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Concentration and asymptotic behavior of solutions for some singularly perturbed mixed problems

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Introduction

The theory of *Partial Differential Equations,* and in particular the *Nonlinear Equations,* is an important tool to describe real models. In fact, many problems coming from science and engineering are described by nonlinear differential equations, which can be notoriously difficult to solve. This is also the case of the equations we study in this thesis.

The aim of this work is to analyze some elliptic equations that are perturbative in nature. We will examine our problem using two tools:

(i) perturbative methods;

(ii) variational methods.

In particular, in this thesis, we are interested in the following perturbed mixed problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$\tilde{P}_\varepsilon$}$$

where Ω is a smooth bounded subset of $\mathbb{R}^n$, $p \in \left(1, \frac{n+2}{n-2}\right)$, $\varepsilon > 0$ is a small parameter, and $\partial_{\mathcal{N}}\Omega$, $\partial_{\mathcal{D}}\Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$. This type of perturbative equations, with Neumann conditions or Dirichlet conditions, were been studied in literature by many authors, see for example [2], [21], [49], [50] and [51].

Nevertheless these problems, with mixed conditions, appear in several situations. Generally the Dirichlet condition is equivalent to impose some *state* on the physical parameter represented by u, while the Neumann conditions give a meaning at the flux parameter crossing $\partial_{\mathcal{N}}\Omega$. Here below there are some common physical applications of such problems:

- *Population dynamics.* Assume that a species lives in a bounded region Ω such that the boundary has two parts, $\partial_{\mathcal{N}}\Omega$ and $\partial_{\mathcal{D}}\Omega$, where the first one is an obstacle that blocks the pass across, while the second one is a killing zone for the population.

- *Nonlinear heat conduction.* In this case $(\tilde{P}_\varepsilon)$ models the heat (for small conductivity) in the presence of a nonlinear source in the interior of the domain, with combined isothermal and isolated regions at the boundary.

- *Reaction diffusion with semi-permeable boundary.* In this framework we have that the meaning of the Neumann part, $\partial_{\mathcal{N}}\Omega$, is an obstacle to the flux of the matter, while the Dirichlet part, $\partial_{\mathcal{D}}\Omega$, stands for a semipermeable region that allows the outwards transit of the matter produced in the interior of the cell Ω by the reaction represented by a general nonlinearity $f(u)$.

Partial differential equations, as the one in $(\tilde{P}_\varepsilon)$, also appear in the study of reaction-diffusion systems. For single equations with Neumann boundary conditions it is known that when Ω is convex the only stable solutions are constants, see [12] and [45]. On the other hand, as noticed in [55], reaction-diffusion *systems* with different diffusivities might lead to non-homogeneous stable steady states. A well-know example is the following one (Gierer-Meinhardt)

$$\begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty), \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \qquad (GM)$$

introduced in [28] to describe some biological experiment. The functions $\mathcal{U}$ and $\mathcal{V}$ represent the densities of some chemical substances, the numbers p, q, r, s are non-negative and such that $0 < \frac{p-1}{q} < \frac{r}{s+1}$, and it is assumed that the diffusivities d_1 and d_2 satisfy $d_1 \ll 1 \ll d_2$. In the stationary case of (GM), as explained in [48], when $d_2 \to +\infty$ the function $\mathcal{V}$ is close to a constant (being nearly harmonic and with zero normal derivative at the boundary), and therefore the equation satisfied by $\mathcal{U}$ resembles the first one in $(\tilde{P}_\varepsilon)$. Clearly a similar reduction procedure could be used when mixed boundary conditions are imposed.

Let us now describe some results which concern singularly perturbed problems, with Neumann or Dirichlet boundary conditions, well studied by different authors (see, for example, [2], [20], [48], [49], [50] and [51]), and specifically

$$(N_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial_{\mathcal{N}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases}$$

$$(D_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega; \\ u = 0 & \text{on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega. \end{cases}$$

These problems arise also as limits of reaction-diffusion systems different from (GM) (with chemotaxis for example, as shown in [48]).

Another motivation comes from the Nonlinear Schrödinger equation (in short NLS)

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^n,$$

where ψ is a complex-valued function (the *wave function*), V is a potential and p is an exponent greater than 1. NLS of this sort are used, for example, in Plasma Physics, but also arise in Nonlinear Optics. Indeed, if one looks for *standing waves*, namely solutions of the form $\psi(x,t) = e^{-\frac{i\omega t}{\hbar}} u(x)$, for some real function u, then the latter will satisfy

$$-\varepsilon^2\Delta u + V(x)u = u^p \quad \text{in } R^n,$$

where we have set $\varepsilon = \hbar$ and we absorbed the constant ω into the potential V. Therefore, up to the potential, we still obtain the equation in (N_ε) or (D_ε): about this subject we refer the reader to the (still incomplete) list of papers [1], [2], [6], [24] and to the bibliographies therein.

The typical concentration behavior of solutions u_ε to the above two problems is via a scaling of the variables in the form $u_\varepsilon(x) \sim U\left(\frac{x-Q}{\varepsilon}\right)$, where Q is some point of $\overline{\Omega}$, and U, see Section 1.1, is a solution of

$$-\Delta U + U = U^p \quad \text{in } \mathbb{R}^n \quad (\text{or in } \mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}), \tag{1}$$

the domain depending on whether Q lies in the interior of Ω or at the boundary; in the latter case Neumann conditions are imposed. When $p < \frac{n+2}{n-2}$ (and indeed only if this inequality is satisfied), such problem (1) admits positive radial solutions which decay to zero at infinity, see Proposition 1.1.1.

Solutions of $(\tilde{P}_\varepsilon)$ which inherit this profile are called *spike layers*, since they are highly concentrated near some point of $\overline{\Omega}$. There is an extensive literature regarding this type of solutions, beginning from the papers [38], [49] and [50]. Indeed their structure is very rich, and we refer for example to the (far from complete) list of references [17], [20], [31], [32], [33], [34], [36], [37], [56] and [57].

Spike layers solving (N_ε) and sitting on the boundary of Ω are peaked near critical points of the mean curvature of $\partial\Omega$. To see this one can exploit the variational structure of the problem, whose Euler functional is

$$\tilde{I}_{\varepsilon,\mathcal{N}}(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) - \frac{1}{p+1}\int_\Omega |u|^{p+1}; \qquad u \in H^1(\Omega). \quad (2)$$

Plugging into $\tilde{I}_{\varepsilon,\mathcal{N}}$ a function of the form $U_{Q,\varepsilon}(x) = U\left(\frac{1}{\varepsilon}(x-Q)\right)$ with $Q \in \partial\Omega$ one sees that

$$\tilde{I}_{\varepsilon,\mathcal{N}}(U_{Q,\varepsilon}) = \tilde{C}_0\varepsilon^n - \tilde{C}_1\varepsilon^{n+1}H(Q) + o(\varepsilon^{n+1}),$$

where $\tilde{C}_0, \tilde{C}_1$ are positive constants depending only on n and p and H denotes the mean curvature.

This expansion can be obtained using the radial symmetry of U and parameterizing $\partial\Omega$ as a normal graph near Q. From the above formula we see that, the bigger is the mean curvature the lower is the energy of this function: roughly speaking, boundary spike layers would tend to move along the gradient of H in order to minimize their energy. Indeed in [49], [50] it was shown that Mountain Pass solutions to (N_ε) (ground states) have only one local maximum over $\overline{\Omega}$ and it concentrates at $\partial\Omega$ near global maxima of the mean curvature.

Concerning instead (D_ε), spike layers with minimal energy concentrate at the interior of the domain, at points which maximize the distance from the boundary, see [51]. The intuitive reason for this is that, if Q is in the interior of Ω and if we want to adapt a function like $U\left(\frac{1}{\varepsilon}(x-Q)\right)$ to the Dirichlet conditions, the adjustment needs an energy which increases as Q becomes closer and closer to $\partial\Omega$. Following the above heuristic argument, we could say that spike layers are *repelled* from the regions where Dirichlet conditions are imposed.

In this thesis, we are interested in finding boundary spike layers for the mixed problem $(\tilde{P}_\varepsilon)$. First, we apply a perturbative approach: the idea is to obtain two compensating effects from the Neumann and the Dirichlet conditions. More precisely, calling $\mathcal{I}_\Omega$ the intersection of the closures of $\partial_\mathcal{D}\Omega$ and $\partial_\mathcal{N}\Omega$, and assuming that the gradient of H at $\mathcal{I}_\Omega$ points toward $\partial_\mathcal{D}\Omega$, a spike layer centered on $\partial_\mathcal{N}\Omega$ will be *pushed* toward $\mathcal{I}_\Omega$ by ∇H and will be *repelled* from $\mathcal{I}_\Omega$ by the Dirichlet condition.

Our main result will show that there exists a solution u_ε to the problem $(\tilde{P}_\varepsilon)$ concentrating at the interface $\mathcal{I}_\Omega$. The general strategy used relies on a finite-dimensional reduction, which is conceptually rather simple and nowadays well understood, see Chapter 1 or, for example, the book [2]. One finds first a manifold Z of approximate solutions to the given problem, which in our case are of the form $U\left(\frac{1}{\varepsilon}(x-Q)\right)$, and solve the equation up to a vector (in the Hilbert space) parallel to the tangent plane of this manifold.

In this way, see Proposition 2.1.3, one generates a new manifold $\tilde{Z}$ close to Z which represents a natural constraint for the Euler functional of $(\tilde{P}_\varepsilon)$, which is

$$\tilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}; \qquad u \in H^1_{\mathcal{D}}(\Omega). \tag{3}$$

Here $H^1_{\mathcal{D}}(\Omega)$ stands for the space of functions in $H^1(\Omega)$ which have zero trace on $\partial_{\mathcal{D}}\Omega$, and by *natural constraint* we mean a set for which constrained critical points of $\tilde{I}_\varepsilon$ are true critical points.

The main difficulty however is to have a good control of $\tilde{I}_\varepsilon|_{\tilde{Z}}$, which is done improving the accuracy of the functions in the original manifold Z: in fact, the better is the accuracy of these functions, the closer is $\tilde{Z}$ to Z, so the main term in the constrained functional will be given by $\tilde{I}_\varepsilon|_Z$, see Proposition 2.2.12 and Lemma 2.3.5 below. To find sufficiently good approximate solutions we start with those constructed in literature for the Neumann problem (N_ε) (see Chapter 2, Section 2.1.2) which reveal the role of the boundary mean curvature, as in the expansion after (2). However these functions are not zero on $\partial_{\mathcal{D}}\Omega$, and if one tries naively to annihilate them using cut-off functions, the corresponding error turns out to be too large. A method which revealed itself to be useful for (D_ε) is to consider the *projection operator* in $H^1(\Omega)$, which consists in associating to some function in this space its closest element in $H^1_{\mathcal{D}}(\Omega)$. In some previous works, see for example [37] and [58], the asymptotic behavior of this projection has been studied in detail when the limit concentration point lies in the interior of Ω, using (see Chapter 1, equations (1.3) and (1.4)) the well known limit behavior of the solution U to (1),

$$\lim_{r\to+\infty} e^r r^{\frac{n-1}{2}} U(r) = \alpha_{n,p},$$

where the positive constant $\alpha_{n,p}$ depends only on n and p, together with

$$\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1; \qquad \lim_{r\to+\infty} \frac{U''(r)}{U(r)} = 1.$$

In our case instead, apart from having mixed conditions, the maxima of the spike-layers tend to the interface $\mathcal{I}_\Omega$, so, to better understand the projection, we need to work at a scale $d \simeq \varepsilon|\log\varepsilon|$, the order of the distance of the peak from $\mathcal{I}_\Omega$. At this scale the boundary of the domain looks nearly flat, so in this step we replace Ω with a non smooth domain $\hat{\Gamma}_D \subseteq \mathbb{R}^n$ such that part of $\partial\hat{\Gamma}_D$ looks like a cut of dimension $n-1$. We choose $\hat{\Gamma}_D$ to be even with respect to the coordinate x_n and we study H^1 projections here (with Dirichlet conditions) which are also even in x_n: as a consequence we will find functions which have zero x_n-derivative on $\{x_n = 0\} \setminus \partial\hat{\Gamma}_D$, which mimics the Neumann boundary condition on $\partial_{\mathcal{N}}\Omega$. After analyzing carefully the projection we define a family of suitable approximate solutions to $(\tilde{P}_\varepsilon)$, which turn out to have a sufficient accuracy for our analysis.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite dimensional one, and study the functional constrained on $\tilde{Z}$. If z_Q^ε denotes (roughly speaking) an approximate solution peaked at Q, with $dist(Q, \mathcal{I}_\Omega) = d_\varepsilon$, then its energy turns out to be the following

$$\tilde{I}_\varepsilon(z_Q^\varepsilon) = \varepsilon^n \left(\tilde{C}_0 - \tilde{C}_1 \varepsilon H(Q) + e^{-2\frac{d_\varepsilon}{\varepsilon}(1+o(1))} + O(\varepsilon^2) \right).$$

The first two terms in this formula are as in the expansion after (2) for (N_ε), while the third one represents a sort of *potential energy* which decreases with the distance of Q from the interface, consistently with the *repulsive effect* which was described before for (D_ε). From the latter formula it follows that, if d_ε remains constant, then we recover the expansion corresponding to the Neumann problem. On the other hand even if $d_\varepsilon \to 0$, that is when the concentration points converge to the interface, one can check that the energy has a critical point when $H|_{\mathcal{I}_\Omega}$ is stationary, provided ∇H points toward $\partial_{\mathcal{D}}\Omega$, and $d_\varepsilon \simeq \varepsilon|\log \varepsilon|$.

Next in the thesis, via variational methods, we analyze also the asymptotic profile of the least energy solutions to the problem $(\tilde{P}_\varepsilon)$ under generic assumptions on the domain and on the interface.

First we show that Mountain Pass solutions are in fact least energy solutions. Then we prove that, given a family of least energy solutions $\{u_\varepsilon\}$, their points of maximum must lie on the boundary of the domain Ω, as in the Neumann case.

We also analyze the rate of convergence to specify better the location of maximum limit points P_ε of the least energy solutions as $\varepsilon \to 0$: we show that the concentration point cannot belong to the interior of Dirichlet boundary part. To obtain these results, we use the moving plane method proving also an important Liouville type result, see Lemma 3.3.2. Next, we characterize the shape of least energy solutions showing that, also in the Mixed case, such solutions can be approximated by the ground state solution U to the problem (1). This fact follows from other results proved in the thesis; in particular we have that, after a scaling, the maximum P_ε (indeed unique, see Theorem 3.3.9) of the solutions u_ε is always bounded away from the interface $\mathcal{I}_\Omega$ as $\varepsilon \to 0$.

Moreover, we prove that, as for (N_ε), the least energy solutions concentrate at boundary points in the closure of $\partial_{\mathcal{N}}\Omega$ where the mean curvature is maximal. When this constrained maximum is attained on the interface (and if ∇H here is non zero), we will be able to show that the Mountain Pass solution has precisely the behavior found in Theorem 2.0.1 by perturbative methods.

In the last part of the thesis we consider the least energy solutions to the problem $(\tilde{P}_\varepsilon)$ and, via numerical algorithm, we construct their shape and we present the related results.

We use a numerical method which allows us to find solutions of Mountain Pass type. Such a method was introduced by Y.S.Choi and P.J.McKenna in [14] for some elliptic equations in square domains. Instead, we consider a particular case of $(\tilde{P}_\varepsilon)$, choosing $p = 3$ and $n = 2$, namely

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^3 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$\tilde{\tilde{P}}_\varepsilon$}$$

where Ω is a bounded domain of $\mathbb{R}^2$.

Such a problem is perturbative one with mixed boundary conditions that are numerically difficult to deal with.

We define problem $(\tilde{\tilde{P}}_\varepsilon)$ in a bounded elliptical domain of $\mathbb{R}^2$ in order to have a non constant mean curvature H to find Mountain Pass type solutions concentrating at the interface $\mathcal{I}_\Omega$. Then, we need to *mesh* Ω in order to describe and define the *discrete differential problem* associated to $(\tilde{\tilde{P}}_\varepsilon)$.

We want to point out that, from the numerical point of view, curved boundary domains, such as the elliptical ones, are generally more difficult to treat than the square ones.

All the algorithm, used to get the shape of least energy (Mountain Pass type) solutions of $(\tilde{\tilde{P}}_\varepsilon)$, was implemented with a MATLAB code.

This thesis is organized as follows.

In **Chapter 1** we will recall some tools of nonlinear analysis which are employed in the rest of the thesis. For a better understanding, we will examine the perturbative approach in critical point theory, in particular referring to the singularly perturbed Neumann problem. Then we will recall some definitions and some theorems about critical point theory. We also will give some notations.

In **Chapter 2** we will deal with the existence results and the asymptotic behavior of some solutions to the singularly perturbed problem $(\tilde{P}_\varepsilon)$ with mixed Dirichlet and Neumann conditions by using perturbative methods. These results were been obtained in [25].

In **Chapter 3** we will analyze the location and the shape of the *least energy solutions* to the problem $(\tilde{P}_\varepsilon)$, when the singular perturbation parameter tends to zero, via variational methods. The results presented in this chapter were been achieved in [26].

In **Chapter 4** we will present some results about the construction of the shape of the *least energy solutions* to the problem $(\tilde{P}_\varepsilon)$ via numerical methods. The results presented in this chapter are part of a work in progress.

Chapter 1

Preliminaries

In this chapter we will recall the fundamental tools and definitions which will be employed in this thesis. We also will give some notations.

1.1 A uniqueness result

Let us consider the following problem

$$-\Delta U + U = U^p \qquad \text{in } \mathbb{R}^n. \tag{1.1}$$

Corresponding to (1.1) we define an *energy* of a function $u \in H^1(\mathbb{R}^n)$:

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} \tag{1.2}$$

Proposition 1.1.1 *If* $p \in \left(1, \frac{n+2}{n-2}\right)$ *problem* (1.1) *admits a solution* U *satisfying*

(i) $U \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ *and* $U > 0$ *in* $\mathbb{R}^n$*;*

(ii) U *is spherically symmetric:* $U(x) = U(r)$ *with* $r = |x|$ *and* $\frac{dU}{dr} < 0$ *for* $r > 0$*;*

(iii) U *and its first derivatives decay exponentially at infinity, i.e., there exists a positive constant* $\alpha_{n,p}$ *depending only on* n *and* p *such that*

$$\lim_{r\to+\infty} e^r r^{\frac{n-1}{2}} U(r) = \alpha_{n,p}, \tag{1.3}$$

and

$$\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1; \qquad \lim_{r\to+\infty} \frac{U''(r)}{U(r)} = 1; \tag{1.4}$$

(iv) for any non-negative solution $u \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ *to* (1.1),

$$0 < I(U) \leq I(u)$$

holds unless $u \equiv 0$.

Remark 1.1.2 *M.K.Kwong [35] showed that radial solutions to* (1.1) *are unique when* $p \in \left(1, \frac{n+2}{n-2}\right)$.

Remark 1.1.3 *Sometimes, through the thesis, we will call* ground state*, the solution given by Proposition 1.1.1 and Remark 1.1.2 .*

1.2 Perturbation methods

In this section, we want to give just a very short outline to the perturbation methods to better understand the basic ideas used in the sequel of the thesis. However for more details about the perturbation methods we refer strongly to the book [2].

There are several elliptic problems on $\mathbb{R}^n$ which are perturbative in nature. For these perturbation problems a specific approach, that takes advantage of such a perturbative setting, seems the most appropriate. Actually, it turns out that such framework can be used to handle a large variety of equations, different in nature. These problems can be studied using a common abstract setting.

Let $\mathcal{H}$ be a Hilbert space. We look for critical point of a sufficiently smooth functional $I_\varepsilon : \mathcal{H} \to \mathbb{R}$ depending on a small *perturbative* parameter $\varepsilon \in \mathbb{R}$, that is solutions of equation

$$I'_\varepsilon(u) = 0, \quad u \in \mathcal{H}. \tag{1.5}$$

Generally, in the perturbative methods, one arranges the functional I_ε associated to the differential problem in a functional like

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u),$$

where I_0 is called *unperturbed functional* and G is a *perturbation.* On the Perturbation theory one supposes that I_0 possesses non-isolated critical points which form a *manifold* Z referred to as a *critical manifold,* that is:

$$Z = \{z \in \mathcal{H} : I'_0(z) = 0\}. \tag{1.6}$$

Let $T_z Z$ be the tangent space to Z at z, we denote the orthogonal complement to $T_z Z$ as

$$W = (T_z Z)^\perp. \tag{1.7}$$

In this case, finding solutions of $I'_\varepsilon(u) = 0$ becomes a kind of bifurcation problem in which $z \in Z$ is the bifurcation parameter and the set $\{0\} \times$

$Z \subset \mathbb{R} \times \mathcal{H}$ is the set of the trivial solutions. One looks for conditions on the perturbation G that generate non-trivial solutions, namely a pair $(\varepsilon, u) \in \mathbb{R} \times \mathcal{H}$, with $\varepsilon \neq 0$, such that $I'_\varepsilon(u) = 0$. In this context, in order to solve equation (1.5) one uses a finite-dimensional reduction procedure. This is the classical Lyapunov-Schmidt method, with appropriate modifications, which allow us to take advantage of the variational nature of our equations. Roughly, under appropriate non degeneracy conditions on Z, it is possible to show that critical points of the perturbation G constrained on Z give rise to critical point of I_ε.

1.2.1 Critical points for the perturbed functional I_ε

In this section we consider functional of the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u)$$

where I_0, G belong to $C^2(\mathcal{H}, \mathbb{R})$. We suppose also that the manifold Z defined in (1.6), is finite dimensional:

$$0 < d = dim(Z) < \infty.$$

If Z is a critical manifold then one has that

$$I'_0(z) = 0 \quad \forall z \in Z.$$

Differentiating this identity, we get

$$(I''_0(z)[v]|\phi) = 0 \qquad \forall v \in T_z Z, \forall \phi \in H,$$

and this shows that every $v \in T_z Z$ is a solution of the linearized equation $I''_0(z)[v] = 0$, that is $I''_0(z)$ has a non-trivial kernel whose dimension is at least d and hence all $z \in Z$ are degenerate critical points of I_0. In this context we shall require that this degeneracy is minimal supposing that

$$(ND) \qquad T_z Z = Ker[I''_0(z)] \quad \forall z \in Z$$

and also that

$$(FR) \qquad \forall z \in Z, I''_0(z) \text{ is an index 0 Fredholm map.}$$

Remark 1.2.1 *A linear map $T \in L(\mathcal{H}, \mathcal{H})$ is Fredholm if the kernel is finite-dimensional and the image is closed and has a finite codimension. The* index *of T is defined as the difference between the dimension of kernel and the codimension of the image of the linear operator T.*

Next, the idea is to apply a finite-dimensional reduction (Lyapunov-Schmidt procedure) and to look for critical points of I_ε in the form $u = z + w$ with $z \in Z$, see equation (1.6), and $w \in W$, see equation (1.7). If $P_z : \mathcal{H} \to W$ denotes the projection onto the orthogonal complement of $T_z Z$, the equation $I'_\varepsilon(z+w) = 0$ is equivalent to the system

$$\begin{cases} P_z I'_\varepsilon(z+w) = 0 & (\textit{auxiliary equation}); \\ (Id - P_z) I'_\varepsilon(z+w) = 0 & (\textit{bifurcation equation}). \end{cases} \tag{1.8}$$

In Chapter 2 we will show how to solve these last equations in the mixed case $(\tilde{P}_\varepsilon)$ considered.

1.2.2 The singular perturbation case

When we deal with singular perturbation problems, as in the case of this thesis, we need to modify a little the abstract setting given before. Unlike the preceding case where we have defined a critical unperturbed manifold Z, see equation (1.6), we have to consider that the functional I_ε associated to the problem $(\tilde{P}_\varepsilon)$ possesses a manifold Z_ε of *pseudo-critical points*. We mean that the norm of $I_\varepsilon(z)$ is small for all $z \in Z_\varepsilon$, in an appropriate uniform way. In our case, in Chapter 2 we will show that $P_z I''_\varepsilon(z)$ is uniformly invertible on $W = (T_z Z)^\perp$. This allows us to apply the contraction mapping theorem to the auxiliary equation and rewrite w tautologically

$$w = -(P_z I''_\varepsilon(z))^{-1} \left[P_z I'_\varepsilon(z) + \left(P_z I'_\varepsilon(z+w) - P_z I'_\varepsilon(z) - P_z I''_\varepsilon(z)[w] \right) \right] := G_{\varepsilon,z}(w).$$

We will show that $G_{\varepsilon,z}(w)$ is a contraction, which maps a ball in W into itself. Then we will provide conditions for solving the *bifurcation equation* and the *auxiliary equation* (2.2).

1.2.3 A technical lemma for the Neumann case

The next lemma is proved in Section 9.2 of [2]. We will use it later in Chapter 2 to construct more accurate approximate solutions to the mixed perturbed problem.

Lemma 1.2.2 *Let $T = (a_{ij})$ be a $(n-1) \times (n-1)$ symmetric matrix, and consider the following problem*

$$\begin{cases} L_U w = -2\langle Tx', \nabla_{x'} \partial_{x_n} U \rangle - (trT) \partial_{x_n} U & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial x_n} w = \langle Tx', \nabla_{x'} U \rangle & \text{on } \partial \mathbb{R}^n_+, \end{cases} \tag{1.9}$$

where L_U is the operator $L_U u = -\Delta u + u - pU^{p-1} u$. Then (1.9) *admits a solution $\overline{w}_T$, which is even in the variables x' and satisfies the following decay estimates*

$$|\overline{w}_T(x)| + |\nabla \overline{w}_T(x)| + |\nabla^2 \overline{w}_T(x)| \le C |T|_\infty (1 + |x|^K) e^{-|x|}, \tag{1.10}$$

where C, K are constants depending only on n and p.

1.2.4 Some estimates for the Neumann problem

The next results collect estimates for the singularly perturbed Neumann problem proved in Chapter 9 in [2]. We will use these estimates in the sequel of the thesis.

We recall that $\Omega_\varepsilon = \dfrac{1}{\varepsilon}\Omega$ and Q is a point belonging to $\partial\Omega_\varepsilon$. Moreover let $z_{\varepsilon,Q}$ be an approximate solution to the problem $(\tilde{P}_\varepsilon)$, depending on (i) U, the ground state solution to the problem (1.1) and on (ii) w, that is the solution given by Lemma 1.2.2. As we will show later in Chapter 2, such a $z_{\varepsilon,Q}$ constitutes a manifold of pseudo-critical points of $I_{\varepsilon,\mathcal{N}}$ (see Subsection 1.2.2).

Lemma 1.2.3 *There exists $C > 0$ such that for ε small there holds*

$$||I'_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q})|| \leq C\varepsilon^2; \qquad \textit{for all } Q \in \partial\Omega_\varepsilon.$$

Lemma 1.2.4 *Let H be the mean curvature of the boundary $\partial\Omega_\varepsilon$. For ε small the following expansion holds*

$$I_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O(\varepsilon^2),$$

where

$$\tilde{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n_+} U^{p+1}dx, \qquad \tilde{C}_1 = \left(\int_0^\infty r^n U_r^2 dr\right)\int_{S^n_+} y_n|y'|^2 d\sigma.$$

Lemma 1.2.5 *For ε small the following expansion holds*

$$\frac{\partial}{\partial Q} I_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2 H'(\varepsilon Q) + o(\varepsilon^2),$$

where $\tilde{C}_1$ and H are given in the preceding Lemma.

1.3 A classical variational theorem

In this section we will recall a classical theorem about critical point theory which will be used in this thesis. Let $\mathcal{B}$ denote a Banach space. In what follows we will denote by $C^1(\mathcal{B},\mathbb{R})$ the set of functionals that are Fréchet differentiable on $\mathcal{B}$ and whose Fréchet derivatives are continuous on $\mathcal{B}$.

Definition 1.3.1 *Let I be a functional belonging to $C^1(\mathcal{B},\mathbb{R})$. $c \in \mathbb{R}$ is called a critical level of I if there exists a critical point u such that $I(u) = c$.*

The Mountain Pass Theorem involves a useful technical assumption, the Palais-Smale condition, that occurs repeatedly in critical point theory.

Definition 1.3.2 *Let $\mathcal{B}$ be a Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$. We will say that I satisfies the Palais-Smale condition* ($(PS)_c$ for short) *if any sequence $\{u_j\}$, $u_j \in \mathcal{B}$, for which*

(i) $I(u_j) \to c$;

(ii) $I'(u_j) \to 0$,

possesses a strongly convergent subsequence.

Remark 1.3.3 *We note that the $(PS)_c$ condition is a compactness condition for the functional I. Such condition implies that the set*

$$K_c := \left\{ u \in \mathcal{B} : I(u) = c, I^{'}(u) = 0 \right\},$$

i.e. the set of critical points having critical value c, is compact for any $c \in \mathbb{R}$.

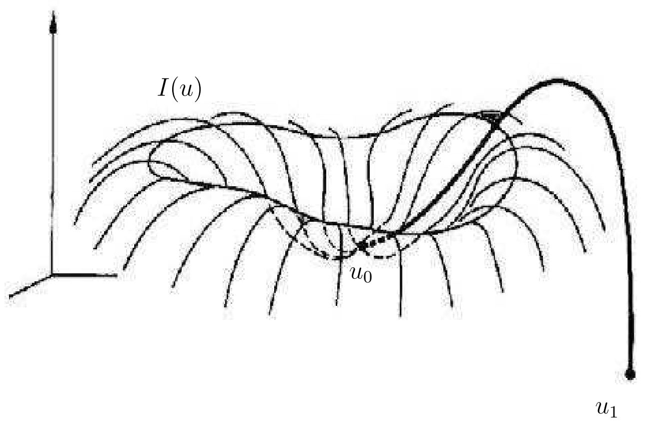

Figure 1.1: The Mountain Pass Geometry.

Theorem 1.3.4 (Mountain Pass) *Let $\mathcal{B}$ be a Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$. Suppose that there exist $u_0, u_1 \in \mathcal{B}$ and $\alpha, r > 0$ such that*

(MP1) $\inf_{\|u-u_0\|=r} I(u) \geq \alpha > I(u_0)$;

(MP2) $\|u_1\| > r$ *and* $I(u_1) \leq I(u_0)$.

Then there exists a sequence $\{u_n\} \subset \mathcal{B}$ such that *(i)* $I(u_n) \to c$, *where*

$$c := \inf_{\gamma\in\tilde{\Gamma}} \max_{t\in[0,1]} I(\gamma(t))$$

$$\tilde{\Gamma} = \{\gamma \in C([0,1],\mathcal{B}) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

(ii) $I'(u_n) \to 0$ *strongly in* $\mathcal{B}^*$.

If the functional I satisfies the $(PS)_c$ condition, then the number c, defined above, is a critical level of I.

1.4 The Perron method

In this thesis, we will use the Perron method to solve some questions of existence of solutions of a Dirichlet problem in a bounded domain, see Section 2.2.1. For more details about the Perron's method we recommend to refer to the book [29].

In the Perron method the study of boundary behaviour of the solution, separated from the existence problem, is connected to the geometric properties of the boundary through the concept of *barrier* function.

Definition 1.4.1 *Let f be a function belonging to $C^0(\overline{\Omega})$ and ξ a point of $\partial\Omega$. Then $f = f_\xi$ is called a* barrier *at ξ relative to Ω if:*

(i) f is superharmonic in Ω;

(ii) $f > 0$ in $\overline{\Omega} - \xi$; $f(\xi) = 0$.

Definition 1.4.2 *A boundary point will be called* regular *if there exists a barrier at that point.*

Theorem 1.4.3 *The classical Dirichlet problem in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.*

1.5 A monotonicity result

The next result, due to H.Berestycki, L.Caffarelli and L.Nirenberg, [11], is about the monotonicity in some directions of positive solutions of elliptic second order boundary value problems of the type:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega; \\ u > 0 & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.11}$$

in the case that Ω is a slab domain. Such result will be useful in Chapter 3 to prove the nonexistence of solutions of boundary value problems like (3.33).

Theorem 1.5.1 ([11]) *Let u be a solution of (1.11) in a slab*

$$\Omega \;=\; \mathbb{R}^{n-1}\times(0,h),$$

$n\geq 2$, satisfying

$$u(x,0)=0, \quad \forall x\in\mathbb{R}^{n-1}. \tag{1.12}$$

Assume that f is Lipschitz and that

$$f(0)\geq 0. \tag{1.13}$$

Then,

$$\frac{\partial u}{\partial y}(x,y)>0, \quad \forall x\in\mathbb{R}^{n-1}, \quad \forall y\in(0,h/2). \tag{1.14}$$

1.6 Some useful results about the least-energy solutions to a semilinear perturbed Neumann problem

In this section we collect a couple of useful results that we will use in Chapter 3.

Let us consider the following problem:

$$\begin{cases} -\varepsilon^2\Delta u+u=f(u) & \text{in } \Omega;\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial_{\mathcal{N}}\Omega;\\ u>0 & \text{in } \Omega. \end{cases} \tag{1.15}$$

Moreover denote $u_\varepsilon(P_\varepsilon)=\max_{x\in\overline{\Omega}}u_\varepsilon(P)$ and $H(\cdot)$ the mean curvature of the boundary $\partial_{\mathcal{N}}\Omega$. The next result was proved by W.M.Ni and I.Takagi in [49].

Theorem 1.6.1 (Ni-Takagi) *Let u_ε be a least-energy solution to the problem* (1.15) *with $f(u)=u^p$, $p\in(1,\frac{n+2}{n-2})$ if $n\geq 3$ and $p\in(1,+\infty)$ if $n=2$. Suppose also that there exists $P_0\in\partial_{\mathcal{N}}\Omega$ such that*

$$P_\varepsilon\to P_0\in\partial_{\mathcal{N}}\Omega.$$

Then

$$\tilde{I}_\varepsilon(u_\varepsilon)=\varepsilon^n\left\{\frac{1}{2}I(U)-(n-1)\varepsilon\gamma H(P_\varepsilon)+o(\varepsilon)\right\} \qquad \text{as } \varepsilon\to 0$$

where U is the ground state solution of problem (1.1)*, $\tilde{I}_\varepsilon(\cdot)$ the Euler functional associated to this problem* (1.2)*, and γ a constant given by*

$$\gamma:=\frac{1}{n+1}\int_{\mathbb{R}^n_+}U'(|z|)z_n dz>0.$$

The following technical result was proved by C.S.Lin, W.M.Ni and I.Takagi in [38].

Lemma 1.6.2 (Li-Ni-Takagi) *Let us consider the problem* (1.15). *Assume that the function $f:\mathbb{R}\to\mathbb{R}$ is of class $C^1(\mathbb{R})$ and satisfies the following classical assumptions:*

(i) $f(t)\equiv 0$ *for* $t\le 0$;

(ii) $f(t)=O(t^r)$ *as* $t\to+\infty$, *with:* $\begin{cases} 2<r+1<\frac{2n}{n-2} & \text{if } n\ge 3, \\ r>1 & \text{if } n=2; \end{cases}$

(iii) $\dfrac{f(t)}{t}$ *is increasing for* $t>0$ *and* $\lim_{t\to+\infty}\dfrac{f(t)}{t}=+\infty$, *while* $f(t)=o(t)$ *for* $t\to 0$;

(iv) there exists a constant $\theta>2$ *such that* $0<\theta F(t)\le tf(t)$ *for* $t\ge 0$, *where* $F(t)$ *is defined by*

$$F(t):=\int_0^t f(s)ds.$$

Moreover let u_ε *be a solution to the problem* (1.15). *Then*

$$\int_\Omega u_\varepsilon^r dx\le C_r\varepsilon^n,$$

where C_r *is a constant independent of* ε *and* r *any exponent greater than zero.*

Next result, proved by W.M. Ni and I.Takagi, locates the maximum point P_ε on the boundary. Indeed it shows that $H(P_\varepsilon)$, the mean curvature of $\partial\Omega$ at P_ε, approaches the maximum of $H(P)$ over $\partial\Omega$ as $\varepsilon\to 0$.

Theorem 1.6.3 *Let* u_ε *be a least-energy solution to problem* (1.15) *and* $P_\varepsilon\in\partial\Omega$ *be the unique point at which* $\max u_\varepsilon$ *is achieved. Then*

$$\lim_{\varepsilon\to 0} H(P_\varepsilon)=\max_{P\in\partial\Omega} H(P),$$

where $H(P)$ *denotes the mean curvature of* $\partial\Omega$ *at* P.

1.7 The Lax-Milgram lemma

In this section we state a very useful lemma that we will utilize in Chapter 4 in connection with the weak formulation of some elliptic problem. Let there be given a normed vector space V with norm $||\cdot||$, a *continuous* bilinear form $a(\cdot,\cdot):V\times V\to\mathbb{R}$ and a *continuous* functional $f:V\to\mathbb{R}$.

Lemma 1.7.1 (Lax-Milgram) *Let V be a Hilbert space, let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous coercive bilinear form and let $f : V \to \mathbb{R}$ be a continuous linear form.*

Then the abstract variational problem: Find an element u such that

$$u \in V \quad \text{and} \quad \forall v \in V, \quad a(u,v) = f(v), \tag{1.16}$$

has one and only one solution.

By this theorem one can affirm that the variational problem (1.16) is well-posed in the sense that its solution exists, is unique and depends continuously on the data f, all other data being fixed.

1.8 Notations

In this section we will introduce some notations used throughout this thesis. In what follows Ω will be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, with a sufficiently smooth boundary $\partial\Omega$. We will suppose that $\partial\Omega$ is union of two disjoint open subsets $\partial_{\mathcal{D}}\Omega$ and $\partial_{\mathcal{N}}\Omega$. We will call $\mathcal{I}_\Omega$ the intersection of the closures of $\partial_{\mathcal{D}}\Omega$ and $\partial_{\mathcal{N}}\Omega$.

We will consider the usual Sobolev space $H_0^1(\Omega)$, i.e. the closure of $C_0^\infty(\Omega)$ with respect to $H^1(\Omega)$, endowed with the norm defined as follows

$$\|u\| = \left(\int_\Omega | \nabla u |^2 \right)^{\frac{1}{2}}.$$

We will also use the Sobolev space $H_{\mathcal{D}}^1(\Omega)$, i.e. the family of functions belonging to $H^1(\Omega)$ with zero trace on $\partial_{\mathcal{D}}\Omega$.

Moreover $\varepsilon > 0$ will denote, throughout the thesis, a perturbation parameter tending to zero.

Generic fixed constants will be denoted by C, and will be allowed to vary within a single line or formula. We will often use the notation $d(1 + o(1))$, where $o(1)$ stands for a quantity which tends to zero as $d \to +\infty$.

Chapter 2

Concentration of solutions and Perturbation Methods

In this chapter we study the asymptotic behavior of some solutions to a singularly perturbed problem with mixed Dirichlet and Neumann boundary conditions. In particular we are interested in the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$\tilde{P}_\varepsilon$}$$

where Ω is a smooth bounded subset of $\mathbb{R}^n$, $p \in \left(1, \frac{n+2}{n-2}\right)$, $\varepsilon > 0$ is a small parameter, and $\partial_{\mathcal{N}}\Omega$, $\partial_{\mathcal{D}}\Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$. We prove that, under suitable geometric conditions on the boundary of the domain, there exist solutions which approach the intersection of the Neumann and the Dirichlet parts as the singular perturbation parameter tends to zero. Denoting $\mathcal{I}_\Omega$ the intersection of closures of $\partial_{\mathcal{D}}\Omega$ and $\partial_{\mathcal{N}}\Omega$ and assuming that the gradient of H at $\mathcal{I}_\Omega$ point toward $\partial_{\mathcal{D}}\Omega$, the main result of this chapter is the following theorem.

Theorem 2.0.1 *Suppose $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a smooth bounded domain, and that $1 < p < \frac{n+2}{n-2}$ ($1 < p < +\infty$ if $n = 2$). Suppose $\partial_{\mathcal{D}}\Omega$, $\partial_{\mathcal{N}}\Omega$ are disjoint open sets of $\partial\Omega$ such that the union of the closures is the whole boundary of Ω and such that their intersection $\mathcal{I}_\Omega$ is an embedded hypersurface. Suppose $\overline{Q} \in \mathcal{I}_\Omega$ is such that $H|_{\mathcal{I}_\Omega}$ is critical and non degenerate at $\overline{Q}$, and that $\nabla H \neq 0$ points toward $\partial_{\mathcal{D}}\Omega$. Then for $\varepsilon > 0$ sufficiently small problem $(\tilde{P}_\varepsilon)$ admits a solution u_ε concentrating at $\overline{Q}$.*

Remark 2.0.2 *(a) The non degeneracy condition in Theorem 2.0.1 can be replaced by the condition that $\overline{Q}$ is a strict local maximum or minimum of*

$H|_{\mathcal{I}_\Omega}$, or by the fact that there exists an open set $\mathcal{V}$ of $\mathcal{I}_\Omega$ containing $\overline{Q}$ such that $H(\overline{Q}) < \inf_{\partial\mathcal{V}} H$ (respectively $H(\overline{Q}) > \sup_{\partial\mathcal{V}} H$).

(b) With more precision, as $\varepsilon \to 0$, the above solution u_ε possesses a unique global maximum point $Q_\varepsilon \in \partial_{\mathcal{N}}\Omega$, and dist$(Q_\varepsilon, \mathcal{I}_\Omega)$ is of order $\varepsilon \log \frac{1}{\varepsilon}$ as ε tends to 0.

To prove Theorem 2.0.1, we will apply the perturbation methods in critical point theory. In the next section, we will start with an abstract outline to prove the existence of critical points of functionals, as in our case, that are perturbative.

2.1 The mixed perturbed problem

We are interested in finding solutions to $(\tilde{P}_\varepsilon)$ with a specific asymptotic profile, so it is convenient to scale the variables like $x \mapsto \varepsilon x$ and to study $(\tilde{P}_\varepsilon)$ in the dilated domain

$$\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega.$$

After this change of variables the problem becomes

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega_\varepsilon & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega_\varepsilon; \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases} \qquad (P_\varepsilon)$$

where $\partial_{\mathcal{N}}\Omega_\varepsilon$ and $\partial_{\mathcal{D}}\Omega_\varepsilon$ stand for the dilations of $\partial_{\mathcal{N}}\Omega$ and $\partial_{\mathcal{D}}\Omega$ respectively.

The Euler functional corresponding to (P_ε) is the following

$$I_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1} dx; \qquad u \in H^1_{\mathcal{D}}(\Omega_\varepsilon), \quad (2.1)$$

where $H^1_{\mathcal{D}}(\Omega_\varepsilon)$ denotes the family of functions in $H^1(\Omega_\varepsilon)$ with zero trace on $\partial_{\mathcal{D}}\Omega_\varepsilon$.

2.1.1 Perturbation in critical point theory

In this subsection, we introduce an abstract perturbation method which takes advantage of the variational structure of the problem, allowing to reduce it to a finite dimensional one. In particular we start treating existence of critical points for a class of functionals which are perturbative in nature.

Given a Hilbert space $\mathcal{H}$ (which might depend on the perturbation parameter ε), we want to consider manifolds embedded smoothly in $\mathcal{H}$, precisely for which

i) there exists a smooth d-dimensional manifold $Z_\varepsilon \subseteq \mathcal{H}$ and $C, r > 0$ such that for any $z \in Z_\varepsilon$, the set $Z \cap B_r(z)$ can be parameterized by a map on $B_1^{\mathbb{R}^d}$ whose C^3 norm is bounded by C.

We are also interested in functionals $I_\varepsilon : \mathcal{H} \to \mathbb{R}$ of class $C^{2,\alpha}$ which satisfy the following properties

ii) there exists a continuous function $f : (0, \varepsilon_0) \to \mathbb{R}$ with $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$ such that $\|I'_\varepsilon(z)\| \le f(\varepsilon)$ for every $z \in Z_\varepsilon$: moreover we require that $\|I''_\varepsilon(z)[q]\| \le f(\varepsilon)\|q\|$ for every $z \in Z_\varepsilon$ and every $q \in T_z Z_\varepsilon$;

iii) there exist C, $\alpha \in (0,1]$, $r_0 > 0$ such that $\|I''_\varepsilon\|_{C^\alpha} \le C$ in the subset $\{u \, : \, dist(u, Z_\varepsilon) < r_0\}$;

iv) letting P_z, $z \in Z_\varepsilon$, denote the projection onto the orthogonal complement of $T_z Z_\varepsilon$, there exists $C > 0$ (independent of z and ε) such that $P_z I''_\varepsilon(z)$, restricted to $(T_z Z_\varepsilon)^\perp$, is invertible from $(T_z Z_\varepsilon)^\perp$ into itself, and the inverse operator satisfies $\|(P_z I''_\varepsilon(z))^{-1}\| \le C$.

First some notation is in order. Let us set $W = (T_z Z_\varepsilon)^\perp$ and let $(q_i)_{1\le i\le d}$ be an orthonormal d-tuple (locally smooth on Z_ε) such that

$$T_z Z_\varepsilon = span\{q_1, \dots, q_d\}.$$

In the sequel we will denote by $z = z_\xi$, $\xi \in \mathbb{R}^d$, a smooth local parameterization of Z_ε as in *i*). Furthermore, we also suppose that $q_i = \partial_{\xi_i} z_\xi / \|\partial_{\xi_i} z_\xi\|$ at a given point of Z_ε.

We will look for critical points of I_ε in the form $u = z + \omega$ with $z \in Z_\varepsilon$ and $\omega \in W$. If $P_z : \mathcal{H} \to W$ is as in *iv*), the equation $I'_\varepsilon(z + \omega) = 0$ is equivalent to the system

$$\begin{cases} P_z I'_\varepsilon(z+\omega) = 0 & (\textit{auxiliary equation}); \\ (Id - P_z) I'_\varepsilon(z+\omega) = 0 & (\textit{bifurcation equation}). \end{cases} \tag{2.2}$$

Proposition 2.1.1 *Let i)-iv) hold true. Then there exists $\varepsilon_0 > 0$ with the following property: for all $|\varepsilon| < \varepsilon_0$ and for all $z \in Z_\varepsilon$, the auxiliary equation in* (2.2) *has a unique solution $\omega = \omega_\varepsilon \in W = (T_z Z_\varepsilon)^\perp$, which is of class C^1 with respect to $z \in Z_\varepsilon$ and such that $\|\omega_\varepsilon(z)\| \le C_1 f(\varepsilon)$ as $|\varepsilon| \to 0$, uniformly with respect to $z \in Z_\varepsilon$. The derivative of ω with respect to z, ω'_ε, satisfies the bound $\|\omega'_\varepsilon(z)\| \le C C_1 f(\varepsilon)^\alpha$.*

Proof. The proof is a refinement of a (by now) standard argument, which can be found for example in [2], Section 2: since however the procedure is rather short, we write here the details for the reader's convenience.

Property *iv*) allows us to apply the contraction mapping theorem to the auxiliary equation. In fact, by the invertibility of $P_z I''_\varepsilon(z)$ we can rewrite it tautologically as

$$\omega = -(P_z I''_\varepsilon(z))^{-1} \left[P_z I'_\varepsilon(z) + \left(P_z I'_\varepsilon(z+\omega) - P_z I'_\varepsilon(z) - P_z I''_\varepsilon(z)[\omega] \right) \right] := G_{\varepsilon,z}(\omega).$$

We claim next that the latter map is a contraction on a suitable metric ball of W. In fact, for the second term of G_ε, using iii) we can write that

$$\begin{aligned} \left\| P_z I'_\varepsilon(z+\omega) - P_z I'_\varepsilon(z) - P_z I''_\varepsilon(z)[\omega] \right\| &= \\ \left\| P_z \int_0^1 (I''_\varepsilon(z+s\omega) - I''_\varepsilon(z))[\omega] ds \right\| &\leq C\|\omega\|^{1+\alpha}, \end{aligned}$$

and therefore by ii) and iv) we have

$$\|G_{z,\varepsilon}(\omega)\| \leq Cf(\varepsilon) + C^2\|\omega\|^{1+\alpha}; \qquad \|\omega\| \leq r_0.$$

Similarly, one also finds

$$\|G_{z,\varepsilon}(\omega_1) - G_{z,\varepsilon}(\omega_2)\| \leq C^2(\|\omega_1\|^\alpha + \|\omega_2\|^\alpha)\|\omega_1 - \omega_2\|.$$

By the last two equations, if we fix $C_1 > 0$ sufficiently large and let

$$B_\varepsilon = \{\omega \in W \ : \ \|\omega\| \leq C_1 f(\varepsilon)\},$$

we can check that $G_{z,\varepsilon}$ is a contraction in B_ε, so for every $z \in Z_\varepsilon$ we obtain a (unique) function ω satisfying the required bound (for brevity, in the sequel the dependence on z will be assumed understood).

Let us now show that also the derivatives of ω with respect to ξ can be controlled by means of $f(\varepsilon)$. Indeed, for the components of ω tangent to Z_ε we can argue as follows: since $(\omega, \partial_\xi z) = 0$ for every ξ, differentiating with respect to ξ we find that $(\partial_\xi \omega, \partial_\xi z) = -(\omega, \partial^2_\xi z)$. Since $\|\omega\| = O(f(\varepsilon))$ and since $\partial^2_\xi z$ is bounded (by i)), the tangent components of $\partial_\xi \omega$ are bounded in norm by $CC_1 f(\varepsilon)$.

About the normal components, we can differentiate the relation $I'_\varepsilon(z+\omega) = \sum_{i=1}^d \overline{\alpha}_i \partial_{\xi_i} z$ (where $\overline{\alpha}_i \in \mathbb{R}$) with respect to ξ to find

$$I''_\varepsilon(z+\omega)[\partial_\xi z] + I''_\varepsilon(z+\omega)[\partial_\xi \omega] = \sum_{i=1}^d (\partial_\xi \overline{\alpha}_i)\partial_{\xi_i} z + \sum_{i=1}^d \overline{\alpha}_i \partial^2_{\xi\xi_i} z. \tag{2.3}$$

Since I''_ε is locally Hölder continuous (by iii)) and since $\|\omega\| = O(f(\varepsilon))$ projecting on W, for ε small by iv) we have that

$$\|P_z \partial_\xi \omega\| \leq C|\overline{\alpha}| + C\|I''_\varepsilon(z)[\partial_\xi z]\| + C\|\omega\|^\alpha \leq Cf(\varepsilon)^\alpha.$$

For the latter inequality we used again $\|\omega\| = O(f(\varepsilon))$ together with the Lipschitzianity of I'_ε (which imply $|\overline{\alpha}| \leq Cf(\varepsilon)$) and ii). ■

Remark 2.1.2 *From formula* (2.3)*, writing* $I''_\varepsilon(z+\omega)$ *as* $(I''_\varepsilon(z+\omega) - I''_\varepsilon(z)) + I''_\varepsilon(z)$ *and using* ii)*, one finds the following more precise estimate*

$$\|P_z \partial_\xi \omega\| \leq Cf(\varepsilon) + C\|(I''_\varepsilon(z+\omega) - I''_\varepsilon(z))[\partial_\xi z]\|.$$

Under further regularity assumptions on ω*, the estimate of* $\|P_z \partial_\xi \omega\|$ *can be improved: this will be useful for us to treat the case* $p \in (1,2)$*, see the proof of Proposition 2.3.5.*

We shall now provide conditions for solving the bifurcation equation in (2.2). In order to do this, let us define the *reduced functional* $\mathbf{I}_\varepsilon : Z \to \mathbb{R}$ by setting

$$\mathbf{I}_\varepsilon(z) = I_\varepsilon(z + \omega_\varepsilon(z)). \tag{2.4}$$

Proposition 2.1.3 *Suppose we are in the situation of Proposition 2.1.1, and let us assume that $\mathbf{I}_\varepsilon$ has, for $|\varepsilon|$ sufficiently small, a stationary point z_ε: then $u_\varepsilon = z_\varepsilon + \omega_\varepsilon(z_\varepsilon)$ is a critical point of I_ε. Furthermore, there exist $\tilde{c}, \tilde{r} > 0$ such that if u is a critical point of I_ε with $dist(u, Z_{\varepsilon,\tilde{c}}) < \tilde{r}$, where*

$$Z_{\varepsilon,\tilde{c}} = \{z \in Z_\varepsilon \ : \ dist(z, \partial Z_\varepsilon) > \tilde{c}\},$$

then u has to be of the form $z_\varepsilon + \omega_\varepsilon(z_\varepsilon)$ for some $z_\varepsilon \in Z_\varepsilon$.

Proof. The first assertion can be proved as follows. Consider the manifold $\tilde{Z}_\varepsilon = \{z + \omega_\varepsilon(z) \ : \ z \in Z_\varepsilon\}$. If z_ε is a critical point of $\mathbf{I}_\varepsilon$, it follows that $u_\varepsilon = z_\varepsilon + \omega(z_\varepsilon) \in \tilde{Z}_\varepsilon$ is a critical point of I_ε constrained on $\tilde{Z}_\varepsilon$ and thus u_ε satisfies $I'_\varepsilon(u_\varepsilon) \perp T_{u_\varepsilon}\tilde{Z}_\varepsilon$. Moreover the definition of ω_ε, see Proposition 2.1.1, implies that $I'_\varepsilon(u_\varepsilon) \in T_{z_\varepsilon}Z_\varepsilon$. Since, for $|\varepsilon|$ small, $T_{u_\varepsilon}\tilde{Z}_\varepsilon$ and $T_{z_\varepsilon}Z_\varepsilon$ are close, which is a consequence of the smallness of ω'_ε, it follows that $I'_\varepsilon(u_\varepsilon) = 0$.

To prove the last statement it is sufficient to notice that the contraction argument in the proof of Proposition 2.1.1 can be performed in the larger ball $\tilde{B} = \{\omega \in W \ : \ \|\omega\| \leq 2\tilde{r}\}$ with $\tilde{r}$ sufficiently small, so one has uniqueness of the solution of the auxiliary equation in this set. The distance condition on $Z_{\varepsilon,\tilde{c}}$ ensures the full applicability of these arguments in $\{dist(u, Z_{\varepsilon,\tilde{c}}) < \tilde{r}\}$, so the conclusion follows. ∎

2.1.2 Approximate solutions for (P_ε) with Neumann conditions

In this subsection we exhibit a family of (known) functions which satisfy the equation in (P_ε) up to an error of order ε^2, and whose normal boundary derivative vanishes, up to the same order. After, introducing some convenient coordinates which stretch the boundary, we recall some results from [2] concerning approximate solutions to the Neumann problem. Then, using a further change of variables which also stretches the interface, we modify these functions conveniently for our purposes.

Let us describe $\partial\Omega_\varepsilon$ near a generic point $Q \in \partial_{\mathcal{N}}\Omega_\varepsilon$. Without loss of generality, we can assume that $Q = 0 \in \mathbb{R}^n$, that $\{x_n = 0\}$ is the tangent plane of $\partial\Omega_\varepsilon$ (or $\partial\Omega$) at Q, and that $\nu(Q) = (0, \dots, 0, -1)$, where $\nu(Q)$ stands for the outer unit normal at Q. In a neighborhood of Q, let $x_n = \psi_Q(x')$ be a local parametrization of $\partial\Omega$, $x' \in \mathbb{R}^{n-1}$. Then on $\partial\Omega$ one has

$$x_n = \psi_Q(x') := \tfrac{1}{2}\langle A_Q x', x'\rangle + O(|x'|^3); \qquad |x'| < \mu_0, \tag{2.5}$$

where A_Q is the Hessian of ψ at 0, and where μ_0 is some small number depending on Ω. We have clearly $H(Q) = \frac{1}{n-1} tr A_Q$. On the other hand, $\partial\Omega_\varepsilon$ is parameterized using the function $\psi_Q^\varepsilon(x') := \frac{1}{\varepsilon}\psi_Q(\varepsilon x')$, for which the following expansions hold

$$\begin{aligned} \psi_Q^\varepsilon(x') &= \frac{\varepsilon}{2}\langle A_Q x', x'\rangle + \varepsilon^2 O(|x'|^3); \\ \partial_i \psi_Q^\varepsilon(x') &= \varepsilon (A_Q x')_i + \varepsilon^2 O(|x'|^2). \end{aligned} \tag{2.6}$$

Concerning the outer normal ν, we have also

$$\nu = \frac{\left(\frac{\partial \psi_Q^\varepsilon}{\partial x_1}, \dots, \frac{\partial \psi_Q^\varepsilon}{\partial x_{n-1}}, -1\right)}{\left(1 + |\nabla \psi_Q^\varepsilon|^2\right)^{\frac{1}{2}}} = \big(\varepsilon(A_Q x'), -1\big) + \varepsilon^2 O(|x'|^2). \tag{2.7}$$

Since $\partial\Omega_\varepsilon$ is almost flat for ε small and since the function U (see the Introduction) is radial, for $Q \in \partial\Omega_\varepsilon$ we have $\frac{\partial}{\partial\nu} U(\cdot - Q) \sim 0$. Thus $U(\cdot - Q)$ is an approximate solution to (P_ε) if we impose pure Neumann boundary conditions. Hence for the latter problem a natural choice of the manifold Z_ε (see Subsection 2.1.1) could be the following

$$Z_\varepsilon = \{U(\cdot - Q) := U_Q \ : \ Q \in \partial\Omega_\varepsilon\}.$$

Indeed we need a more accurate expansion, and we will construct better approximate solutions, in particular improving the condition at the boundary.

Given μ_0 as in (2.5), we introduce a new set of coordinates on $B_{\frac{\mu_0}{\varepsilon}}(Q) \cap \Omega_\varepsilon$. Let

$$\tilde{x}' = x'; \qquad \tilde{x}_n = x_n - \psi_Q^\varepsilon(x'). \tag{2.8}$$

The advantage of these coordinates is that $\partial\Omega_\varepsilon$ identifies with $\{\tilde{x}_n = 0\}$, but the corresponding metric coefficients $\tilde{g}_{ij}$ will not be constant anymore. From (2.6) it follows that

$$\tilde{g}_{ij} = Id + \varepsilon A^Q + O(\varepsilon^2 |\tilde{x}'|^2); \qquad \partial_{\tilde{x}_k}(\tilde{g}_{ij}) = \varepsilon \partial_{\tilde{x}_k} A^Q + O(\varepsilon^2 |\tilde{x}'|), \tag{2.9}$$

with

$$A^Q = \begin{pmatrix} 0 & A_Q \tilde{x}' \\ (A_Q \tilde{x}')^t & 0 \end{pmatrix}.$$

Here the zero in the upper left corner of A^Q stands for the trivial $(n-1) \times (n-1)$ matrix, while $(A_Q\tilde{x}')^t$ stands for the transpose of the column vector $(A_Q \tilde{x}')$. It is also easy to check that the inverse matrix $(\tilde{g}^{ij})$ is of the form $\tilde{g}^{ij} = Id - \varepsilon A^Q + O(\varepsilon^2|\tilde{x}'|^2)$, and that $\partial_{\tilde{x}_k}(\tilde{g}^{ij}) = -\varepsilon \partial_{\tilde{x}_k} A^Q + O(\varepsilon^2 |\tilde{x}'|)$. Moreover, by the expression of the coordinates $\tilde{x}$ one has

$$\det \tilde{g} \equiv 1. \tag{2.10}$$

Recall that the Laplacian with respect to a general metric is given by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_j \left(g^{ij} \sqrt{\det g} \right) \partial_i u + g^{ij} \partial^2_{ij} u,$$

so in our situation, by (2.10), we get

$$\Delta_g u = g^{ij} u_{ij} + \partial_i (g^{ij}) \partial_j u.$$

Using (2.9) we find that, formally

$$\Delta_{\tilde g} u = \Delta_{\mathbb{R}^n} u - \varepsilon \left(2 \langle A_Q \tilde x', \nabla_{\tilde x'} \partial_{\tilde x_n} u \rangle + (tr A_Q) \partial_{\tilde x_n} u \right) + O(\varepsilon^2)(|\nabla u| + |\nabla^2 u|). \tag{2.11}$$

We also give the expression of the unit outer normal to $\partial \Omega_\varepsilon$, $\tilde \nu$, in the new coordinates $\tilde x$. Letting ν_i (resp. $\tilde \nu_i$) be the components of ν (resp. $\tilde \nu$), from $\nu = \sum_{i=1}^n \nu^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n \tilde \nu^i \frac{\partial}{\partial \tilde x^i}$, we have $\tilde \nu_k = \sum_{i=1}^n \nu^i \frac{\partial \tilde x^k}{\partial x^i}$. This implies

$$\tilde \nu^k = \nu^k, \quad k = 1, \dots, n-1; \qquad\qquad \tilde \nu^n = \sum_{i=1}^{n-1} \nu^i \frac{\partial \psi_Q^\varepsilon}{\partial \tilde x^i} + \nu^n.$$

From (2.6) and the subsequent formulas we find

$$\tilde \nu = \left(\varepsilon A_Q \tilde x', -1 \right) + \varepsilon^2 O(|\tilde x'|^2). \tag{2.12}$$

Finally the area-element of $\partial \Omega_\varepsilon$ can be expanded as

$$d\sigma = (1 + O(\varepsilon^2 |\tilde x'|^2)) d\tilde x'. \tag{2.13}$$

Linearizing the equation in (P_ε) near $U(\tilde x)$, one sees that the function $U(y) + \varepsilon w(y)$ solves the equation (P_ε) up to an error $o(\epsilon)$, if w satisfies

$$\begin{cases} L_U w = -2 \langle A_Q \tilde x', \nabla_{\tilde x'} \partial_{\tilde x_n} U \rangle - (tr A_Q)\, \partial_{\tilde x_n} U & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial \tilde x_n} w = \langle A_Q \tilde x', \nabla_{\tilde x'} U \rangle & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

Therefore, if $\overline{w}_{A_Q}$ is given by Lemma 1.2.2 (see Chapter 1) with $T = A_Q$, one sees that, formally at least, the accuracy of the solution improves. To derive rigorous estimates in this spirit, one can choose a cut-off function χ_{μ_0} on $\mathbb{R}^n$ with the properties

$$\begin{cases} \chi_{\mu_0}(\tilde x) = 1 & \text{in } B_{\frac{\mu_0}{4}}; \\ \chi_{\mu_0}(\tilde x) = 0 & \text{in } \mathbb{R}^n \setminus B_{\frac{\mu_0}{2}}; \\ |\nabla \chi_{\mu_0}| + |\nabla^2 \chi_{\mu_0}| \leq C & \text{in } B_{\frac{\mu_0}{2}}(Q) \setminus B_{\frac{\mu_0}{4}}, \end{cases} \tag{2.14}$$

and for any $Q \in \partial \Omega$ define the following function, in the coordinates $(\tilde x', \tilde x_n)$

$$z_{\varepsilon, Q}(\tilde x) = \chi_{\mu_0}(\varepsilon \tilde x)(U(\tilde x) + \varepsilon \overline{w}_{A_Q}(\tilde x)). \tag{2.15}$$

The next result collects estimates in the statements and the proofs of Lemmas 1.2.3, 1.2.4 and 1.2.5 in Subsection 1.2.4.

Proposition 2.1.4 *There exist constants $C, K > 0$, depending only on p, n and Ω such that for ε small the following estimates hold*

$$\left|\frac{\partial z_{\varepsilon,Q}}{\partial \tilde{\nu}}\right|(\tilde{x}') \leq \begin{cases} C\varepsilon^2(1+|\tilde{x}'|^K)e^{-|\tilde{x}'|} & \textit{for } |\tilde{x}'| \leq \frac{\mu_0}{4\varepsilon}; \\ Ce^{-\frac{1}{C\varepsilon}} & \textit{for } \frac{\mu_0}{4\varepsilon} \leq |\tilde{x}'| \leq \frac{\mu_0}{2\varepsilon}, \end{cases}$$

$$\left|-\Delta_g z_{\varepsilon,Q} + z_{\varepsilon,Q} - z_{\varepsilon,Q}^p\right|(\tilde{x}) \leq \begin{cases} C\varepsilon^2(1+|\tilde{x}|^K)e^{-|\tilde{x}|} & \textit{for } |\tilde{x}| \leq \frac{\mu_0}{4\varepsilon}; \\ Ce^{-\frac{1}{C\varepsilon}} & \textit{for } \frac{\mu_0}{4\varepsilon} \leq |\tilde{x}| \leq \frac{\mu_0}{2\varepsilon}, \end{cases}$$

$$I_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O(\varepsilon^2); \qquad \frac{\partial}{\partial Q} I_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2 H'(\varepsilon Q) + o(\varepsilon^2),$$

where

$$\tilde{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n_+} U^{p+1}dx, \qquad \tilde{C}_1 = \left(\int_0^\infty r^n U_r^2 dr\right)\int_{S^n_+} y_n|y'|^2 d\sigma.$$

To improve the estimate, we need to take into account the effect of the Dirichlet boundary condition. To this end, we make next a further change of variables, in order to stretch also the interface: we claim indeed that, in the coordinates $\tilde{x}$, the latter can be parameterized as $\tilde{x}_1 = d + \tilde{\psi}^\varepsilon_Q(\tilde{x}'')$, $\tilde{x}'' = (\tilde{x}_2, \dots, \tilde{x}_{n-1})$, with $d \in \mathbb{R}$ and $\tilde{\psi}^\varepsilon_Q$ satisfying estimates similar to (2.6). To see this, we first claim that the curvature of the interface $\mathcal{I}_{\Omega_\varepsilon}$, in the coordinates $\tilde{x}$, is of order $O(\varepsilon)$. Here we are assuming that the distance of Q from $\mathcal{I}_{\Omega_\varepsilon}$, multiplied by ε, is bounded by a small constant depending on Ω. For showing this, let us consider a curve $\gamma(s)$ in the interface whose geodesic curvature (relative to $\mathcal{I}_{\Omega_\varepsilon}$) vanishes and which is parameterized by arclength: its curvature in $\mathbb{R}^n$ will be therefore of order $O(\varepsilon)$. Let us split γ into its tangent and normal components (with respect to $T_Q\partial\Omega_\varepsilon$) $\gamma = (\gamma^T, \gamma^N)$. Since in our notation $\gamma^N = \psi^\varepsilon_Q(\gamma^T)$, from (2.6) and a Taylor expansion we have that

$$|\dot{\gamma}|^2 = |\dot{\gamma}^T|^2 + |\dot{\gamma}^N|^2 \qquad \Rightarrow \qquad |\dot{\gamma}| = |\dot{\gamma}^T|(1 + O(\varepsilon^2)|\gamma^T|^2).$$

The curvature of γ is $k = \frac{1}{|\dot{\gamma}|}\frac{d}{ds}\left(\frac{\dot{\gamma}}{|\dot{\gamma}|}\right)$, which can be written as

$$k = \frac{(1 + O(\varepsilon^2)|\gamma^T|^2)}{|\dot{\gamma}^T|}\frac{d}{ds}\left(\frac{\dot{\gamma}^T}{|\dot{\gamma}^T|}(1 + O(\varepsilon^2)|\gamma^T|^2)\right).$$

Expanding the above expression we obtain

$$k = k^T(1 + O(\varepsilon^2)|\gamma^T|^2) + O(\varepsilon^2)|\gamma^T|.$$

This formula shows that, since k is of order ε, also k^T is of order ε. Therefore, if the point Q is close to the interface (in the sense specified before) and if we choose the $\tilde{x}_1$ axis to be perpendicular to the projection of $\mathcal{I}_{\Omega_\varepsilon}$ onto $T_Q\partial\Omega_\varepsilon$, if the latter projection is at distance d from Q we have that the points in $\mathcal{I}_{\Omega_\varepsilon}$ satisfy

$$\tilde{x}_1 = d + \tilde{\psi}_Q^\varepsilon(\tilde{x}''); \qquad \tilde{x}'' = (\tilde{x}_2, \dots, \tilde{x}_{n-1}), \tag{2.16}$$

where $\tilde{\psi}_Q^\varepsilon$ (which depends on Q and ε) is such that

$$\tilde{\psi}_\varepsilon(0) = 0; \qquad \nabla\tilde{\psi}_\varepsilon(0) = 0; \qquad \tilde{\psi}_Q^\varepsilon(\tilde{x}'') = \frac{1}{2}\varepsilon\langle \tilde{A}_Q\tilde{x}'', \tilde{x}''\rangle + O(\varepsilon^2|\tilde{x}''|^2),$$

see also (2.6). Below, it will be always understood that the symbol d refers to the distance to the scaled interface $\mathcal{I}_{\Omega_\varepsilon}$ and we will never use subscript ε to stress this fact.

Introducing the new coordinates

$$y_1 = \tilde{x}_1 - \tilde{\psi}_Q^\varepsilon(\tilde{x}''); \qquad y'' = \tilde{x}''; \qquad y_n = \tilde{x}_n, \tag{2.17}$$

we have that the metric coefficients g^y satisfy again

$$\det g^y \equiv 1, \tag{2.18}$$

as in (2.10). Then, similarly to (2.9) we have that

$$g_{ij}^y = Id + \varepsilon\tilde{A}^Q + O(\varepsilon^2|y'|^2); \qquad \partial_{y_k}(g_{ij}^y) = \varepsilon\partial_{y_k}\tilde{A}^Q + O(\varepsilon^2|y'|) \tag{2.19}$$

(and similarly $(g^y)^{ij} = Id - \varepsilon\tilde{A}^Q + O(\varepsilon^2|y'|^2)$, $\partial_{y_k}(g^y)^{ij} = -\varepsilon\partial_{y_k}\tilde{A}^Q + O(\varepsilon^2|y'|)$), where

$$\tilde{A}^Q = \begin{pmatrix} 0 & (\tilde{A}_Q y'')^t & A_Q y' \\ \tilde{A}_Q y'' & 0 & \\ (A_Q y')^t & & 0 \end{pmatrix}.$$

Remark 2.1.5 *(a) We stress that, in the new coordinates y, the origin parameterizes the point Q on $\partial\Omega$, and that functions decaying as $|y| \to +\infty$ will* concentrate *near Q.*

(b) It is also useful to understand how the metric coefficients g_{ij}^y vary with Q. Notice that in (2.19) the deviation from the Kronecker symbols is of order ε, and that we are working in a domain scaled of $\frac{1}{\varepsilon}$, so a variation of order 1 of Q corresponds to a variation of order ε in the original domain. Therefore, a variation of order 1 in Q yields a difference of order ε^2 in g_{ij}^y, and precisely

$$\frac{\partial g_{ij}^y}{\partial Q} = O(\varepsilon^2|y'|^2),$$

with a similar estimate for the derivatives of the inverse coefficients $(g^y)^{ij}$. For more details see the end of Subsection 9.2 in [2].

From the latter formulas, the counterpart of (2.11) becomes

$$\begin{aligned}\Delta_{g^y} u = \Delta_{\mathbb{R}^n} u - \varepsilon \left(2\langle A_Q y', \nabla_{y'} \partial_{y_n} u\rangle + (tr A_Q) \partial_{y_n} u\right) - \\ \varepsilon \left(2\langle \tilde{A}_Q y'', \nabla_{y''} \partial_{y_1} u\rangle + (tr \tilde{A}_Q) \partial_{y_1} u\right) + \\ O(\varepsilon^2) \left(|\nabla_y u| + |\nabla_y^2 u|\right),\end{aligned} \tag{2.20}$$

while the analogues of (2.12) and (2.13) are

$$\nu^y = \left(\varepsilon A_Q y', -1\right) + \varepsilon^2 O(|y'|^2); \qquad d\sigma^y = (1 + O(\varepsilon^2 |y'|^2)) dy'. \tag{2.21}$$

Reasoning as for the above coordinates $\tilde{x}$ (see the comments after (2.13)), one sees that the function $U(y) + \varepsilon \tilde{w}(y)$ solves the equation in (P_ε) up to an error $o(\varepsilon)$ if and only if $\tilde{w}$ satisfies

$$\begin{cases} L_U \tilde{w} = -2\langle A_Q y', \nabla_{y'} \partial_{y_n} U\rangle - (tr A_Q)\, \partial_{y_n} U - & \\ 2\langle \tilde{A}_Q y'', \nabla_{y''} \partial_{y_1} U\rangle - (tr \tilde{A}_Q)\, \partial_{y_1} U & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial y_n} \tilde{w} = \langle A_Q y', \nabla_{y'} U\rangle & \text{on } \partial \mathbb{R}^n_+. \end{cases} \tag{2.22}$$

We have an explicit solution to (2.22) in terms of the above function $\overline{w}_{A_Q}$ through a simple change of variable. Indeed, in the new coordinates y we can write that formally

$$\begin{aligned} z_{\varepsilon,Q}(\tilde{x}) &\simeq U\left(y_1 + \frac{\varepsilon}{2}\langle \tilde{A}_Q y'', y''\rangle, y'', y_n\right) + \varepsilon \overline{w}_{A_Q}\left(y_1 + \frac{\varepsilon}{2}\langle \tilde{A}_Q y'', y''\rangle, y'', y_n\right) \\ &\simeq U(y) + \frac{\varepsilon}{2}\langle \tilde{A}_Q y'', y''\rangle \partial_1 U(y) + \varepsilon \overline{w}_{A_Q}(y) + o(\varepsilon). \end{aligned}$$

Choosing $\tilde{w}(y) = \tilde{w}_Q(y) := \frac{1}{2}\langle \tilde{A}_Q y'', y''\rangle \partial_1 U(y) + \overline{w}_{A_Q}(y)$ and using the fact that $L_U \partial_{y_i} U(y) = 0$, from an explicit computation we find that

$$L_U \tilde{w}_Q(y) = L_U \overline{w}_{A_Q}(y) - (tr \tilde{A}_Q) \partial_1 U(y) + 2\langle \tilde{A} y'', \nabla_{y''} \partial_1 U(y)\rangle,$$

which is exactly the desired equation. Clearly, by (1.10), there exists a constant C'_Ω, depending on Ω and on the curvatures of the interface such that

$$|\tilde{w}_Q(y)| + |\nabla \tilde{w}_Q(y)| + |\nabla^2 \tilde{w}_Q(y)| \leq C'_\Omega (1 + |y|^K) e^{-|y|}. \tag{2.23}$$

In conclusion, choosing a cutoff function as in (2.14) and defining the new approximate solution $\overline{z}_{\varepsilon,Q}$ as

$$\overline{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y) \left(U(y) + \varepsilon \tilde{w}_Q(y)\right), \tag{2.24}$$

the arguments for the proof of Proposition 2.1.3 yield the following variant of the above result.

Proposition 2.1.6 *There exists $C, K > 0$ such that for ε small the following estimates hold*

$$\left|\frac{\partial \overline{z}_{\varepsilon,Q}}{\partial \tilde{\nu}}\right|(y') \leq \begin{cases} C\varepsilon^2(1+|y'|^K)e^{-|y'|} & \text{for } |y'| \leq \frac{\mu_0}{4\varepsilon}, \\ Ce^{-\frac{1}{C\varepsilon}} & \text{for } \frac{\mu_0}{4\varepsilon} \leq |y'| \leq \frac{\mu_0}{2\varepsilon}; \end{cases}$$

$$\left|-\Delta_g \overline{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} - \overline{z}^p_{\varepsilon,Q}\right|(y) \leq \begin{cases} C\varepsilon^2(1+|y|^K)e^{-|y|} & \text{for } |y| \leq \frac{\mu_0}{4\varepsilon}, \\ Ce^{-\frac{1}{C\varepsilon}} & \text{for } \frac{\mu_0}{4\varepsilon} \leq |y| \leq \frac{\mu_0}{2\varepsilon}; \end{cases}$$

$$I_{\varepsilon,\mathcal{N}}(\overline{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O(\varepsilon^2); \qquad \frac{\partial}{\partial Q} I_{\varepsilon,\mathcal{N}}(\overline{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2 H'(\varepsilon Q) + o(\varepsilon^2),$$

where $\tilde{C}_0, \tilde{C}_1$ are as in Proposition 2.1.3.

An immediate consequence of this proposition is that

$$\|I'_{\varepsilon,\mathcal{N}}(\overline{z}_{\varepsilon,Q})\| \leq C\varepsilon^2 \qquad \text{for all } Q \in \partial\Omega_\varepsilon \text{ and for some fixed } C > 0, \tag{2.25}$$

where $I_{\varepsilon,\mathcal{N}}$ is the counterpart of the functional in (2) when we scale Ω to Ω_ε.

2.2 Approximate solutions to (P_ε)

The functions $\overline{z}_{\varepsilon,Q}$ constructed in (2.24) constitute good approximate solutions to (P_ε) when we impose pure Neumann boundary conditions. Nevertheless, we need an expansion which takes into account the parameter $d = \frac{d_\varepsilon}{\varepsilon}$, the distance of the peak point to the interface in the scaled domain (see the notation in the last formula of the introduction), and to this end some relevant modifications are necessary.

As in [51], a useful tool is the *projection operator* onto $H^1_{\mathcal{D}}(\Omega_\varepsilon)$, namely the one which associates to every element in $H^1(\Omega_\varepsilon)$ its closest point in $H^1_{\mathcal{D}}(\Omega_\varepsilon)$. Explicitly, this is constructed subtracting to any given $u \in H^1(\Omega_\varepsilon)$ the solution to

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega_\varepsilon; \\ v = u & \text{on } \partial_{\mathcal{D}}\Omega_\varepsilon; \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial_{\mathcal{N}}\Omega_\varepsilon. \end{cases} \tag{2.26}$$

This solution can be found variationally by looking at the following minimum problem

$$\inf_{v=u \text{ on } \partial_{\mathcal{D}}\Omega_\varepsilon} \left\{ \int_{\Omega_\varepsilon} \left(|\nabla v|^2 + v^2\right) dx \right\}.$$

Instead of studying (2.26) directly, it is convenient to modify the domain in order that the region of the boundary near $\mathcal{I}_{\Omega_\varepsilon}$ becomes flat.

For technical reasons we construct a domain in the following way: first in $\mathbb{R}^{n-1}$ we consider the square

$$S = \{|x_1 + 2| < 2\} \cap \left\{|x''| < 2\right\}; \qquad x'' = (x_2, \dots, x_{n-1}),$$

and then the set $\tilde{S}$ which is obtained from S rounding off smoothly the corners of S, and in order that

$$\begin{aligned} &(S \cap \{|x_1+2| < 1\}) \quad \cup \quad \left(S \cap \{|x''| < 1\}\right) \\ = &\left(\tilde{S} \cap \{|x_1+2| < 1\}\right) \quad \cup \quad \left(\tilde{S} \cap \{|x''| < 1\}\right), \end{aligned} \tag{2.27}$$

see Figure 3.1. Notice that in this way the set $\{x_1 = 0\} \cap \{|x''| < 1\}$ lies on the boundary of $\tilde{S}$. Below, we will identify $\tilde{S}$ with its natural immersion into $\mathbb{R}^n \cap \{x_n = 0\}$.

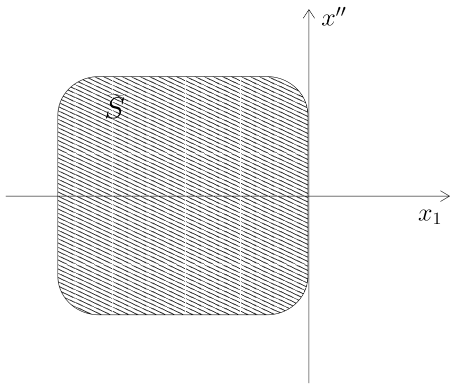

Figure 2.1: The domain $\tilde{S}$.

Next, in the (x_1, x_n) plane, we consider a C^∞ curve γ which runs on the segment $(t, 0)$, $t \in [0, 4(n+2)]$, which hits the x_n axis horizontally, stays in the quadrant $x_1, x_n > 0$ and stays external to the circle $\{x_1^2 + x_n^2 = 16(n+2)^2\}$, see Figure 3.3 (there is no specific reason in choosing the number $4(n+2)$: we just need that a sufficiently large cube is contained in the set $\hat{\Gamma}_D$ defined below).

We then consider the set Γ obtained by intersecting the semi-closed quadrant $\{x_1 \geq 0\} \cap \{x_n > 0\}$ with the interior of the curve, together with its reflection across the x_1 axis. Then, we call $\tilde{\Gamma}$ the set generated by the rotation of Γ around the x_n axis, jointly with $\tilde{S}$. We finally define the burger-shaped domain $\hat{\Gamma}$ as

$$\hat{\Gamma} = \tilde{\Gamma} \cup \check{\Gamma}, \tag{2.28}$$

where $\check{\Gamma}$ is the reflection of $\tilde{\Gamma}$ across the plane $\{x_n = 0\}$, see Figure 3. For

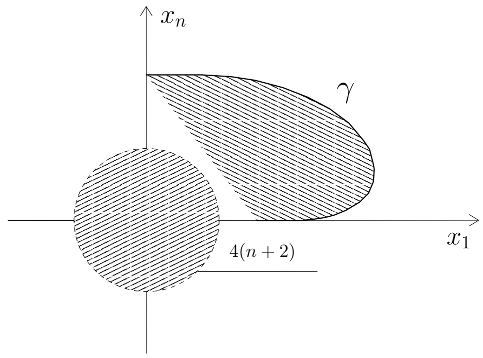

Figure 2.2: The curve γ in the (x_1, x_n) plane.

a large number D, we also consider the scaling

$$\hat{\Gamma}_D = D\hat{\Gamma}. \tag{2.29}$$

The advantage of dealing with this set is that if we solve a Dirichlet problem in $\hat{\Gamma}_D$ with data even in x_n, then for suitable boundary conditions (as in (2.30) below) the solution in the upper part $\hat{\Gamma}_D \cap \{x_n > 0\}$ will be qualitatively similar to that of (2.26). Quantitative estimates on the real accuracy of this substitution will be derived in Proposition 2.2.12, and will turn out to be sufficiently good for our purposes.

Our next goal is to consider the following problem

$$\begin{cases} -\Delta\tilde{\varphi} + \tilde{\varphi} = 0 & \text{in } \hat{\Gamma}_{dD}; \\ \tilde{\varphi} = U(\cdot - dQ_0) & \text{on } \partial\hat{\Gamma}_{dD}, \end{cases} \qquad Q_0 = (-1, 0, \dots, 0). \tag{2.30}$$

The reason for studying (2.30) is that, using the coordinates y introduced in Subsection 2.1.2 (see in particular (2.17)), the function $U(\cdot - dQ_0)$ stands for the main term in the approximate solution $\overline{z}_{\varepsilon,Q}$, up to a translation in the y_1 axis: this is how we are modeling (2.26) (in the above coordinates y), when $u = \overline{z}_{\varepsilon,Q}$. By a scaling of the variables, the latter problem is clearly equivalent to

$$\begin{cases} -\frac{1}{d^2}\Delta\varphi + \varphi = 0 & \text{in } \hat{\Gamma}_D; \\ \varphi = U(d(\cdot - Q_0)) & \text{on } \partial\hat{\Gamma}_D. \end{cases} \tag{2.31}$$

2.2.1 Asymptotic analysis of (2.31)

First of all we notice that (2.31) is solvable: for example one can use the classical method by Perron, provided one knows that barrier functions exist, see Section 1.4. This is guaranteed by the following result. Its proof could

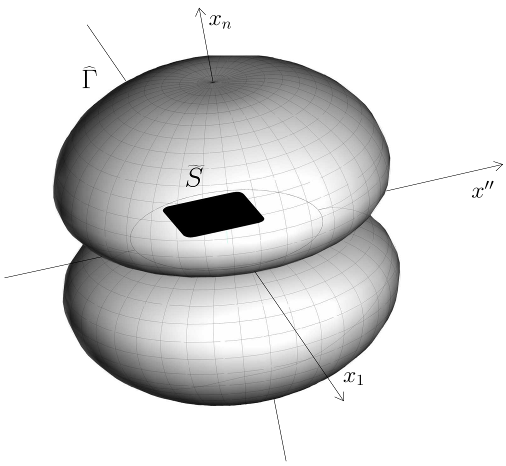

Figure 2.3: The domain $\hat{\Gamma}$.

be probably derived from a modification of some arguments in [30]. Since however we did not find a direct reference for our case, we give a sketch of the construction.

Lemma 2.2.1 *The set $\hat{\Gamma}$ admits barrier functions for the operators Δ and $-\Delta+1$ at all boundary points.*

Proof. It is clearly sufficient to prove the claim only for the points which are not regular. We begin by considering the Green's function $G(\cdot,\cdot)$ of $-\Delta+1$ in $\mathbb{R}^n$: when the singularity is at the origin $G(\cdot,0)$ is radial and by standard ODE analysis one can show that

$$|x|^{n-2}G(x,0)\to c_n \quad |x|\to 0; \qquad e^{|x|}|x|^{\frac{n-1}{2}}G(x,0)\to c_n' \quad |x|\to+\infty, \tag{2.32}$$

where c_n, c_n' are positive dimensional constants (see for instance [54], pages 130-133, for more details). Then, given a smooth radial compactly supported non decreasing mollifier function $\varrho(x)$, we can consider the scaled one $\varrho_\lambda(x)=\lambda^n\varrho(\lambda x)$, and for $z\in\partial\hat{\Gamma}_D$ define

$$f_{z,\lambda}(x)=\int_{\partial\hat{\Gamma}_D}\varrho_\lambda(y-z)G(y,x)d\sigma(y), \qquad x\in\hat{\Gamma}_D.$$

By the first formula in (2.32) and since $\partial\hat{\Gamma}_D$ is $n-1$ dimensional one can show that $f_{z,\lambda}$ is continuous up to the boundary of $\hat{\Gamma}_D$. To sketch the proof of this fact for a singular point, we can ideally substitute $\partial\hat{\Gamma}_D$ with $\mathbb{R}^{n-1}\cap\{y_1'\geq 0\}$ and ϱ_λ with $\chi_{|y'|<1}$. Then the function $f_{z,\lambda}$ becomes

$$f_{z,\lambda}(x)=\int_{\mathbb{R}^{n-1}\cap\{y_1'\geq 0\}}\chi_{B_1(0)}(y')G(y',x)dy',$$

satisfies $-\Delta f_{z,\lambda} + f_{z,\lambda} = 0$ and has clearly a maximum at $x = 0$. Its continuity at the origin follows by dominated convergence: indeed, suppose that $|x_n|^2 + |x'|^2 < R^2$. By (2.32) we have that the integrand is bounded by

$$c_n' \frac{\chi_{|x'-y'|<1}(|y'|)}{(x_n^2 + |y'|^2)^{\frac{n-2}{2}}} \leq c_n' \frac{\chi_{|y'|<R+1}(|y'|)}{|y'|^{n-2}},$$

which is of class L^1 in $\mathbb{R}^{n-1}$, so continuity follows.

Furthermore, by the exponential decay of G (see the second formula in (2.32)), if λ is sufficiently large $f_{z,\lambda}$ will be peaked near some point $P_{z,\lambda} \in \partial\hat{\Gamma}_D$ close to z, and $P_{z,\lambda}$ will be its unique maximum. Finally, fixing any $P \in \partial\hat{\Gamma}_D$, we can vary the position of z, depending on λ and P so that $P_{z,\lambda}$ coincides with P: this choice of z, for λ large, will provide a barrier function which is maximal only at P, and which solves $\Delta = u$ in the interior of the domain. With similar constructions one can find barriers which satisfy the same equation and which are minimal at any given point of the boundary. ■

Remark 2.2.2 *It is worth pointing out that the previous result holds only for domains with cuts since the function in latter formula is only integrable in dimension greater or equal to $n-1$. The above argument does not apply to domains with spines as in Lebesgue's counterexample.*

As in [51], if we consider the function $\phi = -\frac{1}{d}\log\varphi$, then ϕ satisfies

$$\begin{cases} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \hat{\Gamma}_D; \\ \phi = -\frac{1}{d}\log\left(U(d(\cdot - Q_0))\right) & \text{on } \partial\hat{\Gamma}_D, \end{cases} \tag{2.33}$$

where, we recall, $Q_0 = (-1, 0, \dots, 0)$.

Lemma 2.2.3 *For any fixed constant $D > 0$ we have that*

$$-\frac{1}{d}\log U(d(\cdot - Q_0)) \to |x - Q_0| \qquad \textit{uniformly on } \partial\hat{\Gamma}_D \tag{2.34}$$

as $d \to +\infty$.

Proof. By (1.3), the function U satisfies

$$U(x) = \alpha_{n,p} e^{-|x|} \frac{1}{|x|^{\frac{n-1}{2}}} (1 + o_{|x|}(1))$$

as $|x| \to +\infty$, so we find

$$-\frac{1}{d}\log U(d(x - Q_0)) = |x - Q_0| + O\left(\frac{1}{d}|\log(d|x - Q_0|)|\right),$$

$x \in \partial\hat{\Gamma}_D, d \to +\infty$.

For $x \in \partial\hat{\Gamma}_D$, we have by the construction of $\hat{\Gamma}$ that $1 \leq |x - Q_0| \leq CD$, for a fixed constant C, so we obtain the desired claim provided

$$\log(dD) = o(d),$$

which is the case since D is fixed. ■

By Lemma 2.2.3, the boundary datum is everywhere close to the function $|x - Q_0|$, so it is useful for us to consider the following auxiliary problem

$$\begin{cases} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \hat{\Gamma}_D; \\ \phi = |x - Q_0| & \text{on } \partial\hat{\Gamma}_D. \end{cases} \tag{2.35}$$

Lemma 2.2.4 *Suppose $D > 1$ is a fixed constant. Then, when $d \to \infty$, problem (2.35) has a unique solution ϕ^d, which is everywhere positive, and which more precisely satisfies the estimates*

$$1 \leq \phi^d(x) \leq C \qquad \text{in } \hat{\Gamma}_D, \tag{2.36}$$

where C depends only on D and $\hat{\Gamma}$.

Proof. Proving existence and uniqueness is rather easy: one can apply the transformation inverse to the one at the beginning of this subsection, and study

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \hat{\Gamma}_{dD}; \\ \varphi = e^{-|x - dQ_0|} & \text{on } \partial\hat{\Gamma}_{dD}. \end{cases}$$

Existence for the latter problem follows from Lemma 2.2.1, while uniqueness and positivity of ϕ^d follows from the maximum principle.

To prove (2.36) we use suitable barrier functions, exactly as in [51], Lemma 4.2: the barrier argument can be applied to (2.35) even if the domain is singular, by Lemma 2.2.1. Since $dist(Q_0, \partial\hat{\Gamma}_D) = 1$, the function $\phi_-^d \equiv 1$ in $\hat{\Gamma}_D$ is a subsolution to (2.35). On the other hand, for C sufficiently large the function $\phi_+^d(x) = C + 2x_1$ is a supersolution, and then our claim follows. ■

We next show some improved pointwise bounds on ϕ^d, which in particular imply a control on the gradient within some region in the boundary of $\hat{\Gamma}_D$. Here and below, many of the arguments in [51] break down if we try to use them for non smooth domains, and we need to perform suitable adaptations. In fact, we obtain gradient bounds only near smooth parts of the boundary, and in general weaker Sobolev norm bounds, see also the beginning of Subsection 2.2.2.

Lemma 2.2.5 *Suppose $\tilde{S}$ is as at the beginning of this section, and that D is as in Lemma 2.2.4. Then there exists a fixed constant $C > 0$ such that for any $0 < \rho < \frac{1}{8}$ there exists $d_\rho > 0$ so large that*

$$|\phi^d(x) - \phi^d(z_x)| \leq C|x - z_x|,$$
$$z_x \in \partial\hat{\Gamma}_D, dist(z_x, D\tilde{S}) \geq \rho, |x - z_x| \leq \frac{1}{2}, \qquad d \geq d_\rho.$$

In the above formula z_x denotes the point in $\partial\hat{\Gamma}_D$ closest to x (which is unique if D is sufficiently large).

Proof. Let us fix $0 < \rho < \frac{1}{8}$, and let us consider the function $\hat{\phi}^d_+$ defined in this way. For any $x \in \partial\hat{\Gamma}_D$ with $dist(x, D\tilde{S}) \geq \rho$, let n_x denote the unit normal to $\partial\hat{\Gamma}_D$ at x. Since $\partial\hat{\Gamma}_D$ is smooth away from $D\tilde{S}$, since it is flat near this set and since its curvatures tend to zero uniformly as $D \to +\infty$, the sets

$$\Sigma_t := \left\{ z + tn_z \ : \ z \in \partial\hat{\Gamma}_D, dist(z, D\tilde{S}) \geq \rho \right\},$$

where n_z is the inner unit normal at z, constitute a smooth manifold with boundary for any $t \in \left(0, \frac{1}{2}\right]$. Moreover the Σ_t's are flat near their boundary and they are all disjoint as t varies in $\left(0, \frac{1}{2}\right]$. Below in this proof (and only here), we use the convention that the distance function from $\partial D\tilde{S}$ in $\{x_n = 0\}$ is extended inside the interior of $D\tilde{S}$ with negative sign. We then define the function

$$\hat{\phi}^d_+(y) = |z - Q_0| + \theta t; \quad y \in \left\{ z + tn_z \ : \ z \in \partial\hat{\Gamma}_D, dist(z, D\tilde{S}) \geq \rho, t \in \left[0, \frac{1}{2}\right] \right\},$$

where θ is a large constant so that $\hat{\phi}^d_+(y) > \phi^d(y)$ when $y \in \left\{ z + \frac{1}{2}n_z \ : \ z \in \partial\hat{\Gamma}_D, dist(z, D\tilde{S}) \geq \rho \right\}$. The existence of such constant θ is guaranteed by Lemma 2.2.4.

Next we consider a smooth non-increasing function $\chi_\rho : \left[-\frac{1}{2}, \rho\right] \to \mathbb{R}$ satisfying the following properties

$$\begin{cases} \chi_\rho(t) = 0 & \text{for } t \in \left[\frac{5}{6}\rho, \rho\right]; \\ \chi'_\rho(t) < 0 & \text{for } t \in \left[-\frac{1}{2}, \frac{2}{3}\rho\right], \end{cases}$$

and another smooth non decreasing cutoff function $\tilde{\chi}_\rho : \left[-\frac{1}{2}, \rho\right]$ such that

$$\begin{cases} \tilde{\chi}_\rho(t) = 0 & \text{for } t \leq \frac{1}{3}\rho; \\ \tilde{\chi}_\rho(t) = 1 & \text{for } t \geq \frac{2}{3}\rho. \end{cases}$$

Finally we extend the function $\hat{\phi}^d_+$ to the set

$$\left\{ z + tn_z \ : \ z \in \mathbb{R}^{n-1}, \ dist(z, D\tilde{S}) \in \left[-\frac{1}{2}, \rho\right] \right\}$$

(we are using the above convention on the distance function) in the following way

$$\hat{\phi}_+^d(z, x_n) = |z - Q_0| + \theta\, |x_n|\, \tilde{\chi}_\rho\left(dist(z, D\tilde{S})\right) + C_\rho \chi_\rho\left(dist(z, D\tilde{S})\right). \quad (2.37)$$

For $dist(z, D\tilde{S}) \geq \frac{2}{3}\rho$, $\tilde{\chi}_\rho\left(dist(z, D\tilde{S})\right)$ is equal to 1, so the norm of the x_n-component of $\nabla\hat{\phi}_+^d$ is bounded below by θ. Also, for $dist(z, D\tilde{S}) \leq \frac{2}{3}\rho$, $\chi'_\rho\left(dist(z, D\tilde{S})\right)$ is bounded above by a fixed negative constant. It follows that, for θ and C_ρ sufficiently large, the norm of $\nabla\hat{\phi}_+^d$ can be made arbitrarily big on its domain. By (2.36), if θ and C_ρ are large then $\hat{\phi}_+^d$ is everywhere bigger than ϕ^d on $\left\{dist(\cdot, \partial\hat{\Gamma}_D) = \frac{1}{2}\right\}$, so $\hat{\phi}_+^d$ is a supersolution of (2.35) in $\left\{dist(\cdot, \partial\hat{\Gamma}_D) < \frac{1}{2}\right\}$.

On the other hand, we claim that the function $\phi_-^d = |x - Q|$ is a subsolution of (2.35) in the set $\hat{\Gamma}_D \cap \left\{dist(\cdot, \partial\hat{\Gamma}_D) < \frac{1}{2}\right\}$. In fact, consider first $\hat{\Gamma}_D \setminus B_{\tilde{\delta}(\varepsilon)}(Q_0)$, where $\tilde{\delta}(d)$ is a small positive number depending on d. Here ϕ_-^d satisfies

$$d\Delta\phi_-^d - |\nabla\phi_-^d|^2 + 1 = \frac{n-1}{d|x - Q_0|},$$

and moreover, if we choose $\tilde{\delta}(d)$ sufficiently small, we have that $\phi_-^d < \phi^d$, since ϕ^d is positive. Therefore we obtain that $\phi_-^d \leq \phi^d$ in the closure of $\hat{\Gamma}_D \cap \left\{dist(\cdot, \partial\hat{\Gamma}_D) < \frac{1}{2}\right\}$.

Finally, since ϕ_-^d and $\hat{\phi}_+^d$ coincide on $\left\{x \in \partial\hat{\Gamma}_D : dist(x, D\tilde{S}) \geq \rho\right\}$ and have uniform bounds on the gradient here (independently of d), the conclusion is achieved. ■

The gradient estimate which follows from the previous lemma is extended next to a subset of the interior of the domain.

Lemma 2.2.6 *Suppose D, ϕ^d are as in Lemma 2.2.4. Then there exists a fixed constant $C > 0$ such that for any $0 < \rho < \frac{1}{8}$ there exists $d_\rho > 0$ so large that*

$$|\nabla\phi^d(x)| \leq C \qquad in \ \left\{x \in \overline{\hat{\Gamma}}_D : dist(x, D\partial\tilde{S}) \geq \rho\right\}, \quad d \geq d_\rho.$$

PROOF. Recall that $\varphi^d := e^{-d\phi^d}$ satisfies the equation $-\frac{1}{d^2}\Delta\varphi^d + \varphi^d = 0$ in $\hat{\Gamma}_D$, and hence the function $\tilde{\varphi}^d := \varphi^d(\cdot/d)$ solves

$$\begin{cases} -\Delta\tilde{\varphi}^d + \tilde{\varphi}^d = 0 & \text{in } \hat{\Gamma}_{dD}; \\ \tilde{\varphi}^d(\cdot) = e^{-|\cdot - dQ_0|} & \text{on } \partial\hat{\Gamma}_{dD}. \end{cases}$$

Notice that $|\nabla\phi^d| \le C$ is bounded in the desired set if and only if $\frac{|\nabla\tilde{\varphi}^d|}{\tilde{\varphi}^d} \le C$ in its d-dilation.

By the Harnack inequality and by standard elliptic estimates we obtain immediately that

$$\frac{|\nabla\tilde{\varphi}^d|}{\tilde{\varphi}^d} \le C_n \qquad \text{in } \left\{x \in \hat{G}_{dD} : dist(x, \partial\hat{\Gamma}_{dD}) > 1\right\},$$

where C_n depends only on the dimension n. Therefore we only need to show the estimate in a neighborhood of (a subset of) the boundary, which is done using a blow-up argument. Suppose by contradiction to the statement that we have the following condition

for every $m \in \mathbb{N}$ there exists ρ_m such that for every $\varepsilon > 0$ there exist

$Q_m \in \hat{\Gamma}_D$ with $dist(Q_m, D\tilde{S}) \ge \rho_m$ and $d < \varepsilon$ with $|\nabla\phi^d(Q_m)| \ge m$.

If d_{ρ_m} is the constant in Lemma 2.2.5 corresponding to ρ_m, let us choose d_m such that $d_m = \max\{m, 2d_{\rho_m}\}$. Let us also choose $\tilde{Q}_m$ for which

$$M_m := |\nabla\phi^{d_m}(\tilde{Q}_m)| = \sup_{A_m} |\nabla\phi^{d_m}(Q)|; \quad A_m = \{Q \in \hat{\Gamma}_D \ : \ dist(Q, D\tilde{S}) \ge \rho_m\}.$$

Consider the functions $\tilde{\varphi}^{d_m}(\cdot) := e^{-d_m\phi^{d_m}(\cdot/d_m)}$, and the new sequence

$$v_m(x) := \frac{\tilde{\varphi}^{d_m}\left(d_m\tilde{Q}_m + \frac{x}{M_m}\right)}{\tilde{\varphi}^{d_m}(d_m\tilde{Q}_m)}. \tag{2.38}$$

Each v_m satisfies

$$-\Delta v_m + \frac{1}{M_m^2} v_m = 0 \qquad \text{in } \ M_m d_m(\hat{\Gamma}_D - Q_m), \tag{2.39}$$

and moreover

$$v_m > 0; \quad v_m(0) = 1; \quad |\nabla v_m(0)| = 1; \qquad \sup_{x \in M_m d_m(A_m - Q_m)} \frac{|\nabla v_m(x)|}{v_m(x)} \le 1. \tag{2.40}$$

Depending on the asymptotic behavior of $\tilde{Q}_m$, d_m and M_m, we have one of the following two possibilities (up to a subsequence)

(a) the component of $M_m d_m(\hat{\Gamma}_D - Q_m)$ containing the origin converges to $\mathbb{R}^n$;

(b) the component of $M_m d_m(\hat{\Gamma}_D - Q_m)$ containing the origin converges to a half space $\{x_n > B\}$, for some $B \leq 0$. On the other hand, we also have one of the following three cases (still, up to a subsequence)

(c) the component of $M_m d_m(A_m - Q_m)$ containing the origin converges to $\mathbb{R}^n$;

(d) the component of $M_m d_m(A_m - Q_m)$ containing the origin converges to a half space;

(e) the component of $M_m d_m(A_m - Q_m)$ containing the origin converges to a quadrant.

In the above alternatives, by *convergence* we mean Hausdorff convergence once we take the intersection with any fixed compact set of $\mathbb{R}^n$. Case (**c**) can only occur when (**a**) holds, and case (**e**) only when (**b**) holds. Calling $\overline{A}$ the limit of the sets $M_m d_m(A_m - Q_m)$, by (2.40) and by the Ascoli theorem we have convergence (in any smooth sense) of v_m on the compact sets of $\overline{A}$ to some non-negative harmonic function $v : \overline{A} \to \mathbb{R}$ for which $v(0) = 1, |\nabla v(0)| = 1$.

If (**a**) and (**c**) hold, then v must be constant on $\mathbb{R}^n$ by the Liouville theorem, which contradicts the fact that $|\nabla v(0)| = 1$. If (**a**) and (**d**) hold, by (2.39) we have the Harnack inequality for v_m in any fixed compact set of $\mathbb{R}^n$, provided m is sufficiently large: again by the Ascoli theorem we get convergence to a non-negative entire harmonic function and reach a contradiction as in the previous case.

If (**b**) holds, we can use Lemma 2.2.5 (including the notation in its statement), the fact that $dz_{x/d} = z_x$ and $\tilde{\phi}^d(\cdot) = e^{-d\phi^d(\cdot/d)}$ to obtain

$$e^{-|z_x - d_m Q_0|} e^{-C|x - z_x|} \leq \tilde{\varphi}^{d_m}(x) \leq e^{-|z_x - d_m Q_0|} e^{C|x - z_x|}; \qquad |x - z_x| \leq \frac{d_m}{2}.$$

Using (2.38) and the fact that $v_m(0) = 1$, we see that the v_m's converge uniformly to the constant 1 in any given compact set of $\{x_n \geq B\}$. Elliptic regularity results imply indeed convergence in any smooth sense to 1, which is again in contradiction to $|\nabla v_m(0)| = 1$). ■

We are now in position for analyzing the asymptotic behavior of the solutions to (2.33), whose existence (and uniqueness) can be deduced as in Lemma 2.2.4.

Proposition 2.2.7 *For D large but fixed, let Φ^d denote the solution of*

(2.33). *Then, as* $d \to +\infty$, *we have*

$$\Phi^d(x) \to \left(\left(1 + \sqrt{x_1^2 + x_n^2} \right)^2 + |x''|^2 \right)^{\frac{1}{2}}$$

uniformly on the compact sets of $\left[\overline{\hat{\Gamma}_D} \setminus (\partial D\tilde{S}) \right] \cap \overline{B}_{\frac{D}{4}}(0)$.

Proof. Since ϕ^d satisfies (2.35), we have that

$$\phi^d + \sup_{x \in \partial \hat{\Gamma}_D} \left| |x - Q_0| + \frac{1}{d} \log U(d(x - Q_0)) \right|$$

is a supersolution of (2.33), while

$$\phi^d - \sup_{x \in \partial \hat{\Gamma}_D} \left| |x - Q_0| + \frac{1}{d} \log U(d(x - Q_0)) \right|$$

is a subsolution. Since Φ^d lies in between these two, by Lemma 2.2.3 we are reduced to prove the analogous statement for ϕ^d. The proof of the latter fact is a consequence of Lemmas 2.2.8 and 2.2.9 below. ■

Lemma 2.2.8 *If* ϕ^d *is as in Lemma 2.2.4 we have that*

$$\phi^d(x) \to \overline{\phi}(x) := \inf_{z \in \partial \hat{\Gamma}_D} (|x - z| + |z - Q_0|) \qquad \text{as } d \to \infty, \tag{2.41}$$

uniformly on the compact sets of $\overline{\hat{\Gamma}_D} \setminus \partial D\tilde{S}$.

Proof. First of all, by Lemma 2.2.6, in any set compactly contained $\overline{\hat{\Gamma}_D} \setminus \partial D\tilde{S}$ the gradient of ϕ^d is uniformly bounded, provided d is sufficiently large: hence by Ascoli's theorem we know that the ϕ^d's admit limit in the whole closure of $\hat{\Gamma}_D$.

The rest of the proof is a modification of the arguments in Lemma 4.3 of [51]: there it is shown that, when $\hat{\Gamma}_D$ is replaced by a smooth domain Λ, the function on the right-hand side of (2.41) turns out to be the supremum of all the elements of

$$\mathcal{S}_{\Lambda, |\cdot - Q_0|} = \left\{ v \in W^{1,\infty}(\Lambda) \ : \ v(x) \le |x - Q_0| \text{ on } \partial \Lambda, |\nabla v| \le 1 \text{ a.e. in } \Lambda \right\}. \tag{2.42}$$

We can actually reduce ourselves to this situation. First of all, in $\mathbb{R}^{n-1}$, we have this fact once we take Λ to be $D\tilde{S}$, since its boundary is smooth and since the infimum will be attained on $\partial D\tilde{S}$. Let us call $\overline{\phi}_{D\tilde{S}} : D\tilde{S} \to \mathbb{R}$ the function given by

$$\overline{\phi}_{D\tilde{S}}(\mathrm{x}) := \inf_{z \in \partial D\tilde{S}} (|\mathrm{x} - z| + |z - Q_0|) ; \qquad \mathrm{x} \in D\tilde{S},$$

which is a Lipschitz function with Lipschitz constant bounded by 1.

Next, if we restrict ourselves to the set

$$\Lambda_D := \hat{\Gamma}_D \cap \{x_n > 0\},$$

which is now smooth, and if we consider the function $\tilde{\phi}_{D\tilde{S}} : \partial\Lambda_D \to \mathbb{R}$ defined as

$$\tilde{\phi}_{D\tilde{S}}(z) = \begin{cases} |z - Q_0| & z \in \{x_n \geq 0\} \cap \partial\hat{\Gamma}_D; \\ \overline{\phi}_{D\tilde{S}}(z) & z \in D\tilde{S}, \end{cases}$$

we claim that

$$\inf_{z \in \partial\hat{\Gamma}_D} (|x - z| + |z - Q_0|) = \inf_{z \in \partial\Lambda_D} \left(|x - z| + \tilde{\phi}_{D\tilde{S}} \right), \qquad x_n > 0. \tag{2.43}$$

In fact, suppose z_1 realizes the infimum on the right-hand side. If $z_1 \notin D\tilde{S}$ we are done, since $\tilde{\phi}_{D\tilde{S}}$ and $|\cdot - Q_0|$ coincide on $(\partial\Lambda_D) \setminus D\tilde{S}$. If instead $z_1 \in D\tilde{S}$, by the definition of $\tilde{\phi}_{D\tilde{S}}$ there exists $z_2 \in \partial D\tilde{S}$ such that

$$\tilde{\phi}_{D\tilde{S}}(z_1) = |z_1 - z_2| + |z_2 - Q_0|.$$

Therefore we have

$$\begin{aligned} \inf_{z \in \partial\hat{\Gamma}_D} (|x - z| + |z - Q_0|) &\leq |x - z_2| + |z_2 - Q_0| \\ &\leq |x - z_1| + |z_1 - z_2| + |z_2 - Q_0| \\ &= |x - z_1| + \tilde{\phi}_{D\tilde{S}}(z_1) \\ &= \inf_{z \in \partial\hat{\Gamma}_D} (|x - z| + |z - Q_0|), \end{aligned}$$

where the last equality is a consequence of our definition and our assumption on z_1, so the claim follows. Hence we can apply the result in [51] in the smooth domain Λ_D, with boundary datum $\tilde{\phi}_{D\tilde{S}}$, and use (2.43).

Let us now prove next convergence to the function $\overline{\phi}$: similarly to Step 1 in the proof of Lemma 4.3 in [51] (and as noticed before) we can apply Ascoli's theorem (by Lemma 2.2.6) to guarantee existence of a limit for $d \to \infty$, which indeed must belong to the set $\mathcal{S}_{\hat{\Gamma}_D, |\cdot - Q_0|}$ (see the notation in (2.42)). We need then to prove only $\lim_{d\to\infty} \phi^d \geq \overline{\phi}$.

Two comments are in order: first of all the solutions are even (by uniqueness and since the boundary datum in (2.33) is), and we can limit ourselves to take only even functions in the definition of $\overline{\phi}$. Second, we can extend in a Lipschitz manner every $v \in \mathcal{S}_{\hat{\Gamma}_D, |\cdot - Q_0|}$ (even in x_n) to a δ neighborhood of $\hat{\Gamma}_D$: we can for example consider v restricted to the boundary of $\hat{\Gamma}_D$ and extend it constantly in the normal direction (and stop at $x_n = 0$ if we reach this hyperplane at a distance from the boundary smaller than δ). Once we have this extension, we can use convolutions with mollifiers as for Step 2 in the proof of Lemma 4.3 in [51]. ∎

Lemma 2.2.9 *If $\overline{\phi}$ is as in* (2.41), *then*

$$\overline{\phi}(x) = \left(\left(1 + \sqrt{x_1^2 + x_n^2} \right)^2 + |x''|^2 \right)^{\frac{1}{2}} \qquad x \in \overline{B}_{\frac{D}{4}}(0). \tag{2.44}$$

Proof. By construction of $\hat{\Gamma}_D$, the points $z \in \hat{\Gamma}_D$ which realize the infimum will necessarily belong to the set $\{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$, so we can write that

$$\overline{\phi}(x) = \inf_{z \in \{\{x_n = 0\} \cup \{x_1 \geq 0\}\}} \left(|x - z| + |z - Q_0| \right).$$

First of all we claim that the latter infimum can be attained on $\{z_1 = 0\}$. In fact, two possibilities may occur: the first is when $x \in \{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$. In this case the points z realizing the infimum are exactly those which belong to the segment connecting x to Q_0, and we can choose the one for which $z_1 = 0$.

In the second case, when $x \notin \{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$, we can argue as follows: for x and Q_0 fixed, the level sets of the function $z \mapsto |x - z| + |z - Q_0|$ are the axially symmetric ellipsoids with focal points x and Q_0. The smaller is the ellipsoid, the smaller is the value of this function so we are reduced to find the smallest ellipsoid which intersects $\{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$: this will happen at a point where $z_1 = 0$, so we can still reduce ourselves to this situation.

After the above claim has been established, it is sufficient to consider the minimum problem

$$\min_{z'' \in \mathbb{R}^{n-2}} \left((x_1^2 + |z'' - x''|^2 + x_n^2)^{\frac{1}{2}} + (1^2 + |z''|^2)^{\frac{1}{2}} \right). \tag{2.45}$$

By differentiation we obtain that at a minimum point

$$\frac{z'' - x''}{(x_1^2 + |z'' - x''|^2 + x_n^2)^{\frac{1}{2}}} + \frac{z''}{(1^2 + |z''|^2)^{\frac{1}{2}}} = 0,$$

which implies

$$z'' = x'' \frac{\sqrt{x_1^2 + x_n^2} - 1}{x_1^2 + x_n^2 - 1} = \frac{x''}{\sqrt{x_1^2 + x_n^2} + 1}.$$

If we plug in this expression into (2.45) we obtain the desired conclusion. ■

Remark 2.2.10 *Since the convergence in* (2.34) *is indeed uniform in every smooth sense (by* (1.3)*), a reasoning as in the proof of Lemma 2.2.6 hold true for the function Φ^d (the solution of* (2.33)*) as well. Therefore, there still exists a fixed constant $C > 0$ such that for any $0 < \rho < \frac{1}{8}$ there exists $d_\rho > 0$ so large that*

$$|\nabla \Phi^d(x)| \leq C \qquad \text{in } \left\{ x \in \overline{\hat{\Gamma}}_D : dist(x, D\tilde{S}) \geq \rho \right\}, \quad d \geq d_\rho. \tag{2.46}$$

The arguments in [37] (see in particular Proposition 1.4, Lemma 1.5 and Lemma B.1) imply that

$$\nabla \Phi^d \to \nabla \overline{\phi} \qquad \text{uniformly in } \mathcal{O}, \tag{2.47}$$

where $\mathcal{O}$ is any set compactly contained in $\overline{\hat{\Gamma}_D} \setminus (\partial D\tilde{S})$ on which $\nabla \overline{\phi} \neq 0$.

Remark 2.2.11 *It is also useful to make a few comments on the derivative of φ with respect to d, where φ is the solution of* (2.31). *Differentiating* (2.31) *in d we obtain*

$$\begin{cases} -\frac{1}{d^2}\Delta \frac{\partial \varphi}{\partial d} + \frac{\partial \varphi}{\partial d} = -\frac{2}{d^3}\Delta \varphi = -\frac{2}{d}\varphi & \text{in } \hat{\Gamma}_D; \\ \frac{\partial \varphi}{\partial d}(x) = \nabla U(d(x-Q_0)) \cdot (x-Q_0) & \text{on } \partial \hat{\Gamma}_D. \end{cases} \tag{2.48}$$

By the asymptotics in (1.3)-(1.4) *there exists a positive constant C_D such that for d large we have the inequality $\frac{1}{C_D}U(d(x-Q_0)) \leq -\nabla U(d(x-Q_0)) \cdot (x-Q_0) \leq C_D U(d(x-Q_0))$. Hence, from the latter formula, from the fact that $\varphi > 0$,* (2.31) *and the maximum principle we obtain that $\varsigma := -\frac{\partial \varphi}{\partial d} \geq \frac{1}{C_D}\varphi$ in $\hat{\Gamma}_D$. Moreover, as for* (2.33) *one checks that $\Upsilon^d := -\frac{1}{d}\log \varsigma$ satisfies*

$$\begin{cases} \frac{1}{d}\Delta \Upsilon^d - |\nabla \Upsilon^d|^2 + 1 - \frac{\varphi}{d\varsigma} = 0 & \text{in } \hat{\Gamma}_D; \\ \Upsilon^d(x) = -\frac{1}{d}\log\left(-\nabla U(d(x-Q_0)) \cdot (x-Q_0)\right) & \text{on } \partial \hat{\Gamma}_D. \end{cases}$$

Since (as we just remarked) $\frac{\varphi}{\varsigma}$ stays bounded, $\frac{\varphi}{d\varsigma}$ tends to zero as $d \to +\infty$. Moreover by (1.4) *the boundary datum in the latter formula converges in every smooth sense (where $\partial\hat{\Gamma}_D$ is regular) to $|x-Q_0|$ as $d \to +\infty$. As a consequence, the previous analysis adapts to Υ^d and allows to conclude that still*

$$\Upsilon^d \to \overline{\phi} \quad \text{and} \quad \nabla \Upsilon^d \to \nabla \overline{\phi} \qquad \text{uniformly in } \mathcal{O}, \tag{2.49}$$

where $\mathcal{O}$ is as in Remark 2.2.10.

2.2.2 Definition of the approximate solutions and study of their accuracy

In this subsection we construct a manifold of approximate solutions to (P_ε), in order to apply the theory in Subsection 2.1.1.

First of all, we need to collect some preliminary estimates. If Φ^d is the solution of (2.33), the function

$$\Xi_d(y) = e^{-d\Phi^d\left(\frac{y}{d}+Q_0\right)} \tag{2.50}$$

solves the problem

$$\begin{cases} -\Delta \Xi_d + \Xi_d = 0 & \text{in } d(\hat{\Gamma}_D - Q_0); \\ \Xi_d = U(\cdot) & \text{on } d\partial(\hat{\Gamma}_D - Q_0), \end{cases} \qquad Q_0 = (-1, 0, \dots, 0). \tag{2.51}$$

As explained at the beginning of Section 2.2, we can obtain a solution looking at the minimum problem

$$\inf_{v=U \text{ on } d(\partial\hat{\Gamma}_D - Q_0)} \left\{ \int_{\Omega_\varepsilon} \left(|\nabla v|^2 + v^2\right) dy \right\}. \tag{2.52}$$

We can easily derive both norm estimates on Ξ_d from (2.52), and pointwise estimates on Ξ_d, $\nabla\Xi_d$ from Proposition 2.2.7 and Remark 2.2.10 respectively.

To derive a norm estimate, one can take a cutoff function

$$\chi_1 : d(\overline{\hat{\Gamma}_D} - Q_0) \to \mathbb{R}$$

such that

$$\begin{cases} \chi_1(y) = 1 & \text{for } dist(y, d(\partial\hat{\Gamma}_D - Q_0)) \le \frac{1}{2}; \\ \chi_1(y) = 0 & \text{for } y \in d(\partial\hat{\Gamma}_D - Q_0),\ dist(y, d(\partial\hat{\Gamma}_D - Q_0)) \ge 1; \\ |\nabla\chi_1(y)| \le 4 & \text{for all } y, \end{cases}$$

and then consider the function

$$\overline{v}(y) = \chi_1(y) U(y).$$

It is easy to see (using for example polar coordinates centered at the origin and the asymptotic behavior in (1.3)) that $\|\overline{v}\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \le e^{-d(1+o(1))}$, so by (2.52) we also find that

$$\|\Xi_d\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \le \|\overline{v}\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \le e^{-d(1+o(1))}. \tag{2.53}$$

Here and below, we use the notation $d(1 + o(1))$ described at the end of the introduction.

To obtain instead pointwise estimates on Ξ_d, we can use Lemma 2.2.7 to get that

$$\Xi_d(y) = \exp\left[-\left(\left(d + \sqrt{(y_1 - d)^2 + |y_n|^2} \right)^2 + |y''|^2 \right)^{\frac{1}{2}} \right] \times e^{o(d)}; \tag{2.54}$$

$y \in d(\mathcal{O} - Q_0), d \to +\infty$,

where $\mathcal{O}$ is any set compactly contained in $\overline{\hat{\Gamma}_D} \setminus \partial D\tilde{S}$. On the other hand, Remark 2.2.10 implies

$$\begin{aligned} \nabla\Xi_d(y) &= -\exp\left[-\left(\left(d + \sqrt{(y_1 - d)^2 + |y_n|^2} \right)^2 + |y''|^2 \right)^{\frac{1}{2}} \right] \times e^{o(d)} \\ &\times \left(\nabla\overline{\phi}\left(\frac{y}{d} + Q_0\right) + o(1) \right); \quad y \in d(\mathcal{O} - Q_0), d \to +\infty, \end{aligned} \tag{2.55}$$

where $\mathcal{O}$ is as before.

We will obtain next similar bounds and estimates for $\frac{\partial \Xi_d}{\partial d}$ and its gradient: recalling that $\Xi_d(y) = \varphi\left(\frac{y}{d} + Q_0\right)$ (and that also φ depends on d) we have that

$$\frac{\partial \Xi_d}{\partial d}(y) = \frac{\partial \varphi}{\partial d}\left(\frac{y}{d} + Q_0\right) - \frac{y}{d^2} \cdot \nabla \varphi \left(\frac{y}{d} + Q_0\right). \tag{2.56}$$

Using (2.48) and reasoning as for (2.53) one can prove that

$$\left\| \frac{\partial \varphi}{\partial d}\left(\frac{\cdot}{d} + Q_0\right) \right\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \leq e^{-d(1+o(1))}. \tag{2.57}$$

On the other hand, from (2.51) one finds that the function $\varpi := \frac{y}{d^2} \cdot \nabla \varphi\left(\frac{y}{d} + Q_0\right) = \frac{y}{d} \cdot \nabla \Xi_d(y)$ satisfies

$$-\Delta \varpi + \varpi = -\frac{2}{d}\Xi_d \qquad \text{in } d(\hat{\Gamma}_D - Q_0).$$

To control the boundary value of ϖ we divide $\partial d(\hat{\Gamma}_D - Q_0)$ into its intersection with $\{y_n = 0\}$ and its complement. In the first region we have simply that $\varpi = \frac{y}{d} \cdot \nabla U(y)$. In the second instead the estimate in (2.54) holds true, which shows that the L^2 norm of the trace of ϖ on $d(\partial \hat{\Gamma}_D - Q_0)$ is of order $e^{-d(1+o(1))}$. This fact and the latter formula imply that $\|\varpi\|_{H^1(d(\hat{\Gamma}_D - Q_0))}$ satisfies a bound similar to (2.57), and hence from these two we conclude that

$$\left\| \frac{\partial \Xi_d}{\partial d} \right\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \leq e^{-d(1+o(1))}. \tag{2.58}$$

By Remark 2.2.11 (which yields $\varphi \leq C_D \left| \frac{\partial \varphi}{\partial d} \right|$ and (2.49)) together with the Harnack inequality (which implies $|\nabla \varphi| \leq C d \varphi$ in $d(\mathcal{O} - Q_0)$) one also finds

$$\begin{aligned} \frac{\partial \Xi_d}{\partial d}(y) \quad &= \quad -\exp\left[-\left(\left(d + \sqrt{(y_1 - d)^2 + |y_n|^2} \right)^2 + |y''|^2 \right)^{\frac{1}{2}} \right] \times \\ & \qquad e^{o(d)} \times \left(1 + O\left(\frac{|y|}{d} \right) \right) \end{aligned} \tag{2.59}$$

for $y \in d(\mathcal{O} - Q_0)$ and $d \to +\infty$, and moreover

$$\left| \nabla \frac{\partial \Xi_d}{\partial d}(y) \right| \leq \exp\left[-\left(\left(d + \sqrt{(y_1 - d)^2 + |y_n|^2} \right)^2 + |y''|^2 \right)^{\frac{1}{2}} \right] \times e^{o(d)}; \tag{2.60}$$

$y \in d(\mathcal{O} - Q_0), d \to +\infty.$

After these preliminaries, we are now in position to introduce our approximate solutions. Define next two smooth non negative cutoff functions $\chi_D : \mathbb{R}^n \to \mathbb{R}$, $\chi_0 : \mathbb{R} \to \mathbb{R}$ satisfying respectively

$$\begin{cases} \chi_D(y) = 1 & \text{for } |y| \leq \frac{dD}{16}; \\ \chi_D(y) = 0 & \text{for } |y| \geq \frac{dD}{8}; \\ |\nabla \chi_D| \leq \frac{32}{dD} & \text{on } \mathbb{R}^n, \end{cases} \qquad \begin{cases} \chi_0(y) = 0 & \text{for } y \leq -1; \\ \chi_0(y) = 1 & \text{for } y \geq 0; \\ \chi_0 \text{ is non decreasing} & \text{on } \mathbb{R}. \end{cases}$$

Using then the coordinates y in Subsection 2.1.2 (see also (2.16) for the relation between Q and d) we define

$$\hat{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y)\left[(U(y) - \Xi_d(y))\chi_D(y) + \varepsilon \tilde{w}_Q(y)\chi_0(y_1 - d)\right]. \qquad (2.61)$$

Notice that, since $\Xi_d(y)$ coincides with $U(y)$ for $y_n = 0$, $y_1 > d$ and y in the support of χ_D, the function $\hat{z}_{\varepsilon,Q}$ belongs to $H^1_{\mathcal{D}}(\Omega_\varepsilon)$. We also point out that, by our choice of $\hat{\Gamma}_D$, the support of χ_D intersects $d(\partial D\tilde{S} - Q_0)$ only on $\{y_1 = d\}$.

We prove next that the $\hat{z}_{\varepsilon,Q}$'s are good approximate solutions to (P_ε) for suitable conditions on Q.

Proposition 2.2.12 *Let μ_0 be the constant appearing in Proposition 2.1.4. Then there exists another constant $C_\Omega > 0$ (independent of ε) such that, for $C_\Omega \leq d \leq \frac{1}{\varepsilon C_\Omega}$ and for $Dd < \frac{1}{C_\Omega}\frac{\mu_0}{\varepsilon}$, the functions $\hat{z}_{\varepsilon,Q}$ satisfy*

$$\|I'_\varepsilon(\hat{z}_{\varepsilon,Q})\| \leq C\left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right), \quad (2.62)$$

for some fixed $C, K > 0$ and for ε sufficiently small.

PROOF. Using the coordinates y, let us write $\hat{z}_{\varepsilon,Q} = \overline{z}_{\varepsilon,Q} + \check{z}_{\varepsilon,Q}$ (see (2.24)), where

$$\check{z}_{\varepsilon,Q} = \chi_{\mu_0}(\varepsilon y)\left[(\chi_D(y) - 1)U(y) - \chi_D(y)\Xi_d(y) + \varepsilon(1 - \chi_0(y_1 - d))\tilde{w}_Q(y)\right]. \qquad (2.63)$$

With this notation, testing the gradient of I_ε at $\hat{z}_{\varepsilon,Q}$ on any function $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$ we can write that

$$\begin{aligned} I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] &= \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\hat{z}_{\varepsilon,Q}\nabla_{g_y}v + \hat{z}_{\varepsilon,Q}v\right) dy - \int_{\Omega_\varepsilon} \hat{z}^p_{\varepsilon,Q} v\, dy \\ &= \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\overline{z}_{\varepsilon,Q}\nabla_{g_y}v + \overline{z}_{\varepsilon,Q}v\right) dy - \int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q} v\, dy \qquad (2.64) \\ &+ \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\check{z}_{\varepsilon,Q}\nabla_{g_y}v + \check{z}_{\varepsilon,Q}v\right) dy + \int_{\Omega_\varepsilon} (\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q})v\, dy \\ &= I'_\varepsilon(\overline{z}_{\varepsilon,Q})[v] + A_1 + A_2, \end{aligned}$$

where

$$A_1 = \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\check{z}_{\varepsilon,Q}\nabla_{g_y}v + \check{z}_{\varepsilon,Q}v\right) dy; \qquad A_2 = \int_{\Omega_\varepsilon} (\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q})v\, dy.$$

By Proposition 2.1.6 (see (2.25)) we have that $I'_\varepsilon(\overline{z}_{\varepsilon,Q})[v]$ is of order at most ε^2, so we only need to estimate A_1 and A_2 in the last line of (2.64).

To estimate A_1 we divide further $\check{z}_{\varepsilon,Q}$ into the three parts $\check{z}_{\varepsilon,Q,1}, \check{z}_{\varepsilon,Q,2}$ and $\check{z}_{\varepsilon,Q,3}$

$$\check{z}_{\varepsilon,Q,1} = \chi_{\mu_0}(\varepsilon y)(\chi_D(y) - 1)U(y); \qquad \check{z}_{\varepsilon,Q,2} = \chi_{\mu_0}(\varepsilon y)\chi_D(y)\Xi_d(y);$$

$$\check{z}_{\varepsilon,Q,3} = \chi_{\mu_0}(\varepsilon y)\varepsilon(1-\chi_0(y_1-d))\tilde{w}_Q(y),$$

and write $A_1 = A_{1,1} + A_{1,2} + A_{1,3}$, where

$$A_{1,i} = \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\check{z}_{\varepsilon,Q,i}\nabla_{g_y}v + \check{z}_{\varepsilon,Q,i}v\right) dy; \qquad i = 1,2,3.$$

Since $\chi_D(y)$ is identically equal to 1 for $|y| \leq \frac{dD}{16}$ and since $(1-\chi_0(y_1-d)) = 0$ for $y_1 \leq d-1$, from (1.3) and (2.23) one finds

$$|A_{1,1}| \leq e^{-\frac{dD}{16}(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}; \qquad |A_{1,3}| \leq C\varepsilon(1+|d|^K)e^{-|d|}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{2.65}$$

To control $A_{1,2}$ we write that

$$\begin{aligned} A_{1,2} &= \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\check{z}_{\varepsilon,Q,2}\nabla_{g_y}v + \check{z}_{\varepsilon,Q,2}v\right) dy = \int_{\Omega_\varepsilon} \left(g_y^{ij}\partial_i\check{z}_{\varepsilon,Q,2}\partial_j v + \check{z}_{\varepsilon,Q,2}v\right) dy \\ &= \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q,2}\nabla_{\mathbb{R}^n}v + \check{z}_{\varepsilon,Q,2}v)\, dy + \int_{\Omega_\varepsilon} \left(g_y^{ij} - \delta^{ij}\right)\partial_i\check{z}_{\varepsilon,Q,2}\partial_j v\, dy. \end{aligned}$$

From (2.19) (and the subsequent comments) we have that

$$|(g_y)^{ij} - \delta_{ij}| \leq C\varepsilon|y|,$$

and hence

$$\begin{aligned} &\left| A_{1,2} - \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q,2}\nabla_{\mathbb{R}^n}v + \check{z}_{\varepsilon,Q,2}v)\, dy \right| \\ &\leq C\varepsilon \left(\int_{\Omega_\varepsilon} |y|^2|\nabla\check{z}_{\varepsilon,Q,2}|^2 dy \right)^{\frac{1}{2}} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \end{aligned}$$

Since $\check{z}_{\varepsilon,Q,2}$ is supported in $\left\{|y| \leq \frac{dD}{8}\right\}$, we obtain from the above formula and (2.53) that

$$\left| A_{1,2} - \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q,2}\nabla_{\mathbb{R}^n}v + \check{z}_{\varepsilon,Q,2}v)\, dy \right| \leq C\varepsilon dDe^{-d(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

Next, since Ξ_d satisfies (2.51), we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q,2}\nabla_{\mathbb{R}^n}v + \check{z}_{\varepsilon,Q,2}v)\, dy \qquad (2.66) \\ &= \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}(\Xi_d(\chi_{\mu_0}(\varepsilon y)\chi_D - 1))\nabla_{\mathbb{R}^n}v + (\Xi_d(\chi_{\mu_0}(\varepsilon y)\chi_D - 1))v)\, dy. \end{aligned}$$

Since also $Dd < \frac{1}{C_\Omega}\frac{\mu_0}{\varepsilon}$, the function $\chi_{\mu_0}(\varepsilon y)\chi_D - 1$ is identically zero for $|y| \leq \frac{dD}{16}$ (if C_Ω is sufficiently large), so from (2.54), (2.55) and the Hölder inequality one finds (also for D large)

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}(\Xi_d(\chi_{\mu_0}(\varepsilon y)\chi_D - 1))\nabla_{\mathbb{R}^n}v + (\Xi_d(\chi_{\mu_0}(\varepsilon y)\chi_D - 1))v)\, dy \right| \\ &\leq e^{-\frac{dD}{16}(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \qquad (2.67) \end{aligned}$$

The last three formulas imply

$$|A_{1,2}| \leq \left(\varepsilon d D e^{-d(1+o(1))} + e^{-\frac{dD}{16}(1+o(1))}\right) \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

From (2.65) and the latter formula it follows that

$$|A_1| \leq C \left(\varepsilon d D e^{-d(1+o(1))} + e^{-\frac{dD}{16}(1+o(1))} + \varepsilon(1+|d|^K)e^{-|d|}\right) \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{2.68}$$

We can now turn to the estimate of A_2: one has the inequalities

$$|\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}| \leq \begin{cases} C\overline{z}^{p-1}_{\varepsilon,Q}|\check{z}_{\varepsilon,Q}| & \text{for } \check{z}_{\varepsilon,Q} \in \left(0, \frac{1}{2}\overline{z}_{\varepsilon,Q}\right); \\ C|\overline{z}_{\varepsilon,Q}|^{p-1}|\check{z}_{\varepsilon,Q}| + C|\check{z}_{\varepsilon,Q}|^p & \text{otherwise,} \end{cases} \tag{2.69}$$

for a fixed constant C depending only on p. First of all we notice the following: by (1.3) and (2.23) there exists a (small) constant $c_{K,n}$ such that

$$\overline{z}_{\varepsilon,Q}(y) \geq \frac{7}{8}\frac{e^{-|y|}}{1+|y|^{\frac{n-1}{2}}}; \qquad \text{for } |y| \leq \frac{1}{\varepsilon^{c_{K,n}}}.$$

We divide next Ω_ε into the two regions

$$\mathcal{B}_1 = \left\{|y| < \min\left\{\frac{d}{2}, \frac{1}{\varepsilon^{c_{K,n}}}\right\}\right\}; \qquad \mathcal{B}_2 = \Omega_\varepsilon \setminus \mathcal{B}_1.$$

For $y \in \mathcal{B}_1$ we have that $\chi_{\mu_0}(\varepsilon y) \equiv 1$, $\chi_D(y) \equiv 1$, $\chi_0(y_1 - d) \equiv 1$, and hence $\check{z}_{\varepsilon,Q}(y) \equiv -\Xi_d(y)$. By (2.54) one has also that $|\check{z}_{\varepsilon,Q}(y)| = |\Xi_d(y)| \leq e^{-\frac{3}{2}d+o(d)} < \frac{1}{2}\overline{z}_{\varepsilon,Q}$ for $y \in \mathcal{B}_1$. This fact, (2.69) and the Hölder inequality yield

$$\begin{aligned}\int_{\mathcal{B}_1} |\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}||v|\,dy &\leq C\int_{\mathcal{B}_1} \overline{z}^{p-1}_{\varepsilon,Q}|\check{z}_{\varepsilon,Q}||v|\,dy \\ &\leq C\left(\int_{\mathcal{B}_1} e^{-\frac{3(p+1)}{2}d+o(d)}dy\right)^{\frac{1}{p+1}} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \\ &\leq Ce^{-\frac{3}{2}d(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.\end{aligned} \tag{2.70}$$

On the other hand, in $\mathcal{B}_2$ we have that $|\overline{z}_{\varepsilon,Q}| < C\left(e^{-\frac{d}{2}+o(d)} + e^{-\frac{1+o(1)}{\varepsilon^{c_{K,n}}}}\right)$ and that $|\check{z}_{\varepsilon,Q}| \leq e^{-d+o(d)}$, therefore (2.69) and the Hölder inequality imply again

$$\begin{aligned}&\int_{\mathcal{B}_2} |\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}||v|dy \leq \\ &C\left[\left(e^{-\frac{(p-1)d}{2}+o(d)} + e^{-\frac{p-1+o(1)}{\varepsilon^{c_{K,n}}}}\right)e^{-d+o(d)} + e^{-pd+o(d)}\right]\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.\end{aligned}$$

The last formula and (2.70) provide

$$|A_2| \leq C \left(e^{-\frac{(p-1)d}{2}+o(d)} + e^{-\frac{p-1+o(1)}{\varepsilon^c K,n}} + e^{-\frac{d}{2}+o(d)} \right) e^{-d+o(d)} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{2.71}$$

Finally, we obtain the conclusion from (2.64), (2.25), (2.68) and (2.71). ■

We have next a related estimate when we vary the parameters in the definition of $\hat{z}_{\varepsilon,Q}$, in the spirit of condition *ii)* in Subsection 2.1.1.

Proposition 2.2.13 *There exists a constant $C_\Omega > 0$ (independent of ε) such that, for $C_\Omega \leq d \leq \frac{1}{\varepsilon C_\Omega}$ and for $Dd < \frac{1}{C_\Omega}\frac{\mu_0}{\varepsilon}$, the functions $\hat{z}_{\varepsilon,Q}$ satisfy*

$$\|I''_\varepsilon(\hat{z}_{\varepsilon,Q})[q]\| \leq C \left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))} \right) \|q\|, \tag{2.72}$$

for some fixed $C, K > 0$ and for ε sufficiently small. In the above formula q represents a vector in $H^1_{\mathcal{D}}(\Omega_\varepsilon)$ which is tangent to the manifold of the $\hat{z}_{\varepsilon,Q}$'s (when Q varies).

PROOF. Since the arguments are quite similar to those in the proof of Proposition 2.2.12, we will be rather sketchy. Using (2.18) and the first line in (2.64), for any given test function $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$ we can write that

$$I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] = \sum_{i,j} \int_{\mathbb{R}^n_+} \left((g^y)^{ij} \partial_i \hat{z}_{\varepsilon,Q} \partial_j v + \hat{z}_{\varepsilon,Q} v \right) dy - \int_{\mathbb{R}^n_+} \hat{z}^p_{\varepsilon,Q} v \, dy.$$

We want to differentiate next with respect to the parameter Q, taking first a variation q_T of the point Q for which d stays fixed, namely we take the tangential derivative to the level set of the distance d to the interface. In the above formula the dependence on Q is in the metric coefficients $(g^y)^{ij}$ and in the function $\tilde{w}_Q$ appearing in the expression of $\hat{z}_{\varepsilon,Q}$, see (2.61). Therefore we obtain

$$\begin{aligned} \frac{\partial}{\partial Q_T} I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] &= I''_\varepsilon(\hat{z}_{\varepsilon,Q}) \left[\frac{\hat{z}_{\varepsilon,Q}}{\partial Q_T}, v \right] = \sum_{i,j} \int_{\mathbb{R}^n_+} \frac{\partial (g^y)^{ij}}{\partial Q_T} \partial_i \hat{z}_{\varepsilon,Q_T} \partial_j v \, dy \\ &+ \sum_{i,j} \int_{\mathbb{R}^n_+} \left((g^y)^{ij} \partial_i \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_T} \partial_j v + \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_T} v \right) dy \\ &- p \int_{\mathbb{R}^n_+} \hat{z}^{p-1}_{\varepsilon,Q} \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_T} v \, dy. \end{aligned} \tag{2.73}$$

From Remark 2.1.5 (b) we have that $\frac{\partial (g^y)^{ij}}{\partial Q}$ is of order $\varepsilon^2|y|$, while computing the expression of $\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_T}$ we obtain

$$\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_T} = \varepsilon \chi_{\mu_0}(\varepsilon y) \chi_0(y_1 - d) \frac{\partial \tilde{w}_Q(y)}{\partial Q_T}.$$

Since the dependence of $\tilde{w}_Q$ on A_Q, $\tilde{A}_Q$ is linear (see (2.22)) and since the derivatives of the latter quantities with respect to Q are of order ε (by the arguments in Remark 2.1.5) we find that $\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_T}(y) = O\left(\varepsilon^2(1+|y|^K)e^{-|y|}\right)$. Reasoning as in the proof of Proposition 2.2.12 we then have

$$\left\| \frac{\partial}{\partial Q_T} I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] \right\| \leq C\varepsilon^2 \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \qquad \text{for every } v \in H^1_{\mathcal{D}}(\Omega_\varepsilon). \tag{2.74}$$

On the other hand, when we take a variation q_d of Q along the gradient of d, similarly to (2.73) we get

$$\begin{aligned} \frac{\partial}{\partial Q_d} I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] &= I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\frac{\hat{z}_{\varepsilon,Q}}{\partial Q_d}, v\right] = \sum_{i,j}\int_{\mathbb{R}^n_+} \frac{\partial (g^y)^{ij}}{\partial Q_d} \partial_i \hat{z}_{\varepsilon,Q_d} \partial_j v\, dy \\ &+ \sum_{i,j}\int_{\mathbb{R}^n_+} \left((g^y)^{ij} \partial_i \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_d} \partial_j v + \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_d} v \right) dy \\ &- p\int_{\mathbb{R}^n_+} \hat{z}^{p-1}_{\varepsilon,Q} \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_d} v\, dy. \end{aligned} \tag{2.75}$$

Concerning the derivatives of $(g^y)^{ij}$ with respect to Q_d we can argue exactly as for Q_T, to find

$$\left| \sum_{i,j}\int_{\mathbb{R}^n_+} \frac{\partial (g^y)^{ij}}{\partial Q_d} \partial_i \hat{z}_{\varepsilon,Q_d} \partial_j v\, dy \right| \leq C\varepsilon^2 \|v\|.$$

However, $\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_d}$ has a more involved expression compared to the previous case. Assuming that the cutoff function $\chi_D(y)$ is defined as $\overline{\chi}_D\left(\frac{y}{d}\right)$ for some fixed $\overline{\chi}_D$, we obtain

$$\begin{aligned} \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q_d} &= -\chi_{\mu_0}\chi_D \frac{\partial \Xi_d}{\partial d} + \frac{1}{d^2}\chi_{\mu_0}\Xi_d\, y \cdot \nabla \overline{\chi}_D\left(\frac{y}{d}\right) + \varepsilon \chi_{\mu_0} \tilde{w}_Q \frac{\partial \chi_0(y_1 - d)}{\partial d} \\ &+ \varepsilon \chi_{\mu_0}(\varepsilon y)\chi_0(y_1 - d)\frac{\partial \tilde{w}_Q(y)}{\partial Q_d}. \end{aligned} \tag{2.76}$$

The last two terms in the right hand side are easily seen to give contributions to (2.75) of order at most $\varepsilon e^{-d(1+o(1))}\|v\|$ and $\varepsilon^2 e^{-d(1+o(1))}\|v\|$ respectively. Concerning the second one, we can use the fact that $\nabla\chi_D$ is supported in $|y| \geq \frac{Dd}{16}$, together with (2.54), (2.55) to see that the contribution of this term is at most of order $e^{-\frac{dD}{16}(1+o(1))}\|v\|$.

We can then focus on the first term in the right hand side of (2.76), and consider the quantity

$$\begin{aligned} &- \sum_{i,j}\int_{\mathbb{R}^n_+} \left((g^y)^{ij}\partial_i \left(\chi_{\mu_0}\chi_D \frac{\partial \Xi_d}{\partial d}\right) \partial_j v + \chi_{\mu_0}\chi_D \frac{\partial \Xi_d}{\partial d} v \right) dy \\ &+ p\int_{\mathbb{R}^n_+} \hat{z}^{p-1}_{\varepsilon,Q} \chi_{\mu_0}\chi_D \frac{\partial \Xi_d}{\partial d} v\, dy. \end{aligned} \tag{2.77}$$

First of all, by (2.19) and (2.58), if we substitute the coefficients $(g^y)^{ij}$ with the Kronecker symbols we find a difference of order $\varepsilon e^{-d(1+o(1))}$. Next, since Ξ_d satisfies

$$-\Delta \Xi_d + \Xi_d = 0,$$

when we differentiate with respect to d we get the same equation for $\frac{\partial \Xi_d}{\partial d}$, so reasoning as for (2.66), (2.67) (and using the fact that $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$ together with (2.59), (2.60)) we find

$$\left| \int_{\mathbb{R}^n_+} \left(\nabla \left(\chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} \right) \cdot \nabla v + \chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} v \right) dy \right| \leq C e^{-\frac{dD}{16}(1+o(1))} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

For the last term in (2.77) one can use (2.58), (2.59) and the exponential decay of $\hat{z}_{\varepsilon,Q}$ (reasoning with arguments similar to those for (2.71)) to find that it is of order $e^{-d(1+o(1))} \left(e^{-\frac{d}{2}} + e^{-\frac{(p-1)d}{2}} + o(\varepsilon^2) \right) \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}$. All the above comments yield that

$$\begin{aligned} &\left\| \frac{\partial}{\partial Q_d} I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] \right\| \qquad (2.78) \\ &\leq C \left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))} \right) \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \end{aligned}$$

From (2.74) and (2.78) we finally obtain the desired conclusion. ∎

2.3 Proof of Theorem 2.0.1

We can now prove our main theorem. We first derive an accurate expansion of the energy of approximate solutions, and then obtain the existence result using the abstract theory in Subsection 2.1.1.

2.3.1 Energy expansions for the approximate solutions $\hat{z}_{\varepsilon,Q}$

Here we expand $I_\varepsilon(\hat{z}_{\varepsilon,Q})$ in terms of Q and ε, where $\hat{z}_{\varepsilon,Q}$ is the function defined in (2.61).

Proposition 2.3.1 *If $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 2.1.4 and if $d = d(Q)$ is as in Subsection 2.1.2, then we have the following expansion*

$$I_\varepsilon(\hat{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O(\varepsilon^2),$$

as $\varepsilon \to 0$ and $d \to +\infty$.

PROOF. As in the proof of Proposition 2.2.12, let us write $\hat{z}_{\varepsilon,Q} = \overline{z}_{\varepsilon,Q} + \check{z}_{\varepsilon,Q}$, see (2.24) and (2.63). Then, using the coordinates y introduced in Subsection

2.1.2 we find that

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \int_{\Omega_\varepsilon} \left(\nabla_{g_y}\overline{z}_{\varepsilon,Q}\nabla_{g_y}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right) dy \\ &+ \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla_{g_y}\check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q}\right) dy \\ &+ \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy. \end{aligned} \tag{2.79}$$

Using (2.19) (and the subsequent comments) we have that

$$\begin{aligned} &\left|\int_{\Omega_\varepsilon} \left(\nabla_{g_y}\overline{z}_{\varepsilon,Q}\nabla_{g_y}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right) dy - \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right) dy\right| \\ &\leq C\varepsilon \int_{\mathbb{R}^n_+} |y|\,|\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}|\,|\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|dy; \end{aligned} \tag{2.80}$$

$$\begin{aligned} &\left|\int_{\Omega_\varepsilon} \left(|\nabla_{g_y}\check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q}\right) dy - \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q}\right) dy\right| \\ &\leq C\varepsilon \int_{\mathbb{R}^n_+} |y||\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|^2 dy. \end{aligned} \tag{2.81}$$

Concerning (2.80), we can divide the domain of integration into $B_{\frac{d}{2}}(0)$ and its complement and use (1.3), (2.23), (2.53), (2.54), (2.55) to find

$$C\varepsilon \int_{\Omega_\varepsilon} |y|\,|\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}|\,|\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|dy \leq C\varepsilon e^{-\frac{3}{2}d(1+o(1))}.$$

For (2.81), the same estimates yield

$$C\varepsilon \int_{\Omega_\varepsilon} |y||\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|^2 dy \leq C\varepsilon e^{-2d(1+o(1))}.$$

The last two formulas, (2.79), (2.80) and (2.81) imply

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right) dy \\ &+ \frac{1}{2}\int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q}\right) dy \\ &+ \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy \\ &+ O\left(\varepsilon e^{-\frac{3}{2}d(1+o(1))}\right). \end{aligned} \tag{2.82}$$

Using the same notation as in the proof of Proposition 2.2.12, we write $\check{z}_{\varepsilon,Q} = \check{z}_{\varepsilon,Q,1} + \check{z}_{\varepsilon,Q,2} + \check{z}_{\varepsilon,Q,3}$. Formulas (1.3) and (2.23) imply

$$\left| \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,1} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q,1} \right) dy \right| \leq C e^{-\frac{dD}{16}(1+o(1))};$$

$$\left| \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,3} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q,3} \right) dy \right| \leq C \varepsilon e^{-2d(1+o(1))},$$

from which we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q} \right) dy \\ & = \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q,2} \right) dy \\ & + O\left(e^{-\frac{dD}{16}(1+o(1))} + \varepsilon e^{-2d(1+o(1))} \right). \end{aligned}$$

Similar estimates also yield

$$\begin{aligned} \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q} \right) dy & = \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2}|^2 + \check{z}^2_{\varepsilon,Q,2} \right) dy \\ & + O\left(e^{-\frac{dD}{16} - d(1+o(1))} + \varepsilon e^{-2d(1+o(1))} \right). \end{aligned}$$

From a straightforward computation one finds that for any function v

$$\begin{aligned} \nabla \check{z}_{\varepsilon,Q,2} \nabla v + \check{z}_{\varepsilon,Q,2} v & = \nabla \Xi_d \nabla (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) v) + \Xi_d \chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) v \\ & + \nabla (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot)) \left(\Xi_d \nabla v - v \nabla \Xi_d \right). \end{aligned}$$

Applying this relation for $v = \overline{z}_{\varepsilon,Q}$ and $v = \check{z}_{\varepsilon,Q}$ respectively, and using (1.3), (2.23), (2.53), (2.54) and (2.55) we find that

$$\begin{aligned} & \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q} \right) dy \\ & = \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \overline{z}_{\varepsilon,Q}) \nabla_{\mathbb{R}^n} \Xi_d + \chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \overline{z}_{\varepsilon,Q} \Xi_d \right) dy \\ & + O\left(e^{-\frac{dD}{16}(1+o(1))} \right); \end{aligned}$$

$$\begin{aligned} & \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q} \right) dy \\ & = \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \Xi_d)|^2 + (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \Xi_d)^2 \right) dy \\ & + O\left(e^{-\frac{dD}{16}(1+o(1))} \right). \end{aligned}$$

Using now the fact that, by our construction, the function

$$\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\hat{z}_{\varepsilon,Q} = \chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)(\overline{z}_{\varepsilon,Q} + \check{z}_{\varepsilon,Q})$$

vanishes on $d(\hat{\Gamma}_D - Q_0)$, from (2.51) we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n_+} (\nabla_{\mathbb{R}^n}(\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q})\nabla_{\mathbb{R}^n}\Xi_d + \chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q}\Xi_d)\, dy \\
+ & \frac{1}{2}\int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n}(\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\Xi_d)|^2 + (\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\Xi_d)^2\right) dy \\
= & \frac{1}{2}\int_{\mathbb{R}^n_+} (\nabla_{\mathbb{R}^n}(\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q})\nabla_{\mathbb{R}^n}\Xi_d + \chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q}\Xi_d)\, dy.
\end{aligned}$$

From (2.82) and the last eight formulas we find

$$\begin{aligned}
I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \frac{1}{2}\int_{\mathbb{R}^n_+} (\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q})\, dy \\
&+ \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy \\
&+ O\left(e^{-\frac{dD}{16}(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))}\right).
\end{aligned}$$

From (1.3), (2.23), (2.25), (2.53) we then obtain

$$\begin{aligned}
I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \frac{1}{2}\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q} dy \\
&+ \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy \\
&+ O\left(e^{-\frac{dD}{16}(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))} + \varepsilon^2 e^{-d(1+o(1))}\right). \quad (2.83)
\end{aligned}$$

Using a Taylor expansion we can write that

$$\begin{aligned}
& |\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1} \\
= & \begin{cases} -(p+1)\overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q} + O\left(\overline{z}^{p-1}_{\varepsilon,Q}\check{z}^2_{\varepsilon,Q}\right) & \text{for } \check{z}_{\varepsilon,Q} \in \left(0, \frac{1}{2}\overline{z}_{\varepsilon,Q}\right); \\ O\left(|\overline{z}_{\varepsilon,Q}|^p|\check{z}_{\varepsilon,Q}| + |\check{z}_{\varepsilon,Q}|^{p+1}\right) & \text{otherwise.} \end{cases} \quad (2.84)
\end{aligned}$$

As for the estimate of A_2 in (2.71), we divide the domain into the two regions $\mathcal{B}_1, \mathcal{B}_2$, and deduce that

$$\begin{aligned}
& \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy = -\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q} dy \\
& + O\left(e^{-3d(1+o(1))} + e^{-\frac{p+3}{2}d(1+o(1))} + e^{-d(1+o(1))}e^{-\frac{1}{\varepsilon^{c_{K,n}}}}\right).
\end{aligned}$$

Therefore using (2.83) the energy becomes

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) - \frac{1}{2}\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q}dy \\ &+ O\left(e^{-3d(1+o(1))} + e^{-\frac{p+3}{2}d(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))} + \varepsilon^2 e^{-d(1+o(1))}\right). \end{aligned}$$

From (2.54), the expression of $\check{z}_{\varepsilon,Q}$ and estimates in the same spirit as above one finds that

$$\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q}dy = -e^{-2d(1+o(1))},$$

and hence from Proposition 2.1.6 we finally find

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O(\varepsilon^2) + e^{-2d(1+o(1))} \\ &+ O\left(e^{-3d(1+o(1))} + e^{-\frac{p+3}{2}d(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))} + \varepsilon^2 e^{-d(1+o(1))}\right). \end{aligned} \tag{2.85}$$

The conclusion follows from the Schwartz inequality. ∎

We have a related result concerning the derivative of the energy with respect to Q: again, we will be rather quick in the proof since the arguments are quite similar to the previous ones.

Proposition 2.3.2 *If $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 2.1.4 and if $d = d(Q)$ is as in Subsection 2.1.2, then in the same notation as in the previous section we have the following expansion*

$$\frac{\partial}{\partial Q_T} I_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla_T H(\varepsilon Q) + o(\varepsilon^2); \tag{2.86}$$

$$\frac{\partial}{\partial Q_d} I_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o(\varepsilon^2), \tag{2.87}$$

as $\varepsilon \to 0$ and $d \to +\infty$.

Proof. After some elementary calculations, recalling the definition of $\overline{z}_{\varepsilon,Q}$ in (2.24), we can write

$$\begin{aligned} I'_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q}\right] &= \frac{\partial}{\partial Q} I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \int_{\Omega_\varepsilon}\left(\nabla_{g_y}\overline{z}_{\varepsilon,Q}\nabla_{g_y}\frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q} + \overline{z}_{\varepsilon,Q}\frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q}\right)dy \\ &- \int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q}\,dy \\ &+ \int_{\Omega_\varepsilon}\left(\nabla_{g_y}\check{z}_{\varepsilon,Q}\nabla_{g_y}\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} + \check{z}_{\varepsilon,Q}\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q}\right)dy \\ &+ \int_{\Omega_\varepsilon}(\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q})\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q}\,dy, \end{aligned} \tag{2.88}$$

where $\check{z}_{\varepsilon,Q} = \hat{z}_{\varepsilon,Q} - \overline{z}_{\varepsilon,Q}$ was defined in (2.63). The first term on the right hand side is estimated in Proposition 2.1.6. The next two, integrating by parts and using Proposition 2.1.6, can be estimated in terms of a quantity like

$$C\varepsilon^2 \int_{\Omega_\varepsilon} (1 + |y|^K) \left| \frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q} \right|.$$

From the same arguments as in the proof of Proposition 2.2.13 one deduces that the latter integral is of order $\varepsilon^2 e^{-2d(1+o(1))}$. To control the first integral in the last line of (2.88) we can reason as for the estimate of $A_{1,2}$ in the proof of Proposition 2.2.12 to see that this is of order

$$e^{-d(1+o(1))}(\varepsilon + e^{-d}) \left\| \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \right\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

From the proof of Proposition 2.2.13 one can deduce that

$$\left\| \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \right\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \leq C \left(\varepsilon^2 + e^{-d(1+o(1))} \right),$$

and hence the integral under interest is controlled by $o(\varepsilon^2) + e^{-3d(1+o(1))}$.

Finally, the last term in (2.88) can be estimated using a Taylor expansion as for the term A_2 in the proof of Proposition 2.2.12, and up to higher order is given by

$$p \int_{\mathbb{R}^n_+} U(y)^{p-1} \check{z}_{\varepsilon,Q} \nabla U(y) \cdot q \, dy,$$

where q stands either for the variation of Q in the coordinates y. If q preserves d, the latter integral gives a negligible contribution, and we find (2.86). If instead q is directed toward the gradient of d the above estimates (and in particular (2.54)) allow to deduce (2.87). ■

2.3.2 Finite-dimensional reduction and study of the constrained functional

We apply now the abstract setting developed in Subsection 2.1.1. First of all, we state the following two lemmas.

Lemma 2.3.3 *If C_Ω is as in the previous section and if we choose*

$$Z_\varepsilon = \left\{ \hat{z}_{\varepsilon,Q} \ : \ C_\Omega < d < \frac{1}{C_\Omega \varepsilon} \right\}; \qquad \mathcal{H} = H^1_{\mathcal{D}}(\Omega_\varepsilon),$$

then properties i), iii) and iv) in Subsection 2.1.1 holds true, with $\alpha = \min\{1, p-1\}$.

PROOF. The proof of $i)$ and $iii)$ is immediate: the value of α in particular is determined by the standard properties of Nemitski operators. Property $iv)$ can be easily deduced from the (well known) fact that the kernel of the linearization of (1.1) in the half space is spanned by $\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_{n-1}}$, as proved in [52], and from some localization arguments which can be found in Subsections 4.2, 9.2 and 9.3 of [2]. ■

The results in the next lemma could have been collected with the previous one (with some small modification): however, since in the second part of the paper, [26], we are going to need the way it is stated, it was convenient to leave the two of them separated.

Lemma 2.3.4 *For any small positive constant δ, if we take*

$$Z_\varepsilon = \left\{ \hat{z}_{\varepsilon,Q} \ : \ (1-\delta)|\log \varepsilon| < d < \frac{1}{C_\Omega \varepsilon} \right\}; \qquad \mathcal{H} = H^1_D(\Omega_\varepsilon),$$

then also property $ii)$ in Subsection 2.1.1 holds true, with

$$f(\varepsilon) = \varepsilon^{\min\left\{2-2\delta, \frac{p+1}{2}(1-2\delta)\right\}}.$$

PROOF. This lemma simply follows from Propositions 2.2.12 and 2.2.13. ■

As a corollary of the above two lemmas we can apply Propositions 2.1.1 and 2.1.3, so we expand next the reduced functional and its gradient on the natural constraint $\tilde{Z}_\varepsilon$.

Proposition 2.3.5 *With the choice of Z_ε in Lemma 2.3.4, if ω_ε is given by Proposition 2.1.1, then we have*

$$\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) := I_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon(\hat{z}_{\varepsilon,Q})) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O(\varepsilon^2); \tag{2.89}$$

$$\frac{\partial}{\partial Q_T} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_T H(\varepsilon Q) + o(\varepsilon^2); \tag{2.90}$$

$$\frac{\partial}{\partial Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o(\varepsilon^2), \tag{2.91}$$

as $\varepsilon \to 0$, where $\tilde{C}_0$ and $\tilde{C}_1$ are as in Proposition 2.3.1, and where Q_T, Q_d are as in the proof of Proposition 2.2.13.

PROOF. By Propositions 2.1.1 and 2.2.12 we have that

$$\begin{aligned} &\|\omega_\varepsilon(\hat{z}_{\varepsilon,Q})\| \le C_1 \|I'_\varepsilon(\hat{z}_{\varepsilon,Q})\| \\ &\le C\left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right). \end{aligned}$$

From the regularity of I_ε and Proposition 2.3.1 we then have

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon(\hat{z}_{\varepsilon,Q})) &= I_\varepsilon(\hat{z}_{\varepsilon,Q}) + I'_\varepsilon(\hat{z}_{\varepsilon,Q})[\omega_\varepsilon(\hat{z}_{\varepsilon,Q}))] + O\left(\|\omega_\varepsilon(\hat{z}_{\varepsilon,Q})\|^2\right) \\ &= \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O(\varepsilon^2) \\ &+ O\left(\varepsilon^{4-4\delta} + \varepsilon^{(p+1)(1-2\delta)}\right). \end{aligned}$$

This immediately gives (2.89), since $p > 1$ and since δ is small.

The remaining two estimates are also rather immediate for $p \geq 2$: in fact in this case property *iii*) in Subsection 2.1.1 holds true for $\alpha = 1$, so we also have $\|\partial_Q \omega_\varepsilon\| \leq C f(\varepsilon)$ by the last statement in Lemma 2.1.1. This, together with the Lipschitzianity of I'_ε implies that

$$\begin{aligned} \frac{\partial}{\partial Q}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\left[\partial_Q \hat{z}_{\varepsilon,Q} + \partial_Q \omega_\varepsilon\right] = \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) \\ &+ I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\omega_\varepsilon, \partial_Q \hat{z}_{\varepsilon,Q}\right] + I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\omega_\varepsilon, \partial_Q \omega_\varepsilon\right] \\ &+ \|\omega_\varepsilon\|^{1+\alpha}\left(\|\partial_Q \hat{z}_{\varepsilon,Q}\| + \|\partial_Q \omega_\varepsilon\|\right) \\ &= \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) + O(f(\varepsilon)^2) \qquad (2.92) \\ &= \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) + O\left(\varepsilon^{4-4\delta} + \varepsilon^{(p+1)(1-2\delta)}\right), \end{aligned}$$

since $\alpha = 1$. The last two estimates then follow from Proposition 2.3.2.

For the case $1 < p < 2$, we can arrive to the end of the second line in (2.92), but since $\alpha = p-1 \in (0,1)$ we cannot conclude anymore, and we need to use a slightly more technical argument, which requires the improved estimate mentioned in Remark 2.1.2. We only give a brief sketch, and refer the reader for example to Subsection 4.2 in [4], where a similar issue is treated in more detail. The main point is to show that the function ω_ε satisfies suitable pointwise estimates in some regions of the domain. The auxiliary equation in (2.2), after testing on any smooth function and integrating by parts yields

$$-\Delta(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon) + \hat{z}_{\varepsilon,Q} + \omega_\varepsilon - (\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)^p = \overline{\alpha} \cdot \nabla_Q(-\Delta \hat{z}_{\varepsilon,Q} + \hat{z}_{\varepsilon,Q}) \qquad \text{in } \Omega_\varepsilon, \qquad (2.93)$$

where $\overline{\alpha}$ is a vector of $\mathbb{R}^{n-1}$ with norm of order at most $\|I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\|$, which in turn is of order $f(\varepsilon)$.

Using the coordinates y in Subsection 2.1.2 and restricting our attention to $d(\mathcal{O} - Q_0) \cup (\mathbb{R}^n_+ \setminus B_{\frac{dD}{8}}(0))$, see (2.54) and (2.61), we have here pointwise estimates on the right-hand side, by (2.54) and (2.55). This control, together with a bootstrap argument for (2.93) allows to prove that in $d(\mathcal{O} - Q_0) \cup (\mathbb{R}^n_+ \setminus B_{\frac{dD}{8}}(0))$ the function ω_ε is pointwise of order $f(\varepsilon)(U(y) + |\Xi_d|)$. On the other hand, in the complement of this set, still by (2.93) (used in its weak form), the H^1 norm of ω_ε is controlled by

$$\overline{\alpha}(\|\overline{z}_{\varepsilon,Q}\|_{H^1((\mathbb{R}^n_+ \setminus d(\mathcal{O}-Q_0)) \cap B_{\frac{dD}{8}}(0))} + \|\Xi_d\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}).$$

Hence, given any test function $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$, if we use Taylor expansions in the regions where we have pointwise estimates (as for (2.69)) and the Hölder-Sobolev inequalities elsewhere we find that

$$\left| \int_{\Omega_\varepsilon} \left[(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)^{p-1} - \omega_\varepsilon^{p-1} \right] \partial_Q \hat{z}_{\varepsilon,Q} v dx \right| \leq \varepsilon^{1+\tilde{\delta}_p} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \qquad \text{as } \varepsilon \to 0,$$

where $\tilde{\delta}_p$ is a positive constant depending only on p. Remark 2.1.2 then implies that $\|\partial_Q \omega_\varepsilon\| \leq \varepsilon^{1+\tilde{\delta}_p}$ for ε small. Finally, a Taylor expansion of the functional I_ε gives

$$\begin{aligned} \frac{\partial}{\partial Q} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon) \left[\partial_Q \hat{z}_{\varepsilon,Q} + \partial_Q \omega_\varepsilon \right] = \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) \\ &+ O\left(\|I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon) - I'_\varepsilon(\hat{z}_{\varepsilon,Q})\| \right) \\ &+ O\left(\|I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\| \, \|\partial_Q \omega_\varepsilon\| \right) \\ &= \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) + O(f(\varepsilon)^2), \end{aligned} \tag{2.94}$$

provided we choose δ small compared to $\tilde{\delta}_p$. This concludes the proof. ∎

Proof of Theorem 2.0.1 We use degree theory and the previous expansions: first of all, since $\overline{Q}$ is non-degenerate for $H|_{\mathcal{I}_\Omega}$, we can find a small neighborhood $\mathcal{V}$ of $\overline{Q}$ in $\mathcal{I}_\Omega$ such that $\nabla H|_{\mathcal{I}_\Omega} \neq 0$ on $\partial \mathcal{V}$ and such that in some set of coordinates

$$\deg(\nabla H|_{\mathcal{I}_\Omega}, \mathcal{V}, 0) \neq 0.$$

Then, if δ is as in Lemma 2.3.4, we choose $0 < \beta < \frac{\delta}{2}$, and consider the set

$$\mathcal{Y} = \left\{ (d, Q) \ : \ d \in ((1-\beta)|\log \varepsilon|, (1+\beta)|\log \varepsilon|) \, , \varepsilon Q \in \mathcal{V} \right\}.$$

Since $\nabla H|_{\mathcal{I}_\Omega}(Q)$ corresponds to $\nabla_T H(\varepsilon Q)$ in the scaled domain Ω_ϵ, by using (2.90) and our choice of $\mathcal{V}$ we know that, as $\varepsilon \to 0$

$$\nabla_{Q_T} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_T H(\varepsilon Q) + o(\varepsilon^2) \neq 0 \qquad \text{on } \frac{1}{\varepsilon} \partial \mathcal{V}. \tag{2.95}$$

On the other hand, by (2.91) we also have

$$\nabla_{Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\varepsilon^{2(1-\beta)(1+o(1))} \qquad \text{for } d = (1-\beta)|\log \varepsilon|, \tag{2.96}$$

and

$$\nabla_{Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) + o(\varepsilon^2), \qquad \text{for } d = (1+\beta)|\log \varepsilon|. \tag{2.97}$$

Since we are assuming that the gradient of H points toward $\partial_{\mathcal{D}} \Omega$ near the interface $\mathcal{I}_\Omega$, $\nabla_d H(\varepsilon Q)$ is negative and therefore the two d-derivatives in the last two formulas have opposite signs. It follows from the product formula for the degree and (2.95)-(2.97) that

$$\deg(\nabla \mathbf{I}_\varepsilon, \mathcal{Y}, 0) = -\deg(\nabla H|_{\mathcal{I}_\Omega}, \mathcal{V}, 0) \neq 0,$$

which proves the existence of a critical point for $\mathbf{I}_\varepsilon$ in $\mathcal{Y}$. Since we can choose $\mathcal{V}$ and β arbitrarily small, the solution has the asymptotic behavior required by the theorem (and more precisely by Remark 2.0.2 (b): the uniqueness of the global maximum follows from the asymptotics of the solution and standard elliptic regularity estimates). ■

To prove also the assertion in Remark 2.0.2 (a), using (2.89) in the case of local maximum is it easy to construct an open set of Z_ε where the maximum of $\mathbf{I}_\varepsilon$ at the interior is strictly larger than the maximum at the boundary. On the other hand, when we have a local minimum, one can construct a mountain-pass path connecting the two points parameterized by $\left(\frac{1}{\varepsilon}\overline{Q}, (1-\beta)|\log\varepsilon|\right)$ and $\left(\frac{1}{\varepsilon}\overline{Q}, (1+\beta)|\log\varepsilon|\right)$. Using a suitably truncated pseudo-gradient flow, one can prove that the evolution of the path remains inside $\frac{1}{\varepsilon}\mathcal{V} \times ((1-\beta)|\log\varepsilon|, (1+\beta)|\log\varepsilon|)$, and still find a critical point of $\mathbf{I}_\varepsilon$.

Chapter 3

Asymptotics of least energy solutions

In this Chapter we carry on the study of asymptotic behavior of some solutions to a singularly perturbed problem with mixed Dirichlet and Neumann boundary conditions, started in Chapter 2. Here we are mainly interested in the analysis of the location and shape of *least energy solutions*, that is Mountain Pass solutions, when the singular perturbation parameter tends to zero. We will look for positive solutions to the problem

$$\begin{cases} -\varepsilon^2\Delta u + u = f(u) & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$\tilde{P}_\varepsilon$}$$

where Ω is a smooth bounded subset of $\mathbb{R}^n$, $\varepsilon > 0$ a small parameter, and $\partial_{\mathcal{N}}\Omega$, $\partial_{\mathcal{D}}\Omega$ two disjoint subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$. We are interested in the case $f(u) = u^p$, with $p \in \left(1, \frac{n+2}{n-2}\right)$. Singularly perturbed problems with Neumann or Dirichlet boundary conditions have been studied in detail. There is a wide literature regarding the solution of this type of problems (see for example [38], [51], [49], [50]). Many times they have a sharply peaked profile so they are called *spike layers*, since they are highly concentrated near some points of $\overline{\Omega}$, appear in [26].

W.M. Ni and I.Takagi, in the paper [49] studied the homogeneous Neumann boundary problem

$$\begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial_{\mathcal{N}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{N_ε}$$

They showed that *least energy solutions* of (N_ε), that is Mountain Pass solutions, have only one local maximum over $\overline{\Omega}$ achieved at a point that must lie on the boundary, when the perturbation parameter $\varepsilon \to 0$. In the same work, they give also a detailed description of the shape of such a solution. The same authors in a subsequent paper [50] clarified the location of the point where the maximum of least energy solutions is attained. By an asymptotic formula for the smallest positive critical value of the energy they proved that the mean curvature of $\partial\Omega$ attains its maximum where the maximum of least energy solutions concentrate.

W.M.Ni and J.Wei in [51] studied the corresponding Dirichlet problem.

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega. \end{cases} \tag{D_ε}$$

A least energy solution of (D_ε) is a function u_ε which minimizes the energy

$$\tilde{I}_{\varepsilon,\mathcal{D}}(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) - \frac{1}{p+1}\int_\Omega |u|^{p+1}; \qquad u \in H_0^1(\Omega).$$

The existence and characterization of a least positive critical value of $\tilde{I}_{\varepsilon,\mathcal{D}}$ is provided by the Mountain Pass theorem (1.3.4). They proved that *least energy solutions* of (D_ε) have only one local maximum over $\overline{\Omega}$ achieved at the *most centered* part of Ω when the perturbation parameter goes to zero.

Later P.L.Felmer and M.Del Pino [21] generalized the works of Ni-Takagi and Ni-Wei by enlarging a lot the class of the nonlinearity in both problems (N_ε) and (D_ε).

Mixed Dirichlet-Neumann boundary value problems present some particular features. This can be easily seen in the one dimensional case:

If we analyze the one-dimensional problem

$$\begin{cases} -\varepsilon^2 u'' + u = u^p & \text{in } (-1,1) \\ u(-1) = 0 = u'(1) \end{cases} \tag{3.1}$$

then it is immediate to see that positive solutions of this problem correspond with trajectories in the phase plane (see Figure 3.1) given by

$$\begin{aligned} x' &= y \\ y' &= \varepsilon^{-2}(x - x^p) \end{aligned}$$

This phase plane has $(0,0)$ and $(1,0)$ as critical points in the admissible region (since we are looking for positive solutions, we must take $x > 0$). The first one is a saddle point, and the second one is a center. There are three types of trajectories:

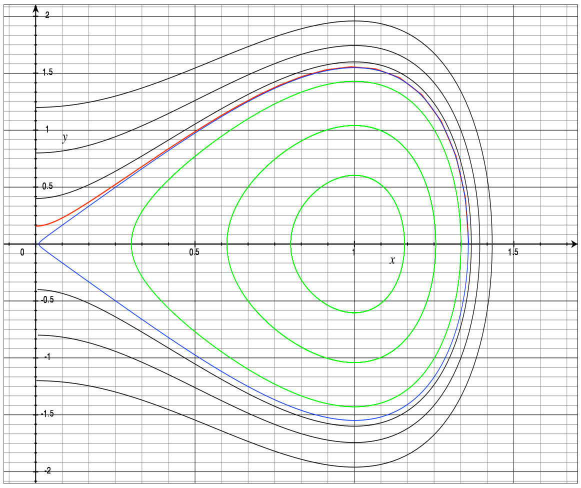

Figure 3.1: Phase plane.

- There is a homoclinic trajectory, which starts and ends at $(0, 0)$. This trajectory corresponds, up to a rescaling (ε), to a ground state.
- There are infinitely many closed trajectories around the point (1.0). This means that we have periodic solutions, which correspond with solutions with several "bumps", and boundary condition $u'(-1) = u(1) = 0$.
- There are infinitely many trajectories passing through a point $(0, c)$, turning around the homoclinic trajectory, and passing from $(0, -c)$. One of these arcs covers exactly the interval $(-1, 1)$ and correspond to a solution with Dirichlet boundary condition $u(-1) = 0 = u(1)$, and maximum at the middle point $t = 0$. Moreover, there is another solution which passes from $(0, c)$ to $(m, 0)$ as t goes from -1 to $+1$. This arc corresponds to a solution with mixed boundary condition $u(-1) = 0 = u'(1)$.

However, the n-dimensional case is more involved, in particular when there exists an *interface* which separates the Dirichlet and the Neumann parts, namely $\mathcal{I}_\Omega := \overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega} \neq \emptyset$.

In these mixed problems we have to solve two questions:

Question 1: Determine the geometric conditions that a point P has to satisfy, to have some sequence of solutions $\{u_\varepsilon\}$ whose maximum points converge to P.

Question 2: Given a precise sequence $\{u_\varepsilon\}$, determine (up to subsequences) the limit of the maximun points.

Question 1 has been analyzed in Chapter 2 of this thesis, and question 2 is the subject of this chapter. In particular we want to study Mountain

Pass solutions to the mixed problem $(\tilde{P}_\varepsilon)$. We will characterize the shape of least-energy solutions and we will show that, as in the Neumann case, the concentration point must lie on the boundary of Ω. In particular, as $\varepsilon \to 0$, there exist least-energy solutions reaching first, the interior of Neumann boundary part and then, concentrating at the interface $\mathcal{I}_\Omega$. Moreover we will show that, under suitable geometric assumptions on Ω and $\mathcal{I}_\Omega$, the ground state solution u_ε of (P_ε) belongs to the set $Z_{\varepsilon,\tilde{c}}$ (see the notation in Proposition 2.1.3), where

$$Z_\varepsilon = \{\hat{z}_{\varepsilon,Q} \ : \ d_{1,\varepsilon} \leq d \leq d_{2,\varepsilon}, Q \in \mathcal{V}\},$$

where $d_{1,\varepsilon} \to +\infty$, $\varepsilon d_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$, and where $\mathcal{V}$ is a neighborhood of a point $\overline{Q}$ which realizes $\max_{Q \in \mathcal{I}_{\Omega_\varepsilon}} H$. If ∇H points inside $\partial_{\mathcal{D}}\Omega_\varepsilon$ we can then apply the second statement in Proposition 2.1.3 (with $f(\varepsilon) = \left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right)$, by Proposition 2.2.12) and (2.91). These two yield that $u_\varepsilon = \hat{z}_{\varepsilon,Q} + \omega(\hat{z}_{\varepsilon,Q})$, where the distance of Q from $\mathcal{I}_{\Omega_\varepsilon}$ is of order $\varepsilon|\log\varepsilon|$, as in Theorem 2.0.1, see Remark 2.0.2 (*b*).

3.1 The least energy solution

Generally a *least energy solution* for the above problems $(\tilde{P}_\varepsilon)$ exists if the function $f : \mathbb{R} \to \mathbb{R}$ is of class $C^1(\mathbb{R})$ and satisfies the following classical assumptions:

(i) $f(t) \equiv 0$ for $t \leq 0$;

(ii) $f(t) = O(t^r)$ as $t \to +\infty$, with: $\begin{cases} 2 < r+1 < \frac{2n}{n-2} & \text{if } n \geq 3, \\ r > 1 & \text{if } n = 2; \end{cases}$

(iii) $\dfrac{f(t)}{t}$ is increasing for $t > 0$ and $\displaystyle\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$, while $f(t) = o(t)$ for $t \to 0$;

(iv) There exists a constant $\theta > 2$ such that $0 < \theta F(t) \leq t f(t)$ for $t \geq 0$, where $F(t)$ is defined by

$$F(t) := \int_0^t f(s)ds.$$

Remark 3.1.1 *In our case all the above assumptions hold, being the non-linearity of the form $f(u) = u^p$ with $p \in \left(1, \frac{n+2}{n-2}\right)$.*

The differential problem $(\tilde{P}_\varepsilon)$ presents a variational structure and the associated Euler functional $\tilde{I}_\varepsilon : X \to \mathbb{R}$ is defined by

$$\tilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx; \qquad u \in X \qquad (3.2)$$

where

$$X = \{u \in H^1_{\mathcal{D}}(\Omega) : u \not\equiv 0 \quad \text{and } u \geq 0 \text{ in } \Omega\} \tag{3.3}$$

and $H^1_{\mathcal{D}}(\Omega)$ (see Section 1.8) stands for the space of functions in $H^1(\Omega)$ which have zero trace on $\partial_{\mathcal{D}}\Omega$:

$$H^1_{\mathcal{D}}(\Omega) = \{u \in H^1(\Omega) : u|_{\partial_{\mathcal{D}}\Omega} = 0\} \tag{3.4}$$

In general, critical levels can be found by min-max procedures. The least energy level comes from the Mountain-Pass Theorem (1.3.4) due to Ambrosetti-Rabinowitz.

Remark 3.1.1 *We want to point out that in our case, since the exponent p is subcritical, the* Palais-Smale condition *is easily fulfilled by the functional $\tilde{I}_\varepsilon$ (3.2).*

Below, in the next Proposition, we simplify the characterization of the energy of *Least Energy Solutions* namely the Mountain Pass solutions of the problem $(\tilde{P}_\varepsilon)$, in a different way that will be useful throughout the paper.

Proposition 3.1.2 *Let $\tilde{I}_\varepsilon$ and X defined in* (3.2) *and* (3.3)*. Then the Mountain Pass critical level c_ε can be characterized as*

$$c_\varepsilon = \min_{v \in X} \max_{t>0} \tilde{I}_\varepsilon(tv) \tag{3.5}$$

Proof. We can follow the ideas contained in ([49], Lemma 3.1), then we will give here a sketchy proof.

First, we note that in our case, the function $h(t) = \tilde{I}_\varepsilon(tv)$ attains an unique maximum for such a $t^* > 0$ because $h'(t^*) = 0$, that is

$$\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dx = \frac{1}{t^*}\int_\Omega u f(t^*u) dx,$$

with $f(t) = t^p$, has an unique solution since the right-hand side

$$\int_\Omega u^2 \frac{f(t^*u)}{t^*u} dx$$

is strictly increasing in $t^* > 0$ by virtue of (iii) and the nonlinear term of problem $(\tilde{P}_\varepsilon)$ is in the form u^p, implying that $h(t) > 0$ for $t > 0$ small and $h(t) \to -\infty$ as $t \to +\infty$. Then if we fix a non-negative function $\widetilde{v} \in X$, $\widetilde{v} \not\equiv 0$, $\tilde{I}_\varepsilon(\widetilde{v}) = 0$, by Theorem (1.3.4) we get a critical point v^* such that $\tilde{I}_\varepsilon(v^*) = c_\varepsilon$. By the above considerations we have that $c_\varepsilon = \max_{t>0} \tilde{I}_\varepsilon(tv^*)$. Then it is obvious that

$$c_\varepsilon \geq \min_{v \in X} \max_{t>0} \tilde{I}_\varepsilon(tv).$$

By contradiction suppose that the strict inequality holds. Then there exists some fixed $v \in X$ such that

$$c_\varepsilon > \max_{t>0} \tilde{I}_\varepsilon(tv).$$

We can construct a path $\gamma \in \tilde{\Gamma}$ (see (i), Theorem (1.3.4)) where $\tilde{I}_\varepsilon$ is positive only on the line segment joining 0 and $\widehat{tv}$, where $\tilde{I}_\varepsilon(\widehat{tv}) = 0$. Then we find

$$c_\varepsilon > \max_{v \in \gamma} \tilde{I}_\varepsilon(v).$$

From (i) of Theorem (1.3.4), we get the desired contradiction. Then the equality holds true, showing that c_ε is the *least positive* critical value of $\tilde{I}_\varepsilon$.
■

It is interesting to point out an important and simple fact that comes from the characterization of the critical value c_ε given in equation (3.5). We have the following

Corollary 3.1.3 *If we denote with $c_\varepsilon^{\mathcal{N}}$ the critical value of the functional $\tilde{I}_{\varepsilon,\mathcal{N}}$ associated to the semilinear Neumann problem (N_ε), $c_\varepsilon^{\mathcal{D}}$ the critical value of the functional $\tilde{I}_{\varepsilon,\mathcal{D}}$ associated to the semilinear Dirichlet problem (D_ε) and with $c_\varepsilon^{\mathcal{M}}$ the critical value of the functional $\tilde{I}_\varepsilon$ associated to the mixed semilinear problem $(\tilde{P}_\varepsilon)$, then one has that:*

$$c_\varepsilon^{\mathcal{N}} \leq c_\varepsilon^{\mathcal{M}} \leq c_\varepsilon^{\mathcal{D}} \tag{3.6}$$

Proof. It is sufficient to observe the definition (3.5) and that

$$H^1(\Omega) \supset H^1_{\mathcal{D}}(\Omega) \supset H^1_0(\Omega)$$

■

To study the concentration of the solutions u_ε to problems (N_ε) or (D_ε), one usually employs a blow-up argument consisting in a suitable scaling of the variables, witch allows to prove that $u_\varepsilon(x) \sim U(\frac{x-Q}{\varepsilon})$, where U, see Section 1.1, denotes a solution of the following problem:

$$-\Delta U + U = U^p \qquad \text{in } \mathbb{R}^n \quad (\text{or in } \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \, : \, x_n > 0\}),$$

the domain depending on whether Q lies in the interior of Ω or at the boundary; in the latter case Neumann conditions are imposed. As we have seen in Remark 1.1.2, when $p < \frac{n+2}{n-2}$ (and indeed only if this inequality is satisfied), problem (1.1) admits positive radial solutions which decay (exponentially fast) to zero at infinity, see Proposition 1.1.1:

- $\lim_{r \to +\infty} e^r r^{\frac{n-1}{2}} U(r) = \alpha_{n,p}$,

- $\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1; \qquad \lim_{r\to+\infty} \frac{U''(r)}{U(r)} = 1,$

for some positive constant $\alpha_{n,p}$ depending only on n and p. We denote the energy associated to the problem (1.1) as made in (1.2):

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1} dx.$$

3.2 Straightening the boundary

As in Section 2.1, it is convenient to change variable $x \to \varepsilon x$ and to study the problem $(\tilde{P}_\varepsilon)$ in the dilated domain Ω_ε. Then the problem becomes

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega_\varepsilon & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega_\varepsilon; \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{P_ε}$$

In the blow-up argument a crucial point is to analyze the behavior of solutions on the boundary. Here we introduce some preliminary material that will be used later. Letting $P_0 \in \partial\Omega \subset \mathbb{R}^n$, we set $P_0 = (x', x_n)$ with $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. To simplify the notation, without loss of generality, we can assume that $P_0 = 0 \in \mathbb{R}_n$. Since we assume that Ω is regular, we describe the $\partial\Omega$ near a generic point $P_0 \in \partial\Omega$: we know that there exists a constant $\delta > 0$ and a smooth function ψ_{P_0} defined for $|x'| < \delta$ such that

- $\psi_{P_0}(0) = 0\,, \quad \frac{\partial \psi_{P_0}}{\partial x_j} = 0\,,\ 1 \le j \le n-1;$
- Given a neighborhood of P_0, denoted by U, then

  $\Omega \cap U = \{(x', x_n) | x_n > \psi_{P_0}(x')\}$ and $\partial\Omega \cap U = \{(x', x_n) | x_n = \psi_{P_0}(x')\}$.

The first condition implies that $\{x_n = 0\}$ is the tangent plane of $\partial\Omega$ at P_0. We frequently need to change coordinates near a point of $\partial\Omega$ to flatten the boundary, in fact, there exists a constant $\delta' > 0$ such that if $|y| < \delta'$ then we can define $x = \Phi(y) = (\Phi_1(y), \cdots, \Phi_n(y))$ in the following way:

$$\begin{aligned} \Phi_j(y) &= y_j - y_n \frac{\partial \psi_{P_0}}{\partial x_j}(y')\,, \quad 1 \le j \le n-1 \\ \Phi_n(y) &= y_n + \psi_{P_0}(y'). \end{aligned}$$

In particular, $D\Phi(0) = Id$ and Φ has a inverse that we denote by $\Psi \equiv \Phi^{-1}$, defined in a neighborhood of the origin. Then if we change variables, and define

$$v_\varepsilon(y) = u_\varepsilon(x) = u_\varepsilon(\Phi(y))$$

then v_ε satisfies

$$\varepsilon^2\left(\sum_{i,j}^{n} a_{ij}(y)\frac{\partial^2 v_\varepsilon}{\partial y_i y_j}+\sum_{j}^{n} b_j(y)\frac{\partial v_\varepsilon}{\partial y_j}\right)-v_\varepsilon(y)+v_\varepsilon^p(y)=0 \quad \text{in} \quad \{y_n>0\}\cap B_\rho \tag{3.7}$$

where B_ρ stands for the open ball centred at zero with radius ρ small enough and boundary conditions that depend on where P_0 lies in the boundary:

$$\begin{cases} v_\varepsilon = 0 \text{ on } \{y_n=0\}\cap B_\rho, \\ \text{ if the point } P_0 \text{ lies on the interior of the Dirichlet boundary part } \partial_{\mathcal{D}}\Omega, \text{ or} \\ \frac{\partial v_\varepsilon}{\partial y_n}=0 \text{ on } \{y_n=0\}\cap B_\rho, \\ \text{ if the point } P_0 \text{ lies on the interior of the Neumann boundary part } \partial_{\mathcal{N}}\Omega, \text{ or} \\ v_\varepsilon=0 \text{ on } \{y_1>0, y_n=0\}\cap B_\rho \text{ and } \frac{\partial v_\varepsilon}{\partial y_n}=0 \text{ on } \{y_1<0, y_n=0\}\cap B_\rho, \\ \text{ if the point } P_0 \text{ lies on the interior of the interface } \mathcal{I}_\Omega := \overline{\partial_{\mathcal{D}}\Omega}\cap\overline{\partial_{\mathcal{N}}\Omega}. \end{cases} \tag{3.8}$$

The coefficients in equation (3.7) are given by

$$\begin{cases} a_{ij}(y)=\sum_k \frac{\partial \Psi_i}{\partial x_k}(\Phi(y))\cdot\frac{\partial \Psi_j}{\partial x_k}(\Phi(y)), \quad 1\le i,j\le n; \\ b_j(y)=(\Delta\Psi_j)(\Phi(y)), \quad 1\le j\le n. \end{cases} \tag{3.9}$$

In particular, since $D\Psi(0)=Id$, then

$$a_{ij}(0)=\delta_{ij}. \tag{3.10}$$

Remark 3.2.1 (i) *To better understand the notation in the sequel, in Figure 3.2 we show that, (a-b) when the limit point concentrates at the boundary $\partial\Omega$, first we flatten the domain and then we scale; instead, (c) if the limit point concentrates in the interior of Ω, obviously we only scale .*

3.2.1 Convergence of scaled function

Here we want to point out that the proof of convergence of scaled function is a little different if the points P_ε converge to a point P_0 in the interior of domain or on the boundary of domain. Below we analyze the different cases. Let u_ε be a *least energy solution* to the problem $(\tilde{P}_\varepsilon)$, suppose that P_ε a maximum point of u_ε and f the nonlinear term appearing in problem $(\tilde{P}_\varepsilon)$. We have two cases:

Case 1: Up to a subsequence, $P_\varepsilon \to P_0 \in \Omega$.

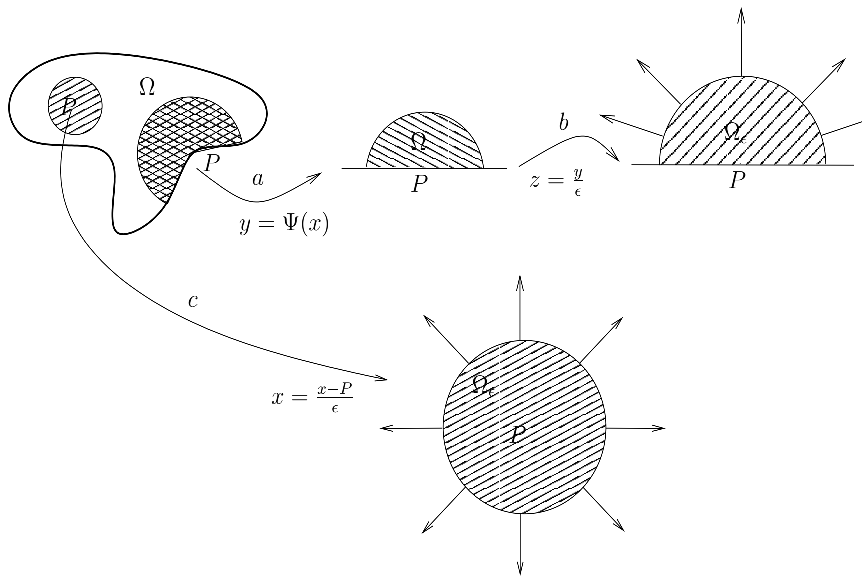

Figure 3.2: Flattening & Scaling

We define the scaled function w like in (3.51).

$$w_{\varepsilon_j}(x) = u_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j x) \tag{3.11}$$

We consider two increasing sequences $j, m \in \mathbb{N}$ such that $\varepsilon_j \to 0$ as $j \to +\infty$ and $r_m \to +\infty$ when $m \to +\infty$. In (3.11) we consider $y \in B_{\rho_j}$. For each r_m we consider a number j_m such that

$$2r_m < \rho_j \quad \text{if} \quad j \geq j_m. \tag{3.12}$$

As starting point, see Lemma 1.6.2, we use the following:

$$\int_\Omega u_\varepsilon^r dx \leq C_r \varepsilon^n \tag{3.13}$$

where C_r is a constant independent from ε and r any exponent greater than zero. We want to remark that this estimate is also valid under mixed boundary conditions since, for the functional space X defined in (3.3), we have also classical results about Sobolev embedding theorems and because hypothesis (iv) (see Section 3.1) holds true also in our case. Hence the proof Lemma 2.3 in [38] applies in our case as well.

From equations (3.11) and (3.13) we have

$$\int_{B_{\rho_j}} |w_{\varepsilon_j}|^r dx = \int_{B_{\rho_j}} |u_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j x)| < C_r$$

From this last equation and (3.12) we get

$$\int_{B_{2r_m}} |w_{\varepsilon_j}|^r dx \le \int_{B_{\rho_j}} |w_{\varepsilon_j}|^r dx \le C_r. \tag{3.14}$$

Then, if we use the classical L^p estimate and Sobolev Embedding theorem we obtain (see [29]):

$$\|w_j\|_{W^{2,r}(B_{rm})} \le C(\|u\|_{L^r} + \|f\|_{L^r}).$$

By Sobolev embedding we know that if

$$kr > n \quad \text{and} \quad n + r(h+\alpha) < kr$$

we get

$$W^{2,r}(B_{r_m}) \subset C^{h,\alpha}(\overline{B_{rm}})$$

Therefore it readily follows $w_{\varepsilon_j} \in C^{0,\alpha}(\overline{B_{r_m}})$. The functions w_{ε_j} satisfy the following differential equation

$$-\Delta w_{\varepsilon_j} + w_{\varepsilon_j} = w_{\varepsilon_j}^p,$$

where $f \in C^1(\overline{B_{r_m}})$ and $w_{\varepsilon_j} \in C^{0,\alpha}(\overline{B_{r_m}})$. By Schauder interior estimate in B_{r_m} one has

$$\|w_{\varepsilon_j}\|_{C^{2,\alpha}(\overline{B_{r_m}})} < C \tag{3.15}$$

From (3.15) we get that$\{w_{\varepsilon_j}\}_{j\ge j_m}$ is a relatively compact set in $C^2(\overline{B_{r_m}})$. Therefore we can extract a convergent subsequence by a diagonal process. We choose a monotone increasing sequence $\{R_k\}$ such that

- $R_k \to +\infty$ as $k \to +\infty$,
- $B_{R_1} \subset B_{R_2} \subset \cdots \subset B_{R_k}$

We have

$$\|w_{\varepsilon_j}\|_{C^{2,\alpha}(\overline{B_{rm}})} < C \quad \forall B_{R_k} \in \mathbb{R}_n \text{ bounded in the Euclidean topology.}$$

Let us choose

$\{w^1_{\varepsilon_j}\} \subset \{w_{\varepsilon_j}\}$ (Ascoli Arzelà Theorem) $\Rightarrow \exists w^1 : w^1_{\varepsilon_j} \to w^1$ uniformly on B_{R_1},

$\{w^2_{\varepsilon_j}\} \subset \{w^1_{\varepsilon_j}\}$ (Ascoli Arzelà Theorem) $\Rightarrow \exists w^2 : w^2_{\varepsilon_j} \to w^2$ uniformly on B_{R_2},

$\vdots$

$\{w^k_{\varepsilon_j}\} \subset \{w^{k-1}_{\varepsilon_j}\}$ (Ascoli Arzelà Theorem) $\Rightarrow \exists w^k : w^k_{\varepsilon_j} \to w^k$ uniformly on B_{R_k}.

Remark 3.2.2 *The subsequence w^k is an* extension *of w^{k-1}, since $w^k \equiv w^{k-1}$ on $B_{R_{k-1}}$ because subsequences of the same convergent sequence.*

Then we can consider the sequence $\{w^j_{\varepsilon_j}\}$ that converges uniformly to the function

$$w(x) = w^k(x) \quad \text{if } x \in B_{R_k} \forall R > 0.$$

Therefore we have

$$w \in C^2(\mathbb{R}_n) \cap W^{1,2}(\mathbb{R}^n)$$

and in particular convergence in $C^2_{loc}(\mathbb{R}^n)$. From equation (3.11) we get that $w_{\varepsilon_j}(0) > c$, where c is some positive constant. This implies that $w \geq 0$. By the strong maximum principle we obtain that $w > 0$ in $\mathbb{R}^n$, solution of the following PDE

$$-\Delta w + w = w^p. \tag{3.16}$$

Remark 3.2.3 *We can also obtain a Hölder estimate in a different way, following [53] or [15]. We can write the mixed problem $(\tilde{P}_\varepsilon)$ as*

$$-\Delta u_\varepsilon = \varepsilon^{-2} f_\varepsilon,$$

where $f_\varepsilon \equiv u^p_\varepsilon - u_\varepsilon$. We know uniform a priori bounds for $||u_\varepsilon||_{L^r}$, for any r, see equation (3.13)*; in particular it follows*

$$||f_\varepsilon||_{L^r} \leq C_r \varepsilon^{n/r}.$$

We will look for a uniform C^α estimate in the rescaled problem that is when we take the sequence $\{w_\varepsilon\}$, defined by equation (3.11)*. We follow the proof of the C^α regularity result by G. Stampacchia in [53], to get the dependence on ε of the final estimate. Roughly, the proof consists in a careful estimate of the measure of the sets*

$$A(k, R) \equiv \{x \in B_R(x_0) \cap \Omega \,|\, u(x) > k\}, \quad x_0 \in \Omega, \quad K \in \mathbb{R},$$

which leads to a Caccioppoli type inequality. This inequality, combined with an iterative argument, allows us to control the oscillation of u, proving the Hölder estimate. However, in our case we need an explicit control of the dependence with respect to ε of the constants which appear in all the process, since we need to extend the Hölder estimates to the rescaled problem. Then we get the following result:

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_\varepsilon |x - y|^\alpha,$$

where:

- $C_\varepsilon \approx ||f_\varepsilon/\varepsilon^2||_{m/2}$.
- $\alpha = \min(\alpha_0, -2(\frac{n}{m} - 1))$,

where α_0 is a constant which depends only on Ω.

In particular, if we take $m > n$, but $m \approx n$, then we have that

$$\alpha = -2(\frac{n}{m} - 1).$$

Let us fix such a p from now on. Therefore, in our case we have

$$C_\varepsilon \approx C_n \varepsilon^{2(\frac{n}{m}-1)}.$$

We note that the exponent is negative, and therefore

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_n \varepsilon^{2(\frac{n}{m}-1)} |x-y|^{-2(\frac{n}{m}-1)} \tag{3.17}$$

In particular, taking the rescaled functions v_ε we get

$$|w_\varepsilon(r) - w_\varepsilon(s)| = |u_\varepsilon(\varepsilon r + P_\varepsilon) - u_\varepsilon(\varepsilon s + P_\varepsilon)| \leq \ldots \leq C_n |r-s|^{-2(\frac{n}{m}-1)}, \tag{3.18}$$

that is Uniform Hölder estimate for the sequence $\{w_\varepsilon\}$. In particular, taking a subsequence we can assume that w_ε converges uniformly on compact sets to a continuous function w.

Otherwise as said in Remark 3.2.1, we can get also a uniform (on compact sets) convergence result for the solutions of the scaled problem with the boundary *locally rectified*, namely:

Case 2: Up to a subsequence $P_\varepsilon \to P \in \partial_{\mathcal{N}}\Omega$.

We know that there exists a neighborhood U of this point and a regular map, see Subsection 3.2, $\Psi = \Phi^{-1}$ such that $\Psi(U \cap \Omega) \subset \mathbb{R}^n_+$. We can define as before the functions

$$v_\varepsilon(y) \equiv u_\varepsilon(\Phi(y)) \quad y \in \overline{B^+_{\rho_j}} := \{y \in \overline{B_{\rho_j}}, y_n \geq 0\},$$

and then scaling

$$w_\varepsilon(z) = \tilde{v}_\varepsilon(Q_\varepsilon + \varepsilon z), \qquad Q_\varepsilon = \psi(P_\varepsilon) \tag{3.19}$$

which solve an elliptic problem in some (rescaled) domain with boundary conditions on a flat boundary. In this case the function $\tilde{v}_\varepsilon$ is defined by reflection as

$$\tilde{v}_\varepsilon := \begin{cases} v_\varepsilon(y) & if \quad y \in \overline{B^+_{\rho_j}} \\ v_\varepsilon(y', -y_n) & if \quad y \in B^-_{\rho_j} \end{cases} \tag{3.20}$$

where as usual $y' = (y_1, y_2, \ldots, y_{n-1})$ and $B^-_{\rho_j} := \{y \in B_{\rho_j}, y_n < 0\}$. Then, we can write the differential equation satisfied from the scaled function w defined as above (where $Q_\varepsilon = \Psi(P_\varepsilon)$) and P_ε is the maximum point of the solution of the problem $(\tilde{P}_\varepsilon)$. After some computations, from equation (3.7) we obtain

$$\varepsilon^2\left(\sum_{i,j}^{N}\frac{1}{\varepsilon^2}a_{ij}^{\varepsilon}(z)\frac{\partial^2 w_\varepsilon}{\partial z_i z_j}+\varepsilon\sum_{j}^{N}\frac{1}{\varepsilon^2}b_j(z)^{\varepsilon}\frac{\partial w_\varepsilon}{\partial z_j}\right)-w_\varepsilon(z)+w_\varepsilon(z)^p=0 \quad (3.21)$$

where the coefficients a_{ij}^{ε} and b_j^{ε} depend on the subsequences $\{\varepsilon\}_r$ and $\{Q_\varepsilon\}_r$ and on the coefficients a_{ij} and b_j defined before by equation (3.9). It turns out that these coefficients are Lipschitz continuous with constant uniformly bounded in r, where with r we are denoting the index of subsequence. Then, if we take the limit $\varepsilon \to 0$ as in the first case above ($P_\varepsilon \to P_0$), with the same argument we get a convergent subsequence

$$w_r \to w \in C^2(\mathbb{R}_n)\cap W^{1,2}(\mathbb{R}_n) \quad (3.22)$$

with w solution to the problem (3.16). We note also that if $P_\varepsilon \to P_0 \in \partial_{\mathcal{N}}\Omega$, the previous estimates in Remark 3.2.3, in particular equation (3.18), can be applied concluding that there exists a Hölder continuous function w such that $w_\varepsilon \to w$ uniformly on compact sets on $\mathbb{R}^n_+$.

Since we are dealing with problems which in general have many solutions, we need to fix the sequence of solutions that we consider. A natural choice is to take the sequence of *least energy solutions*, that is, the solutions that come from the *Mountain Pass Lemma.* In this work, from now on, we will always consider these solutions. Let $\{u_\varepsilon\}$ be a sequence of least energy solutions to the problems $(\tilde{P}_\varepsilon)$, and suppose that P_ε a maximum point of u_ε .
Then as $\varepsilon \to 0$, depending on the location of the limit point $P_0 = \lim_{\varepsilon\to 0} P_\varepsilon$ (for a subsequence) we could arrive to a problem in all $\mathbb{R}^n$ or to a problem in $\mathbb{R}^n_+$. The first case is easily excluded, as the following theorem shows.

Theorem 3.2.4 *Let $\{u_\varepsilon\}$ be the family of least energy solutions to problem $(\tilde{P}_\varepsilon)$ and $\{P_\varepsilon\}$ their points of maximum. Then, up to a subsequence, $P_\varepsilon \to P_0 \in \partial\Omega$.*

Given the family of least energy solutions $\{u_\varepsilon\}$, let us denote by P_ε a point where u_ε attains a local maximum.

Theorem 3.2.5 *Let $\{u_\varepsilon\}$ be the family of least energy solutions to problem $(\tilde{P}_\varepsilon)$ and $\{P_\varepsilon\}$ their points of maximum. Then, up to a subsequence, $P_\varepsilon \to P_0 \in \partial\Omega$.*

Proof. We will work out the proof in two steps. In the first one we will get an upper bound for the critical level c_ε while in the second, using this upper bound, we will show that the concentration point P_0 must lie on the boundary. Then let c_ε be as in (3.5) and define (see [49])

$$M[u] = \sup_{t>0} \tilde{I}_\varepsilon(tu) \qquad u \in X$$

Step 1. First we claim that, if $\partial_{\mathcal{N}}\Omega \neq \emptyset$, then

$$c_\varepsilon \le \varepsilon^n \left\{ \frac{I(U)}{2} + o(\varepsilon) \right\}.$$

Choose $P \in \partial_{\mathcal{N}}\Omega$ and define the set

$$A_{iR} := B_R(x-P) \cap \Omega = \{x \in \Omega : |x-P| < iR\} \quad i = 1, 2$$

where throughout the paper $B_\rho(\cdot - c)$ stands for the open ball centred at c and with radius ρ. We want to remark that the radius R is such that $\overline{A_{2R}} \cap \overline{\partial_{\mathcal{D}}\Omega} = \emptyset$.

Define next a smooth non increasing and non negative radial cutoff functions $\chi_R : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \chi_R(y) = 1 & \text{in } B_R; \\ \chi_R(y) = 0 & \text{in } \mathbb{R}^n \backslash B_{2R}; \\ |\nabla \chi_R(y)| < C & \text{in } B_{2R} \backslash B_R. \end{cases} \tag{3.23}$$

From the definition of the Mountain Pass level, see Proposition 3.1.2, we can argue that, after a flattening and a scaling,

$$c_\varepsilon \le M \left[\chi_{\frac{R}{\varepsilon}} \left(\frac{x-Q}{\varepsilon} \right) U \left(\frac{x-Q}{\varepsilon} \right) \right],$$

where U is the ground state solution of problem (1.1) and $Q = \Psi(P)$. Then, if

$$\Omega_\varepsilon := \left\{ y \in \mathbb{R}^n : \exists x \in \Omega \ s.t. \ y = \frac{x-Q}{\varepsilon} \right\}$$

denote the scaled domain and

$$\varphi_\varepsilon := \chi_{\frac{R}{\varepsilon}} \left(\frac{x-Q}{\varepsilon} \right) U \left(\frac{x-Q}{\varepsilon} \right),$$

we have

$$M[\varphi_\varepsilon] = \frac{1}{2} t_*^2 \varepsilon^n \int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^2 + \varphi_\varepsilon^2 - \frac{\varepsilon^n}{p+1} \int_{\Omega_\varepsilon} (t_* \varphi_\varepsilon)^{p+1}$$

where t^* is such that $g(t) \equiv I(t\varphi_\varepsilon)$ attains its unique maximum. From (3.23), we get

$$M[\varphi_\varepsilon] = \frac{1}{2} t_*^2 \varepsilon^n \int_{A_{2R_\varepsilon}} |\nabla \varphi_\varepsilon|^2 + \varphi_\varepsilon^2 dy - \frac{\varepsilon^n}{p+1} \int_{A_{2R_\varepsilon}} (t_* \varphi_\varepsilon)^{p+1} = I_1 + I_2 - I_3.$$

where

$$R_\varepsilon = \frac{R}{\varepsilon}, \qquad A_{iR_\varepsilon} = \{ y \in \Omega_\varepsilon : |y| < iR_\varepsilon \},$$

$$
\begin{aligned}
I_1 &= \frac{1}{2}t_*^2\varepsilon^n \int_{A_{2R_\varepsilon}} |\nabla\varphi_\varepsilon|^2, \\
I_2 &= \frac{1}{2}t_*^2\varepsilon^n \int_{A_{2R_\varepsilon}} \varphi_\varepsilon^2, \\
I_3 &= \frac{\varepsilon^n}{p+1} \int_{A_{2R_\varepsilon}} (t_*\varphi_\varepsilon)^{p+1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_1 &= \frac{1}{2}t_*^2\varepsilon^n \int_{A_{2R_\varepsilon}} |\nabla\varphi_\varepsilon(y)|^2 dy \\
&= \frac{1}{2}\varepsilon^n t_*^2 \left(\int_{A_{R_\varepsilon}} |\nabla U(y)|^2 dy + \int_{A_{2R_\varepsilon}\setminus A_{R_\varepsilon}} |\nabla\left(\varphi(y)U(y)\right)|^2 dy \right) \\
&= \frac{1}{2}\varepsilon^n t_*^2 \left(\int_{R^n_+} |\nabla U(y)|^2 dy + o(1) \right) \\
&= \frac{1}{4}\varepsilon^n t_*^2 \left(\int_{R^n} |\nabla U(y)|^2 dy + o(1) \right) \quad \text{as } \varepsilon \to 0. \qquad (3.24)
\end{aligned}
$$

where in the last equality we have used the asymptotic behavior of U (1.3), and polar coordinates centered at the origin and (3.23). For the same reasons we obtain the follow two equations:

$$
\begin{aligned}
I_2 &= \frac{1}{2}\varepsilon^n t_*^2 \left(\int_{R^n_+} u_0^2(y) dy + o(1) \right) \\
&= \frac{1}{4}\varepsilon^n t_*^2 \left(\int_{R^n} u_0^2(y) dy + o(1) \right) \quad \text{as } \varepsilon \to 0. \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \frac{\varepsilon^n}{p+1} \left(\int_{R^n_+} (t_*\varphi_\varepsilon)^{p+1} dy + o(1) \right) \\
&= \frac{\varepsilon^n}{2(p+1)} \left(\int_{R^n} (t_*\varphi_\varepsilon)^{p+1} dy + o(1) \right) \quad \text{as } \varepsilon \to 0. \qquad (3.26)
\end{aligned}
$$

From the uniqueness of the maximum of $I(tU(\cdot))$ for $t > 0$, it follows that

$$t^* - 1 = o(1) \qquad \text{as } \varepsilon \to 0 \qquad (3.27)$$

Then from equations (3.24), (3.25), (3.26) and(3.27) we get

$$c_\varepsilon \le \varepsilon^n \left\{ \frac{I(U)}{2} + o(1) \right\}. \qquad (3.28)$$

Step 2. Since Ω is bounded, for some subsequence we can assume $P_\varepsilon \to P_0 \in \overline{\Omega}$. Suppose $P_0 \in \Omega$. In this case, since $dist\{P_0, \partial\Omega\} = \rho > 0$,

$$dist\{P_\varepsilon, \partial\Omega\}/\varepsilon \to \infty \qquad \text{as } \varepsilon \to 0.$$

Then $u_\varepsilon(P_\varepsilon + \varepsilon y)$, up to a subsequence, would converge uniformly to a limit U belonging to $C^2(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$ (see Subsection 3.2), solution to the problem

$$-\Delta U + U = U^p \qquad \text{in } \mathbb{R}^n.$$

We want to point out that by maximum principle $U > 0$ and moreover using argument in [27], we get

$$\lim_{r \to +\infty} e^r r^{\frac{n-1}{2}} U(r) = \alpha_{n,p},$$

where, as above, $\alpha_{n,p}$ is a positive constant depending only on n and p.

By the uniqueness result in [35] it follows that

$$c_\varepsilon = \varepsilon^n \left\{ I(U) + o(1) \right\}. \tag{3.29}$$

This is a contradiction from equation (3.28). Then the concentration necessarily occurs on the boundary $\partial\Omega$. ■

Remark 3.2.6 *With the same arguments in the proof of the last theorem, getting the same contradiction, see* (3.29)*, we can show that*

$$\lim_{\varepsilon \to 0} \frac{dist(P_\varepsilon, \partial\Omega)}{\varepsilon} \neq \infty,$$

that is, in the scaled problem the limit point P_0 *(or* Q_0*, if we flatten before) is inside a neighborhood of* $\partial\mathbb{R}^n_+$ *:*

$$P_0 \in B_R(x')$$

for some number R *positive and some* $x' = (x_1, x_2, \ldots, x_{n-1}) \in \partial\mathbb{R}^n_+$*.*

Case 3: We suppose, up to a subsequence, that $P_\varepsilon \to P_0 \in \partial_{\mathcal{D}}\Omega$.

We want to point out that in this case we can have only that

$$\frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \to C, \tag{3.30}$$

where $0 < C < +\infty$. This follow by Remark 3.2.6 and from a general property of the least-energy solution to problem $(\tilde{P}_\varepsilon)$ at the local maximum points, as shown below (see equation (3.32)). Then, straightening a portion of boundary around the point P_0, as usual we define $v_\varepsilon(y) = u_\varepsilon(\Phi(y))$, with y defined in an open set containing the ball $B_{2s}(Q_0)$ with $s > 0$ sufficiently small and $Q_0 = \Psi(P_0)$. Now, denote

$$w_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z) \qquad z \in B_{\frac{s}{\varepsilon}}$$

with $Q_\varepsilon = (q'_\varepsilon, \varepsilon c_\varepsilon)$, $q'_\varepsilon \in \mathbb{R}^{n-1}$ and $c_\varepsilon > 0$ bounded via hypothesis (3.30). Moreover w_ε satisfies

$$\begin{cases} \varepsilon^2 \left(\sum_{i,j}^{N} \frac{1}{\varepsilon^2} a^\varepsilon_{ij}(z) \dfrac{\partial^2 w_\varepsilon}{\partial z_i z_j} + \varepsilon \sum_j^N \dfrac{1}{\varepsilon^2} b_j(z)^\varepsilon \dfrac{\partial w_\varepsilon}{\partial z_j} \right) - w_\varepsilon(z) + w_\varepsilon(z)^p = 0 \\ \text{in } B_{\frac{s}{\varepsilon}} \cap \{z_n > -c_\varepsilon\}, \\ \\ w_\varepsilon(z) = 0 \\ \text{in } B_{\frac{s}{\varepsilon}} \cap \{z_n = -c_\varepsilon\}. \end{cases}$$

where the coefficients a^ε_{ij} and b^ε_j, depend on the subsequences $\{\varepsilon\}_r$ and $\{Q_\varepsilon\}_r$ and on the coefficients a_{ij} and b_j defined before by equation (3.9). From equation (3.13), following the previous arguments we get

$$w_\varepsilon \to w \in C^2(\mathbb{R}^n_+) \cap W^{1,2}(\mathbb{R}^n_+)$$

and w solution of (3.16).

Case 4: We suppose, up to a subsequence, that $P_\varepsilon \to P_0 \in \mathcal{I}_\Omega$.

We can repeat the arguments of **Case 3**, supposing, for the same reason that

$$\frac{dist(P_\varepsilon, \mathcal{I}\Omega)}{\varepsilon} \to C$$

with C positive constant getting

$$w_\varepsilon \to w \in C^2(\mathbb{R}^n_+) \cap W^{1,2}(\mathbb{R}^n_+),$$

where w is a solution to the problem (3.16). The same conclusions of this two last cases can be obtained applying the arguments in Remark 3.2.3. Therefore by Theorem 3.2.5 and Remark 3.2.6, we can exclude that the limit point P_0 is in the interior of domain Ω. Then as $\varepsilon \to 0$, from the previous cases, we reach the limit problem

$$-\Delta w + w = w^p \qquad \text{in } \mathbb{R}^n_+$$

and, in according to equation (3.8), with one of the following boundary condition depending on the limit point P_0. We have three cases:

$$\begin{cases} (1) \quad w = 0 \text{ on } \partial\mathbb{R}^+_n \text{ if } P_\varepsilon \to P_0 \in \partial_{\mathcal{D}}\Omega, \\ \\ (2) \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^+_n \text{ if } P_\varepsilon \to P_0 \in \partial_{\mathcal{N}}\Omega, \\ \\ (3) \quad w = 0 \text{ on } \{y_1 > 0 \cap \partial\mathbb{R}^+_n\} \text{ and } \frac{\partial w}{\partial \nu} = 0 \text{ on } \{y_1 < 0 \cap \partial\mathbb{R}^+_n\} \text{ if } P_\varepsilon \to P_0 \in \mathcal{I}_\Omega. \end{cases} \tag{3.31}$$

Next we try to determine the part of the boundary where the concentration takes place.

3.3 Analysis of the limit points P_ε

In this section we want to specify the limit behaviour of the point P_ε. We will establish where, up to a subsequence, this point can be attained and what is the rate of convergence, see Corollary 3.3.6. This last result is very important for the sequel because it will allow us to complement the present results with those in the first part of the paper, [26]. We start showing a useful property of the solution u_ε to the problem $(\tilde{P}_\varepsilon)$ in a local maximum point. Denoting $u_\varepsilon(P_\varepsilon) = \max_{x\in\overline{\Omega}} u_\varepsilon(P)$, we claim that:

$$u_\varepsilon(P_\varepsilon) \ \geq \ 1. \tag{3.32}$$

This follows by using elementary arguments about subharmonic functions. In fact if we suppose that the maximum point $P_\varepsilon \in \Omega$ then, bearing in mind problem $(\tilde{P}_\varepsilon)$, in a neighborhood of P_ε we have that

$$u(P_\varepsilon)^p - u(P_\varepsilon) = -\varepsilon^2 \Delta u(P_\varepsilon) \geq 0,$$

that imply (3.32).

Now, we suppose that $P_\varepsilon \in \partial_{\mathcal{N}}\Omega$ and by contradiction that $u_\varepsilon(P_\varepsilon) < 1$. Then we get

$$-\varepsilon^2 \Delta u = u^p - u \leq 0$$

in a neighborhood of P_ε. From the boundary Hopf lemma follows that $\dfrac{\partial u_\varepsilon}{\partial \nu} > 0$ for some point in $\partial_{\mathcal{N}}\Omega$ that give us the desired contradiction. We do not have to consider the case $P_\varepsilon \in \overline{\partial_{\mathcal{D}}\Omega}$ since we are looking for nontrivial positive solutions to the problem $(\tilde{P}_\varepsilon)$.
By Theorem 3.2.5, we can affirm that the limit point P_0 of a least-energy solution cannot belong to the interior of Ω. To understand better the location of such point, the idea is to analyze the rate of convergence. Our first result about the location of the concentration points is the following:

Theorem 3.3.1 *Let u_ε be a least-energy solution to the problem $(\tilde{P}_\varepsilon)$ and $\{P_\varepsilon\}$ its local maximum points. As $\varepsilon \to 0$, up to a subsequence, we suppose that $P_\varepsilon \to P_0 \in \partial\Omega$. Then the limits of concentration points satisfy*

$$P_0 \notin \mathit{Int}(\partial_{\mathcal{D}}\Omega).$$

Proof. Denote $\rho_\varepsilon = \operatorname{dist}(P_\varepsilon, \partial\Omega)$. Assume by contradiction $P_0 \in \operatorname{Int}(\partial_{\mathcal{D}}\Omega)$, and let $\widetilde{P}_\varepsilon \in \operatorname{Int}(\partial_{\mathcal{D}}\Omega)$ be such that

$$\rho_\varepsilon = \operatorname{dist}(P_\varepsilon, \tilde{P}_\varepsilon).$$

Taking a subsequence, we have two cases:

1. $\frac{\rho_\varepsilon}{\varepsilon} \to \infty \quad$ as $\varepsilon \to 0$;

2. $\frac{\rho_\varepsilon}{\varepsilon} \to c < \infty \quad$ as $\varepsilon \to 0$.

In the case (1) after the usual scaling $u_\varepsilon(P_\varepsilon + \varepsilon y)$, we arrive to a problem in all $\mathbb{R}^n$ and, as in the second step of the proof of Theorem 3.2.5, we can show that the energy solution

$$c_\varepsilon = \varepsilon^n \{I(U) + o(1)\}$$

which is a contradiction for equation (3.28) (see also Remark 3.2.6).
In the case (2), we can consider two different subcases:

(i) $c = 0$;

(ii) $c > 0$.

If $c = 0$ we immediately get the contradiction. In fact in the scaled problem, the maximum of the *least energy solution* is attained on the Dirichlet boundary part and then the contradiction follows from equation (3.32).
Otherwise if $c > 0$, as usual, after straightening the boundary and scaling (see Figure 3.2), we arrive to a solution $v > 0$ with $v(Q_0) \geq 1$, of the problem

$$\begin{cases} -\Delta v + v = v^p & \text{in} \quad \mathbb{R}^n_+; \\ v(0,\alpha) > 1; & \\ v(\bar{x}, 0) = 0 & \text{on} \quad \partial\mathbb{R}^n_+ = 0. \end{cases} \tag{3.33}$$

Here we will show, by contradiction, that the problem (3.33) does not have any solution. Without loss of generality, we can consider the limit maximum point $Q_0 = \Psi(P_0)$ located in the x_n axis at some positive distance of the limit boundary $\partial\mathbb{R}^n_+$. In fact when $\varepsilon \to 0$ the limit maximum point Q_0 will be achieved at some point of the type $(q_0, \alpha) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$. Being the problem (3.33) invariant under translation, we can assume $Q_0 \to (0, \alpha)$ as $\varepsilon \to 0$. On the other hand, since we assume that ($\frac{\rho_\varepsilon}{\varepsilon} \to c$, with $0 < c < \infty$) then x_n remains bounded away from zero. Now, we choose a positive number

$$h > 2\alpha \tag{3.34}$$

and we consider a slab domain $\Omega = \mathbb{R}^{n-1} \times (0, h)$. The contradiction readily follows from a theorem proved by H. Berestycki, L. Caffarelli, L. Nirenberg, see Theorem 1.11 in Section 1.5. Since our limit problem (3.33) satisfies all the hypothesis of Theorem 1.11, then we obtain that the limit solution is increasing in the x_n direction in the $(0, h/2)$ interval. The hypothesis (3.34) imply that the limit maximum point Q_0 is inside this interval and then we have the contradiction. ∎

From the previous convergence results, see Theorem 3.2.5 and Theorem 3.3.1, we can estimate more precisely the convergence $P_\varepsilon \to P_0$.

- If P_0 is in the interior of the Neumann boundary part, after straightening the boundary and scaling, we get a limit problem with Neumann boundary conditions, and then, since the argument is local, we can use the methods in [50], to obtain the same asymptotic expansion getting a term which involves the mean curvature H, where as usual, $H(\cdot)$ stands for the mean curvature of $\partial\Omega$ and it is clear that

$$H(P) = \frac{1}{n-1} tr A_P$$

where A_P is the hessian of ψ at 0 (see Subsection 3.2). In this case, the result, due to W.M.Ni and I.Takagi, gives us the desired expansion of the energy, see Theorem 1.6.1:

$$\tilde{I}_\varepsilon(u_\varepsilon) = \varepsilon^n \left\{ \frac{1}{2} I(U) - (n-1)\varepsilon\gamma H(P_\varepsilon) + o(\varepsilon) \right\} \qquad \text{as } \varepsilon \to 0$$

- The remaining case is $P_0 \in \overline{\partial_N \Omega} \cap \overline{\partial_D \Omega}$.

  In the sequel, we try to extend some of the main results for the Neumann case to the Mixed boundary value problem. For example, it is possible to prove that, for ε small, necessarily $P_\varepsilon \in \partial_N\Omega$, as in the Neumann problem. The proof follows from various lemmas. In the first one we get an important Liouville type result:

Lemma 3.3.2 *Let us consider the problem*

$$\begin{cases} -\Delta u + u = u^p & in \quad \mathbb{R}^n_+; \\ u \geq 0 & in \quad \mathbb{R}^n_+; \\ u = 0 & on \quad \Gamma_0; \\ \frac{\partial u}{\partial \nu} = 0 & on \quad \Gamma_1, \end{cases} \tag{3.35}$$

where

$$\begin{aligned} \Gamma_0 &= \{x = (x^1, \cdots, x^n) \,:\, x^n = 0\,,\, x^1 > 0\}, \\ \Gamma_1 &= \{x = (x = (x^1, \cdots, x^n) \,:\, x^n = 0\,,\, x^1 < 0\} \end{aligned}$$

and

$$1 < p < \frac{n+2}{n-2}.$$

Define

$$\tilde{X} = \left\{ \phi \in W^{1,2}(\mathbb{R}^n_+) : \phi = 0 \ on \ \Gamma_0, \frac{\partial \phi}{\partial \nu} = 0 \ on \ \Gamma_1 \right\}$$

Then if $u \in \tilde{X}$ is a solution, $u \equiv 0$.

Proof. Here we want to obtain a Liouville type result for the problem (3.35). Previously Damascelli and Gladiali in [16] obtained a similar Liouville type result for a mixed problem that unfortunately does not apply to our case. They considered the differential problem $-\triangle u = f(u)$, getting a nonexistence result when the nonlinear term $f(u)$ satisfies different conditions. Some of them

(i) $g(t) := \frac{f(t)}{t^{\frac{n+2}{n-2}}}$ is nonincraesing in $(0, +\infty)$;

(ii) $f(t) > 0$ for every $t > 0$,

do not apply in our case, being $f(t) = t^p - t$.

However we can use some of the ideas in [16], getting a nonexistence result. In the proof we use the moving plane method, therefore we introduce some notation. For $\lambda > 0$ we let

$$T_\lambda = \{x \in \mathbb{R}^n_+ : x_1 = \lambda\} \quad \Sigma_\lambda = \{x \in \mathbb{R}^n_+ : x_1 > \lambda\},$$

and also we define

$$R_\lambda(x) = x_\lambda = (2\lambda - x_1, x_2, \cdots, x_n) \quad x \in \mathbb{R}^n_+ \qquad u_\lambda(x) = u(x_\lambda).$$

Note the x_λ is the reflection trough the plane T_λ and $u_\lambda(x)$ the reflected function.

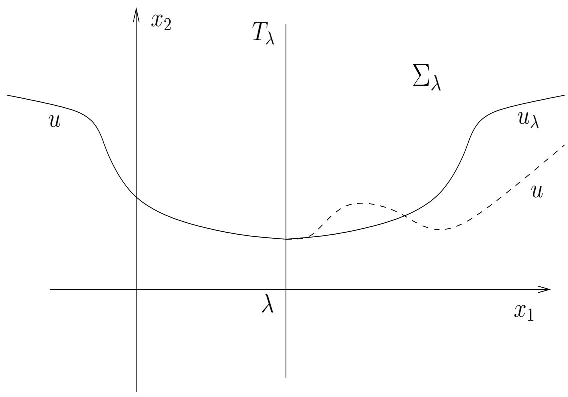

Figure 3.3: Moving plane

We note that, for any $\lambda \in \mathbb{R}$, the function $(u - u_\lambda)^+$ vanishes in the set $\Sigma_\lambda \cap \Gamma_0$. In particular, if $\lambda \geq 0$ then $\Sigma_\lambda \cap \{x_n = 0\} \subset \Gamma_0$. If $\lambda < 0$, then $(\Sigma_\lambda \cap \{x_n = 0\}) \cap \Gamma_1 \neq \emptyset$, but notice that in this part of the boundary the normal derivatives of u and u_λ vanishes. Then if we consider the problem (3.35) in Σ_λ and the similar one satisfied by the reflected function u_λ (see

[16]), we multiply both problems by the function $(u-u_\lambda)^+$, and we integrate by parts and subtract the equations we obtain:

$$\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 dx + \int_{\Sigma_\lambda} [(u-u_\lambda)^+]^2 dx = \int_{\Sigma_\lambda} [u^p - u_\lambda^p](u-u_\lambda)^+ dx$$
$$\leq p \int_{\Sigma_\lambda} \left(u^{p-1} + u_\lambda^{p-1}\right) [(u-u_\lambda)^+]^2 dx.$$

Then we have to consider separately two cases:

Case 1 $\frac{4}{n} + 1 \leq p < \frac{n+2}{n-2}$

Case 2 $1 < p \leq \frac{4}{n} + 1$

If we are in **Case 1**, via Holder and Sobolev inequalities, we get:

$$\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2 dx$$
$$\leq p\Big(\int_{\Sigma_\lambda \cap \{u \geq u_\lambda\}} (u_\lambda^{p-1} + u^{p-1})^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_{\Sigma_\lambda} [(u-u_\lambda)^+]^{2^*}\Big)^{\frac{2}{2^*}}$$

$$\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2 dx$$
$$\leq pC\Big(\int_{\Sigma_\lambda \cap \{u \geq u_\lambda\}} (u_\lambda^{p-1} + u^{p-1})^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2\Big)$$

Then

$$\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2 dx \leq \tag{3.36}$$
$$\leq pC\Big(\int_{\Sigma_\lambda \cap \{u \geq u_\lambda\}} (u_\lambda^{p-1} + u^{p-1})^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2\Big)$$

where C is the Sobolev constant.

On the other hand, in **Case 2**, using again Holder and Sobolev inequalities, we are led to consider the following

$$\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2 dx$$
$$\leq p\Big(\int_{\Sigma_\lambda \cap \{u \geq u_\lambda\}} (u_\lambda^{p-1} + u^{p-1})^{\frac{2}{p-1}}\Big)^{\frac{p-1}{2}} \Big(\int_{\Sigma_\lambda} [(u-u_\lambda)^+]^{\frac{4}{3-p}}\Big)^{\frac{3-p}{2}}$$

Then

$$\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2 dx \leq \tag{3.37}$$
$$\leq pC\Big(\int_{\Sigma_\lambda \cap \{u \geq u_\lambda\}} (u_\lambda^{p-1} + u^{p-1})^{\frac{2}{p-1}}\Big)^{\frac{p-1}{2}} \Big(\int_{\Sigma_\lambda} |\nabla(u-u_\lambda)^+|^2 + [(u-u_\lambda)^+]^2\Big)$$

In both cases we get

$$\|(u-u_\lambda)^+\|_{W^{1,2}(\Sigma_\lambda)} \le pC\left(\int_{\Sigma_\lambda\cap\{u\ge u_\lambda\}}(u_\lambda^{p-1}+u^{p-1})^r\right)^{\frac{1}{r}}\|(u-u_\lambda)^+\|_{W^{1,2}(\Sigma_\lambda)} \tag{3.38}$$

where $r>0$ is some number such that $(p-1)r\in[2,2^*]$.

We want to show, that $u\le u_\lambda$ in Σ_λ $\forall\lambda\in\mathbb{R}$ i.e. the solution is decreasing in the x_1 direction and this implies that the solution cannot have finite energy. Now, it is easy to see that there is at least one $\widetilde{\lambda}$ such that $u\le u_\lambda$ in Σ_λ $\forall\lambda>\widetilde{\lambda}$. In fact if λ is big enough, then it readily follows that

$$\int_{\Sigma_\lambda\cap\{u\ge u_\lambda\}}(u_\lambda^{p-1}+u^{p-1})^r<1$$

and then

$$\|(u-u_\lambda)^+\|_{W^{1,2}(\mathbb{R}^n_+)}\le 0 \quad \forall\lambda>\widetilde{\lambda}.$$

Then we can infer that $\forall\lambda>\widetilde{\lambda}$ $\quad u\le u_\lambda$ in Σ_λ. From the Hopf lemma and the strong maximum principle, we can claim that $u<u_\lambda$ in Σ_λ $\quad\forall\lambda>\widetilde{\lambda}$. In fact if $u\equiv u_\lambda$ in Σ_λ, then u would be a nontrivial solution of problem (3.35) with $\dfrac{\partial u}{\partial\nu}<0$ on $R_\lambda(\Gamma_1)$. This is a contradiction since $\dfrac{\partial u_\lambda}{\partial\nu}=0$ on $R_\lambda(\Gamma_1)$. Then by maximum principle we have the conclusion.
Now, we will show that

$$\inf\left\{\widetilde{\lambda}: u\le u_\lambda \text{ in } \Sigma_\lambda \text{ if } \lambda>\widetilde{\lambda}\right\}=-\infty.$$

We suppose by contradiction that such infimum $\widehat{\lambda}$ is finite. We write a generic point $x\in\mathbb{R}^n_+$ in the form $x=(x_1,y)$ where $x_1\in\mathbb{R}$ and $y\in\mathbb{R}^{n-2}\times\mathbb{R}_+$. We define a change of variable such that

$$u_{\widehat{\lambda}}(x_1,y)=u(2\widehat{\lambda}-x_1,y):=u(z,y).$$

Then if we consider $\mu<\widehat{\lambda}$ we have

$$u_\mu(x_1,y)=u(2\mu-x_1,y)=u(z-2(\widehat{\lambda}-\mu),y)=u(z-\delta,y).$$

where $\delta:=2(\widehat{\lambda}-\mu)$ is bigger than zero. Here, we want to prove the following

Claim 3.3.3 *If* $2\le g\le 2^*$

$$\int_{\mathbb{R}^n_+\cap\{z>\widehat{\lambda}\}}u^g(z-\delta,y)dzdy\longrightarrow\int_{\mathbb{R}^n_+\cap\{z>\widehat{\lambda}\}}u^g(z,y)dzdy \qquad as \quad \delta\to 0.$$

We know that $\forall\alpha>0$ there exists a compact set K (see Figure 3.4), such that:

$$\int_{\mathbb{R}^n_+\setminus K}u^g(z,y)dzdy<\frac{\alpha}{2}. \tag{3.39}$$

By a direct computation we find

$$\int_{\mathbb{R}^n_+\cap\{z>\widehat{\lambda}\}} u^g(z-\delta,y)-u^g(z,y)dzdy =$$

$$\int_{\mathbb{R}^n_+\cap\{s>\widehat{\lambda}-\delta\}} u^g(s,y)-\int_{\mathbb{R}^n_+\cap\{z>\widetilde{\lambda}\}} u^g(z,y).$$

From (3.39), we have

$$\int_{\mathbb{R}^n_+\cap\{\widehat{\lambda}-\delta<z<\widehat{\lambda}\}} u^g(z,y)<\frac{\alpha}{2}+Md\delta<\alpha \tag{3.40}$$

if we choose δ such that $Md\delta<\frac{\alpha}{2}$. This concludes the proof of the Claim.

Notice that the constant M that appears in (3.40) is the absolute maximum of u^g over the compact K whereas d is a constant that depends on the diameter of the compact K. It is important to note that the two constants are independent of δ.

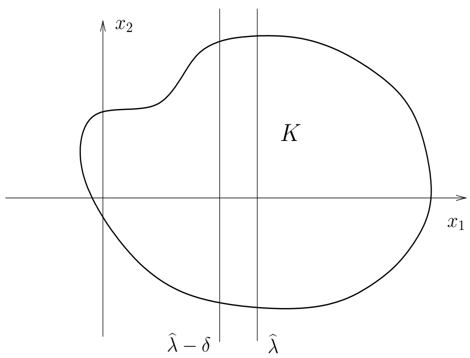

Figure 3.4: The compact K

Now we fix another compact set K_0, such that $K_0\in\mathbb{R}^n_+\cap\Sigma_{\widehat{\lambda}}$ and

$$\int_{\Sigma_{\widehat{\lambda}}\setminus K_0} u^g_{\widehat{\lambda}}(x)dx<\rho.$$

Over the compact K_0, we have

$$u_{\widehat{\lambda}}-u>c_{K_0}>0$$

where c_{K_0} is some constant depending on K_0. The continuity of the reflection implies

$$u_\mu-u>\frac{c_{K_0}}{2} \tag{3.41}$$

in fact if $\mu < \widehat{\lambda}$ we have

$$|u_{\widehat{\lambda}} - u_\mu| < \varepsilon \Longleftrightarrow |\widehat{\lambda} - \mu| < \widetilde{\delta} \tag{3.42}$$

$$u_{\widehat{\lambda}} - u = |u_{\widehat{\lambda}} - u| = |u_\mu + u_{\widehat{\lambda}} - u_\mu - u| \geq c_{K_0};$$

$$|u_\mu - u| = u_\mu - u \geq c_{K_0} - \varepsilon > \frac{c_{K_0}}{2} \tag{3.43}$$

for some $\widetilde{\delta} > 0$. The inequality (3.43) is satisfied only if $u_\mu - u \geq c_{K_0} - \varepsilon$. In fact, by contradiction, if we suppose that $u_\mu - u < \varepsilon - c_{K_0}$ then

$$u_\mu - u_{\widehat{\lambda}} + c_{K_0} < u_{\widehat{\lambda}} - u + u_\mu - u_{\widehat{\lambda}} + < \varepsilon - c_{K_0}. \tag{3.44}$$

From (3.44) and by the continuity of the reflection we get the contradiction:

$$u_{\widehat{\lambda}} - u_\mu > 2c_{K_0} - \varepsilon > \varepsilon.$$

By continuity, we can choose $\widetilde{\delta}$ in (3.42) such that, on the compact K_0, $u_\mu - u > \dfrac{c_{K_0}}{2}$. Then Claim 3.3.3 (note that the compact K_0 does not depend on μ) implies

$$\int_{\Sigma_\mu \setminus K_0} u_\mu^g \longrightarrow \int_{\Sigma_{\widehat{\lambda}} \setminus K_0} u_\mu^g < \rho. \tag{3.45}$$

Then from equations (3.45) and (3.41) we obtain:

$$\int_{\Sigma_\mu \cap \{u \geq u_\mu\}} u_\mu^g \leq \int_{\Sigma_\mu \setminus K_0} u_\mu^g \leq 2\rho. \tag{3.46}$$

Finally if we choose $\rho << \frac{1}{2}$, via equations (3.38) and (3.46) we get the contradiction concluding the proof of the lemma. ■

Remark 3.3.4 *We want to point out that if* $\frac{4}{n} + 1 \leq p < \frac{n+2}{n-2}$ *then* $g := (p-1)\frac{n}{2}$ *belongs to* $[2, 2^*]$ *and similarly if* $1 < p \leq \frac{4}{n} + 1$ *then* $g := \frac{4}{3-p}$ *belongs to* $[2, 2^*]$.

Theorem 3.3.1 guarantees that the limit point P_0 cannot be inside $\partial_{\mathcal{D}}\Omega$. However, it could be possible that $P_0 \in \overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega}$. In any case, we get the estimate

Lemma 3.3.5 *Suppose we are in the hypothesis of Theorem 3.2.5, then*

$$\frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0.$$

Proof. We assume by contradiction that there exists a constant $C \geq 0$ such that for ε small and for some subsequence we have

$$\frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \rightarrow \ C.$$

We distinguish two cases:

- Case 1.
$$\frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$
  Then if we consider $Q_\varepsilon = \psi(\varepsilon P_\varepsilon)$ (i.e., the maximum point in the scaled problem with rectified boundary, where the functions are denoted by w_ε), up to a subsequence, $\{Q_\varepsilon\}$ converges to a point on the boundary of $\mathbb{R}^n_+$ with Dirichlet boundary data.

  On the other hand $w_\varepsilon(Q_\varepsilon) \geq 1$. Hence, we get a contradiction with the continuity of the uniform limit w_0.

- Case 2.
$$\frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \rightarrow C > 0 \text{ as } \varepsilon \rightarrow 0.$$
  Here, we will distinguish two subcases:

  – Subcase 2.1
$$\frac{dist(P_\varepsilon, \overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega})}{\varepsilon} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0.$$
    Then, passing to the problem satisfied by w_ε, we get that the limit w_0 solves an equation in $\mathbb{R}^n_+$, with Dirichlet boundary data on the boundary, and with an interior maximum point $w_0(Q_0) > 1$:
$$\begin{cases} -\Delta w_0 + w_0 = w_0^p & \text{in} \quad \mathbb{R}^n_+; \\ w_0(0,\alpha) > 1; & \\ w_0(\bar{x},0) = 0 & \text{on} \quad \partial\mathbb{R}^n_+ = 0. \end{cases}$$
    The contradiction follows from the same reasons seen in the proof of Theorem 3.2.5 (see case 2, subcase (ii)).

  – Subcase 2.2
$$\frac{dist(P_\varepsilon, \overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega})}{\varepsilon} \rightarrow M > 0 \text{ as } \varepsilon \rightarrow 0.$$
    Notice that we cannot have $M = 0$, since we are assuming that $\dfrac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \rightarrow C > 0.$
    In this case, in the limit, the problem which is solved by w_0, is a PDE with mixed boundary conditions, and an interior maximum

point $w_0(Q_0) > 1$. We note that $Q_0(0,\alpha), 0 \in \mathbb{R}^{n-1}, \alpha > 0$, because $M > 0$.

$$\begin{cases} -\Delta w_0 + w_0 = w_0^p & \text{in} \quad \mathbb{R}^n_+; \\ w_0 \geq 0 & \text{in} \quad \mathbb{R}^n_+; \\ w(0,\alpha) > 1; & \\ w_0 = 0 & \text{on} \quad \Gamma_0; \\ \frac{\partial w_0}{\partial n} = 0 & \text{on} \quad \Gamma_1. \end{cases} \tag{3.47}$$

Then the contradiction follows by using the Lemma 3.3.2.

This completes the proof. ■

In particular since $\overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega} \subset \overline{\partial_{\mathcal{D}}\Omega}$, it follows:

Corollary 3.3.6 *If we denote $\mathcal{I}_\Omega \equiv \overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega}$, then*

$$\frac{dist(P_\varepsilon, \mathcal{I}_\Omega)}{\varepsilon} \to \infty \ \textit{as} \ \varepsilon \to 0.$$

Finally, assuming that the concentration point is located on the boundary at the Dirichlet-Neumann interface, from this last Corollary we can show that, when ε is small, the point where the least-energy solution u_ε of the differential problem $(\tilde{P}_\varepsilon)$ attains the maximum, must lie on the boundary. That is, if the limit point P_0 lies at the interface $\mathcal{I}_\Omega$, then the sequence $\{P_\varepsilon\}$ for ε small enough, lies in the Neumann boundary part. In the next theorem we consider only the case that the limit point $P_0 \in \mathcal{I}_\Omega$. Otherwise if the limit point $P_0 \in \partial_{\mathcal{N}}\Omega$, since the argument is local we can join with the Neumann case studied by W.M. Ni and I.Takagi in [49]. They proved a similar result (see Theorem 2.1), showing that in the Neumann case (Problem (N_ε)), if a least energy solution u_ε attains a local maximum at $P_\varepsilon \in \overline{\Omega}$, then the distance from P_ε to $\partial\Omega$ is $O(\varepsilon)$ which imply that, for ε sufficiently small, the sequence $\{P_\varepsilon\}$ must lie on the boundary. The next theorem extends this result to the mixed case.

Theorem 3.3.7 *Suppose that $P_\varepsilon \to P_0 \in \mathcal{I}_\Omega$. Then for ε small enough one has*

$$P_\varepsilon \in \partial_{\mathcal{N}}\Omega.$$

PROOF. It is sufficient to apply Lemma 3.3.5 and Corollary 3.3.6. In fact, by virtue of Lemma 3.3.5, we can exclude that the point P_ε is on $\partial_{\mathcal{D}}\Omega$ when $\varepsilon \to 0$. So we suppose that up to a subsequence $P_\varepsilon \to P_0$ such that $\{P_\varepsilon\} \in \Omega$ for ε sufficiently small. Corollary 3.3.6 implies

$$\frac{dist(P_\varepsilon, \mathcal{I}_\Omega)}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0.$$

Then, after scaling, we reach the limit problem

$$-\Delta U + U = U^p \qquad \text{in } \mathbb{R}^n.$$

As in the proof of Theorem 3.2.5 by a cut-off function, we can show that the non-trivial solution of the rescaled problem has an energy

$$c_\varepsilon = \varepsilon^n \left\{ I(U) + o(\varepsilon) \right\},$$

where U stand for the ground state solution. This is a contradiction, since $c_\varepsilon \leq \varepsilon^n \left\{ \frac{I(U)}{2} + o(1) \right\}$ (see Theorem 3.2.5). ■

In particular it follows

Corollary 3.3.8 *Let us consider problem $(\tilde{P}_\varepsilon)$ and suppose that up to a subsequence $P_\varepsilon \to P_0 \in \overline{\partial_{\mathcal{N}}\Omega}$. Then for ε small enough P_ε lies on the Neumann boundary part.*

Proof. It's sufficient to apply Theorem 3.3.7 and Theorem 2.1 in [49]. ■

In the next two theorems we want to better characterize the shape of least energy solutions to problem $(\tilde{P}_\varepsilon)$. Since before we have proved Corollary 3.1.3 and Corollary 3.3.6, we can strongly rely results and ideas of the proofs of W.M. Ni and I.Takagi in [49]. Therefore here we will be very short and rather sketchy.

Theorem 3.3.9 *Let u_ε be a least-energy solution of problem $(\tilde{P}_\varepsilon)$. Then for ε sufficiently small, there exists at most one local maximum P_ε, achieved at some point that must lie in the Neumann boundary part.*

Proof. By contradiction, suppose that there exist two local maxima for u_ε achieved at points $\widetilde{P}_\varepsilon$ and P_ε. Call $p_\varepsilon = \left| \frac{\widetilde{P}_\varepsilon - P_\varepsilon}{\varepsilon} \right|$ then, up to a subsequence, we distinguish three cases

1. $p_\varepsilon \to 0$ as $\varepsilon \to 0$,
2. $p_\varepsilon \to c$ with $0 < c < +\infty$ as $\varepsilon \to 0$,
3. $p_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

We know, see Corollary 3.3.8, that for ε sufficiently small the local maximum point is attained on the boundary Neumann part. Let us consider the following PDE

$$\begin{cases} -\Delta w_0 + w_0 = w_0^p & \text{in} \quad \mathbb{R}^n_+; \\ w_0 > 0 & \text{in} \quad \mathbb{R}^n_+; \\ w_0(0) = \max_{\partial \mathbb{R}^n_+} w_0; & \\ \frac{\partial w_0}{\partial \nu} = 0 & \text{on} \quad \partial \mathbb{R}^n_+. \end{cases} \tag{3.48}$$

Problem (3.48) possesses a least energy solution w_0, decreasing and vanishing exponentially at infinity. As usual, consider the diffeomorphism $y = \Psi(x)$, introduced before that flattens the boundary around at some point on $\partial\Omega$. After rescaling and straightening the boundary around the any of the two maxima points, the limit problem satisfied by w_ε is the same one (see (3.48) above), both maxima are in the interior boundary Neumann part, and not in in the set $\mathcal{I}_\Omega$, since Corollary 3.3.6 holds true. If we are in the second or third case, then we get readily the contradiction, since the solution w_0 of the limit problem is exponentially decreasing. Then suppose that

$$p_\varepsilon = \left| \frac{\widetilde{P}_\varepsilon - P_\varepsilon}{\varepsilon} \right| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.49}$$

Hence we have that there exists some $\varepsilon_0 > 0$, sufficiently small, such that for all $\varepsilon \le \varepsilon_0$, the maximum points $\widetilde{P}_\varepsilon$ and P_ε, after straightening with the diffeomorphism $y = \Psi(x)$ and scaling, are together contained in a ball of radius $\tilde{\delta}$ sufficiently small, since

$$|\widetilde{P}_\varepsilon - P_\varepsilon| = \left| \frac{\widetilde{P}_\varepsilon - P_\varepsilon}{\varepsilon} \right| \varepsilon \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

We assume that $\Phi = \Psi^{-1}$ is defined in some open set containing a closed ball $B_{\frac{\varrho}{\varepsilon}}(\cdot)$. Then we define the reflection $\widetilde{v}$ of v with respect to the y_n coordinate. Moreover if we rescale, we obtain

$$w_{\varepsilon_k}(y) = \widetilde{v}_{\varepsilon_k}(P_{\varepsilon_k} + \varepsilon_k y) \qquad z \in \overline{B_{\frac{\varrho}{\varepsilon}}}.$$

From (3.49) we may assume that there exists a closed ball $B_{\nu(\varepsilon)}$, $\nu(\varepsilon)$ sufficiently small, with more than maximum points inside. Then the contradiction follows by Lemma 4.2 in [49]. ■

3.4 Location of the least energy solution

In this section we use some results proved in Chapter 2. First of all to better understand the sequel, we will recall a Lemma due to P.C.Fife, see [23], stated as in cite [49].

Lemma 3.4.1 *Let $\phi(x)$ be a C^2 function satisfying the elliptic equation*

$$\varepsilon^2 \left(\Sigma a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \Sigma b_j(x) \frac{\partial \phi}{\partial x_j} \right) - c(x)\phi = 0$$

in a bounded domain D, where (i) the coefficient a_{ij}, b_j are bounded and (ii) $c(x) \ge c_0 > 0$ for all $x \in D$. Then there is a positive constant μ depending only on a_{ij}, b_j, c_0 and D such that

$$|\phi(x)| \le (\sup |\phi(x)|) \exp\{-\mu\delta(x)/\varepsilon\},$$

where $\delta(x)$ is the distance from x to ∂D.

Lemma 3.4.2 *Let u_ε be a least-energy solution to the problem $(\tilde{P}_\varepsilon)$. Let $K \subset \Omega_\varepsilon$ be a compact containing the origin and $B_r^+ \subset \Omega_\varepsilon$ the set $\{z \in B_r, z_n > 0\}$. Moreover let $y = \Psi(x)$ be the diffeomorphism that straightens a boundary portion around P_ε, where P_ε is the unique maximum point of u_ε. Let w_ε be, as usual, the scaling of the least-energy solution u_ε after straightening the boundary. Then for any $\alpha > 0$, there exists ε_0 such that for $\varepsilon < \varepsilon_0$ we have two sets $\widehat{\Omega}_\varepsilon \subset \widetilde{\Omega}_\varepsilon \subset \Omega$, such that*

1. *$diam(\widehat{\Omega}_\varepsilon) < C_1\varepsilon$,*
2. $\left\| u_\varepsilon(x) - U(\frac{\Psi(x)}{\varepsilon}) \right\|_{C^2(\widehat{\Omega}_\varepsilon)} < C\frac{\alpha}{\varepsilon^2}, \qquad \|w_\varepsilon(z) - U(z)\|_{C^2(K)} < \alpha,$
3. $|u_\varepsilon(x)| < C\alpha e^{-\mu_1 \delta(x)/\varepsilon}$ *if* $x \in \Omega \backslash \widehat{\Omega}_\varepsilon$,
4. $\|u_\varepsilon\|_{H^1_{\mathcal{D}}(\Omega\backslash\widetilde{\Omega}_\varepsilon)} \le \left(C\alpha \frac{e^{-\mu_1\delta(x)/\varepsilon}}{\varepsilon^2}\right)^{1/2}, \qquad \|w_\varepsilon\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon\backslash B_r^+)} \le \left(C\alpha \frac{e^{-\mu_1\delta_0/\varepsilon}}{\varepsilon^n}\right)^{1/2}.$

where $\delta(x) = dist(x, \partial\widehat{\Omega}_\varepsilon \setminus \partial\Omega)$ with C_1, C, μ_1, δ_0 are positive constants that depending only on Ω.

Proof. We carry on the proof in two steps.

Step 1. From Corollary 3.3.8 we know that, for ε small enough $P_\varepsilon \in \partial_{\mathcal{N}}\Omega$. Moreover if we denote

$$d = \frac{dist(P_\varepsilon, \mathcal{I}_\Omega)}{\varepsilon} \tag{3.50}$$

that is, the distance of the point $Q_\varepsilon = \Psi(P_\varepsilon)$ from the interface $\mathcal{I}_{\Omega_\varepsilon}$ in the scaled domain after a straightening of the boundary around the point P_ε, by Corollary 3.3.6 we have that $d \to +\infty$ for ε sufficiently small. Then we define

$$v_\varepsilon(y) \equiv u_\varepsilon(\Phi(y)) \quad y \in \overline{B_s^+} := \{y \in \overline{B_s(Q_\varepsilon)}, y_n \ge 0\},$$

with s sufficiently small. Then the scaled function

$$w_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z) \tag{3.51}$$

solves an elliptic problem in some (rescaled) domain with Neumann boundary conditions on a flat boundary. In this case the function $\tilde{w}_\varepsilon$ is defined by reflection as

$$\tilde{w}_\varepsilon := \begin{cases} w_\varepsilon(z) & if \quad z \in \overline{B^+_{\frac{s}{\varepsilon}}} \\ w_\varepsilon(z', -z_n) & if \quad z \in B^-_{\frac{s}{\varepsilon}}. \end{cases} \tag{3.52}$$

Then, denoting for the sake of simplicity $\tilde{w}_\varepsilon$ as w_ε, by equations (3.21), (3.22), we know that, given a compact $K \subset \Omega_\varepsilon$ containing the origin and $\alpha > 0$ there exist ε_0 such that,

$$\|w_\varepsilon(z) - U(z)\|_{C^2(K)} < \alpha \qquad \forall \varepsilon < \varepsilon_0 \tag{3.53}$$

where, as usual, U the ground state solution of Problem (1.1). Then, since $\nabla\Psi(\cdot), \nabla^2\Psi(\cdot)$ are bounded, from equation (3.53) we get also

$$\left\| u_\varepsilon(x) - U(\frac{\Psi(x)}{\varepsilon}) \right\|_{C^2(\tilde{\Omega}_\varepsilon^r)} \leq C\frac{\alpha}{\varepsilon^2} \qquad \forall \varepsilon < \varepsilon_0$$

where $\tilde{\Omega}_\varepsilon^r = \Phi(B_{r\varepsilon}^+)$, $\tilde{\Omega}_\varepsilon^r \in \Omega$, with $diam(\tilde{\Omega}_\varepsilon^r) \leq C_1\varepsilon$, $B_r^+ \subset K$ as $\varepsilon \to 0$, and C, C_1 are constants that depend on the domain Ω, proving 1. and 2. We note that if $P_\varepsilon \to P_0 \in \mathcal{I}_\Omega$, from Corollary 3.3.6, we have that the size of $\tilde{\Omega}_\varepsilon^r$ becomes small faster than the rate of convergence of P_ε to P_0.
Then choose $B_r^+ \subset\subset K$. From equation (3.53) we have

$$w_\varepsilon(z) \leq U(z) + \alpha. \tag{3.54}$$

We know that in the flattened and scaled domain Ω_ε, $d \to +\infty$, by (3.50). Then from equation (1.3), we can choose the compact K and therefore r sufficiently big, such that the following estimate holds

$$U(z) \leq \alpha_{n,p} e^{-\frac{r}{2}} < \alpha \qquad \text{if } \frac{r}{2} \leq |z| \leq r.$$

From equation (3.54) we get

$$w_\varepsilon(z) \leq 2\alpha \qquad \text{if } \frac{r}{2} \leq |z| \leq r.$$

Since the scaling that we are using further throughout the paper does not modify the values of the function but only the *size* of the domain, it follows that

$$u_\varepsilon(x) \leq 2\alpha \quad \text{if } x \in \Phi(\tilde{B}) \cap \Omega$$

where $\tilde{B} := B_{\varepsilon r}^+ \backslash B^+{}_{\frac{\varepsilon r}{2}}$.
We know that $\int_\Omega |\nabla u_\varepsilon|^2 < +\infty$. Moreover from Theorem 3.3.9, for ε sufficiently small, we can affirm that u_ε has one local maximum. Then it follows that the set $\{x \in \Omega : u_\varepsilon(x) > 2\alpha\}$ has one connected component. Then we have that

$$u_\varepsilon(x) \leq 2\alpha \qquad \text{if } x \in \Omega\backslash\widehat{\Omega}_\varepsilon,$$

where, as above, $\widehat{\Omega}_\varepsilon = \Phi(B_{\frac{\varepsilon r}{2}}^+)$. Now as in [49], we can apply Lemma 3.4.1 to u_ε in $(\Omega \setminus \widehat{\Omega}_\varepsilon)$ getting that

$$u_\varepsilon(x) < k_1 \alpha e^{-\mu\delta(x)/\varepsilon}, \tag{3.55}$$

for some positive constant k_1, since $u_\varepsilon(x) < 2\alpha$ in this set. We want to note that the estimate (3.55) keeps an exponential decay for all the interior points of the set $(\Omega \setminus \widehat{\Omega}_\varepsilon)$, that is where $\delta(x) > 0$, see Lemma 3.4.1. Now, since it is important for the sequel, we want to get exponential decay also for the boundary points $x \in \partial(\Omega \setminus \widehat{\Omega}_\varepsilon) \cap \partial\Omega$, that is where $\delta(x) = 0$ in the equation (3.55). For the boundary Neumann part, after straightening (around each point), denoting $\tilde{v}_\varepsilon$ the reflected function around these points we get

$$\varepsilon^2\left(\sum_{i,j}^{n} a_{ij}(y)\frac{\partial^2 \tilde{v}_\varepsilon}{\partial y_i y_j} + \sum_{j}^{n} b_j(y)\frac{\partial \tilde{v}_\varepsilon}{\partial y_j}\right) - \tilde{v}_\varepsilon(y) + \tilde{v}_\varepsilon^p(y) = 0 \qquad \text{in } \tilde{S},$$

where $\tilde{S}$ is a strip around some Neumann boundary point.

Then we use Lemma 3.4.1 again for this problem in a strip around $\partial(\Omega \backslash \widehat{\Omega}_\varepsilon) \cap \partial_{\mathcal{N}}\Omega$, getting

$$\tilde{v}_\varepsilon(y) < k_2 \alpha e^{-\mu\delta(y)/\varepsilon}, \tag{3.56}$$

for some positive constant k_2 and where $\delta(y)$ is the distance from $\partial\tilde{S}$. Then, from (3.55) and since $u_\varepsilon \equiv 0$ on $\overline{\partial_{\mathcal{D}}\Omega}$, by continuity, also for all the points $x \in \partial(\Omega\backslash\widehat{\Omega}_\varepsilon) \cap \overline{\partial_{\mathcal{D}}\Omega}$ we have

$$u_\varepsilon(x) < k_1 \alpha e^{-\mu\delta(x)/\varepsilon}, \tag{3.57}$$

where $\delta(x)$ is the distance from $\partial\widehat{\Omega}_\varepsilon \setminus \partial\Omega$. Finally from equations (3.55), (3.56) and (3.57), we prove (3).
This means that in the scaled domain Ω_ε we have a similar estimate for the scaled function $w_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z)$, that is

$$|w_\varepsilon(z)| < C\alpha e^{-\mu_1\delta(z)} \qquad \text{if } z \in \Omega_\varepsilon \setminus B_{\frac{r}{2}} \tag{3.58}$$

where $\delta(z)$ is the distance from $\partial B_{\frac{r}{2}} \cap \Omega_\varepsilon$.

We are assuming that μ_1 is uniform on ε (see [23], Lemma 4.2).

Step 2. Since from the boundary conditions u_ε vanishes on an open set $\partial_{\mathcal{D}}\Omega \subset \partial\Omega$, a Poincaré inequality holds true, that is

$$\|u_\varepsilon\|_{L^2(\Omega)} \le C_p \|\nabla u_\varepsilon\|_{L^2(\Omega)}.$$

The last inequality is very important in our case because it implies the equivalence with the gradient norm in $L^2(\Omega)$, in fact

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)} \le \sqrt{\|u_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}} \le \sqrt{C_p^2 + 1}\|\nabla u_\varepsilon\|_{L^2(\Omega)}. \tag{3.59}$$

Now, let r^* be such that $\frac{r}{2} < r^* < r$ and $\widetilde{\Omega_\varepsilon} = \Phi(B^+_{\varepsilon r^*})$. From equation $(\tilde{P}_\varepsilon)$, integrating by parts we can write

$$\varepsilon^2 \int_{\Omega\backslash\widetilde{\Omega}_\varepsilon} |\nabla u_\varepsilon|^2 \le \int_{\Omega\backslash\widetilde{\Omega}_\varepsilon} \left(u_\varepsilon^{p+1} + u_\varepsilon^2\right) + \int_{\partial(\Omega\backslash\widetilde{\Omega}_\varepsilon)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial\nu} \le C\alpha e^{-\mu_1\delta(x)/\varepsilon} \tag{3.60}$$

where $\delta(x)$ is the distance from $\partial\widehat{\Omega}_\varepsilon \setminus \partial\Omega$ and ν is the outer normal at the boundary of the domain $\Omega\backslash\widetilde{\Omega}_\varepsilon$. As $\varepsilon \to 0$, from Corollary 3.3.6 we know that $dist(P_\varepsilon, \mathcal{I}_\Omega)$ goes to zero slower than ε, and that, as seen above, $diam(\widetilde{\Omega}_\varepsilon) = O(\varepsilon)$. Then we get, in the domain $\Omega\backslash\widetilde{\Omega}_\varepsilon$, for all ε sufficiently small, an equivalent norm as equation (3.59) states. In fact a Poincaré inequality holds, since an open set $\Upsilon \subset \partial_{\mathcal{D}}\Omega$ on which $u_\varepsilon \equiv 0$, is always contained in the boundary of $\Omega\backslash\widetilde{\Omega}_\varepsilon$. Then from equation (3.60) and from the equivalence of the norm we have that

$$\|u_\varepsilon\|_{H^1_{\mathcal{D}}(\Omega\backslash\tilde{\Omega}_\varepsilon)} \le \left(C\alpha \frac{e^{-\mu_1\delta(x)/\varepsilon}}{\varepsilon^2} \right)^{1/2}$$

Now we want to obtain a similar estimate for the scaled function w_ε. From equation (3.60), by a straight calculation we have

$$\int_{\Omega_\varepsilon\backslash B_r^+} |\nabla w_\varepsilon(z)|^2 \le \frac{\mathcal{K}}{\varepsilon^{n-2}} \int_{\Omega\backslash\tilde{\Omega}_\varepsilon^r} |\nabla u_\varepsilon|^2 \le C\alpha \frac{e^{-\mu_1\delta(x)/\varepsilon}}{\varepsilon^n}, \tag{3.61}$$

for some constant $\mathcal{K}$ and where, as above, $\tilde{\Omega}_\varepsilon^r = \Phi(B_{r\varepsilon}^+)$. We want to point out that in the last equation, in the domain $\Omega \setminus \tilde{\Omega}_\varepsilon^r$, $\delta(x) \ge \delta_0 > 0$. We note also that in the scaled domain $\dfrac{dist(P_\varepsilon, \mathcal{I}_\Omega)}{\varepsilon} = +\infty$, see Corollary 3.3.6; then again a Poincaré inequality holds in the scaled domain. From equation (3.61) we get

$$\|w_\varepsilon\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon\backslash B_r^+)} \le \left(C\alpha \frac{e^{-\mu_1\delta_0/\varepsilon}}{\varepsilon^n} \right)^{1/2} \to 0 \quad \text{as } \varepsilon \to 0.$$

proving (4). ■

Remark 3.4.3 *This Lemma is very important because it states that the least-energy solutions to the Mixed problem can be approximated, via a scaling, by the ground state solution U in a subset of Ω and outside this subset can be made arbitrarily small. This follows because Corollary 3.3.6 holds, that is, in the straightened and scaled domain the Mixed problem is similar to the Neumann one.*

Lemma 3.4.4 *Let u_ε a least energy solution to the problem $(\tilde{P}_\varepsilon)$. Suppose $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is a smooth bounded domain, $\mathcal{I}_\Omega$ denote the interface, $H|_{\mathcal{I}_\Omega}$ has a strict local maximum and ∇H points inside $\partial_{\mathcal{D}}\Omega_\varepsilon$. Define, as in the first part of paper, the manifold*

$$Z_\varepsilon = \{\hat{z}_{\varepsilon,Q} \;:\; d_{1,\varepsilon} \le d \le d_{2,\varepsilon}, Q \in \mathcal{V}\},$$

there are $d_{1,\varepsilon}$, $d_{2,\varepsilon}$, such that $d_{1,\varepsilon} \to +\infty$, $\varepsilon d_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$, $\hat{z}_{\varepsilon,Q}$ as in the equation (2.61) *and where $\mathcal{V}$ is a neighborhood of a point $\overline{Q}$ which realizes* $\max_{Q\in\mathcal{I}_\Omega} H$ *. Then as $\varepsilon \to 0$*

$$dist(w_\varepsilon, Z_\varepsilon) < \alpha,$$

where w_ε is the scaling of the least-energy solution u_ε after straightening the boundary and α a sufficiently small constant.

Proof. Recall that

$$\hat{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y)\left[(U(y) - \Xi_d(y))\chi_D(y) + \varepsilon\tilde{w}_Q(y)\chi_0(y_1 - d)\right].$$

Then we evaluate the difference

$$\begin{aligned} & \|\hat{z}_{\varepsilon,Q}(y) - w_\varepsilon(y)\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} && (3.62)\\ = \;& \|\chi_{\mu_0}(\varepsilon y)\left[(U(y) - \Xi_d(y))\chi_D(y) + \varepsilon\tilde{w}_Q(y)\chi_0(y_1 - d)\right] - w_\varepsilon(y)\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \\ \le \;& \|\left[(U(y) - \Xi_d(y)) + \varepsilon\tilde{w}_Q(y)\right] - w_\varepsilon(y)\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} < \alpha \to 0 \quad \text{if } \varepsilon \to 0. \end{aligned}$$

■

By this lemma we have showed that the least-energy solution belongs to the set Z_ε, that is, as said before, when ∇H points inside $\partial_{\mathcal{D}}\Omega_\varepsilon$ we can then apply the second statement in Proposition 2.1.3 with

$$f(\varepsilon) = \left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right),$$

by Proposition 2.2.12 and (2.91). These two yield that $u_\varepsilon = \hat{z}_{\varepsilon,Q} + \omega(\hat{z}_{\varepsilon,Q})$, where the distance of Q from $\mathcal{I}_\Omega$ is of order $\varepsilon|\log\varepsilon|$, as in Theorem 2.0.1, see Remark 2.0.2 (b).

It is well known, see Theorem 1.6.3 in Chapter 1, that in the Neumann case (1.15) if the unique maximum point P_ε of least-energy solutions u_ε concentrate on the boundary of the domain at a point P_0, then as $\varepsilon \to 0$ one has

$$H(P_0) = \lim_{\varepsilon\to 0} H(P_\varepsilon) = \max_{P\in\partial\Omega} H(P) \tag{3.63}$$

In the following we want to go ahead in such direction to find other results for the Mixed case $(\tilde{P}_\varepsilon)$. In the next theorem, we will show that if a least-energy solution concentrate at a point of the interface, then here the curvature H is maximal.

Theorem 3.4.5 *Let u_ε be a least energy solution to the Mixed problem $(\tilde{P}_\varepsilon)$. We suppose that, up to a subsequence, as $\varepsilon \to 0$, $P_\varepsilon \to \widehat{P} \in \mathcal{I}_\Omega$. Then $\widehat{P}$ is such that*

$$H(\widehat{P}) = \max_{P\in\overline{\partial_{\mathcal{N}}\Omega}} H(P)$$

Proof. Here we use also some results proved in the first part of this paper (see [26]). In particular from Lemma 3.4.4 and Proposition 2.3 in [26] we have an *uniqueness result.* From Proposition 4.1 in [26], we have the following expansion for the functional $\tilde{I}_\varepsilon(u)$ associated to the Mixed problem:

$$\tilde{I}_\varepsilon(u_\varepsilon) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(P_\varepsilon) + e^{-2d(1+o(1))} + O(\varepsilon^2), \tag{3.64}$$

as $\varepsilon \to 0$ and $d \to +\infty$. We want to remember that, see [26]:

$$d = dist(Q_\varepsilon, \mathcal{I}_{\Omega_\varepsilon}),$$

and $\tilde{C}_0$, $\tilde{C}_1$ are constants such that

$$\tilde{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n_+} U^{p+1} dx, \qquad \tilde{C}_1 = \left(\int_0^\infty r^n U_r^2 dr\right) \int_{S^n_+} y_n |y'|^2 d\sigma.$$

We note also that with P_ε we refer to a point in the domain Ω and otherwise with Q_ε we refer, after a flattening, to a point in the scaled domain Ω_ε, see Figure 3.2.

By contradiction, we suppose that there exists a point $P^* \in \overline{\partial_{\mathcal{N}} \Omega}$ such that

$$H(\widehat{P}) < H(P^*) \tag{3.65}$$

Then we evaluate equation (3.64), over another function, say $\hat{z}_{\varepsilon,P^*}$, getting

$$\tilde{I}_\varepsilon(\hat{z}_{\varepsilon,P^*}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(P^*) + e^{-2d(1+o(1))} + O(\varepsilon^2).$$

We remember that $\hat{z}_{\varepsilon,Q}$ is represented by

$$\hat{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y) \left[(U(y) - \Xi_d(y)) \chi_D(y) + \varepsilon \tilde{w}_Q(y) \chi_0(y_1 - d) \right]. \tag{3.66}$$

We suggest to refer the Subsection 3.2 of [26] to better understand the meaning of all terms.

Then if we arrange the last equation, we have

$$\tilde{I}_\varepsilon(\hat{z}_{\varepsilon,P^*}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(P_\varepsilon) - A_\varepsilon(P^*, P_\varepsilon) + e^{-2d(1+o(1))} + O(\varepsilon^2) \tag{3.67}$$

where

$$A_\varepsilon(P^*, P_\varepsilon) = \tilde{C}_1 \varepsilon H(P^*) - \tilde{C}_1 \varepsilon H(P_\varepsilon),$$

and $A_\varepsilon(P^*, P_\varepsilon) \to 0$ as $\varepsilon \to 0$. Next we can show that, as $\varepsilon \to 0$

$$\begin{cases} A_\varepsilon(P^*, P_\varepsilon) > 0 \\ |A_\varepsilon(P^*, P_\varepsilon)| = O(\varepsilon^s) \quad 0 < s < 2 \end{cases} \tag{3.68}$$

The former of (3.68) easily follow from equation (3.65). To prove the latter one, we note that

$$\varepsilon^2 \tilde{C}_1 \frac{|P^* - P_\varepsilon|}{\varepsilon} \frac{1}{|\, H \,|_\infty} \leq |A_\varepsilon(P^*, P_\varepsilon)| \leq \varepsilon^2 \tilde{C}_1 \,|\, H \,|_\infty \frac{|P^* - P_\varepsilon|}{\varepsilon}$$

From Corollary 3.3.6, we know that for ε sufficiently small, $\dfrac{|P^* - P_\varepsilon|}{\varepsilon} \to +\infty$, showing the last of (3.68). Then for ε sufficiently small, from equation (3.64), (3.67) (3.68) we have

$$\tilde{I}_\varepsilon(\hat{z}_{\varepsilon,P^*}) < \tilde{I}_\varepsilon(u_\varepsilon).$$

This obviously is a contradiction, since, by hypothesis, u_ε is a least energy solution. ■

Next we will prove another important result

Corollary 3.4.6 *Let u_ε be a least-energy solution to the Mixed problem $(\tilde{P}_\varepsilon)$ and P_ε such that $u(P_\varepsilon) = \max_{x\in\overline{\Omega}} u_\varepsilon(x)$. Moreover suppose that there exists a point $P^* \in \partial_{\mathcal{D}}\Omega$, where P^* is such that*

$$H(P^*) = \max_{P\in\partial\Omega} H(P),$$

and let $\widehat{P}$ be the unique point, such that

$$H(\widehat{P}) = \max_{P\in\overline{\partial_{\mathcal{N}}\Omega}} H(P).$$

Then

$$P_\varepsilon \to \widehat{P} \tag{3.69}$$

Proof. By Theorem 3.3.1 we can affirm that, up to a subsequence, the point P_ε cannot concentrate on the Dirichlet boundary part. Then P_ε must lie on the Neumann boundary part $\partial_{\mathcal{N}}\Omega$. We have two cases:

(i) if P_ε concentrate on the interior of boundary Neumann part $\partial_{\mathcal{N}}\Omega$ we can joint with the Neumann case and we know (see [49]) that $\widehat{P}$ maximize the mean curvature $H(\cdot)$;

(ii) if P_ε concentrate on the interface $\mathcal{I}_\Omega$ then from Theorem 3.4.5, $\widehat{P}$ maximize the mean curvature $H(\cdot)$.

In both cases (i) and (ii), hypothesis (3.69) is satisfied, concluding the proof. ■

Remark 3.4.7 *In the Corollary 3.4.6 we require that the point $\widehat{P}$ is unique because, otherwise, we need of more terms in the expansion* (3.64) *that contain a higher order of the derivatives of the boundary. However we can consider a particular case, very interesting.*

Proposition 3.4.8 *Let u_ε be a least-energy solution to the problem $(\tilde{P}_\varepsilon)$. Suppose that the maximum of $H(\cdot)$ on the closure of the Neumann boundary part $\overline{\partial_{\mathcal{N}}\Omega}$, is achieved both at the interior and at the interface $\mathcal{I}_\Omega$ that is, there exist $\widetilde{P} \in \partial_{\mathcal{N}}\Omega$ and $\widehat{P} \in \mathcal{I}_\Omega$ such that*

$$H_{\max}(\widetilde{P}) = H_{\max}(\widehat{P}).$$

Then the least-energy solution u_ε peaks at point $\widetilde{P}$.

PROOF. We know that, up to a subsequence, $P_\varepsilon \to P \in \overline{\partial_{\mathcal{N}}\Omega}$ and that one has the following energy expansions:

$$\tilde{I}_\varepsilon(u_\varepsilon) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(P_\varepsilon) + e^{-2\frac{d_\varepsilon}{\varepsilon}(1+o(1))} + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0. \tag{3.70}$$

Peaking at the interior, d_ε remains a constant recovering the expansion corresponding to the Neumann problem, see Theorem 1.6.1. Otherwise if the concentration points converge to the interface, we have that $d_\varepsilon \simeq \varepsilon|\log\varepsilon|$. It follows that the least-energy solution u_ε peaks at the point $\widetilde{P}$, since the boundary expansion (3.70) has an extra $e^{-2\frac{d_\varepsilon}{\varepsilon}}$ term which is positive, concluding the proof. ■

Chapter 4

The shape of the least-energy solution

In this chapter we are interested in the construction of the shape of positive least-energy solutions (Mountain Pass Type Solutions) for a mixed semilinear Dirichlet-Neumann problem. We analyze the case $n = 2$ and $p = 3$, that is:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^3 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}} \Omega; & u = 0 \text{ on } \partial_{\mathcal{D}} \Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \qquad (\widetilde{\tilde{P}}_\varepsilon)$$

where Ω is an elliptical domain of $\mathbb{R}^2$. We want to point out that in the two dimensional case the interface $\mathcal{I}_\Omega$ is a point. Figure 4.1 shows how Dirichlet and Neumann conditions are imposed.

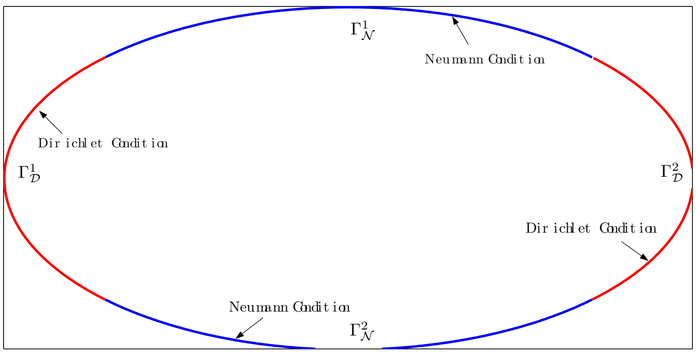

Figure 4.1: The Elliptical Domain Ω.

In order to construct the shape of the positive least energy solution of problem $(\tilde{\tilde{P}}_\varepsilon)$ we take advantage of Mountain Pass Algorithm due to Y.S. Choi and P.J. McKenna, see [14], taking into account the intrinsic differences.

Indeed in [14] the authors applied such method to study some elliptic equations in square domains in order to show the capability of this algorithm. On the contrary, in our case, we consider a perturbative problem with mixed boundary conditions and defined in an elliptical domain contained in $\mathbb{R}^2$. In fact we need to consider the problem in a domain with non constant mean curvature H because our goal is finding a spike layer solution concentrating at the interface.

From numerical point of view a curved boundary domain is difficult *to mesh* in order to define all the *discrete variables* that the algorithm requires. Moreover we impose mixed conditions on four disjoint subsets of the boundary $\partial\Omega$, as shown in Figure 4.1, in such a way that Dirichlet conditions are imposed on the two boundary parts, $\Gamma^1_{\mathcal{D}}$ and $\Gamma^2_{\mathcal{D}}$, where the curvature H reaches its maximum value. We need Dirichlet conditions on $\Gamma^1_{\mathcal{D}}$ and $\Gamma^2_{\mathcal{D}}$ because this implies that least energy solutions do not concentrate where the mean curvature is maximal. In this way we get solutions to $(\tilde{\tilde{P}}_\varepsilon)$ concentrating at the interface $\mathcal{I}_\Omega$ as shown in Chapter 3.

Both from the mathematical point of view and from the numerical one, all these motivations made difficult to implement the code in some *programming language*.

Solutions of problem $(\tilde{\tilde{P}}_\varepsilon)$ correspond to critical points of the functional

$$\tilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \frac{1}{4}\int_\Omega u^4; \qquad u \in X \tag{4.1}$$

where

$$X = \{u \in H^1_{\mathcal{D}}(\Omega) : u \not\equiv 0 \quad \text{and } u \geq 0 \ \text{ in } \Omega\}$$

and $H^1_{\mathcal{D}}(\Omega)$ (see Section 1.8) stands for the space of functions in $H^1(\Omega)$ which have zero trace on $\partial_{\mathcal{D}}\Omega$, i.e.

$$H^1_{\mathcal{D}}(\Omega) = \{u \in H^1(\Omega) : u|_{\partial_{\mathcal{D}}\Omega} = 0\}.$$

To apply the method of Y.S. Choi and P.J. McKenna is necessary to have the Mountain Pass geometry: it is clear that $u \equiv 0$ is a solution of problem $(\tilde{\tilde{P}}_\varepsilon)$ and that in fact this solution corresponds to a local minimum of the functional $\tilde{I}_\varepsilon$. If we pick another positive function $v \in H^1_{\mathcal{D}}(\Omega)$, then we have that $\tilde{I}_\varepsilon(tv) \to -\infty$ as $t \to +\infty$. Since for $\tilde{I}_\varepsilon$ hypothesis $(i), (ii), (iii)$ and (iv) in Section 3.1 hold together with a $(PS)_c$ condition, then we can apply the Mountain Pass Theorem (1.3.4) to get the existence of a least energy solution (Proposition 3.1.2) to problem $(\tilde{\tilde{P}}_\varepsilon)$.

In this chapter, following [14], we implement an algorithm that will always converge to a solution with the Mountain Pass (least-energy) property.

On a finite-dimensional approximating subspace, one takes a piecewise linear path from the local minimum to some point v belonging to $H^1_{\mathcal{D}}(\Omega)$ such that the value $\tilde{I}_\varepsilon(v)$ is lower than that of the minimum. One finds the maximum of $\tilde{I}_\varepsilon$ along that path and then one deforms the path by pushing the point at which the local maximum is located in the *steepest descent direction.* This direction is located by minimizing the Fréchet derivative of $\tilde{I}_\varepsilon$ (with an appropriate norm to use) at the local maximum point. One repeat this step, stopping only when the critical point is reached, that is when no further lowering of the local maximum is possible.

4.1 The Mountain Pass geometry

Since it is relevant for the application of the algorithm, in this section we show that the functional associated to the differential problem $(\tilde{\tilde{P}}_\varepsilon)$ has the Mountain Pass geometry. A crucial point for the proof is based on the use of a Poincaré inequality.

We know that $\tilde{I}_\varepsilon(0) = 0$. We are looking for positive solutions of problem $(\tilde{\tilde{P}}_\varepsilon)$, with a nonlinear term $f(t) = t^3$, and F given by

$$F(t) = \int_0^t f(s)ds.$$

We have also that (see *(ii)* and *(iii)*, Section 3.1)

$$\begin{aligned} f(t) &= O(t^p) \quad \text{as } t \to +\infty, \\ f(t) &= o(t) \quad \text{for } t \to 0. \end{aligned} \tag{4.2}$$

Then by equations (4.2), we get that $\forall M > 0 \quad \exists \delta = \delta_M$ such that

$$\begin{aligned} f(u) &\le Mu + \delta_M u^p \qquad \forall u \ge 0, \\ F(u) &\le \frac{M}{2}u^2 + \frac{\delta_M}{p+1}u^{p+1}, \qquad \forall u \ge 0. \end{aligned} \tag{4.3}$$

By the Poincaré inequality, it follows that $\int_\Omega u^2 \le C_p^2 \int_\Omega |\nabla u|^2$. Then we have:

$$\varepsilon^2 \int_\Omega |\nabla u|^2 \le \int_\Omega \varepsilon^2 |\nabla u|^2 + u^2 \le \varepsilon^2(1 + C_p^2) \int_\Omega |\nabla u|^2. \tag{4.4}$$

Remark 4.1.1 *These last inequalities imply that*

$$\|u\|_{H^1_{\mathcal{D}}} = \left(\int_\Omega \varepsilon^2 |\nabla u|^2 \right)^{\frac{1}{2}}$$

is an equivalent norm to

$$\|u\|_{H^1_{\mathcal{D}}} = \left(\int_\Omega \varepsilon^2 |\nabla u|^2 + u^2 \right)^{\frac{1}{2}}$$

for the boundary problem $(\widetilde{\widetilde{P}}_\varepsilon)$.

Then by equation (4.3) and Remark 4.1.1, if we choose $M < 1$, we get:

$$\begin{aligned}
\tilde{I}_\varepsilon(u) &= \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int_\Omega F(u) \\
&\geq \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \frac{M}{2}\int_\Omega u^2 - \frac{\delta_M}{p+1}\int_\Omega u^{p+1} \\
&= \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + \frac{1}{2}(1-M)\int_\Omega u^2 - \frac{\delta_M}{p+1}\int_\Omega u^{p+1} \\
&\geq \frac{1}{2}||u||^2_{H^1_{\mathcal{D}}} - \frac{\delta_M}{p+1}||u||^{p+1}_{H^1_{\mathcal{D}}}. \qquad (4.5)
\end{aligned}$$

Equation (4.5) shows that $u \equiv 0$ is a local minimum of $\tilde{I}_\varepsilon$ satisfying *(MP1)* of Theorem 1.3.4.

We know also (see (iv), Section 3.1) that there exists a constant $\theta > 2$ such that, if $f(u) = u^3$

$$0 < \theta F(t) \leq t f(t) \quad \forall t \geq 0 \quad \textit{(Ambrosetti-Rabinowitz condition).}$$

From this inequality, trivially:

$$\frac{d}{du}\left(\frac{F(u)}{u^\theta}\right) = \frac{f(u)u^\theta - \theta u^{\theta-1}F(u)}{u^{\theta-1}u^{\theta+1}} \geq 0. \qquad (4.6)$$

Equation (4.6) implies:

$$\begin{aligned}
\frac{F(u)}{u^\theta} &\geq \frac{F(R)}{R^\theta} \qquad \forall u \geq R > 0, \\
F(u) &\geq Bu^\theta - c \qquad \forall u \geq 0, \qquad (4.7)
\end{aligned}$$

where

$$B = \frac{F(R)}{R^\theta}$$

and c is a positive constant. Then by (4.7) and (4.4) we get

$$\begin{aligned}
\tilde{I}_\varepsilon(u) &= \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int_\Omega F(u) \\
&\leq \frac{1}{2}(1 + C_p^2)\int_\Omega \varepsilon^2|\nabla u|^2 - \int_\Omega Bu^\theta + \int_\Omega c. \qquad (4.8)
\end{aligned}$$

If we pick a function $u \geq 0$ and a constant $t > 0$, we obtain:

$$\tilde{I}_\varepsilon(tu) \leq \frac{t^2}{2}(1 + C_p^2)\int_\Omega \varepsilon^2|\nabla u|^2 - t^\theta\int_\Omega Bu^\theta + \int_\Omega c. \qquad (4.9)$$

Being $\theta > 2$ one has that

$$I(tu) \to -\infty \qquad \text{as } t \to +\infty. \qquad (4.10)$$

Equation (4.10) implies that *(MP2)* of Theorem 1.3.4 holds.

The assumptions *(MP1)* and *(MP2)* together imply the Mountain Pass geometry. Then, in order to obtain a Mountain Pass critical point, we have to prove that a Palais-Smale condition holds for the functional $\tilde{I}_\varepsilon$.

4.1.1 The Palais-Smale $(PS)_c$ condition

Following [7], we first prove that every $(PS)_c$ sequence of $\tilde{I}_\varepsilon$ is bounded. Take $\{u_n\}$ Palais-Smale sequence at level c. We know that

$$\tilde{I}_\varepsilon(u_n) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u_n|^2 + u_n^2 - \int_\Omega F(u_n) \to c, \tag{4.11}$$

and

$$\tilde{I}'_\varepsilon(u_n)(u_n) \to 0. \tag{4.12}$$

From equation (4.12), it follows that:

$$\tilde{I}'_\varepsilon(u_n)(u_n) = \int_\Omega \varepsilon^2|\nabla u_n|^2 + u_n^2 - \int_\Omega f(u_n)u_n = o(1)\|u_n\|. \tag{4.13}$$

We multiply equation (4.13) by $q := \frac{1}{\theta}$, $q \in (0, \frac{1}{2})$ (see the *Ambrosetti-Rabinowitz condition*) and then we substract this from equation (4.11) getting:

$$\begin{aligned} &\frac{1}{2}\int_\Omega \varepsilon^2|\nabla u_n|^2 + u_n^2 - \int_\Omega F(u_n) - q\int_\Omega \varepsilon^2|\nabla u_n|^2 + u_n^2 + q\int_\Omega f(u_n)u_n \\ &= c + o(1)\|u_n\|. \end{aligned} \tag{4.14}$$

Then by the *Ambrosetti-Rabinowitz condition*

$$\begin{aligned} &(\frac{1}{2} - q)\int_\Omega \varepsilon^2|\nabla u_n|^2 + u_n^2 + q\int_\Omega f(u_n)u_n - \int_\Omega F(u_n) \\ &\quad \geq (\frac{1}{2} - q)\int_\Omega \varepsilon^2|\nabla u_n|^2 + u_n^2 \geq (\frac{1}{2} - q)\int_\Omega \varepsilon^2|\nabla u_n|^2. \end{aligned} \tag{4.15}$$

Equations (4.4), (4.14) and (4.15) imply:

$$c + o(1)\|u_n\|_{H^1_{\mathcal{D}}} \geq (\frac{1}{2} - q)||u_n\|^2_{H^1_{\mathcal{D}}} \tag{4.16}$$

Since $q < \frac{1}{2}$ equation (4.16) shows that the norm is bounded, that is

$$\|u_n\|_{H^1_{\mathcal{D}}} \leq const. \tag{4.17}$$

By equation (4.17)

$$u_n \rightharpoonup u_0, \tag{4.18}$$

and moreover, from equation (4.12) one has that

$$0 \leftarrow \tilde{I}'_\varepsilon(u_n)(u_n - u_0) = \int_\Omega \varepsilon^2 \nabla u_n \nabla (u_n - u_0) + u_n(u_n - u_0) - \int_\Omega f(u_n)(u_n - u_0). \tag{4.19}$$

By Sobolev Embedding we have that $u_n \to u_0$ in $L^{p+1}(\Omega)$. Then

$$\int_\Omega |f(u_n)||u_n - u_0| \le \|f(u_n)\|_{L^{\frac{p+1}{p}}} \|u_n - u_0\|_{L^{p+1}} \to 0.$$

Equation (4.19) becomes:

$$\int_\Omega \varepsilon^2 |\nabla u_n|^2 + u_n^2 - \int_\Omega \varepsilon^2 \nabla u_n \nabla u_0 - \int_\Omega u_n u_0 \to 0. \tag{4.20}$$

By equation (4.20) and by weakly convergence (4.18) $u_n \rightharpoonup u_0$ we get:

$$\|u_n\|^2 \to \|u_0\|^2. \tag{4.21}$$

By Lemma 4.1.1 below and equations (4.18) and (4.21), *Palais-Smale* $(PS)_c$ *condition* holds for $\tilde{I}_\varepsilon$.

Lemma 4.1.1 *Let H be a Hilbert space. Let u_n be a sequence belonging to H and u_0 a point of H. We suppose that:*

(i) $u_n \rightharpoonup u_0$,

(ii) $\|u_n\| \to \|u_0\|$.

Then $u_n \to u_0$ in H.

PROOF. A straight calculation implies:

$$\|u_n - u_0\|^2 = \|u_n\|^2 + \|u_0\|^2 - 2 < u_n, u_0 > \longrightarrow \|u_0\|^2 + \|u_0\|^2 - 2\|u_0\|^2 = 0. \tag{4.22}$$

■

4.2 The steepest descent direction

In this section we consider the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^3 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = g \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{4.23}$$

where $g \not\equiv 0$ and $g \in L^2(\partial_{\mathcal{N}}\Omega)$. This is important to better understand the boundary problem that we have to solve to finding the steepest descent direction v. Then, later, we will set $g \equiv 0$.

The steepest descent method is a numerical method that tracks down a local minimum of a functional without a good initial guess provided that some growth conditions of the functional at infinity is satisfied. The steepest descent direction at any $u \in H^1_{\mathcal{D}}(\Omega)$ is the direction that a given functional I decreases most rapidly per unit distance with respect to the norm in $H^1_{\mathcal{D}}(\Omega)$. With such method, we have to analyze the perturbed mixed problem $(\tilde{\tilde{P}}_\varepsilon)$ and then drawing the different shapes of the least-energy solution u_ε as $\varepsilon \to 0$.

In our case, the steepest descent direction at $u \in H^1_{\mathcal{D}}(\Omega)$ corresponds to finding the *direction* $v \in H^1_{\mathcal{D}}(\Omega)$ with $||v||_{H^1_{\mathcal{D}}} = 1$ such that

$$\frac{\tilde{I}_\varepsilon(u+\delta v) - \tilde{I}_\varepsilon(u)}{\delta}$$

will be as negative as possible when $\delta \to 0$, namely to finding the minimum of Fréchet derivative $\tilde{I}'_\varepsilon(u)v$ subject to the constraint that $||v||_{H^1_{\mathcal{D}}} = 1$. Introducing the Lagrange multiplier λ, we look for the unconstrained minimum of the functional $J_\varepsilon : H^1_{\mathcal{D}}(\Omega) \to \mathbb{R}$ defined as:

$$J_\varepsilon(v) = \int_\Omega \varepsilon^2 \nabla u \nabla v + uv - f(u)v + \lambda|\nabla v|^2 - \int_{\partial_{\mathcal{N}}\Omega} \varepsilon^2 g v \tag{4.24}$$

where $u, v \in H^1_{\mathcal{D}}(\Omega)$.

From this last equation we see that we are minimizing over $\int_\Omega |\nabla v|^2 = 1$, since by Remark 4.1.1 this quantity, for a fixed small ε, is an equivalent norm. This fact is very important because in this way is more simple to deal with equation (4.24).

Now we find the direction $v \in H^1_{\mathcal{D}}(\Omega)$ that minimizes the *Fré*chet derivative of functional J_ε defined in equation (4.24). We have

$$J'_\varepsilon(v)w = \int_\Omega \varepsilon^2 \nabla u \nabla w + uw - f(u)w + 2\lambda \nabla v \nabla w - \int_{\partial_{\mathcal{N}}\Omega} \varepsilon^2 g w \tag{4.25}$$

where $u, w \in H^1_{\mathcal{D}}$. If such a function v is a minimum of the functional J_ε (4.24), it must satisfy

$$J'_\varepsilon(v) = 0 \qquad \text{i.e.} \tag{4.26}$$

$$\int_\Omega \varepsilon^2 \nabla u \nabla w + uw - f(u)w - \int_{\partial_{\mathcal{N}}\Omega} \varepsilon^2 g w = -\int_\Omega 2\lambda \nabla v \nabla w. \tag{4.27}$$

To finding the direction v, we have to write the partial differential equation associated to the weak problem (4.27). To do that we denote $\overline{v}$ as:

$$\overline{v} := 2\lambda v + \varepsilon^2 u. \tag{4.28}$$

Note that $\overline{v} \in H^1_{\mathcal{D}}(\Omega)$ and that by linearity one has

$$\nabla \overline{v} = 2\lambda \nabla v + \varepsilon^2 \nabla u. \tag{4.29}$$

Moreover we define a function l as

$$l(u) := f(u) - u. \tag{4.30}$$

Now we arrange equation (4.27) in this way:

$$\int_\Omega \nabla w(\varepsilon^2\nabla u + 2\lambda\nabla v) + w(u - f(u)) - \int_{\partial_\mathcal{N}\Omega} \varepsilon^2 g w = 0. \tag{4.31}$$

By equations (4.28), (4.29) and (4.30), we get:

$$\int_\Omega \nabla w \nabla \overline{v} - l(u)w - \int_{\partial_\mathcal{N}\Omega} \varepsilon^2 g w = 0, \tag{4.32}$$

and integrating by parts the equation (4.32) becomes:

$$\int_\Omega -\triangle\overline{v}w - l(u)w + \int_{\partial_\mathcal{N}\Omega} \frac{\partial\overline{v}}{\partial n} w - \int_{\partial_\mathcal{N}\Omega} \varepsilon^2 g w = 0. \tag{4.33}$$

Then $\overline{v} \in H^1_\mathcal{D}(\Omega)$ corresponds to a weak solution for the problem:

$$\begin{cases} -\Delta\overline{v} = l(u) & \text{in } \Omega; \\ \frac{\partial\overline{v}}{\partial\nu} = 0 \text{ on } \partial_\mathcal{N}\Omega; & \overline{v} = 0 \text{ on } \partial_\mathcal{D}\Omega; \\ \overline{v} > 0 & \text{in } \Omega, \end{cases} \tag{$P_{\overline{v}}$}$$

where we have set $g \equiv 0$. If we find the weak solution $\overline{v}$ to problem $(P_{\overline{v}})$, by equation (4.28) we get the steepest descent direction v. From equation (4.28), $|\lambda|$ is determined so that $||v||_{H^1_\mathcal{D}} = 1$. To find the sign of λ we have to consider

$$\begin{aligned} \frac{\tilde{I}_\varepsilon(u+\delta v) - \tilde{I}_\varepsilon(u)}{\delta} &= \int_\Omega \varepsilon^2\nabla u\nabla v + uv - f(u)v - \int_{\partial_\mathcal{N}\Omega} \varepsilon^2 g v + o(\delta) \\ &= \int_\Omega -2\lambda\nabla v\nabla v + o(\delta) \\ &= -2\lambda + o(\delta) \end{aligned}$$

In the last equality, we have used the fact that v is a weak solution of equation (4.27) and that $g \equiv 0$ on $\partial_\mathcal{N}\Omega$. As $\delta \to 0$ the first term on right-hand side dominates and λ has to be chosen positive so that v represents the steepest descent direction.

4.3 Solution of problem $(P_{\overline{v}})$

In order to solve problem $(P_{\overline{v}})$ we need to find a numerical approximation of the solution $\overline{v}$, with $l(u)$ known at the mesh points. We use the finite element method which is the most appropriate for approximating the solutions of second-order problem posed in variational form over a given functional space V. A well-known approach to solve such problems is Galerkin's method, which consists in defining similar problems, called *discrete problems*, over finite-dimensional subspace V_h of the space V.

The finite element method is a Galerkin's method characterized by three basic aspects in the construction of the space V_h:

1. a *triangulation* $\mathcal{T}_h$ is established over the set $\overline{\Omega}$, that is, the set $\overline{\Omega}$ is written as a finite union of finite elements $K \in \mathcal{T}_h$;
2. the function $v_h \in V_h$ are *piecewise polynomials*, i.e., the spaces $P_K = \{v_{h|K} : v_h \in V_h\}$ consist of polynomials;
3. there should exist a basis in the space V_h whose functions have a compact support.

4.3.1 Numerical approximation

To solve problem $(P_{\overline{v}})$, we consider the linear abstract variational problem:

Find $u \in V$ such that

$$\forall v \in V, \qquad a(u, v) = f(v), \tag{4.34}$$

where the space V, the bilinear form $a(\cdot, \cdot)$ and the linear form f satisfy the assumptions of the Lax-Milgram lemma, see Section 1.7. In our case

- $V = H^1_{\mathcal{D}}(\Omega)$,
- $a(u, v) = \displaystyle\int_\Omega \nabla u \nabla v$,
- $f(v) = \displaystyle\int_\Omega l v$,

where l is defined in (4.30).

Then we have to construct a similar problem in finite-dimensional subspace of the space V. With any V_h of V we associate the *discrete problem*:

Find $u_h \in V_h$ such that

$$\forall v_h \in V_h, \qquad a(u_h, v_h) = f(v_h). \tag{4.35}$$

By Lax-Milgram lemma , we infer that such a problem has only one solution u_h. We denote N_h the number of all mesh points of the domain without consider the mesh points belonging to the Dirichlet boundary. We use a piecewise linear basis functions φ_j satisfying

$$\varphi_j(N_i) = \delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

where N_i is a generic mesh point and $i, j = 1, 2, \ldots, N_h$. Hence we express the *discrete solution* u_h as

$$u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x), \qquad \forall x \in \Omega \cup \partial_{\mathcal{N}}\Omega, \quad u_j = u_h(N_j),$$

and $v_h \in V_h$ as

$$v_h(x) = \sum_{i=1}^{N_h} v_i \varphi_i(x), \qquad \forall x \in \Omega \cup \partial_{\mathcal{N}}\Omega., \quad v_i = v_i(N_i).$$

Remark 4.3.1 *We note that in the numeration N_h of mesh points, we do not have to consider the mesh points belonging to the Dirichlet boundary $\partial_{\mathcal{D}}\Omega$ because at such points the discrete solution u_h must be set equal to zero.*

Then we have to find the coefficients u_j such that they are solutions of the linear system

$$\sum_{j=1}^{N_h} a(\varphi_j, \varphi_i) u_j = f(\varphi_i), \qquad i = 1, 2, \ldots, N_h. \tag{4.36}$$

From equation (4.36) we get

$$A\mathbf{u} = \mathbf{b} \tag{4.37}$$

where A is a $N_h \times N_h$ matrix given by

$$A = [a_{ij}], \qquad \text{with} \qquad a_{ij} = \int_\Omega \nabla\varphi_j \nabla\varphi_i,$$

and $\mathbf{u}$ and $\mathbf{b}$ are vectors defined as

$$\mathbf{u} = [u_j] \quad \text{with} \quad u_j = u_h(N_j), \qquad \mathbf{b} = [b_i] \quad \text{with} \quad \mathrm{b}_i = \int_\Omega f\varphi_i.$$

All the integrals in the above expressions are been approximated by Gaussian 2-d quadrature rule and 2-d composite trapezoidal rule.

Proposition 4.3.2 *Let u_h be the discrete solution to the problem (4.35). Then the H^1 norm is given numerically by*

$$||u_h||_{H^1} = (\mathbf{u}^T A\mathbf{u})^{\frac{1}{2}}, \tag{4.38}$$

where A and $\mathbf{u}$ are as in equation (4.37).

Proof. By a straight calculation, one has that:

$$\begin{aligned}
\int_\Omega \nabla u_h \nabla u_h &= \\
a\left(\sum_{j=1}^{N_h} u_j\varphi_j, \sum_{i=1}^{N_h} u_i\varphi_i\right) &= \sum_{j=1}^{N_h}\sum_{i=1}^{N_h} a(u_j\varphi_j, u_i\varphi_i) \\
\sum_{j=1}^{N_h}\sum_{i=1}^{N_h} u_i a(\varphi_j, \varphi_i) u_j &= \sum_{j=1}^{N_h}\sum_{i=1}^{N_h} u_i a_{ij} u_j \\
&= \mathbf{u}^T A \mathbf{u}.
\end{aligned}$$

■

By Proposition (4.3.2) we can find the H^1 norm of approximated functions, solutions, and also the H^1 distance between any two given functions.

Now, another important goal is to create a mesh for the domain Ω which is elliptical. With respect to the domains considered in [14], in our case the main difficulty is that the boundary is curved. In general a mesh requires a

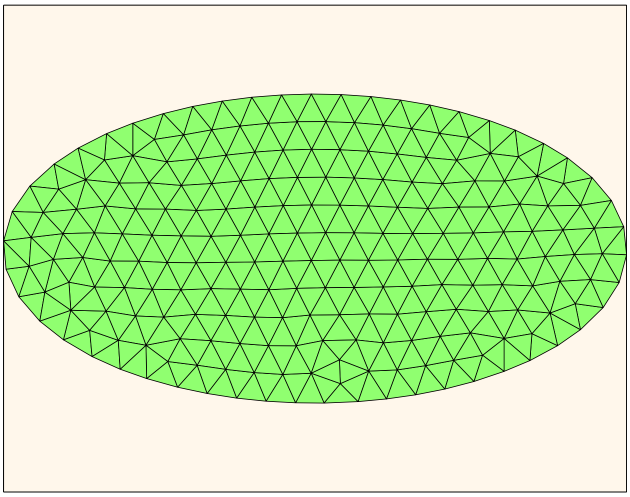

Figure 4.2: Subdivision of the domain Ω into triangles.

choice of mesh points (vertex nodes) and a triangulation, see Figure 4.2. The mesh should have small elements to resolve the fine details of the geometry, but larger sizes, where possible, to reduce the total number of nodes. We have chosen triangular elements because triangles are obviously better at approximating a curved boundary. We subdivide the basic region Ω into triangles. The union of these triangles will be a polygon Ω^h. In general, if $\partial\Omega$ is a curved boundary, there will be a nonempty *skin* $\Omega \setminus \Omega^h$. We assume that Ω^h is a subset of Ω and that no vertex of any triangles lies along the edge of another.

4.4 The numerical algorithm

In this section, since we have used the same algorithm proposed by Y.S.Choi and P.J.McKenna to get the shape of the least energy solution to the problem $(\tilde{\tilde{P}}_\varepsilon)$, we outline and rewrite, for completeness and a better understanding, the main steps in finding an approximate critical point as made by the authors in [14].

First, we initialize a path from the local minimum w to the point of lower altitude e. Second, we find the maximum of $\tilde{I}_\varepsilon$ along this path. Third, we deform the path in such a way as to make the maximum along the path decrease as fast as possible. Finally, in the last step we decide whether to stop, if the maximum turns out to have been a critical point, or repeat the process again.

4.4.1 Step 1 of the algorithm

Since for our problem $(\tilde{\tilde{P}}_\varepsilon)$ the Mountain Pass geometry holds, there exist the local minimum w and the point of lower altitude e. We prescribe a fixed number N, which determine the number of linear segments in the piecewise linear path Γ_0. One possible starting path would be simply to take a straight line path from e, to w. It is clear that the initial point e was not optimal. Thus we are led to make an improvement in the initial path by choosing

$$g(i) = w + \frac{i}{N}(e - w), \qquad 1 \leq i \leq N$$

and then evaluating $\tilde{I}_\varepsilon(g(i))$ for $i \geq 1$ until we find at $i = i_0$ that

$$\tilde{I}_\varepsilon(g(i)) \leq \tilde{I}_\varepsilon(w).$$

Having found such an i_0, we replace the original e with $g(i_0)$, shortening the distance between w and e. We now choose our initial Γ_0 to be the straight line between w and e, and the corresponding $g(i)$ to be the equally spaced points along this path.

4.4.2 Step 2 of the algorithm

With a given discretized path $g(i)$, the value of $i = i_m$ at which a maximum of $\tilde{I}_\varepsilon(g(i))$ occurs is computed. If the number of segments N is large enough, then $1 \leq i_m \leq N - 1$, because w is a local minimum and $\tilde{I}_\varepsilon(e) \leq \tilde{I}_\varepsilon(w)$. This first estimate of the maximum along the path is a little crude. We try to refine it by the procedures below. Since the path is a discretized one, we will try to locate a higher maximum between $i = i_{m-1}$ and $i = i_{m+1}$. Now, we join $g(i_m)$ and $g(i_{m+1})$ by a straight line path. In theory we seek $\alpha \geq 0$ such that $I(g(i_m) + \alpha z)$ stops increasing, where $z = g(i_{m+1}) - g(i_m)$. In practice, we test successively

$$g_1 = \frac{1}{2}(g(i_m) + g(i_{m+1})), \quad g_2 = \frac{1}{2}(g(i_m) + g(i_1)) \quad g_3 = \frac{1}{2}(g(i_m) + g(i_2))$$

etc. until either

(a) g_i is found such that $I(g_i) > I(g(i_m))$ or;

(b) $||g_i - g(i_m)||_{H^1_{\mathcal{D}}} = (1/2^i)||g_{i_{m+1}} - g(i_m)||_{H^1_{\mathcal{D}}}$ is smaller than a prescribed tolerance.

In case (a), we evaluate $g_{mid} = \frac{1}{2}(g(i_m) + g_i)$, and perform a quadratic polynomial interpolation to find a better value for the maximum location of $\tilde{I}_\varepsilon$. In case (b), we do not change $g(i_m)$.

Having checked whether we can improve the estimate of the maximum in the direction of $g(i_{m+1})$, we now perform a similar task with $g(i_{m-1})$ and our revised location of $g(i_m)$, so that $g(i_m)$ will be moved to attain a higher maximum. We now refine the path in the neighborhood of the local maximum, by moving some of the nearby nodes closer. If the H^1 norm distance between $g(i_m)$ and $g(i_{m+1})$ is larger than a prescribed value, we then move a prescribed number of mesh points (around this local maximum) to avoid the *missing* of saddle points.

4.4.3 Step 3 of the algorithm

The steepest descent direction at the local maximum location is computed by finding a solution of the problem $(P_{\overline{v}})$ via finite element method, solving equation (4.37). Thus, a numerical approximation of $2\lambda v = \overline{v} - \varepsilon^2 u$ can be computed.

If the difference on the right-hand side is smaller than a prescribed tolerance in the H^1 norm, we stop the algorithm and find an approximate critical point of the functional $\tilde{I}_\varepsilon$. Otherwise, up to a prescribed maximum, the maximum point is moved *downhill* for a certain descent distance. This distance is determined by a similar procedure as in step 2. If this distance or the difference in $\tilde{I}_\varepsilon$ before and after the descent is smaller than a prescribed tolerance, then we have a numerical approximation of a critical point and stop the algorithm. If not, repeat steps 2 and 3.

4.5 The spike-layer solution

In this section we present the results obtained applying the numerical method described in this chapter. The algorithm was implemented using a MATLAB code.

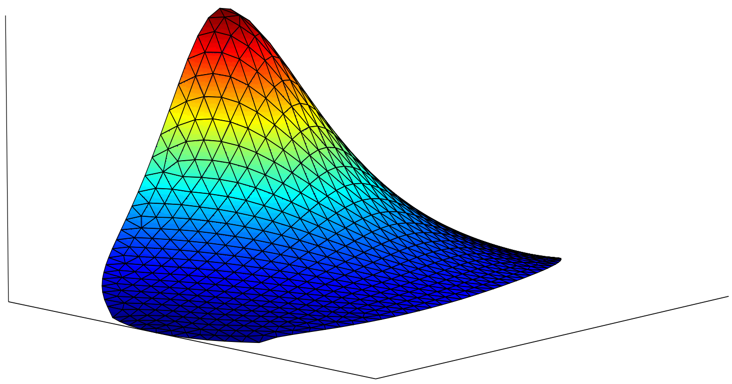

Figure 4.3: Three-dimensional view with $\varepsilon = 0.7$.

Below, in the different figures, we analyze the behavior of the least energy solution when the perturbation parameter ε goes to zero. We run the algorithm fixing, every time, a smaller different ε-value. As Figure 4.1 shows, we have considered an elliptical domain with boundary subdivided in four parts, that is, we set:

$$\begin{aligned} \partial_{\mathcal{N}}\Omega &= \Gamma^1_{\mathcal{N}} \cup \Gamma^2_{\mathcal{N}}, \\ \partial_{\mathcal{D}}\Omega &= \Gamma^1_{\mathcal{D}} \cup \Gamma^2_{\mathcal{D}}, \end{aligned}$$

with $\Gamma^1_{\mathcal{N}} \cap \Gamma^2_{\mathcal{N}} = \emptyset$, $\Gamma^1_{\mathcal{D}} \cap \Gamma^2_{\mathcal{D}} = \emptyset$ and $\partial\Omega = \overline{\partial_{\mathcal{N}}\Omega} \cup \overline{\partial_{\mathcal{D}}\Omega}$.

Since we are working in a domain $\Omega \subset \mathbb{R}^2$, the interface $\mathcal{I}_\Omega$ consists in four points belonging to $\mathbb{R}^2$ space. On the two boundary parts, where the curvature reaches its maximum value we impose Dirichlet conditions, see Figure 4.1. We know, by Theorem 3.3.1, that the maximum point P_ε cannot concentrate on the Dirichlet boundary part. Then, since the two points where the curvature maximize belong to $\Gamma^1_{\mathcal{D}}$ and $\Gamma^2_{\mathcal{D}}$, the maximum point P_ε should concentrate at a point $P_0 \in \mathcal{I}_\Omega$, as Corollary 3.4.6 states.

For each ε-value considered, we have a three-dimensional view, see for example Figure 4.3, and a two-dimensional view, see for example Figure 4.4.

The three-dimensional view shows better the profile of such a solution: in fact that is called *spike layer*, since it is highly concentrated near some point of Ω. As the perturbation parameter ε becomes smaller, see Figure 4.5, Figure 4.7 and Figure 4.9, the Mountain Pass type solutions u_ε tend to zero uniformly on every compact set of $\overline{\Omega} \setminus \{P_0\}$, while there exist $\delta \geq 1$ and P_ε such that $u_\varepsilon(P_\varepsilon) \geq \delta$, see Section 3.3. As above, the point P_ε is such that $u_\varepsilon(P_\varepsilon) = \max_{x\in\overline{\Omega}} u_\varepsilon(P)$ and, up to a subsequence, $P_\varepsilon \to P_0$.

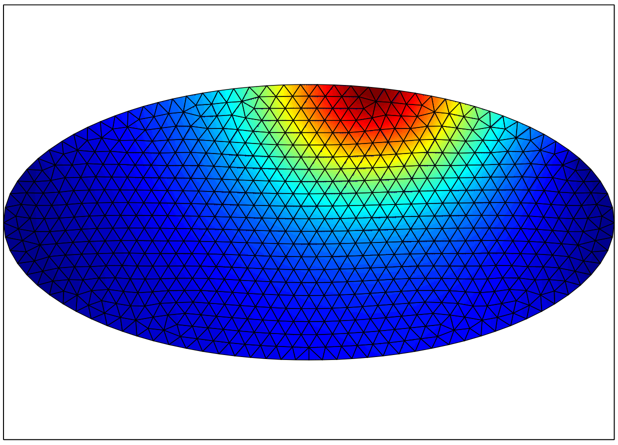

Figure 4.4: Two-dimensional view with $\varepsilon = 0.7$.

Instead, in the two dimensional view, see Figure 4.6, Figure 4.8 and Figure 4.10, we note that the maximum point P_ε belongs to the Neumann boundary part (see Corollary 3.3.8) and, as ε goes to zero, this point reaches the interface $\mathcal{I}_\Omega$. Moreover the support of the least energy solution becomes smaller when ε goes to zero. In fact, for example, if we give a look at Figure 4.9 and Figure 4.10, we note that the maximum point P_ε concentrates at the interface $\mathcal{I}_\Omega$ and the spike layer's profile is concentrated around the maximum point in a small support, as Lemma 3.4.2 shows.

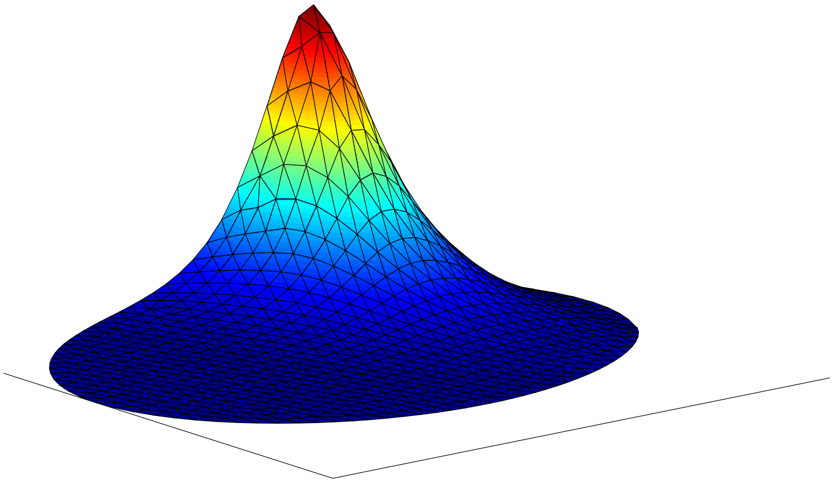

Figure 4.5: Three-dimensional view with $\varepsilon = 0.4$.

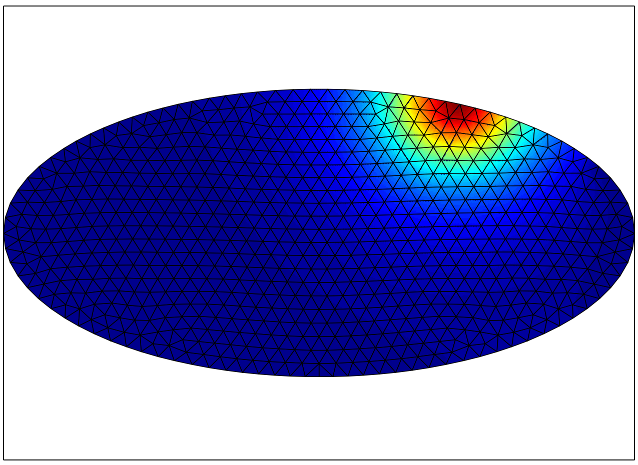

Figure 4.6: Two-dimensional view with $\varepsilon = 0.4$.

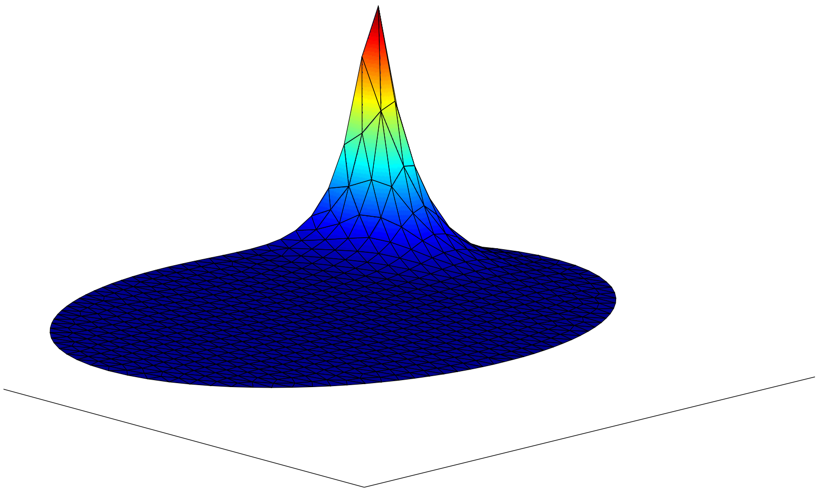

Figure 4.7: Three-dimensional view with $\varepsilon = 0.2$.

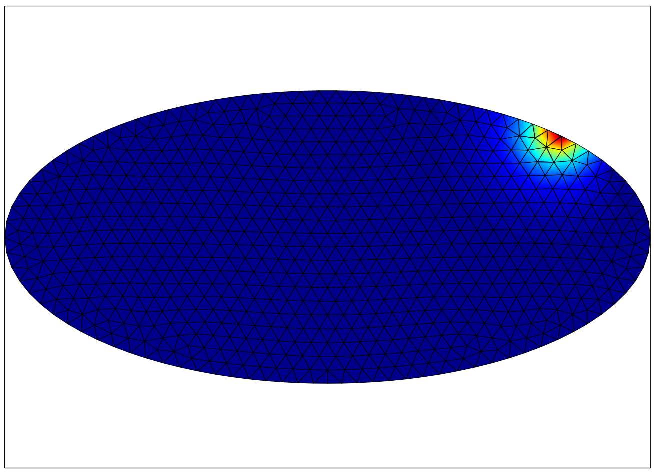

Figure 4.8: Two-dimensional view with $\varepsilon = 0.2$.

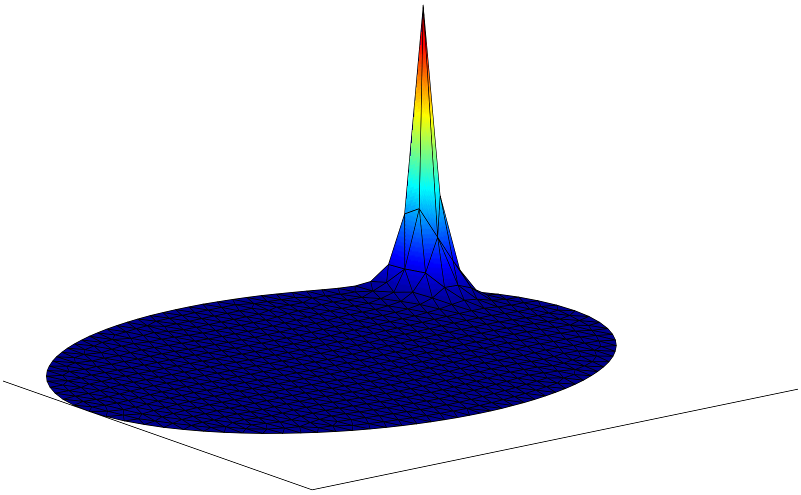

Figure 4.9: Three-dimensional view with $\varepsilon = 0.1$.

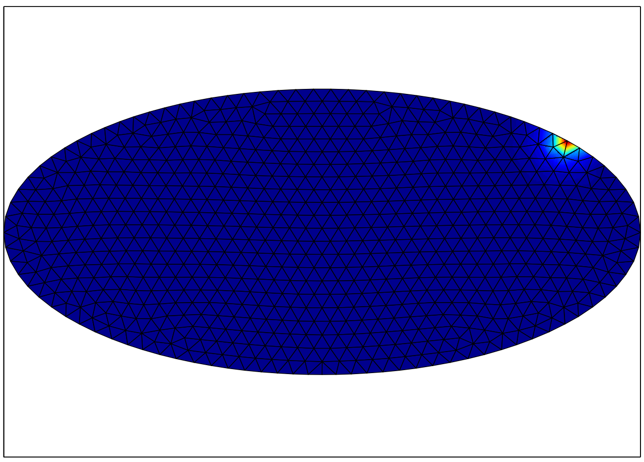

Figure 4.10: Two-dimensional view with $\varepsilon = 0.1$.

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