Università della Calabria

Dipartimento di Matematica

Dottorato di Ricerca in Matematica ed Informatica

XXIII CICLO

Settore Disciplinare MAT/05 – ANALISI MATEMATICA

Tesi di Dottorato

Convex Tomography in dimension two and three

Marina Dicosta

Supervisore
Prof. Aljoša Volčič

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A.A. 2009 – 2010

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To my love, Marco

Abstract

In this thesis we investigate a problem which is part of *Geometric Tomography.* Geometric Tomography is a branch of Mathematics which deals with the determination of a convex body (or other geometric objects) in $\mathbb{R}^n$ from the measure of its sections, projections, or both. In particular, it is focused on finding conditions sufficient to establish the minimum number of point X-rays needed to determine uniquely a convex body in $\mathbb{R}^n$.
The interest for this particular mathematical subject has its roots in the studies related to *Tomography.*
Nowadays it is rare that someone has never heard of CAT scan (Computed Assisted Tomography). This medical diagnostic technique, born in the late 70s, allows us to reconstruct the image of a three-dimensional object by a large number of projections at different directions. It is useful to emphasize that CAT is a direct application of a pure mathematical instrument known as *'the Radon transform".*
From the mathematical point of view, the question of when a convex body, a compact convex set with nonempty interior, can be reconstructed by means of its X-ray, arose from problems (published in 1963) posed by Hammer in 1961 during a Symposium on convexity:

> « Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord along an X-ray line. (No diffusion please). How many pictures must be taken to permit exact reconstruction of the body if:
>
> **(a)** The X-rays issue from a finite point source?
>
> **(b)** The X-rays are assumed parallel? »

A convex body is a convex compact set with nonempty interior.
We distinguish between two problems, according to the X-rays are at a finite point

source or at infinity.
We are searching which properties a set of directions U must fulfill, in order to determine uniquely a convex body K by means of its (either parallel or point) X-rays, in the directions of U. When U is an infinite set then this reconstruction is possible and this follows from general theorems regarding the inversion of the Radon transform.
This thesis consists of two parts. The first (Chapter 3) is concerned with the determination of a planar convex body from its i-chord functions, while the second part (Chapter 4) generalizes the planar results to the three-dimensional case. The main results provide a partial answer to the problem posed by R. J. Gardner:

> "How many point X-rays are needed to determine a convex body in $\mathbb{R}^n$?"

We use two analytic tools both considered in geometric tomography for the first time by K. J. Falconer. One is the i-chord function, which is related through the Funk theorem to the measures of the i-dimensional sections, when i is a positive integer. The other tool, inspired by a paper by D. C. Solmon, suggested the introduction of a kind of "*Cavalieri Principle*" for point X-rays, and has been later on extended to other dimensions and real values of i.
The i-chord functions allow to translate in analytical form the information given by the ith section functions. Therefore, from an analytical point of view we have the following problem:

> "Suppose that $K \subset \mathbb{R}^n$ is a convex body and let p_h be some noncollinear points (some of them are possibly at infinity). Suppose, moreover, that we are given the i-chord functions at the points p_h, with $i \in \mathbb{R}$. Is K then uniquely determined among all convex bodies?"

The i-chord functions $\rho_{i,K}$ can be seen as a generalization of the radial function of the convex body K. The latter is the function that gives the signed distance from the origin to the boundary of K. For integer values of the parameter i, the i-chord function is closely linked to the ith section function of a convex body, that is the function assigning to each subspace of dimension i the i-dimensional measure. When $i = 1$, the 1st section function coincides with the 1-chord function, that is the point X-ray of the convex body at the origin.
In Chapter 3 we consider two planar problems. One problem consists of the determination of a planar convex body K from the i-chord functions, for $i > 0$, at two points when the line l passing through p_1 and p_2 meets the interior of K and the two points p_1 and p_2 are both exterior or interior to K. If the line l supports K, then the results hold for $i \geq 1$. The second result concerns the determination of a planar

convex body K from its i-chord functions at three noncollinear points for $0 < i < 2$. Chapter 4 deals with the problem of determining a three-dimensional convex body K from the i-chord functions at three noncollinear points non belonging to K. Also in this case we search a sort of "Cavalieri Principle", for a suitable measure involving i-chord functions for $1 < i < 3$.
We are not able to extend this result to generic convex bodies when $i = 1$. In this case we have to assume that the convex body is of class $\mathcal{C}^{1+\alpha}$ with $\alpha \in]0, 1[$.

Sommario

Il lavoro di tesi si inserisce prevalentemente nell'ambito della ricostruzione delle immagini, in particolare nel settore della *Tomografia Geometrica*, la quale si occupa di trovare il minor numero di radiografie necessarie per ricostruire univocamente un corpo. La teoria generale, basata sul concetto di *Trasformata di Radon* trova una sua significativa applicazione in medicina nella Tomografia Computerizzata, che consente la ricostruzione dell'immagine di una sezione del corpo umano mediante l'uso dei raggi X presi in diverse direzioni. Dal punto di vista matematico, il problema di quando un corpo convesso (o un altro oggetto geometrico), può essere ricostruito per mezzo delle sue radiografie, si è sviluppato grazie ai problemi posti nel 1961 da P.C. Hammer all'*American Mathematical Society Symposium* sulla Convessità:

> "Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord along an X-ray line. (No diffusion please). How many pictures must be taken to permit exact reconstruction of the body if:
>
> **(a)** The X-rays issue from a finite point source?
>
> **(b)** The X-rays are assumed parallel?"

Un corpo convesso è un insieme convesso compatto dotato di punti interni. Ci si trova dunque di fronte due problemi, a seconda che le radiografie siano prese da sorgenti finite, o all'infinito. Ci chiediamo allora quali proprietà deve soddisfare un insieme di direzioni U, affinché un corpo convesso K sia univocamente determinato dalle sue radiografie (parallele o puntuali) nelle direzioni in U. Nel caso in cui U è un insieme infinito questo è vero e sussistono dei teoremi di unicità basati sull'inversione della Trasformata di Radon. Il lavoro di tesi è suddiviso fondamentalmente in due parti. Una prima parte (Capitolo 3) si occupa dello studio della determinazione di un corpo convesso planare per mezzo delle funzioni i-cordali, mentre la seconda

parte (Capitolo 4) generalizza quanto visto nel caso planare, al caso tridimensionale. I risultati ottenuti forniscono una parziale risposta al problema posto da R. J. Gardner:

> "How many point X-rays are needed to determine a convex body in $\mathbb{R}^n$?"

Vengono utilizzati due strumenti analitici, entrambi introdotti per la prima volta da K. J. Falconer nell'ambito della tomografia geometrica. Il primo è la funzione i-cordale, la quale è legata per mezzo del Teorema di Funk alle misura della sezione i-dimensionale, quando i è un numero intero positivo. Il secondo strumento, ispirato da un articolo di D. C. Solmon, è stato la ricerca di una opportuna misura che fornisse una specie di *"Principio di Cavalieri"* per le funzioni i-cordali.
Le funzioni i-cordali costituiscono uno strumento essenziale che permette di tradurre in forma analitica le informazioni date dalle funzioni di i-sezione. Pertanto, dal punto di vista analitico si ha il seguente problema:

> "Supponiamo che $K \subset \mathbb{R}^n$ sia un corpo convesso e siano p_h dei punti non allineati (alcuni dei quali eventualmente all'infinito). Supponiamo inoltre di conoscere le funzioni i-cordali in p_h per $i \in \mathbb{R}$. K è univocamente determinato tra tutti i corpi convessi?"

Le funzioni i-cordali $\rho_{i,K}$ possono essere viste come una generalizzazione della funzione radiale del corpo convesso K. Quest'ultima è la funzione che fornisce la distanza con segno dall'origine al bordo di K. Per valori interi del parametro i, le funzioni i-cordali sono strettamente collegate alle funzioni i-sezione di un corpo convesso, ovvero la funzione che fornisce la misura i-dimensionale dell'intersezione con un sottospazio avente dimensione i. Quando $i = 1$, la funzione 1-sezione coincide con la funzione 1-cordale, ovvero con la radiografia puntuale nell'origine.
Nel Capitolo 3 affrontiamo due problemi planari. Uno consiste nella determinazione di un corpo convesso K prendendo le funzioni i-cordali, per $i > 0$, in due punti nel caso in cui la retta l passante per p_1 e p_2 interseca la parte interna di K e i punti sono entrambi esterni o interni a K. Se la retta l supporta K, i risultati ottenuti valgono solo per $i \geq 1$. Il secondo risultato riguarda la determinazione di un corpo convesso planare K dalle funzioni i-cordali in tre punti non allineati per $0 < i < 2$.
Nel Capitolo 4 affrontiamo il problema di determinare un corpo convesso tridimensionale K prendendo le funzioni i-cordali per tre punti non allineati ed esterni a K. Anche in questo caso si cerca una specie di "Principio di Cavalieri" per un'opportuna misura che coinvolga le funzioni i-cordali in tre punti non allineati per $1 < i < 3$. Non è stato possibile estendere questo risultato a corpi convessi generici per $i = 1$. In questo caso si deve richiede che il corpo convesso sia di classe $\mathcal{C}^{1+\alpha}$ con $\alpha \in]0, 1[$.

Contents

Introduction

"Stay hungry, stay foolish."

Steve Jobs, 5th June 2005

In this thesis we will investigate a problem which belongs to *Geometric Tomography.* Geometric Tomography is a branch of Mathematics which deals with the determination of a convex body in $\mathbb{R}^n$ from the measure of its sections, projections, or both. In particular, it is focused on finding conditions sufficient to establish the minimum number of point X-rays needed to determine uniquely a convex body in $\mathbb{R}^n$.
The interest for this particular mathematical subject has its roots in the studies related to *Tomography.* The word "tomography" is derived from the Greek $\tau \acute{o} \mu o \varsigma$ (tómos –slice) and $\gamma \rho \acute{\alpha} \varphi \epsilon \iota \nu$ (gráphein – to write).
Nowadays it is rare that someone has never heard of CAT scan (Computed Assisted Tomography). This medical diagnostic technique, born in the late 70s, allows us to reconstruct the image of a three-dimensional object by a large number of projections in different directions. The first CT-scanner, was conceived for EMI in 1967 by Sir Godfrey Newbold Hounsfield, an English electrical engineer at Atkinson Morley Hospital in Wimbledon, London. This device, called "EMI-scanner" was the first able to display cross-sections of the human body, particularly of the skull (1 October 1971). Though extremely innovative, this tomograph took many hours to get data, and several days for producing the images. This marks the beginning of a new frontier of medicine just called Computed Tomography, in fact, the term "computed tomography" refers to the computation of tomography from X-ray pictures.
A few years later, in 1979 Hounsfield along with Allan McLeod Cormack, a South African-born American physicist, were awarded the Nobel Prize in Medicine.
Later improvements have been possible thanks to the increased power of computers, a better technology in data collection and better reconstruction algorithms.
After this brief introduction, it is useful to emphasize that CAT is a direct application

of a pure mathematical instrument known as *"the Radon transform"*, named after the Austrian mathematician who in 1917 introduced this kind of transform during his research in measure theory.
In 1917 Johann Radon published his paper, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, (to which little attention was paid for a long time) with which Radon can be recognized as the *father of tomography*. Radon did not mean his transform to be used for a medical application.
We will give next a brief description of the process implemented in the CAT, as well as its existing relationship with the Radon transform.
To perform a radiography, the patient is placed between a X-ray generator and a film sensitive to X-rays or a digital detector. According to the density and composition of the different areas of the object a proportion of X-rays are absorbed by the object. The X-rays that pass through are then captured behind the object by the detector, which gives a two-dimensional representation of all the structures superimposed on each other.
From the mathematical point of view, the question of when a convex body, a compact convex set with nonempty interior, can be reconstructed by means of its X-ray, arose from problems posed by Hammer in 1961 during a Symposium on convexity, [24].

> « Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord along an X-ray line. (No diffusion please). How many pictures must be taken to permit exact reconstruction of the body if:
>
> **(a)** The X-rays issue from a finite point source?
>
> **(b)** The X-rays are assumed parallel? »

It is rather surprising how clear was Hammer's idea on the reconstruction of objects from X-rays so many years before anybody figured out to use the Radon transform for CT-scanners.
We distinguish between two problems, according to the X-rays are at a finite point source or at infinity.
We are now searching which properties a set of directions U must fulfill, in order to determine a convex body K in a unique way by means of its (either parallel or point) X-rays, at the directions of U. When U is an infinite set then this reconstruction is possible and there are a lot of uniqueness theorems based on the inversion of the Radon transform, [23] and [42]. In particular Zalcman, [50, Section 2], states that

K is uniquely determined if the lengths of the chords are given from infinitely many directions.

State of art

The problems considered in our thesis are of a different nature and are probably of less practical application.
There are several paper in the literature which study the reconstruction of a convex body in $\mathbb{R}^n$, for $n \geq 2$, from the measures of their intersections with affine subspaces of dimension k, with $1 \leq k \leq n-1$. There is a first group of papers that gives conditions on the position and number of points $p_1, p_2, \ldots, p_m$ (some possibly at infinity) which guarantee the uniquely determination of a convex body K in $\mathbb{R}^n$ by the measures of the intersections of K with all the affine subspaces, of a given dimension k, passing through p_h, for $1 \leq h \leq m$. For example, [17], [11], [9], [10], [44], [14], [13], [18], [39], [44], [46] and [20].
A second group of papers, on the other hand, fixes a point p and ensure that K is uniquely determined by the measures of the intersections of K with the affine subspaces through p of (at least) two different dimensions, [18] and [49]. The earliest papers concern Hammer's question (b). The (parallel) X-ray of a convex body K in the direction u is the function giving the lengths of all the chords of K parallel to u. The uniqueness aspect of question (b) is equivalent to asking which finite sets of directions are such that the corresponding X-rays distinguish between different convex bodies. Simple examples showed that there are arbitrarily large finite sets of directions that do not have this desirable property and that no set of three directions does it. In fact, three directions never suffices, since each triangle is affinely equivalent to an equilateral triangle, and, by a rotation of $\frac{\pi}{3}$ about the barycentre we can build a different equilateral triangle with equal X-rays along the three directions connecting the corresponding vertices.
The first studies were made, independently, by Falconer [11] and Gardner [13]. A complete solution was found by Gardner and McMullen, [17], (see also [16, Chapter 1]). A corollary of their result is that there are sets of four directions in S^1, the unit disk in $\mathbb{R}^2$ such that the X-rays of any planar convex body in these directions determine it uniquely among all planar convex bodies. It also was shown in [17] that a suitable set of four directions is one such that the corresponding set of slopes has a transcendental cross-ratio. Clearly this is an impractical choice of directions.

However, Gardner and Gritzmann, [19] showed that further suitable sets of four directions are those whose set of slopes, in increasing order, have a rational cross-ratio not equal to $\frac{3}{2}$, $\frac{4}{3}$, 2, 3, or 4.
The corresponding uniqueness problem in higher dimensions can be solved by taking four directions, as specified above, all lying in the same two-dimensional plane. Since the corresponding X-rays determine each two-dimensional section of a convex body parallel to this plane, they determine the whole body.
The point X-ray of a convex body K at a point p is the function giving the lengths of all the chords of K lying on lines through p.
The position of the sources with respect to the convex body is important.
Parallel X-rays can be viewed as a limit case of point X-rays, where the sources lay on the line at infinity, namely the set of points at infinity of each line in the plane.
It is easy to see that a single source p does not suffice. The only known result in the opposite direction is provided by a paper of Lam and Solmon [28] which proves that, with some exceptions, convex polygons are uniquely determined *among convex polygons* by one point X-ray. In fact, they in [28] show an algorithm for the reconstruction of a convex polygon from one directed X-ray at the origin.
One of the problems is that *Cavalieri's principle* does not hold for point X-rays, since the area increases for increasing distance from the sources. Consequently, new measures has been introduced,for which a kind of Cavalieri principle holds.
The uniqueness aspect of Hammer's question (a) is not completely solved, but it is known that a planar convex body K is determined uniquely among all planar convex bodies by its X-rays taken at

(i) two points such that the line through them intersects K and it is known whether or not K lies between the two points. (see Falconer, [9] and [11] and Gardner, [16, Theorem 5.3.3]);

(ii) three points such that K lies in the triangle having these three points as vertices. (see Gardner [16, Theorem 5.3.6]);

(iii) any set of four collinear points whose cross ratio is restricted as in the parallel X-ray case above. (see Gardner, [14]);

(iv) any set of four points in general position. (see Volčič, [44]).

Except in the case of (i), little is known about the uniqueness problem for point X-rays in higher dimensions. Many of these results have been extended to spaces of constant curvature. For example, Dulio and Peri in [2] and in [3] establish a version of [16, Theorem 5.3.3]) that holds in spaces of constant curvature. In particular, in [4], they show that convex bodies in a plane of constant curvature are determined (up to reflection in the origin, in the case of the sphere) by point X-rays at four points in general position. The main result in [17] (see also [16, Theorem 1.2.11]) is that a planar convex body is determined, among all planar convex bodies, by its parallel X-rays in a finite set U of directions if and only if U is not a subset of the directions of edges of an affinely regular polygon. This gives many choices for sets U with the uniqueness property. For example, Gardner and Gritzmann, [19] showed that U can be any set of at least seven directions with rational slopes, but as mentioned above, the minimum number of directions in U is four.

Most of the existing literature provide uniqueness theorems and does not address the problem of actual reconstruction. Only recently a paper of Gardner and Kiderlen, in [20], proposed algorithms for reconstructing a planar convex body K from either its parallel X-rays taken in a fixed finite set of directions or its point X-rays taken at a fixed finite set of points. The two algorithms construct a convex polygon P_k whose X-rays approximate (in the least squares sense) k equally spaced noisy X-ray measurements in each of the directions or at each of the points. P_k, almost surely, tends to K in the Hausdorff metric as k tends to infinity. This result provides a solution, in the strongest sense, of the Hammer's X-ray problems.
Hammer actually asked his questions in 1961, a year before (independently) the results obtained by Giering. In [21], he studied one of these problems proving that given a planar convex body K chord lengths in three appropriate directions (depending on K) are enough to distinguish from any other planar convex body the Steiner Symmetrals of K about lines in those directions. Volčič in [46] provides a new proof of this result, and permits a wider choice of triplets of directions, possibly also on a finite line, and also Gardner in [13] gives a shorty version of Giering's result. Furthermore, Giering has shown that two directions are in general not enough.

Zuccheri in [51] describes a method for computing jth derivatives of the polar representations of the boundary of a convex body K, from generalized i-chord functions at two points not lying on the boundary of K, ∂K, but such that the line l passing through these two points meets ∂K at two distinct points. On the other hand, Falconer in [11] and in [9] provides methods for computing these intersection points for i-chord functions for $i \in \mathbb{N}$. L. Zuccheri extends these methods to the values of $i < 1$, and observes that these intersection points can be obtained by solving a system of algebraic equations.

Main results

The aim of this thesis is to extend to the case of i-chord functions the main results obtained for $i = 1$ by Volčič in [44] and by Gardner in [16].
The notion of i-chord functions has been introduced by Falconer in [11] for integer values $0 < i < n$, where n is the dimension of the Euclidean space $\mathbb{R}^n$ in which the problem is handled.
The i-chord functions $\rho_{i,K}$ are generalizations of the radial function of the convex body K. The latter is the function that gives the signed distance from the origin to the boundary of K. The i-chord functions are particularly useful when i is an integer strictly between 0 and n, but other values are also relevant, as in the various forms of the notorious equichordal problem. The i-chord functions has been extended to all integer values by Gardner in [14] and to all real numbers in [16].
For integer values of i, the i-chord function is closely related (via Funk theorem) to the ith section function of a convex body, the function giving the i-dimensional volumes of its intersections with i-dimensional subspaces.
When $i = 1$, the ith section function coincides with the 1-chord function, also known as the point X-ray at the origin (or, in the computer tomography, the fan-beam X-ray at the origin). The $(n-1)$th section function is simply called the section function.
The i-chord function is a technical tool which allows to translate in analytical form the information given by the ith section function. Therefore, from an analytical point of view we have the following problem:

> « Suppose that $K \subset \mathbb{R}^n$ is a convex body and let p_h be some noncollinear points (some of them are possibly at infinity). Suppose, moreover, that we are given the i-chord functions at the points p_h, with $i \in \mathbb{R}$. Is K then uniquely determined among all convex bodies? »

A second tool used throughout this thesis is a kind of "*Cavalieri Principle*" for a measure ν_i having an appropriate invariance property involving the i-chord functions. The Cavalieri principle, introduced for $i = 1$ in [44] and for all integers in [14], can be in fact extended to all real values as shown in [16].

The main contribution of the work reported in this thesis are summarized in the following.

First of all, we will show the determination of a planar convex body K from the i-chord functions, for $i > 0$, at two points when the line l passing through p_1 and p_2 meets the interior of K and the two points p_1 and p_2 are both exterior or interior to K, and it is known where K meets the line l. If the line l supports K, then the results will hold for $i \geq 1$.
A second relevant result will concern the determination of a planar convex body K from its i-chord functions at three noncollinear points for $0 \leq i < 2$. In particular, when the convex body is contained in the interior of the triangle formed by the three points the result will hold for $i > 0$.
Finally we will tackle the problem of determining a three-dimensional convex body K from the i-chord functions at three noncollinear points non belonging to K using a sort of "Cavalieri Principle", for a suitable measure involving i-chord functions for $1 < i < 3$.
Moreover we will give an uniqueness result for convex bodies of class $\mathcal{C}^{1+\alpha}$ with $\alpha \in]0, 1]$.

Outline of this thesis

The thesis is organised as follows.
First, basic notations and definitions on convexity are introduced in Chapter 1, moreover, integral transforms and X-ray transform are also discussed and basic notions on differentiability are provided.

Then, a wider presentation of the notion of i-chord function is introduced in Chapter 2, where its relationship with the concepts of ith section function and of X-ray of order i is also discussed. In particular, the important tool of the corresponding components is provided.

After that, the determination of a planar convex body from its i-chord functions is discussed in Chapter 3.

Afterwards, uniqueness results regarding the three-dimensional case are presented in Chapter 4.

Finally, conclusions and open problems are reported in Chapter 5.

Chapter 1

Preliminary Topics

This chapter aims at introducing the main topics this thesis is about, giving concepts necessary to better understand the following chapters.
In particular, we will give concepts regarding convexity, integral transforms and their equivalent X-ray transforms, and finally we will recall some basics on differentiability. References for this material are [16], [32], [5, 6, 7], [33, 25], [23, 37, 26] and [36].

1.1 Notations and definitions

Let us now introduce some basic definitions and notations.
We denote the n-dimensional Euclidean space by $\mathbb{R}^n$ with the origin o.

Definition 1.1.1 (Convex set)**.**
A set C in $\mathbb{R}^n$ is called convex if the line segment joining any pair of points of C lies entirely in C.

Definition 1.1.2 (Convex body)**.**
A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior.

We denote by $\mathcal{K}^n$ the class of nonempty compact convex subsets of $\mathbb{R}^n$, and by $\mathcal{K}_0^n$ the class of convex bodies.
We write $\operatorname{int} K$ and ∂K, respectively, for the interior and the boundary of K. We use the symbol $K - p$ to denote the translated convex body:

$$K - p = \{x - p : x \in K\}\,.$$

We denote by $\mathscr{G}(n, k)$ the set of all k-dimensional linear subspaces of $\mathbb{R}^n$, while by $\mathscr{G}(n, k, u)$ we denote the set of all k-dimensional affine subspaces of $\mathbb{R}^n$ parallel to the vector u belonging to the unit sphere S^{n-1}. With λ_k we denote the k-dimensional Lebesgue measure. We employ the symbol l_u for the line passing through o and parallel to $u \in S^{n-1}$. For any $A \subset \mathbb{R}^n$, we use the symbols $\operatorname{lin} A$ and $\operatorname{aff} A$ to denote, respectively, the smallest linear subspace containing A and the smallest affine subspace containing A. Moreover, for every $x \in \mathbb{R}^n \setminus \{o\}$, and for every $A \subset \mathbb{R}^n$, we define

$$\operatorname{pos} x := \{\lambda x : \lambda \geq 0\}$$

and

$$\operatorname{pos} A := \operatorname{conv} \bigcup_{x \in A} \{\operatorname{pos} x : A \cap \operatorname{pos} x \neq \emptyset\}.$$

We write

$$\operatorname{pos}_p A := p + \operatorname{pos}(A - p)$$

for the convex positive half-cone with vertex p generated by A. Let $E_1, E_2 \subset \mathbb{R}^n$ such that $\operatorname{pos}_p E_1 = \operatorname{pos}_p E_2$. Let l be any half-line issuing from p and intersecting E_1 (and so also E_2). We say that E_1 is between p and E_2 if for any point $x_1 \in E_1 \cap l$ and any $x_2 \in E_2 \cap l$, x_1 is between p and x_2.
A simplex with vertices $a_1, a_2, \cdots, a_k$ will be denoted by $\triangle(a_1, a_2, \cdots, a_k)$. Hence, a simplex $\triangle(a_1, a_2)$ is the segment $[a_1; a_2]$ with endpoints a_1 and a_2, while a simplex $\triangle(a_1, a_2, a_3)$ is the triangle with vertices a_1, a_2 and a_3.

Definition 1.1.3 (Support function, supporting hyperplane)**.**
Let $K \in \mathcal{K}^n$, the support function h_K of K is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$.
If $u \in S^{n-1}$, the supporting hyperplane to K with outer normal vector is defined by

$$H_u = \{x : x \cdot u = h_K(u)\}.$$

The support function $h_K(u)$ at a unit vector u gives the signed distance from the origin o to the supporting hyperplane H_u, (see Figure 1.1).

Since if K and K' are two nonempty compact convex sets, $K \subset K'$ if and only if $h_K \leq h_{K'}$, it follows that a compact convex set is uniquely determined by its support function.

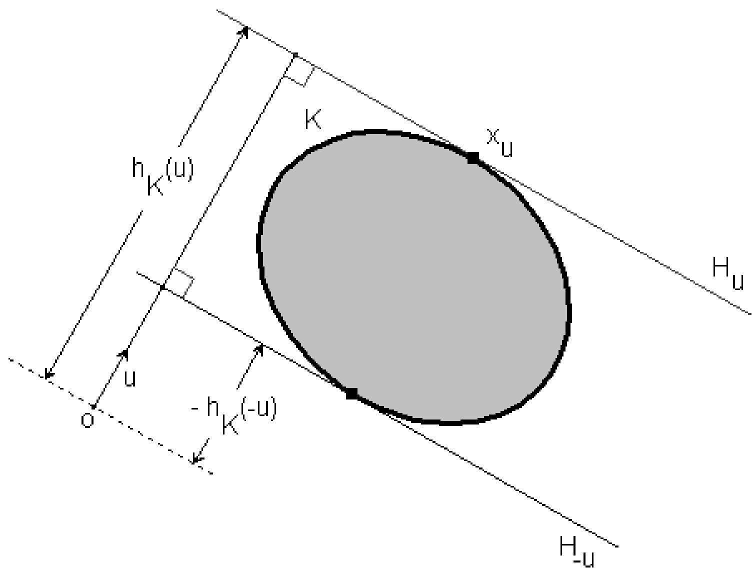


Figure 1.1: The support function

Definition 1.1.4 (Star-shaped set).
Let $L \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$. The set L is star-shaped at a point p if its intersection with every line through p is either empty or connected.

1.2 Integral transforms

Although the integral transforms can be defined more generally, it is assumed throughout this section that $f \in L^1_0(\Omega)$, i.e., that f is integrable and vanishes outside Ω, where Ω is a bounded open subset of $\mathbb{R}^n$.

Definition 1.2.1 (Directed X-ray of f at a point p).
Let $p \in \mathbb{R}^n$. The divergent beam transform, or directed X-ray, of f at the point p is defined for each $u \in S^{n-1}$ by

$$D_p f(u) = \int_0^{+\infty} f(p + tu)dt.$$

Physically the function f is the X-ray attenuation coefficient, i.e., the density of the object X-rayed, the point p is the X-ray source and the unit vector u is an X-ray detector. Therefore $D_p f(u)$ represents the attenuation in the X-ray beam, i.e., the total mass of the object along the half-line issuing from p and having direction u. The role of computed tomography is to reconstruct this function f from a number of X-rays. The word tomography refers to the two-dimensional problem of the reconstruction of the cross sections of f. Historically, computed tomography began with parallel beam X-rays in which the photons travel along lines with a fixed direction rather than along rays emanating from a fixed point source.

Definition 1.2.2 (Parallel X-ray of f in the direction u)**.**
Let $u \in S^{n-1}$. The X-ray transform, or parallel X-ray, of f in direction u is defined for each $x \in u^{\perp}$ and $t \in \mathbb{R}$ by

$$X_u f(x) = \int_{-\infty}^{+\infty} f(x + tu)dt$$

where dt denotes integration with respect to λ_1.

Definition 1.2.3 (Radon transform)**.**
The Radon transform of f is defined for $t \in \mathbb{R}$ and $u \in S^{n-1}$ by

$$\tilde{f}(t, u) = \int_{u^{\perp}+tu} f(x)dx.$$

For each $u \in S^{n-1}$, Fubini's theorem guarantees that $\tilde{f}(t, u)$ exists for almost all t. Moreover, by the orthogonal decomposition theorem, the set $H = u^{\perp} + tu$ represents a hyperplane in $\mathbb{R}^n$, therefore the Radon transform can be rewritten in the following way

$$\tilde{f}(H) = \int_H f(x)dx$$

so it is a function defined on the space of hyperplanes. The invertibility of the Radon transform makes possible bodies to be reconstructed completely, if all their X-ray pictures are given.

Definition 1.2.4 (X-ray of f at a point p)**.**
Let $p \in \mathbb{R}^n$. The line transform, or X-ray, of f at the point p is given by

$$X_p f(u) = \int_{-\infty}^{+\infty} f(p + tu)dt.$$

Observe that, the X-ray of f at the point p is linked to the directed X-ray of f in the directions u and $-u$ by the following relation

$$X_p f(u) = D_p f(u) + D_p f(-u)$$

for each $u \in S^{n-1}$, [36].

Definition 1.2.5 (k-dimensional X-ray).
Let $1 \leq k \leq n-1$ and $S \in \mathscr{G}(n,k)$. The k-dimensional X-ray, or k-plane transform, of f parallel to the subspace S is defined for each $y \in S^{\perp}$ by

$$X_S f(y) = \int_S f(x+y)dx.$$

Definition 1.2.6 (X-ray of order i).
Let $p \in \mathbb{R}^n$ and $i \in \mathbb{R}$. The X-ray of order i of f at p is defined for each $u \in S^{n-1}$ by

$$X_{i,p} f(u) = \int_{-\infty}^{+\infty} f(p+tu)|t|^{i-1}dt.$$

The most interesting aspects of computed tomography are the questions of uniqueness and stability. Regarding uniqueness we have the following theorems proved in [16, 26, 37], and [23].

Theorem 1.2.7.
Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $f \in L_0^1(\Omega)$. Suppose that $D \subset S^{n-1}$ is an infinite set. If $X_u f = 0$ for all $u \in D$, then $f = 0$, λ_n-almost everywhere.

Theorem 1.2.8.
Let Ω be a bounded open subset of the unit ball B in $\mathbb{R}^n$ and let $f \in L_0^1(\Omega)$. Suppose that $P \subset \mathbb{R}^n \setminus B$ is an infinite set. If $D_p f = 0$ for all $p \in P$, then $f = 0$, λ_n-almost everywhere.

These two uniqueness theorems for the divergent beams and line transforms, 1.2.7 and 1.2.8, state that any object, even with varying density, is uniquely determined by an infinite set of point X-rays. On the other hand, no finite set of point X-rays suffice.

1.3 X-ray transforms

For characteristic functions of bounded measurable sets the integral transforms, defined in the previous section, boil down to the following functions.

Definition 1.3.1 (Directed X-ray).
Let p be a point and let E be a bounded measurable set in $\mathbb{R}^n$. The directed X-ray of

E at p is defined for $\lambda_{n-1}-$almost all $u \in S^{n-1}$ by

$$D_pE(u) = D_p\mathbb{1}_E(u) = \lambda_1(E \cap (r_u + p)),$$

where r_u is the ray emanating through o parallel to u.

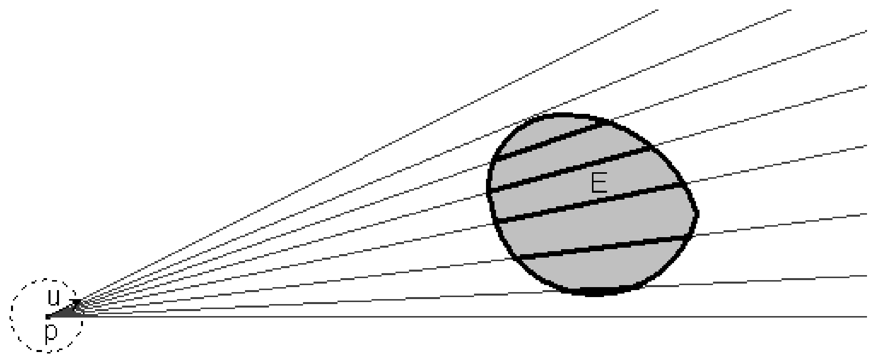


Figure 1.2: The directed X-ray of E at p

This function gives us information about the "lengths" of all the intersections of the set E with all the half-lines issuing from p, (see Figure 1.2).

Definition 1.3.2 (X-ray)**.**
Let p be a point and let E be a bounded measurable set in $\mathbb{R}^n$. The X-ray of E at p is defined for $\lambda_{n-1}-$almost all $u \in S^{n-1}$ by

$$X_pE(u) = X_p\mathbb{1}_E(u) = \lambda_1(E \cap (l_u + p)).$$

This function provides us the lengths of all the intersections of the set E with all the lines through p, (see Figure 1.3).
When E is a Borel set, $D_p(E)$ and $X_p(E)$ are defined everywhere on S^{n-1}. With X-ray, we consider the rays issuing from p in both the directions u and $-u$ as a single beam. For this reason, X-rays do not exist in nature but are merely a mathematical idealization.
The X-ray gives us less information than directed X-ray. For example, Figure 1.4 shows that two congruent disks have the same X-rays at the middle point of the

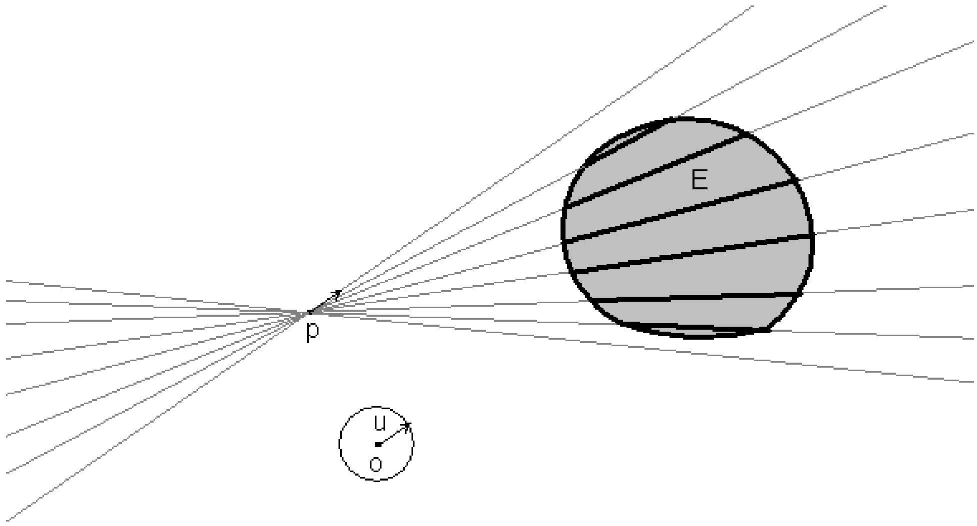


Figure 1.3: The X-ray of E at p

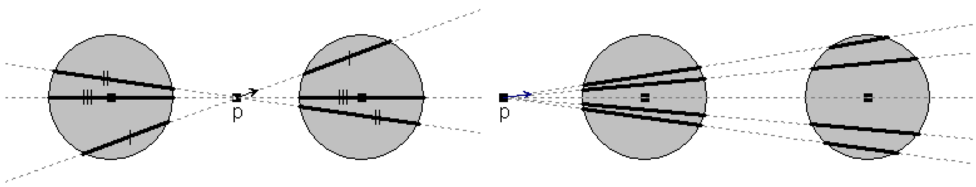


Figure 1.4: Directed X-ray and X-ray

segment joining their centers, but they have different directed X-rays.
In general

$$X_pE(u) = X_pE(-u)$$

for each $u \in S^{n-1}$, i.e., $X_p(E)$ is even, while $D_p(E)$ is not necessarily an even function.
If E is a body star-shaped at p, each ray issuing from p meets E in a (possibly degenerate) segment, so the directed X-rays give the length of each line segment, while the X-rays give the length of the "two" line segments.

Definition 1.3.3 (X-ray of order i)**.**
Let $i \in \mathbb{R}$. Let p be a point and E a bounded measurable set in $\mathbb{R}^n$. The X-ray of order i of E at p is defined by

$$X_{i,p}E(u) = \int_{-\infty}^{\infty} \mathbb{1}_E(p+tu)|t|^{i-1}dt,$$

for $u \in S^{n-1}$ for which the integral exists.

Note that when $i = 1$ we retrieve the definition of the X-ray of K at p.

Definition 1.3.4 (k-plane transform)**.**
Let $\leq k \leq n-1$ and $S \in \mathscr{G}(n,k)$. The k-plane transform, or the X-ray, of f parallel to the subspace S is defined for each $y \in S^{\perp}$ by

$$X_S f(y) = \int_S f(x+y)dx.$$

Definition 1.3.5 (k-dimensional X-ray)**.**
Let p be a point and let E be a bounded λ_n−measurable set in $\mathbb{R}^n$. If $1 \leq k \leq n-1$, we define the k-dimensional X-ray of E at p as a function on $\mathscr{G}(n,k)$ such that to each $G \in \mathscr{G}(n,k)$ assigns the measure $\lambda_k(E \cap (G+p))$.

The X-ray of a convex body is related to the notion of *Steiner symmetral*, introduces by Jacob Steiner [41].

Definition 1.3.6 (Steiner symmetral)**.**
Let K be a convex body in $\mathbb{R}^n$. Let $u \in S^{n-1}$ and let l_u be the line through the origin and parallel to u. For each $x \in u^{\perp}$, let $c(x)$ be defined as follows. If $K \cap (l_u + x)$ is empty, let $c(x) = \emptyset$. Otherwise, let $c(x)$ be the segment on $l_u + x$ centered at x whose length is equal to $\lambda_1(K \cap (l_u + x))$. The union of all the line segments $c(x)$ is called

Steiner symmetral of K and it is denoted by S_uK. The mapping S_u from $\mathcal{K}^n$ into itself is called Steiner Symmetrization.

Notice that S_uK has the same X-rays as K in the direction u, for this reason the Steiner Symmetral S_uK is immediately determined by the X-rays of K in direction u, (see Figure 1.5). Consequently, we shall identify the X-rays with the Steiner Symmetral, [17] and [21].

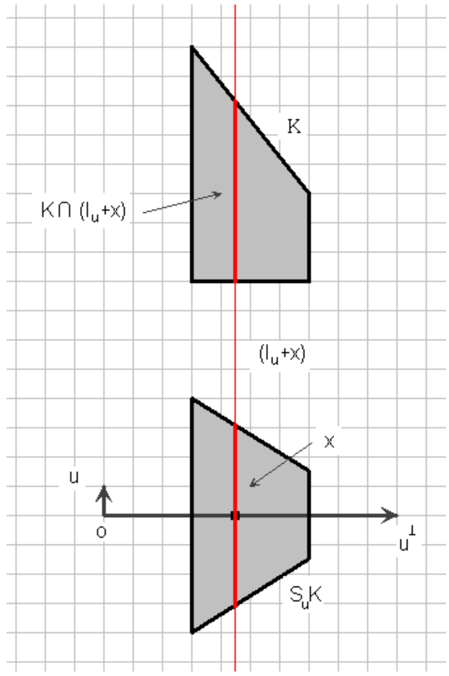


Figure 1.5: The Steiner symmetral of K

1.4 Basic notions on differentiability

Recall now briefly some preliminary properties about differentiability. A real-valued function on an open subset U on $\mathbb{R}^n$ is said to be of class $\mathscr{C}^k$ if it is k-times continuously differentiable, that is, all partial derivatives of order k exist and are continuous. The class of such functions is signified by $\mathscr{C}^k(U)$.
The class $\mathscr{C}^\infty(U)$ consists of those real-valued functions belonging to $\mathscr{C}^k(U)$ for all

$k \in \mathbb{N}$.
We say that a convex body K is of class $\mathscr{C}^k$ or of class $\mathscr{C}^\infty$ if the boundary of K, ∂K is of class $\mathscr{C}^k$ or of class $\mathscr{C}^\infty$, respectively.

Definition 1.4.1 (Function Hölder continuous).
A function f is called Hölder continuous at a if there is $\alpha \in (0,1)$, a constant $H > 0$, and an interval I containing a, such that

$$|f(x) - f(a)| \leq H\,|x-a|^\alpha$$

for all $x \in I$. We write $f \in \mathscr{C}^{n+\alpha}$ at a if $f \in \mathscr{C}^n$ at a and $f^{(n)}$ is Hölder continuous at a with α as above.

Definition 1.4.2 (Taylor polynomial of degree n for f at x_0).
Let f be n-times differentiable on an open interval containing the point x_0. Then the Taylor polynomial of degree n for f at x_0 is the polynomial

$$T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

Theorem 1.4.3 (Taylor's theorem).
Let f be $(n+1)$-times differentiable on an open interval containing the points x_0 and x. Then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

and ξ is some point between x_0 and x.

Corollary 1.4.4 (Remainder Estimate).
Let f be $(n+1)$-times differentiable on an open interval containing the points x_0 and x. If

$$|f^{(n-1)}(\xi)| \leq M$$

for all ξ between x_0 and x, then

$$f(x) = T_n(x) + R_n(x),$$

where

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

Lemma 1.4.5.
Let f and g be two convex functions on $[a, b]$, and let $\{x_n\}$ be a sequence in $[a, b]$ converging to c such that $x_n < c$ for all $n \in \mathbb{N}$. If $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$ and $f(c) = g(c)$ then $f'(c) = g'(b)$.

Proof.
Since g is a convex function on $[a, b]$, g is continuous in (a, b) and moreover g is differentiable in all but at most countably many points of (a, b).

$$\begin{aligned} f'(c) &= = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{x_n \to c} \frac{f(x_n) - f(c)}{x_n - c} \\ &= \lim_{x_n \to c} \frac{g(x_n) - g(c)}{x_n - c} \\ &= \lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = g'(c). \end{aligned}$$

$\square$

Chapter 2

The i-chord functions

In this chapter we will address an important tool used throughout this thesis. The i-chord functions have been defined so far in the literature for $u \in S^{n-1}$. We find it appropriate to extend the definition to all of $\mathbb{R}^n$, since the i-chord functions are defined in terms of the radial function.

2.1 Radial function and i-chord function

We begin this chapter with the key definition of *radial function*.

Definition 2.1.1 (Radial function)**.**
If L is nonempty, compact, and star-shaped at the origin o in $\mathbb{R}^n$. Its radial function ρ_L is defined by

$$\rho_L(x) = \max\{c : cx \in L\},$$

for $x \in \mathbb{R}^n \setminus \{o\}$ such that the line through x and o intersects L.

This definition, introduced by Gardner and Volčič, [18], differs from the usual definition of radial function in which the maximum is taken only over nonnegative c and shows the duality with the definition 1.1.3 of the support function defined in Chapter 1.
The radial function is positively homogeneous of degree -1, i.e.,

$$\rho_L(cx) = \frac{1}{c}\rho_L(x) \text{ for } c > 0$$

and this allows us to work with the restriction of the radial function ρ_L to the unit sphere, in fact we mostly use this restriction to the unit sphere. The radial function

$\rho_L(u)$ at a unit vector $u \in S^{n-1}$ gives the signed distance from o to the boundary of L along the line l_u, (see Figure 2.1).

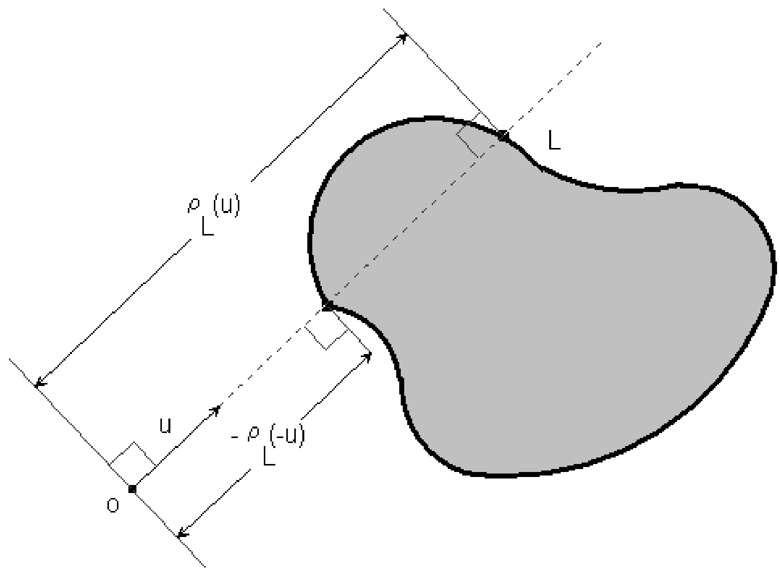


Figure 2.1: The radial function

Denote by D_L and S_L the domain and the support of the restriction of the radial function ρ_L to the unit sphere S^{n-1}, respectively.
A *star body* is a body such that ρ_L, restricted to S_L, is continuous.
A *star set* is a set that is a star body in its linear hull.
The class of star bodies contains the class of convex bodies.
The definition of the X-ray of a body E star-shaped at a point p can be reformulated in terms of its radial function ρ_E. For each $u \in S^{n-1}$, we have,

$$X_pE(u) = \begin{cases} \rho_{E-p}(u) + \rho_{E-p}(-u) & \text{if } p \in E, \\ \big||\rho_{E-p}(u)| - |\rho_{E-p}(-u)|\big| & \text{if } p \notin E. \end{cases}.$$

The notion of i-chord functions for $i \neq 1$ arise naturally from a certain generalization of Hammer's problems to higher dimensions.
In fact, this has been introduced by Falconer in [11] for integer values $0 < i < n$, where n is the dimension of the Euclidean space $\mathbb{R}^n$ in which the problem is handled.
The i-chord functions $\rho_{i,K}$ can be seen as a generalization of the radial function of

a convex body K. The i-chord functions are particularly useful when i is an integer strictly between 0 and n, but they have been extended to all integer values by Gardner in [14] and to all real numbers in [16].

Definition 2.1.2 (i-chord function)**.**
Let $i \in \mathbb{R}$, and suppose that L is a star set in $\mathbb{R}^n$. If $i \leq 0$, we assume that $o \in \operatorname{relint} L$ or $o \notin L$. The i-chord function $\rho_{i,L}$ of L at o is defined for $u \in S^{n-1}$ as follows. If the line through o parallel to u does not intersect L we define $\rho_{i,L} = 0$. Otherwise, if $i \neq 0$, we let

$$\rho_{i,L}(u) = \begin{cases} \rho_L(u)^i + \rho_L(-u)^i & \text{if } o \in L, \\ \left||\rho_L(u)|^i - |\rho_L(-u)|^i\right| & \text{if } o \notin L \end{cases}.$$

For $i = 0$, we define the 0-chord function of L at o for $u \in S^{n-1}$ by

$$\rho_{0,L}(u) = \begin{cases} \rho_L(u)\rho_L(-u) & \text{if } o \in L, \\ \exp\left|\log|\frac{\rho_L(u)}{\rho_L(-u)}|\right| & \text{if } o \notin L \end{cases}.$$

If p is a point in $\mathbb{R}^n$ such that $L - p$ is a star set, then the i-chord function of L at p is simply the i-chord function at the origin o of $L - p$, in symbol

$$\rho_{i,p,L}(u) = \rho_{i,L-p}(u).$$

In order to better understand this definition, consider the distances from a point $p \in \mathbb{R}^n$ to the boundary of L along a line through p and parallel to a direction $u \in S^{n-1}$. If p belongs to L, then the i-chord function of L at p gives the sum of the ith powers of these distances or the product of these distances, according as $i \neq 0$ or $i = 0$, respectively. While, if p does not belong to L, then it renders the difference (the greater less the smaller) of the ith powers of these distances or the quotient (the greater over the smaller) of these distances, according as $i \neq 0$ or $i = 0$, respectively. The assumption that for $i \leq 0$ the point p has to belong to the relative boundary of L or $p \notin L$ is necessary in order to avoid singularities.
Note that for $i = 1$ we retrieve the X-ray of L at p.
The analogous notion of a directed i-chord function at a point p, not in the interior of a body, can be obtained from the i-chord function by setting it equal to zero at $u \in S^{n-1}$ if the ray issuing from p in the direction u does not meet the body in a point other that p itself.
The definition of 0-chord function is motivated by the following proposition.

Proposition 2.1.3.
Let $i \in \mathbb{R}$, and suppose that L is a star set in $\mathbb{R}^n$. Then

$$\rho_{0,L}(u) = \lim_{i\to 0}\left(\frac{1}{2}\rho_{i,L}(u)\right)^{\frac{2}{i}} \tag{2.1}$$

if $o \in K$ and

$$\rho_{0,L}(u) = \lim_{i\to 0}\exp\left(\frac{\rho_{i,L}(u)}{|i|}\right) \tag{2.2}$$

if $o \notin K$.

Proof.

$$\begin{aligned}
\rho_{0,L}(u) &= \lim_{i\to 0}\left(\frac{1}{2}\rho_{i,L}(u)\right)^{\frac{2}{i}} \\
&= \lim_{i\to 0}\left[\frac{1}{2}\left(\rho_L(u)^i + \rho_L(-u)^i\right)\right]^{\frac{2}{i}} \\
&= \lim_{i\to 0}\exp\left\{\log\left[\frac{1}{2}\left(\rho_L(u)^i + \rho_L(-u)^i\right)\right]^{\frac{2}{i}}\right\} \\
&= \exp\left\{\lim_{i\to 0}\frac{2\log\left[\frac{1}{2}\left(\rho_L(u)^i + \rho_L(-u)^i\right)\right]}{i}\right\} \quad \text{by De l'Hopital} \\
&= \exp\left\{\lim_{i\to 0} 2\cdot\frac{\frac{1}{2}\left(\rho_L(u)^i\log\rho_L(u) + \rho_L(-u)^i\log\rho_L(-u)\right)}{\frac{1}{2}\left(\rho_L(u)^i + \rho_L(-u)^i\right)}\right\} \\
&= \exp\left\{\log\rho_L(u) + \log\rho_L(-u)\right\} \\
&= \exp\left\{\log\rho_L(u)\rho_L(-u)\right\} \\
&= \rho_L(u)\rho_L(-u).
\end{aligned}$$

The equation (2.2) can be written in the following way.

$$\rho_{0,L}(u) = \lim_{i\to 0}\exp\left(\frac{\left||\rho_L(u)|^i - |\rho_L(-u)|^i\right|}{|i|}\right),$$

so, we have to distinguish two cases.

If both numerator and denominator are positive (or negative) we get

$$
\begin{aligned}
\rho_{0,L}(u) &= \lim_{i\to 0}\exp\left(\frac{\left||\rho_L(u)|^i - |\rho_L(-u)|^i\right|}{|i|}\right) \\
&= \exp\left\{\lim_{i\to 0}\frac{|\rho_L(u)|^i - |\rho_L(-u)|^i}{i}\right\} \text{ by De l'Hopital} \\
&= \exp\left\{\lim_{i\to 0}\left(|\rho_L(u)|^i\log|\rho_L(u)| - |\rho_L(-u)|^i\log|\rho_L(-u)|\right)\right\} \\
&= \exp\left\{\log|\rho_L(u)| - \log|\rho_L(-u)|\right\} \\
&= \exp\left\{\log\frac{|\rho_L(u)|}{|\rho_L(-u)|}\right\}.
\end{aligned}
$$

Otherwise, if numerator is positive and denominator negative (or viceversa), we have

$$
\begin{aligned}
\rho_{0,L}(u) &= \lim_{i\to 0}\exp\left(\frac{\left||\rho_L(-u)|^i - |\rho_L(u)|^i\right|}{|i|}\right) \\
&= \exp\left\{\lim_{i\to 0}\frac{|\rho_L(-u)|^i - |\rho_L(u)|^i}{i}\right\} \text{ by De l'Hopital} \\
&= \exp\left\{\lim_{i\to 0}\left(|\rho_L(-u)|^i\log|\rho_L(-u)| - |\rho_L(u)|^i\log|\rho_L(u)|\right)\right\} \\
&= \exp\left\{\log|\rho_L(-u)| - \log|\rho_L(u)|\right\} \\
&= \exp\left\{\log\frac{|\rho_L(-u)|}{|\rho_L(u)|}\right\} \\
&= \exp\left\{\log\left(\frac{|\rho_L(u)|}{|\rho_L(-u)|}\right)^{-1}\right\} \\
&= \exp\left\{-\log\frac{|\rho_L(u)|}{|\rho_L(-u)|}\right\}.
\end{aligned}
$$

Therefore

$$
\rho_{0,L}(u) = \exp\left|\log\left|\frac{\rho_L(u)}{\rho_L(-u)}\right|\right|.
$$

□

Note that in the equation (2.1),

$$\lim_{i \to 0} \left(\frac{1}{2} \rho_{i,L}(u) \right)^{\frac{1}{i}} = \sqrt{\rho_{0,L}(u)}$$

the quantity in the limit on the left is the ith mean of $\rho_L(u)$ and $\rho_L(-u)$, while that on the right is the geometric mean of $\rho_L(u)$ and $\rho_L(-u)$.

Falconer in [9] was the first to define i-chord functions, also called generalized chord functions, for integer values of i, and gave some uniqueness results, for $i \geq 1$, by use of a version of the stable manifold theorem of differentiable dynamics. Gardner in [16] generalizes the notion of i-chord function to real values of the parameter i, while Soranzo in [39] extends the definition of i-chord function to $i = \pm\infty$, when K is a convex body, by

$$\rho_{+\infty,K}(u) = \max \{ |\rho_K(u)|, |\rho_K(-u)| \}$$

and

$$\rho_{-\infty,K}(u) = \min \{ |\rho_K(u)|, |\rho_K(-u)| \}$$

and found some results about the determination of convex bodies by these functions.

2.2 The ith section function

For integer values of i, the i-chord function is closely related (via Funk theorem) to the ith section function of a convex body, the function giving the i-dimensional volumes of its intersections with i-dimensional subspaces.

Definition 2.2.1 (ith section function)**.**
Let $p \in \mathbb{R}^n$ and let K be a convex body in $\mathbb{R}^n$ and let i be an integer, $1 \leq i \leq n-1$. The i-section function of K at p is defined on $\mathscr{G}(n,i)$ by

$$G \mapsto \lambda_i(K \cap G).$$

The i-section function of K in a direction $u \in S^{n-1}$ is defined on $\mathscr{G}(n,i,u)$ by

$$G \mapsto \lambda_i(K \cap G).$$

where $\mathscr{G}(n,i)$ denote the Grassmannian manifold of i-dimensional linear subspaces of $\mathbb{R}^n$ and λ_i the i-dimensional Lebesgue measure, while $\mathscr{G}(n,i,u)$ is the manifold of all the i-dimensional affine subspaces of $\mathbb{R}^n$ parallel to the vector u.

We are in front of the following geometric problem:

> « Suppose that $K \subset \mathbb{R}^n$ is a convex body and let p_h be some non-collinear points (some of them are possibly at infinity). Suppose, moreover, that we are given at the points p_h the ith section functions for $i \in \{1,2,\cdots,n-1\}$. Is K then uniquely determined among all convex bodies? »

The ith section functions are a particular case of the notion of dual mixed volumes introduced by Lutwak in [31] and generalized in [18]. Let L be a star set in $\mathbb{R}^n$, and let $i \in \mathbb{R}$ be non-zero. If $1 \le k \le n-1$, the *dual volume* $\tilde{V}_{i,k}(L \cap S)$ is given for $S \in \mathscr{G}(n,k)$ by

$$\tilde{V}_{i,k}(L \cap S) = \frac{1}{2k}\int_{S^{n-1}\cap S} \rho_{i,L}(u)du. \tag{2.3}$$

The function $\tilde{V}_{i,k}(L \cap \cdot)$ is called a *section function.* When $i = k$, the function $\tilde{V}_{i,i}(L \cap \cdot)$ is called *ith section function.*
Observe that

$$\tilde{V}_{i,i}(L \cap S) = \lambda_i(L \cap S), \tag{2.4}$$

for each $S \in \mathscr{G}(n,i)$.
This means that the ith section function is nothing other than the i-dimensional X-ray of L at the origin.

Note that the relation (2.4) expresses the equivalence between the ith section function and the X-ray of order i at the origin of the body K, moreover for $k = i = 1$ the 1-chord function of K at a point p and the 1st section function are nothing other than the ordinary X-ray of K.

2.3 i-chord function and ith section function

In this section we state a lemma and two propositions that show the relationship between i-chord function and ith section function. For further details we refer to Gardner's book [16]. First of all, the notion of the i-chord function is, in general, closely linked to the notion of the X-ray of order i, and this relationship is established by the following lemma.

Lemma 2.3.1.
Let E be a body in $\mathbb{R}^n$ star-shaped at o. Suppose either that $o \notin E$ or that $i \in \mathbb{R}_+$. If $i \neq 0$, then for each $u \in S^{n-1}$,

$$X_{i,o}E(u) = \begin{cases} \frac{1}{i}\left(\rho_E{}^i(u) + \rho_E{}^i(-u)\right) & \text{if } o \in E, \\ \\ \frac{1}{i}\left|\left|\rho_E{}^i(u)\right| - \left|\rho_E{}^i(-u)\right|\right| & \text{if } o \notin E. \end{cases}$$

If $i = 0$ and $o \notin E$, then for each $u \in S^{n-1}$,

$$X_{0,o}E(u) = \left|\log\left|\frac{\rho_E(u)}{\rho_E(-u)}\right|\right|.$$

Therefore, the X-ray of order i is the i-chord function divided by i or the natural logarithm of the i-chord function, according as $i \neq 0$ or $i = 0$, respectively.

We now state the following useful proposition that gives a link between i-chord functions and ith section functions for $1 \leq i \leq n-1$, even if the latter is defined only for integer values of i. This result relies on Funk's theorem [12] and is discussed in detail in [16].

Proposition 2.3.2.
Let K, K' be two convex bodies in $\mathbb{R}^n$, and let $i \in \mathbb{N}$ be such that $1 \leq i \leq n-1$. Then

$$\lambda_i(K \cap S) = \lambda_i(K' \cap S)$$

for all $S \in \mathscr{G}(n,i)$, if and only if

$$\rho_{i,K}(u) = \rho_{i,K'}(u)$$

for all $u \in S^{n-1}$.

This observation allows us to treat the geometric problem of studying the reconstruction of a convex body from X-rays or ith section functions within the more general problem of retrieving a convex body from its i-chord functions. In fact, the concept of chord function is not so appealing from the geometric point of view as much as that of section function.

The next proposition (proved in [40], Proposition 2.2) establishes an important relationship between the kth section function in direction u and the ordinary X-ray in the same direction.

Proposition 2.3.3.
Let $u \in S^{n-1}$ be a fixed direction, and suppose that K and K' are two convex bodies in $\mathbb{R}^n$. Let $k \in \mathbb{N}$ be such that $1 \leq k \leq n-1$. Then

$$\lambda_k(K \cap S) = \lambda_k(K' \cap S)$$

for all $S \in \mathscr{G}(n,k,u)$ if and only if

$$\lambda_1(K \cap l) = \lambda_1(K' \cap l)$$

for every line l parallel to u.

Proof.
Let $l_u + x \in \mathscr{G}(n,k,u)$ be the line passing through $x \in u^{\perp}$ and parallel to $u \in S^{n-1}$. By Fubini's theorem we have

$$\lambda_k(K_j \cap G) = \int_{G \cap u^{\perp}} \lambda_1(K_j \cap (l_u + x)) d\lambda_{k-1}(x)$$

for $j = 1, 2$, and this proves the theorem in one direction.
Now, consider the mapping on $\mathscr{G}(n,k,u)$ such that to each $G \in \mathscr{G}(n,k,u)$ assigns the measure $\lambda_k(K_j \cap G)$. This is the $(k-1)$-dimensional Radon transform of the function $f_j(x) = \lambda_1(K_j \cap (l_u + x))$ for $j = 1, 2$, defined on the $(n-1)$-dimensional subspace $u^{\perp}$. Since the Radon transform is injective it follows that $f_1 = f_2$, that is

$$\lambda_1(K_1 \cap (l_u + x)) = \lambda_1(K_2 \cap (l_u + x)),$$

and this completes the proof. □

Proposition 2.3.3 shows that, when the point source is at the infinity, the kth section function in direction u has as its counterpart the ordinary X-ray. By using Propositions 2.3.2 and 2.3.3 we can reformulate the problem in analytic form:

> « Suppose that $K \subset \mathbb{R}^n$ is a convex body and let p_h be some noncollinear points (some of them are possibly at infinity). Suppose, moreover, that we are given at the points p_h the i-chord functions, with $i \in \mathbb{R}$. Is K then uniquely determined among all convex bodies? »

2.4 The corresponding components

We can establish a correspondence between components of int$(K \triangle K')$, whenever K and K' have the same i-chord function, for $i > 0$, at a point p. Consider two convex bodies K and K' with the same i-chord function at a point $p \in \partial K \cup \partial K'$. Let l be a line through p. Then two things may happen

1. $l \cap \operatorname{int}(K \triangle K')$ has two components;

2. $l \cap \operatorname{int}(K \triangle K')$ is empty.

Moreover, since $i > 0$, $l \cap K$ and $l \cap K'$ are two closed segments which may have a nonempty intersection and in addition no one includes the other.

Definition 2.4.1 (Corresponding components).
Let K and K' be two convex bodies having the same i-chord functions at $p \notin \partial K \cup \partial K'$. If A is a component of $int(K \triangle K')$ then we define

$$A' = \bigcup_{z \in A} \{pz \cap int(K \triangle K')\} \setminus A$$

and we shall say that A and A' correspond to each other through p.

Observe that if $A \subset (K \setminus K')$ then $A' \subset (K' \setminus K)$. In addition, the corresponding components are star-shaped at p but not necessarily convex.

Lemma 2.4.2.
Let K and K' be two convex bodies having the same i-chord functions at $p \notin \partial K \cup \partial K'$. If A and A' are two corresponding components of $int(K \triangle K')$ and correspond to each other through p then A and A' have the same i-chord functions at p.

Proof.
To simplify the notation assume that $p = o$. Let z be a point in A. Denote by u the direction identified by the point z, i.e. $u = \frac{z}{||z||}$, and call l the line issuing from the origin with direction u. Since $z \in A$, $A \cap l$ is a segment. To see this, suppose by contrary that $A = \{z\}$, then $A \cap l = \{z\}$ and $-u\rho_K(-u) = -u\rho_{K'}(-u) = \{z\}$, but this means that $z \in \partial K \cap \partial K'$ and this contradicts the assumptions that $A \subset \operatorname{int}(K \setminus K')$ Since K and K' have the same i-chord functions at o, we have

(a)
$$\left||\rho_K(u)|^i - |\rho_K(-u)|^i\right| = \left||\rho_{K'}(u)|^i - |\rho_{K'}(-u)|^i\right| \tag{2.5}$$

if $o \notin K \cup K'$, while

(b)

$$\rho_K(u)^i + \rho_K(-u)^i = \rho_{K'}(u)^i + \rho_{K'}(-u)^i \tag{2.6}$$

if $o \in K \cap K'$.

(a) Without loss of generality we may assume that o, $-u\rho_K(-u)$ and $u\rho_K(u)$ are in that order on l as well as o, $-u\rho_{K'}(-u)$ and $u\rho_{K'}(u)$, and moreover that

$$0 < \rho_K(u) < \rho_{K'}(u) \tag{2.7}$$

for every $z \in A$.

The relation (2.5) can be rewritten in the following way

$$\rho_K(u)^i - (-\rho_K(-u))^i = \rho_{K'}(u)^i - (-\rho_{K'}(-u))^i$$

and from this follows that

$$(-\rho_{K'}(-u))^i - (-\rho_K(-u))^i = \rho_{K'}(u)^i - \rho_K(u)^i.$$

The last identity tells us that A and A' have the same i-chord functions.

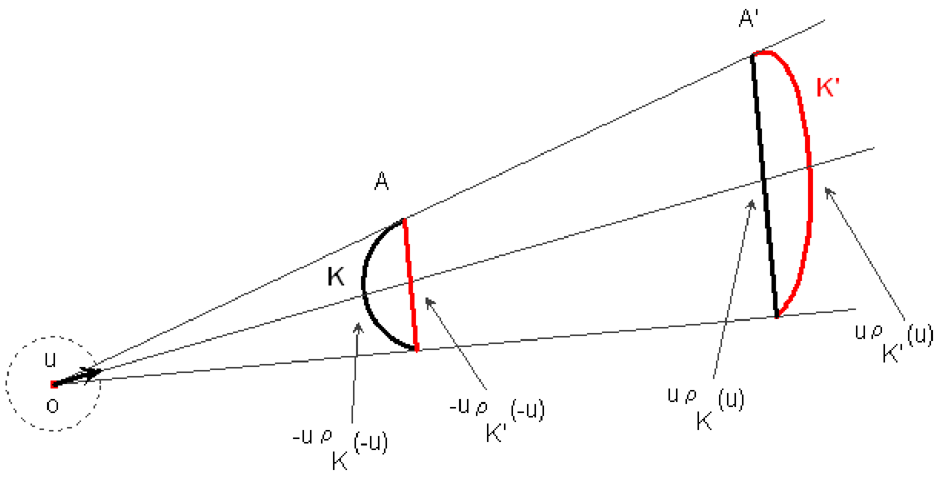


Figure 2.2

(b) Without loss of generality we may suppose that $-u\rho_K(-u)$, o and $u\rho_K(u)$ are in the same order on l as well as $-u\rho_{K'}(-u)$, o, $u\rho_{K'}(u)$.
Assume also that
$$0 < \rho_{K'}(u) < \rho_K(u) \tag{2.8}$$
holds for every $z \in A$.
The relation (2.6) can be rewritten in the following way
$$\rho_{K'}(u)^i - \rho_K(u)^i = \rho_K(-u)^i - \rho_{K'}(-u)^i$$
and this means that A and A' have the same i-chord functions at o.

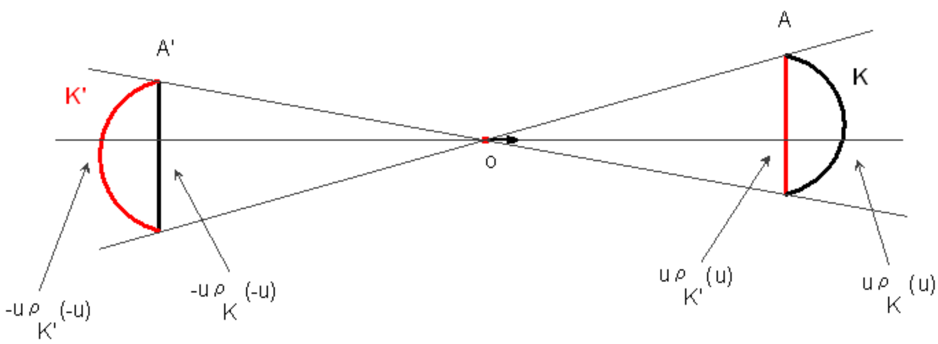


Figure 2.3

Moreover observe that, from (2.8) follows that
$$\rho_{K'}(u)^i < \rho_K(u)^i$$
that is
$$\rho_K(u)^i - \rho_{K'}(u)^i > 0$$
and this means that $A' \cap l$ is a non-degenerate segment, too. $\square$

Now recall some useful topological preliminaries.

Definition 2.4.3 (Space path-connected)**.**
A topological space X is said to be path-connected (or pathwise connected) if there is a path joining any two points in X.

Theorem 2.4.4.
Let X and Y be topological spaces and let $f : X \mapsto Y$ be a continuous function. If X is path-connected then the image $f(X)$ is path-connected.

Proof.
Let $A \subset X$ be path-connected. We want to prove that $f(A)$ is path-connected. In fact, let $y_1, y_2 \in f(A)$, then there exist $x_1, x_2 \in A$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since A is path-connected there exists a path $\gamma : [0,1] \mapsto X$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$ and $\gamma([0,1]) \subset A$. Now, the composition of two continuous functions is continuous, so the function $f \circ \phi : [0,1] \mapsto Y$ is continuous and moreover

$$(f \circ \gamma)(0) = f(\gamma(0)) = f(x_1) = y_1, \ (f \circ \gamma)(1) = f(\gamma(1)) = f(x_2) = y_2$$

and

$$(f \circ \gamma)([0,1]) = f(\gamma([0,1])) \subset f(A)$$

then $f(A)$ is path-connected. ☐

Theorem 2.4.5.
If A is a nonempty connected open subset of $\mathbb{R}^n$ then A is path-connected.

Proof.
Let x, y be two points of A and suppose that y is not reachable from x. Divide A in two subsets X and Y, where X is the set of all the points that are reachable from x, while Y contains all the points that are not reachable from x. Therefore $\{X, Y\}$ is a partition of A, $A = X \dot{\cup} Y$ with X and Y nonempty because at least $x \in X$ and $y \in Y$. Let us show that X and Y are open. Let $x^* \in X \subset A$. Since A is open there exist an open ball $B(x^*)$ centered at x^* such that $x^* \in B(x^*) \subset A$. Since $B(x^*)$ is path-connected, every $z \in B(x^*)$ is reachable from x^* and so also from x.

Consequently $x^* \in B(x^*) \subset X$. Analogously, let $y^* \in Y \subset A$, then there exists an open ball $B(y^*)$ centered at y^* such that $y^* \in B(y^*) \subset A$. Since $B(y^*)$ is path-connected, every $w \in B(y^*)$ is not reachable from x hence $y^* \in B(y^*) \subset Y$ and so Y is open. But this contradicts our assumption because A would not be connected. Therefore $Y = \emptyset$ and $A = X$ is path-connected. $\square$

Proposition 2.4.6.
If K and K' have the same i-chord functions at $p \notin \partial K \cup \partial K'$, and suppose that A is a component of $int(K \triangle K')$, then the set

$$A' = \bigcup_{z \in A} \{pz \cap int(K \triangle K')\} \setminus A$$

is another component of $int(K \triangle K')$.

Proof.
To simplify the notation assume that $p = o$. It follows immediately from Lemma 2.4.2 that if $z \in A$, then $A \cap pz$ and $A' \cap pz$ are non-degenerate segment. On the other hand, if $z \in \partial K \cap \partial K'$, then also the other endpoint of the segment $pz \cap K$ belongs to $\partial K \cap \partial K'$.
We have to distinguish two cases:

(a) A is between the origin o and A';

(b) o is between A and A'.

(a) In this case $\text{pos}(A) = \text{pos}(A')$.
The set

$$U = S^{n-1} \cap \text{pos}(A)$$

is open and connected. Therefore we can represent the component A as

$$A = \left\{ z : -\rho_K \left(-\frac{z}{||z||} \right) < ||z|| < -\rho_{K'} \left(-\frac{z}{||z||} \right), \; \frac{z}{||z||} \in U \right\}$$

with $\rho_K(-v) = \rho_{K'}(-v)$ when v belongs to $\overline{U}$.
Similarly,

$$A' = \left\{ w : \rho_K \left(\frac{w}{||w||} \right) < ||w|| < \rho_{K'} \left(\frac{w}{||w||} \right), \; \frac{w}{||w||} \in U \right\}.$$

If v belongs to $\overline{U}$ then $\rho_K(v) = \rho_{K'}(v)$ and moreover,

$$-\rho_K(-u) < -\rho_{K'}(-u)$$

if and only if

$$\rho_K(u) < \rho_{K'}(u).$$

Let z_1, $z_2 \in \mathrm{pos}(A)$ and suppose that $y_1, y_2 \in A'$. Then

$$y_j = \frac{z_j}{||z_j||}\left[t_j\rho_K\left(\frac{z_j}{||z_j||}\right) + (1-t_j)\rho_{K'}\left(\frac{z_j}{||z_j||}\right)\right]$$

for $j = 1, 2$. If $\frac{z_1}{||z_1||} = \frac{z_2}{||z_2||}$, then y_1 and y_2 are aligned and so they are the endpoints of a segment contained in A'. Otherwise, let

$$x_j = \frac{z_j}{||z_j||}\left[t_j\rho_K\left(-\frac{z_j}{||z_j||}\right) + (1-t_j)\rho_{K'}\left(-\frac{z_j}{||z_j||}\right)\right]$$

for $j = 1, 2$. Since A is a component, A is a nonempty connected open subset of $\mathbb{R}^n$, then A is path-connected, therefore there exists a path $x(s)$ joining x_1 and x_2 interior to A. We may represent it with two mapping f and g given by

$$f : [0,1] \mapsto [0,1] \quad \text{and} \quad g : [0,1] \mapsto U$$

such that

$$f(0) = t_1, \ f(1) = t_2$$

and

$$g(0) = \frac{z_1}{||z_1||}, \ g(1) = \frac{z_2}{||z_2||}.$$

According to these assumptions,

$$x(s) = g(s)\left[f(s)\rho_K(-g(s)) + (1-f(s))\rho_{K'}(-g(s))\right].$$

Then

$$y(s) = g(s)\left[f(s)\rho_K(g(s)) + (1-f(s))\rho_{K'}(g(s))\right]$$

is a path contained in A' which connect y_1 to y_2, therefore A' is path-connected and so connected, and this implies that A' is a component.

(b) Since the origin is between A and A', $\mathrm{pos}(A) = -\mathrm{pos}(A')$. We can follow the same argument in order to obtain the same conclusion. The only difference is that this time the set U is given by

$$U = S^{n-1} \cap \mathrm{pos}(A').$$

With this assumption the expression of A and A' remain unchanged. Moreover, choosing z_1 and z_2 in posA', we can consider y_j and x_j, for $j = 1, 2$, in the same way of the previous case and this completes the proof.

□

Consider two corresponding components A and A' through a point p. If A is nearer to p than A' or, in other words, A is between p and A', we shall say that A is "visible" from p and write $p(A) = A'$, whereas if A' is nearer to p, we write $p^{-1}(A) = A'$; in this way, either $p(A)$ or $p^{-1}(A)$ is defined.

Chapter 3

Determination of planar convex bodies by i-chord functions

In this chapter our goal is to investigate the tomography of convex bodies in the plane.

3.1 i-chord functions of planar convex bodies

Lemma 3.1.1 (Cavalieri principle)**.**
Let $u \in S^{n-1}$ be a direction and let E and E' be two measurable sets such that for each line l parallel to u, $l \cap E$ and $l \cap E'$ are segments of equal length. Then $\lambda_n(E) = \lambda_n(E')$.

The Cavalieri principle is substituted, in modern measure theory, by the Fubini's theorem.
From now on we will restrict our attention to the planar case. The three-dimensional case will be treated in Chapter 4. We are not interested, in this exposition, in higher dimensions.
The two-dimensional Cavalieri principle states that if two measurable sets have the same parallel X-rays, then they have the same area.
For X-rays issuing from a point there is a substitute for the Cavalieri principle. This fact was for the first time observed and exploited by Volčič in [44], where the author introduced an appropriate measure which is preserved if two measurable sets have the same point X-rays. The idea of replacing Lebesgue measure by the measure ν_i of Definition 3.1.2, which has seeds in work of Finch, Smith and Solmon [23] and of Falconer (see [11, Lemma2]), is due to Volčič. This idea has been extended by

Gardner ([14]) to i-chord functions for $i \in \mathbb{Z}$ and later on the same author in his book [16] to all real values of i.

Definition 3.1.2 (The measure ν_k).
Let $\mathscr{L}_2$ be the class of bounded Lebesgue measurable subset of $\mathbb{R}^2$. Let l be a line chosen as the x-axis of a Cartesian coordinate system in $\mathbb{R}^2$. If $E \in \mathscr{L}_2$, define for each $k \in \mathbb{R}$,

$$\nu_k(E) = \iint_E |y|^{k-2} dxdy.$$

ν_k is a measure on $\mathscr{L}_2$ and the line l will be called the base line *for ν_k.*

Observe that

$$\nu_2(E) = \lambda_2(E).$$

If $k > 1$, then ν_k is a finite measure, but if $k \leq 1$, then ν_k is a σ-finite measure in $\mathbb{R}^2$, which is finite on sets having positive distance from l.

Lemma 3.1.3.
The set $F = \{(x, y) : a|x - x_0| \leq y \leq b\}$, with $a > 0$ has finite ν_k-measure for every positive value of k.

Proof.
Since the integrand $f(x, y) = |y|^{k-2}$ is an unbounded function for $y = 0$, this is an improper integral of second kind. Moreover f is an even function therefore

$$\nu_k(F) = \iint_F |y|^{k-2} dxdy = 2 \lim_{\varepsilon \to 0} \iint_{T_\varepsilon} y^{k-2} dxdy$$

where T_ε is the trapezium in the first quadrant (see Figure 3.1)

$$T_\varepsilon = \left\{(x, y) : -\frac{y}{a} + x_0 \leq x \leq \frac{y}{a} + x_0,\ \varepsilon \leq y \leq b\right\}.$$

Since T_ε is a normal domain with respect to the y-axis, the computation of this

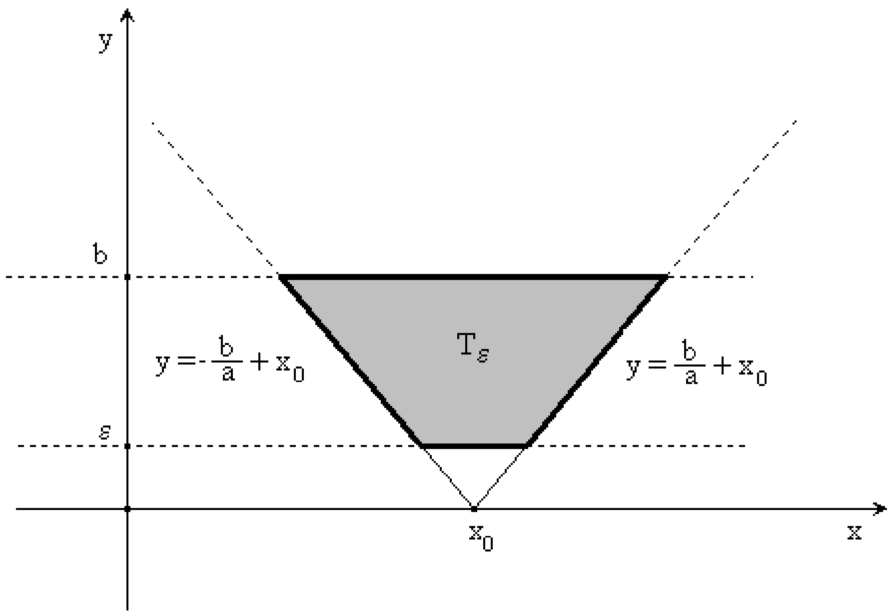


Figure 3.1

double integral reduces to:

$$
\begin{aligned}
\nu_k(F) &= 2\lim_{\varepsilon\to 0}\iint_{T_\varepsilon} y^{k-2}dxdy = 2\lim_{\varepsilon\to 0}\int_\varepsilon^b \left(\int_{-\frac{y}{a}+x_0}^{\frac{y}{a}+x_0} y^{k-2}dx\right)dy \\
&= 2\lim_{\varepsilon\to 0}\int_\varepsilon^b y^{k-2}\left(\int_{-\frac{y}{a}+x_0}^{\frac{y}{a}+x_0} dx\right)dy = 2\lim_{\varepsilon\to 0}\int_\varepsilon^b y^{k-2}\left(\frac{2y}{a}\right)dy \\
&= \frac{4}{a}\lim_{\varepsilon\to 0}\int_\varepsilon^b y^{k-1}dy = \frac{4}{a}\lim_{\varepsilon\to 0}\left[\frac{y^k}{k}\right]_\varepsilon^b \\
&= \frac{4}{a}\lim_{\varepsilon\to 0}\left[\frac{b^k-\varepsilon^k}{k}\right] = \frac{4b^k}{ak} < \infty.
\end{aligned}
$$

□

Let $E_1, E_2 \in \mathscr{L}_2$ with $\lambda_2(E_j) > 0$, $j = 1, 2$. Let $p = (x_0, y_0) \in \mathbb{R}^2$, and suppose that E_1 and E_2 are bodies star-shaped at p, with the same i-chord functions at p, for some i. Moreover, suppose that $E_1 \cup E_2$ is contained in the half-plane $\{(x, y) : y > 0\}$

and that E_1 is between p and E_2. Let (r, θ) be polar coordinates centered at p

$$\begin{cases} x = x_0 + r\cos\theta & r \in [0, +\infty) \\ y = y_0 + r\sin\theta & \theta \in [0, \pi] \end{cases} . \tag{3.1}$$

Let $0 \le \alpha < \beta \le \pi$, and let

$$E_j = \{(r, \theta) : r_j(\theta) \le r \le s_j(\theta),\ \alpha \le \theta \le \beta\},$$

for $j = 1, 2$.

Since E_1 and E_2 have, by assumptions, the same i-chord functions at p, and $p \notin E_1 \cup E_2$ we have that

$$s_1(\theta)^i - r_1(\theta)^i = s_2(\theta)^i - r_2(\theta)^i \tag{3.2}$$

or

$$\frac{s_1(\theta)}{r_1(\theta)} = \frac{s_2(\theta)}{r_2(\theta)} \tag{3.3}$$

hold for $\alpha \le \theta \le \beta$ when $i \neq 0$ or $i = 0$, respectively.
For $j = 1, 2$, the expression of ν_k in polar coordinates is

$$\nu_k(E_j) = \int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} |y_0 + r\sin\theta|^{k-2} r\,dr\,d\theta.$$

If $i \neq 0$ we put $t = r^i$ and we get

$$\nu_k(E_j) = \int_\alpha^\beta \int_{r_j(\theta)^i}^{s_j(\theta)^i} \frac{1}{i} \cdot t^{\frac{2-i}{i}} \left(y_0 + t^{\frac{1}{i}} \sin\theta\right)^{k-2} dt\,d\theta, \tag{3.4}$$

while if $i = 0$ making the substitution $t = \log r$ we obtain

$$\nu_k(E_j) = \int_\alpha^\beta \int_{\log r_j(\theta)}^{\log s_j(\theta)} e^{2t} \left(y_0 + e^t \sin\theta\right)^{k-2} dt\,d\theta. \tag{3.5}$$

For the next three lemmas we have the same assumptions just described.

Lemma 3.1.4.
Let $i \in \mathbb{R}$. Suppose that E_1 and E_2 are defined as above, with finite ν_i-measure and the same i-chord functions at $p = (x_0, 0)$. Then $\nu_i(E_1) = \nu_i(E_2)$.

Proof.
If $i \neq 0$, substituting $k = i$ and $y_0 = 0$ in (3.4) for $j = 1, 2$ we get

$$\begin{aligned} \nu_i(E_j) &= \int_\alpha^\beta \int_{r_j(\theta)^i}^{s_j(\theta)^i} \frac{1}{i} \cdot t^{\frac{2-i}{i}} (t^{\frac{1}{i}} \sin\theta)^{i-2} dt d\theta \\ &= \int_\alpha^\beta (\sin\theta)^{i-2} \left(\int_{r_j(\theta)^i}^{s_j(\theta)^i} \frac{1}{i} dt \right) d\theta \\ &= \int_\alpha^\beta (\sin\theta)^{i-2} \left(\frac{s_j(\theta)^i - r_j(\theta)^i}{i} \right) d\theta. \end{aligned}$$

While if $i = 0$, substituting $k = i$ and $y_0 = 0$ in (3.5) for $j = 1, 2$ we get

$$\begin{aligned} \nu_0(E_j) &= \int_\alpha^\beta \int_{\log r_j(\theta)}^{\log s_j(\theta)} e^{2t} (e^t \sin\theta)^{-2} dt d\theta \\ &= \int_\alpha^\beta (\sin\theta)^{-2} \int_{\log r_j(\theta)}^{\log s_j(\theta)} dt d\theta \\ &= \int_\alpha^\beta (\sin\theta)^{-2} \left(\log \frac{s_j(\theta)}{r_j(\theta)} \right) d\theta \end{aligned}$$

and relations (3.2) and (3.3) complete the proof. □

For i-chord functions at a point p we have to distinguish between equality and equality almost everywhere. This difference does not exist for convex bodies not containing p in their boundaries.
For example, let K be the upper half of the unit disk and let K' be the centered disk of radius $\frac{1}{2}$, (see Figure 3.2). These two convex bodies, K and K', have 1-chord functions at the origin equal almost everywhere, but not everywhere, since $o \in \partial K$.
If the i-chord functions of two star bodies at a point p agree almost everywhere, and p is either contained in the interior of the bodies or exterior to them, then the i-chord functions at p agree everywhere. Moreover, if the point p is exterior to both bodies, then the equality of the i-chord functions at p implies that the bodies have common supporting lines through p.

Lemma 3.1.5.
Let $i \in \mathbb{R}$. Suppose that E_1 and E_2 are defined as above, with finite ν_{i-1}-measure and the same i-chord functions at $p = (x_0, 0)$. Then the centroids of E_1 and E_2, with respect to the measure ν_{i-1}, lie on the same line through p.

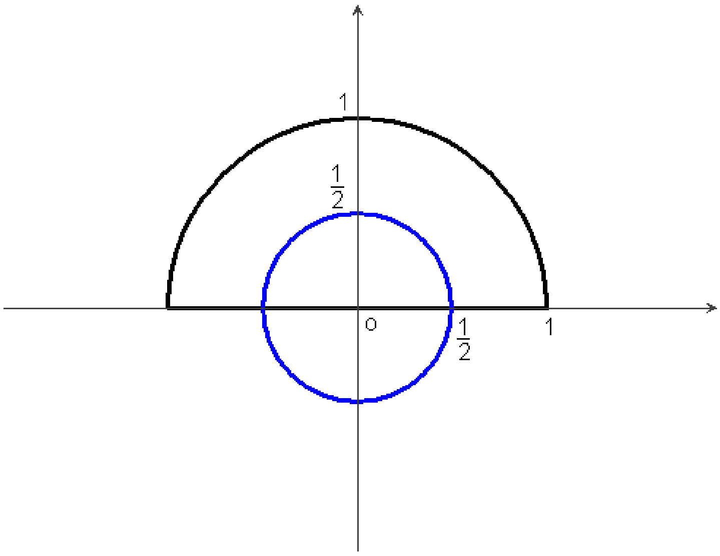


Figure 3.2

Proof.
Denote by $c_j = (x_j, y_j)$ the coordinates of the centroid of E_j, $j = 1, 2$, with respect to the measure ν_{i-1}. Then for $j = 1, 2$ we have

$$x_j = \frac{1}{\nu_{i-1}(E_j)} \iint_{E_j} xy^{i-3} dxdy$$

and

$$y_j = \frac{1}{\nu_{i-1}(E_j)} \iint_{E_j} y^{i-2} dxdy = \frac{\nu_i(E_j)}{\nu_{i-1}(E_j)}.$$

Since, by assumption, $\nu_{i-1}(E_j)$ is finite for $j = 1, 2$ then also $\nu_i(E_j)$ is finite for $j = 1, 2$. Now, compute the slope m_j of the line through p and (x_j, y_j), for $j = 1, 2$.

$$\begin{aligned} m_j = \frac{y_j}{x_j - x_0} &= \frac{\frac{\nu_i(E_j)}{\nu_{i-1}(E_j)}}{\frac{1}{\nu_{i-1}(E_j)} \iint_{E_j} xy^{i-3} dxdy - x_0} \\ &= \frac{\frac{\nu_i(E_j)}{\nu_{i-1}(E_j)}}{\frac{\iint_{E_j} xy^{i-3} dxdy - x_0 \iint_{E_j} y^{i-3} dxdy}{\nu_{i-1}(E_j)}} \\ &= \frac{\nu_i(E_j)}{\iint_{E_j} (x - x_0) y^{i-3} dxdy}. \end{aligned}$$

Using polar coordinates centered at p we get

$$\begin{aligned}
m_j &= \frac{\nu_i(E_j)}{\int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} r\cos\theta (r\sin\theta)^{i-3} r dr d\theta} \\
&= \frac{\nu_i(E_j)}{\int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} r^{i-1}\cos\theta \sin\theta^{i-3} dr d\theta} \\
&= \frac{\nu_i(E_j)}{\int_\alpha^\beta \cos\theta \sin\theta^{i-3}\left(\int_{r_j(\theta)}^{s_j(\theta)} r^{i-1} dr\right) d\theta} \\
&= \begin{cases} \frac{\nu_i(E_j)}{\int_\alpha^\beta \cos\theta\sin\theta^{i-3}\log\frac{s_j(\theta)}{r_j(\theta)} d\theta} & \text{if } i = 0 \\ \\ \frac{\nu_i(E_j)}{\int_\alpha^\beta \cos\theta\sin\theta^{i-3}\frac{s_j(\theta)^i - r_j(\theta)^i}{i} d\theta} & \text{if } i \neq 0 \end{cases}.
\end{aligned}$$

By Lemma 3.1.4 and by relations (3.2) and (3.3), the equality of the i-chord functions at p implies that the expression of the slopes m_1 and m_2 are equal and this means that the line through p containing the centroid of E_1 and the line through p containing the centroid of E_2, are the same. □

Lemma 3.1.6.
Let $i \in \mathbb{R}$. Suppose that E_1 and E_2 are defined as above, with finite ν_i-measure and the same i-chord functions at $p = (x_0, y_0)$. Then

(a) *If $y_0 = 0$ and $k > \max\{i, 1\}$, then $\nu_k(E_1) < \nu_k(E_2)$, if $i \geq 0$, and $\nu_k(E_1) > \nu_k(E_2)$, if $i < 0$.*

(b) *If $y_0 < 0$, and E_1 has finite ν_i-measure, then $\nu_i(E_1) < \nu_i(E_2)$, if $i > 2$, and $\nu_i(E_1) > \nu_i(E_2)$, if $i < 2$; if $y_0 > 0$, these inequalities are reversed.*

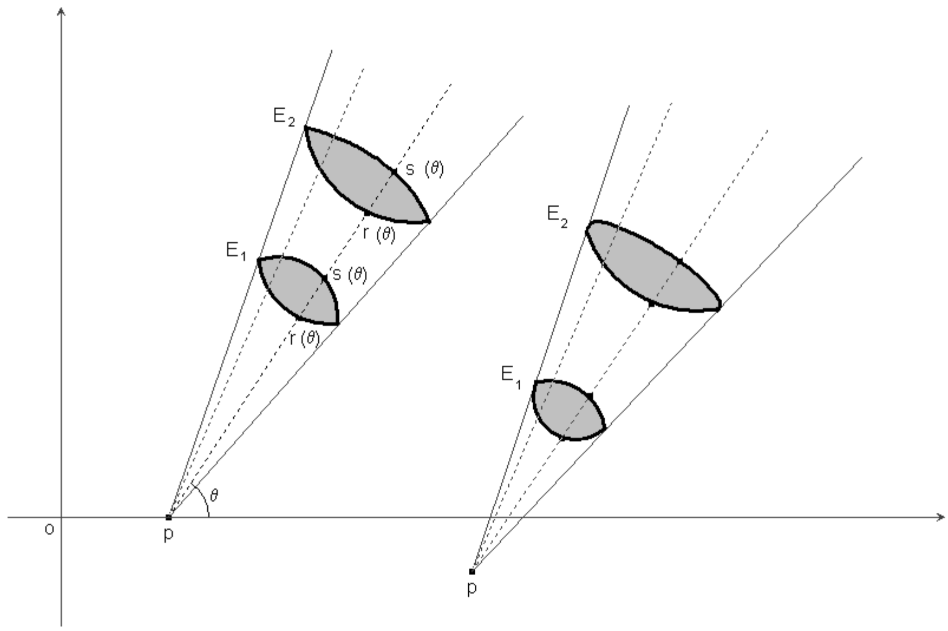


Figure 3.3

Proof.

(a) Assume $y_0 = 0$. If $i \neq 0$, the expression (3.4) of ν_k becomes

$$\nu_k(E_j) = \int_\alpha^\beta \int_{r_j(\theta)^i}^{s_j(\theta)^i} \frac{1}{i} \cdot t^{\frac{k-i}{i}} (\sin\theta)^{k-2} dt d\theta. \tag{3.6}$$

By assumption $k > max\{1, i\}$, so in particular $k > i$, therefore the integrand increases with t.
Similarly, if $i = 0$, the expression (3.5) of ν_k becomes

$$\nu_k(E_j) = \int_\alpha^\beta \int_{\log r_j(\theta)}^{\log s_j(\theta)} e^{kt} (\sin\theta)^{k-2} dt d\theta. \tag{3.7}$$

Again, since $k > 0$ the integrand increases with t. In both cases, the range of the inner integral is of the same length for $j = 1, 2$, so if $i \geq 0$, $\nu_k(E_1) < \nu_k(E_2)$. If $i < 0$, the integrands decrease and moreover we have that $s_j(\theta)^i < r_j(\theta)^i$ for $j = 1, 2$, so by interchanging the limits of the inner integral we obtain

$\nu_k(E_1) > \nu_k(E_2)$.

(b) If $i \neq 0$, substituting $k = i$ the derivative with respect to t of the integrand in (3.4) is

$$-\frac{i-2}{i^2}\left(t^{\frac{1}{i}} \sin\theta + y_0\right)^{i-3} t^{\frac{2-2i}{i}} y_0.$$

Suppose that $y_0 < 0$. If $i > 2$, the integrand increases with t, and the equality of i-chord functions at p implies $\nu_i(E_1) < \nu_i(E_2)$. If $0 < i < 2$, the integrand decreases with t, so $\nu_i(E_1) > \nu_i(E_2)$. If $i < 0$, the integrand decreases but $s_j(\theta)^i < r_j(\theta)^i$ for $j = 1, 2$, so by interchanging the limits of the inner integral we obtain $\nu_i(E_1) > \nu_i(E_2)$. In the same way we treat the case $i = 0$. Substituting $k = i = 0$, the derivative with respect to t of the integrand in (3.5) is

$$2\left(e^t \sin\theta + y_0\right)^{-3} e^{2t} y_0$$

and this decreases with t.

The case when $y_0 > 0$ is dealt with similarly.

$\square$

Note that the Lemma 3.1.4 and the Lemma 3.1.6 (a) hold for all values of i, while the Lemma 3.1.6 (b) holds for all $i \neq 2$, for this reason Lemma 3.1.6 (b) is unavailable for $i = 2$.

Lemma 3.1.7.
Let $i > 0$, and let $a, b, c, d \in \mathbb{R}_+$ such that $0 \leq a < b \leq c < d$. If

$$b^i - a^i = d^i - c^i$$

then

(a) $b^k - a^k > d^k - c^k$ *if* $k < i$;

(b) $b^k - a^k < d^k - c^k$ *if* $k > i$.

Proof.
Letting

$$\gamma = b^i - a^i = d^i - c^i > 0$$

we can write

$$b = (\gamma + a^i)^{\frac{1}{i}}$$

hence

$$b^k - a^k = (\gamma + a^i)^{\frac{k}{i}} - a^k.$$

Consider now the function f defined by

$$f(t) = (\gamma + t^i)^{\frac{k}{i}} - t^k.$$

Since

$$\begin{aligned} f'(t) &= kt^{i-1}(\gamma + t^i)^{\frac{k-i}{i}} - kt^{k-1} \\ &= kt^{i-1}\left[(\gamma + t^i)^{\frac{k-i}{i}} - t^{k-i}\right], \end{aligned}$$

it follows that f is constant if $k = i$, while it is decreasing for $k < i$ and is increasing for $k > i$. This implies that for $a < c$ we have

$$\begin{cases} f(a) > f(c) & \text{if } k < i \\ f(a) < f(c) & \text{if } k > i \end{cases}$$

and this completes the proof. □

Lemma 3.1.8.
Let $a, b, c, d \in \mathbb{R}_+$ such that $0 \leq a < b \leq c < d$. If

$$b - a = d - c$$

then

$$\frac{b}{a} > \frac{d}{c}.$$

Proof.
Letting

$$\gamma = b - a = d - c > 0$$

we can write $b = \gamma + a$ hence

$$\frac{b}{a} = \frac{\gamma + a}{a} = \frac{\gamma}{a} + 1.$$

Consider now the function f defined by $f(t) = \frac{\gamma}{t} + 1$. Since

$$f'(t) = -\frac{\gamma}{t^2} < 0$$

it follows that f is decreasing. Then for $a < c$ we have $f(a) > f(c)$, that is

$$\frac{b}{a} > \frac{d}{c}.$$

$\square$

Lemma 3.1.9.
Let $i > 0$. Suppose that E_1 and E_2 are defined as above, with finite area and the same i-chord functions at $p = (x_0, y_0)$. Then

(a) $\lambda_2(E_1) = \lambda_2(E_2)$ *if $i = 2$;*

(b) $\lambda_2(E_1) > \lambda_2(E_2)$ *if $i < 2$.*

Proof.
Compute $\lambda_2(E_j)$ using polar coordinates centered at p.

$$\begin{aligned} \lambda_2(E_j) &= \iint_{E_j} dx dy = \int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} r dr d\theta \\ &= \int_\alpha^\beta \left[\frac{r^2}{2}\right]_{r_1(\theta)}^{s_j(\theta)} d\theta \\ &= \int_\alpha^\beta \frac{s_j(\theta)^2 - r_j(\theta)^2}{2} d\theta. \end{aligned}$$

Equality (3.2) of i-chord functions at p implies that $\lambda_2(E_1) = \lambda_2(E_2)$ for $i = 2$. If $i < 2$, by putting $a = r_1(\theta)$, $b = s_1(\theta)$, $c = r_2(\theta)$ and $d = s_2(\theta)$ and using the Lemma 3.1.7 (a) we obtain the conclusion. $\square$

Observe that the previous lemma holds independently on the position of p in the Euclidean plane, since y_0 does not appear in the expression for the integral.

Lemma 3.1.10.
Let $i > 0$. Assume that E_1 and E_2 are bounded measurable subset with the same 1-chord function at a direction parallel to the base line. Then $\nu_i(E_1) > \nu_i(E_2)$ for $i < 2$.

Proof.
By assumption, the equality of 1-chord function implies that E_1 and E_2 can be parametrized in this way: $E_j = \{(x,y) : a \le x \le b, r_j(x) \le y \le s_j(x)\}$, for $j = 1, 2$ with

$$s_1(x) - r_1(x) = s_2(x) - r_2(x).$$

Compute now

$$\begin{aligned} \nu_i(E_1) &= \iint_{E_1} y^{i-2} dx dy = \int_a^b \left(\int_{r_1(x)}^{s_1(x)} y^{i-2} dy \right) dx \\ &= \begin{cases} \int_a^b \left(\frac{s_1(x)^{i-1} - r_1(x)^{i-1}}{i-1} \right) dx & i \neq 1 \\ \\ \int_a^b \ln \frac{s_1(x)}{r_1(x)} dx & i = 1 \end{cases} . \end{aligned}$$

By Lemma 3.1.7 we have that

$$s_1(x)^{i-1} - r_1(x)^{i-1} > s_2(x)^{i-1} - r_2(x)^{i-1} \quad \text{for } i < 2$$

while

$$s_1(x)^{i-1} - r_1(x)^{i-1} < s_2(x)^{i-1} - r_2(x)^{i-1} \quad \text{for } i > 2$$

and this implies that

$$\nu_i(E_1) > \nu_i(E_2) \quad \text{for } i < 2$$

while

$$\nu_i(E_1) < \nu_i(E_2) \quad \text{for } i > 2.$$

If $i = 1$ we apply Lemma 3.1.8 and we get

$$\frac{s_1(x)}{r_1(x)} > \frac{s_2(x)}{r_2(x)}$$

therefore

$$\nu_1(E_1) > \nu_1(E_2)$$

and this complete the proof. $\square$

3.2 A two-point solution

Recall the definition of corresponding components.

Definition 3.2.1 (Corresponding components).
Let K and K' be two convex bodies having the same i-chord functions at $p \notin \partial K \cup \partial K'$. If A is a component of $int(K \triangle K')$ then we define

$$A' = \bigcup_{z \in A} \{pz \cap int(K \triangle K')\} \setminus A$$

and we shall say that A and A' correspond to each other through p.

Consider two corresponding components A and A' through a point p. If A is nearer to p than A' or, in other words, A is between p and A', we shall say that A is "visible" from p and write $p(A) = A'$, whereas if A' is between p and A, we write $p^{-1}(A) = A'$; in this way, either $p(A)$ or $p^{-1}(A)$ is defined.

From now on, we shall suppose that i-chord functions, at two distinct points, of a convex body K are given.
In the following proofs we use the measure introduced in Definition 3.1.2 and some of the lemmas from the previous section.
Here, and for the remainder of this chapter, we only need these for $i > 0$, however.

Theorem 3.2.2.
Let $i > 0$ and suppose that K, K' are planar convex bodies and that p_1, p_2 are distinct points in $\mathbb{R}^2$ such that K and K' have the same i-chord functions at p_1 and p_2. Moreover, suppose that the line l through p_1 and p_2 meets $int K$, p_1 and p_2 do not belong to $int K$, and K and K' either both meet the segment $[p_1, p_2]$ or are both disjoint from $[p_1, p_2]$. Then $K = K'$.

Proof.
Assume that $K' \neq K$. Consider the measure ν_k with the line l as base line.
First let us prove that no component C of $\text{int}(K \triangle K')$ has its closure $\bar{C}$ meeting the line l.

- K intersects the line l between p_1 and p_2.
  Suppose on contrary that there exists such a component C visible from p_1 so that $C' = p_1(C)$ is defined. Moreover, C' is visible from p_2 and $C'' = p_2(C')$ is defined and intersects C, but by definition of connected component necessarily $C = C''$, therefore $p_2(C') = C$, (see Figure 3.4). By construction C and C' have the same i-chord functions at p_1 and p_2. In particular, the equality of i-chord function at p_1, by Lemma 3.1.6 (a), gives $\nu_{i+1}(C) < \nu_{i+1}(C')$, while the equality of i-chord functions at p_2 gives the opposite inequality $\nu_{i+1}(C') < \nu_{i+1}(C'') = \nu_{i+1}(C)$, a contradiction.

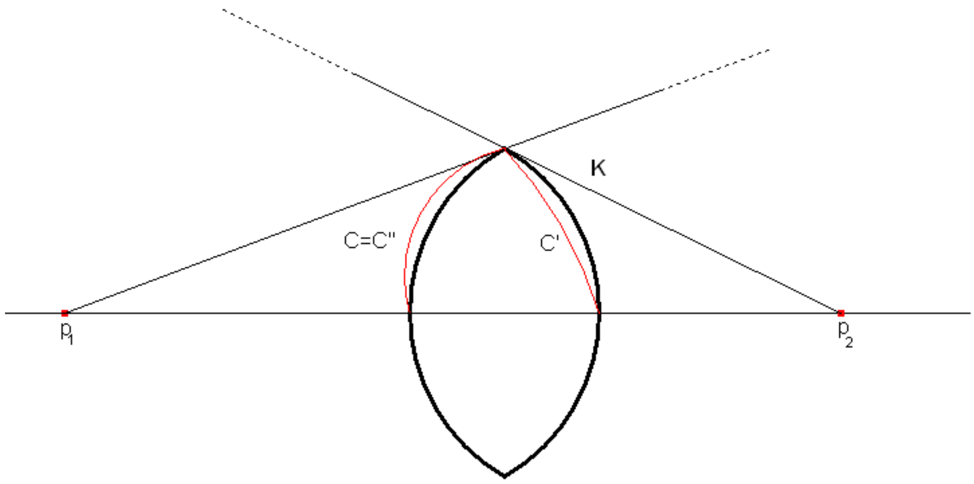


Figure 3.4

- p_1, p_2 and $K \cap l$ are in that order on l.
  Let C be a component visible from both p_1 and p_2. Then $C' = p_2(C)$ and $C'' = p_1(C)$ intersect, so again by definition of connected component C' and C'' have to coincide, $C' = C''$. Consider any line t through p_2 separating p_1 from $K \cup K'$ and let ν_i be the measure having the line t as base line, so p_1 and p_2 correspond, respectively, to negative y-coordinate and zero y-coordinate, (see Figure 3.5). By Lemma 3.1.4 $C' = p_2(C)$ implies $\nu_i(C) = \nu_i(C')$, but by Lemma 3.1.6 (b) $C' = p_1(C)$ implies $\nu_i(C) < \nu_i(C')$ if $i > 2$ or $\nu_i(C) > \nu_i(C')$ if $1 \le i < 2$, a contradiction.

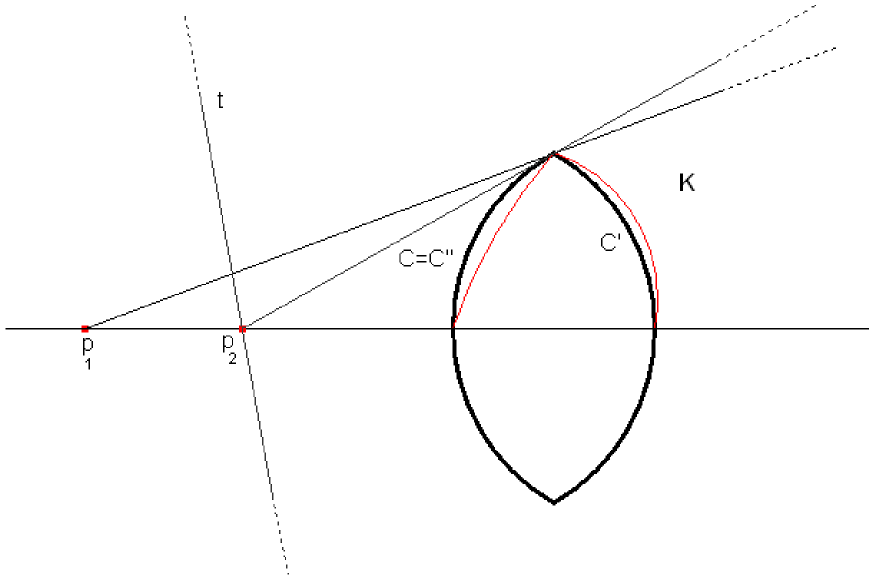


Figure 3.5

Denote by u the direction of the line l. When $i = 2$ assume that $l \cap C \neq \emptyset$, and also that ∂K and $\partial K'$ meet l at points at distances $r_1 = -\rho_{K-p_1}(-u)$, $s_1 = \rho_{K-p_1}(u)$ and $s_1 = \rho_{K-p_1}(u)$, $r_2 = -\rho_{K'-p_1}(-u)$ and $s_2 = \rho_{K'-p_1}(u)$, respectively, from p_1 with $r_j < s_j$, for$j = 1$, 2. The equality of 2-chord functions of K and K' at p_1 implies that

$$s_1{}^2 - r_1{}^2 = s_2{}^2 - r_2{}^2.$$

If the distance between p_1 and p_2 is b, then the equality of 2-chord functions at p_2 implies

$$(s_1 + b)^2 - (r_1 + b)^2 = (s_2 + b)^2 - (r_2 + b)^2.$$

Observe that for $j = 1$, 2

$$(s_j + b)^2 - (r_j + b)^2 = \int_{r_j{}^2}^{s_j{}^2} (1 + bt^{-\frac{1}{2}})dt.$$

For $j = 1, 2$, the interval of integration is of the same length, and the integrand decreases as t increases. Therefore $r_1 = r_2$ and $s_1 = s_2$, contradicting the assumption on C. These arguments prove that there is no component which meets l. There is a component C' of $\mathrm{int}\,(K \triangle K')$, disjoint from C, such that C and C' have the same 2-chord functions at p_1 and p_2. By Lemma 3.1.3, C and C' have finite ν_1-measure , so by Lemma 3.1.5 C and C' have their centroids c and c', respectively, with respect to the measure ν_1, lying on the same line $l_j \neq l$ through p_j, for $j = 1, 2$. But this implies that $c = c' = l_1 \cap l_2$, which is impossible since C and C' are disjoint.

As seen before, there is a nonempty component C_1 of $\mathrm{int}\,(K \triangle K')$ such that its closure $\overline{C_1}$ does not meet l, then C_1 must be away enough from l.

- p_1, p_2, $K \cap l$ are in that order on l;
  Moreover suppose that C_1 is visible from neither from p_1 nor from p_2. Then $C_2 = {p_2}^{-1}(C_1)$ is defined and consequently C_2 is visible from p_1 thus $C_3 = p_1(C_2)$ and $C_4 = {p_2}^{-1}(C_3)$ are also defined. We generate so a sequence of

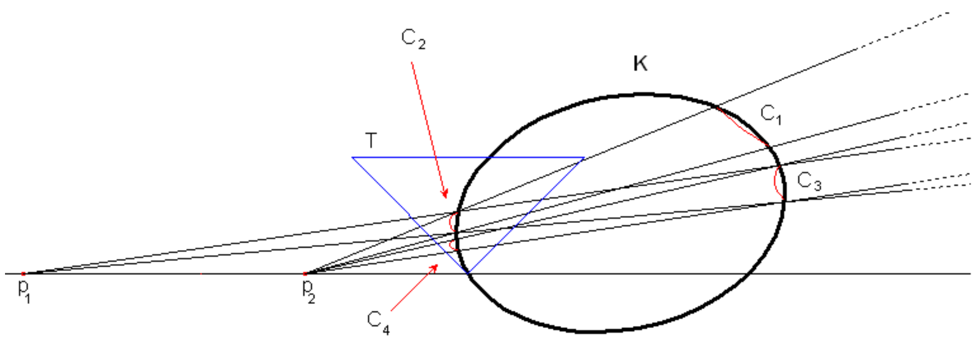


Figure 3.6

disjoint components of $\mathrm{int}\,(K \triangle K')$

$$C_{2n+1} = p_1(C_{2n}) \text{ and } C_{2n+2} = {p_2}^{-1}(C_{2n+1})$$

for $n \in \mathbb{N}$.

Observe that the Lebesgue measure of these components is decreasing, but they have equal ν_i-measure, therefore by Lemma 3.1.4

$$\nu_i(C_1) = \nu_i(C_2) = \cdots = \nu_i(C_n) = \cdots = \delta > 0$$

for $n \in \mathbb{N}$. From the convexity of K, there exists a subsequence $\{C_{2n}\}_{n\in\mathbb{N}}$ contained in a triangle T with a vertex on l and having basis parallel to l, (see Figure 3.6). By Lemma 3.1.3, the triangle T has finite ν_i-measure

$$C_{2n} \subset T \quad \forall n \in \mathbb{N} \implies \bigcup_{n\in\mathbb{N}} C_{2n} \subset T.$$

By countable additivity and monotonicity of the measure ν_i we get

$$\sum_{n\in\mathbb{N}} \nu_i(C_{2n}) = \nu_i\left(\bigcup_{n\in\mathbb{N}} C_{2n}\right) < \nu_i(T) < \infty,$$

but, on the other hand

$$\sum_{n\in\mathbb{N}} \nu_i(C_{2n}) = \sum_{n\in\mathbb{N}} \delta = \infty,$$

a contradiction.
Similarly, if C_1 is visible from p_2 and so also from p_1, then we can consider the sequence of disjoint components of int $(K \triangle K')$ given by

$$C_{2n+1} = {p_2}^{-1}(C_{2n}) \text{ and } C_{2n+2} = p_1(C_{2n+1})$$

for $n \in \mathbb{N}$ and reach the same conclusion.

- p_1, $K \cap l$, p_2 are in that order on l;
  In this case, when C_1 is not visible from p_1, we consider the sequence is

  $$C_{2n+1} = {p_2}^{-1}(C_{2n}) \text{ and } C_{2n+2} = {p_1}^{-1}(C_{2n+1})$$

  for $n \in \mathbb{N}$.
  While, if C_1 is visible from p_1, then C_1 is not visible from p_2 and we consider the following sequence

  $$C_{2n+1} = {p_1}^{-1}(C_{2n}) \text{ and } C_{2n+2} = {p_2}^{-1}(C_{2n+1})$$

  for $n \in \mathbb{N}$.

In all these situations, we can find a triangle having finite ν_i-measure which contains a subsequence of C_n reaching the same contradiction as in the first case. $\square$

Theorem 3.2.3.
Let $i \geq 1$ and suppose that K and K' are planar convex bodies and that p_1, p_2 are distinct points (possibly one at infinity) such that K and K' have the same i-chord functions at p_1 and p_2. Moreover, suppose that the line l through p_1 and p_2 supports K and that p_1 and p_2 do not belong to K. Then $K = K'$.

Proof.
We can use the same argument followed in the previous theorem in order to show that no component C of $\operatorname{int}(K \triangle K')$ has its closure $\overline{C}$ meeting the line l. This means that the components of $\operatorname{int}(K \triangle K')$ have positive distance from the line l and therefore $K \cap l = K' \cap l$ if these sets are line segments. Suppose that $K \cap l$ and $K' \cap l$ are single points. Without loss of generality we may assume that l is the x-axis, p_1 the origin and $p_2 = (x_2, 0)$, with $x_2 > 0$, and suppose that $K \subset \{(x, y) : y \geq 0\}$. Consider the functions $g_1(u)$ and $g_2(v)$ defined as follows:

$$g_1(u) = \nu_i(K_1(u)), \quad u \in \left]0, \frac{\pi}{2}\right[,$$

$$g_2(v) = \nu_i(K_2(v)), \quad v \in \left]\frac{\pi}{2}, \pi\right[,$$

where

$$K_1(u) = \{(x, y) : y \geq x \tan u\} \cap K$$

and

$$K_2(v) = \{(x, y) : y \geq (x - x_2) \tan v\} \cap K.$$

Both functions are continuous on their supports and strictly monotone. In particular g_1 is decreasing, while g_2 is increasing. Moreover, they are such that

$$\lim_{u \to 0} g_1(u) = \nu_i(K)$$

and

$$\lim_{v \to \pi} g_2(v) = \nu_i(K).$$

For u small enough, there exists a unique angle $v(u)$ such $g_1(u) = g_2(v(u))$ and such that the corresponding lines $y = x \tan u$ and $y = (x - x_2) \tan v(u)$ intersect at a point

$\overline{q}(u)$ interior to K. Therefore

$$\lim_{u\to 0} \overline{q}(u) = \overline{q} \in \partial K \cap l,$$

that is, $\overline{q}$ is uniquely determined by i-chord functions of K, and this means that $K \cap l = K' \cap l$.
Since l supports K, we have to distinguish two cases:

(a) there exist a line t distinct from l, supporting K at some point $q \in \partial K \in l$;

(b) l is the only line supporting K at every point in $\partial K \cap l$.

In both cases, if l (or t) is a supporting line for K, then l (or t) is a supporting line for K', too. In fact, if l is the only line supporting K at every point in $\partial K \cap l$ and suppose by contrary that there exists a line t through q, distinct from l, supporting K', then $t \cap \operatorname{int} K' \neq \emptyset$. This implies that there is a component C of $\operatorname{int}(K \setminus K')$ such that $\bar{C} \cap l \neq \emptyset$, a contradiction.

Case **(a)**

Let $t \neq l$ be a line supporting K at $q \in \partial K \cap l$. Take the sequence of components of $\operatorname{int}(K \triangle K')$ defined as follows.

$$C_{2n+1} = p_2(C_{2n}) \text{ and } C_{2n+2} = {p_1}^{-1}(C_{2n+1})$$

for all $n \in \mathbb{N}$. Again if we consider the subsequence $\{C_{2n}\}_{n\in\mathbb{N}}$ contained in a triangle with an edge on the line t separating p_1 from $K \cup K'$ we get a contradiction.
If $[p_1, p_2] \cap K \neq \emptyset$, then we can obtain the same contradiction using the sequence

$$C_{2n+1} = {p_2}^{-1}(C_{2n}) \text{ and } C_{2n+2} = {p_1}^{-1}(C_{2n+1})$$

for all $n \in \mathbb{N}$ or the sequence

$$C_{2n+1} = {p_1}^{-1}(C_{2n}) \text{ and } C_{2n+2} = {p_2}^{-1}(C_{2n+1})$$

for all $n \in \mathbb{N}$.

Case **(b)**

If l is the only line supporting K at every point of $K \cap l$, it may happen that $\nu_i(K \triangle K') = \infty$, so the prevision method does not work. For this reason we use a different technique called "chord chasing". Suppose that $\partial K \cap l = \partial K' \cap l = [q_1, q_2]$ (where possibly $q_1 = q_2$). If $K \neq K'$, there exists a line l', such that $K \cap l' \neq K' \cap l'$ and such that every supporting line or chord of K or K' between l and l' makes an angle smaller than $\frac{\pi}{4}$ with the line l. We may also assume that for any $x \in \partial K \cup \partial K'$ between l and l', the lines through x and p_j, $j = 1, 2$, make an angle with l smaller than $\frac{\pi}{4}$. We now consider the various positions of the points p_1 and p_2 with respect to the segment $[q_1, q_2]$. Suppose initially that p_2, p_1, q_1 and q_2 are in that order on l. Take $y_1 \in \partial K' \cap l'$ such that $[p_1, y_1] \cap K \neq \emptyset$ and $y_1 \notin K$. Let p_1, x_2, x_1, and y_1 be in that order on the line through p_1 and y_1, with $x_j \in \partial K$, $j = 1, 2$, and let y_2 be the other endpoint of the chord of K' on that line. By assumption K and K' have the same i-chord function at p_1, i. e. $\rho_{i,K,p_1} = \rho_{i,K',p_1}$ then

$$||y_1 - p_1||^i - ||y_2 - p_1||^i = ||x_1 - p_1||^i - ||x_2 - p_1||^i. \tag{3.8}$$

Now let the line through p_2 and y_2 meet ∂K (respectively $\partial K'$) in points z_2, z_3 (respectively y_2, y_3), with p_2, z_2, y_2, z_3, y_3 in that order. Consider a line m supporting K at z_2. Let T be the triangle determined by m, l, and the line through p_1 and x_2. From the assumptions it follows that the angle at the vertex v of T not belonging to l is larger than $\frac{\pi}{2}$. The same angle is opposite to $[z_2, y_2]$ in the triangle with vertices z_2, y_2, and v, and therefore $||y_2 - z_2|| > ||v - y_2||$. Then by convexity of K we have $||v - y_2|| > ||x_2 - y_2||$ and hence

$$||y_2 - z_2|| > ||x_2 - y_2|| \tag{3.9}$$

and the equality of i-chord function at p_2, i. e. $\rho_{i,K,p_2} = \rho_{i,K',p_2}$ implies

$$||y_3 - p_2||^i - ||y_2 - p_2||^i = ||z_3 - p_2||^i - ||z_2 - p_2||^i. \tag{3.10}$$

Next, let the line through p_1 and y_3 meet ∂K (respectively $\partial K'$) in points x_3, x_4 (respectively y_3, y_4), with p_1, x_4, y_4, x_3, y_3 in that order. From the convexity of K we have

$$||y_3 - z_3|| < ||x_3 - y_3||. \tag{3.11}$$

The equations (3.8) and (3.10) can be rewritten in the following way:

$$||y_2 - p_1||^i - ||x_2 - p_1||^i = ||y_1 - p_1||^i - ||x_1 - p_1||^i \tag{3.8'}$$

$$||y_3 - p_2||^i - ||z_3 - p_2||^i = ||y_2 - p_2||^i - ||z_2 - p_2||^i. \tag{3.10'}$$

Consider the function $f(t) = t^i$. Its derivative is $f'(t) = it^{i-1}$.
By the Lagrange theorem there exist (and are unique, since f is strictly convex) $s \in\,]x_2, y_2[$ and $w \in\,]x_1, y_1[$ such that

$$||y_2 - p_1||^i - ||x_2 - p_1||^i = \big(||y_2 - p_1|| - ||x_2 - p_1||\big)\ i\ ||s - p_1||^{i-1}$$

and

$$||y_1 - p_1||^i - ||x_1 - p_1||^i = \big(||y_1 - p_1|| - ||x_1 - p_1||\big)\ i\ ||w - p_1||^{i-1}.$$

By (3.8') we get

$$\begin{aligned} ||y_2 - x_2||\ i\ ||s - p_1||^{i-1} &= ||y_1 - x_1||\ i\ ||w - p_1||^{i-1} \\ ||y_2 - x_2|| &= ||y_1 - x_1|| \left(\frac{||w - p_1||}{||s - p_1||}\right)^{i-1}. \end{aligned} \tag{3.12}$$

Analogously, there exist (and are unique) $t \in\,]z_3, y_3[$ and $r \in\,]z_2, y_2[$ such that

$$||y_3 - p_2||^i - ||z_3 - p_2||^i = \big(||y_3 - p_2|| - ||z_3 - p_2||\big)\ i\ ||t - p_2||^{i-1}$$

and

$$||y_2 - p_2||^i - ||z_2 - p_2||^i = \big(||y_2 - p_2|| - ||z_2 - p_2||\big)\ i\ ||r - p_2||^{i-1}.$$

As before, by (3.10') we get

$$\begin{aligned} ||y_3 - z_3||\ i\ ||t - p_2||^{i-1} &= ||y_2 - z_2||\ i\ ||r - p_2||^{i-1} \\ ||y_3 - z_3|| &= ||y_2 - z_2|| \left(\frac{||r - p_2||}{||t - p_2||}\right)^{i-1}. \end{aligned} \tag{3.13}$$

Now we have

$$\begin{aligned} ||y_3 - x_3|| &\overset{(3.11)}{>} ||y_3 - z_3|| \overset{(3.13)}{=} ||y_2 - z_2|| \left(\frac{||r - p_2||}{||t - p_2||}\right)^{i-1} \\ &\overset{(3.9)}{>} ||y_2 - x_2|| \left(\frac{||r - p_2||}{||t - p_2||}\right)^{i-1} \\ &\overset{(3.12)}{>} ||y_1 - x_1|| \left(\frac{||w - p_1||}{||s - p_1||}\right)^{i-1} \left(\frac{||r - p_2||}{||t - p_2||}\right)^{i-1}. \end{aligned}$$

Putting

$$c' = \left(\frac{||p_1 - w||\ ||p_2 - r||}{||p_1 - s||\ ||p_2 - t||} \right)^{i-1}$$

we get

$$||y_3 - x_3|| > c'||y_1 - x_1||.$$

If we consider the points a and b in which the two segments $[r, s]$ and $[t, w]$, extended, meet l then we obtain four triangles $\triangle(s, p_1, a)$, $\triangle(w, p_1, b)$, $\triangle(r, p_2, a)$ and $\triangle(t, p_2, b)$. We denote by $\alpha = \measuredangle(s, p_1, a) = \measuredangle(w, p_1, b)$, $\beta = \measuredangle(r, p_2, a) = \measuredangle(t, p_2, b)$, $\gamma = \measuredangle(w, b, p_1) = \measuredangle(t, b, p_2)$, $\delta = \measuredangle(s, a, p_1) = \measuredangle(r, a, p_2)$, therefore by the *Law of Sines*, applied to the four triangles mentioned above, we have that

$$\begin{aligned} ||p_1 - w|| &= ||b - w||\ \frac{\sin\gamma}{\sin\alpha},\ ||p_1 - s|| = ||a - s||\ \frac{\sin\delta}{\sin\alpha} \\ ||p_2 - r|| &= ||a - r||\ \frac{\sin\delta}{\sin\beta},\ ||p_2 - t|| = ||b - t||\ \frac{\sin\gamma}{\sin\beta}. \end{aligned}$$

So c' can be rewritten in the following way

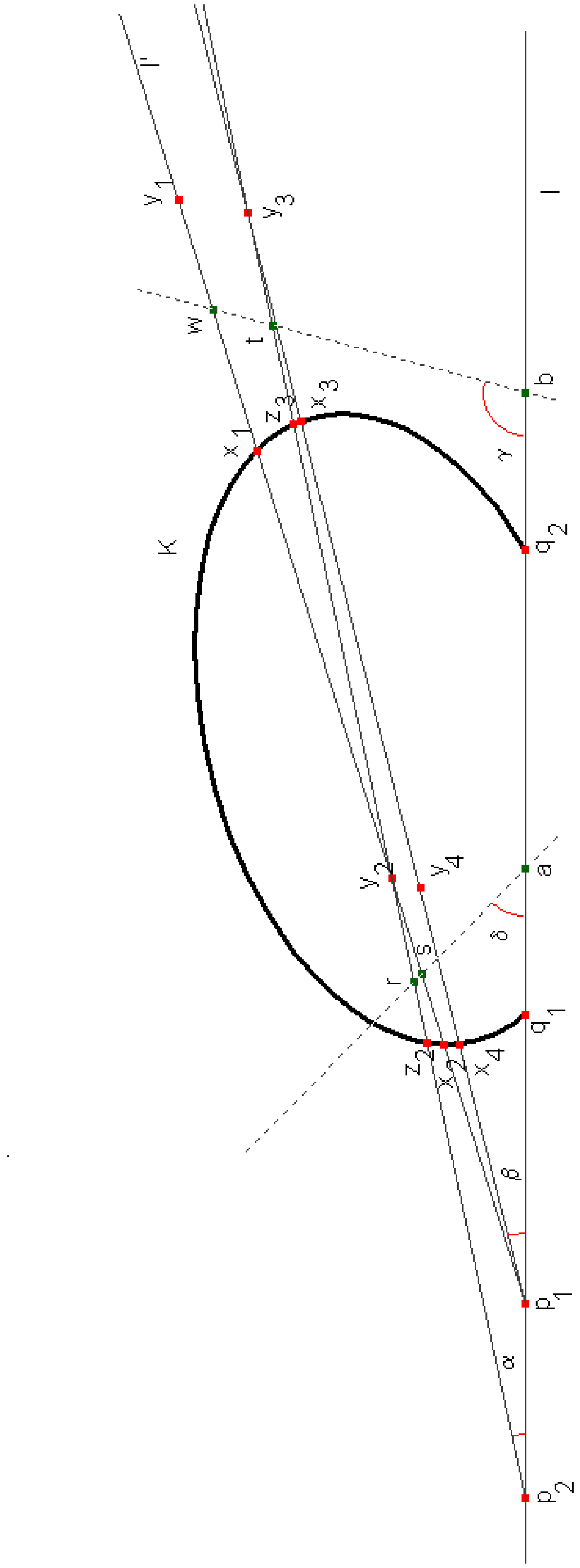
$$\begin{aligned} c' &= \left(\frac{||b - w||\frac{\sin\gamma}{\sin\alpha} \cdot ||a - r||\frac{\sin\delta}{\sin\beta}}{||a - s||\frac{\sin\delta}{\sin\alpha} \cdot ||b - t||\frac{\sin\gamma}{\sin\beta}} \right)^{i-1} \\ &= \left(\frac{||b - w||\ ||a - r||}{||a - s||\ ||b - t||} \right)^{i-1} \\ &= \left(\frac{||b - w||}{||b - t||} \right)^{i-1} \left(\frac{||a - r||}{||a - s||} \right)^{i-1} \geq 1. \end{aligned}$$

This holds because by assumption $i \geq 1$. This implies that

$$||y_3 - x_3|| > ||y_1 - x_1||.$$

Iterating inductively this construction, we obtain two sequences $\{x_{2n+1}\}$ and $\{y_{2n+1}\}$ such that

$$||y_{2n+1} - x_{2n+1}|| > ||y_{2n-1} - x_{2n-1}||$$


l'
y1
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r
q1
z2
x2
x4
β
p1
α
p2

for every n, and on the other hand

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = q_2$$

which is impossible.

Now suppose that p_1, $[q_1, q_2]$ and p_2 are in that order on l. Take $y_1 \in (\partial K' \cap l') \setminus K$ such that the line through p_1 and y_1 intersects $\operatorname{int} K$. Let p_1, x_2, y_2, x_1, and y_1 be in that order, with $x_j \in \partial K$ and $y_j \in \partial K'$ for $j = 1, 2$. By assumption K and K' have the same i-chord function at p_1 then

$$||y_1 - p_1||^i - ||y_2 - p_1||^i = ||x_1 - p_1||^i - ||x_2 - p_1||^i$$

or equivalently

$$||y_1 - p_1||^i - ||x_1 - p_1||^i = ||y_2 - p_1||^i - ||x_2 - p_1||^i. \tag{3.14}$$

Now let the line through p_2 and y_2 meet ∂K (respectively $\partial K'$) in points z_2, z_3 (respectively y_2, y_3), with z_2, y_2, z_3, y_3 and p_2 in that order. Then by convexity

$$||y_2 - z_2|| > ||y_2 - x_2|| \tag{3.15}$$

and the equality of i-chord function at p_2 implies

$$||z_2 - p_2||^i - ||z_3 - p_2||^i = ||y_2 - p_2||^i - ||y_3 - p_2||^i.$$

or equivalently

$$||z_3 - p_2||^i - ||y_3 - p_2||^i = ||z_2 - p_2||^i - ||y_2 - p_2||^i. \tag{3.16}$$

Next, let the line through p_1 and y_3 meet ∂K (respectively $\partial K'$) in points x_3, x_4 (respectively y_3, y_4), with p_1, x_4, y_4, x_3, y_3 in that order. Then by convexity

$$||y_3 - x_3|| > ||y_3 - z_3||. \tag{3.17}$$

By the Lagrange theorem, applied to the function $f(t) = t^i$, there exist (and are unique) $r \in]x_1, y_1[$ and $s \in]x_2, y_2[$ such that

$$||y_1 - p_1||^i - ||x_1 - p_1||^i = \big(||y_1 - p_1|| - ||x_1 - p_1||\big)\ i\ ||r - p_1||^{i-1}$$

and

$$||y_2 - p_1||^i - ||x_2 - p_1||^i = \big(||y_2 - p_1|| - ||x_2 - p_1||\big)\ i\ ||s - p_1||^{i-1}.$$

By (3.14) we get

$$\begin{aligned} ||y_1 - x_1||\ i\ ||r - p_1||^{i-1} &= ||y_2 - x_2||\ i\ ||s - p_1||^{i-1} \\ ||y_2 - x_2|| &= ||y_1 - x_1|| \left(\frac{||r - p_1||}{||s - p_1||}\right)^{i-1} \\ ||y_2 - x_2|| &> ||y_1 - x_1||. \end{aligned} \tag{3.18}$$

Analogously, there exist (and are unique) $t \in \,]y_2, z_2[$ and $w \in \,]y_3, z_3[$ such that

$$||z_2 - p_2||^i - ||y_2 - p_2||^i = \big(||z_2 - p_2|| - ||y_2 - p_2||\big)\ i\ ||t - p_2||^{i-1}$$

and

$$||z_3 - p_2||^i - ||y_3 - p_2||^i = \big(||z_3 - p_2|| - ||y_3 - p_2||\big)\ i\ ||w - p_2||^{i-1}.$$

By (3.16) we get

$$\begin{aligned} ||z_3 - y_3||\ i\ ||u - p_2||^{i-1} &= ||z_2 - y_2||\ i\ ||t - p_2||^{i-1} \\ ||z_3 - y_3|| &= ||z_2 - y_2|| \left(\frac{||t - p_2||}{||w - p_2||}\right)^{i-1} \\ ||z_3 - y_3|| &> ||z_2 - y_2||. \end{aligned} \tag{3.19}$$

Now we have

$$\begin{aligned} ||y_3 - x_3|| &\overset{(3.17)}{>} ||y_3 - z_3|| \overset{(3.19)}{>} ||y_2 - z_2|| \\ &\overset{(3.15)}{>} ||y_2 - x_2|| \overset{(3.18)}{>} ||y_1 - x_1||. \end{aligned}$$

Therefore $||y_3 - x_3|| > ||y_1 - x_1||$.
Continuing in this fashion, we construct two sequences $\{x_{2n+1}\}$ and $\{y_{2n+1}\}$ such that

$$||y_{2n+1} - x_{2n+1}|| > ||y_{2n-1} - x_{2n-1}||$$

for every n, and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = q_2$$

which is impossible.
Finally, suppose that p_2 is at infinity. Let p_1, q_1 and q_2 be in that order on l, and p_2 is the point at infinity of l. Take $y_2 \in (\partial K' \cap l') \setminus K$ such that the line through p_1 and y_2 intersects $\operatorname{int} K$. Let x_1, y_1, x_2, and y_2 be in that order on l', with $x_j \in \partial K \cap l'$ and $y_j \in \partial K' \cap l'$ for $j = 1, 2$. By assumption K and K' have the same X-ray in the direction of l therefore

$$||y_1 - y_2|| = ||x_1 - x_2||$$

and also

$$||y_1 - x_1|| = ||y_2 - x_2||. \tag{3.20}$$

Now let the line through p_1 and y_2 meet ∂K (respectively $\partial K'$) in points z_2, z_3 (respectively y_2, y_3), with p_1, z_3, y_3, z_2 and y_2 in that order. Then by convexity

$$||y_2 - z_2|| > ||x_2 - y_2|| \tag{3.21}$$

and the equality of i-chord function at p_1 implies

$$||y_2 - p_1||^i - ||y_3 - p_1||^i = ||z_2 - p_1||^i - ||z_3 - p_1||^i$$

or equivalently

$$||y_2 - p_1||^i - ||z_2 - p_1||^i = ||y_3 - p_1||^i - ||z_3 - p_1||^i. \tag{3.22}$$

Next, let the line through y_3 parallel to l meet ∂K (respectively $\partial K'$) in points x_3, x_4 (respectively y_3, y_4), with x_3, y_3, x_4, y_4 in that order. Then by convexity

$$||y_3 - x_3|| > ||y_3 - z_3|| \tag{3.23}$$

and the equality of X-ray in the direction of l implies

$$||y_3 - y_4|| = ||x_3 - x_4||$$

and also

$$||y_3 - x_3|| = ||y_4 - x_4||. \tag{3.24}$$

Again, by the Lagrange theorem there exist (and are unique) $r \in \,]z_2, y_2[$ and

$s \in\,]z_3, y_3[$ such that

$$||y_2 - p_1||^i - ||z_2 - p_1||^i = \big(||y_2 - p_1|| - ||z_2 - p_1||\big)\ i\ ||r - p_1||^{i-1}$$

and

$$||y_3 - p_1||^i - ||z_3 - p_1||^i = \big(||y_3 - p_1|| - ||z_3 - p_1||\big)\ i\ ||s - p_1||^{i-1}.$$

By (3.22) we get

$$\begin{aligned} ||y_2 - z_2||\ i\ ||r - p_1||^{i-1} &= ||y_3 - z_3||\ i\ ||s - p_1||^{i-1} \\ ||y_3 - z_3|| &= ||y_2 - z_2|| \left(\frac{||r - p_1||}{||s - p_1||}\right)^{i-1}. \end{aligned} \tag{3.25}$$

Now we have

$$\begin{aligned} ||y_4 - x_4|| &\overset{(3.24)}{=} ||y_3 - x_3|| \overset{(3.23)}{>} ||y_3 - z_3|| \\ &\overset{(3.25)}{=} ||y_2 - z_2|| \left(\frac{||r - p_1||}{||s - p_1||}\right)^{i-1} > ||y_2 - z_2||. \end{aligned}$$

Therefore $||y_4 - x_4|| > ||y_2 - x_2||$, and continuing inductively to construct the sequences $\{x_{2n}\}$ and $\{y_{2n}\}$ with $x_{2n} \in \partial K$ and $y_{2n} \in \partial K'$ such that

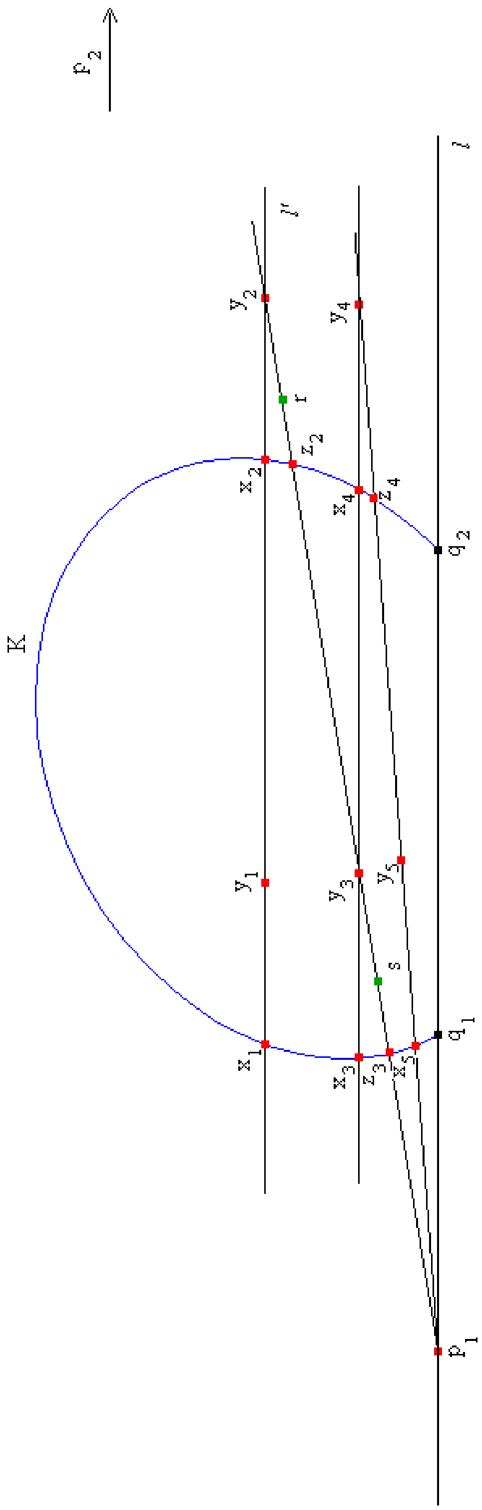
$$||y_{2n+2} - x_{2n+2}|| > ||y_{2n} - x_{2n}||$$

for every n, while on the other hand

$$\lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} y_{2n+2} = q_2,$$

a contradiction.

$\square$



p 2
l
l′
y 2
y 4
r
z 2
x 2
x 4
z 4
q 2
K
y 1
y 3
y 5
s
q 1
x 1
x 3
z 3
x 5
p 1

Remark 3.2.4.
Observe that this result holds for every i belonging to $[1, +\infty)$. In particular for $i = 1$ we retrieve ([16], Theorem 5.3.1) and for $i = 2$ ([14], Theorem 1).

The essential idea of the *chord chasing* seems to have first been employed by Süss in [38], and also used by Rogers in [34].

Theorem 3.2.5.
Let $i > 0$. Suppose that K is a planar convex body and that p_1, p_2 are distinct points belonging to $\operatorname{int} K$. If K' is a planar convex body with the same i-chord functions as K at p_1 and p_2, then $K = K'$.

Proof.
By assumption p_1 and p_2 are interior to K.
First we show that $(\partial K \cap \partial K') \setminus l \neq \emptyset$, that is, the boundaries of K and K' intersect in some other point not belonging to the line l. If we suppose the contrary, then $\operatorname{int}(K \triangle K')$ has exactly two components C and C'. Consider the two lines t_1 and t_2 through p_1 and p_2 respectively, orthogonal to l. Denote by E_j and E'_j, $j = 1, 2$, the open half-planes determined by the two lines t_1 and t_2, with $p_1 \in E_2$ and $p_2 \in E'_1$, (see Figure 3.7) Lemma 3.1.3 and convexity imply that C and C' both have finite

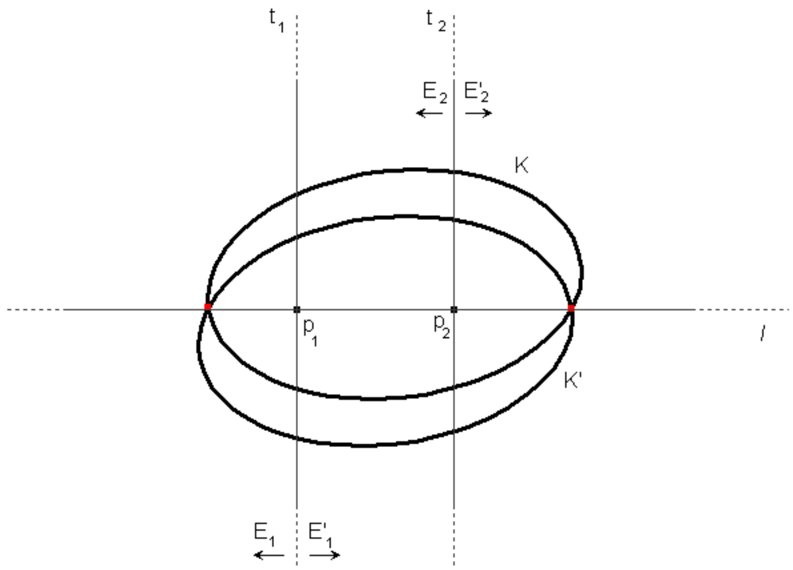


Figure 3.7

ν_i-measure. Since $E_j \cap C$ and $E'_j \cap C'$ have the same i-chord functions at p_j, $j = 1, 2$, by Lemma 3.1.4 we have

$$\nu_i(E_1 \cap C) = \nu_i(E'_1 \cap C') \tag{3.26}$$

and

$$\nu_i(E_2 \cap C) = \nu_i(E'_2 \cap C'). \tag{3.27}$$

But on the other hand $E_1 \cap C \subset E_2 \cap C$ and $E'_2 \cap C' \subset E'_1 \cap C'$, therefore by the monotonicity of the measure ν_i we have

$$\nu_i(E_1 \cap C) < \nu_i(E_2 \cap C) \overset{(3.26)}{=} \nu_i(E'_2 \cap C') < \nu_i(E'_1 \cap C') \overset{(3.27)}{=} \nu_i(E_1 \cap C),$$

a contradiction. Assume now that there exists a component C of $\operatorname{int}(K \triangle K')$ with one endpoint $x \in \partial K \cap \overline{C}$ not belonging to l. We will show now that $\overline{C} \cap l = \emptyset$. Suppose, by contradiction, that the other endpoint y belongs to l. We may suppose that p_1, p_2 and y are in that order on l. Let $x_1 \in \partial K \cap xp_1$ and $x_2 \in \partial K \cap x_1p_2$, with $x \neq x_1 \neq x_2 \neq x$.

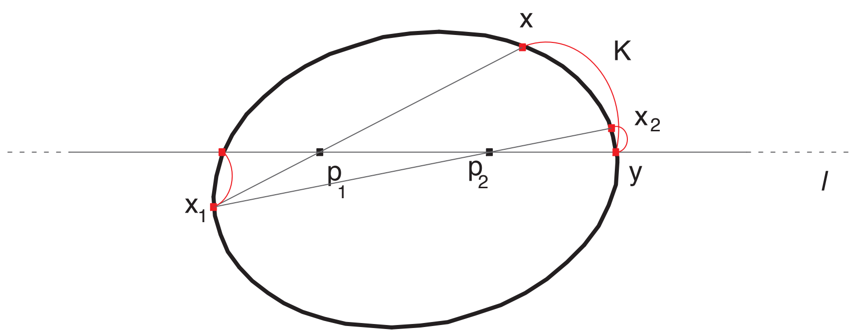


Figure 3.8

Then x_1, x_2 belong to $\partial K \cap \partial K'$, but x_2 must lie strictly between x and y in ∂K and this means that the closures of their two correspondent components intersect each other and this is impossible, (see Figure 3.8). Suppose now that C_1 is a component of $\operatorname{int}(K \triangle K')$. Having excluded that $\overline{C_1} \cap l \neq \emptyset$, we can construct a sequence of disjoint components

$$C_{2n} = {p_1}^{-1}(C_{2n-1}) \text{ and } C_{2n+1} = {p_2}^{-1}(C_{2n})$$

for all $n \in \mathbb{N}$, (see Figure 3.9). They all have the same ν_i-measure and are contained in $\text{int}\,(K \triangle K')$ which is of finite ν_i-measure, therefore K is uniquely determined.

$\square$

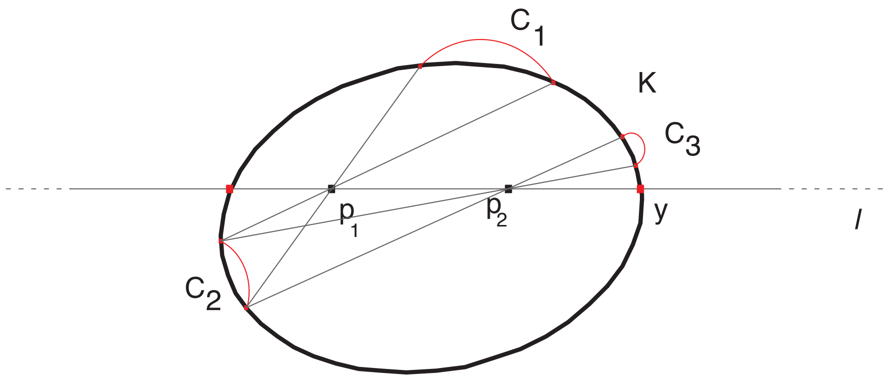


Figure 3.9

Observe that the knowledge of the position with respect to the segment $[p_1, p_2]$ of the intersection of K with the line l is a fundamental hypothesis, because it guarantees the uniqueness of the convex body K, in fact, as can be seen from Figure 3.10, we cannot exclude the existence of a convex body $K' \neq K$ having the same i-chord functions at p_1 and p_2 but not having the same component of $l \setminus \{p_1, p_2\}$ as K.

Falconer in [11], referring to this situation depicted in Figure 3.10, stated that if p_1 and p_2 are two different exterior points "*it seems unlikely that two distinct sets could have the same chord functions but it seems difficult to produce a general method for eliminating one of the two possibilities.*"

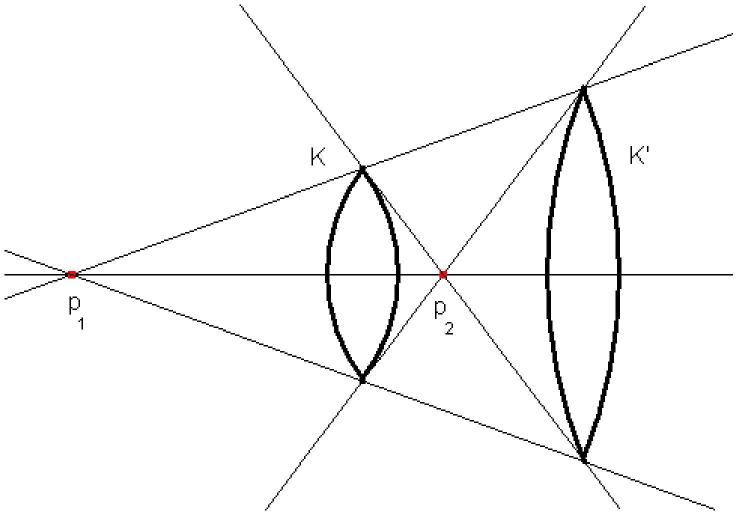


Figure 3.10

If one of the points is at infinity, then an easy example of two convex bodies having the same chord functions is the following, (see Figure 3.11).

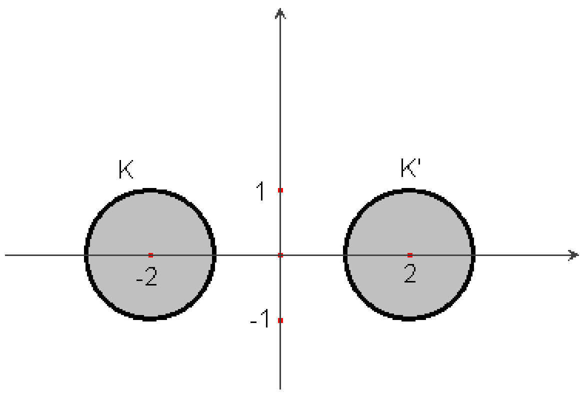


Figure 3.11

Let p_1 be the origin of a cartesian system, and let p_2 be the point at infinity of the x-axis, then

$$K = \left\{(x, y) : (x-2)^2 + y^2 \leq 1\right\}$$

and

$$K' = \left\{(x, y) : (x+2)^2 + y^2 \leq 1\right\}$$

have the same X-rays at the origin and at the infinity.

However, some interesting computer studies carried at by Volčič with the help of Michelacci show that it is possible the existence of examples similar to those depicted in Figure 3.10.

3.3 A three-point solution

Adding a further point source, the ambiguity showed in Figure 3.10 disappears, and we have the following uniqueness theorem depending on the position of K with respect to the three noncollinear points.

Theorem 3.3.1.
Let $i > 0$. i-chord functions at three noncollinear points determine a convex body K in the interior of the triangle T formed by the points.

Proof.
Let $K' \neq K$ be a convex body having the same i-chord functions as K at the three points p_1, p_2 and p_3. Then K' must be also contained in the interior of the triangle T with vertices the three points p_j, for $j = 1, 2, 3$. Let A be a component of $\mathrm{int}\,(K \triangle K')$ of maximal ν_{i+1}-measure. Denote by E the open half-plane containing A and bounded by the line through the endpoints of A in $\partial K \cap \partial K'$. For some j, $p_j \in E$ and this implies that A is visible from p_j. Without loss of generality we may assume that A is visible from p_1. But then $p_1(A)$ is defined, and taking $p_1 p_2$ as base line, by Lemma 3.1.6 (a), $\nu_{i+1}(p_j(A)) > \nu_{i+1}(A)$, a contradiction. $\square$

Theorem 3.3.2.
Let $0 < i < 2$ and let p_1, p_2 and p_3 be three noncollinear point in $\mathbb{R}^2$. Denote by $T = \Delta(p_1, p_2, p_3)$ the triangle with vertices in these three points, and let $T'(p_j)$ be the image of T under reflection at p_j. Then i-chord functions at p_j for $j = 1, 2, 3$ determine a convex body K in each of the following situations.

(a) $K \subset int(pos_{p_j} T) \setminus T$;

(b) $K \subset int(pos_{p_j} T'(p_j))$.

Proof.
For reasons of symmetry we may assume that K (and therefore also K') is contained in the cone $\mathrm{pos}_{p_1} T$ determined by lines through p_1 and p_j, $j = 2, 3$, and containing the triangle T. So let $T'(p_1) = T'$ be the image of T under reflection at p_1. Then the two cases that we have to consider are:

(a) $K \subset \mathrm{int}(\mathrm{pos}_{p_1} T) \setminus T$;

(b) $K \subset \mathrm{int}(\mathrm{pos}_{p_1} T')$.

We shall assume that $K' \neq K$ and conclude with a contradiction.

(a) In this case the segment $[p_2, p_3]$ separates T from K (and so K'). Take a coordinate system in which the line p_2p_3 has equation $y = 0$ and consider p_2p_3 as the *base line* for the measure ν_i. We can assume that $K \subset \{(x, y) : y > 0\}$ so that the point p_1 has negative y-coordinate. Let B_1 be a component of $\mathrm{int}\,(K \triangle K')$ having maximal ν_i-measure. Since $0 < i < 2$ the maximality of $\nu_i(B_1)$ and Lemma 3.1.6 (b) imply that all components having maximal measure must be visible from p_1. Therefore B_1 is visible from p_1, so by convexity B_1 is visible also from either p_2 or p_3. Without loss of generality, we may assume that $B_2 = p_2(B_1)$ is defined, and by Lemma 3.1.4 $\nu_i(B_2) = \nu_i(B_1)$. Again, from the maximality of $\nu_i(B_2)$ it follows that B_2 is visible from p_1 and by convexity also from either p_2 or p_3, but by construction B_1 is not visible from p_2. We may assume that $B_3 = p_3(B_2)$ is defined and $\nu_i(B_3) = \nu_i(B_2)$. By construction B_3 is not visible from p_3 and if it were not visible from p_2 also, it would not be visible from p_1, obtaining again the same contradiction, since $\nu_i({p_1}^{-1}(B_3)) > \nu_i(B_1)$. So this procedure can be iterated obtaining a sequence of disjoint components given by $B_{2n} = p_2(B_{2n-1})$ and $B_{2n+1} = p_3(B_{2n})$ having the same ν_i-measure and contained in the set $K \cup K'$. Since $i > 0$, and $K \cup K' \subset \{(x, y) : y > 0\}$, $\nu_i(K \cup K') < \infty$, we have

$$\nu_i\Bigg(\bigcup_{j\in\mathbb{N}} B_j\Bigg) = \sum_{j\in\mathbb{N}} \nu_i(B_j) = \infty,$$

but, on the other hand,

$$\infty = \nu_i\Bigg(\bigcup_{j\in\mathbb{N}} B_j\Bigg) \leq \nu_i(K \cup K') < \infty,$$

a contradiction.

(b) Also in this case, we consider a component C_1 of int $(K\triangle K')$ having maximal measure ν_i. Let us take now the line p_1p_2 separating p_3 from K as *base plane* for the measure ν_i. Assume that $K \subset \{(x,y) : y > 0\}$ so that the point p_3 has negative y-coordinate. The maximality of $\nu_i(C_1)$ and Lemma 3.1.6 (b) imply that C_1 is visible from p_3, so by convexity C_1 is visible also from p_1 consequently $C_2 = p_1(C_1)$ is defined and, by Lemma 3.1.4 $\nu_i(C_1) = \nu_i(C_2)$. Now C_2 is not visible from p_2, thus $C_3 = {p_2}^{-1}(C_2)$ is defined and by Lemma 3.1.4 $\nu_i(C_3) = \nu_i(C_2)$. Obviously C_3 is visible from p_2 and by convexity C_3 is also visible from p_1, so $C_4 = p_1(C_3)$ is defined, and by Lemma 3.1.4 $\nu_i(C_4) = \nu_i(C_3)$. Iterating this process, we construct the sequence $C_{2n} = p_1(C_{2n-1})$ and $C_{2n+1} = {p_2}^{-1}(C_{2n})$. As in the previous case, we get a contradiction because this components are disjoint, have the same ν_i-measure, and are contained in $K \cup K'$ of finite ν_i-measure.

□

Theorem 3.3.3.
Let $i > 0$. Suppose that K, K' are two planar convex bodies with the same i-chord functions at three noncollinear points p_1, p_2, and p_3 not belonging to $K \cup K'$. If $\mathrm{int}\,K$ and $\mathrm{int}\,K'$ meet one or more lines p_jp_h, for $j \neq h$ and $j,h \in \{1,2,3\}$, then $K = K'$.

Proof.
If K intersects one or more lines p_jp_h, then the supports of two i-chord functions at p_j and p_h determine one bounded quadrangle and two unbounded regions which may contain K but the support of the i-chord function at the remaining third point determines the position of K with respect to the segment $[p_j, p_h]$. Thus, K and K' belong to the same component of $p_jp_h \setminus \{p_jp_h\}$, then either both meet the segment $[p_j, p_h]$ or are both disjoint from $[p_j, p_h]$. We can therefore apply Theorem 3.2.2 to conclude that $K = K'$.
In fact, without loss of generality we may assume that $\mathrm{int}\,K$ and $\mathrm{int}\,K'$ meet the line p_1p_2.

- Assume that p_1, $K \cap p_1p_2$ and p_2 are in that order on p_1p_2
  Let A be a component of int $(K\triangle K')$ visible from p_1 such that its closure meets the segment $[p_1, p_2]$, then $A' = p_1(A)$ is visible from p_2 and $A'' = p_2(A')$ is such that $A \subset A'$ but this contradicts the definition of connected component, therefore it must be $p_2(A') = A$. So we have only two components of int $(K\triangle K')$,

one totally visible from p_1 and one totally visible from p_2. The equality of the i-chord functions at p_3 also implies that the bodies have common supporting lines through p_3, so the i-chord functions distinguish the convex body, because the line supporting K and K' are distinct. In fact, there is a direction $u \in S^1$ such that $\rho_{K',p_3}(u) \neq \rho_{K,p_3}(u)$. This implies that $K \cap [p_1, p_2] = K' \cap [p_1, p_2] = \{q\}$. Take as base line for the measure ν_i a line through p_1 or p_2 such that it does not meet K and so K'. Without loss of generality we can choose p_2. Take a coordinate system in which this line has equation $y = 0$. We can assume that $K \subset \{(x, y) : y > 0\}$ so that the point p_3 has negative y-coordinate. By Lemma 3.1.4, the equality of i-chord functions at p_2 implies $\nu_i(A') = \nu_i(A)$, while by Lemma 3.1.6, the equality of i-chord functions at p_3 implies $\nu_i(A') < \nu_i(A)$ if $i > 2$ or $\nu_i(A') > \nu_i(A)$ if $i < 2$, a contradiction.

- Assume that p_1, p_2 and $K \cap p_1p_2$ are in that order on p_1p_2.
  Suppose now that A is a component of $\operatorname{int}(K \triangle K')$ such that its closure meets $p_1p_2 \setminus [p_1, p_2]$. Let A be visible from p_1 and consequently also from p_2, then $A' = p_2(A)$ is not visible from p_1 thus $A'' = {p_1}^{-1}(A')$ is such that $A \subset A''$ but this contradicts the definition of connected component, therefore it must be ${p_1}^{-1}(A') = A$. Also in this case we have only two components of $\operatorname{int}(K \triangle K')$. Again the i-chord function at p_3 distinguish uniquely the convex body, because the line supporting K and K' are distinct. In fact, there is a direction $u \in S^1$ such that $\rho_{K',p_3}(u) \neq \rho_{K,p_3}(u)$. This implies that the intersection of K with the line p_1p_2 is a singleton $\{q\}$. Take as base line for the measure ν_i the line p_2p_3. Take a coordinate system in which this line has equation $y = 0$. We can assume that $K \subset \{(x, y) : y > 0\}$ so that the point p_1 has negative y-coordinate. By Lemma 3.1.4, the equality of i-chord functions at p_2 implies $\nu_i(A') = \nu_i(A)$, while by Lemma 3.1.6, the equality of i-chord functions at p_1 implies $\nu_i(A) < \nu_i(A')$ if $i > 2$ or $\nu_i(A) > \nu_i(A')$ if $i < 2$, a contradiction.

In both cases, when $i = 2$ we follow the same argument used in Theorem 3.2.2, showing that the centroids with respect to the measure ν_i of the two disjoint components must be the same, contradicting the existence of such components. □

Theorem 3.3.4. *Let $i > 0$. Suppose that K, K' are two planar convex bodies with the same i-chord functions at three noncollinear points p_1, p_2, and p_3 not belonging to $K \cup K'$. If one or more lines p_jp_h, for $j \neq h$ and $j, h \in \{1, 2, 3\}$, support K and K', then $K = K'$.*

Proof.
If one or more lines p_jp_h support K, then, as in the previous theorem, the supports of two i-chord functions at p_j and p_h determine one bounded quadrangle and two unbounded regions containing K but the support of the i-chord function at the remaining third point determines the position of K with respect to the segment $[p_j, p_h]$. Thus, K and K' belong to the same component of $p_jp_h \setminus \{p_jp_h\}$,then either both meet the segment $[p_j, p_h]$ or are both disjoint from $[p_j, p_h]$. We can therefore apply Theorem 3.2.3 to conclude that $K = K'$.
In fact, without loss of generality we may assume that the line p_1p_2 supports K and K'.

- p_1, $K \cap p_1p_2$ and p_2 are in that order on the line p_1p_2. Let A be a component of $\operatorname{int}(K \triangle K')$ such that its closure meets the segment $[p_1, p_2]$. Without loss of generality, we may assume that A is visible from p_1, then $A' = p_1(A)$ is visible from p_2, and $A'' = p_2(A')$ is such that $A \subsetneq A''$ but this contradicts the definition of connected component, therefore it must be $p_2(A') = A$. Also in this case we have only two components of $\operatorname{int}(K \triangle K')$. Again the i-chord function at p_3 distinguishes uniquely the convex body, because the line supporting K and K' are distinct. In fact, there is a direction $u \in S^1$ such that $\rho_{K',p_3}(u) \neq \rho_{K,p_3}(u)$. The remaining situation is when the intersection of K with the segment $[p_1p_2]$ is a singleton $\{q\}$, and so $p_3(A') = A$, too. Take as base line for the measure ν_i a line through p_1 separating p_3 from K. Take a coordinate system in which this line has equation $y = 0$. We can assume that $K \subset \{(x, y) : y > 0\}$ so that the point p_3 has negative y-coordinate. By Lemma 3.1.4, the equality of i-chord functions at p_1 implies $\nu_i(A') = \nu_i(A)$, while by Lemma 3.1.6, the equality of i-chord functions at p_3 implies $\nu_i(A) < \nu_i(A')$ if $i > 2$ or $\nu_i(A) > \nu_i(A')$ if $i < 2$, a contradiction.

- p_1, p_2 and $K \cap p_1p_2$ are in that order on the line p_1p_2. Let A be a component of $\operatorname{int}(K \triangle K')$ such that its closure meets $p_2p_3 \setminus [p_2, p_3]$ and such that A is visible from p_2 and also from p_3 and p_1, then $A' = p_2(A)$ and $A'' = {p_1}^{-1}(A')$ are such that $A \subset A''$ but this contradicts the definition of connected component, therefore it must be ${p_1}^{-1}(A') = A$. Also in this case we have only two components of $\operatorname{int}(K \triangle K')$. Again the i-chord function at p_3 distinguishes uniquely the convex body, because the line supporting K and K' are distinct. Necessarily $K \cap [p_1p_2]$ must be a singleton $\{q\}$, and so $p_3(A') = A$, too. Take as base line for the measure ν_i a line through p_1 separating p_2 from p_3 (and so also

p_3 from K. Take a coordinate system in which this line has equation $y = 0$. We can assume that $K \subset \{(x, y) : y > 0\}$ so that the point p_3 has negative y-coordinate. By Lemma 3.1.4, the equality of i-chord functions at p_1 implies $\nu_i(A') = \nu_i(A)$, while by Lemma 3.1.6, the equality of i-chord functions at p_3 implies $\nu_i(A) < \nu_i(A')$ if $i > 2$ or $\nu_i(A) > \nu_i(A')$ if $i < 2$, a contradiction.

In both cases, when $i = 2$ we follow the same argument used in Theorem 3.2.2, showing that the centroids with respect to the measure ν_i of the two disjoint components must be the same, contradicting the existence of such components. □

Remark 3.3.5.

If K contains in its interior two or three points then, it is uniquely determined by its i-chord functions at these three points, because two interior points are enough as seen in Theorem 3.2.5. But, if K contains in its interior only a point then K might be non uniquely determined. Figure 3.12 shows that in this case the knowledge of i-chord functions at three points does not guarantee the uniqueness of K.

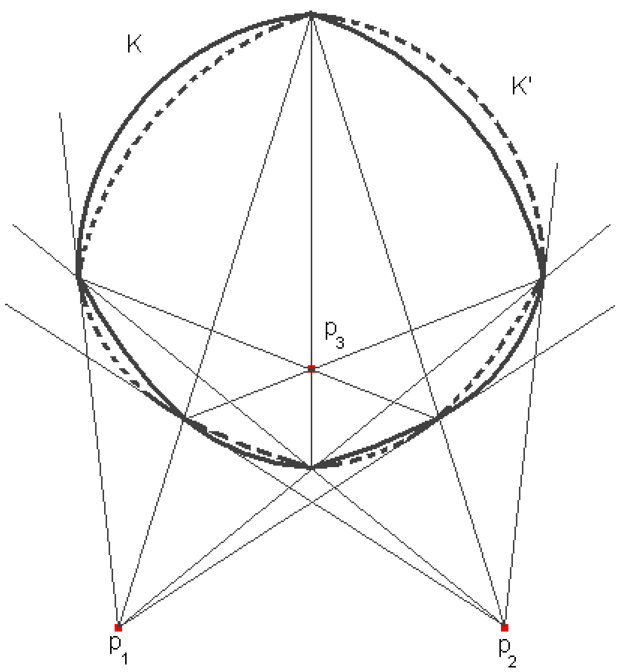


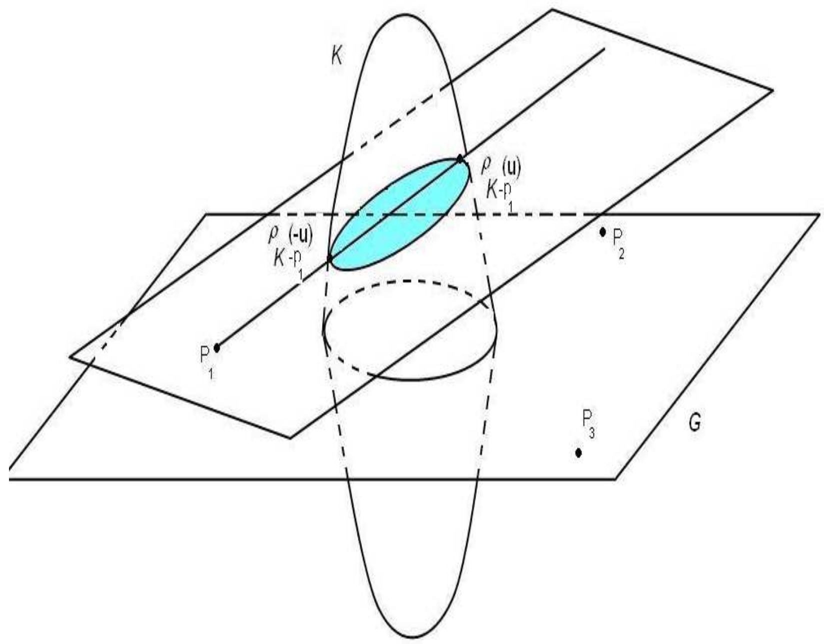
Figure 3.12

Chapter 4

Three-dimensional case

In this chapter we will propose to generalize what we have seen in the previous chapter, in particular, we want to extend to the three-dimensional case the reconstruction theorem of convex body from i-chord functions at three exterior points. Assume that we have three noncollinear points p_1, p_2 and p_3 in the three-dimensional Euclidean space and a convex body K such that its interior does not contain any of them, but intersects the plane G determined by these three points. The question is which kind of information is sufficient to determine K uniquely.
The knowledge of the intersections of a convex body K with a one-dimensional subspace, i.e., a line l through the origin, is equivalent to the knowledge of the length of this intersection, $\lambda_1(K \cap l)$.
If we measure the lengths of the intersections with all the line through the origin, then we have exactly the X-ray at the origin of the body K. This means that in the Euclidean plane there is analogy between chord function (or section function) and X-ray. This analogy is still valid in Euclidean space by the generalizations of these two concepts to 2-chord function (or 2-section function) and two-dimensional X-ray, respectively. The problem for $i = 1$ has been considered by D. C. Solmon and by Volčič in the late 80s and at about the same time it has been also analyzed by Gardner.

We assume that $K \subset \mathbb{R}^3$ is a convex body whose interior intersects a plane G containing through three noncollinear points p_1, p_2 and p_3. Furthermore we assume that we know the lengths (areas) of the intersections of K with any line l (any plane T) belonging to the pencil of lines (planes) based in p_j, $j = 1, 2, 3$. We assume $0 < i < 3$. This condition is justified by the fact that we are interested in



particular in the geometric problems corresponding to the values $i = 1$ and $i = 2$. The intersections with lines correspond to the case $i = 1$, while the intersections with planes correspond to $i = 2$.

Since we consider a three-dimensional problem, $i = 3$ does not have any geometric meaning. The method of chord functions provides a uniqueness result for any $i > 1$, but is not strong enough to handle the case $i = 1$ which corresponds to ordinary X-rays and is therefore of particular interest from the geometric point of view.

In this chapter, by u_x, u_y, u_z we indicate the unit vectors parallel to the axes x, y and z and, to be short, by l_x, l_y and l_z we denote, respectively, the x-axis, the y-axis and the z-axis.

4.1 The measure μ_k and its properties

Definition 4.1.1.

Let $\mathscr{L}_3$ be the class of bounded Lebesgue measurable subset of $\mathbb{R}^3$. Fix a Cartesian coordinate system in $\mathbb{R}^3$, if $E \in \mathscr{L}_3$, define for each $k \in \mathbb{R}$,

$$\mu_k(E) = \iiint_E |z|^{k-3} dxdydz.$$

μ_k is a measure on $\mathscr{L}_3$ and the plane $(l_z)^{\perp} = \{(x, y, z) : z = 0\}$ will be called the base plane *for μ_k.*

Observe that

$$\mu_3 = \lambda_3.$$

Suppose now that $E_1, E_2 \in \mathscr{L}_3$ and $\lambda_3(E_j) > 0,\ j = 1, 2$.
The next Lemma shows how it is possible to obtain information about the measure μ_k of two star bodies E_1 and E_2 with the same i-chord functions at a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$.

Lemma 4.1.2.
Let $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ and let E_1, E_2 be star bodies at p having the same i-chord functions at p for some $i \in (0, 3)$. Let C be the half-cone with vertex p generated by E_1 and E_2, i.e.

$$C := pos_p\, E_1 = pos_p\, E_2.$$

Suppose that $E_1 \cup E_2$ is contained in the half-space $\{(x, y, z) : z > 0\}$ and

$$-\rho_{E_1-p}(-u) \leq \rho_{E_1-p}(u) \leq -\rho_{E_2-p}(-u) \leq \rho_{E_2-p}(u) \quad \forall u \in S^2 \cap (C - p)$$

(which implies that E_1 is between p and E_2).
Then

(a) *If $z_0 = 0$ and $k > \max\{i, 1\}$, then $\mu_k(E_1) < \mu_k(E_2)$, if $i \geq 0$, and $\mu_k(E_1) > \mu_k(E_2)$, if $i < 0$.*

(b) *If $z_0 = 0$, $\mu_i(E_1) = \mu_i(E_2)$.*

(c) *If $z_0 < 0$, and E_1 has finite μ_i-measure, then $\mu_i(E_1) < \mu_i(E_2)$, if $i > 3$, and $\mu_i(E_1) > \mu_i(E_2)$, if $i < 3$; and if $z_0 > 0$, these inequalities are reversed.*

Proof.
For explicit computation of $\mu_k(E_j)$ we have to use a spherical coordinate system centered at the point p.

$$\begin{cases} x = x_0 + r \sin\theta \cos\varphi & r \in [0, +\infty) \\ y = y_0 + r \sin\theta \sin\varphi & \theta \in [0, \pi] \\ z = z_0 + r \cos\theta & \varphi \in [0, 2\pi) \end{cases} . \tag{4.1}$$

We can write

$$E_j = \{(r,\theta,\varphi) : r_j(\theta,\varphi) \le r \le s_j(\theta,\varphi), (\theta,\varphi) \in C\}$$

for $j = 1, 2$. Since E_1 and E_2 have, by assumption, the same i-chord functions at p, and $p \notin E_1 \cup E_2$ we have that

$$s_1(\theta,\varphi)^i - r_1(\theta,\varphi)^i = s_2(\theta,\varphi)^i - r_2(\theta,\varphi)^i \tag{4.2}$$

$$\frac{s_1(\theta,\varphi)}{r_1(\theta,\varphi)} = \frac{s_2(\theta,\varphi)}{r_2(\theta,\varphi)} \tag{4.3}$$

hold for every $(\theta,\varphi) \in C$ according as $i \neq 0$ or $i = 0$, respectively.
For $j = 1, 2$, the expression of μ_k with respect to spherical coordinates is

$$\mu_k(E_j) = \iint_C \int_{r_j(\theta,\varphi)}^{s_j(\theta,\varphi)} \frac{r^2 \sin\theta}{(r\cos\theta + z_0)^{3-k}} dr d\theta d\varphi.$$

Note that we eliminated the absolute value in the denominator of the integrand, having assumed that E_1 and E_2 are contained in the half-plane $(l_z)^\perp$.
If $i \neq 0$ we put $t = r^i$ and we get

$$\begin{aligned} \mu_k(E_j) &= \iint_C \int_{r_j(\theta,\varphi)^i}^{s_j(\theta,\varphi)^i} \frac{t^{\frac{2}{i}} \sin\theta}{(t^{\frac{1}{i}} \cos\theta + z_0)^{3-k}} \cdot t^{\frac{1-i}{i}} dt d\theta d\varphi \\ &= \iint_C \int_{r_j(\theta,\varphi)^i}^{s_j(\theta,\varphi)^i} \frac{1}{i} \cdot \frac{t^{\frac{3-i}{i}} \sin\theta}{(t^{\frac{1}{i}} \cos\theta + z_0)^{3-k}} dt d\theta d\varphi. \end{aligned} \tag{4.4}$$

While if $i = 0$ putting $t = \log r$ we obtain

$$\begin{aligned} \mu_k(E_j) &= \iint_C \int_{\log r_j(\theta,\varphi)}^{\log s_j(\theta,\varphi)} \frac{e^{2t} \sin\theta}{(e^t \cos\theta + z_0)^{3-k}} \cdot e^t dt d\theta d\varphi \\ &= \iint_C \int_{\log r_j(\theta,\varphi)}^{\log s_j(\theta,\varphi)} \frac{e^{3t} \sin\theta}{(e^t \cos\theta + z_0)^{3-k}} dt d\theta d\varphi. \end{aligned} \tag{4.5}$$

(4.6)

(a) Assume $z_0 = 0$. If $i \neq 0$, the expression (4.4) of μ_k becomes

$$\begin{aligned}\mu_k(E_j) &= \iint_C \int_{r_j(\theta,\varphi)^i}^{s_j(\theta,\varphi)^i} \frac{1}{i} \cdot \frac{t^{\frac{3-i}{i}} \sin\theta}{(t^{\frac{1}{i}} \cos\theta)^{3-k}} dt d\theta d\varphi \\ &= \iint_C \int_{r_j(\theta,\varphi)^i}^{s_j(\theta,\varphi)^i} \frac{1}{i} \cdot \frac{t^{\frac{k-i}{i}} \sin\theta}{\cos\theta)^{3-k}} dt d\theta d\varphi \\ &= \iint_C \frac{1}{i} \frac{\sin\theta}{\cos\theta^{3-k}} \left(\int_{r_j(\theta,\varphi)^i}^{s_j(\theta,\varphi)^i} t^{\frac{k-i}{i}} dt \right) d\theta d\varphi.\end{aligned}$$

By assumption $k > \max\{1, i\}$, so in particular $k > i$, therefore the integrand increases with t.
Similarly, if $i = 0$, the expression (4.5) of μ_k becomes

$$\begin{aligned}\mu_k(E_j) &= \iint_C \int_{\log r_j(\theta,\varphi)}^{\log s_j(\theta,\varphi)} \frac{e^{3t} \sin\theta}{(e^t \cos\theta)^{3-k}} dt d\theta d\varphi \\ &= \iint_C \frac{\sin\theta}{\cos\theta^{3-k}} \left(\int_{\log r_j(\theta,\varphi)}^{\log s_j(\theta,\varphi)} e^{kt} dt \right) d\theta d\varphi.\end{aligned}$$

Again, since $k > 0$ the integrand increases with t. In both cases, the range of the inner integral is of the same length for $j = 1, 2$, so if $i \geq 0$, $\mu_k(E_1) < \mu_k(E_2)$. If $i < 0$, the integrand decreases and moreover we have that $s_j(\theta,\varphi)^i < r_j(\theta,\varphi)^i$ for $j = 1, 2$, so by interchanging the limits of the inner integral we obtain $\mu_k(E_1) > \mu_k(E_2)$.

(b) Assume again $z_0 = 0$. If $k = i \neq 0$, then the expression of μ_i is given by

$$\begin{aligned}\mu_i(E_j) &= \iint_C \frac{1}{i} \frac{\sin\theta}{\cos\theta^{3-i}} \left(\int_{r_j(\theta,\varphi)^i}^{s_j(\theta,\varphi)^i} dt \right) d\theta d\varphi \\ &= \iint_C \frac{1}{i} \frac{\sin\theta}{\cos\theta^{3-k}} \left(s_j(\theta,\varphi)^i - r_j(\theta,\varphi)^i \right) d\theta d\varphi\end{aligned}$$

while if $k = i = 0$, the measure μ_0 is given by

$$\begin{aligned}\mu_0(E_j) &= \iint_C \frac{\sin\theta}{\cos\theta^3} \left(\int_{\log r_j(\theta,\varphi)}^{\log s_j(\theta,\varphi)} dt \right) d\theta d\varphi \\ &= \iint_C \frac{\sin\theta}{\cos\theta^3} \left(\log \frac{s_j(\theta,\varphi)}{r_j(\theta,\varphi)} \right) d\theta d\varphi,\end{aligned}$$

and relations (4.2) and (4.3) complete the proof.

(c) If $i \neq 0$, substituting $k = i$ the derivative with respect to t of the integrand in (4.4) is

$$-\frac{i-3}{i^2}\left(t^{\frac{1}{i}}\cos\theta + z_0\right)^{i-4} t^{\frac{3-2i}{i}} z_0 \cos\theta.$$

Suppose that $z_0 < 0$. If $0 < i < 3$, the integrand decreases with t, and the equality of i-chord functions at p implies $\mu_i(E_1) > \mu_i(E_2)$. If $i < 0$, the integrand decreases but $s_j(\theta,\varphi)^i < r_j(\theta,\varphi)^i$ for $j = 1, 2$, so by interchanging the limits of the inner integral we obtain $\mu_i(E_1) > \mu_i(E_2)$. In the same way we treat the case $i = 0$. Substituting $k = i = 0$, the derivative with respect to t of the integrand in (4.5) is

$$3\left(e^t\cos\theta + z_0\right)^{-4} e^{3t} z_0 \sin\theta$$

and this decreases with t.

The case when $z_0 > 0$ is dealt with similarly.

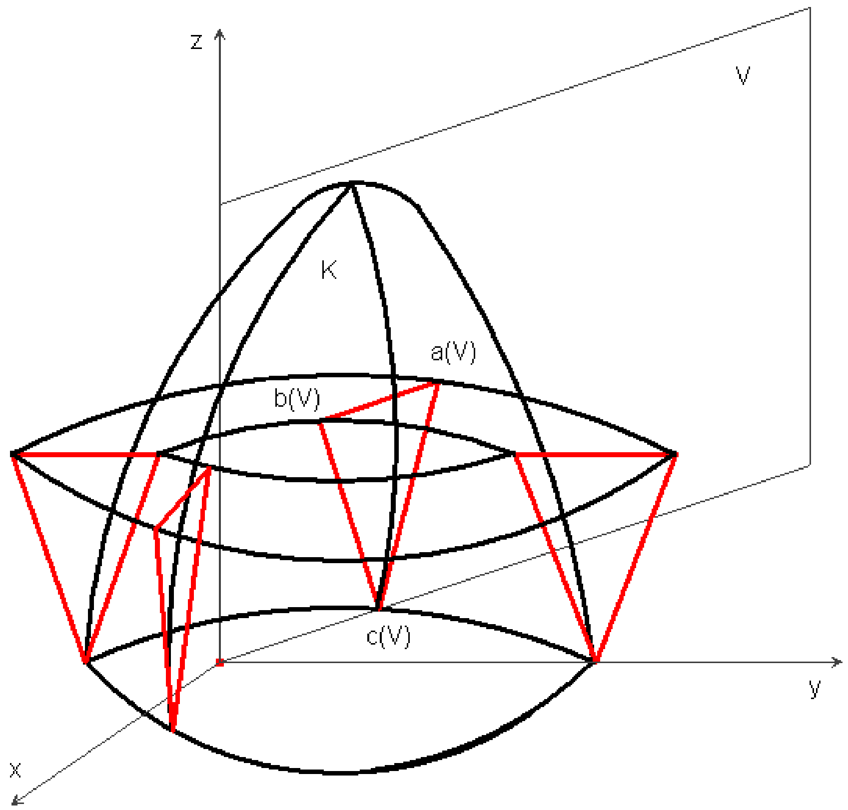
$\square$

4.2 The *Groove*

Let K be a three-dimensional convex body which intersects the plane $G = (l_z)^{\perp}$ and let o belong to the interior of K. Denote by G^+ one of the half spaces determined by G. Without loss of generality, we may assume that $G^+ = \{(x, y, z) : z > 0\}$. Consider a set $S_{\alpha,\delta}(K) \subset G^+$ such that for any half-plane V whose boundary is l_z, the set $S_{\alpha(V),\delta(V)}(K) \cap V$ is an isosceles triangle $\triangle(a(V), b(V), c(V))$ with one vertex $c(V)$ on $\partial K \cap (l_z)^{\perp} \cap V$, and basis $\triangle(a(V), b(V))$ perpendicular to l_z. Moreover, assume that the length of the two equal sides is $\delta(V) > 0$ and that the angle $\measuredangle(b(V), c(V), a(V)) = \alpha(V)$ is constant. Hence $\alpha(V)$ is the infimum of angles in $(0, \pi)$ between $a(V) - c(V)$ and $b(V) - c(V)$ for $b(V) \in intK \cap V$ and $a(V) \in V \setminus K$. We will call such a set a "groove".

In symbols,

$$S_{\alpha(V),\delta(V)}(K) := \bigcup_{\partial V = l_z} \Big\{ \triangle(a(V), b(V), c(V)) | a(V) \in V \setminus K, b(V) \in V \cap intK,$$
$$b(V) - a(V) \perp l_z, c(V) \in \partial K \cap (l_z)^{\perp} \cap V,$$
$$||a(V) - c(V)|| = ||b(V) - c(V)|| = \delta(V) > 0,$$
$$\measuredangle(b(V), c(V), a(V)) = \alpha(V) \Big\}.$$



Let now $\alpha_0 = \max_V \alpha(V)$ and let $\delta_0 = \frac{h_K(u)}{\cos\frac{\alpha_0}{2}}$, where $h_K(u)$ is the value of the support function of K in the direction u parallel to l_z, that is, $h_K(u)$ is the distance between the plane G and the parallel plane supporting K in G^+.

If $k > 2$, then the measure μ_k is a finite measure, but if $k \leq 2$, then μ_k is a σ-finite measure in $\mathbb{R}^3$, which is finite on sets having positive distance from the base

plane $G = (l_z)^{\perp}$.
In particular, we have the following lemma.

Lemma 4.2.1.
Let $k, \delta_0 \in \mathbb{R}_+$ with $k > 1$, and let $\alpha_0 \in (0, \pi)$. Then the groove $S_{\alpha_0,\delta_0}(K)$ has finite measure μ_k.

Proof.
The groove $S_0 := S_{\alpha_0,\delta_0}(K)$ can be parametrized by using cylindrical coordinate in the following way

$$\begin{aligned} S_{\alpha_0,\delta_0}(K) = \Big\{ &(r\cos\theta, r\sin\theta, z) : 0 \le r(\theta) - f_z(\theta) \le r \le r(\theta) + g_z(\theta), \\ &0 \le \theta \le 2\pi, 0 \le z \le \delta_0 \cos\frac{\alpha_0}{2} \Big\} \end{aligned}$$

with f_z, g_z positive functions such that $f_z(\theta) + g_z(\theta) < 2z\tan\frac{\alpha_0}{2}$, for each $\theta \in [0, 2\pi]$. It follows that $r_1(\theta) := r(\theta) - f_{z_0}(\theta)$ and $r_2(\theta) := r(\theta) + g_{z_0}(\theta)$, with $r_1(\theta) < r_2(\theta)$, are bounded star-shaped curves in the plane $z = z_0$, this implies that there exists a positive constant R such that $r_2(\theta) < R$ for every $(\theta, z_0) \in [0, 2\pi] \times [0, \delta_0 \cos\frac{\alpha_0}{2}]$.
Now compute $\mu_k(S_0)$.

$$\begin{aligned} \mu_k(S_0) &= \iiint_{S_0} \frac{dxdydz}{z^{3-k}} \\ &= \int_0^{\delta_0\cos\frac{\alpha_0}{2}} \left[\int_0^{2\pi} \left(\int_{r(\theta)-f_z(\theta)}^{r(\theta)+g_z(\theta)} \frac{r}{z^{3-k}} dr \right) d\theta \right] dz \\ &= \frac{1}{2} \int_0^{\delta_0\cos\frac{\alpha_0}{2}} \frac{1}{z^{3-k}} \left\{ \int_0^{2\pi} \Big[r^2 \Big]_{r(\theta)-f_z(\theta)}^{r(\theta)+g_z(\theta)} d\theta \right\} dz. \end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi} \left[r^2\right]_{r(\theta)-f_z(\theta)}^{r(\theta)+g_z(\theta)} d\theta &= \int_0^{2\pi} \left[(r(\theta)+g_z(\theta))^2-(r(\theta)-f_z(\theta))^2\right] d\theta \\
&= \int_0^{2\pi} (g_z(\theta)+f_z(\theta))\,(2r(\theta)+g_z(\theta)-f_z(\theta))\,d\theta \\
&= \int_0^{2\pi} (g_z(\theta)+f_z(\theta))\,(r(\theta)+g_z(\theta)+r(\theta)-f_z(\theta))\,d\theta \\
&\leq \int_0^{2\pi} 2z\tan\frac{\alpha_0}{2}\cdot(r_2(\theta)+r_1(\theta))\,d\theta \\
&< \int_0^{2\pi} 2z\tan\frac{\alpha_0}{2}\cdot 2R d\theta \\
&< 8\pi R\tan\frac{\alpha_0}{2} z.
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_k(S_0) &< \frac{1}{2}\left(8\pi R\tan\frac{\alpha_0}{2}\right)\int_0^{\delta_0\cos\frac{\alpha_0}{2}} \frac{z}{z^{3-k}}dz \\
&= 4\pi R\tan\frac{\alpha_0}{2}\int_0^{\delta_0\cos\frac{\alpha_0}{2}} \frac{1}{z^{2-k}}dz.
\end{aligned}$$

Observe that:

$$\int_0^{\delta_0\cos\frac{\alpha_0}{2}} \frac{1}{z^{2-k}}dz = \begin{cases} +\infty & \text{if } k=1 \\ \frac{(\delta_0\cos\frac{\alpha_0}{2})^{k-1}}{k-1} & \text{if } k>1 \end{cases}.$$

So we can conclude that

$$\mu_k(S_0) = \begin{cases} +\infty & \text{if } k=1 \\ \frac{4\pi R\tan\frac{\alpha_0}{2}(\delta_0\cos\frac{\alpha_0}{2})^{k-1}}{k-1} < +\infty & \text{if } k>1 \end{cases}.$$

□

4.3 Uniqueness results

Now consider a three-dimensional convex body K and three noncollinear points sources p_1, p_2 and p_3 not belonging to K. Suppose that the plane G, which they identify, intersects the interior of K. The next lemma is a first result about the uniqueness of the planar convex body obtained from the intersection of K with the plane G.

Lemma 4.3.1.
Let K and K' be convex bodies in $\mathbb{R}^3$ whose interior intersect a given plane G, and let $0 < i < 3$. If p_1, p_2, p_3 are three noncollinear points of G not belonging to $K \cup K'$ and the i-chord functions of K and K' at p_j are equal for $j = 1, 2, 3$, then $K \cap G = K' \cap G$.

Proof.
Let $K_0 = K \cap G$ and $K_0' = K' \cap G$.
If intK_0 intersects one or more lines $p_j p_h$, for $j \neq h$ and $j, h \in \{1, 2, 3\}$, then the supports of two i-chord functions at p_j and p_h determine one bounded quadrangle and two unbounded regions which may contain K_0. But the support of the i-chord function at the remaining third point determines the position of K_0 with respect to the segment $[p_j, p_h]$. Thus, K_0 and K_0' belong to the same component of $p_j p_h \setminus \{p_j p_h\}$, then either both meet the segment $[p_j, p_h]$ or are both disjoint from $[p_j, p_h]$. We can therefore apply Theorem 3.2.2 to conclude that $K_0 = K_0'$.

If one or more lines $p_j p_h$ for $j \neq h$ and $j, h \in \{1, 2, 3\}$ support K_0, then the support of the i-chord function at p_j and the support of the i-chord function at p_h determine one bounded quadrangle and two unbounded regions that may contain K_0. But the support of the i-chord function at the remaining third point determines the position of K_0 with respect to the segment $[p_j, p_h]$. Thus, K_0 and K_0' belong to the same component of $p_j p_h \setminus \{p_j p_h\}$, then either both meet the segment $[p_j, p_h]$ or are both disjoint from $[p_j, p_h]$. We can therefore apply Theorem 3.2.3 to conclude that $K_0 = K_0'$.
Let $T = \triangle(p_1, p_2, p_3)$. For reasons of symmetry we may assume that K_0 (and therefore also K_0') is contained in a half-cone $\mathrm{pos}_{p_1} T$ determined by lines through p_1 and p_j, $j = 2, 3$, and containing the triangle T. Let T' be the image of T under reflection at p_1. We have three cases to consider:

(a) $K_0 \subset \mathrm{int}\, T$;

(b) $K_0 \subset \mathrm{int}(\mathrm{pos}_{p_1} T \setminus T)$;

(c) $K_0 \subset \mathrm{int}(\mathrm{pos}_{p_1} T')$.

We shall assume that $K_0' \neq K_0$ and conclude with a contradiction.
Let $G := \mathrm{aff}(p_1, p_2, p_3)$. Let us show that no component of $\mathrm{int}\,(K \triangle K')$ is such that its closure meets the plane G.

(a) Suppose that there are such components and take one, call it A, having maximal μ_{i+1}-measure. By definition of component, A has to be visible from one point, say p_1. But then $p_1(A)$ is defined and, by Lemma 4.1.2 (a),

$$\mu_{i+1}(p_1(A)) > \mu_{i+1}(A),$$

a contradiction.

(b) Let H be a plane through the line p_2p_3 separating p_1 from K_0 (and so K_0'). Take a coordinate system in which the plane H has equation $z = 0$ and consider H as *base plane* for the measure μ_i. We can assume that $K_0 \subset \{(x, y, z) : z > 0\}$ so that the point p_1 has negative z-coordinate. Among the components such that their closures meet the plane G take one, call it B, having maximal measure μ_i. If B is visible from p_1 then B is visible also from p_2 and p_3, so $B' = p_2(B)$ is defined and, by Lemma 4.1.2 (b), $\mu_i(B) = \mu_i(B')$. Now B' is not visible from p_1, thus $B'' = {p_1}^{-1}(B')$ is defined and by Lemma 4.1.2 (c)

$$\mu_i(B'') > \mu_i(B') = \mu_i(B),$$

a contradiction.
On the other hand, if B is not visible from p_1, then ${p_1}^{-1}(B)$ is defined and by Lemma 4.1.2 (c)

$$\mu_i({p_1}^{-1}(B)) > \mu_i(B),$$

again a contradiction.

(c) Let us take now the plane H' through the line p_1p_3 separating p_2 from K, (and so from K_0'), as *base plane* for the measure μ_i. Assume that $K_0 \subset \{(x, y, z) : z > 0\}$ so that the point p_2 has negative z-coordinate. Consider a component C such that its closure meets the plane G and having maximal measure μ_i. If C is visible from p_2 then C is visible also from p_1 consequently $C' = p_1(C)$ is defined and, by Lemma 4.1.2 (b), $\mu_i(C) = \mu_i(C')$. Now C' is not visible from p_2, thus $C'' = {p_2}^{-1}(C')$ is defined and by Lemma 4.1.2 (c)

$$\mu_i(C'') > \mu_i(C') = \mu_i(C),$$

a contradiction.
Otherwise, if C is not visible from p_2, then ${p_2}^{-1}(C)$ is defined and by Lemma 4.1.2 (c)

$$\mu_i({p_2}^{-1}(C)) > \mu_i(C),$$

again a contradiction.

Consequently, in each of these three cases no component of int $(K \triangle K')$ is such that its closure intersects the plane G contrary to the assumptions. This means that the intersection of a three-dimensional body with the plane G is uniquely determined, therefore $K_0 = K'_0$. □

Theorem 4.3.2.
Let $i > 1$. Let K be a convex body in $\mathbb{R}^3$ and let $G := aff(p_1, p_2, p_3)$, $T := \triangle(p_1, p_2, p_3)$ for some noncollinear points p_1, p_2, p_3 in $\mathbb{R}^3$. If $(G \cap int K) \subset T$, then K is uniquely determined by i-chord functions at p_1, p_2, p_3.

Proof.
Let $K_0 := K \cap G$. By Lemma 4.3.1, the set K_0 is uniquely determined.
Suppose that K' is another convex body in $\mathbb{R}^3$ with the same i-chord functions as K at p_j for $j = 1, 2, 3$. Obviously, $K' \cap G = K_0$. We shall assume that $K' \neq K$ and derive a contradiction. Let A_1 be a component of int $(K \triangle K')$ having positive distance from the plane G. The component A_1 is not visible from at least one of the three points. In fact, if this is not true, then we may suppose that $A_1 \subset$ int $(K \setminus K')$. Denote by A^j the component of int $(K \triangle K')$ defined by $A^j = p_j(A)$. Then there exist $q \in \overline{A_1} \cap \partial K$ and $q_j \in \overline{A^j} \cap \partial K$ such that p_j, q and q_j are collinear and in that order on the line $p_j q_j$. By convexity, the triangle $\tilde{T}$ with vertices q_1, q_2 and q_3 is contained in K. When varying the point r in $\tilde{T}$, the line rq intersects the plane G in the triangle T determined by the p_j's. Therefore there exists a point $r_0 \in \tilde{T}$ such that $r_0 q$ intersects G in a point which is interior to K. But this contradicts, by convexity, the fact that $q \in \partial K$. Without loss of generality we may assume that A_1 is not visible from p_2 thus $A_2 = {p_2}^{-1}(A_1)$ is defined. Similarly, there is a point different from p_2 from which A_2 is not visible, say p_1, so $A_3 = {p_1}^{-1}(A_2)$ is defined, and so on.

Denoting with c_j the distance from G of the point A_j closest to G, we infer that the sequence $\{c_j\}_{j\in\mathbb{N}}$ is strictly decreasing. This implies that the components B_j are all disjoint, i.e. $A_l \cap A_m = \emptyset$ for $l \neq m$. In addition, if we consider the plane G as the *base plane* for the measure μ_i, this components are all contained in the half-space $G^+ = \{(x, y, z) : z > 0\}$ and have all the same measure μ_i, i.e.

$$\mu_i(A_k) = \beta \qquad \forall\ k \in \mathbb{N}.$$

In particular they are all contained in the groove $S_{\alpha_0,\delta_0}(K)$ whose base is the convex curve ∂K_0 and height $\delta_0 \cos\frac{\alpha_0}{2}$, i.e.

$$\bigcup_{k\in\mathbb{N}} A_k \subset S_{\alpha_0,\delta_0}(K).$$

By countable additivity and monotonicity of the measure μ_i, we have

$$\mu_i\Big(\bigcup_{k\in\mathbb{N}} A_k\Big) = \sum_{k\in\mathbb{N}} \mu_i(A_k) = \sum_{k\in\mathbb{N}} \beta = \infty,$$

but, on the other hand,

$$\infty = \mu_i\Big(\bigcup_{k\in\mathbb{N}} A_k\Big) \leq \mu_i\big(S_{\alpha_0,\delta_0}(K)\big).$$

But by Lemma 4.2.1, the groove has finite measure for $i > 1$, a contradiction. Therefore no such K' exists. $\square$

Theorem 4.3.3.
Let K be a convex body in $\mathbb{R}^3$ and let $G := aff(p_1, p_2, p_3)$, $T := \triangle(p_1, p_2, p_3)$ for some noncollinear points p_1, p_2, p_3 in $\mathbb{R}^3$. Let $T'(p_j)$ be the image of T under reflection at p_j. If $G \cap intK \neq \emptyset$, then K is uniquely determined by i-chord functions at p_1, p_2, p_3 for some $i \in (0; 3)$ in each of the following cases:

(a) $(\partial K \cap G) \subset int(pos_{p_j} T) \setminus T$;

(b) $(\partial K \cap G) \subset int(pos_{p_j} T'(p_j))$.

Proof.
Let $K_0 := K \cap G$. By Lemma 4.3.1, the set K_0 is uniquely determined.

Suppose that K' is another convex body in $\mathbb{R}^3$ with the same i-chord functions as K at p_j for $j = 1, 2, 3$. Obviously, $K' \cap G = K_0$. We shall assume that $K' \neq K$ and derive a contradiction. For reasons of symmetry we may assume that K_0 is contained in the cone $\mathrm{pos}_{p_1} T$ determined by lines through p_1 and p_j, $j = 2, 3$, and containing the triangle T. So let $T'(p_1) = T'$ be the image of T under reflection at p_1. Then the two cases that we have to consider are:

(a) $K_0 \subset \mathrm{int}(\mathrm{pos}_{p_1} T) \setminus T$;

(b) $K_0 \subset \mathrm{int}(\mathrm{pos}_{p_1} T')$.

(a) In this case, there is a plane H through the line p_2p_3 separating p_1 from K. Take a coordinate system in which the plane H has equation $z = 0$, and consider H as *base plane* for the measure μ_i. We can assume that $K \subset \{(x, y, z) : z > 0\}$ and consequently the point p_1 has negative z-coordinate. Let B_1 be a component of $\mathrm{int}\,(K \triangle K')$ having maximal measure μ_i. B_1 is visible from p_1, since otherwise by Lemma 4.1.2 (c),

$$\mu_i({p_1}^{-1}(B_1)) > \mu_i(B_1),$$

a contradiction. Then B_1 is visible either from p_2 or p_3. Without loss of generality we may assume that B_1 is visible from p_2, therefore $C_2 = p_2C_1$ is defined and by Lemma 4.1.2 (b) we have

$$\mu_i(B_2) = \mu_i(B_1).$$

Again if B_2 is not visible from p_1 then by Lemma 4.1.2 (c)

$$\mu_i({p_1}^{-1}B_2) > \mu_i(B_2) = \mu_i(B_1)$$

contrary to the maximality of $\mu_i(B_1)$. Necessarily B_2 has to be visible from p_1. So B_2 is a component not visible from p_2, but visible from p_1, thus by convexity of K, B_2 is visible from p_3 and hence $p_3(B_2) = B_3$ is defined and

$$\mu_i(B_3) = \mu_i(B_2) = \mu_i(B_1).$$

Iterating this construction, we define

$$B_{2n} = p_2(B_{2n-1})$$

$$B_{2n+1} = p_3(B_{2n}).$$

These components are disjoint, have the same measure μ_i by construction and are contained in $K \cup K'$ that has finite measure μ_i, a contradiction.

(b) Let us take now the plane H' through the line $p_1 p_2$ separating p_3 from K as *base plane* for the measure μ_i. Assume that $K \subset \{(x, y, z) : z > 0\}$ so the point p_3 has negative z-coordinate. Let C_1 be a component of $\mathrm{int}\,(K \triangle K')$ with maximal measure μ_i. The maximality of $\mu_i(C_1)$ and Lemma 4.1.2 (c) imply that C_1 has to be visible from p_3, and by convexity C_1 is visible also from p_1, consequently $C_2 = p_1(C_1)$ is defined. By Lemma 4.1.2 (b)

$$\mu_i(C_2) = \mu_i(C_1).$$

If C_2 is not visible from p_3 then, as before, by Lemma 4.1.2 (c)

$$\mu_i({p_3}^{-1}(C_2)) > \mu_i(C_2) = \mu_i(C_1)$$

contrary to the maximality of $\mu_i(C_1)$. Then necessarily C_2 is visible from p_3, but C_2, by construction, is not visible from p_1, then C_2 is not visible from p_2. This means that $C_3 = {p_2}^{-1}(C_2)$ is defined and by Lemma 4.1.2 (b)

$$\mu_i(C_3) = \mu_i(C_2) = \mu_i(C_1).$$

Obviously C_3 is visible from p_2 and has to be visible from p_3, then by convexity C_3 is visible from p_1, therefore $C_4 = p_1(C_3)$ is defined and

$$\mu_i(C_4) = \mu_i(C_3).$$

We define iteratively

$$C_{2n} = p_1(C_{2n-1})$$

$$C_{2n+1} = {p_2}^{-1}(C_{2n}).$$

These components are pairwise disjoint, they have the same measure μ_i, and are contained in $K \cup K'$ that has finite measure μ_i, a contradiction.

□

The method of i-chord functions fails for $i = 1$ when $\partial K \cap G$ is contained in the interior of the triangle determined by the three noncollinear points because the sequence of components of the symmetric difference of the two three-dimensional convex bodies is contained in the groove $S_{\alpha_0,\delta_0}(K)$ having μ_1-measure infinite. In order to obviate to this problem we have to make further assumption on the convex body K.

Consider two different three-dimensional convex bodies K and K' and three noncollinear points p_1, p_2 and p_3 not belonging to $K \cup K'$. Suppose that the plane G which they identify, intersects the interior of K and the interior of K'. Assume that K and K' have the same i-chord functions at p_j, for $j = 1, 2, 3$. Consider a plane F perpendicular to the plane G and intersecting both the interior of K and K'. Since, by assumption, K and K' are convex, the curves $\partial K \cap F$ and $\partial K' \cap F$ are convex, too. Moreover, by Lemma 4.1.2, $\partial K \cap G = \partial K' \cap G$ consequently $(\partial K \cap G) \cap F = (\partial K' \cap G) \cap F$ are two distinct points uniquely determined. Denote these two points by q_1 and q_2 and denote by G^+ one of the half-spaces determined by G.
From now on, we shall call "*upper tangent*" to the boundary of K at a point q, the restriction of the line tangent to $\partial K \cap G^+$ at q, which always exist by convexity.
The following lemma shows that the upper tangent in the plane F to the boundary of K and to the boundary of K' are the same.

Lemma 4.3.4.
Suppose K and K' have the same i-chord functions at three noncollinear points p_1, p_2 and p_3 and K intersects the plane G determined by the three points in the interior of the triangle $\triangle(p_1, p_2, p_3)$. Let F be a plane orthogonal to G intersecting $K \cap G$ in its relative interior. Then $F \cap K$ and $F \cap K'$ have the same upper tangent at the points q_1, q_2 of $F \cap \partial K \cap G$.

Proof.
By Lemma 4.3.1 we know that $\partial K \cap G = \partial K' \cap G$.
We distinguish two cases. Either $(\partial K \cap \partial K') \cap F$ has positive distance from q_1 and q_2, respectively, or its closure contains q_1 and q_2, respectively.
In the first case $\partial K \cap F$ and $\partial K' \cap F$ coincide close to q_1 and q_2 and therefore the upper tangents at q_1 and q_2 are pairwise equal.
In the second case, since the upper tangent exists, and since ∂K and $\partial K'$ intersects infinitely often close to q_1 and q_2, by Lemma 1.4.5 the same conclusion holds. ☐

From now on, assume that $K \in \mathcal{C}^{1+\alpha}$ with $\alpha \in (0,1)$. Let o be interior to the planar convex body $K_0 = K \cap G$. For any $\theta \in [0, 2\pi]$ let F_θ be the half-plane orthogonal to G, contained in G^+ whose boundary is the line $l_\theta \subset G$ through o and parallel to θ. Denote by h_0 the distance between G and the parallel plane G' contained in G^+ supporting K.
Let $t_1(\theta)$ and $t_2(\theta)$ be the polar representation of the upper tangents of K at the two points q_1 and q_2, respectively, of $\partial K \cap F_\theta$.
For any $\theta \in [0, 2\pi]$ let $Q(\theta)$ be the quadrangle in F_θ determined by $t_1(\theta)$, $t_2(\theta)$, $G \cap F_\theta$ and $G' \cap F_\theta$, (see Figure 4.1).

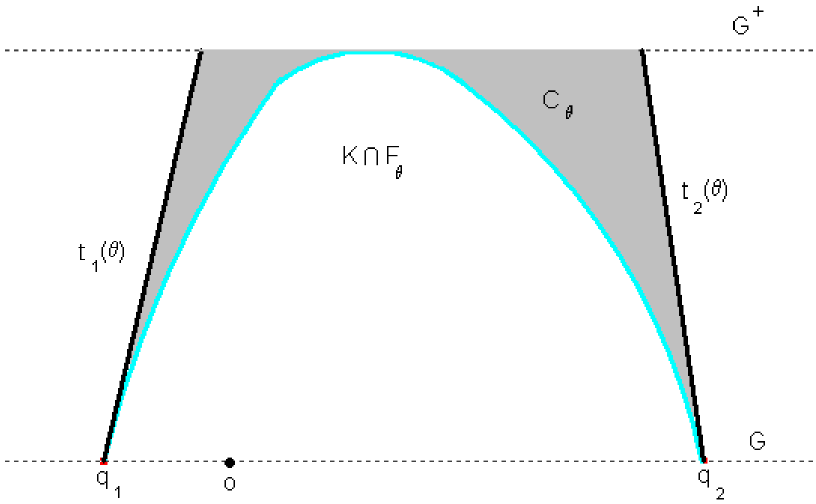


Figure 4.1

Consider now the "*Claw*"

$$C = \bigcup_{\theta \in [0,2\pi]} \left(Q(\theta) \smallsetminus (K \cap F_\theta) \right).$$

Looking at $Q(\theta) \smallsetminus (K \cap F_\theta)$, we see that it has two components, so it is enough if we consider one of them, call it $C(\theta)$. Given θ, let $g(\theta, z)$ be the function which represents the boundary of K and by $f(\theta, z)$ its upper tangent at $z = 0$.
Denote by g_z the partial derivative of g with respect to z, then by Taylor's theorem (see Theorem 1.4.3, Chapter 1) we have

$$f(\theta, z) = g(\theta, 0) + g_z(\theta, 0)z$$

and

$$g(\theta, z) = g(\theta, 0) + g_z(\theta, \xi)z$$

with $\xi \in]0, z[$.
Therefore

$$f(\theta, z) - g(\theta, z) = [g_z(\theta, 0) - g_z(\theta, \xi)]\, z$$

and

$$f(\theta, z) + g(\theta, z) = 2g(\theta, 0) + [g_z(\theta, 0) + g_z(\theta, \xi)]\, z. \tag{4.7}$$

Since $g \in \mathcal{C}^{1+\alpha}$, with $\alpha \in (0, 1)$ then

$$|f(\theta, z) - g(\theta, z)| \leq H|\xi|^{\alpha} z. \tag{4.8}$$

Furthermore, since $\xi \in (0, z)$ we have

$$|f(\theta, z) - g(\theta, z)| \leq H z^{\alpha+1}. \tag{4.9}$$

Exploiting the concepts we have introduced we can demonstrate the following lemma.

Lemma 4.3.5.
The claw C, defined above, has finite measure μ_1.

Proof.
We use cylindrical coordinates in order to compute the measure μ_1 of the set C.

$$\begin{aligned}
\mu_1(C) &= \iiint_C \frac{dxdydz}{|z|^2} = \int_0^{2\pi} \left(\iint_{C(\theta)} \frac{rdrdz}{z^2} \right) d\theta \\
&= \int_0^{2\pi} \left(\lim_{\varepsilon \to 0} \iint_{C_\varepsilon(\theta)} \frac{rdrdz}{z^2} \right) d\theta
\end{aligned}$$

where

$$C_\varepsilon(\theta) = C(\theta) \cap \{z : z \geq \varepsilon\}.$$

$$
\begin{aligned}
\iint_{C_\varepsilon(\theta)} \frac{rdrdz}{z^2} &= \int_\varepsilon^{h_0} \frac{1}{z^2}\left(\int_{f(\theta,z)}^{g(\theta,z)} rdr\right) dz \\
&= \int_\varepsilon^{h_0} \frac{1}{z^2}\left(g(\theta,z)^2 - f(\theta,z)^2\right) dz \\
&= \int_\varepsilon^{h_0} \frac{1}{z^2}\left[\left(g(\theta,z) - f(\theta,z)\right)\left(g(\theta,z) + f(\theta,z)\right)\right] dz \\
&\overset{4.9,\,4.7}{=} \int_\varepsilon^{h_0} \frac{1}{z^2}\left(\left(Hz^{\alpha+1}\right)\left(2|g(\theta,0)| + |g_z(\theta,0) + g_z(\theta,\xi)|\, z\right)\right) dz \\
&= \int_\varepsilon^{h_0} \frac{1}{z^2}\left[2|g(\theta,0)|Hz^{\alpha+1} + |g_z(\theta,0) + g_z(\theta,\xi)|\, Hz^{\alpha+2}\right] dz \\
&\leq 2\gamma_1 H \int_\varepsilon^{h_0} z^{\alpha-1} dz + \gamma_2 H \int_\varepsilon^{h_0} z^\alpha dz \\
&= 2\gamma_1 H \frac{h_0^\alpha - \varepsilon^\alpha}{\alpha} + \gamma_2 H \frac{h_0^{\alpha+1} - \varepsilon^{\alpha+1}}{\alpha+1},
\end{aligned}
$$

where we have denoted by γ_1 and γ_2 the quantities

$$\gamma_1 = |g(\theta,0)|$$

and

$$\gamma_2 = |g_z(\theta,0) + g_z(\theta,\xi)|$$

that are both finite. Therefore

$$
\begin{aligned}
\mu_1(C) &= \iiint_C \frac{dxdydz}{|z|^2} = \int_0^{2\pi} \lim_{\varepsilon\to 0}\left(2\gamma_1 H \frac{h_0^\alpha - \varepsilon^\alpha}{\alpha} + \gamma_2 H \frac{h_0^{\alpha+1} - \varepsilon^{\alpha+1}}{\alpha+1}\right) d\theta \\
&= \int_0^{2\pi} H{h_0}^\alpha \left(\frac{2\gamma_1}{\alpha} + \frac{\gamma_2}{\alpha+1} h_0\right) \\
&= 2\pi H{h_0}^\alpha \left(\frac{2\gamma_1}{\alpha} + \frac{\gamma_2}{\alpha+1} h_0\right) < \infty.
\end{aligned}
$$

□

If we consider the components of int $(K \triangle K')$ their union is contained in the Claw C and therefore $\mu_1(K' \triangle K) < \infty$. As we have seen, the finiteness of the measure μ_1 guaranteed if the boundary ok K is represent by a function which satisfies a Hölder condition with exponent $\alpha \in (0,1)$.
So, we have the following.

Theorem 4.3.6.
Let $\alpha \in (0,1)$ and let p_1, p_2 and p_3 be three noncollinear points. Let $G := aff(p_1, p_2, p_3)$ be a plane and $T := \triangle(p_1, p_2, p_3)$ be a triangle. If $K \in \mathcal{C}^{1+\alpha}$ is a convex body such that $(G \cap int K) \subset T$, then K is uniquely determined by X-rays at p_1, p_2 and p_3.

Proof.
Let $K_0 := K \cap G$. By Lemma 4.3.1, the set K_0 is uniquely determined.
Suppose that K' is another convex body in $\mathbb{R}^3$ with the same X-rays as K at p_j for $j = 1, 2, 3$. Obviously, $K' \cap G = K_0$. We shall assume that $K' \neq K$ and derive a contradiction. Let A_1 be a component of $\operatorname{int}(K \triangle K')$ having positive distance from the plane G. Following the same argument used in the proof of Theorem 4.3.2 we have that A_1 is not visible from at least one of the three points. Without loss of generality we may assume that A_1 is not visible from p_2 thus $A_2 = {p_2}^{-1}(A_1)$ is defined. Similarly, there is a point different from p_2 from which A_2 is not visible, say p_1, so $A_3 = {p_1}^{-1}(A_2)$ is defined, and so on. Denoting with c_j the distance from G of the point A_j closest to G, we infer that the sequence $\{c_j\}_{j\in\mathbb{N}}$ is strictly decreasing. This implies that the components B_j are all disjoint, i.e. $A_l \cap A_m = \emptyset$ for $l \neq m$. In addition, if we consider the plane G as the *base plane* for the measure μ_i, this components are all contained in the half-space $G^+ = \{(x, y, z) : z > 0\}$ and have all the same measure μ_i, i.e.

$$\mu_i(A_k) = \beta \qquad \forall\ k \in \mathbb{N}.$$

The components contained in K' are all contained in the claw C having finite measure μ_1. Therefore $K' \setminus K$ is empty and so is then $K \setminus K'$.
In conclusion, K is uniquely determined. $\square$

So we can conclude with the following important uniqueness result for convex body of class $\mathcal{C}^{1+\alpha}$.

Corollary 4.3.7.
Let $\alpha \in (0,1)$. If $K \in \mathcal{C}^{1+\alpha}$, then X-rays at three noncollinear points determine uniquely K among all convex bodies.

Chapter 5

Conclusions and open problems

The main results obtained provide a partial answer to the problem 5.7 posed by Gardner in [16]:

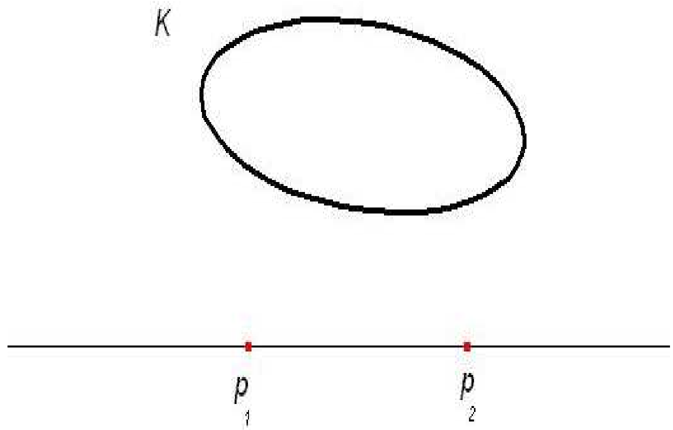
> « How many point X-rays are needed to determine a convex body in $\mathbb{R}^n$? »

and are summarized in the following.
First of all, we have shown the determination of a planar convex body K by taking the i-chord functions, for $i > 0$, at two points when the line l passing through p_1 and p_2 meets the interior of K and the two points p_1 and p_2 are both exterior or interior to K. If the line l supports K, then the results hold for $i \geq 1$.
A second relevant result concerns the determination of a planar convex body K from its i-chord functions at three noncollinear points for $0 < i < 2$. In particular, when the convex body is contained in the interior of the triangle formed by the three points we have shown that the result holds for $i > 0$.

Finally we have tackled the problem of determining a three-dimensional convex body K from the i-chord functions at three noncollinear points not belonging to K using a sort of "Cavalieri Principle", for a suitable measure involving i-chord functions for $1 < i < 3$.
Since we have not been able to extend this result to generic convex bodies for $i = 1$ we had to assume that the convex body is of class $\mathcal{C}^{1+\alpha}$ with $\alpha \in]0, 1[$.

Nevertheless numerous problems remain still open.

First of all, the case when the line l through the two point sources p_1 and p_2 does not intersect K is the major open problem in this part of geometric tomography.



Suppose that the line l through the two point sources p_1 and p_2 meets the interior of K.
Is K determined by i-chord functions at p_1 and p_2 if either

(i) p_1 and p_2 are not in int$K$, (see Figure 3.10 in Chapter 3)

or

(ii) $p_1 \notin K$ and $p_2 \in \operatorname{int} K$?(see Figure 5.1).

These questions are unresolved at present. In fact, it is unknown whether these can actually occur.

If we consider another point source, we then have the following open problem.
Is the planar convex body K determined by i-chord functions at three noncollinear points p_1, p_2 and p_3 if $p_3 \in \operatorname{int} K$ and the line through p_1 and p_2 misses K? (see Figure 3.12 in Chapter 3).

These problems are quite interesting and might be a possible topic for future works.

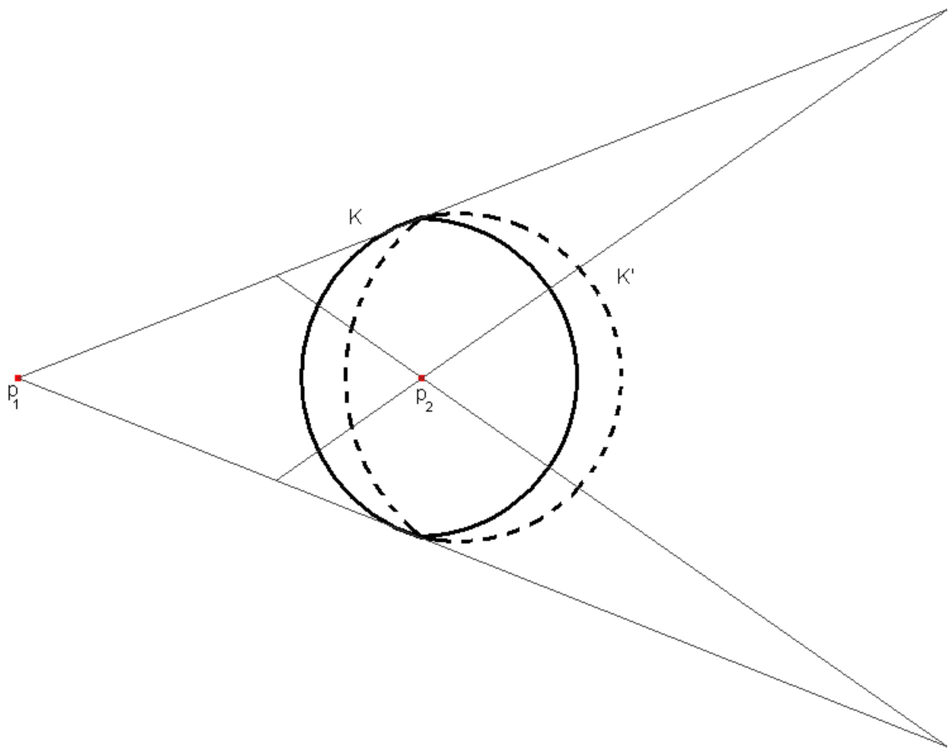


Figure 5.1

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Marina Dicosta

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