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Total and Convex Total Boundedness in F-seminormed Spaces.
Implicit Functions in Locally Convex Linear Spaces.

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Ph.D. Thesis

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Contents

Introduction

This thesis consists of two independent parts.
The first (Chapter 1, 2 and 3) is concerned with the following two problems of functional analysis:

Problem 1 : *The estimate of the measure of non convex total boundedness in terms of simpler quantitative characteristics in the space* L_0 *of measurable functions;*

Problem 2 : *The comparison of the measure of non equiabsolute continuity with the Hausdorff noncompactness measure and the non convex total boundedness measure in concrete function spaces.*

The second part (Chapter 4) deals with the study of the following topic of Non Linear Operator Theory:

Implicit and inverse function theorems.

Throughout the first part of the thesis all linear spaces are real and all groups are assumed to be additive and commutative. Moreover, we use the convention $\inf \emptyset := +\infty$.
In 1988 Idzik [25] proved that the answer to the well-known Schauder's problem [23, Problem 54]: *does every continuous self-map* f *defined on a convex compact subset* M *of a Hausdorff topological linear space have a fixed point?* is affirmative if M is convexly totally bounded.

A subset M of a topological linear space X is said to be *convexly totally bounded* (*ctb* for short) if for every 0-neighborhood U there are points $f_1, ..., f_n \in M$ and a finite number of convex subsets $C_1, ..., C_n$ of U such that $M \subseteq \bigcup_{i=1}^{n} (f_i + C_i)$. If X is locally convex every totally bounded subset of X is ctb. This is not true, in general, if X is nonlocally convex (see [17],[43] and [47]).
De Pascale, Trombetta and Weber [17] defined the measure of nonconvex total bound-

edness, modelled on Idzik's concept, that may be regarded as the analogue of the well-known notion of Hausdorff measure of noncompactness in nonlocally convex linear spaces. The above notions of ctb set and non convex total boundedness are especially useful working in the setting of nonlocally convex topological linear spaces (see e.g. [30], [36], [35]).
Let $(G, \|\cdot\|_G)$ be a normed group, Ω a non empty set and $\mathcal{P}(\Omega)$ the power set of Ω. The space $L_0 := L_0(\Omega, \mathcal{A}, G, \eta)$ is a space of $G-$valued functions defined on Ω which depends on an algebra $\mathcal{A}$ in $\mathcal{P}(\Omega)$ and a submeasure $\eta : \mathcal{P}(\Omega) \longrightarrow [0, +\infty]$. In particular, if $(G, \|\cdot\|_G) = (E, \|\cdot\|_E)$ is a Banach space, $\mu : \mathcal{A} \longrightarrow \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is the extend real number system, is a finitely additive measure and η its total variation, then the space $L_0 = L_0(\mathcal{A}, \Omega, E, \eta)$ coincides with the space of measurable functions introduced in [21, chapter III], in order to develop the integration theory with respect to finitely additive measures.

In Chapter 2, under the hypothesis that $(G, \|\cdot\|_G) = (E, \|\cdot\|_E)$ is a normed space we estimate the measure of nonconvex total boundedness in $L_0(\mathcal{A}, \Omega, E, \eta)$ and we characterize the convexly totally bounded subsets of L_0. For a subset M of L_0 we introduce two quantitative characteristics $\lambda_{0,w}(M)$ and $\omega_{0,w}(M)$ involving convex sets, which measure, respectively, the *degrees of nonconvex equal quasi-boundedness* and *of nonconvex equal measurability* of M. Then, let $\gamma_{0,w}(M)$ be the measure of non convex total boundedness of M, we establish some inequalities between $\gamma_{0,w}(M)$, $\lambda_{0,w}(M)$ and $\omega_{0,w}(M)$ that give, as a special case, a Fr$\acute{e}$chet-$\check{S}$mulian type convex total boundedness criterion in the space L_0. This generalizes previous results of Trombetta [42]. Moreover, we extend our result to the space $\mathcal{L}_0$ that is a generalization of the space L_0 (see [7]).

Measures of noncompactness are very useful tools in various problems of functional analysis and operator theory. They are very often used in the theory of functional equations, including ordinary equations, equations with partial derivatives, integral and integro-differential equation, in the optimal control theory, fixed point theory, approximation theory and geometric theory of Banach space. There exist a considerable literature devote to this subject (cf. [9], [8], [4], [2]). In several applications it is necessary to know the degree of noncompactness of a set of functions. For this various authors introduced other quantitative characteristics (cf. [46],[32],[7]) and they compared such quantitative characteristics with the classical Hausdorff or Kuratowski measures of noncompactness. By these comparisons they obtained some inequalities, that give, as a special case the classical compactness criteria of Arzelà-Ascoli, Fr$\acute{e}$chet-$\check{S}$mulian and Vitali.
Assume that $(G, \|\cdot\|_G) = (E, \|\cdot\|_E)$ is an $F-$normed space and let E^Ω be the linear space of all $E-$valued functions defined on Ω.

Motivated by the above considerations, in Chapter 3 we define a quantitative characteristic, which measure the degree of nonequiabsolute continuity for subsets of certain F-seminormed subspaces of E^{Ω}.
We then compare the above quantitative characteristic with the Hausdorff noncompactness measure and with the non convex total boundedness measure. By these comparisons we establish some inequalities. In particular, as a special case of these inequalities, we get sufficient conditions for the total boundedness of a set of functions and for the convex total boundedness of a convex set of functions.
From our results we derive a Vitali-type compactness criterion and a convex total boundedness criterion in a particular class of F-seminormed subspaces of E^{Ω}, which contains the classical vector-valued Orlicz's spaces.
Finally we point out that it is not so clear if the Schauder's problem has been solved in its generality. In particular, the proof given by Cauty in [10] contains some unsolved gaps (see [11], [12],[13]). However, the results of this thesis are meant to be independent from the Schauder's problem.

Implicit function theorems are an important tool in nonlinear analysis. They have significant applications in the theory of nonlinear integral equations. One of the most important results is the classic Hildebrandt-Graves theorem. The main assumption in all its formulations is some differentiability requirement. Applying this theorem to various types of Hammerstein integral equations in Banach spaces, it turned out that the hypothesis of existence and continuity of the derivative of the operators related to the studied equation is too restrictive. In [48] it is introduced an interesting linearization property for parameter dependent operators in Banach spaces. Moreover, it is proved a generalization of the Hildebrandt-Graves theorem which implies easily the second averaging theorem of Bogoljubov for ordinary differential equations on the real line.

Let $X = (X, \|\cdot\|_X)$ and $Y = (Y, \|\cdot\|_Y)$ be Banach spaces, Λ an open subset of the real line $\mathbb{R}$ or of the complex plane $\mathbb{C}$, A an open subset of the product space $\Lambda \times X$ and $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X into Y. An operator $\Phi : A \longrightarrow Y$ and an operator function $L : \Lambda \longrightarrow \mathcal{L}(X, Y)$ are called *osculating* at $(\lambda_0, x_0) \in A$ if there exists a function $\sigma : \mathbb{R}^2 \to [0, +\infty)$ such that

$$\lim_{(\rho,\, r)\to(0,0)} \sigma(\rho, r) = 0$$

and

$$\|\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)\|_Y \leq \sigma(\rho, r)\,\|x_1 - x_2\|_X\,,$$

when $|\lambda - \lambda_0| \leq \rho$ and $\|x_1 - x_0\|_X\,, \|x_2 - x_0\|_X \leq r$.
The notion of osculating operators has been considered from different points of view (see [40], [41]).

In Capter 4 we prove an implicit function theorem in locally convex topological linear spaces, where the classical conditions of differentiability, are replaced by the above linearization property, suitably reformulated. Moreover, as an example of application, we study the stability of the solutions of an Hammerstein equation depending on a parameter.

Chapter 1

Preliminary Topics

This chapter contains some basic concepts and background used throughout this thesis. We first recall the definitions of pseudonormed group, $F-$seminormed space, Riesz pseudonormed group and Riesz $F-$seminormed space. Then we recall the construction of the spaces L_0 and the notions of Hausdorff noncompactness, of non strongly convex and of non convex total boundedness measures. References for this material are [1], [27], [34] for pseudonormed groups and for $F-$seminormed spaces; [7], [46], [16], [5], [6] for the space L_0; [9],[8], [4], [2] for the Hausdorff measure of noncompactness; [17] and [42], as well as papers [19], [47], [20] for the measures of non strongly convex and of non convex total boundedness.

1.1 Notations and definitions

Throughout this thesis we shall use the following notations. We will denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the set of natural, real and complex numbers, respectively. Moreover Ω will denote a non empty set and $\mathcal{P}(\Omega)$ the power set of Ω. For two non empty sets A and B we denote by B^A the set of all maps from A to B.

Definition 1.1.1 *Let G be a group. The map $\|\cdot\|_G : G \longrightarrow [0, +\infty]$ is a group pseudonorm, if*

(1) $\|0\|_G = 0,$

(2) $\|-f\|_G = \|f\|_G,$

(3) $\|f + g\|_G \leq \|f\|_G + \|g\|_G,$

for all $f, g \in G$.
If $\|f\|_G = 0 \Longrightarrow x = 0$ the group pseudonorms $\|\cdot\|_G$ is called a group norm.

Let $(G, \|\cdot\|_G)$ be a pseudonormed group. For $M \subseteq G$. We denote by $\overline{M}^{\|\cdot\|_G}$ the closure of M in $(G, \|\cdot\|_G)$. Further, we will use the notation

$$B_{r,f}(G) := \{g \in G : \|f - g\|_G \leq r\}$$

for the closed ball of center f and radius $r > 0$ in G.
In the case $f = 0$ we simple write $B_{r,0}(G) := B_r(G)$.
Let $X = X(+, \cdot)$ be a linear space and M be a subset of X. We denote by coM the convex hull of M. Moreover, if $(X(+), \|\cdot\|_X)$ is a pseudonormed group, we call $\overline{co}M^{\|\cdot\|_X} := \overline{coM}^{\|\cdot\|_X}$ the closed convex hull of M.

Definition 1.1.2 *(cf. [27], p.38). Let X be a linear space. An F-seminorm on X is a map $\|\cdot\|_X : X \longrightarrow [0, +\infty[$, which has the following properties:*

$$\|\lambda f\|_X \leq \|f\|_X, \qquad \forall f \in X, \forall \lambda \in \mathbb{R} \text{ with } |\lambda| \leq 1; \tag{1.1.1}$$

$$\lim_{n\to\infty} \|\frac{1}{n} f\|_X = 0, \qquad \forall f \in X; \tag{1.1.2}$$

$$\|f + g\|_X \leq \|f\|_X + \|g\|_X, \qquad \forall f, g \in X. \tag{1.1.3}$$

Notice that 1.1.2 implies $\|0\|_X = 0$. If the converse,

$$\|f\|_X = 0 \Longrightarrow f = 0, \tag{1.1.4}$$

also holds, then the $F-$ seminorm $\|\cdot\|_X$ is called an $F-$ norm.

Let $G := (G, \|\cdot\|_G)$ be a normed group and G^Ω the group of all $G-$valued functions on Ω.
For $f \in G^\Omega$ we denote by $\|f\|_G$ the function $x \to \|f(x)\|_G$.

Definition 1.1.3 *Let K be a subgroup of G^Ω. A Riesz group pseudonorm on K is a map $\|\cdot\|_K : K \longrightarrow [0, +\infty]$ such that:*

(1) $\|0\|_K = 0$,

(2) $\|f + g\|_K \leq \|f\|_K + \|g\|_K$,

(3) $\|f\|_G \leq \|g\|_G \Longrightarrow \|f\|_K \leq \|g\|_K$,

for all $f, g \in K$.

We observe that $\|f\|_K = \|-f\|_K \; \forall f \in K$ it is a consequence of (3). Let $(E, \|\cdot\|_E)$ be an $F-$ normed space and E^Ω the linear space of all $E-$valued functions on Ω.

Definition 1.1.4 *Let L be a linear subspace of E^Ω. A Riesz $F-$seminorm on L is a map $\|\cdot\|_L : L \longrightarrow [0, +\infty[$ such that:*

$$\lim_{n\to\infty} \|\frac{1}{n} f\|_L = 0, \qquad \forall f \in L; \tag{1.1.5}$$

$$\|f+g\|_L \le \|f\|_L + \|g\|_L, \qquad \forall f, g \in L; \tag{1.1.6}$$

$$\|f\|_E \le \|g\|_E \Longrightarrow \|f\|_L \le \|g\|_L, \qquad \forall f, g \in L. \tag{1.1.7}$$

We note that, if $\lambda \in [-1, 1]$ then for all $f \in L$

$$\|\lambda f(x)\|_E \le \|f(x)\|_E \quad \forall x \in \Omega.$$

Then by 1.1.7 follows that

$$\|\lambda f\|_L \le \|f\|_L \qquad \forall f \in L, \forall \lambda \in \mathbb{R} \text{ with } |\lambda| \le 1. \tag{1.1.8}$$

Definition 1.1.5 *A subset M of a topological group X is said to be totally bounded if for all neighborhood V of 0 there exist a finite number of elements $f_1, ..., f_n$ from X such that $f_1 + V, ..., f_n + V$ cover M.*

Definition 1.1.6 *The map $\alpha : \mathcal{A} \longrightarrow [0, +\infty]$ is a submeasure if*

(1) $\alpha(\emptyset) = 0$;

(2) $A \subseteq B \Longrightarrow \alpha(A) \le \alpha(B) \quad \forall A, B \in \mathcal{A}$ *(monotonicity);*

(3) $\alpha(A \cup B) \le \alpha(A) + \alpha(B) \quad \forall A, B \in \mathcal{A}$ *(subadditivity).*

1.2 The space L_0

In this section we recall briefly the definition of the space L_0.
Let $(G, \|\cdot\|_G)$ be a normed group and let $\eta : \mathcal{P}(\Omega) \longrightarrow [0, +\infty]$ be a submeasure.
In this general setting it is possible to give a natural generalization of the topology of the convergence in measure using a pseudonorm. This topology is generated

on the group G^{Ω} of all $G-$valued functions on Ω by the Riesz group pseudonorm $\|\cdot\|_0 : G^{\Omega} \longrightarrow [0,+\infty]$ defined by

$$\|f\|_0 := \inf\{a > 0 : \eta(\|f\|_G \geq a) \leq a\}$$

where $\{\|f\|_G \geq a\} := \{x \in \Omega : \|f(x)\|_G \geq a\}$.
Let $\mathcal{A}$ be an algebra in the power set $\mathbb{P}(\Omega)$ of Ω and

$$S := S(\Omega, \mathcal{A}, G) = span\{y\chi_A : y \in G \text{ and } A \in \mathcal{A}\}$$

the corresponding group of all $G-$valued simple functions on Ω, where χ_A denotes the characteristic function defined on Ω.
Then the closure $L_0 := L_0(\Omega, \mathcal{A}, G, \eta) = \overline{S}^{\|\cdot\|_0}$ of S in $(G^{\Omega}, \|\cdot\|_0)$ is a subgroup of L_0. In a more special setting the functions of L_0 are called measurable.
Let $(G, \|\cdot\|_G) := (E, \|\cdot\|_E)$ be an $F-$normed space. Then $S(\Omega, \mathcal{A}, E)$ is a linear subspace of E^{Ω} and the restriction of $\|\cdot\|_0$ on $S(\Omega, \mathcal{A}, E)$ is a Riesz $F-$seminorm, hence also the restriction of $\|\cdot\|_0$ on $L_0(\Omega, \mathcal{A}, E, \eta)$ is a Riesz $F-$seminorm. Identification of function $f, g \in E^{\Omega}$ for which $\|f-g\|_0 = 0$ turns $(L_0, \|\cdot\|_0)$ into a F-normed linear space.
Let $(E, \|\cdot\|_E) := (G, \|\cdot\|_G)$ be a Banach space, $\mu : \mathcal{A} \longrightarrow \overline{\mathbb{R}}$ a finitely additive measure and η its total variation. Then the space $L_0(\Omega, \mathcal{A}, E, \eta)$ coincide with the space of *measurable functions* introduced in [21].
Let Ω be a Lebesgue-measurable subset of $\mathbb{R}^n$, $\mathcal{A}$ the $\sigma-$algebra of all Lebesgue-measurable subsets of Ω and $\eta/_{\mathcal{A}}$ the Lebesgue measure. If $\eta(\Omega) < +\infty$, then $L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)$ coincides with the space $M(\Omega)$ of all real valued measurable functions defined on Ω. If $\eta(\Omega) = +\infty$, then $L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)$ coincides with the space $T_0(\Omega)$ of all real valued totally measurable functions defined on Ω (see [21]).

1.3 The Hausdorff noncompactness measure .The measures of non convex and of non strongly convex total boundedness

The first to consider a quantitative characteristic $\alpha(A)$ measuring the degree of noncompactness of a subset M in a metric space was K. Kuratowski in 1930, in connection with certain problems of General Topology. G. Darbo [15] used this measure to generalized Schauder's fixed point theorem to a wide class of operators, called $k-$set-contractive operators, which satisfy the condition $\alpha(T(A)) \leq k\alpha(A)$ for some $k \in [0, 1)$. Other measure of noncompactness have been defined since then. The most important ones are the Hausdorff measure of noncompactness introduced by

Gohberg, Goldénshtein and Markus [24] and the separation measure of noncompactness considered by Istrăţescu [26], Sadovskiĭ [38] and other authors.
Let G be a group and $\|\cdot\|_G : G \longrightarrow [0,+\infty]$ be a group pseudonorm.
Recall that, if $M \subseteq G$, the Hausdorff measure of noncompactness of M is defined by:

$$\gamma_G(M) := \inf\{\epsilon > 0 : \text{ There is a finite subset } F \text{ of } G \text{ such that } M \subseteq F + B_\epsilon(G)\}$$

A subset M of G is totally bounded if and only if $\gamma_G(M) = 0$.
Proposition 1.3.1 below summarizes some properties of γ_G.

Proposition 1.3.1 *Let $M, N \subseteq G$.*

(a) Regularity*: $\gamma_G(M) = 0$ iff M is totally bounded;*

(b) Monotonicity*: $M \subseteq N$ implies $\gamma_G(M) \leq \gamma_G(N)$;*

(c) Nonsingularity*:γ_G is equal to zero on every one-element set;*

(d) Semi-additivity*: $\gamma_G(M \cup N) = \max\{\gamma_G(M), \gamma_G(N)\}$;*

(e) Invariance under translations*: $\gamma_G(x + M) = \gamma_G(M)$ for any $x \in G$.*

We can also define the internal measure of noncompactness β_G of M by

$$\beta_G(M) := \inf\{\epsilon > 0 : \text{ There is a finite subset } F \text{ of } M \text{ such that } M \subseteq F + B_\epsilon(G)\}$$

It is easy to prove that the following inequalities hold:

$$\gamma_G(M) \leq \beta_G(M) \leq 2\gamma_G(M) \qquad (M \subseteq G)$$

Assume that $G(+,\cdot)$ is a linear space and that $(G(+), \|\cdot\|_G)$ is a pseudonorm group . Then we can define the measure of non convex total boundedness $\gamma_{G,w}$ of a subset M of G.

Definition 1.3.2 *Let $M \subset G$.*

$$\begin{aligned}\gamma_{G,w}(M) : &= \inf\{\epsilon > 0 : \textit{There is } f_1, ..., f_n \in G \textit{ and convex subsets} \\ &\qquad C_1, ..., C_n \textit{ of } B_\epsilon(G) \textit{ such that } M \subseteq \bigcup_{i=1}^{n}(f_i + C_i)\}\end{aligned}$$

A subset M of G is called convexly totally bounded if and only if $\gamma_{G,w}(M) = 0$.

Analogously we can define the internal measures of non convex total boundedness $\beta_{G,w}$ of M.

Definition 1.3.3 *Let $M \subset G$.*

$$\beta_{G,w}(M) : \;=\; \inf\{\epsilon > 0 : \textit{ There is } f_1, ..., f_n \in M \textit{ and convex subsets } C_1, ..., C_n \textit{ of } B_\epsilon(G) \textit{ such that } M \subseteq \bigcup_{i=1}^{n}(f_i + C_i)\}$$

Proposition 1.3.4 below summarizes some properties of the parameters just defined.

Proposition 1.3.4 *Let $M, N \subseteq G$.*

(a) $\gamma_G(M) \leq \gamma_{G,w}(M)$;

(b) M *is ctb iff* $\gamma_{G,w}(M) = 0$

(c) *If* $M \subset N$, *then* $\gamma_{G,w}(M) \leq \gamma_{G,w}(N)$;

(d) $\gamma_{G,w}(M) \leq \beta_{G,w}(M) \leq 2\gamma_{G,w}(M)$;

(e) $\gamma_{G,w}(M + N) \leq \gamma_{G,w}(M) + \gamma_{G,w}(N)$

Chapter 2

The Measure of Non Convex Total Boundedness in the Space L_0

In this chapter we deal with Problem 1 of the introduction in the space $L_0 := L_0(\Omega, \mathcal{A}, E, \eta)$, where E is a normed space. To this end we introduce and study two quantitative characteristics which measure the degree of nonconvex equal quasi-boundedness and the degree of nonconvex equal measurability of a subset M of L_0. By the comparison of this with the measure of nonconvex total boundedness of a subset M of L_0 we obtain some inequalities, that give, as a special case, a Fr*é*chet-$\check{S}$mulian type convex total boundedness criterion in the spaces L_0. In the last section we extend our results to the space $\mathcal{L}_0$ that is a generalization of the space L_0.

2.1 Preliminaries

In this section we present some definitions and known results which will be needed throughout this chapter. We set $\gamma_{L_0} := \gamma_0$ and $\gamma_{L_0,w} := \gamma_{0,w}$. Let $(G, \|\cdot\|_G)$ be a normed group and $A_1, ..., A_m \in \mathcal{A}$ be finite a partition of Ω. We set

$$S(A_1, ..., A_m) := \{s \in S : s = \textstyle\sum_{i=1}^m y_i \chi_{A_i}, \text{ where } y_i \in G \text{ for } i = 1, ..., m\}.$$

In [46] the following two quantitative characteristic λ_0 and ω_0 are used to estimate γ_0 in $L_0(\Omega, \mathcal{A}, G, \eta)$:

$$\begin{aligned} \lambda_0(M) \quad : \quad &= \inf\{\varepsilon > 0 : \text{there exists a finite subset } F \text{ of } G \text{ such that} \\ &\qquad M \subseteq \big(F^\Omega \cap S\big) + B_\varepsilon(L_0)\} \end{aligned}$$

and

$$\begin{aligned} \omega_0(M) \quad := \quad &\inf\{\varepsilon > 0 : \text{there exists a finite partition } A_1, ..., A_m \in \mathcal{A} \text{ of } \Omega \quad \text{such that} \\ &M \subseteq S(A_1, ..., A_m) + B_\varepsilon(L_0)\}. \end{aligned}$$

A subset $M \subseteq L_0$ is called *equally quasi-bounded* iff $\lambda_0(M) = 0$, and *equally measurable* iff $\omega_0(M) = 0$.
In [46, Theorem 2.2.2] is proved the following theorem

Theorem 2.1.1 *Let $M \subseteq L_0$. Then*

$$\max\{\lambda_0(M), \omega_0(M)\} \leq \gamma_0(M) \leq \lambda_0(M) + 2\omega_0(M).$$

In particular M is ctb if and only if $\lambda_0(M) = \omega_0(M) = 0$
Here we give a different prove of the above theorem with that assumption that $(G, \|\cdot\|_G) = (E, \|\cdot\|_E)$ is a normed space. For this we require the following lemma

Lemma 2.1.2 *Let $H \subseteq S(A_1, ..., A_m)$ with $\gamma_0(H) < +\infty$. Then*

$$\gamma_0(H) = \lambda_0(H)$$

Proof. We prove that $\gamma_0(H) \geq \lambda_0(H)$.
Since $\omega_0(H) = 0$ and $max\{\lambda_0(H), \omega_0(H)\} \leq \gamma_0(H)$ it follows that

$$\lambda_0(H) \leq \gamma_0(H)$$

We prove now that $\gamma_0(H) \leq \lambda_0(H)$
Let $\alpha > \lambda_0(H)$ then by definition of λ_0 there exist $F = \{z_1, ..., z_n\} \subseteq E$ such that

$$H \subseteq (F^\Omega \cap S) + B_\alpha(L_0).$$

Let $s \in H$ then $s = \sum_{i=1}^m x_i \chi_{A_i}$ with $x_i \in E$ for $i = 1, .., m$.
Let $\varphi \in (F^\Omega \cap S)$ then $\varphi = \sum_{j=1}^k z_j \chi_{B_j}$ with $z_j \in F, \quad j \leq m$
Since $H \subseteq (F^\Omega \cap S) + B_\alpha(L_0)$ it follows that $\forall s \in H \quad \exists \varphi \in (F^\Omega \cap S)$ and $\exists b_\alpha \in B_\alpha(L_0)$ such that $s - \varphi = b_\alpha \Rightarrow \|s - \varphi\|_0 \leq \alpha$.
For all $s \in H$ we define the function

$$\bar{\varphi}_s = \sum_{i=1}^m \bar{z}_i \chi_{A_i}$$

where $\bar{z}_i \in \{z_j : B_j \cap A_i \neq \emptyset\}$ is such that $\|x_i - z_j\|_E = min$.
We have that $\|s - \bar{\varphi}_s\|_0 \leq \|s - \varphi_s\|_0 \leq \alpha$.
Let $K := \{\bar{\varphi}_s : s \in H\}$ then K is a finite subset of S and

$$H \subseteq K + B_\alpha(L_0).$$

This prove that $\gamma_0(H) \leq \lambda_0(H)$. □

Now we are in position to prove the theorem 2.1.1

Proof.

We prove the left inequality. It is trivially if $\gamma_0(M) = +\infty$.
Assume that $\gamma_0(M) < \alpha < +\infty$.
By definition of γ_0 there are function $s_1, ..., s_n \in S$ such that

$$M \subseteq \bigcup_{i=1}^{n} s_i + B_\alpha(L_0).$$

Put $F := \bigcup_{i=1}^{n} s_i(\Omega)$ and $A_1, ..., A_m \in \mathcal{A}$ be a partition of Ω such that $s_i|_{A_j}$ is constant for $i = 1, ..., n$ and $j = i, ..., m$.
Then

$$M \subseteq (F^\Omega \cap S(A_1, ..., A_m)) + B_\alpha(L_0) \quad and \quad M \subseteq S(A_1, ..., A_m) + B_\alpha(L_0).$$

Therefore $max\{\lambda_0(M), \omega_0(M)\} \leq \gamma_0(M)$.
We now prove the right inequality. Clearly, it is true if $\lambda_0(M) = +\infty$ or $\omega_0(M) = +\infty$.
Assume that $\lambda_0(M) < \alpha < +\infty$ and $\omega_0(M) < \beta < +\infty$.
By definition of λ_0 there are a finite subset F of E such that

$$M \subseteq (F^\Omega \cap S) + B_\alpha(L_0).$$

By definition of ω_0 there is a partition $A_1, ..., A_m \in \mathcal{A}$ of Ω such that

$$M \subseteq S(A_1, ..., A_m) + B_\beta(L_0).$$

Set

$$H := (M - B_\beta) \cap S(A_1, ..., A_m).$$

Then $M \subseteq H + B_\beta(L_0)$ and therefore $\gamma_0(M) \leq \gamma_0(H) + \beta$. By Lemma 2.1.2 $\gamma_0(H) = \lambda_0(H)$ therefore $\gamma_0(M) \leq \lambda_0(H) + \beta \leq \lambda_0(M) + \beta + \beta \leq \alpha + 2\beta$.
So

$$\gamma_0(M) \leq \lambda_0(M) + 2\omega_0(M).$$

□

In particular, M is totally bounded iff $\lambda_0(M) = \omega_0(M) = 0$.
Moreover, in [7] it is defined the following quantitative characteristic $\sigma(M)$ which is useful for the calculation of $\lambda_0(M)$:

$$\begin{aligned} \sigma(M) \quad &:= \{\varepsilon > 0 : \text{there exists a finite subset } F \text{ of } E \text{ such that for all } f \in M \\ &\qquad \text{there is } D_f \subseteq \Omega \text{ with } \eta(D_f) \leq \varepsilon \text{ and } f(\Omega \backslash D_f) \subseteq F + B_\varepsilon(E)\}. \end{aligned}$$

The following result was established in [7, Proposition 2.1].

Proposition 2.1.3 *Let $M \subseteq L_0$. Then*

$$\lambda_0(M) = \sigma(M)$$

.

We list below some properties, to be used later, of the quantitative characteristics γ_0 and $\gamma_{0,w}$.

Proposition 2.1.4 *Let $M, N \subseteq L_0$. Then*

a. $\gamma_0(M) \leq \gamma_{0,w}(M)$;

b. *if* $M \subseteq N$, *then* $\gamma_0(M) \leq \gamma_0(N)$ *and* $\gamma_{0,w}(M) \leq \gamma_{0,w}(N)$;

c. $\gamma_0(M \cup N) = max\{\gamma_0(M), \gamma_0(N)\}$ *and* $\gamma_{0,w}(M \cup N) = max\{\gamma_{0,w}(M), \gamma_{0,w}(N)\}$;

d. $\gamma_0(M + N) \leq \gamma_0(M) + \gamma_0(N)$ *and* $\gamma_{0,w}(M + N) \leq \gamma_{0,w}(M) + \gamma_{0,w}(N)$.

Proposition 2.1.5 *Let $M, N \subseteq L_0$. Then*

$$\begin{aligned}\gamma_{0,w}(M) \quad = \quad & \inf\{\varepsilon > 0 : \textit{there exist functions } s_1, ..., s_n \in S \textit{ and convex subsets} \\ & C_1, ..., C_n \textit{ of } B_\varepsilon(L_0) \textit{ such that } M \subseteq \bigcup_{i=1}^{n} (s_i + C_i)\}.\end{aligned}$$

Proof. Set $a := \inf\{\epsilon > 0 :$ there are $s_1, ..., s_n \in S$ and convex sets $C_1, ..., C_n$ of $B_\epsilon(L_0)$ such that $M \subseteq \bigcup\limits_{i=1}^{n} (s_i + C_i)\}$. Obviously $\gamma_{0,w}(M) \leq a$. Clearly $\gamma_{0,w}(M) = +\infty$ implies $a = +\infty$. Assume that $\gamma_{0,w}(M) < \alpha < +\infty$. Then there exist $f_1, ..., f_n \in L_0$ and convex subsets $C_1, ..., C_n$ of $B_\alpha(L_0)$ such that $M \subseteq \bigcup\limits_{i=1}^{n} (f_i + C_i)\}$. Let $\delta > 0$, we choose $s_i \in S$ such that $\|f_i - s_i\| \leq \frac{\delta}{n}$ for $i \leq n$. Then $C_\delta := co\{0, f_1 - s_1, ..., f_n - s_n\}$ is a convex subset of $B_\delta(L_0)$ and $M \subseteq \bigcup\limits_{i=1}^{n} (s_i + C_i + C_\delta)\}$. Therefore $a \leq \alpha + \delta$. So $a \leq \gamma_{0,w}(M)$. □

2.2 The non convex total boundedness in the space L_0

Definition 2.2.1 *Let $M \subset L_0$. We define:*

$$\begin{aligned}\lambda_{0,w}(M) \quad : \quad & = \inf\{\varepsilon > 0 : \textit{ there exist a finite subset } F \textit{ of } E \textit{ and convex subsets} \\ & C_1, ..., C_n \textit{ of } \ B_\varepsilon(L_0) \textit{ such that } M \subseteq (F^\Omega \cap S) + \bigcup_{i=1}^{n} C_i \Big\}\end{aligned}$$

and

$$\omega_{0,w}(M) \quad : \quad = \inf\{\varepsilon > 0 : \textit{there exists a partition } A_1, ..., A_m \in \mathcal{A} \textit{ of } \Omega \textit{ and convex subsets } C_1, ..., C_n \textit{ of } B_\varepsilon(L_0) \textit{ such that } M \subseteq S(A_1, ..., A_m) + \bigcup_{i=1}^{n} C_i\}.$$

We call M convexly equally quasi-bounded if $\lambda_{0,w}(M) = 0$ and convexly equally measurable if $\omega_{0,w}(M) = 0$.

We observe that if $E = \mathbb{R}$, then the quantitative characteristics $\lambda_{0,w}$ and $\omega_{0,w}$ coincide with those introduced in [42].

Proposition 2.2.2 below summarizes some properties of the quantitative characteristics just defined that follows immediately by the definitions.

Proposition 2.2.2 *Let $M,\ N \subseteq L_0$.*

a. $\lambda_0(M) \leq \lambda_{0,w}(M)$ *and* $\omega_0(M) \leq \omega_{0,w}(M)$*;*

b. if $M \subseteq N$ *then* $\lambda_{0,w}(M) \leq \lambda_{0,w}(N)$ *and* $\omega_{0,w}(M) \leq \omega_{0,w}(N)$*;*

c. $\lambda_{0,w}(M \cup N) = \max\{\lambda_{0,w}(M), \lambda_{0,w}(N)\}$ *and* $\omega_{0,w}(M \cup N) = \max\{\omega_{0,w}(M), \omega_{0,w}(N)\}$*;*

d. $\lambda_{0,w}(M + N) \leq \lambda_{0,w}(M) + \lambda_{0,w}(N)$ *and* $\omega_{0,w}(M + N) \leq \omega_{0,w}(M) + \omega_{0,w}(N)$*.*

The following lemma is crucial in the proof of Theorem 2.2.4.

Lemma 2.2.3 *Let $A_1, ..., A_n \in \mathcal{A}$ be a partition of Ω and $H \subseteq S(A_1, ..., A_n)$. Then*

$$\gamma_0(H) = \gamma_{0,w}(H)$$

.

Proof. By proposition 2.1.4 (a), $\gamma_0(H) \leq \gamma_{0,w}(H)$. Since $\omega_0(H) = 0$, it follows by theorem 2.1.1 and proposition 2.1.3 that $\gamma_0(H) = \sigma(H)$. Then it is sufficient to show that $\gamma_{0,w}(H) \leq \sigma(H)$. Observe that the last inequality is trivial if $\sigma(H) = \eta(\Omega)$. Hence we can assume that $\sigma(H) < \eta(\Omega)$. Let $\alpha \in\,]\sigma(H), \eta(\Omega)[$. By the definition of σ we can find a finite set $F = \{0, z_1, ..., z_p\} \subseteq E$ and sets $D_0 =: \emptyset, D_1, ..., D_r \subseteq \Omega$, with $\eta(D_j) \leq \alpha$ and $D_j = \bigcup_{l=1}^{k} A_{i_l}^j$ for $j = 1, ..., r$, where $\left\{A_{i_1}^j, ..., A_{i_k}^j\right\}$ is a proper subfamily of the partition $A_1, ..., A_n$, such that for all $s \in H$ there is $j \in \{0, 1, ..., r\}$ with $s(\Omega \setminus D_j) \subseteq F + B_\alpha(E)$. Put

$$\left\{A_{i_{k+1}}^j, ..., A_{i_n}^j\right\} =: \{A_1, ..., A_n\} \setminus \left\{A_{i_1}^j, ..., A_{i_k}^j\right\}$$

for $j = 1, ..., r$. We define

$$C_\alpha^0 := \{s \in S(A_1, ..., A_n) : s(\Omega) \subseteq B_\alpha(E)\}$$

and

$$C_\alpha^j := \{s \in S(A_1, ..., A_n) : s(A_{i_{k+1}}^j \cup ... \cup A_{i_n}^j) \subseteq B_\alpha(E)\}.$$

Then it is easy to check that C_α^j $(j = 0, 1, ..., r)$ are convex subsets of $B_\alpha(L_0)$.

Set $H_0 := \{s \in H : s\,(\Omega) \subseteq F + B_\alpha(E)\}$ and $H_j := \{s \in H : s\,(\Omega \backslash D_j) \subseteq F + B_\alpha(E)\}$ for $j = 0, 1, ..., r$. We will prove that if $H_j \neq \emptyset$ then

$$H_j \subseteq \left[F^\Omega \cap S(A_1, ..., A_n)\right] + C_\alpha^j \ \text{ for } j = 0, 1, ..., r.$$

Hence

$$H = \bigcup_{j=0}^{r} H_j \subseteq \left[F^\Omega \cap S(A_1, ..., A_n)\right] + \bigcup_{j=0}^{r} C_\alpha^j.$$

It follows that $\gamma_{0,w}(H) \leq \alpha$, and therefore $\gamma_{0,w}(H) \leq \gamma_0(H)$.

Let $s = \sum_{i=1}^n y_i \chi_{A_i} \in H$. If $s \in H_0$, we have that

$$s(\Omega) = \bigcup_{i=1}^{n} y_i \subseteq F + B_\alpha(E).$$

Hence for all $i \in \{1, ..., n\}$ there is $z_i \in \{0, z_1, ..., z_p\}$ such that $y_i - z_i \in B_\alpha(E)$. Then $s = \varphi + h$, where

$$\varphi := \sum_{i=1}^{n} z_i \chi_{A_i},$$

and

$$h := \sum_{i=1}^{n} x_i \chi_{A_i} + \sum_{i=1}^{n} (y_i - z_i) \chi_{A_i}.$$

Therefore $s = \varphi + h \in \left[F^\Omega \cap S(A_1, ..., A_n)\right] + C_\alpha^0$.

Suppose $s \in H \backslash H_0$. Then there is $j \in \{1, ..., r\}$ such that $s \in H_j$, so

$$s(A_{i_{k+1}} \cup ... \cup A_{i_n}) = \bigcup_{l=k+1}^{n} y_{i_l} \subseteq F + B_\alpha(E).$$

Hence for all $l \in \{k+1, ..., n\}$ there is $z_{i_l} \in \{0, z_1, ..., z_p\}$ such that $y_{i_l} - z_{i_l} \in B_\alpha(E)$. Then $s = \varphi + h$, where

$$\varphi := \sum_{l=k+1}^{n} z_{i_l} \chi_{A_{i_l}},$$

and

$$h := \sum_{l=1}^{k} x_{i_l} \chi_{A_{i_l}} + \sum_{l=k+1}^{n} (y_{i_l} - z_{i_l}) \chi_{A_{i_l}}.$$

Therefore $s = \varphi + h \in \left[F^\Omega \cap S(A_1, ..., A_n)\right] + C_\alpha^j$.

We are now in a position to prove the main result of this chapter.

Theorem 2.2.4 *Let $M \subseteq L_0$. Then*

$$\max\{\lambda_{0,w}(M), \omega_{0,w}(M)\} \leq \gamma_{0,w}(M) \leq \lambda_{0,w}(M) + 2\omega_{0,w}(M).$$

Proof. We first prove the left inequality which is trivial if $\gamma_{0,w}(M) = +\infty$. Assume that $\gamma_{0,w}(M) < \alpha < +\infty$. By proposition 2.1.5 there are functions $s_1, ..., s_n \in S$ and convex sets $C_1, ..., C_n$ in $B_\alpha(L_0)$ such that

$$M \subseteq \bigcup_{i=1}^{n}(s_i + C_i).$$

Put $F := \bigcup_{i=1}^{n} s_i(\Omega)$ and let $\{A_1, ..., A_m\} \subseteq \mathcal{A}$ be a partition of Ω such that $s_i|_{A_j}$ is constant for $i = 1, ..., n$ and $j = i, ..., m$.Then

$$M \subseteq (F^\Omega \cap S(A_1, ..., A_m)) + \bigcup_{i=1}^{n} C_i,$$

and

$$M \subseteq S(A_1, ..., A_m) + \bigcup_{i=1}^{n} C_i.$$

Therefore

$$\max\{\lambda_{0,w}(M), \omega_{0,w}(M)\} \leq \gamma_{0,w}(M) \qquad (*)\,.$$

We now prove the right inequality. Clearly, it is true if $\lambda_{0,w}(M) = +\infty$ or $\omega_{0,w}(M) = +\infty$. Assume that $\lambda_{0,w}(M) < \alpha < +\infty$ and $\omega_{0,w}(M) < \beta < +\infty$. By the definition of $\lambda_{0,w}$ there are a finite subset F of E and convex sets $C_1, ..., C_n$ in $B_\alpha(L_0)$ such that

$$M \subseteq (F^\Omega \cap S) + \bigcup_{i=1}^{n} C_i.$$

Moreover, we can find a partition $\{A_1, ..., A_m\} \subseteq \mathcal{A}$ of Ω and a convex sets $K_1, ..., K_m$ of $B_\beta(L_0)$ such that

$$M \subseteq S(A_1, ..., A_m) + \bigcup_{j=1}^{m} K_j.$$

Set

$$H := (M - \bigcup_{j=1}^{m} K_j) \cap S(A_1, ..., A_m),$$

we have $\lambda_{0,w}(H) \leq \lambda_{0,w}(M) + \beta$.
Finally, we prove that

$$\lambda_{0,w}(H) = \gamma_{0,w}(H).$$

Of course $\omega_0(H) = 0$. So by the above inequality $(*)$, it follows that $\lambda_{0,w}(H) \leq \gamma_{0,w}(H)$. Therefore

$$\lambda_{0,w}(H) \leq \gamma_{0,w}(H) = \gamma_0(H) = \lambda_0(H) \leq \lambda_{0,w}(H),$$

which implies $\lambda_{0,w}(H) = \gamma_{0,w}(H)$. By the definition of H it follows that

$$M \subseteq H + \bigcup_{j=1}^{m} K_j.$$

Hence

$$\begin{aligned} \gamma_{0,w}(M) &\leq \gamma_{0,w}(H) + \beta = \lambda_{0,w}(H) + \beta \leq \lambda_{0,w}(M) + \beta + \beta \\ &\leq \alpha + 2\beta, \end{aligned}$$

and

$$\gamma_{0,w}(M) \leq \lambda_{0,w}(M) + 2\omega_{0,w}(M).$$

The proof is complete.

As a corollary of Theorem 2.2.4 we obtain the following Fr$\acute{e}$chet-$\check{S}$mulian type criterion of convex total boundedness.

Corollary 2.2.5 *A subset M of L_0 is ctb if and only if $\lambda_{0,w}(M) = \omega_{0,w}(M) = 0$.*

Corollary 2.2.6 *Let $M \subseteq L_0$. Then*

$$\begin{aligned} \max\{\lambda_{0,w}(M), \omega_{0,w}(M)\} &\leq \lambda_{0,w}(\bar{M}) + 2\omega_{0,w}(\bar{M}) \\ &\leq \lambda_{0,w}(M) + 2\omega_{0,w}(M). \end{aligned}$$

In particular, $\lambda_{0,w}(M) = \lambda_{0,w}(\bar{M})$ and $\omega_{0,w}(M) = \omega_{0,w}(\bar{M})$ if M is *ctb*.

Proposition 2.2.7 *Let M be subset of L_0 and suppose that $\lambda_0(M) = 0$.Then $\gamma_{0,w}(M) \leq 2\omega_{0,w}(M)$ and $\lambda_{0,w}(M) \leq 2\omega_{0,w}(M)$.*

Proof. Let $\alpha > \omega_{0,w}(M)$. Then there are a partition $A_1, ..., A_n \in \mathcal{A}$ and convex sets $C_1, ..., C_n$ of $B_\alpha(L_0)$ such that $M \subseteq S(A_1, ..., A_n) + \bigcup_{j=1}^{m} C_j$.

Set $H := \left(M - \bigcup_{j=1}^{m} C_j\right) \cap S(A_1, ..., A_n)$. Then $M \subseteq H + \bigcup_{j=1}^{m} C_j$ and therefore $\gamma_{0,w}(M) \leq \gamma_{0,w}(H) + \alpha$.

We now prove that $\gamma_{0,w}(H) \leq \alpha$. By 2.1.1 and 2.2.3 we have that $\gamma_{0,w}(H) = \gamma_0(H) = \lambda_0(H)$. Hence, since $H \subseteq M - \bigcup_{j=1}^{m} C_j$ and $\lambda_0(M) = 0$ we have that $\lambda_0(H) \leq \alpha$. So that $\gamma_{0,w}(M) \leq 2\alpha$ and $\gamma_{0,w}(M) \leq 2\omega_{0,w}(M)$.

By theorem 2.2.4 it follows then $\lambda_{0,w}(M) \leq 2\omega_{0,w}(M)$. □

Therefore we have that M is ctb iff $\lambda_0(M) = \omega_{0,w}(M) = 0$.

2.3 The non convex total boundedness in the space $\mathcal{L}_0$.

Let (E, σ) be a Hausdorff locally convex linear space and τ a Frèchet-Nicodym topology on $\mathcal{P}(\Omega)$ [14]. Moreover, let $\{\|\cdot\|_i, \quad i \in I\}$ be a defining family of seminorms for the topology σ and $\{\eta_j, j \in J\}$ a family of submeasures which generates τ (see[31, Theorem 15, p. 188] and [14, Theorem 2.7]) .
For $(i,j) \in I \times J$, let $E^{\Omega}_{i,j} := \left(E^{\Omega}, \|\cdot\|_{i,j}\right)$ where for
$f \in E^{\Omega}$,

$$\|f\|_{i,j} := \inf\{a > 0 : \eta_j(\{x \in \Omega : \|f(x)\|_i \geq a\}) \leq a\}.$$

Let τ_0 be the topology generated by the family of group seminors $\{\|\cdot\|_{i,j} : (i,j) \in I \times J\}$. Moreover, let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be an algebra and $S := S(\mathcal{A}, \Omega, E)$ the space of all $E-$valued $\mathcal{A}-$simple functions of E^{Ω}. We denote by $\mathcal{A}_0$ (the space of measurable functions) the closure of S in (E^{Ω}, τ_0), by $L_{i,j}$ the closure of S in $(E^{\Omega}, \|\cdot\|_{i,j})$. Obviously, we have $\mathcal{L}_0 \subseteq L_{i,j}$ for all $(i,j) \in I \times J$, because the topology $\tau_{i,j}$ generated by $\|\cdot\|_{i,j}$ is less fine then τ_0. We denote by $\lambda^{i,j}_w$ and $\omega^{i,j}_w$ the quantitative characteristics (constructed by means of $\|\cdot\|_{i,j}$) corresponding to $\lambda_{0,w}$ and $\omega_{0,w}$. For $M \subseteq \mathcal{L}_0$ we denote $\gamma^{i,j}_w(M)$ the measure of nonconvex total boundedness of M with respect to $\|\cdot\|_{i,j}$, and we define $\gamma_w(M)$, $\lambda_w(M)$ and $\omega_w(M)$ as functions from $I \times J$ to $[0, \infty]$ by

$$\begin{aligned} \gamma_w(M)(i,j) &:= \gamma^{i,j}_w(M), \\ \lambda_w(M)(i,j) &:= \lambda^{i,j}_w(M), \\ \omega_w(M)(i,j) &:= \omega^{i,j}_w(M). \end{aligned}$$

We consider the natural partial ordering in the set of all functions from $I \times J$ to $[0, \infty]$, i.e. $h_1 \leq h_2$ if and only if $h_1(i,j) \leq h_2(i,j), \forall (i,j) \in I \times J$. Since $\mathcal{L}_0$ is dense in $(L_{i,j}, \|\cdot\|_{i,j})$, the measure of nonconvex total boundedness of a set M in $\mathcal{L}_0$ calculated in $(L_{i,j}, \|\cdot\|_{i,j})$ coincides with $\gamma^{i,j}_w(M)$ calculated in $(\mathcal{L}_0, \|\cdot\|_{i,j})$. Therefore by Theorem 2.2.4 we get the following inequalities.

Theorem 2.3.1 *Let $M \subseteq \mathcal{L}_0$. Then*

$$\max\{\lambda_w(M), \omega_w(M)\} \leq \gamma_w(M) \leq \lambda_w(M) + 2\omega_w(M).$$

2.4 Open Problem

In [17] has been introduced the notion of strongly convex total boundedness - a somewhat stronger condition than convex total boundedness - and a quantitative characteristic, which measures the degree of nonstrongly convex total boundedness. This notion and this quantitative characteristic are - in contrast to convex total boundedness - invariant when one passes to the convex hull of a set. That admits the formulation of a fixed point theorem of Darbo type [15] in nonlocally convex linear spaces. The notion of strongly convexly totally bounded sets and the corresponding noncompactness measure is the main tool in [19] to get a Fan's best approximation result in nonlocally convex linear spaces. In [47] strongly convex total boundedness is linked with affine embeddability in locally convex linear spaces. The measure of nonstrongly convex total boundedness for a set M in $L_0(\Omega, \mathcal{A}, E, \eta)$ is defined by

$$\begin{aligned}\gamma_{0,s}(M): \quad = \quad & \inf\{\epsilon > 0: \text{ There is a finite subset } F \subset L_0(\Omega, \mathcal{A}, E, \eta) \text{ and a convex} \\ & \text{subset } C \text{ of } B_\epsilon(L_0(\Omega, \mathcal{A}, E, \eta)) \text{ such that } M \subseteq F + C\}\end{aligned}$$

A set M is called strongly convexly totally bounded (*sctb* for short) if $\gamma_{0,s}(M) = 0$. Sufficient condition for the strongly convex total boundedness of a set in $L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)$, where $\eta : \Omega \to [0, +\infty[$, was given in [17]. In [42] the following two quantitative characteristics λ_s and ω_s are used to estimate the nonstrongly convex total boundedness measure in $L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)$:

$$\begin{aligned}\lambda_s(M): \quad = \quad & \inf\{\epsilon > 0: \text{ There is an } a \in [0, +\infty) \text{ and a convex} \\ & \text{subset } C \text{ of } B_\epsilon(L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)) \text{ such that } M \subseteq [-a, a] + C\}\end{aligned}$$

$$\begin{aligned}\omega_s(M): \quad = \quad & \inf\{\epsilon > 0: \text{ There is a partition } A_1, ..., A_n \in \mathcal{A} \text{ of } \Omega \text{ and a convex} \\ & \text{subset } C \text{ of } B_\epsilon(L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)) \text{ such that } M \subseteq S(A_1, ..., A_n) + C\}\end{aligned}$$

Theorem 2.4.1 *[42, theorem 4.6]. Let $M \subseteq L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)$. Then*

$$\max\{\lambda_s(M), \omega_s(M)\} \leq \gamma_{0,s}(M) \leq \lambda_s(M) + 2\omega_s(M).$$

In particular, M is sctb if and only if $\lambda_s(M) = \omega_s(M) = 0$

To estimate the measure of nonstrongly convex total boundedness in the space $L_0(\Omega, \mathcal{A}, E, \eta)$ it seems natural to introduce two quantitative characteristics $\lambda_{0,s}$ and $\omega_{0,s}$ which are the natural generalizations of λ_s and ω_s.

Definition 2.4.2 *Let $M \subset L_0(\Omega, \mathcal{A}, E, \eta)$. We define:*

$$\begin{aligned}\lambda_{0,s}(M) \quad : \quad & = \inf\{\varepsilon > 0: \textit{ there exist a finite subset } F \textit{ of } E \textit{ and a convex subset} \\ & C \textit{ of } B_\varepsilon(L_0(\Omega, \mathcal{A}, E, \eta)) \textit{ such that } M \subseteq (F^\Omega \cap S) + C\}\end{aligned}$$

and

$$\omega_{0,s}(M) \; : \; = \inf\{\varepsilon > 0 : \textit{there exists a partition } A_1, ..., A_m \in \mathcal{A} \textit{ of } \Omega \textit{ and a convex subset } C \textit{ of } B_\varepsilon(L_0(\Omega, \mathcal{A}, E, \eta)) \textit{ such that } M \subseteq S(A_1, ..., A_m) + C\}.$$

In particular, we call M strongly convexly equally quasi-bounded if $\lambda_{0,s}(M) = 0$ and strongly convexly equally measurable if $\omega_{0,s}(M) = 0$

As in the proof of Theorem 2.2.4 it is easy to obtain the following inequality for a subset M of $L_0(\Omega, \mathcal{A}, E, \eta)$:

$$\max\{\lambda_{0,s}(M), \omega_{0,s}(M)\} \leq \gamma_{0,s}(M).$$

Moreover let $A_1, ..., A_n \in \mathcal{A}$ be a partition of Ω and $H \subseteq S(A_1, ..., A_n)$ if

$$\gamma_{0,w}(H) = \gamma_{0,s}(H)$$

holds, with a proof similar to the one of Theorem 2.2.4 we obtain that

$$\gamma_{0,s}(M) \leq \lambda_{0,s}(M) + 2\omega_{0,s}(M).$$

Therefore the following characterization of the strongly convex totally bounded subsets of $L_0(\Omega, \mathcal{A}, E, \eta)$:

$$M \text{ is sctb if and only if } \lambda_{0,s}(M) = \omega_{0,s}(M) = 0$$

Chapter 3

Total and Convex Total Boundedness in $F-$ seminormed spaces

Let $(E, \|\cdot\|_E)$ be an $F-$normed space. In this chapter our purpose is to study the Problem 2 of the introduction in a class of $F-$seminormed subspaces of the linear space E^Ω, which contains a large class of Riesz $F-$seminormed linear spaces.

3.1 The spaces L

Let $(E, \|\cdot\|_E)$ be an $F-$normed space, $\mathcal{A}$ an algebra in $\mathcal{P}(\Omega)$, $\eta : \mathcal{A} \longrightarrow [0, +\infty]$ a submeasure and $\eta_0 : \mathcal{P}(\Omega) \longrightarrow [0, +\infty]$ the submeasure defined by

$$\eta_0(B) := \inf\{\eta(A) : B \subset A \text{ and } A \in \mathcal{A}\}.$$

Set $\mathcal{A}_0 := \inf\{A \in \mathcal{A} : \eta_0(A) < +\infty\}$.

We will denote by

$$S_0(\mathcal{A}, E) := span\{y\chi_A : y \in E \text{ and } A \in \mathcal{A}_0\}$$

and by

$$\mathcal{A}_{0,L} := \{A \in \mathcal{A}_0 : \text{there exists } y \in E \setminus \{0\} \text{ such that } y\chi_A \in L\},$$

which is an ideal of $\mathcal{A}_0$.
Moreover for all $f \in E^\Omega$ we denote by

$$\|f\|_0 := \inf\{a > 0 : \eta_0(\|f\|_E \geq a) \leq a\}$$

Throughout this chapter $L := (L, \|\cdot\|_L)$ stands for an F-seminormed subspace of the linear space E^Ω) with the following properties:

(a) $\chi_A f \in L$ for all $A \in \mathcal{A}$ and for all $f \in L$;

(b) $S_0 \cap L$ is dense in $(L, \|\cdot\|_L)$;

(c) if $A \in \mathcal{A}_0$, then $\lim\limits_{\substack{\eta(B)\to 0 \\ A \supseteq B \in \mathcal{A}}} \|\chi_B f\|_L = 0$ for all $f \in L$.

(d) there is $k > 0$ such that $\|f\|_E \leq \|g\|_E$ implies $\|f\|_L \leq k\|g\|_L$ for all $f, g \in L$

Clearly, if the above condition is satisfies for $k = 1$, L is a Riesz F-seminormed space.

3.2 The Hausdorff measure of non compactness and the measure of non equiabsolute continuity

We start to introduce a quantitative characteristic which measure the degree of non equiabsolute continuity of a subset of the space L.

Definition 3.2.1 *Let $M \subseteq L$. We define for $A \in \mathcal{A}_{0,L}$ and $\delta > 0$:*

$$\begin{aligned} \Pi_L(M, A, \delta) &:= \max\left\{ \sup_{f \in M} \left\|\chi_{\Omega \setminus A} f\right\|_L, \sup_{f \in M} \sup_{\substack{A \supseteq B \in \mathcal{U} \\ \eta(B) \leq \delta}} \|\chi_B f\|_L \right\}, \\ \Pi_L(M, A) &:= \lim_{\delta \to 0} \Pi_L(M, A, \delta), \\ \Pi_L(M) &:= \inf_{A \in \mathcal{A}_{0,L}} \Pi_L(M, A). \end{aligned}$$

A subset M of L is called $\|\cdot\|_L$ $-$equiabsolutely continuous ($\|\cdot\|_L - eac$ for short) if $\Pi_L(M) = 0$.

The space L is called *regular* if each singleton $M = \{f\}$ with $f \in L$ is $\|\cdot\|_L$$-$eac.

Remark 3.2.2 *Let Ω be a subset of $\mathbb{R}^n$ with finite Lebesgue measure, $\mathcal{A}$ the $\sigma-$ algebra of all Lebesgue measurable subsets of Ω and $\mu : \mathcal{A} \to [0, +\infty[$ the Lebesgue measure. For $p \in [1, +\infty[$, let $L := L_p(\Omega, \mathcal{A}, \mathbb{R}, \mu)$ be the classical Lebesgue space. Then, in this setting, the set function Π_L coincides with the parameter Π_p introduced by Appell and De Pascale in[[3],Definition 3.1, p.509].*

Proposition 3.2.3 *The space L is a regular space.*

Proof. Let $f \in L$ and $\epsilon > 0$. By (b) we can choose $s \in S \cap L$ such that $\|f - s\|_L \leq \epsilon/k$. Set $A_s := supp(s) := \{x \in \Omega : s(x) \neq 0\}$. Then $A_s \in \mathcal{A}_{0,L}$ and $\|\chi_{\Omega\setminus A_s} f\|_L = \|\chi_{\Omega\setminus A_s}(f - s)\|_L \leq k\|f - s\|_L \leq \epsilon$. Hence $\inf\limits_{A\in\mathcal{A}_{0,L}} \|\chi_{\Omega\setminus A} f\|_X = 0$. On the other hand, for all $A \in \mathcal{A}_{0,L}$ by (c) we have $\Pi_L(\{f\}, A) := \lim\limits_{\delta\to 0}\Pi_L(\{f\}, A, \delta) = \|\chi_{\Omega\setminus A} f\|_L$. Then $\Pi_L(\{f\}) = 0$. □

Theorem 3.2.4 *Let $M \subseteq L$. Then*

$$\Pi_L(M) \leq k\gamma_L(M).$$

Proof. Let $\alpha > \gamma_L(M)$. By (a) there are $s_1, ..., s_n \in S_0 \cap L$ such that $M \subseteq \bigcup\limits_{i=1,\ldots,n} (s_i + B_\alpha(L))$. Set $A_i := supp(s_i)$ for $i = 1, ..., n$. We have $A_i \in \mathcal{A}_{0,L}$ for $i = 1, ..., n$ and therefore $A := \bigcup\limits_{i=1,\ldots,n} A_i \in \mathcal{A}_{0,L}$. Let $f \in M$. Choose $i \in \{1, ..., n\}$ such that $\|f - s_i\|_L \leq \alpha$. Then

$$\|\chi_{\Omega\setminus A} f\|_L \leq \|\chi_{\Omega\setminus A}(f - s_i)\|_L + \|\chi_{\Omega\setminus A} s_i\|_L \leq k\|f - s_i\|_L \leq k\alpha. \tag{3.2.1}$$

Moreover, let $\delta > 0$. For $B \subseteq A, B \in \mathcal{A}$ and $\eta(B) \leq \delta$, we have

$$\begin{aligned} \|\chi_B f\|_L \leq\ & \|\chi_B(f - s_i)\|_L + \|\chi_B s_i\|_L \leq k\|f - s_i\|_L + \|\chi_B s_i\|_L \\ \leq\ & k\alpha + \max_{j=1,\ldots,n} \Pi_L(\{s_j\}, A, \delta) \end{aligned} \tag{3.2.2}$$

By (3.2.1) and (3.2.2) and Proposition 3.2.3, it follows that

$$\Pi_L(M) \leq\ k\alpha + \max_{j=1,\ldots,n} \Pi_X(\{s_j\}) = k\alpha,$$

and therefore $\Pi_L(M) \leq k\gamma_L(M)$. □

The following example shows that the hypothesis (d) on the space in the above theorem cannot be cancelled.

Example 3.2.5 *Let $L := L_1\,[0,1,] \subseteq \mathbb{R}^{[0,1]}$ be the space of all real Lebesgue-integrable functions defined on the interval $[0,1]$. Then $\|f\|_L := \big|\int\limits_{[0,1,]} f\big|$ is a seminorm on L_1 which does not satisfy condition (d). In fact if we consider the subset $M = \{f_n = (n + 1/n)\,\chi_{[0,1/n]} : n \in \mathbb{N}\}$ of L, then $\gamma_L(M) = 0$ and $\Pi_L(M) = 1$.*

Theorem 3.2.6 *Let $M \subseteq L$ and $A \in \mathcal{A}_{0,L}$. Suppose $\beta_0(\chi_A M) < c := \sup_{y\in E} \|y\|_E$ and let $\beta_0(\chi_A M) < \delta < c$. Then*

$$\gamma_L(M) \leq 3\Pi_L(M, A, \delta) + k\,\|y_0\chi_A\|_L, \tag{3.2.3}$$

where $y_0 \in E$ and $\|y_0\|_E = \beta_0(\chi_A M)$.

Proof. Let $M \subseteq L$ and $A \in \mathcal{A}_{0,L}$. Then there are functions $f_1, ..., f_n \in M$ such that

$$\chi_A M \subseteq \overset{n}{\underset{i=1}{\cup}} (\chi_A f_i + B_{0,\delta}).$$

Fix $f \in M$ and let $i \in \{1, ..., n\}$ such that $\|\chi_A(f - f_i)\|_0 < \delta$. Set D_f :=$\{\|\chi_A(f - f_i)\|_E > \delta\}$, then $D_f \subseteq A$ and $\eta(D_f) < \delta$. Hence, by definition of η_0 there exists $C_f \in \mathcal{A}$ such that $D_f \subseteq C_f$ and $\eta(C_f) \leq \delta$. Then $B_f := A \cap C_f \subseteq A$, $\eta(B_f) \leq \delta$ and $\chi_{B_f}\chi_A f_i = \chi_{B_f} f_i$. Set $a := \Pi_L(M, A, \delta)$. By definition of $\Pi_L(M, A, \delta)$ it follows $\|\chi_{\Omega\setminus A} f\|_L \leq a$, $\|\chi_{Bf} f\|_L \leq a$ and $\|\chi_{B_f}\chi_A f_i\|_L = \|\chi_{B_f} f_j\|_L \leq a$. Therefore

$$\begin{aligned}
\|f - \chi_A f_i\|_L &\leq \|\chi_{\Omega\setminus A} f\|_L + \|\chi_A(f - f_i)\|_L \\
&\leq \|\chi_{\Omega\setminus A} f\|_L + \|\chi_{A\setminus B_f}(f - f_i)\|_L + \|\chi_{B_f} f\|_L + \|\chi_{B_f} f_i\|_L \\
&\leq 3\Pi_L(M, A, \delta) + \|\chi_{A\setminus B_f}(f - f_i)\|_L.
\end{aligned}$$

Let $y \in E$ such that $\|y\|_E > \delta$. Then the function $t \to \|ty\|_E$ ($t \in [0, 1]$) is continuous and $\lim\limits_{t\to 0}\|ty\|_E = 0$. Hence there are t_0 and $t_\delta \in [0, 1]$ such that $\|t_0 y\|_E = \beta_0(\chi_A M)$ and $\|t_\delta y\|_E = \delta$. We set $y_0 := t_0 y$ and $y_\delta := t_\delta y$, then $y_0,\ y_\delta \in E$. Since $A \in \mathcal{A}_{0,L}$, the simple function $\chi_A y_\delta \in L$. For each $f \in M$ we have

$$\|\chi_{A\setminus B_f}(f - f_i)(x)\|_E \leq \delta = \|y_\delta\|_E = \|y_\delta \chi_A(x)\|_E \text{ for all } x \in A.$$

Hence $\|\chi_{A\setminus B_f}(f - f_i)\|_E \leq \|y_\delta \chi_A\|_E$, so that

$$\|\chi_{A\setminus B_f}(f - f_i)\|_L \leq k\ \|y_\delta \chi_A\|_L.$$

Therefore

$$\gamma_L(M) \leq 3\Pi_L(M, A, \delta) + k\ \|y_\delta \chi_A\|_L,$$

Since $\lim\limits_{t_\delta \to t_0} \|t_\delta y\|_E = \|t_0 y\|_E$, by the continuity of $\|\cdot\|_L$ it follows that

$$\gamma_L(M) \leq 3\Pi_L(M, A, \delta) + k\ \|y_0 \chi_A\|_L.$$

$\square$

In the case in which the space L is endowed with a q-norm $\|\cdot\|_L$ (for $0 < q \leq 1$), i.e., $\|\alpha f\|_L = |\alpha|^q \|f\|_L$, then (3.2.3) assumes the following simpler form

$$\gamma_L(M) \leq 3\Pi_L(M, A, \delta) + k\ \beta_0(\chi_A M)^q \|y_0 \chi_A\|_L.$$

Remark 3.2.7 *If $c < \infty$ then the case $\beta_0(\chi_A M) = c$ can be considered. Then we have*

$$\gamma_L(M) \leq \Pi_L(M, A, \beta_0(\chi_A M)) + k\ \sup_{f\in M} \|\chi_A f\|_L$$

and

$$\gamma_L(M) \leq \Pi_L(M, A, \beta_0(\chi_A M)) + k\ \|y_0 \chi_A\|_L$$

if $\beta_0(\chi_A M) = \|y_0\|$ for $\|y_0\| \in E$.

As corollary of the theorem 3.2.6 we obtain a sufficient condition for the total boundedness of a subset of the space L.

Corollary 3.2.8 *Let M be an $\|\cdot\|_L$-eac subset of X and suppose that $\chi_A M$ is totally bounded in $(E^\Omega, \|\cdot\|_0)$ for all $A \in \mathcal{A}_{0,L}$. Then M is totally bounded in $(L, \|\cdot\|_L)$.*

Proposition 3.2.9 *Let M be a totally bounded subset of L. Assume that given a sequence $\{f_n\}$ in L, then $f_n \xrightarrow{\|\cdot\|_L} 0$ implies $\chi_A f_n \xrightarrow{\|\cdot\|_0} 0$ for all $A \in \mathcal{A}_{0,L}$. Then $\chi_A M$ is totally bounded in $(E^\Omega, \|\cdot\|_0)$ for all $A \in \mathcal{A}_{0,L}$.*

Proof. Let M be a totally bounded subset of $(L, \|\cdot\|_L)$ and let $A \in \mathcal{A}_{0,L}$. We prove first the following statement:

$$\text{for all } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } \chi_A B_\delta(L) \subset B_\epsilon(E^\Omega) \qquad (*)$$

Indeed, fixed $\epsilon > 0$, we can choose $k \in \mathbb{N}$ such that $\frac{1}{k} \leq \epsilon$. Suppose that for all $n \geq k$ we have $\chi_A B_{\frac{1}{n}}(L) \not\subseteq B_\epsilon(E^\Omega)$ and choose $f_n \in \left(\chi_A B_{\frac{1}{n}}(L) \setminus B_\epsilon(E^\Omega)\right)$. Thus we have $f_n \xrightarrow{\|\cdot\|_L} 0$, and therefore,$\chi_A f_n = f_n \xrightarrow{\|\cdot\|_0} 0$, a contradiction.
Fix now $\epsilon > 0$. By (*) there exist $\delta > 0$ such that $\chi_A B_\delta(L) \subset B_\epsilon(E^\Omega)$. Since M is totally bounded in $(L, \|\cdot\|_L)$, there exist a finite subset F of L such that $M \subset F + B_\delta(L)$. Hence $\chi_A M \subset \chi_A F + \chi_A B_\delta(L) \subset \chi_A F + B_\epsilon(E^\Omega)$, and therefore $\chi_A M$ is totally bounded in $(E^\Omega, \|\cdot\|_0)$. ☐

Combining Theorem 3.2.6 and Proposition3.2.9, we obtain the following Vitali-type total boundedness criterion

Theorem 3.2.10 *Assume that given a sequence $\{f_n\}$ in L, then $\|f_n\|_L \to 0$ implies $\|\chi_A f_n\|_0 \to 0$ for all $A \in \mathcal{A}_{0,L}$. Then a subset M of L is totally bounded in $(L, \|\cdot\|_L)$ iff it is $\|\cdot\|_L$-eac in L and $\chi_A M$ is totally bounded in $(E^\Omega, \|\cdot\|_0)$ for all $A \in \mathcal{A}_{0,L}$.*

3.3 The measure of non convex total boundedeness and the measure of non equiabsolute continuity

The next theorem shows the relationship between the measure of non equiabsolute continuity and the measure of non convex total boundedness.

Theorem 3.3.1 *Let $M \subset L$.*

(a) $\Pi_L(M) \leq \gamma_{w,L}(M)$;

(b) *Suppose that M is convex. Let $A \in \mathcal{A}_{0,L}$ and $\delta > \beta_{0,w}(\chi_A M)$. Then*

$$\gamma_{L,w}(M) \leq 3\Pi_L(M, A, \delta) + k \, \|y_0 \chi_A\|_L, \tag{3.3.1}$$

where $y_0 \in E$ and $\|y_0\|_E = \beta_{0,w}(\chi_A M)$.

Proof. (a) follows immedialely from Theorem 3.2.4 and Proposition 1.3.4 (a).
(b) Let $\delta > \beta_{0,w}(\chi_A M)$. Then there are functions $f_1, ..., f_n \in M$ and convex sets $C_1, ..., C_n$ of $B_\delta(E^\Omega)$ such that $\chi_A M \subset \bigcup_{i=1}^{n} (f_i \chi_A + C_i \chi_A)$. Since $M \subset \chi_A M + \chi_{\Omega \setminus A} M$, by properties (c) and (e) of Proposition 1.3.4, it is sufficient to show that

$$\gamma_{w,L}(\chi_A M) + \gamma_{w,L}(\chi_{\Omega \setminus A} M) \leq k \, \|y_0 \chi_A\|_L + 3\Pi_L(M, A, \delta). \tag{3.3.2}$$

Put $a := 3\Pi_L(M, A, \delta)$. Then $\|\chi_{\Omega \setminus A} f\|_L \leq a$ and $\|\chi_B f\|_L \leq a$ for all $B \in \mathcal{A}$, with $B \subset A$ and $\eta(B) < \delta$, and for all $f \in M$. Since M is convex, $K_i := [\chi_A M \cap (f_i \chi_A + C_i \chi_A)] - f_i \chi_A \quad (i = 1, ..., n)$ is convex. Clearly $\chi_A M \subset \bigcup_{i=1}^{n} (f_i \chi_A + K_i)$. Moreover, if we fix i and $g \in K_i$ then $g = f\chi_A - f_i \chi_A$ and as in the proof of theorem 3.2.6, it follows that $\|\chi_A (f - f_i)\|_L \leq \|y_\delta \chi_A|_L + 2a$ and so

$$\gamma_{w,L}(\chi_A M) \leq 2a + k \, \|y_0 \chi_A\|_L. \tag{3.3.3}$$

By the convexity of M we have that $\chi_{\Omega \setminus A} M$ is convex. Hence, since

$$\|\chi_{\Omega \setminus A} M\|_L \leq a \text{ for all } f \in M, \gamma_{w,L}(\chi_{\Omega \setminus A} M) \leq a. \tag{3.3.4}$$

Finally 3.3.2 follows immediately from 3.3.3 and 3.3.4. □

The following corollary of Theorem 3.3.1 (b) gives a sufficient condition for the convex total boundedness of a convex set of L.

Corollary 3.3.2 *Let M be a convex subsets of L. If M is $\| \cdot \|_L -$eac and $\chi_A M$ ctb in $(E^\Omega, \| \cdot \|_0)$ for all $A \in \mathcal{A}_{0,L}$, then M is ctb in $(L, \| \cdot \|_L)$.*

Remark 3.3.3 *Suppose that the condition $f_n \xrightarrow{\|\cdot\|_L} 0 \implies \chi_A f_n \xrightarrow{\|\cdot\|_0} 0$ for all $A \in \mathcal{A}_{0,L}$ holds in L. Then M ctb in $(L, \| \cdot \|_L)$ implies $\chi_A M$ ctb in $(E^\Omega, \| \cdot \|_0)$ for all $A \in \mathcal{A}_{0,L}$*

By Corollary 3.3.2 and Remark 3.3.3 we get the next convex total boundedness criterion.

Theorem 3.3.4 *Suppose that the condition* $f_n \xrightarrow{\|\cdot\|_L} 0 \Longrightarrow \chi_A f_n \xrightarrow{\|\cdot\|_0} 0$ *for all* $A \in \mathcal{A}_{0,L}$ *holds in* L. *A convex subset of* L *is ctb in* $(L, \|\cdot\|_L)$ *iff it is* $\|\cdot\|_L-$*eac in* L *and* $\chi_A M$ *is ctb in* $(E^\Omega, \|\cdot\|_0)$ *for all* $A \in \mathcal{A}_{0,L}$.

The following example [32, example 4.9] shows that the hypothesis of convexity in the theorem 3.3.1 cannot be cancelled.

Example 3.3.5 *Let* $\Omega = [1, \frac{\pi^2}{6}]$, $\mathcal{A}$ *the* $\sigma-$ *algebra of Lebesgue-measurable subsets of* Ω *and* μ *the Lebesgue measure on* $\mathcal{A}$. *Let* $L_{\frac{1}{2}}(\Omega, \mathcal{A}, \mu)$ *be the corrisponding Lebesgue spaces defined in [21, Chapter III], where* $\|f\|_{\frac{1}{2}} := \int\limits_\Omega |f|^p$. *Then* $L_{\frac{1}{2}}$ *is a space of type* L.

Fix a real number $\alpha \in]3, 4[$ *and define for* $n \in \mathbb{N}, n \neq 1$ *such that* $A_n := \left[\sum\limits_{k=1}^{n-1} \frac{1}{k^2}, \sum\limits_{k=1}^{n} \frac{1}{k^2}\right]$, $f_n := \chi_{A_n} n^\alpha$ *and* $M := \{f_n : n \in \mathbb{N}\}$. *Then*

(a) M *is ctb in* $(E^\Omega, \|\cdot\|_0)$;

(b) M *is equiabsolutely continuous in* $\left(L_{\frac{1}{2}}, \|\cdot\|_{\frac{1}{2}}\right)$;

(c) M *is not ctb in* $\left(L_{\frac{1}{2}}, \|\cdot\|_{\frac{1}{2}}\right)$.

3.4 Example

In this section we give an example of a class of $F-$seminormed spaces of type L. We start to consider the space L_N introduced in [16] in the same way as Dunford and Schwartz [21, p.112] define the space of integrable functions and the integral for integrable functions. We briefly recall the definition of the space L_N.
Assume that $(E, \|\cdot\|_E)$ is a complete $F-$normed space. Set $S_0(\mathcal{A}, \mathbb{R}) := span\{\chi_A : A \in \mathcal{A}_0\}$ and let $\|\cdot\| : S_0(\mathcal{A}, \mathbb{R}) \longrightarrow [0, +\infty[$ be a Riesz $F-$seminorm (a Riesz pseudonorm in the terminology of [1, p.39]) such that $\eta(A) = \|\chi_A\|$ for all $A \in \mathcal{A}_0$.
Let $\bar{n} \in \mathbb{N}$ and $N : [0, +\infty) \to [0, +\infty)$ be a continuous, strictly increasing function such that

(1) $N(0) = 0$

(2) $N(s + t) \leq \bar{n}(N(s) + N(t))$ for all $s, t \geq 0$.

The latter condition holds if and only if N satisfies the Δ_2-condition, that is, there is a constant $c \in [0, +\infty[$ with $N(2t) \leq cN(t)$ for all $t \geq 0$ (see [16, p. 90]).

For $s \in S_0(\mathcal{A}, E)$, $\|s\|_N$ is defined by

$$\|s\|_N = \|N \circ \|s\|_E\|,$$

and the space $L_N := L_N(E)$ is defined as [16, p. 92] the space of all functions $f \in L_0(\Omega, \mathcal{A}, E, \eta_0)$, for which there is a $\|\cdot\|_N$- Cauchy sequence (s_n) in $S_0(\mathcal{A}, E)$ converging to f with respect to $\|\cdot\|_0$, and

$$\|f\|_N = \lim_n \|s_n\|_N,$$

the sequence (s_n) is said to determine f.
We recall [16, Proposition 2.10 (b)], that $\|f\|_N = \|N \circ \|f\|_E\|_1$, for all $f \in L_N$.
The function $\|\cdot\|_N : L_N \longrightarrow [0, +\infty[$ has the following properties:

(1) $\|0\|_N = 0$;

(2) $\|f + g\|_N \leq 2\bar{n} \max\{\|f\|_N,\ \|g\|_N\}\ \forall f, g \in L_N$;

(3) $\|\alpha f\|_N \leq \|f\|_N,\ \forall \alpha \in \mathbb{R}$ with $|\alpha| \leq 1$ and $\forall f \in L_N$;

(4) $\lim_{\alpha \to 0} \|\alpha f\|_N = 0\ \forall f \in L_N$.

Therefore $\|\cdot\|_N$ is a $\Delta-$seminorm in the sense of [29, p. 2].
Moreover

$$\|f\|_E \leq \|g\|_E \text{ implies } \|f\|_N \leq \|g\|_N \quad \text{for all } f, g \in L_N \tag{3.4.1}$$

Proposition 3.4.1 *The space $L_N := (L_N, \|\cdot\|_N)$ satisfies the properties (a) - (b) - (c) and (d) of section 3.1.*

Proof.(see [16, Proposition2.6]) □

Set $L := L_N$. By [29, Theorem 1.2] if we choose p so that $2^{\frac{1}{p}} = 2\bar{n}$ then the formula

$$\|f\|_L = \inf\Big\{\sum_{i=1}^n \|f_i\|_N^p : \sum_{i=1}^n f_i = f\Big\}$$

defines an F-seminorm on L generating the same topology of the $\Delta-$seminorm $\|\cdot\|_N$.
Moreover, being $\frac{1}{4}\|f\|_N^p \leq \|f\|_L \leq \|f\|_N^p$ [29, see proof Theorem 1.2] using (3.4.1) we obtain that

$$\|f\|_E \leq \|g\|_E \text{ implies } \|f\|_L \leq 4\|g\|_L \quad \text{for all} f, g \in L_N. \tag{3.4.2}$$

By Proposition 3.4.1 and by 3.4.2 follows immediately that $(L, \|\cdot\|_L)$ is an $F-$seminormed space of type L.

Proposition 3.4.2 *[16, 2.2] The $\|\cdot\|_0-$topology on $S_0(\mathcal{A}, E)$ is coarser than $\|\cdot\|_N-$topology.*

As consequence of Proposition 3.4.2 we have the following

Proposition 3.4.3 *Let $\{f_n\}$ be a sequence in L, then $\|f_n\|_L \to 0$ implies $\|\chi_A f_n\|_0 \to 0$ for all $A \in \mathcal{A}_{0,L}$.*

Finally we obtain the following characterizations of totally bounded and convex totally bounded subsets in the spaces L.

Theorem 3.4.4 *A subset M of L is totally bounded in $(L, \|\cdot\|_L)$ iff it is $\|\cdot\|_L$-eac in L and $\chi_A M$ is totally bounded in $(E^\Omega, \|\cdot\|_0)$ for all $A \in \mathcal{A}_{0,L}$.*

Theorem 3.4.5 *A convex subset M of L_N is convexly totally bounded in $(L, \|\cdot\|_L)$ iff it is $\|\cdot\|_L$-eac in L and $\chi_A M$ is convexly totally bounded in $(E^\Omega, \|\cdot\|_0)$ for all $A \in \mathcal{A}_{0,L}$.*

Chapter 4

Implicit Functions in Locally Convex Spaces

In the last chapter we present a new implicit function theorem in locally convex spaces substituting the differentiability assumptions by the condition of the existence of a family of linear operator osculating with a given non linear map. As application we consider the problem of stability with respect to a parameter of solutions to Hammerstein type equation in a locally convex space.

4.1 Preliminaries.

Before providing the main results, we need to introduce some basic facts about locally convex topological linear spaces. We give these definitions following [28] , [33], [34]. Let X be a Hausdorff locally convex topological vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A family of continuous seminorms P which induces the topology of X is called a *calibration* for X. Denote by $\mathcal{P}(X)$ the set of all calibrations for X. A *basic calibration* for X is $P \in \mathcal{P}(X)$ such that the collection of all

$$U(\varepsilon, p) = \{x \in X : p(x) \leq \varepsilon\}, \varepsilon > 0, p \in P,$$

is a neighborhood base at 0. Observe that $P \in \mathcal{P}(X)$ is a basic calibration for X if and only if for each $p_1, p_2 \in P$ there is $p_0 \in P$ such that $p_i(x) \leq p_0(x)$ for $i = 1, 2$ and $x \in X$. Given $P \in \mathcal{P}(X)$, the family of all maxima of finite subfamily of P is a basic calibration.

A linear operator L on X is called *P-bounded* if there exists a constant $C > 0$ such that

$$p(Lx) \leq Cp(x), x \in X, p \in P.$$

Denote by $\mathcal{L}(X)$ the space of all continuous linear operators on X and by $\mathcal{B}_P(X)$ the space of all P-bounded linear operators L on X. We have $\mathcal{B}_P(X) \subset \mathcal{L}(X)$.

Moreover, the space $\mathcal{B}_P(X)$ is a unital normed algebra with respect to the norm

$$\|L\|_P = \sup\{p(Lx) : x \in X, p \in P \text{ and } p(x) = 1\}.$$

We say that a family $\{L_\alpha : \alpha \in I\} \subset \mathcal{B}_P(X)$ is *uniformly P-bounded* if there exists a constant $C > 0$ such that

$$p(L_\alpha x) \leq Cp(x), x \in X, p \in P,$$

for any $\alpha \in I$.

In the following we will assume that X is a complete Hausdorff locally convex topological linear space and that $P \in \mathcal{P}(X)$ is a basic calibration for X.

4.2 An Implicit Function Theorem in Locally Convex Spaces

Let Λ be an open subset of the real line $\mathbb{R}$ or of the complex plane $\mathbb{C}$. Consider the product space $\Lambda \times X$ of Λ and X provided with the product topology. Let A be an open subset of $\Lambda \times X$ and $(\lambda_0, x_0) \in A$. Consider a nonlinear operator $\Phi : A \longrightarrow X$ and the related equation

$$\Phi(\lambda, x) = 0. \tag{1}$$

Assume that (λ_0, x_0) is a solution of the above equation. A fundamental problem in nonlinear analysis is to study solutions (λ, x) of the equation (1) for λ close to λ_0.

We say that an operator $\Phi : A \longrightarrow X$ and an operator $L : \Lambda \longrightarrow \mathcal{L}(X)$ are called *P-osculating* at $(\lambda_0,\, x_0)$ if there exist a function $\sigma : \mathbb{R}^2 \to [0, +\infty)$ and $q \in P$ such that

$$\lim_{(\rho,\, r) \to (0,0)} \sigma(\rho, r) = 0$$

and for any $p \in P$

$$p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)) \leq \sigma(\rho, r)\, p(x_1 - x_2),$$

when $|\lambda - \lambda_0| \leq \rho$ and $x_1, x_2 \in x_0 + U(r, q)$.

Now we prove our main result.

Theorem 4.2.1 *Suppose that $\Phi : A \longrightarrow X$ and (λ_0, x_0) satisfy the following conditions:*

(a) *(λ_0, x_0) is a solution of the equation (1) and the operator $\Phi(\cdot, x_0)$ is continuous at λ_0;*

(b) *there exists an operator function $L : \Lambda \longrightarrow \mathcal{L}(X)$ such that Φ and L are P-osculating at (λ_0, x_0);*

(c) *the linear operator $L(\lambda)$ is invertible and $L(\lambda)^{-1} \in \mathcal{B}_P(X)$ for each $\lambda \in \Lambda$. Moreover the family $\left\{L(\lambda)^{-1} : \lambda \in \Lambda\right\}$ is uniformly P-bounded.*

Then there are $\varepsilon > 0$, $q \in P$ and $\delta > 0$ such that, for each $\lambda \in \Lambda$ with $|\lambda - \lambda_0| \leq \delta$, the equation (1) has a unique solution $x(\lambda) \in x_0 + U(\varepsilon, q)$.

Proof. Let Φ and $L : \Lambda \longrightarrow L(X)$ be P-osculating at (λ_0, x_0). Consider the operator $T : A \longrightarrow X$ defined by

$$T(\lambda, x) = x - L(\lambda)^{-1}\Phi(\lambda, x).$$

Let $p \in P$. By the assumption (c) there exist $C > 0$ such that

$$p(T(\lambda, x_1) - T(\lambda, x_2)) \leq C\, p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)),$$

for any $(\lambda, x_1), (\lambda, x_2) \in A$. Moreover, since Φ and L are P-osculating at (λ_0, x_0), there are a function $\sigma : R^2 \to [0, +\infty)$ and $q \in P$ such that

$$p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)) \leq \sigma(\rho, r)p(x_1 - x_2),$$

for $|\lambda - \lambda_0| \leq \rho$ and $x_1, x_2 \in x_0 + U(r, q)$. Hence

$$p(T(\lambda,\ x_1) - T(\lambda,\ x_2)) \leq C\sigma(\rho, r)p(x_1 - x_2),$$

for $|\lambda - \lambda_0| \leq \rho$ and $x_1, x_2 \in x_0 + U(r, q)$.

Choose $\varepsilon > 0$ such that

$$p(T(\lambda, x_1) - T(\lambda, x_2)) \leq \frac{1}{2}p(x_1 - x_2),$$

for $|\lambda - \lambda_0| \leq \varepsilon$ and $x_1, x_2 \in x_0 + U(\varepsilon, q)$. Therefore, for each $\lambda \in \Lambda$ such that $|\lambda - \lambda_0| \leq \varepsilon$, the operator $T(\lambda, \cdot\,)$ from $x_0 + U(\varepsilon, q)$ into X is a contraction in the sense of [39].

Since $\Phi(\,\cdot\,, x_0)$ is continuous at λ_0, we may further find $\delta' > 0$ such that

$$p(\Phi(\lambda, x_0)) \leq \frac{\varepsilon}{2C},$$

and

$$p(T(\lambda, x_0) - x_0) \leq Cp(\Phi(\lambda, x_0)) \leq \frac{\varepsilon}{2},$$

for $|\lambda - \lambda_0| \leq \delta'$. Set $\delta := \min\{\varepsilon, \delta'\}$ we have

$$p(T(\lambda, x) - x_0) \leq p(T(\lambda, x) - T(\lambda, x_0)) + p(T(\lambda, x_0) - x_0) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for $|\lambda - \lambda_0| \leq \delta$ and $x \in x_0 + U(\varepsilon, q)$. This shows that

$$T(\lambda, \cdot)(x_0 + U(\varepsilon, q)) \subseteq x_0 + U(\varepsilon, q),$$

for each λ such that $|\lambda - \lambda_0| \leq \delta$. Then, by [39, Theorem 1.1], when $|\lambda - \lambda_0| \leq \delta$, the operator $T(\lambda, \cdot\,)$ has a unique fixed point $x(\lambda) \in x_0 + U(\varepsilon, q)$, which is obviously a solution of the equation (1). $\square$

4.3 An application

As an example of application of our main result, we study the stability of the solutions of an operator equation with respect to a parameter.

Consider in X the Hammerstein equation

$$x = \lambda KFx, \tag{2}$$

containing a parameter $\lambda \in \Lambda$. In our case K is a continuous linear operator on X and $F : X \to X$ is the so-called superposition operator. We have the following:

Theorem 4.3.1 *Let K be P-bounded. Suppose that for each $x \in X$ there exists $q \in P$ such that the operator F satisfies the Lipschitz condition*

$$p\left(Fx_1 - Fx_2\right) \leq \omega\left(r\right) p\left(x_1 - x_2\right),$$

for any $p \in P$ and $x_1, x_2 \in x + U\left(r, q\right)$, where $\lim_{r \to 0} \omega(r) = 0$. If $x_0 \in X$ is a solution of the equation (2) for $\lambda = \lambda_0$, then there exist $\varepsilon > 0$ and $\delta > 0$ such that, for each $\lambda \in \Lambda$ with $|\lambda - \lambda_0| \leq \delta$, the equation (2) has a unique solution $x(\lambda) \in x_0 + U\left(\varepsilon, q\right)$.

Proof. Since the linear operator K is P-bounded we can find a constant $C > 0$ such that

$$p\left(Kx\right) \leq Cp\left(x\right), x \in X, p \in P.$$

If $\lambda = 0$ then $x_0 = 0$ is clearly a solution of the equation (2). Consider the operator $\Phi_0 : \Lambda \times X \longrightarrow X$ defined by

$$\Phi_0(\lambda, x) = x - \lambda KFx,$$

and set $L_0(\lambda)x = x$ for any $\lambda \in \Lambda$ and $x \in X$. Clearly the operator $\Phi\left(\cdot, 0\right)$ is continuous at 0. By the hypothesis made on the operator F, there exists $q \in P$ such that

$$p\left(\Phi_0\left(\lambda, x_1\right) - \Phi_0\left(\lambda, x_2\right) - L_0(\lambda)(x_1 - x_2)\right) \leq C\rho\omega\left(r\right) p\left(x_1 - x_2\right),$$

for any $p \in P$, when $|\lambda| \leq \rho$ and $x_1, x_2 \in U\left(r, q\right)$, the operators Φ_0 and L_0 are P-osculating at $(0, 0)$. Moreover, for each $\lambda \in \Lambda$, we have $L_0(\lambda)^{-1} = L_0(\lambda)$ and $p\left(L_0(\lambda)^{-1}x\right) = p\left(x\right)$, for any $x \in X$ and $p \in P$. Then the result follows by Theorem 4.2.1.

Now assume that $x_0 \in X$ is a solution of the equation (2) for some $\lambda_0 \neq 0$. Let $\Phi : \Lambda \times X \longrightarrow X$ be defined by

$$\Phi(\lambda, x) = \frac{x}{\lambda} - KFx,$$

and set $L(\lambda)x = \frac{x}{\lambda}$ for any $\lambda \in \Lambda$ and $x \in X$. The operator $\Phi(\cdot, x_0)$ is continuous at λ_0 and there exists $q \in P$ such that

$$p\left(\Phi\left(\lambda, x_1\right) - \Phi\left(\lambda, x_2\right) - L(\lambda)(x_1 - x_2)\right) \leq C\omega\left(r\right) p\left(x_1 - x_2\right),$$

for any $p \in P$, when $\lambda \in \Lambda$ and $x_1, x_2 \in x_0 + U(r, q)$. So the operators Φ and L are P-osculating at (λ_0, x_0). Further, assuming $|\lambda - \lambda_0| \leq a$ for some $a > 0$, we can find $b > 0$ such that $p\left(L(\lambda)^{-1}x\right) \leq bp\left(x\right)$, for any $p \in P$ and $x \in X$. As before, the proof is completed by appealing to Theorem 4.2.1. $\square$

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