[\sigma(kT)]\frac{E_p}{T}(t - kT), \quad kT \leq t \leq kT + T, \tag{6.2}$$

where $E_p > 0$ represents the amplitude of the sawtooth. In case of natural sampling, the relation which allows to get the intersection instant τ_k is:

$$\sigma(kT + \tau_k) = sgn[\sigma(kT)]E_p\frac{\tau_k}{T} \tag{6.3}$$

where it is evident that τ_k is due to the error signal behavior in the interval $[kT, kT + \tau_k)$.

6.3 Local stability and simplification hypothesis

The state-space representation of the system to be controlled is the following one:

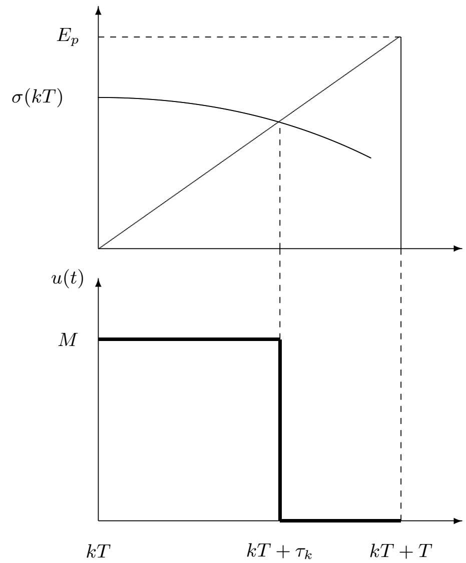

Fig. 6.2. Lead-type PWM modulator

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t), \\ y(t) = c^T x(t), \\ \sigma(t) = r - y(t) \end{cases} \tag{6.4}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ e $u \in \mathbb{R}$ (described by eq. (6.1)) are the input and output of the plant respectively, and $r \in \mathbb{R}$ is the reference signal assumed to be constant. The state transition matrix A is assumed to be invertible.

Furthermore, it is assumed that eq. (6.3) has only a solution τ_k within the period T. Although this hypothesis, from a formal point of view, can be possible only considering a particular class on input functions, from a practical point of view, it is not a limitation if the period T takes into consideration the time constants of the controlled process which provides only a solution. In the implementation of the PWM logic, we will assume $\tau_k = T$, if there are no solutions in the interval $[kT, kT + T]$.

The solution of the system represented in the state space is:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\xi)}bu(\xi)d\xi. \tag{6.5}$$

For $t \in [kT, kT + \tau_k]$ we have:

$$x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\xi)}bu(\xi)d\xi. \tag{6.6}$$

Substituting the expression $u(t)$ given in eq. (6.1), through algebra, it follows that:

$$x(t) = e^{A(t-kT)}x(kT) + Msgn\left[\sigma(kT)\right]A^{-1}\left[e^{A(t-kT)} - I\right]b, \tag{6.7}$$

where I is the identity matrix of n size.
In the second subinterval $t \in [kT + \tau_k, kT + T)$, the system state is described as:

$$x(t) = e^{A(t-kT-\tau_k)}x(kT + \tau_k). \tag{6.8}$$

Therefore from eq. (6.7) it is possible to obtain the state value in the switching instant $kT + \tau_k$,

$$x(kT + \tau_k) = e^{A\tau_k}x(kT) + Msgn\left[\sigma(kT)\right]A^{-1}\left[e^{A\tau_k} - I\right]b, \tag{6.9}$$

which can be substituted in eq. (6.8) having:

$$x(kT + T) = e^{AT}x(kT) + Msgn\left[\sigma(kT)\right]A^{-1}\left[e^{AT} - e^{A(T-\tau_k)}\right]b. \tag{6.10}$$

As

$$\sigma(t) = r - c^T x(t), \tag{6.11}$$

eq. (6.3) can be rewritten as

$$c^T x(kT + \tau_k) + \frac{E_p}{T}\tau_k sgn\left[\sigma(kT + \tau_k)\right] - r = 0. \tag{6.12}$$

As regards the hypothesis of the existence of a unique solution in the interval $[kT, kT + T)$, must be worth the identity $sgn\left[\sigma(kT + \tau_k)\right] = sgn\left[\sigma(kT)\right]$; for this reason eq. (6.12) becomes.

$$c^T e^{A\tau_k}x(kT) + Msgn[\sigma(kT)]c^T A^{-1}\left[e^{A\tau_k} - I\right]b + \frac{E_p}{T}\tau_k sgn[\sigma(kT)] - r = 0. \tag{6.13}$$

Eqs. (6.10) and (6.13) describe the dynamic behavior of the PWM feedback system.
Supposing that there is an equilibrium point, it means that after a transient, the state at the beginning of the period coincides to that at the end of the period. Moreover, if r is a constant reference, the value of τ_k, the value of the error signal in the initial period $\sigma(kT)$, and therefore $sgn[\sigma(kT)]$, must be constant for each period. Giving with (x_∞, τ_∞), the equilibrium point, we have:

$$\begin{array}{ll} x(kT + T) = x(kT) = x_\infty, & \\ \tau_k = \tau_\infty, & \forall k \\ sgn[\sigma(kT)] = sgn_\infty. & \end{array} \tag{6.14}$$

Eqs. (6.10) and (6.13) taking into consideration eq. (6.14) become respectively:

$$x_\infty = Msgn_\infty A^{-1} \left[I - e^{AT} \right]^{-1} (e^{AT} - e^{A(T-\tau_\infty)})b, \tag{6.15}$$

and

$$Mc^T A^{-1} \left[I - e^{AT} \right]^{-1} (e^{A\tau_\infty} - I)b + \frac{E_p}{T}\tau_\infty - rsgn_\infty = 0. \tag{6.16}$$

In order to obtain the sufficient and necessary condition for the local stability, let us consider the discrete model linearized around the equilibrium point (x_∞, τ_∞) [118, 52]

$$\begin{cases} dx(kT+T) = Fdx(kT), \\ F = e^{AT} - \frac{1}{Q} e^{AT} bc^T, \end{cases} \tag{6.17}$$

where

$$Q = c^T \left[I - e^{AT} \right]^{-1} (e^{A\tau_\infty} - e^{AT})b + \frac{E_p}{TM}. \tag{6.18}$$

The equilibrium point is locally asymptotically stable, only if the self values of the matrix F belong to the one radius circle. Therefore from eqs. (6.17)-(6.18) it is possible to obtain the limit value of E_p as function of T and τ_∞. Furthermore, the value of E_p, must assure the local stability for every value of τ_k in $[0, T]$. This value should be compared to that obtained by the method of the dual input describing function, which will be valid for every value of τ_k in the interval $[0, T]$, and for each reference r.

6.4 Dual input describing function and stability criterion

Taking into consideration the dual input descriptive function and the system in fig. 6.1, let us assume that a limit cycle for this system will be verified.
On the base of the well known distinctive features of the subharmonics modulation techniques, applied to the inverter, the PWM input and output $\sigma(t)$ and $u(t)$ respectively, are periodic period NT, with positive integer N [108, 119]. Therefore as input in the PWM modulator, let us consider a sinusoidal input with μE_p amplitude and pulsation $\omega_c = \frac{2\pi}{NT}$ added to a constant $\mu_0 E_p$:

$$\sigma(t) = \mu_0 E_p + \mu E_p \sin(\omega_c t + \phi). \tag{6.19}$$

with $\mu < \mu_0 < 1 - \mu$ so that the modulation saturation is avoided.
Furthermore, we suppose that the system is in a cycle limit condition around the equilibrium point and that this fluctuation does not mean a change of sign in the modulator output. Therefore it is assumed that the sign $sgn[\sigma(kT)]$ is equal to 1. Although this hypothesis can be restrictive, as we can see from the simulations, it does not damage the method prevision capacity and moreover

it allows to simplify the expression of the PWM describing function, giving the opportunity of further theoretical analysis.
Through Fourier series, the signal $u(t)$ can be expressed as:

$$u(t) = b_0 + \sum_{n=1}^{\infty} \left[b_n \cos(n\omega_c t) + a_n \sin(n\omega_c t) \right] \tag{6.20}$$

where the coefficients can be determined by the following expressions::

$$\begin{aligned} b_0 &= \tfrac{1}{NT} \int_0^{NT} u(t)dt, \quad b_n = \tfrac{2}{NT} \int_0^{NT} u(t)\cos(n\omega_c t)\, dt, \\ a_n &= \tfrac{2}{NT} \int_0^{NT} u(t)\sin(n\omega_c t)\, dt. \end{aligned} \tag{6.21}$$

The $u(t)$ fundamental component is

$$u(t) \approx b_0 + b_1 \cos(\omega_c t) + a_1 \sin(\omega_c t) = b_0 + U_1 \sin(\omega_c t + \phi_1) \tag{6.22}$$

where $U_1 = \sqrt{a_1^2 + b_1^2}$ and $\phi_1 = tan^{-1}(b_1/a_1)$.
The dual input describing function of the non linear element is the complex ratio between the output fundamental component of the non linear element and the input, or rather:

$$\begin{cases} D_0 = \frac{b_0}{\mu_0 E_p}, \\ D_1 = \frac{U_1 e^{j\phi_1}}{\mu E_p e^{j\phi}} = \frac{a_1 + jb_1}{\mu E_p e^{j\phi}}. \end{cases} \tag{6.23}$$

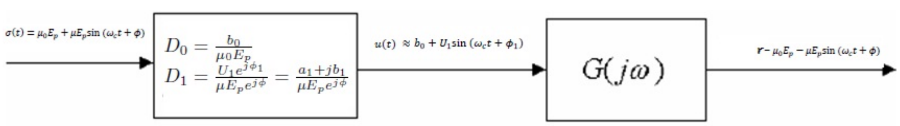

Fig. 6.3. Dual-input describing function

Eq. (6.3), particularized to the input $\sigma(t)$ of eq. (6.19), gives the following expression of the switching instant τ_k in the generic interval $[kT, kT + T)$:

$$\frac{\tau_k}{T} = \mu_0 + \mu \sin\left(\frac{2\pi}{N} k + \frac{2\pi}{N} \frac{\tau_k}{T} + \phi \right). \tag{6.24}$$

The last equation, (6.24), can be brought back to the following form:

$$E_k - \epsilon \sin(E_k) = M_k \tag{6.25}$$

where

$$E_k = \frac{2\pi}{N} k + \frac{2\pi}{N} \frac{\tau_k}{T} + \phi, \tag{6.26}$$

and

$$\epsilon = \frac{2\pi}{N} \mu \tag{6.27}$$

and

$$M_k = \frac{2\pi}{N} \mu_0 + \frac{2\pi}{N} k + \phi. \tag{6.28}$$

The eq. (6.25) is formally isomorph in connection to the Kepler equation (2.2). Therefore from eq. (2.3), taking into account positions (6.26), (6.27) and (6.28), it is possible to write the solution of (6.24) with respect to τ_k/T in terms of Kapteyn series

$$\frac{\tau_k}{T} = \mu_0 + \frac{N}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} J_n(n\epsilon) \sin(nM_k), \tag{6.29}$$

or by eq. (4.15) in integral form:

$$\frac{\tau_k}{T} = \mu_0 + \frac{N}{2\pi^2} \int_0^{\pi} \left\{ \arctan \left[\tan \left(\frac{\pi - M_k + \theta - \epsilon \sin(\theta)}{2} \right) \right] + \right.$$
$$\left. + \arctan \left[\tan \left(\frac{\pi - M_k - \theta + \epsilon \sin(\theta)}{2} \right) \right] \right\} d\theta. \tag{6.30}$$

Taking into consideration the expression $u(t)$, which can be rewritten in a more compact way as :

$$u(t) = M \sum_{k=0}^{N-1} \left(\delta_{-1}(t - kT) - \delta_{-1}(t - kT - \tau_k) \right), \tag{6.31}$$

where

$$\delta_{-1}(t) = \begin{cases} 1, \ t \geq 0, \\ 0, \ \text{otherwise} \end{cases}, \tag{6.32}$$

the mean value b_0 and the coefficient of the fundamental b_1 e a_1 di $u(t)$ become:

$$b_0 = \frac{M}{N} \sum_{k=0}^{N-1} \frac{\tau_k}{T} = M\mu_0 + \frac{M}{\pi} \sum_{p=1}^{\infty} \frac{1}{p} J_{pN}(pN\epsilon) \sin\left[p(N\phi + 2\mu_0\pi)\right], \tag{6.33}$$

and

$$b_1 = \frac{2M}{NT}\sum_{k=0}^{N-1}\int_{kT}^{kT+\tau_k}\cos\left(\frac{2\pi}{NT}t\right)dt = \frac{2M}{\pi}\sum_{k=0}^{N-1}\sin\left(\frac{\pi\tau_k}{NT}\right)\cos\left(\frac{\pi}{NT}(2kT+\tau_k)\right), \tag{6.34}$$

and

$$a_1 = \frac{2M}{NT}\sum_{k=0}^{N-1}\int_{kT}^{kT+\tau_k}\sin\left(\frac{2\pi}{NT}t\right)dt = \frac{2M}{\pi}\sum_{k=0}^{N-1}\sin\left(\frac{\pi\tau_k}{NT}\right)\sin\left(\frac{\pi}{NT}(2kT+\tau_k)\right). \tag{6.35}$$

Let us consider the complex variable $c_1 = a_1 + jb_1$, which can be rewritten as:

$$c_1 = -\frac{M}{\pi}e^{j\phi}\sum_{k=0}^{N-1}\left[\cos(E_k) - j\sin(E_k)\right]. \tag{6.36}$$

Considering Kapteyn series in equations (2.4) and (2.6) related with the Kepler problem, eq. (6.36) becomes:

$$c_1 = -\frac{N\epsilon}{2} + \sum_{k=0}^{N-1}\sum_{n=1}^{\infty}\left\{\frac{1}{n}\left[J_{n-1}(n\epsilon) - J_{n+1}(n\epsilon)\right]\cos(nM_k) + \right.$$
$$\left. -j\left[J_{n-1}(n\epsilon) + J_{n+1}(n\epsilon)\right]\sin(nM_k)\right\}. \tag{6.37}$$

From the following relations (whose demonstrations are similar to those of Lemma 5.1 in the previous chapter)

$$\begin{aligned}
&\sum_{k=0}^{N-1}\cos(nM_k) = N\cos(p(N\phi + 2\mu_0\pi)),\ n = pN,\ p\in\mathbb{N},\\
&\sum_{k=0}^{N-1}\sin(nM_k) = N\sin(p(N\phi + 2\mu_0\pi)),\ n = pN,\ p\in\mathbb{N},
\end{aligned} \tag{6.38}$$

eq. (6.37) can be reduced as:

$$c_1 = \frac{M}{\pi}e^{j\phi}\left\{\frac{N}{2}\epsilon + \sum_{p=1}^{\infty}\frac{1}{p}J_{pN+1}(pN\epsilon)e^{jp(2\mu_0\pi+N\phi)} + \right.$$
$$\left. -\sum_{p=1}^{\infty}\frac{1}{p}J_{pN-1}(pN\epsilon)e^{-jp(2\mu_0\pi+N\phi)}\right\}. \tag{6.39}$$

Lastly, from equation (6.39) and from Bessel functions properties in eq. (1.41), the following expression of D_1 can be obtained

$$D_1 = \frac{2M}{E_p}\left\{\frac{1}{2} + \sum_{p=1}^{\infty} \frac{1}{2pN+2}\left(J_{pN}(2\pi\mu p) + J_{pN+2}(2\pi\mu p)\right) e^{jp(2\mu_0\pi + N\phi)} + \right.$$
$$\left. - \sum_{p=1}^{\infty} \frac{1}{2pN-2}\left(J_{pN-2}(2\pi\mu p) + J_{pN}(2\pi\mu p)\right) e^{-jp(2\mu_0\pi + N\phi)}\right\}. \tag{6.40}$$

The following theorem provides a sufficient condition on the ripple-stability on the PWM feedback control system.

Theorem 6.1. *A sufficient condition which grants the absence of solutions as regards the describing function balance equation, and therefore the absence of the expectable oscillations with the method of the dual input describing function of period NT with $N \geq 2$, for the PWM feedback control system, is:*

$$\left|G\left(j\frac{\pi}{T}\right)\right| < \frac{E_p}{2M}. \tag{6.41}$$

Proof. Let us consider the auxiliary variables $\alpha = 2\pi\mu$ and $\beta = 2\mu_0\pi + N\phi$. The function D_1 can be rewritten as

$$D_1 = u(\alpha, \beta) + jv(\alpha, \beta), \tag{6.42}$$

where

$$u(\alpha,\beta) = \frac{2M}{E_p}\left\{\frac{1}{2} + \sum_{p=1}^{\infty}\cos(p\beta)\left[\frac{J_{pN}(p\alpha) + J_{pN+2}(p\alpha)}{2pN+2} + \right.\right.$$
$$\left.\left. - \frac{J_{pN-2}(p\alpha) + J_{pN}(p\alpha)}{2pN-2}\right]\right\}, \tag{6.43}$$

and

$$v(\alpha,\beta) = \frac{2M}{E_p}\left\{\sum_{p=1}^{\infty}\sin(p\beta)\left[\frac{J_{pN}(p\alpha) + J_{pN+2}(p\alpha)}{2pN+2} + \right.\right.$$
$$\left.\left. + \frac{J_{pN-2}(p\alpha) + J_{pN}(p\alpha)}{2pN-2}\right]\right\}. \tag{6.44}$$

Let $|D_1|_{max}$ be the absolute maximum value of D_1, a sufficient condition for the system to have no oscillations with period NT is:

$$\left|G\left(j\frac{2\pi}{NT}\right)\right| < \frac{1}{|D_1|_{max}}. \tag{6.45}$$

Given that $|D_1| = \sqrt{u(\alpha,\beta)^2 + v(\alpha,\beta)^2}$, the values of α e β maximizing $|D_1|$ can be obtained, finding the solutions of these two following equations:

$$u\frac{\partial u(\alpha,\beta)}{\partial\alpha}+v\frac{\partial v(\alpha,\beta)}{\partial\alpha}=0, \tag{6.46}$$

and

$$u\frac{\partial u(\alpha,\beta)}{\partial\beta}+v\frac{\partial v(\alpha,\beta)}{\partial\beta}=0. \tag{6.47}$$

We obtain the following non trivial solutions:

$$\begin{cases} N=2:(\alpha,\beta)=(0,\pi), \quad H=\frac{M}{E_p}\begin{pmatrix}-\frac{1}{3} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}, \quad |D_1|=\frac{2M}{E_p}, \\ N=3:(\alpha,\beta)_1=(0,\frac{\pi}{2}), \ H=\frac{M}{E_p}\begin{pmatrix}\frac{1}{16} & \frac{1}{4}\\ \frac{1}{4} & 0\end{pmatrix}, \\ \qquad (\alpha,\beta)_2=(0,\frac{3\pi}{2}), H=\frac{M}{E_p}\begin{pmatrix}\frac{1}{16} & -\frac{1}{4}\\ -\frac{1}{4} & 0\end{pmatrix}, \\ N=4:(\alpha,\beta)=(0,-), \quad H=\frac{M}{E_p}\begin{pmatrix}-\frac{\cos(\beta)}{12} & 0\\ 0 & 0\end{pmatrix}, \\ N\geq 5:(\alpha,\beta)=(0,-), \quad H=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}. \end{cases} \tag{6.48}$$

where, H is the Hessian matrix, obtained for each stationary point, after standard manipulations and using the Bessel function property in eq. (1.42) and calculating the partial derivative of second order of $|D_1|$ with respect to α e β, valued in the relative stationary point
So, the only maximum of $|D_1|$ is given for $N=2$ with $(\alpha,\beta)=(0,\pi)$, and holds $|D_1|=\frac{2M}{E_p}$, and therefore taking into account the eq. (6.45), the theorem has been demonstrated.

Remark 6.2. The expression D_0 in eq. (6.23) for $N=2$, $\alpha=0$ and $\beta=\pi$ provides the value of μ_0:

$$\mu_0=\frac{r}{E_p+G(0)M} \tag{6.49}$$

where $G(0)=-c^TA^{-1}b$ is the static gain of the plant.

6.5 Numerical simulations

In this paragraph, the validity of the proposed criterion has been verified through a comparison with the conditions of the local stability. In the following two examples, without loss of generalization, let us consider $M=1$ and the time constants of the process have been normalized in the interval $[0,1]$.

6.5.1 Example 1

The describing function has been sampled with a first order plant:

$$G(s) = \frac{1}{s\gamma + 1}.$$

The condition in eq. (6.41), solved with respect to E_p, provides the critic value of E_p, defined as $E_{p_{df}}$:

$$E_{p_{df}} = \frac{2M}{\sqrt{\pi^2 \frac{\gamma^2}{T^2} + 1}}. \tag{6.50}$$

From equations (6.17) and (6.18) we have that Q and F respectively

$$Q = \frac{1 - e^{\frac{T-\tau_\infty}{\gamma}}}{\gamma\left(1 - e^{\frac{T}{\gamma}}\right)} + \frac{E_p}{TM}, \tag{6.51}$$

and

$$F = e^{-\frac{T}{\gamma}} \left(1 - \frac{1}{\frac{1-e^{\frac{T-\tau_\infty}{\gamma}}}{1-e^{\frac{T}{\gamma}}} + \frac{E_p\gamma}{TM}}\right). \tag{6.52}$$

The derivative of F with respect to τ_∞ is:

$$\frac{\partial F}{\partial \tau_\infty} = -\frac{M^2T^2}{\gamma} \frac{e^{\frac{\tau_\infty}{\gamma}}\left(e^{\frac{T}{\gamma}} - 1\right)}{\left(E_p\gamma e^{\frac{T+\tau_\infty}{\gamma}} + MTe^{\frac{T}{\gamma}} - (E_p\gamma + MT)e^{\frac{\tau_\infty}{\gamma}}\right)^2} \tag{6.53}$$

And it is negative for each $\tau_\infty \in [0, T]$.
Therefore the stationary point is locally stable only if the two following conditions are satisfied:

$$\lim_{\tau_\infty \to 0} F = \frac{E_p\gamma}{E_p\gamma + MT} e^{-\frac{T}{\gamma}} < 1, \tag{6.54}$$

and

$$\lim_{\tau_\infty \to T} F = \frac{E_p\gamma - MT}{E_p\gamma} e^{-\frac{T}{\gamma}} > -1. \tag{6.55}$$

As the condition in eq. (6.54) is satisfied $\forall \gamma > 0$, from condition in eq. (6.55) the result is that the critic value of E_p, which assures the local stability of the equilibrium point ($E_{p_{ls}}$) is:

$$E_{p_{ls}} = \frac{MT}{\gamma\left(1 + e^{\frac{T}{\gamma}}\right)}. \qquad (6.56)$$

Taking into consideration that the ratio between $E_{p_{ls}}$ and $E_{p_{df}}$ is the following one:

$$\frac{E_{p_{ls}}}{E_{p_{df}}} = \frac{x\sqrt{1 + \frac{\pi^2}{x^2}}}{2\left(1 + e^x\right)} \qquad (6.57)$$

where $x = \frac{T}{\gamma}$, it can be easy demonstrated that $E_{p_{ls}} < E_{p_{df}}$ for each $x \geq 0$; for this reason the proposed criterion grants the local stability of the equilibrium for each ratio $\frac{\tau_\infty}{T} \in [0, 1]$.

6.5.2 Example 2

Consider a second order system, characterized by the following transfer function

$$G(s) = \frac{s\xi_3 + 1}{(s\xi_1 + 1)(s\xi_2 + 1)}$$

with $\xi_1, \xi_2, \xi_3 \in [0, 1]$ e $\xi_1 > \xi_2$. The value of the modulator sampling period T has been randomly chosen in the interval $[0, \min(\xi_2, \xi_3)/2]$ according to the fact that the PWM systems have been planned with a sampling period lower than the time response of the system linear part [108].
Ten Montecarlo simulations have been conducted. In every simulation a value of $E_p = \rho E_{p_{df}}$ with $\rho = i/10, (i = 1, 2, ..10)$ has been chosen and 10^3 systems have been considered and the number of the experiments in which the feedback PWM system is locally stable, has been counted.
In table 6.1 the possibility of having a locally stable system illustrated for each of the 10 Montecarlo simulation has been illustrated. The obtained results confirm that choosing $E_p = E_{p_{df}}$ is the method to avoid the cycle limits and to assure the local stability.

ρ	0.1	0.2	0.3	0.4	0.5
Prob.	0.5806	0.6325	0.7119	0.7764	0.8467
ρ	0.6	0.7	0.8	0.9	1.0
Prob.	0.9064	0.9652	0.9936	0.9997	1.0000

Table 6.1. Probability to have a local stable process for different values ofi ρ

7

Models of the energy spectrum of fronts and other applications

In this chapter, the proposed approach is used in other applications present in literature, related to Kapteyn series. In particular, firstly Meissel's expansions, shown in chapter 2 in equation (2.11) and (2.12) have been considered; later on taking into consideration [31] and [46] respectively, Kapteyn series useful in elettromagnetic radiations and in atmospheric disturbances.

7.1 Meissel's expansions

Kapteyn series in eq. (2.11) with $\epsilon \in (0, 1]$, as already mentioned in chapter 2, can be expressed with polynomial in variable ϵ, which Meissel himself finds in cases $m = 1, 2, 3, 4, 5$.
With the approach proposed in chapter 3, it is possible to generalize this result through an integral expression. Indeed, the considered expression is equivalent to $\gamma(1, 0, 0, 1, 2, 2m, \epsilon, 0, 0)$ in eq. (3.6), and for this reason, taking into account its integral expression in eq. (3.21) and its linking property with Lerch trascendent function with the polylog function in eq. (3.24) and the property in eq. (3.26), which connects this last function to the Bernoulli's polynomials is possible to obtain the following integral representation:

$$P_m(\epsilon) = \sum_{n=1}^{\infty} \frac{J_{2n}(2n\epsilon)}{n^{2m}} = \frac{(-1)^{m+1}(2\pi)^{2m-1}}{(2m)!} \int_0^{\pi} B_{2m}\left(\frac{\theta - \epsilon \sin\theta}{\pi}\right) d\theta \quad (7.1)$$

Through this one, it is possible to have polynomials for various values of m: e.g. for $m = 6$ and $m = 7$ we have:

$$P_6(\epsilon) = \frac{\epsilon^2}{2} - \frac{341\epsilon^4}{2048} + \frac{69553\epsilon^6}{3359232} - \frac{32197\epsilon^8}{23887872} + \frac{5269\epsilon^{10}}{103680000} - \frac{\epsilon^{12}}{1036800}, \quad (7.2)$$

and

$$P_7(\epsilon) = \frac{\epsilon^2}{2} - \frac{1365\epsilon^4}{8192} + \frac{2515513\epsilon^6}{120932352} - \frac{4741165\epsilon^8}{3439853568} + \\ + \frac{20881861\epsilon^{10}}{373248000000} - \frac{5369\epsilon^{12}}{3732480000} + \frac{\epsilon^{14}}{50803200}. \tag{7.3}$$

7.2 Radiation problems

In [31] Kapteyn series, described by eq. (2.24) and (2.25) are presented in electromagnetic radiation problems, in particular in the description of the radiations produced by a fair distributions of point charges at equal distance, which evenly move on a circular trajectory. As already shown in chapter 2, the integral representations of these series, illustrated in eq. (2.26) and (2.27) are provided.
Taking into consideration the properties of Bessel function in eq. (1.45), these Kapteyn series can be rewritten, respectively as:

$$F^+ = \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} \frac{1}{n} J_{2n}[2nb\cos(\psi)]d\psi, \tag{7.4}$$

and

$$F^- = \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} \frac{(-1)^n}{n} J_{2n}[2nb\cos(\psi)]d\psi. \tag{7.5}$$

From which, using the proposed approach, considering $\gamma(1,0,0,1,2,1,b\cos(\psi),0,0)$ for F_+ and $\gamma(-1,0,0,1,2,1,b\cos(\psi),0,0)$ for F^-, the following integral representations can be found:

$$F^+ = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\psi \int_0^{\pi} d\theta \ln[2\sin(\theta - b\cos\psi\sin\theta)] \tag{7.6}$$

and

$$F^- = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\psi \int_0^{\pi} d\theta \ln[2|\cos(\theta - b\cos\psi\sin\theta)|]. \tag{7.7}$$

7.3 Atmospheric disturbance models

Within meteorology, how atmospheric disturbations are caused, is a fully investigated problem.
In 1980, basing his search on those of Hoskins and Betherthon [43], Blumen

illustrated that the atmospheric disturbance genesis could be mold starting from Burger's equation [44], taken into account without viscosity:

$$u_t + uu_x = 0. \tag{7.8}$$

In which the u_δ notation means the partial derivative of u with respect to δ, to be more precise:

$$u_\delta = \frac{\partial u}{\partial \delta}. \tag{7.9}$$

This equation was studied by Andrews and Hoskins [128], who by using the methodology introduced by Platzman [45] in 1964, pointed out that:

$$\begin{aligned} u(x, t = 0) = -\sin(x) \Rightarrow \\ u(x,t) = -2\textstyle\sum_{k=1}^{\infty} \frac{J_k(kt)}{kt} \sin(kx), \quad t \leq 1 \end{aligned} \tag{7.10}$$

In this solution, it is really important to notice what happens to the instant $t = t_B = 1$; not only because it describes the superior limit of the model validity, but also because the solution $u(x, 1)$ carries information, which can be useful in weather forecast out of the model time validity [128, 44, 129, 130]. The solution (7.10), expressed in form of Kapteyn series, requires long time calculations, which increases when t approaches to 1.
Because of these problems of numeric convergence of Kapteyn series, Boyd in 2001 [46] introduced an approximated formula for the calculation of eq. (7.10):

$$u(x, 1) \simeq -(6x)^{\frac{1}{3}}, \quad |x| \ll 1. \tag{7.11}$$

The poor preciseness of the result, we obtain using this formula, induces to find an alternative way to calculate the solution of Burger's equation.
This is possible through the proposed approach, indeed the series

$$\sum_{k=1}^{\infty} \frac{J_k(kt)}{k} \sin(kx) \tag{7.12}$$

is the particular case $\gamma(1, 0, \frac{x}{2\pi}, 1, 1, 1, t, 0, 0)$ of the class of Kapteyn series in eq. (3.6), therefore considering the proposed approach and taking into account the definition of the polylogarithm in eq. (1.52) and its link with Lerch transcendental function (3.24) and eqs. (4.13) and (4.14), the result is:

$$u(x,t) = -\frac{1}{\pi t} \int_0^{\pi} \left\{ \arctan\left[\tan\left(\frac{\pi - \theta - x + t\sin(\theta)}{2} \right) \right] + \right. \tag{7.13}$$

$$\left. - \arctan\left[\tan\left(\frac{\pi - \theta + x + t\sin(\theta)}{2} \right) \right] \right\} d\theta. \tag{7.14}$$

Considerations on the integrand of Kepler's problem solution in section (4.3) in chapter 4 appear to be the same for this integrand.

7.4 Numerical experiments

To value the correctness of the proposed solution two numerical experiments have been conducted.

7.4.1 First numerical experiment

The first experiment aims to value the behavior of (7.13) with respect to the equations (7.10) and (7.11).
Taken $t = 1$ and chosen $x = \frac{1}{10}$, we repetitive calculate the solution of the equation (7.10), in which the number of the considered Bessell's function gradually increases in connection with every repetition.
In detail, to begin with the truncation of the sum at 100 terms, and increasing this value of the number of considered terms of 100 for each iteration until a maximum of 4000, many numerical convergence problems have been detected. For this reason, in this experiment has been considered as true the value of the solution calculated by applying (7.13) with accuracy of 32. To sum up, taking into consideration the fact that the integrand as function of θ presents only a discontinuity and it is a piecewise constant function, the integral has been calculated using a bisection algorithm to find firstly the discontinuity point and later the area under the curve.

Method		Relative error
Kapteyn series (7.10)	100	$2.1885 - 2$
	200	$4.2500 - 3$
	300	$9.4215 - 4$
	400	$2.7288 - 3$
	500	$2.8983 - 3$
	600	$2.2100 - 3$
	700	$1.1618 - 3$
	800	$1.2322 - 4$
	900	$6.5424 - 4$
	1000	$1.0443 - 3$
Integral form(7.13) (machine precision)		$2.6591 - 4$
Approximated formula (7.11)		$1.1898 - 1$
Bisection algorithm (54 itarations)		$7.9666 - 17$

Table 7.1. Obtained results with $t = 1$ and $x = \frac{1}{10}$

7.4.2 Second numerical experiment

The second numerical experiment has been conducted considering $t = \frac{7}{10}$ and $x = \frac{3}{10}$.
The procedures of the experiment are the same of the experiment below, except for the fact that the Boyd formula is not used because, as already mentioned, it is worth only for $t = 1$.
Even in this case, Kapteyn series which appears in eq. (7.10) presents numerical convergence problems, but it has been observed that it converges considering almost 500 terms.
In this case, the true value is identified by the Kapteyn series, truncated at 500 terms. The detailed results are presented in the following chart:

Method	Relative error
Integral form (32 digits)	$1.2483 - 32$
Integral form (Machine precision)	$2.0513 - 4$
Bisection algorithm	$7.5155 - 17$

8

Conclusions

In this work of thesis, an integral representation of a class of Kapetyn series has been introduced.
The validity of this formulation, as regards computational aspects and accuracy in the obtained results, has been object of a critic and methodic search. What appears as evident, is that the validity of the proposed integral representation, is strictly connected to the specific structure of the integrand function, which in turn depends on the application context.
For example, within Kepler's problem and model of the energy spectrum of fronts, the given integral representations are very useful.
Indeed, starting from the conducted experiments, it is evident that the Burgers equation solution in break time $t_B = 1$ and Kepler's problem solution for $\epsilon = 1$ determine several numerical convergence problems of the involved Kapteyn series.
Vice versa, using the proposed integral forms, not only the result is calculated in fast times, but also with high accuracy.
It is important to underline that the integral representation of the Kapteyn series, illustrated in this work, is not the only one: in literature there are a series of other formulations completely different from each other, coming from different approach methods related to these series and linked to specific applications, such as in contexts of electromagnetic radiations and Kepler's problem.
The latter has deserved special consideration in this work, due to the problem itself, but also because of the interesting isomorphism between Kepler's equation and equations used to determine the PWM output signal switching instants.
Possible further developments could be those of considering wider classes of Kaptyen series and those of finding a systematic way to solve the obtained integral forms.

References

1. L.Gatteschi, *Funzioni Speciali*, Utet, 1973.
2. M.Abramowitz, I.Stegun, *Handbook of Mathematical Functions*, Dover, 1972.
3. G.N.Watson, *A treatise on the theory of Bessel functions* , Cambridge Mathematical Library, 1992.
4. D.Bernoulli, *Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae*, Comm. Acad. Sci. Imp. Petrop. VI, 1738.
5. D.Bernoulli, *Demonstrationes theorematum suorum de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae*, Comm. Acad. Sci. Imp. Petrop. VII, 1740.
6. L.Eulero, *De oscillationibus minimis funis libere suspensi*, Acat Acad. Pretop. V. pars 1, 1784.
7. L.Eulero, *De pertubatione motus chordarum ab earum pondere oriunda*, Acat Acad. Pretop. V. pars 1, 1784.
8. J.L.Lagrange, *Sur le probléme de Kepler*, M'emoires de l'Acadèmie Royale des Sciences, 1771.
9. F.Carlini, *Ricerche sulla convergenza della serie che serva alla soluzione del problema di Keplero*, 1817: Traslated into German by Jacobi, Astr. Nach. XXX, 1850.
10. S.D.Poisson, *Sur le développement des coordonnées d'une planéte dans son mouvement elliptique et la fonction pertubatrice de ce mouvement*, Connaissance des Tems, 1833.
11. M.N.Vrahatis, O.Ragos, T.Skiniotis, F.A.Zafiropoulos, T.N.Grapsa, *RFSFNS: A portable package for the numerical determination of the number and the calculation of roots of Bessel functions*, Computer Physics Communications, 117, 3, 1995
12. A.Elbert, P.Kosik, A.Laforgia, *Monotonicity properties of the zeros of derivative of Bessel functions*, Studia Sci. Math. Hungarica 25, 377-385, 1990.
13. L.J.Landau, *Ratios of Bessel Functions and Roots of* $\alpha J_\nu(x) + xJ\nu'(x) = 0$, Journal of Mathematical Analysis and Applications 240, 174-204, 1999.
14. J.Harrison, *Fast and accurate Bessel function computation*, Proceedings of ARITH19, the 19th IEEE Conference on Computer Arithmetic, IEEE Computer Society Press, 2009.
15. S.D.Poisson, *Mémoire sur l'intégration des équations linéaires aux différences partielles*, Journal de l'École Polytechnique, 1823.

16. H.J. Eom, *Electromagnetic wave theory for boundary-value problems: an advanced course*, Springer, 2004.
17. L.Eulero, *De resolutione aequationis* $dy+\alpha yydx = bx^m dx$, Novi Comm. Acad. Petrop. IX, 1764.
18. S.D.Poisson, *La théorie de la chaleur*, 1835.
19. B.LeMéhauté, S.Wang, *Water waves generated by underwater explosion*, Hoepli, 1996.
20. J.A.Rial, *Pacemaking the Ice Ages by Frequency Modulation of Earth's Orbital Eccentricity*, Science, 1999.
21. G.Voronoi, *Sur une fonction transcendante et ses applications á la sommation de quelques séries*, Ann. sci. de l' École norm. sup. XXI, 207-268, 1904.
22. F.Bessel, *Untersuchung des Theils der planetarischen Störungen*, Berlin Abhandlungen, 1826.
23. C.G.J.Jacobi, *Formula trasformationis integralium definitorum*, Journal für Math. XV, 1836.
24. Schläfli, *Sopra un teorema di Jacobi recato a forma più generale ed applicata alla funzione cilindrica*, Ann. di Math. (2) v., 1873.
25. J.H.Graf, *Beitrag zur Aufflösung von Differentiagleichungen zweiter Ordnung denen gewisse bestimmte Integrale genfügen*, Math. Ann. LVI, 1903.
26. A.S.Chessin, *On the expression of Bessel's Function in Form of Definite Integrals*, Johns Hopkins Univ. Circulars, 1895.
27. N.Nielsen, *Handbuck der Theorie der Cylinderfunktionem*,1904.
28. E.C.J.Von.Lommel, *Studien über die Bessel'schen Functionen*, 1868.
29. B.Draščič, T.K. Pogány, *On integral representation of Bessel function of the first kind*, Journal of Mathematical Analysis and Applications, 2005.
30. W.Kapteyn, *Recherches sur le fonctions de Fourier-Bessel*, Ann.Sci. École norm. Sup. 3, 1893.
31. I.Lerche, R.C.Tautz, *Kapteyn series arising in radiation problems*, Journal of physics A: Mathematical and theoretical, 2008.
32. C.Brezinski, *Convergence acceleration during the 20th century*, J. Comput. Appl. Math. 122, 2000.
33. C.E.Siewert, E.E.Burniston, *An exact analytical solution of Kepler's equation*', Celestial Mechanics, 6, 1972.
34. N.I.Ioakimidis, K.E.Papadakis, *A new simple method for the analytical solution of Kepler's equation'*, Celestial Mechanics, 35, 1984.
35. W.Neutsch, E.Schrüfer, *Simple integrals for solving of Kepler's equation*', Astrophysics and Space Science, 125, 1986.
36. A.Eisinberg, G.Fedele, A.Ferrise, D.Frascino, *On an integral representation of a class of Kapteyn (Fourier-Bessel) series: Kepler's equation, radiation problems and Meissel's expansion*, Applied Mathematics Letters, 2010.
37. J.Keplero, *Astronomia Nova ΑΙΤΙΟΛΟΓΗΤΟΣ seu physica coelestis, tradita commentariis de motibus stellae Martis ex observationibus G.V. Tychonis Brahe*, Bibliothòque nationale de France, 1609.
38. G.A.Schott, *Electromagnetic Radiation* , Cambridge , 1912.
39. R.F. Budden, *The summation of certain Kapteyn series of bessel functions, with an application to electromagnetic radiation*, Proc. London Math. Soc., 1926.
40. I.Lerche, R.C.Tautz, *A note on summation of Kapteyn series in astrophysical problems*, The Astrophysical Journal, 2007.

41. D.E.Dominici, *An application of Kapteyn series to a problem from queueing theory*, Proceedings in Applied Mathematics and Mechanics, 2007.
42. C.Knessl, C.Tier, *Heavy traffic analysis of a Markov-modulated queque with finite capacity and general service times*, SIAM J. Appl. Math. , 1998.
43. B.J.Hoskins, F.P. Bretherton. *Atmospheric frontogenesis models: mathematical formulation and solution*, Journal of the atmospheric sciences, 1972.
44. W.Blumen,*A comparison between Hoskins-Bretherton model of frontogenesis and the analysisof an intense surface frontal zone*, Journal of the atmospheric sciences, 1980.
45. G.W. Platzman, *An exact integral of complete spectral equations for unsteady one-dimensional flow*, Tellus, 1964.
46. J.Boyd, *The energy spectrum of fronts: time evolution of shocks in Burger's equation*, Journal of the atmospheric sciences, 1991.
47. S.V.Alekseenko, P.A.Kuibin, L.O. Valerii, *Theory of Concentrated Vortices: An Introduction*, Springer, 2007.
48. R.C.Tautz, I.Lerche, *A Review of Procedures for Summing Kapteyn Series in Mathematical Physics*, Hindawi Publishing Corporation Advances in Mathematical Physics, 2009.
49. I.Lerche, R.C.Tautz, D.S.Citrin, *Terahertz-sideband spectra involving Kapteyn series*, Journal of physics A: Mathematical and theoretical, 2009.
50. D.S.Citrin, *Optical analogue for phase-sensitive measurements in quantum-transport experiments*, physical review b, 1999.
51. G.Fedele, D.Frascino, *Spectral analysis of a class of DC-AC PWM inverters by Kapteyn series*, IEEE Transactions on Power Electronics, 25, 2010.
52. A.Eisinberg, G.Fedele, D.Frascino, *Keplers equation and limit cycles in a class of PWM feedback control systems*, Nonlinear Dynam, 2010.
53. N.Herz, *Bemerkungen zur Theorie der Bessel'schen Functionen*, Astr. Nach. CVII, 1884.
54. D.F.F.Meissel, *Neue Entwickelungen über die Bessel'schen Functionen*, Astr. Nach CXXIX, 1892.
55. D.F.F.Meissel, *Weitere Entwickelungen über die Bessel'schen Functionen*, Astr. Nach CXXX, 1892.
56. K.T.Whittaker, G.N. Watson. *A course of modern analysis*, Cambridge Univ. Press, 1935.
57. H.Bateman, A.Erdély, *Higher Transcendental Functions*, McGraw-Hill, New York 1953.
58. A.Jonquière, *A note sur la sèrie* $\sum_{n=1}^{n=\infty} \frac{x^n}{n^s}$, Bull. Soc. Math. France, 1889.
59. A.W.Odell, R.H.Gooding, *Procedures for solving Kepler's equation*, Celestial Mechanics, 1986.
60. T. Fukushima, *A fast procedure solving Kepler's equation for elliptic case*, The Astronomical Journal, 1996.
61. E.D.Charles, J.B.Tatum, *The convergence of Newton-Raphson iteration with Kepler's equation*, Celestial Mechanics and Dynamical Astronomy, 1998.
62. M.Palacios, *Kepler equation and accelerated Newton method*, Journal of Computational and Applied Mathematics, 2002.
63. T.Fukushima, *A method solving Kepler's equation without transcendental function evaluations*, Celestial Mechanics and Dynamical Astronomy, 1997.
64. S.A.Feinstein, C.A.McLaughlin, *Dynamic discretization method for solving Kepler's equation*, Celestial Mechanics and Dynamical Astronomy, 2006.

65. K.Stumpff, *On the application of Lie-series to the problems of celestial mechanics*,National Aeronautics and Space Administration, TN D-4460, 168.
66. B.I.Kaminskii, *Expansion of the coordinates of elliptical motion with respect to the Levi-Civita parameter*, Sov. Astron., 1982.
67. J.P.Boyd, *Rootfinding for a trascendental equation without a first guess: Polynomialization of Kepler's equation through Chebyshev polynomial expansion of the sine*, Applied Numerical Mathematics, 2007.
68. D.Mortari, A.Clocchiatti, *Solving Kepler's equation using Bézier curves*, Celestial Mechanics and Dynamical Astronomy, 2007.
69. J.A.López, M.Barreda, *A formulation to obtain semi-analytical planetary theories using true anomalies as temporal variables*, Journal of Computational and Applied Mathematics 2007.
70. P.Sengupta, *The Lambert W function and solutions to Kepler's equation*, Celestial Mechanics and Dynamical Astronomy, 2007.
71. J.P.Boyd, *Chebyshev expansion on intervals with branch points with application to the root of Kepler's equation: A Chebyshev-Hermite-Padé method*, Journal of Computational and Applied Mathematics, 2009.
72. A.E.Dubinov, I.N.Galidakis, *Explicit solution of the Kepler equation*, Physics of Particles and Nuclei Letters, 2007.
73. J.Bernoulli, *Methodus generalis construendi omnes aequationes differentiales primi gradus*, Acta Eruditorum publicata Lipsiae, 1694.
74. J.Bernoulli, *Leibnizens gesamellte Werke, Dritte Folge (Mathematik), II*, Halle, 1855.
75. C.Riccati, *Animadversationes in aequationes differentiates secundi gradus*, Actorum Eruditorurum quae Lipsiae publicantur Supplementa VIII, 1724.
76. D.Bernoulli, *Notata in praecedens schediasma III. Co. Jacobi Riccati*, Actorum Eruditorurum quae Lipsiae publicantur Supplementa VIII, 1724.
77. D.Bernoulli, *Exercitationes quaedam mathematicae*, 1725.
78. L.Eulero, *Institutiones Calculi Integralis*, 1769.
79. M.W Feise, D.S.Citrin, *Semiclassical theory of terahertz multiple-harmonic generation in semiconductor superlattices*, Appl. Phys. Lett, 1999.
80. A.A.Ignatov, Yu.A.Romanov, *Nonlinear electromagnetic properties of semiconductors with a superlattice*, physica status solidi (b), 1976.
81. J.Holtz, *Pulsewidth Modulation - A survey*, IEEE Trans. Ind. Electr., 29,5, 410-420, 1992.
82. M.B.Sandler, *Digital-to-analogue conversion using pulse width modulation*, IEE Electronics Communication & Engineering Journal, 1993.
83. S.Y.Suh, *Pulse Width Modulation for Analog Fiber-Optic Communications*, Journal of Lightwawe Technology, 1987.
84. R.F.Griswold, *Pulse width modulation control valves*, United States Patent, Appl. No. 430,151, 1975.
85. R.F.Smedley, *Digital, pulse width modulation audio power amplifier with noise and ripple shaping*, United States Patent, Appl. No. 379,800, 1996.
86. H.J.Chizeck, P.E.Crago, L.S.Kofman, *Robust Closed-Loop Control of Isometric Muscle Force Using Pulsewidth Modulation*, IEEE Transaction on Biomedical Engineering, 1988.
87. Z.Lai, K.M.Smedley, *A General Constant-Frequency Pulsewidth Modulator and Its Applications*, IEEE Transactions on Circuits and Systems-I, 1998.
88. D.G.Holmes, T.A.Lipo. *Pulse Width Modulation for Power Converters.* IEEE Press, 2003.

89. J.Holtz, L.Springob. *Reduced Harmonics PWM Controlled Line-Side Converter for Electric Drives*, IEEE Trans. Ind. Electr., 1993.
90. S.R.Bowes, *Novel Real-Time Harmonic Minimized PWM Control for Drives and Static Power Converters*, IEEE Trans. Power Electr., 1994.
91. S.R.Bowes, P.R.Clark, *Regular-Sampled Harmonic-Elimination PWM Control of Inverter Drives"*, IEEE Trans. Power Electr., 1995.
92. N.Mohan, T.M.Undeland, W.P.Robbins, *Elettronica di potenza (Convertitori e applicazioni)*, Hoepli , 2005.
93. IEEE-519, *Standard Pratices and Requirements for Harmonic Control in Electrical Power System*, 1992.
94. D.H.Jang, G.H.Choe, M.Ehsani, *Asymmetrical PWM Technique with Harmonic Elimination and Power Factor Control in AC Choppers*, IEEE Trans. Power Electr., 1995.
95. D.Czarkowski, D.V.Chudnovsky, G.V.Chudnovsky, I.W.Selesnick, *Solving the Optimal PWM Problem for Single-Phase Inverters*, IEEE Trans. Circ. and Systems-I, 2002.
96. P.N.Enjeti, P.D.Ziogas, J.F.Lindsay, *Programmed PWM Techniques to Eliminate Harmonics: A Critical Evaluation*, IEEE Trans. Ind. Electr., 1990.
97. Z.Song, D.V.Sarwate, *The frequency spectrum of pulse width modulated signals*, Signal Processing, 2003.
98. X.Jiang, J.King, A.Emadi, *A Power Harmonics Detection Approach Based on the Least Squares Energy Minimization Principle*, 30th Annual Conf. of the IEEE Ind. Electr. Soc., 2004.
99. E. Balestrieri, S.Rapuano, *Defining DAC performance in the frequency domain*, Measurement, 2007.
100. H.S.Black, emphModulation Theory. New York: Van Nostrand, 1953.
101. L.Asiminoaei, P.Rodríguez, F.Blaabjerg,*Application of Discontinuous PWM Modulation in Active Power Filters*, IEEE Trans. Power Electr., 2008.
102. J.Hobraiche, J.P.Vilain, P.Macret, N.Patin, *A New PWM Strategy to Reduce the Inverter Input*, IEEE Trans. Power Electr., 2009.
103. R.A.Guinee, C.Lyden, *A Novel Fourier Series Time Function for Modeling and Simulation of PWM*, IEEE Tran. on Circuits and Systems - I, 2005.
104. J.Hamman, A.E.Van Der Merwe, F.S.Van Der Merwe, *Voltage Harmonics Generated by Voltage-Fed Inverters Using PWM Natural Sampling*, IEEE Tran. Power Electr., 1988.
105. Y.Z.Tsypkin, A.N.Churilov, *Relay Control Systems*, Cambridge Univ. Press., 1984.
106. R.D.Middlebrook, S.Cuk, *A general unified approach to modeling switching-converter power stages*, IEEE Power Electron. Specialists Conf. Record, 1976.
107. S.C.Chung, S.R.Huang, C.I.Lin, *Applications of describing functions to estimate the continuous and discontinuous conduction mode for a DC-to-DC buck converter*, IEEE. Proc. Electr. Power Appl., 2000.
108. A.Gelb, W.E.Vander Velde, *Multiple-input describing functions and nonlinear system design*, McGraw-Hill, 1968.
109. J.H.Taylor, *Describing Functions, in Electrical Engineering Encyclopedia*, John Wiley & Sons, Inc., New York, 1999.
110. K.H.Kienitz, *On the Implementation of the Eigenvalue Method for Limit Cycle Determination in Nonlinear Systems*, Nonlinear Dynamics, 2005.
111. G.Somieski, *An Eigenvalue Method for Calculation of Stability and Limit Cycles in Nonlinear Systems*, Nonlinear Dynamics, 2001.

112. Y.G.Sun, *Existence and uniqueness of limit cycle for a class of nonlinear discrete-time systems*, Chaos, Solitons & Fractals, 2008.
113. T.Zhang, M.O.Tadé, Y.C.Tian, *Linear estimate of the number of limit cycles for a class of non-linear systems*, Chaos, Solitons & Fractals, 2007.
114. F.R.Delfeld, G.J.Murphy, *Analysis of pulse-width-modulated control systems*, IRE Trans. Aut. Control, 1961.
115. A.V.Peterchev, S.R.Sanders, *Quantization resolution and limit cycling in digitally controlled PWM converters*, IEEE Trans. Power Electron., 2003.
116. G.Abdelnour, J.Cheung, C.Chang, G.Tinetti, *Application of describing function in the transient response analysis of a three term fuzzy controller*, IEEE Trans. Syst. Man Cybern., 1993.
117. L.J.Heyns, J.J.Kruger, *Describing function based analysis of a nonlinear hydraulic transmission line*, IEEE Trans. Contr. Syst. Technology, 1994.
118. A.Eisinberg, G.Fedele, D.Frascino, *Local stability of a class of PWM feedback control systems*, Tech. Rep., n. 1/08, Lab. Logistica, Univ. Calabria, Italy, 2008.
119. A.Balestrino, G.De Maria, L.Sciavicco, *On the ordinary and modified subharmonic control*, II IFAC Symposium on control in power and electric drives, 1977.
120. A.K.Mahalanabis, A.K.Nath, *On the dual-input describing function of a nonlinear element*, IEEE Trans. Aut. Control, 1965.
121. I.S.Gradshteyn, I.M.Ryzhik, *Table of Integrals, Series and Products*, third ed. Academic Press, New York, 1965.
122. A.Bellini, S.Bifaretti, S.Costantini. *Elettronica di Potenza*, Aracne, 2004.
123. N.J.Higham. *Accuracy and stability of numerical algorithms*, Siam, 1996.
124. D.M.Young, R.T.Gregory. *A survey of numerical mathematics*, Addison-Wesley Publishing Company, 1973.
125. P.Colwell, *Solving Kepler's equation*, Willmann-Bell, Inc., 1993.
126. R.L.Graham, D.E.Knuth, O.Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd Edition Addison-Wesley, 1994.
127. G.Szegö, *Orthogonal Polynomials*, 4th ed. Providence, RI: Amer. Math. Soc., 1975.
128. D.G.Andrews, B.J.Hoskins, *Energy spectra predicted by semigestrophic theories of frontogenesis*, Journal of the atmospheric sciences, 1978.
129. A.Hollongsworth, P.Lomberg, *The statisctical structure of short-range forecast errors as determined from radiosonde data-Part I*, Tellus, 1986.
130. R.L. Gall, R.T. Williams, T.L. Clark. *On the minimum of surface fronts*, Journal of the atmospheric sciences, 1987.
131. S.Wolfram, *Mathematica: a System for Doing Mathematics by Computers*, Second. ed. Addison-Wesley, 1991.