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Interpolation problems and applications

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Introduction

The study of real phenomenons leads, very often, to the analysis of functions not easily usable, that because of their complicated analytic expressions which are too complex to evaluate efficiently or because of the functions studied are known only in a number of data points. From that the necessity of approximate a function with a simpler mathematical expression taking account of its known values and minimizing the remainder. Interpolation is a method of approximating a given function using data that are available at a distribution of data points. The data can be consist, for example, of the function values or some of its derivatives. The general approach is to construct an interpolating function, the interpolant, which fits the available data perfectly. Among the different classes of functions in which find the interpolants, the polynomial one is the most studied because of its known characteristics. In the following, at first we deal with the problem from a general point of view, that is in an arbitrary linear space of finite dimension, then we focus our study on two different polynomial interpolation problems of which we examine some applications. In Chapter 1 we consider interpolation, remainder theory, convergence theorems for interpolatory processes and some classic problems of finite interpolation. We also present a number of basic definitions and elementary properties that will be of use in the later chapters. In Chapter 2 we introduce the sequences of Appell polynomials for which we provides an algebraic approach with the aim of giving a unifying theory for all classis of Appell polynomials and a their very natural generalization. We establish the equivalence of the new approach with the previous characterizations through a circular theorem. Moreover, we give general properties of Appell polynomials by employing basic tools of linear algebra and we consider their expansion in Fourier series. We also propose an efficient and stable Gaussian algorithm for the computation of the coefficients for particular sequences of Appell polynomials and consider classic examples, in particular Bernoulli, Euler, normalized Hermite and Laguerre polynomials and their possible generalizations not studied in the literature so far. Finally, the proposed algebraic approach allows the solution, expressed by a determinantal form using a basis of Appell polynomials, of a remarkable new general linear interpolation problem. In Chapter 3 we examine this interpolation problem related to the Appell polynomials and we provide an explicit solution of it and a study of the remainder. Moreover, we consider classic examples, in particular Taylor, Bernoulli, Euler, normalized Hermite and Laguerre polynomials and their possible generalizations. We also present certain numerical examples to test the method proposed. In Chapter 4 we consider series related to particular interpolation problems, as Bernoulli and Lidstone, and we prove that, for real entire functions of exponential type, these expansions coincide with the known Fourier series. In Chapter 5 we introduce a new interpolation problem of Birkhoff kind. Birkhoff, or lacunary, interpolation appears whenever observation gives scattered or irregular information about a function and its derivatives. Lacunary interpolation differs radically from the more familiar of Lagrange and Hermite interpolation for which the interpolation polynomial always exists, is unique, and can be given by an explicit formula. In particular our study deals with an interpolation problem in which the data are the function values at boundary points of a given interval $[a, b]$ and its second derivative at $n-1$ internal points of the interval $[a, b]$. We provide an explicit solution of the interpolation problem considered and an estimation error. Moreover, we study the convergence of the method and we give the examples of equidistant interpolation points and Chebyshev points. Finally, as application of this interpolation problem, we consider a new class of interpolatory type quadrature formulae, called extended Gauss-Birkhoff quadratures with respect to a weight function w.

In Chapter 6 we examine a boundary value problem related to the interpolatory problem proposed in the previous chapter. We provide a general procedure to determine collocation methods for this problem and we give an a-priori estimation error. We also propose a numerical algorithm for the calculation and, finally, present certain numerical examples to compare the method proposed with the Matlab build-in function bvp4c.

Chapter 1

Interpolation

In this chapter we present a number of basic definitions, main theorems and elementary properties from interpolation theory that will be of use in the later chapters.

1.1 General problem of finite interpolation

The general problem of finite interpolation ([21]) is the following:

Let X be a linear space of dimension n and let $L_1, L_2, ..., L_n$ be n given linear functionals defined on X. For a given set of values $\omega_1, \omega_2, ..., \omega_n$, find, if it exists, an element $x \in X$ such that

$$L_i(x) = \omega_i, \quad i = 0, ..., n. \tag{1.1}$$

For the solution we consider the following

Lemma 1 *Let X have dimension n. If $x_1, x_2, ..., x_n$ are independent in X and $L_1, L_2, ..., L_n$ are independent in the dual space X^* then*

$$|L_i(x_j)| \neq 0. \tag{1.2}$$

Conversely, if either $x_1, x_2, ..., x_n$ *or* $L_1, L_2, ..., L_n$ *are independent and (1.2) holds then the other set is also independent.*

Proof. Suppose that $|L_i(x_j)| = 0$. Then also $|L_j(x_i)| = 0$. The linear system

$$\begin{cases} a_1 L_1(x_1) + a_2 L_2(x_1) + ... + a_n L_n(x_1) = 0 \\ \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_1 L_1(x_n) + a_2 L_2(x_n) + ... + a_n L_n(x_n) = 0 \end{cases}$$

would have a nontrivial solution $a_1, a_2, ..., a_n$.

The property of linearity of functional L_i implies that

$$(a_1 L_1 + a_2 L_2 + ... + a_n L_n)(x_i) = 0, \qquad i = 1, ..., n.$$

Since $x_1, x_2, ..., x_n$ form a basis for X,

$$(a_1 L_1 + a_2 L_2 + ... + a_n L_n)(x) = 0, \qquad x \in X,$$

and hence $a_1 L_1 + a_2 L_2 + ... + a_n L_n = 0$.

Therefore, $L_1, L_2, ..., L_n$ are dependents contrary to our assumption.

To show the converse, we may trace the argument backwards. ■

Theorem 2 *Let a linear space* X *have dimension* n *and let* $L_1, L_2, ..., L_n$ *be* n *elements of* X^*. *The interpolation problem (1.1) possesses a solution for arbitrary values* $\omega_1, \omega_2, ..., \omega_n$ *if and only if the* L_i *are independent in* X^*. *The solution will be unique.*

Proof. Let $x_1, x_2, ..., x_n$ be a basis for X. If $L_1, L_2, ..., L_n$ are independent, then, by lemma 1,

$$|L_i(x_j)| \neq 0.$$

Hence the system

$$L_i(a_1x_1 + a_2x_2 + ... + a_nx_n) = \omega_i, \quad i = 1, 2, ...n$$

or

$$a_1L_i(x_1) + a_2L_i(x_2) + ... + a_nL_i(x_n) = \omega_i \tag{1.3}$$

possesses a solution $a_1, a_2, ..., a_n$ and the element

$$a_1x_1 + a_2x_2 + ... + a_nx_n$$

solves the interpolation problem (1.1).

Conversely, if the problem has a solution for arbitrary ω_i, then the system (1.3) has a solution for arbitrary ω_i. By a known theorem of linear algebra, this implies that

$$|L_i(x_j)| \neq 0$$

and hence by 1, the L_i are independent. ∎

The determinant $|L_i(x_j)|$ is a generalized Gram determinant and its nonvanishing is synonymous with the possibility of solution of the interpolation problem.

We may speak of independent systems of functionals as having the "interpolation property".

1.2 Systems possessing the "interpolation property"

We study, now, some spaces and functionals for which the interpolation problem can be solved.

Example 3 (Interpolation at discrete points)

$$X = P_n. \qquad L_0(f) = f(z_0), L_1(f) = f(z_1), ..., L_n(f) = f(z_n).$$

We assume that $z_i \neq z_j, i \neq j$.

Example 4 (Taylor Interpolation)

$$X = P_n. \qquad L_0(f) = f(z_0), L_1(f) = f'(z_0), ..., L_n(f) = f^{(n)}(z_0).$$

Example 5 (Abel-Gontscharoff Interpolation)

$$X = P_n. \qquad L_0(f) = f(z_0), L_1(f) = f'(z_1), ..., L_n(f) = f^{(n)}(z_n).$$

Example 6 (Lidstone Interpolation)

$$X = P_{2n+1}. \qquad \begin{array}{ll} L_1(f) = f(z_0), & L_2(f) = f(z_1) \\ L_3(f) = f''(z_0), & L_4(f) = f''(z_1) \\ \vdots & \vdots \\ L_{2n+1}(f) = f^{(2n)}(z_0), & L_{2n+2}(f) = f^{(2n)}(z_1), \qquad (z_0 \neq z_1). \end{array}$$

Example 7 (Lidstone of Second Type Interpolation)

$$X = P_{2n+1}. \qquad \begin{array}{ll} L_1(f) = f'(z_0), & L_2(f) = f'(z_1) \\ L_3(f) = f'''(z_0), & L_4(f) = f'''(z_1) \\ \vdots & \vdots \\ L_{2n+1}(f) = f^{(2n+1)}(z_0), & L_{2n+2}(f) = f^{(2n+1)}(z_1), \qquad (z_0 \neq z_1). \end{array}$$

Example 8 (Simple Hermite or Osculatory Interpolation)

$$X = P_{2n-1}. \qquad L_1(f) = f(z_1), \quad L_2(f) = f'(z_1)$$

$$L_3(f) = f(z_2), \quad L_4(f) = f'(z_2)$$

$$\vdots \qquad\qquad \vdots$$

$$L_{2n-1}(f) = f(z_n), L_{2n}(f) = f'(z_n), \qquad (z_i \neq z_j, i \neq j).$$

Example 9 (Full Hermite Interpolation)

$X = P_n$. To avoid indexing difficulties, we list the functional information employed without using the symbol L.

$$f(z_0), f'(z_0), ..., f^{(m_0)}(z_0)$$

$$f(z_1), f'(z_1), ..., f^{(m_1)}(z_1)$$

$$\vdots$$

$$f(z_n), f'(z_n), ..., f^{(m_n)}(z_n)$$

$$(z_i \neq z_j, \quad N = m_0 + m_1 + ... + m_n + n).$$

Example 10 (Generalized Taylor Interpolation)

X consists of the linear combination of the $n+1$ linearly independent functions

$$\varphi_0(z), \varphi_1(z), ..., \varphi_n(z)$$

that are analytic at z_0.

$$L_0(f) = f(z_0), L_1(f) = f'(z_0), ..., L_n(f) = f^{(n)}(z_0).$$

$$\left|\varphi_i^{(j)}(z_0)\right| \neq 0.$$

Example 11 (Trigonometric Interpolation)

A linear combination of $1, \cos x, ..., \cos nx, \sin x, ..., \sin nx$ *is known as a trigonometric polynomial of degree* $\leq n$. *The corresponding linear space will be designate by* T_n. *It has dimension* $2n+1$.

$$X = T_n. \qquad L_0(f) = f(x_0), L_1(f) = f'(x_0), ..., L_{2n}(f) = f^{(2n)}(x_0),$$

$$-\pi \leq x_0 < x_1 < \ldots < x_{2n} < \pi.$$

Example 12 (Fourier Series)

$$X = T_n. \qquad L_{2k}(f) = \textstyle\int_{-\pi}^{\pi} f(x) \cos kx dx, \qquad k = 0, 1, ..., n.$$

$$L_{2k-1}(f) = \textstyle\int_{-\pi}^{\pi} f(x) \sin kx dx, \qquad k = 1, ..., n.$$

Before demonstrating that these functionals are independent over the respective spaces, a few remarks are in order. Ex. 3 is, of course, Theorem 2. Exs. 3, 4, 8 are special cases of Ex. 9 . Ex. 4, is a special case of Ex. 10 if we select $\varphi_k(z) = z^k$.

To show that the interpolation problem formed from these examples has a solution it suffices to show that $det(L_i(x_j))$ does not vanish, or to apply the following

Theorem 13 (Alternative Theorem) *Consider the system of* n *linear equations in* n *unknowns* $x_1, x_2, ..., x_n$

$$\sum_{J=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, ...n) \tag{1.4}$$

the homogeneous system

$$\sum_{J=1}^{n} a_{ij} x_j = 0 \qquad (i = 1, 2, ...n) \tag{1.5}$$

possesses a non-trivial solution if and only if $|A| = |a_{ij}| = 0$. *If for a fixed* $A = (a_{ij})$ *there are solutions to the non-homogeneous system (1.4) for every selection of the quantities* b_i, *then* $|A| \neq 0$ *and the homogeneous system has only the trivial solution.*

Solution 14 (Example 4) *We show that if $p \in P_N$ and satisfies*

$$\begin{aligned}
&p(z_0), p'(z_0), ..., p^{(m_0)}(z_0) = 0 \\
&p(z_1), p'(z_1), ..., p^{(m_1)}(z_1) = 0 \\
&\quad \vdots \\
&p(z_n), p'(z_n), ..., p^{(m_n)}(z_n) = 0
\end{aligned} \tag{1.6}$$

where $N = m_0 + m_1 + ... + m_n + n$, then p must vanish identically. By the Factorization Theorem, if p satisfies all conditions of (1.6) with the exception of the last, i.e., $p^{(m_n)}(z_n) = 0$, then we must have

$$\begin{aligned}
&p(z) = A(z)(z - z_0)^{m_0+1}(z - z_1)^{m_1+1}...(z - z_{n-1})^{m_{n-1}+1}(z - z_n)^{m_n}, \\
&A(z) = polinomial.
\end{aligned}$$

By examining the degree of this product, it appears that $A =$ constant. Since, moreover,

$$p^{(m_n)}(z_n) = A(m_n)!(z_n - z_0)^{m_0+1}(z_n - z_1)^{m_1+1}...(z_n - z_{n-1})^{m_{n-1}+1} = 0$$

and $z_i \neq z_j$, $i \neq j$, we have $A = 0$ and therefore $p \equiv 0$. The homogeneous interpolation problem has the zero solution only and hence the nonhomogeneous problem possesses a unique solution.

Solution 15 (Example 5) *The generalized Gram determinant is*

$$\begin{vmatrix}
1 & z_0 & z_0^2 & \cdots & z_0^n \\
0 & 1 & 2z_1 & \cdots & nz_1^{n-1} \\
0 & 0 & 2 & \cdots & n(n-1)z_2^{n-2} \\
\vdots & & & & \vdots \\
0 & 0 & 0 & \cdots & n!
\end{vmatrix} = 1!2! \cdots n! \neq 0.$$

Solution 16 (Example 6) *Let $p \in P_{2n+1}$. If $p^{(2j)}(z_0) = 0$ for $j = 0, 1, ..., n$, then*

$$p(z) = a_1(z - z_0) + a_3(z - z_0)^3 + ... + a_{2n+1}(z - z_0)^{2n+1}.$$

If now, $p^{(2n)}(z_1) = 0$ then $a_{2n+1} = 0$ e $p^{(2j)}(z_1) = 0$, $j = n-1, n-2, ...0$ implies, by recurrence, that the remaining coefficients are 0. The homogeneous interpolation problem has the zero solution only and hence the nonhomogeneous problem has a solution and it is unique.

Solution 17 (Example 7) *With the same techniques used for the Ex. 6 we can prove that the homogeneous interpolation problem has the zero solution only and hence the nonhomogeneous problem has a solution and it is unique.*

Solution 18 (Example 10) *No proof is required, for condition (1.2) has been built into the hypothesis. In this example the crucial determinant reduces to the Wronskian of the functions $\phi_0, ..., \phi_n$ and we postulate that it does not vanish at z_0.*

Solution 19 (Example 11) *The generalized Gram determinant is*

$$G = \begin{vmatrix} 1 & \cos x_0 & \sin x_0 & \cos 2x_0 & \sin 2x_0 & \cdots & \cos nx_0 & \sin nx_0 \\ 1 & \cos x_1 & \sin x_1 & \cos 2x_1 & \sin 2x_1 & \cdots & \cos nx_1 & \sin nx_1 \\ & & & & & & & \\ 1 & \cos x_{2n} & \sin x_{2n} & \cos 2x_{2n} & \sin 2x_{2n} & \cdots & \cos nx_{2n} & \sin nx_{2n} \end{vmatrix}.$$

To evaluate G we reduce its element to complex form. Multiply the 3rd, 5th,... columns by $\imath$ and add them respectively to the 2nd, 4th,... columns. We obtain

$$G = \begin{vmatrix} 1 & e^{\imath x_j} \sin x_j & e^{2\imath x_j} \sin 2x_j & \cdots & e^{n\imath x_j} \sin nx_j \end{vmatrix}.$$

Multiply the 3rd, 5th,... columns by $-2\imath$ and to them add the 2nd, 4th,... columns respectively:

$$(-2\imath)^{(n)}G = \begin{vmatrix} 1 & e^{\imath x_j} & e^{-\imath x_j} & e^{2\imath x_j} & e^{-2\imath x_j} & \dots & e^{n\imath x_j} & e^{-n\imath x_j} \end{vmatrix}.$$

Interchange the columns:

$$(-1)^{n(n+1)}(-2\imath)^{(n)}G = \begin{vmatrix} e^{-n\imath x_j} & e^{-(n-1)\imath x_j} & \dots & 1 & \dots & e^{(n-1)\imath x_j} & e^{n\imath x_j} \end{vmatrix}.$$

Multiply the $j-th$ row by $e^{n\imath x_j}$, $\quad j = 0, ..., 2n$:

$$e^{n\imath(x_0+x_1+\dots+x_{2n})}(-1)^{n(n+1)}(-2\imath)^{(n)}G = \begin{vmatrix} 1 & e^{\imath x_j} & e^{2\imath x_j} & \dots & e^{n\imath x_j} \end{vmatrix}.$$

The determinant in the last line is a Vandermonde. Hence,

$$e^{n\imath(x_0+x_1+\dots+x_{2n})}(-1)^{n(n+1)}(-2\imath)^{(n)}G = \prod_{j>k}^{2n}(e^{\imath x_j} - e^{\imath x_k}).$$

In view of the conditions on the x_j, $e^{\imath x_j} \neq e^{\imath x_k}$, $j \neq k$ and so $G \neq 0$.

Solution 20 (Example 12) *In view of the orthogonality of the sines and cosines, the crucial determinant has positive quantities on the main diagonal and 0's elsewhere and hence does not vanish.*

1.3 Representation of the solution of the interpolation problem in linear spaces

Theorem 21 *Let X be a linear space of dimension n. Let $L_1, L_2, ...L_n$ be n independent functional in X^*. Then, there are determined uniquely n independent elements of X, $x_1^*, x_2^*, ..., x_n^*$, such that*

$$L_i(x_j^*) = \delta_{ij}, \qquad i = 1, 2, ..., n \tag{1.7}$$

where with δ_{ij} we denoted the Kronecker symbol

$$\delta_{ij} = \begin{cases} 0 & se \quad i \neq j \\ \\ 1 & se \quad i = j \end{cases}.$$

For any $x \in X$ we have

$$x = \sum_{i=1}^{n} L_i(x) x_i^*. \tag{1.8}$$

For every choice of $\omega_1, \omega_2, ..., \omega_n$, the element

$$x = \sum_{i=1}^{n} \omega_i x_i^* \tag{1.9}$$

is the unique solution of the interpolation problem

$$L_i(x) = \omega_i, \qquad i = 1, 2, ..., n. \tag{1.10}$$

Proof. Let $x_1, x_2, ..., x_n$ be a basis for X. By Lemma 1, $|L_i(x_j)| \neq 0$. If we set $x_i^* = a_{j1}x_1 + \ldots + a_{jn}x_n$, $j = 1, ..., n$, then this determinant condition guarantees that the system (1.7) can be solved for a_{ji} to produce a set of elements $x_1^*, x_2^*, ..., x_n^* \in X$. By Theorem 2 the solution to the interpolation problem (1.7) is unique, for each j, and by Lemma 1 the x_i^* are independent. To prove the (1.8) denote

$$y = \sum_{i=1}^{n} L_i(x) x_i^*$$

and for the property of linearity of functionals L_j we have

$$L_j(y) = \sum_{i=1}^{n} L_i(x) L_j(x_i^*).$$

Hence, by (1.7), $L_j(y) = L_j(x)$, $j = 1, 2, ..., n$. Again, since the interpolation with n condition L_i is unique, y=x and this establishes (1.8). Equation (1.9) is established similarly.

■

We will say that the elements $x_i^* \in X$ and the functionals L_i are biorthonormal if they satisfy the (1.7). For a given set of independent functional, we can always find a related biorthonormal set of polynomials.

The solution of the interpolation problem (1.10) can be given in determinantal form.

Theorem 22 *Let the hypotheses of Theorem 21 hold and let $x_1, x_2, ..., x_n$ be a basis for X. If $\omega_1, \omega_2, ..., \omega_n$ are arbitrary numbers then the element*

$$x = -\frac{1}{G}\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n \\ \omega_1 & L_1(x_1) & L_1(x_2) & \cdots & L_1(x_n) \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \omega_n & L_n(x_1) & L_n(x_2) & \cdots & L_n(x_n) \end{vmatrix}, \tag{1.11}$$

where $G = |L_i(x_j)|$, satisfies $L_i(x) = \omega_i$, $i = 1, 2, ..., n$.

Proof. It is clear that x is a linear combination of $x_1, x_2, ..., x_n$ and hence is in X. Furthermore, we have

$$L_i(x) = -\frac{1}{G}\begin{vmatrix} 0 & L_i(x_1) & L_i(x_2) & \cdots & L_i(x_n) \\ \omega_1 & L_1(x_1) & L_1(x_2) & \cdots & L_1(x_n) \\ \vdots & & & & \vdots \\ \omega_i & L_i(x_1) & L_i(x_2) & \cdots & L_i(x_n) \\ \vdots & & & & \vdots \\ \omega_n & L_n(x_1) & L_n(x_2) & \cdots & L_n(x_n) \end{vmatrix}. \tag{1.12}$$

Expand this determinant by minors of the $1st$ column. The minor of each nonzero element, with the exception of ω_i, is zero, for it contains two identical rows. The cofactor of ω_i is $-G$. Hence, $L_i(x) = \omega_i,\ i = 1, 2, ..., n.$ ■

Example 23 (Taylor Interpolation)

The polynomials $\frac{z^n}{n!}$, $n = 0, 1, ...,$ *and the functionals* $L_n(f) = f^{(n)}(0)$, $n = 0, 1, ...,$ *are biorthonormal, for* $L_i(\frac{z^j}{j!}) = \delta_{ij}$.

Example 24 (Osculatory Interpolation)

Set $\omega(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$, $l_k(z) = \dfrac{\omega(z)}{(z - z_k)\omega'(z_k)}$.
The polynomials $\left[1 - \dfrac{\omega''(z_k)}{\omega'(z_k)}(z - z_k)\right] l_k^2(z), \quad (z - z_k)l_k^2(z)$ *of degree* $2n - 1$ *and the functionals*

$$L_k(f) = f(z_k), \quad M_k(f) = f'(z_k), \qquad k = 1, 2, ...n$$

are biorthonormal. The resulting expansion of type (1.9) , is therefore,

$$p_{2n-1}(z) = \sum_{k=1}^{n} \omega_k \left[1 - \frac{\omega''(z_k)}{\omega'(z_k)}(z - z_k)\right] l_k^2(z) + \sum_{k=1}^{n} \omega_k'(z - z_k)l_k^2(z),$$

and produces the unique element of $\mathcal{P}_{2n-1}$ *which solves the"osculatory" interpolation problem.*

$$\begin{array}{ll} p(z_k) = \omega_k & \\ & k = 1, 2, ..., n. \\ p'(z_k) = \omega_k' & \end{array}$$

Example 25 (Two Point Taylor Interpolation)

Let a and b be distinct point. The polynomial

$$p_{2n-1}(z) = (z - a)^n \sum_{k=0}^{n-1} \frac{B_k(z - b)^k}{k!} + (z - b)^n \sum_{k=0}^{n-1} \frac{A_k(z - a)^k}{k!}$$

$$A_k = \frac{d^k}{dz^k}\left[\frac{f(z)}{(z-b)^n}\right]_{z=a}$$

$$B_k = \frac{d^k}{dz^k}\left[\frac{f(z)}{(z-a)^n}\right]_{z=b}$$

is the unique solution in $\mathcal{P}_{2n-1}$ *of the interpolation problem*

$$p_{2n-1}(a) = f(a), p'_{2n-1}(a) = f'(a), ..., p_{2n-1}^{(n-1)}(a) = f^{(n-1)}(a)$$

$$p_{2n-1}(b) = f(b), p'_{2n-1}(b) = f'(b), ..., p_{2n-1}^{(n-1)}(b) = f^{(n-1)}(b).$$

Example 26 (General Hermite Interpolation)

Let $z_1, z_2, ..., z_n$ *be* n *distinct points,* $\alpha_1, \alpha_2, ..., \alpha_n$ *be* n *integers* ≥ 1 *and* $N = \alpha_1 + \alpha_2 + ... + \alpha_n - 1$. *Set*

$$\omega_z = \prod_{i=1}^{n}(z - z_i)^{\alpha_i}$$

and

$$l_{ik}(z) = \omega_z \frac{(z-z_i)^{k-\alpha_i}}{k!}\frac{d^{(\alpha_i-k-1)}}{dz^{(\alpha_i-k-1)}}\left[\frac{(z-z_i)^{\alpha_i}}{\omega(z)}\right]_{z=z_i}$$

$$p_N(z) = \sum_{i=1}^{n} r_i l_{i0}(z) + \sum_{i=1}^{n} r'_i l_{i1}(z) + ... + \sum_{i=1}^{n} r_i^{(\alpha_i-1)} l_{i\alpha_i-1}(z)$$

is the unique member of $\mathcal{P}_n$ *for which*

$$\begin{aligned} p_N(z_1) &= r_1, p'_N(z_1) = r'_1, ..., p_N^{(\alpha_i-1)}(z_1) = r_1^{(\alpha_i-1)} \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ p_N(z_n) &= r_n, p'_N(z_n) = r'_n, ..., p_N^{(\alpha_n-1)}(z_n) = r_n^{(\alpha_n-1)}. \end{aligned}$$

Example 27 *Given the* $2n+1$ *points*

$$-\pi \leq x_0 < x_1 < \cdots < x_{2n} < \pi.$$

Construct the functions

$$t_j(x) = \frac{\prod_{\substack{k=0\\k\neq j}}^{2n} \sin \frac{1}{2}(x - x_k)}{\prod_{\substack{k=0\\k\neq j}}^{2n} \sin \frac{1}{2}(x_j - x_k)}, \qquad j = 0, 1, ..., 2n.$$

If $L_j(f) = f(x_j)$, then t_j and L_j are biorthonormal. Each function $t_j(x)$ is a linear combination of $1, \cos x, ..., \cos nx$, $\sin x, ..., \sin nx$ and hence is an element of $\mathcal{T}_n$.

To show this, observe that the numerator of t_j is the product of $2n$ factors of the form $\sin \frac{1}{2}(x - x_k) = \alpha e^{\frac{\imath x}{2}} + \beta e^{-\frac{\imath x}{2}}$ for appropriate constants α and β.

The product is therefore of the form $\sum_{k=-n}^{n} c_k e^{\imath kx}$, and is a combination of the required form. The function

$$T(x) = \sum_{k=0}^{2n} \omega_k t_k(x) \tag{1.13}$$

is therefore an element of $\mathcal{T}_n$ and is the unique solution of the interpolation problem $T(x_k) = \omega_k$, $k = 0, 1, ..., 2n$. Formula (1.13) is known as the Gauss formula of trigonometric interpolation.

Example 28 *Given $n + 1$ distinct points*

$$0 \leq x_0 < x_1 < \cdots < x_n < \pi.$$

Set

$$C_j(x) = \frac{\prod_{\substack{k=0\\k\neq j}}^{n} (\cos x - \cos x_k)}{\prod_{\substack{k=0\\k\neq j}}^{n} (\cos x_j - \cos x_k)}, \qquad j = 0, 1, ..., n.$$

Then C_j *is a cosine polynomial of order* $\leq n$ *(i.e., a function of the form* $\sum_{k=0}^{n} a_k \cos kx$*) for which* $C_j(x_k) = \delta_{jk}$*. Given* $n+1$ *distinct values* $\omega_0, \omega_1, ..., \omega_n$ *there is a unique cosine polynomial of order* $\leq n$*,* $C(x)$*, for which* $C(x_k) = \omega_k$*,* $k = 0, 1, ..., n$*. It is*

$$C(x) = \sum_{k=0}^{n} \omega_k C_k(x).$$

Example 29 *Given* n *distinct points*

$$0 < x_1 < \cdots < x_n < \pi.$$

Set

$$S_j(x) = \sin x \frac{\prod_{\substack{k=1 \\ k \neq j}}^{n} (\cos x - \cos x_k)}{\prod_{\substack{k=1 \\ k \neq j}}^{n} (\cos x_j - \cos x_k)}, \qquad j = 0, 1, ..., n.$$

Then S_j *is a sine polynomial of order* $\leq n$ *for which* $S_j(x_k) = \delta_{jk}$*. Given* n *distinct values* $\omega_1, ..., \omega_n$ *there is a unique sine polynomial of order* $\leq n$*,* $S(x)$*, for which* $S(x_k) = \omega_k$*,* $k = 1, ..., n$ *and it is*

$$S(x) = \sum_{k=1}^{n} \omega_k S_k(x).$$

Example 30 *Let* $z_0, z_1, ..., z_n$*, be* $n+1$ *distinct real(or complex) points. Let* $\omega_0, \omega_1, ..., \omega_m$*, be a second such set of* $m+1$ *points. Set*

$$P(z) = (z - z_0) \cdots (z - z_n),$$

$$Q(\omega) = (\omega - \omega_0) \cdots (\omega - \omega_m),$$

$$P_j(z) = P(z)/(z - z_j),$$

$$Q_k(z) = Q(z)/(\omega - \omega_k).$$

The $(m+1)(n+1)$ *polynomials*

$$l_{jk}(z,\omega) = \frac{P_j(z)Q_k(\omega)}{P_j(z_j)Q_k(\omega_k)}$$

satisfy

$$l_{jk}(z_r,\omega_s) = \delta_{jr}\delta_{ks}.$$

Hence

$$p(z,\omega) = \sum_{j=0}^{n}\sum_{k=0}^{m} \mu_{jk} l_{jk}(z,\omega)$$

is a polynomial of degree $\leq mn$ *which satisfies the* $(m+1)(n+1)$ *interpolation conditions*

$$p(z_j,\omega_k) = \mu_{jk} \qquad \begin{array}{l} j = 0,1,...,n \\ k = 0,1,...,m. \end{array}$$

1.4 Representation of the remainder for polynomial interpolation

Let $x_0, ..., x_n$ be $n+1$ distinct points. The numbers $\omega_0, ..., \omega_n$ are frequently the values of some function $f(x)$ at the points $x_i : \omega_i = f(x_i)$.

Definition 31 *We shall designate the unique polynomial of class* $\mathcal{P}_n$ *that coincides with* f *at* $x_0, ..., x_n$ *by* $p_n(f;x)$.

At the points $x \in [a,b] \backslash \{x_0, ..., x_n\}$ generally we have

$$p_n(f;x) \neq f(x).$$

In order to estimate the difference between the function $f(x)$ and the interpolation polynomial $p_n(f;x)$ we introduce the remainder

$$R_n(x) \equiv f(x) - p_n(f;x) \tag{1.14}$$

with $x \in [a, b]$. The following theorem provides an exact approximation of the remainder, from which is possible to obtain a-priori estimates.

Theorem 32 *Let $f(x) \in C^n[a, b]$ and suppose that $f^{(n+1)}(x)$ exists at each point of (a, b). If $a \leq x_0 < x_1 < ... < x_n \leq b$, then*

$$f(x) - p_n(f; x) = \frac{1}{(n+1)!}\omega_n(x) f^{(n+1)}(\xi) \tag{1.15}$$

where $min(x, x_0, x_1, ...x_n) < \xi < max(x, x_0, x_1, ...x_n)$ and $\omega_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$. The point ξ depends upon $x, x_0, x_1, ...x_n$ and f.

Proof. Let x be fixed and $\neq x_0, x_1, ...x_n$. Set

$$F(t) = R_n(f; t)\omega_n(x) - \omega_n(t)R_n(f; x)$$

where $R_n(f; x)$ is defined by (1.14). We can observe that F vanishes at x, and at the $n+1$ points $x_0, x_1, ..., x_n$, for

$$R_n(f; x_i) = f(x_i) - p_n(f; x_i) = 0.$$

A repeated application of Rolle's Theorem implies that the function $F^{(n+1)}(t)$ must vanish at a point ξ with

$$min(x, x_0, x_1, ...x_n) \ < \xi < max(x, x_0, x_1, ...x_n).$$

But we have

$$F^{(n+1)}(t) = R_n^{(n+1)}(f; t)\omega_n(x) - \omega_n^{(n+1)}(t)R_n(f; x) = f^{(n+1)}(t)\omega_n(x) - (n+1)!R_n(f; x)$$

so that

$$0 = F^{(n+1)}(\xi) = f^{(n+1)}(\xi)\omega_n(x) - (n+1)!R_n(f;x)$$

and therefore

$$R_n(f;x) = \frac{1}{(n+1)!}\omega_n(x)f^{(n+1)}(\xi).$$

■

This theorem provides a qualitative analysis of the remainder for polynomial interpolation. Generally the function f is not known and therefore is very complicated to have information about the $(n+1)-$th derivative of the function interpolated. Moreover the derivative is calculated in an opportune point in (a,b) of which is guaranteed only the existence. However, by (1.15), is possible, many times, to obtain upper bounds of the remainder.

1.5 Peano's Theorem and its consequence

If we examine the Cauchy remainder for polynomial interpolation (1.15) we may note the prominent role played by the portion $f^{(n+1)}(\xi)$. If, for instance, $f \in P_n$, then $f^{(n+1)} \equiv 0$, and the remainder vanishes identically as it should. For a fixed x, we may consider the remainder $R_n(f;x) = f(x) - p_n(f;x)$ as a linear functional which operates on f and which annihilates all elements of $\mathcal{P}_n$. Peano observed that if a linear functional has this property, then it must also have a simple representation in terms of $f^{(n+1)}$.

Let $L : C^n[a,b] \to \mathbb{R}$ be a linear functional of the type

$$L(f) = \int_a^b \left[a_0(x)f(x) + a_1(x)f'(x) + ... + a_n(x)f^{(n)}(x)\right] dx +$$
$$+ \sum_{i=1}^{j_0} b_{i0}f(x_{i0}) + \sum_{i=1}^{j_1} b_{i1}f'(x_{i1}) + ... + \sum_{i=1}^{j_n} b_{in}f^{(n)}(x_{in})$$

with $a_i(x) \in C[a,b]$, $i = 0, ..., n$ and $x_{ik} \in [a,b]$ $\forall i,k$.

Theorem 33 (Peano) *Let $L(p) = 0$ $\forall p \in P_n$. Then, $\forall f \in C^{(n+1)}[a,b]$*

$$L(f) = \int_a^b f^{(n+1)}(t)K(t)dt \tag{1.16}$$

where

$$K(t) = \frac{1}{n!}L_x\left[(x-t)_+^n\right], \tag{1.17}$$

and

$$(x-t)_+^n = \begin{cases} (x-t)^n & se\ x \geq t \\ 0 & se\ \ x \leq t \end{cases} \tag{1.18}$$

The notation $L_x\left[(x-t)_+^n\right]$ means that the functional L is applied to $(x-t)^n$, considered as a function of x.

Proof. Consider the Taylor's Theorem with the exact remainder

$$f(x) = f(a) + (x-a)f'(a) + ... + \frac{(x-a)^n}{n!} + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt. \tag{1.19}$$

By (1.18) we may evidently write the (1.19) as

$$f(x) = f(a) + (x-a)f'(a) + ... + \frac{(x-a)^n}{n!} + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(x-t)_+^n dt. \tag{1.20}$$

Now apply L to both sides of this expansion and recall that $L(p) = 0$ when p is a polynomial of degree $\leq n$. This yields

$$L(f) = \frac{1}{n!}L\left[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt\right]$$

and under hypotheses we may interchange the functional L with the integral. Hence,

$$L(f) = \frac{1}{n!}\int_a^b f^{(n+1)}(t)L\left[(x-t)_+^n\right]dt.$$

■

The function $K(t)$ is called the Peano Kernel associated with the functional L.

Corollary 34 *If, in addition to the above hypotheses, the kernel $K(t)$ does not change its sign on $[a,b]$, then $\forall f \in C^{(n+1)}[a,b]$,*

$$L(f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}L(x^{n+1}), \quad a \leq \xi \leq b. \tag{1.21}$$

Proof. From (1.16) and applying the Mean Value Theorem for Integrals, we have

$$L(f) = f^{(n+1)}(\xi)\int_a^b K(t)dt, \quad \text{con} \quad a \leq \xi \leq b. \tag{1.22}$$

Insert $f = x^{n+1}$ in (1.22) and obtain

$$L(x^{n+1}) = (n+1)!\int_a^b K(t)dt. \tag{1.23}$$

Combining these yields (1.21). ■

We provides, now, some examples of error functionals of this type.

Example 35 (Kowalewski's Exact Remainder for Polynomial Interpolation)

Let $x_0, x_1, ..., x_n$ be fixed in $[a,b]$.

Let $L(f) = R_n(f;x) = f(x) - \sum_{k=0}^{n} f(x_k)l_k(x)$ where $l_k(x) = \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}$, $k = 0,...,n$.

Then,

$$K(t) = \frac{1}{n!} L_x \left[(x-t)_+^n\right] = \frac{1}{n!}\left[(x-t)_+^n - \sum_{k=0}^{n}(x_k-t)_+^n l_k(x)\right] =$$
$$= \frac{1}{n!}\sum_{k=0}^{n}\left[(x-t)_+^n - (x_k-t)_+^n\right] l_k(x).$$

The last equality follows since the polinomials $l_k(x)$ satisfy

$$\sum_{k=0}^{n} l_k(x) = 1.$$

Fo fixed k, we have by (1.18)

$$\int_a^b \left[(x-t)_+^n - (x_k-t)_+^n\right] f^{(n+1)}(t)dt =$$
$$= \int_a^x \left[(x-t)^n - (x_k-t)^n\right] f^{(n+1)}(t)dt +$$
$$+ \int_{x_k}^x (x_k-t)^n f^{(n+1)}(t)dt.$$

Hence,

$$K(t)f^{(n+1)}(t)dt =$$
$$= \frac{1}{n!}\int_a^x f^{(n+1)}(t)\sum_{k=0}^{n}\left[(x-t)^n - (x_k-t)^n\right] l_k(x)dt +$$
$$+ \frac{1}{n!}\sum_{k=0}^{n} l_k(x)\int_{x_k}^x (x_k-t)^n f^{(n+1)}(t)dt$$

Since $\sum_{k=0}^{n}(x_k-t)^n l_k(x) = p_n((x-t)^n; x) = (x-t)^n$, *we have*

$$\sum_{k=0}^{n}\left[(x-t)^n - (x_k-t)^n\right] l_k(x) = (x-t)^n - \sum_{k=0}^{n}(x_k-t)^n l_k(x) = 0.$$

Thus, finally,

$$L(f) = f(x) - p_n(f;x) = \frac{1}{n!}\sum_{k=0}^{n} l_k(x) \int_{x_k}^{x} (x_k - t)^n f^{(n+1)}(t)dt = \tag{1.24}$$

$$= -\frac{1}{(n+1)!}\sum_{k=0}^{n} l_k(x)(x_k - x)^{n+1} f^{(n+1)}(\xi_x), \qquad f \in C^{n+1}[a,b], \quad \xi_x \in [a,b]\,. \tag{1.25}$$

Example 36 (Integral Remainder for Linear Interpolation)

The case $n = 1$, $x_0 = a$, $x_1 = b$ is particularly noteworthy. Then $l_0(x) = \dfrac{x-b}{a-b}$, $l_1(x) = \dfrac{x-a}{b-a}$. From (1.24),

$$f(x) - \frac{x-b}{a-b}f(a) - \frac{x-a}{b-a}f(b) =$$

$$= \frac{x-b}{b-a}\int_a^x (t-a)f''(t)dt + \frac{x-a}{b-a}\int_x^b (t-b)f''(t)dt \tag{1.26}$$

Introduce the following function defined over the square $a \le x \le b$, $a \le t \le b$

$$G(x,t) = \begin{cases} \dfrac{(t-a)(x-b)}{b-a} & t \le x \\ \dfrac{(x-a)(t-b)}{b-a} & x \le t. \end{cases}$$

Then we may write (1.26) in the form

$$R_1(f;x) = \int_a^b G(x,t)f''(t)dt.$$

The function $G(x,t)$ is, for fixed x,the Peano kernel for $R_1(f)$.

Example 37 *Let*

$$x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ x_3 = x_0 + 3h \quad \text{and} \quad L(f) = -f(x_0) + 3f(x_1) - 3f(x_2) + f(x_3). \tag{1.27}$$

Observe that $L(p) = 0\ \forall p \in P_2$. *Hence,* $n = 2$ *and*

$$K(t) = \frac{1}{2!} L(x-t)_+^2.$$

If we write this out explicitly we find

$$2K(t) = \begin{cases} (x_3-t)^2 - 3(x_2-t)^2 + 3(x_1-t)^2 = (t-x_0)^2, & x_0 \le t \le x_1 \\ (x_3-t)^2 - 3(x_2-t)^2, & x_1 \le t \le x_2 \\ (x_3-t)^2, & x_2 \le t \le x_3 \end{cases}$$

The kernel $K(t)$ *consists of 3 parabolic arches and is of the class* $C^1[x_0, x_3]$.

Thus, for $f \in C^3[x_0, x_3]$,

$$L(f) = \int_{x_0}^{x_3} K(t) f^{(3)}(t)dt.$$

Note that $K(t) \ge 0$. *We may apply (1.21) yielding*

$$L(f) = \frac{f^{(3)}(\xi)}{3!} L(x^3) = h^3 f^{(3)}(\xi) \qquad x_0 \le \xi \le x_3.$$

Example 38 (Remainder in Trapezoidal Rule) *Let*

$$L(f) = \int_a^b f(x)dx - \frac{b-a}{2}\left[f(a) + f(b)\right]$$

be the error in estimating the definite integral $\int_a^b f(x)dx$ *by the trapezoidal rule* $\frac{b-a}{2}\left[f(a)+f(b)\right]$. *The rule is exact for linear function, and, in particular, for constants. If we select* $n = 0$, *we have*

$$L_x[(x-t)_+^0] = \int_a^b (x-t)_+^0 dx - \frac{b-a}{2}\left[(a-t)_+^0 + (b-t)_+^0\right] =$$

$$= \int_t^b dx - \frac{b-a}{2}[0+1] = \frac{1}{2}(a+b) - t, \qquad t > a.$$

Therefore

$$L(f) = -\int_a^b (t - \frac{1}{2}(a+b)) f'(t)dt. \tag{1.28}$$

Consider, next, the extended trapezoidal rule,

$$L(f) = \int_a^b f(x)dx - \frac{b-a}{n}\left[\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(a+jh) + \frac{f(b)}{2}\right], \qquad h = \frac{b-a}{n}.$$

An expression analogous to (1.28) is most conveniently obtained by adding expressions of this form for each subinterval

$$L(f) = -\sum_{j=0}^{n-1}\int_{a+jh}^{a+(j+1)h}(t-(a+(j+\frac{1}{2})h))f'(t)dt. \tag{1.29}$$

Example 39 (Remainder in Simpson's Rule) *Let*

$$L(f) = \int_{-1}^{+1} f(x)dx - \frac{1}{3}f(-1) - \frac{4}{3}f(0) - \frac{1}{3}f(1) \tag{1.30}$$

be the error in estimating the definite integral $\int_{-1}^{1} f(x)dx$ *by the Simpson's rule* $L(p) = 0\ \forall p \in P_3$. *Applying* $K(t) = \frac{1}{3!}L(x-t)_+^3$ *we find*

$$\begin{aligned} K(t) &= \frac{1}{3!}L(x-t)_+^3 = \\ &= \int_{-1}^{+1}(x-t)_+^3 dx - \frac{1}{3}(-1-t)_+^3 - \frac{4}{3}(-t)_+^3 - \frac{1}{3}(1-t)_+^3 = \\ &= \begin{cases} \frac{1}{3!}\left[\frac{(1-t)^4}{4} - \frac{(1-t)^3}{3}\right] = -\frac{1}{72}(1-t)^3(3t+1), & 0 \le t \le 1 \\ \frac{1}{3!}\left[\frac{(1-t)^4}{4} - \frac{4}{3}(-t)^3 - \frac{1}{3}(1-t)^3\right] = -\frac{1}{72}(1+t)^3(-3t+1), & -1 \le t \le 0 \end{cases} \end{aligned}$$

Note that $K(t) \le 0$ *in* $[-1,1]$, *so the Corollary (1.21) is applicable:*

$$L(f) = \frac{f^{(4)}(\xi)}{4!}L(x^4) = \frac{f^{(4)}(\xi)}{4!}(-\frac{4}{15}) = -\frac{f^{(4)}(\xi)}{90}, \quad -1 \le \xi \le 1.$$

This leads to the following error for Simpson's rule:

$$\int_{-1}^{+1} f(x)dx = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) - \frac{f^{(4)}(\xi)}{90}, \ -1 \le \xi \le 1. \tag{1.31}$$

1.6 General remainder theorem for interpolation in linear spaces

Given an element x in a linear space X, we interpolate to x by an appropriate linear combination of $x_1, ..., x_n$ such that $L_i(a_1x_1 + ... + a_nx_n) = L_i(x)$, $i = 1, ..., n$. Let

$$x_R = x - (a_1x_1 + ... + a_nx_n). \tag{1.32}$$

Then $L_i(x_R) = 0$, $i = 1, 2, ..., n$.

Theorem 40 *Under the assumption that* $|L_i(x_j)| \neq 0$, *we have*

$$x_R = \begin{vmatrix} x & x_1 & \cdots & x_n \\ L_1(x) & L_1(x_1) & \cdots & L_1(x_n) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ L_n(x) & L_n(x_1) & \cdots & L_n(x_n) \end{vmatrix} \div \begin{vmatrix} L_1(x_1) & L_1(x_2) & \cdots & L_1(x_n) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ L_n(x_1) & L_n(x_2) & \cdots & L_n(x_n) \end{vmatrix} \tag{1.33}$$

Proof. It is clear by expanding the numerator of (1.33) by the minors of its first row that the right hand side of (1.33) is a linear combination of $x, x_1, ..., x_n$, and that the coefficients of x is precisely 1. Applying L_i to the right hand side, we see that this row is identical with the $(i+1)-th$ row and hence $L_i(x_R) = 0$, $i = 1, 2, ..., n$. Thus, the expression (1.33) has all the properties the remainder x_R should have. ■

Chapter 2

Appell polynomials

In this chapter we consider a wide class of polynomials introduced by Appell in 1880 ([3]) and we give an algebraic theory by means of determinantal forms. The new definition proposed provides a very natural generalization related to a linear functional. These polynomials will be the basis for the solution of a new interpolation problem that will be introduced in chapter 3.

2.1 Introduction

In 1880 [3] P.E. Appell introduced and widely studied sequences of n-degree polynomials

$$A_n(x), \; n = 0, 1, ... \tag{2.1}$$

satisfying the recursive relations

$$\frac{dA_n(x)}{dx} = nA_{n-1}(x), \; n = 1, 2, ... \tag{2.2}$$

In particular, Appell noticed the one-to-one correspondence of the set of such sequences $\{A_n(x)\}_n$ and the set of numerical sequences $\{\alpha_n\}_n$, $\alpha_0 \neq 0$ given by the explicit representation

$$A_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \binom{n}{2}\alpha_{n-2}x^2 + \cdots + \alpha_0 x^n,\ n = 0, 1, ... \tag{2.3}$$

Equation (2.3), in particular, shows explicitly that for each $n \geq 1$ $A_n(x)$ is completely determined by $A_{n-1}(x)$ and by the choice of the constant of integration α_n. Furthermore Appell provided an alternative general method to determine such sequences of polynomials, that satisfy (2.2). In fact, given the power series:

$$a(h) = \alpha_0 + \frac{h}{1!}\alpha_1 + \frac{h^2}{2!}\alpha_2 + \cdots + \frac{h^n}{n!}\alpha_n + \cdots, \quad \alpha_0 \neq 0 \tag{2.4}$$

with α_i $i = 0, 1, ...$ real coefficients, a sequence of polynomials satisfying (2.2) is determined by the power series expansion of the product $a(h)e^{hx}$, i.e.:

$$a(h)e^{hx} = A_0(x) + \frac{h}{1!}A_1(x) + \frac{h^2}{2!}A_2(x) + \cdots + \frac{h^n}{n!}A_n(x) + \cdots \tag{2.5}$$

The function $a(h)$ is said 'generating function' of the sequence of polynomials $A_n(x)$.

Well known examples of sequences of polynomials verifying (2.2) or, equivalently (2.3) and (2.5), now called Appell Sequences, are:

1. the sequences of growing powers of variable x

$$1, x, x^2, ..., x^n, ...,$$

   as already stressed in [3];

2. the Bernoulli sequence $B_n(x)$ ([5],[34]);

3. the Euler sequences $E_n(x)$ ([23], [34]);

4. the Hermite normalized sequences $H_n(x)$ [6];

5. the Laguerre sequences $L_n(x)$ [6];

Moreover, further generalizations of above polynomials have been considered ([6], [8], [10]).

Sequences of Appell polynomials have been well studied because of their remarkable applications in Mathematical and Numerical Analysis, as well as in Number theory, as both classic literature ([3], [47], [49], [6]) and more recent one ([22], [30], [20], [9], [32], [10], [8]) testify.

In a recent work [12], a new approach to Bernoulli polynomials was given, based on a determinantal definition. The authors, through basic tools of linear algebra, have recovered the fundamental properties of Bernoulli polynomials; moreover the equivalence, with a triangular theorem, of all previous approaches is given.

Now we want to propose a similar approach for more general Appell polynomials and to establish its equivalence with previous characterizations, through a circular theorem.

2.2 A determinantal definition

Let us consider $P_n(x)$, $n = 0, 1, ...$ the sequence of polynomials of degree n defined by

$$\begin{cases} P_0(x) = \frac{1}{\beta_0} \\ P_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, ... \end{cases} \tag{2.6}$$

where $\beta_0, \beta_1, ..., \beta_n \in R$, $\beta_0 \neq 0$.

Then we have

Theorem 41 *The following relation holds*

$$P_n'(x) = nP_{n-1}(x) \quad n = 1, 2, ... \tag{2.7}$$

Proof. Using the properties of linearity we can differentiate the determinant (2.6), expand the resulting determinant with respect to the first column and recognize the factor $P_{n-1}(x)$ after multiplication of the i-th row by $i - 1$ $i = 2, ..., n$ and the j-th column by $\frac{1}{j}$ $j = 1, ..., n$. ∎

Theorem 42 *With the previous notations we have*

$$P_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \binom{n}{2}\alpha_{n-2}x^2 + \cdots + \alpha_0 x^n,\ n = 0, 1, ... \tag{2.8}$$

where

$$\alpha_0 = \frac{1}{\beta_0}, \tag{2.9}$$

$$\alpha_i = \frac{(-1)^i}{(\beta_0)^{i+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1 \end{vmatrix}, \quad i = 1, 2, ..., n \tag{2.10}$$

Proof. Expanding the determinant $P_n(x)$ with respect to the first row we obtain

$$P_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \sum_{j=0}^{n} (-1)^j x^j \begin{vmatrix} \binom{j+1}{j}\beta_1 & \binom{j+2}{j}\beta_2 & \cdots & \cdots & \binom{n-1}{j}\beta_{n-j-1} & \binom{n}{j}\beta_{n-j} \\ \beta_0 & \binom{j+2}{j+1}\beta_1 & \cdots & \cdots & \binom{n-1}{j+1}\beta_{n-j-2} & \binom{n}{j+1}\beta_{n-j-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{j+2}\beta_{n-j-3} & \binom{n}{j+2}\beta_{n-j-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}.$$

After multiplication of the $i-th$ column by $\binom{j+i}{j}$, $i = 1, ..., n-j$ and $k-th$ row by $\frac{1}{\binom{j+k-1}{j}}$,

$k = 2, ..., n$ we have

$$P_n(x) = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(\beta_0)^{n-j+1}} \binom{n}{j} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1 \end{vmatrix} x^j.$$

that proves the thesis. ■

Corollary 43 *For the polynomials $P_n(x)$ we have*

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j} P_{n-j}(0)\, x^j, \; n = 0, 1, ... \tag{2.11}$$

Proof. Taking into account

$$P_i(0) = \alpha_i, \quad i = 0, 1, ..., n, \tag{2.12}$$

relation (2.11) is a consequence of (2.8). ■

For computation we can observe that α_n is a n-order determinant of a particular upper Hessemberg form and it's known that the algorithm of Gaussian elimination without pivoting for computing the determinant of an upper Hessemberg matrix is stable [31, p. 27].

Theorem 44 *For the coefficients α_i in (2.8) the following relations hold*

$$\alpha_0 = \frac{1}{\beta_0}, \tag{2.13}$$

$$\alpha_i = -\frac{1}{\beta_0} \sum_{k=0}^{i-1} \binom{i}{k} \beta_{i-k} \alpha_k, \quad i = 1, 2, ..., n. \tag{2.14}$$

Proof. Set $\overline{\alpha}_i = (-1)^i (\beta_0)^{i+1} \alpha_i$ for $i = 1, 2, ..., n$. From (2.9) α_i is a determinant of an upper Hessemberg matrix of order i and for that ([12]) we have

$$\overline{\alpha}_i = \sum_{k=0}^{i-1} (-1)^{i-k-1} h_{k+1,i} q_k(i) \overline{\alpha}_k, \tag{2.15}$$

where

$$h_{l,m} = \begin{cases} \beta_m & \text{for } l = 1, \\ \binom{m}{l-1}\beta_{m-l+1} & \text{for } 1 < l \leq m+1, \\ 0 & \text{for } l > m+1, \end{cases} \qquad l, m = 1, 2, ..., i, \tag{2.16}$$

and

$$q_k(i) = \prod_{j=k+2}^{i} h_{j,j-1} = (\beta_0)^{i-k-1}, \quad k = 0, 1, ..., i-2, \tag{2.17}$$

$$q_{i-1}(i) = 1. \tag{2.18}$$

By virtue of the previous setting, (2.15) implies

$$\begin{aligned} \overline{\alpha}_i &= \sum_{k=0}^{i-2} (-1)^{i-k-1} \binom{i}{k} \beta_{i-k} (\beta_0)^{i-k-1} \overline{\alpha}_k + \binom{i}{i-1} \beta_1 \overline{\alpha}_{i-1} = \\ &= (-1)^i (\beta_0)^{i+1} \left(-\frac{1}{\beta_0} \sum_{k=0}^{i-1} \binom{i}{k} \beta_{i-k} \frac{1}{(-1)^k (\beta_0)^{k+1}} \overline{\alpha}_k \right) = \\ &= (-1)^i (\beta_0)^{i+1} \left(-\frac{1}{\beta_0} \sum_{k=0}^{i-1} \binom{i}{k} \beta_{i-k} \alpha_k \right) \end{aligned}$$

and the proof is concluded. ■

Theorem 45 *Let $P_n(x)$ be the sequence of Appell polynomials with generating function $a(h)$ as in (2.4) and (2.5). If $\beta_0, \beta_1, ..., \beta_n$, with $\beta_0 \neq 0$, are the coefficients of Taylor series* *expansion of function* $\frac{1}{a(h)}$ *we have*

$$P_0(x) = \frac{1}{\beta_0} \tag{2.19}$$

$$P_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, ... \tag{2.20}$$

Proof. Let $P_n(x)$ be the sequence of Appell polynomials with generating function $a(h)$ i.e.

$$a(h) = \alpha_0 + \frac{h}{1!}\alpha_1 + \frac{h^2}{2!}\alpha_2 + \cdots + \frac{h^n}{n!}\alpha_n + \cdots \tag{2.21}$$

and

$$a(h)e^{hx} = \sum_{n=0}^{\infty} P_n(x)\frac{h^n}{n!}. \tag{2.22}$$

Let $b(h)$ be such that $a(h)b(h) = 1$. We can write $b(h)$ as its Taylor series expansion (in h) at the origin that is

$$b(h) = \beta_0 + \frac{h}{1!}\beta_1 + \frac{h^2}{2!}\beta_2 + \cdots + \frac{h^n}{n!}\beta_n + \cdots \tag{2.23}$$

Then, according to the Cauchy-product rules, we find

$$a(h)b(h) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_k\beta_{n-k}\frac{h^n}{n!}$$

by which

$$\sum_{k=0}^{n}\binom{n}{k}\alpha_k\beta_{n-k}=\begin{cases}1 & \text{for } n=0,\\ 0 & \text{for } n>0.\end{cases}$$

Hence

$$\begin{cases}\beta_0=\frac{1}{\alpha_0},\\ \beta_n=-\frac{1}{\alpha_0}\left(\sum\limits_{k=1}^{n}\binom{n}{k}\alpha_k\beta_{n-k}\right), \quad n=1,2,...\end{cases} \tag{2.24}$$

Let us multiply both hand sides of equation (2.22) for $\frac{1}{a(h)}$ and, in the same equation, replace functions e^{hx} and $\dfrac{1}{a\left(h\right)}$ by their Taylor series expansion at the origin; then (2.22) becomes

$$\sum_{n=0}^{\infty}\frac{x^n h^n}{n!}=\sum_{n=0}^{\infty}P_n\left(x\right)\frac{h^n}{n!}\sum_{n=0}^{\infty}\frac{h^n}{n!}\beta_n. \tag{2.25}$$

By multiplying the series on the left hand side of (2.25) according to the Cauchy-product rules, previous equality leads to the following system of infinite equations in the unknown $P_n\left(x\right),\ n=0,1,...$

$$\begin{cases}P_0\left(x\right)\beta_0=1,\\ P_0\left(x\right)\beta_1+P_1\left(x\right)\beta_0=x,\\ P_0\left(x\right)\beta_2+\binom{2}{1}P_1\left(x\right)\beta_1+P_2\left(x\right)\beta_0=x^2,\\ \vdots\\ P_0\left(x\right)\beta_n+\binom{n}{1}P_1\left(x\right)\beta_{n-1}+...+P_n\left(x\right)\beta_0=x^n,\\ \vdots\end{cases} \tag{2.26}$$

The special form of the previous system (lower triangular) allows us to work out the unknown $P_n(x)$ operating with the first $n+1$ equations, only by applying the Cramer rule:

$$P_n(x) = \frac{\begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 1 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & x \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & x^2 \\ \vdots & & & \ddots & & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & x^{i-1} \\ \beta_n & \binom{n}{1}\beta_{n-1} & \cdots & \cdots & \binom{n}{n-1}\beta_1 & x^i \end{vmatrix}}{\begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 0 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & 0 \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & 0 \\ \beta_n & \binom{n}{1}\beta_{n-1} & \cdots & \cdots & \binom{n}{n-1}\beta_1 & \beta_0 \end{vmatrix}} =$$

$$= \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 1 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & x \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & x^2 \\ \vdots & & & \ddots & & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & x^{i-1} \\ \beta_n & \binom{n}{1}\beta_{n-1} & \cdots & \cdots & \binom{n}{n-1}\beta_1 & x^i \end{vmatrix}.$$

By transposition of the previous, we have

$$P_n(x) = \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \\ 1 & x & x^2 & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad n = 1, 2, ... \tag{2.27}$$

that is exactly (2.19) after n circular row exchanges: more precisely, the i-th row moves to the $(i+1)$-th position for $i = 1, \ldots, n-1$, the n-th row goes to the first position. ■

Theorems 41, 42 and 45 concur to assert the validity of following

Theorem 46 (Circular [16]) *For Appell polynomials we have*

$$\begin{array}{ccc} \textit{(2.2 and 2.3)} & \longrightarrow & \textit{(2.4 and 2.5)} \\ \nwarrow & & \swarrow \\ & \textit{(2.6)} & \end{array} \tag{2.28}$$

Proof.

(2.2 and 2.3)⇒(2.4 and 2.5): Follows from Appell proof [3].

(2.4 and 2.5)⇒(2.6): Follows from Theorem 45.

(2.6)⇒(2.2 and 2.3): Follows from Theorems 41 and 42.

■

Therefore we can give, now, the following

Definition 47 *The Appell polynomial of degree n, denoted by $A_n(x)$, is defined by*

$$A_0(x) = \frac{1}{\beta_0} \tag{2.29}$$

$$A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, ... \tag{2.30}$$

where $\beta_0, \beta_1, ..., \beta_n \in R$, $\beta_0 \neq 0$.

2.3 General properties of Appell polynomials

By tools of elementary algebra we can prove the general properties of Appell polynomials.

Theorem 48 (Recurrence) *For Appell sequence $A_n(x)$ we have*

$$A_n(x) = \frac{1}{\beta_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x)\right), \quad n = 1, 2, ... \tag{2.31}$$

Proof. Set $\overline{A}_{-1}(x) = 1$ and $\overline{A}_n(x) = (-1)^n (\beta_0)^{n+1} A_n(x)$ for each $n \geq 0$. From (2.30) $A_n(x)$ is a determinant of an upper Hessemberg matrix of order $n+1$ and for that

([12]) we have

$$\overline{A}_n(x)=\sum_{k=0}^{n}(-1)^{n-k}q_k(n+1)h_{k+1,n+1}\overline{A}_{k-1}(x), \tag{2.32}$$

where

$$h_{i,j}=\begin{cases} x^{j-1} & se \quad i=1, \\ \binom{j-1}{i-2}\beta_{j-i+1} & se \quad 1<i\leq j+1, \qquad i,j=1,2,...,n+1, \\ 0 & se \quad i>j+1, \end{cases} \tag{2.33}$$

and

$$q_k(n+1)=\prod_{j=k+2}^{n+1}h_{j,j-1}=(\beta_0)^{n-k}, \;\; k=0,1,...,n-1,$$

$$q_n(n+1)=1.$$

By virtue of previous setting, (2.32) implies

$$\begin{aligned}\overline{A}_n(x) &= (-1)^n(\beta_0)^n x^n+\sum_{k=1}^{n}(-1)^{n-k}(\beta_0)^{n-k}\binom{n}{k-1}\beta_{n-k+1}\overline{A}_{k-1}(x)= \\ &= (-1)^n(\beta_0)^n x^n+\sum_{k=0}^{n-1}(-1)^{n-k-1}(\beta_0)^{n-k-1}\binom{n}{k}\beta_{n-k}\overline{A}_k(x)= \\ &= (-1)^n(\beta_0)^{n+1}\left(\frac{x^n}{\beta_0}-\frac{1}{\beta_0}\sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}\frac{(-1)^k}{(\beta_0)^{k+1}}\overline{A}_k(x)\right)= \\ &= (-1)^n(\beta_0)^{n+1}\left(\frac{x^n}{\beta_0}-\frac{1}{\beta_0}\sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x)\right)\end{aligned}$$

and the proof is concluded. ■

Corollary 49 *If $A_n(x)$ is an Appell polynomial then*

$$x^n=\sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}A_k(x), \quad n=0,1,... \tag{2.34}$$

Proof. The result follows from (2.31). ∎

Let's consider two sequences of Appell polynomials

$$A_n(x), B_n(x) \quad n = 0, 1, \ldots$$

and indicate with $(AB)_n(x)$ the polynomial that is obtained replacing in $A_n(x)$ powers $x^0, x^1, \ldots, x^n$, respectively, with the polynomials $B_0(x), B_1(x), \ldots, B_n(x)$. The following theorem can be proven.

Theorem 50 *The sequences*

i) $\lambda A_n(x) + \mu B_n(x), \quad \lambda, \mu \in R,$

ii) $(AB)_n(x)$

are sequences of Appell polynomials again.

Proof.

i) follow from the property of linearity of determinant.

ii) by definition we have

$$(AB)_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} B_0(x) & B_1(x) & B_2(x) & \cdots & \cdots & B_{n-1}(x) & B_n(x) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}.$$

Expanding the determinant $(AB)_n(x)$ with respect to the first row we obtain

$$\begin{aligned}(AB)_n(x) &= \frac{(-1)^n}{(\beta_0)^{n+1}} \sum_{j=0}^{n} (-1)^j (\beta_0)^j \binom{n}{j} \overline{\alpha}_{n-j} B_j(x) \\ &= \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(\beta_0)^{n-j+1}} \binom{n}{j} \overline{\alpha}_{n-j} B_j(x), \end{aligned} \tag{2.35}$$

where

$$\overline{\alpha}_0 = 1,$$

$$\overline{\alpha}_i = \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1 \end{vmatrix}, \quad i = 1, 2, ..., n.$$

We observe that

$$A_i(0) = \frac{(-1)^i}{(\beta_0)^{i+1}} \overline{\alpha}_i, \quad i = 1, 2, ..., n$$

and hence (2.35) becomes

$$(AB)_n(x) = \sum_{j=0}^{n} \binom{n}{j} A_{n-j}(0) B_j(x). \tag{2.36}$$

Differentiating both hand sides of (2.36) and since $B_j(x)$ is a sequence of Appell polynomials, we deduce

$$\begin{aligned} ((AB)_n(x))' &= \sum_{j=0}^{n} \binom{n}{j} A_{n-j}(0) B'_j(x) = \\ &= \sum_{j=1}^{n} j \binom{n}{j} A_{n-j}(0) B_{j-1}(x) \\ &= n \sum_{j=1}^{n} \binom{n-1}{j-1} A_{n-j}(0) B_{j-1}(x) = \\ &= n \sum_{j=0}^{n-1} \binom{n-1}{j} A_{n-1-j}(0) B_j(x) = \\ &= n (AB)_{n-1}(x). \end{aligned}$$

■

Theorem 51 *[45, p. 27] For Appell polynomials $A_n(x)$ we have*

$$A_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) y^{n-i}, \quad n = 0, 1, ... \tag{2.37}$$

Proof. Starting by the definition in (2.30) and using the identity

$$(x+y)^i = \sum_{k=0}^{i} \binom{i}{k} y^k x^{i-k}, \tag{2.38}$$

we infer

$$A_n\left(x+y\right)=\frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} 1 & (x+y)^1 & \cdots & (x+y)^{n-1} & (x+y)^n \\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \beta_0 & \beta_1\binom{n}{n-1} \end{vmatrix}=\sum_{i=0}^{n}\frac{(-1)^n\, y^i}{(\beta_0)^{n+1}}$$

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & \binom{i}{i} & \cdots & \binom{n-1}{i}x^{n-1-i} & \binom{n}{i}x^{n-i} \\ \beta_0 & \beta_1 & \cdots & \beta_{i-1} & \beta_i & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \beta_{i-2}\binom{i-1}{1} & \beta_{i-1}\binom{i}{1} & \cdots & \beta_{n-2}\binom{n-1}{1} & \beta_{n-1}\binom{n}{1} \\ \vdots & \ddots & \ddots & & & & & \vdots \\ \vdots & & \ddots & \beta_0 & \beta_1\binom{i}{i-1} & \cdots & & \beta_{n-i+1}\binom{n}{i-1} \\ \vdots & & & 0 & \beta_0 & \cdots & & \beta_{n-i}\binom{n}{i} \\ \vdots & & & & 0 & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n}{n-1} \end{vmatrix}=$$

$$=\sum_{i=0}^{n} y^i\,\frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}}\begin{vmatrix} \binom{i}{i} & \binom{i+1}{i}x^1 & \binom{i+2}{i}x^2 & \cdots & \binom{n-1}{i}x^{n-i-1} & \binom{n}{i}x^{n-i} \\ \beta_0 & \beta_1\binom{i+1}{i} & \beta_2\binom{i+2}{i} & \cdots & \beta_{n-i-1}\binom{n-1}{i} & \beta_{n-i}\binom{n}{i} \\ 0 & \beta_0 & \beta_1\binom{i+2}{i+1} & \cdots & \beta_{n-i-2}\binom{n-1}{i+1} & \beta_{n-i-1}\binom{n}{i+1} \\ \vdots & & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n}{n-1} \end{vmatrix}.$$

We divide, now, each $j-$th column, $j=2,...,n-i+1$, for $\binom{i+j-1}{i}$ and multiply each $h-$th row, $h=3,...,n-i+1$, for $\binom{i+h-2}{i}$. Thus we finally obtain

$$A_n(x+y) = \\ = \sum_{i=0}^{n} \frac{\binom{i+1}{i}\cdots\binom{n}{i}}{\binom{i+1}{i}\cdots\binom{n-1}{i}} y^i \frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}} \begin{vmatrix} 1 & x^1 & x^2 & \cdots & x^{n-i-1} & x^{n-i} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-i-1} & \beta_{n-i} \\ 0 & \beta_0 & \beta_1\binom{2}{1} & \cdots & \beta_{n-i-2}\binom{n-i-1}{1} & \beta_{n-i-1}\binom{n-i}{1} \\ \vdots & & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & ... & ... & 0 & \beta_0 & \beta_1\binom{n-i}{n-i-1} \end{vmatrix} = \\ = \sum_{i=0}^{n} \binom{n}{i} A_{n-i}(x)\, y^i = \sum_{i=0}^{n} \binom{n}{i} A_i(x)\, y^{n-i}.$$

■

Corollary 52 (Forward difference) *For Appell polynomials $A_n(x)$ we have*

$$\Delta A_n(x) \equiv A_n(x+1) - A_n(x) = \sum_{i=0}^{n-1} \binom{n}{i} A_i(x), \quad n=0,1,... \tag{2.39}$$

Proof. The desired result follows from (2.37) with $y=1$. ■

Corollary 53 (Multiplication Theorem) *For Appell polynomials $A_n(x)$ we have*

$$A_n(mx) = \sum_{i=0}^{n} \binom{n}{i} A_i(x)\,(m-1)^{n-i}\, x^{n-i}, \qquad \begin{array}{l} n=0,1,..., \\ m=1,2,... \end{array} \tag{2.40}$$

Proof. The desired result follows from (2.37) with $y=x(m-1)$. ■

Theorem 54 (Symmetry) *For Appell polynomials $A_n(x)$ the following relation holds*

$$\left(A_n(h-x) = (-1)^n A_n(x)\right) \Leftrightarrow \left(A_n(h) = (-1)^n A_n(0)\right), \qquad \begin{matrix} h \in \mathbb{R} \\ n = 0, 1, ... \end{matrix} \tag{2.41}$$

Proof.

$\Rightarrow$) Follows from the hypothesis with $x = 0$

$\Leftarrow$) Using (2.37) we find

$$\begin{aligned} A_n(h-x) &= \sum_{i=0}^{n} \binom{n}{i} A_i(h) (-x)^{n-i} = \\ &= (-1)^n \sum_{i=0}^{n} \binom{n}{i} A_i(h) (-1)^i x^{n-i} = \\ &= (-1)^n \sum_{i=0}^{n} \binom{n}{i} A_{n-i}(h) (-1)^{n-i} x^i. \end{aligned}$$

Therefore, using the assumptions and (2.11), we have

$$\begin{aligned} A_n(h-x) &= (-1)^n \sum_{i=0}^{n} \binom{n}{i} A_{n-i}(0) x^i = \\ &= (-1)^n A_n(x). \end{aligned}$$

■

Lemma 55 *For the numbers α_{2n+1} and β_{2n+1} we have*

$$(\alpha_{2n+1} = 0) \Longleftrightarrow (\beta_{2n+1} = 0), \quad n = 0, 1, ... \tag{2.42}$$

Proof. As in (2.24), we know that

$$\begin{cases} \beta_0 = \frac{1}{\alpha_0}, \\ \beta_n = -\frac{1}{\alpha_0} \left(\sum\limits_{k=1}^{n} \binom{n}{k} \alpha_k \beta_{n-k} \right), \quad n = 1, 2, ... \end{cases}$$

Hence

$$\begin{cases} \beta_1 & = -\frac{1}{\alpha_0}\alpha_1\beta_0, \\ \beta_{2n+1} & = -\frac{1}{\alpha_0}\binom{2n+1}{1}\alpha_1\beta_{2n} - \\ & -\frac{1}{\alpha_0}\left(\sum_{k=1}^{n}\left[\binom{2n+1}{2k}\alpha_{2k}\beta_{2(n-k)+1} + \binom{2n+1}{2k+1}\alpha_{2k+1}\beta_{2(n-k)}\right]\right), \quad n = 1, 2, ... \end{cases}$$

and

$$\alpha_{2n+1} = 0, \quad n = 0, 1, ... \Rightarrow$$

$$\Rightarrow \begin{cases} \beta_1 & = 0 \\ \beta_{2n+1} & = -\frac{1}{\alpha_0}\sum_{k=1}^{n}\binom{2n+1}{2k}\alpha_{2k}\beta_{2(n-k)+1}, \quad n = 1, 2, ... \end{cases} \Rightarrow$$

$$\Rightarrow \beta_{2n+1} = 0, \quad n = 0, 1, ...$$

In the same way, again from (2.24), we have

$$\begin{cases} \alpha_0 = \frac{1}{\beta_0} \\ \alpha_n = -\frac{1}{\beta_0}\left(\sum_{k=0}^{n-1}\binom{n}{k}\alpha_k\beta_{n-k}\right), \quad n = 1, 2, ... \end{cases}$$

As a consequence

$$\begin{cases} \alpha_1 & = -\frac{1}{\beta_0}\alpha_0\beta_1, \\ \alpha_{2n+1} & = -\frac{1}{\beta_0}\left(\sum_{k=0}^{n-1}\left[\binom{2n+1}{2k}\alpha_{2k}\beta_{2(n-k)+1} + \binom{2n+1}{2k+1}\alpha_{2k+1}\beta_{2(n-k)}\right]\right) - \\ & -\frac{1}{\beta_0}\binom{2n+1}{2n}\alpha_{2n}\beta_1, \quad n = 1, 2, ... \end{cases}$$

and

$$\beta_{2n+1} = 0, \quad n = 0, 1, ... \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_1 & = 0, \\ \alpha_{2n+1} & = -\frac{1}{\beta_0}\sum_{k=0}^{n-1}\binom{2n+1}{2k+1}\alpha_{2k+1}\beta_{2(n-k)}, \quad n = 1, 2, ... \end{cases} \Rightarrow$$

$$\Rightarrow \alpha_{2n+1} = 0, \quad n = 0, 1, ...$$

■

Theorem 56 *For Appell polynomials $A_n(x)$ the following relation holds*

$$(A_n(-x) = (-1)^n A_n(x)) \Longleftrightarrow (\beta_{2n+1} = 0), \quad n = 0, 1, ... \tag{2.43}$$

Proof. By Theorem 54 with $h = 0$ and Lemma 55, we find

$$\begin{aligned}(A_n(-x) = (-1)^n A_n(x)) &\Longleftrightarrow (A_n(0) = (-1)^n A_n(0)) \\ &\Longleftrightarrow (A_{2n+1}(0) = 0) \Longleftrightarrow (\alpha_{2n+1} = 0) \Longleftrightarrow (\beta_{2n+1} = 0).\end{aligned}$$

■

Theorem 57 *For each $n \geq 1$ it is true that*

$$\int_0^x A_n(x)\,dx = \frac{1}{n+1}\left[A_{n+1}(x) - A_{n+1}(0)\right] \tag{2.44}$$

and

$$\int_0^1 A_n(x)\,dx = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}A_i(0). \tag{2.45}$$

Proof. Equality (2.44) follows from (2.7). Moreover, for $x = 1$ we find

$$\int_0^1 A_n(x)\,dx = \frac{1}{n+1}\left[A_{n+1}(1) - A_{n+1}(0)\right] \tag{2.46}$$

and, using (2.37) with $x = 0$ and $y = 1$, we obtain

$$A_{n+1}(1) = \sum_{i=0}^{n+1}\binom{n+1}{i}A_i(0), \tag{2.47}$$

so, by (2.47), relation(2.46) becomes

$$\begin{aligned}\int_0^1 A_n(x)\,dx &= \frac{1}{n+1}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}A_i(0) - A_{n+1}(0)\right] = \\ &= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}A_i(0).\end{aligned}$$

■

2.4 Fourier series expansion of Appell polynomials

We consider, now, the Fourier expansion of Appell polynomials.

Theorem 58 *The Appell polynomials of even index have the following Fourier expansion on the interval [0,1]*

$$A_{2n}(x)=\frac{1}{2n+1}\left[A_{2n+1}(1)-A_{2n+1}(0)\right]+\sum_{k=1}^{\infty}a_k\cos 2\pi kx+b_k\sin 2\pi kx \tag{2.48}$$

with

$$a_k=\sum_{i=0}^{n-1}(-1)^i\,2\frac{\left[A_{2(n-i)-1}(1)-A_{2(n-i)-1}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n)!}{(2(n-i)-1)!},\quad k=1,2,..., \tag{2.49}$$

$$b_k=\sum_{i=0}^{n}(-1)^{i+1}\,2\frac{\left[A_{2(n-i)}(1)-A_{2(n-i)}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n)!}{(2(n-i))!},\quad k=1,2,... \tag{2.50}$$

Proof. Let us consider the Fourier expansion of Appell polynomial of even index $A_{2n}(x)$, $n\geq 1$ on the interval [0,1]

$$A_{2n}(x)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}a_k\cos 2\pi kx+\sum_{k=1}^{\infty}b_k\sin 2\pi kx \tag{2.51}$$

where

$$a_k=2\int_0^1 A_{2n}(x)\cos 2\pi kx dx,\quad k=0,1,... \tag{2.52}$$

$$b_k=2\int_0^1 A_{2n}(x)\sin 2\pi kx dx,\quad k=1,2,... \tag{2.53}$$

From (2.52), for k=0 we have

$$a_0=2\int_0^1 A_{2n}(x)dx=\frac{2}{2n+1}\left[A_{2n+1}(1)-A_{2n+1}(0)\right].$$

For k>0, by integration by part,

$$\frac{1}{2}a_k=\int_0^1 A_{2n}(x)\cos 2\pi kx dx=\left[\frac{\sin 2\pi kx}{2\pi k}A_{2n}(x)\right]_0^1-2n\int_0^1\frac{\sin 2\pi kx}{2\pi k}A_{2n-1}(x)dx$$

The quantity in brackets is equal to zero, and the integral at second member, after an other integration by part, becomes

$$\begin{aligned}\frac{1}{2}a_k &= \int_0^1 A_{2n}(x)\cos 2\pi kx dx = \\ &= \frac{2n}{(2\pi k)^2}\left[A_{2n}(1) - A_{2n}(0)\right] - \frac{(2n)\,(2n-1)}{(2\pi k)^2}\int_0^1 \cos 2\pi kx A_{2n-2}(x)dx.\end{aligned}$$

After further integrations by part, finally, we obtain

$$\begin{aligned}\frac{1}{2}a_k &= \int_0^1 A_{2n}(x)\cos 2\pi kx dx = \\ &= \sum_{i=0}^{n-1}(-1)^i\,\frac{\left[A_{2(n-i)-1}(1) - A_{2(n-i)-1}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n)!}{(2(n-i)-1)!} + \\ &+ (-1)^n\,\frac{(2n)!}{(2\pi k)^{2n}}\int_0^1 A_0(x)\cos 2\pi kx dx\end{aligned}$$

the integral at second member is equal to zero and so we have

$$\frac{1}{2}a_k = \sum_{i=0}^{n-1}(-1)^i\,\frac{\left[A_{2(n-i)-1}(1) - A_{2(n-i)-1}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n)!}{(2(n-i)-1)!}. \tag{2.54}$$

With the same techniques we can expand the coefficients b_k

$$\begin{aligned}\frac{1}{2}b_k &= \int_0^1 f(x)\sin 2\pi kx dx = \left[-\frac{\cos 2\pi kx}{2\pi k}A_{2n}(x)\right]_0^1 + 2n\int_0^1 \frac{\cos 2\pi kx}{2\pi k}A_{2n-1}(x)dx = \\ &= -\frac{1}{2\pi k}\left[A_{2n}(1) - A_{2n}(0)\right] + \frac{2n}{2\pi k}\int_0^1 A_{2n-1}(x)\cos 2\pi kx dx.\end{aligned}$$

By a second integration by part we obtain

$$\begin{aligned}\frac{1}{2}b_k &= -\frac{1}{2\pi k}\left[A_{2n}(1) - A_{2n}(0)\right] + \\ &+ \frac{2n}{(2\pi k)^2}\left[A_{2n-1}(x)\sin 2\pi kx\right]_0^1 - \frac{(2n)\,(2n-1)}{(2\pi k)^2}\int_0^1 A_{2n-2}(x)\sin 2\pi kx dx.\end{aligned}$$

At second member the second term is equal to zero. And the integral at second member, after an other integration by part, becomes

$$\begin{aligned}\frac{1}{2}b_k &= \int_0^1 A_{2n}(x)\sin 2\pi kx dx = \\ &= \sum_{i=0}^{n-1}(-1)^{i+1}\frac{\left[A_{2(n-i)}(1)-A_{2(n-i)}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n)!}{(2(n-i))!}+ \\ &+(-1)^n\frac{(2n)!}{(2\pi k)^{2n}}\int_0^1 A_0(x)\sin 2\pi kx dx = \\ &= \sum_{i=0}^{n-1}(-1)^{i+1}\frac{\left[A_{2(n-i)}(1)-A_{2(n-i)}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n)!}{(2(n-i))!}+ \\ &+(-1)^{n+1}(2n)!\frac{\left[A_0(1)-A_0(0)\right]}{(2\pi k)^{2n+1}}.\end{aligned}$$

Hence,

$$\frac{1}{2}b_k = \sum_{i=0}^{n}(-1)^{i+1}\frac{\left[A_{2(n-i)}(1)-A_{2(n-i)}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n)!}{(2(n-i))!}. \tag{2.55}$$

■

With the same techniques we can expand the Appell polynomials of odd index, for which the following theorem holds.

Theorem 59 *The Appell polynomials of odd index have the following Fourier expansion on the interval [0,1]*

$$A_{2n-1}(x) = \frac{1}{2n}\left[A_{2n}(1)-A_{2n}(0)\right]+\sum_{k=1}^{\infty}a_k\cos 2\pi kx + b_k\sin 2\pi kx \tag{2.56}$$

with

$$a_k = \sum_{i=0}^{n-1}(-1)^i\, 2\frac{\left[A_{2(n-1-i)}(1)-A_{2(n-1-i)}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n-1)!}{(2(n-1-i))!}, \quad k=1,2,..., \tag{2.57}$$

$$b_k = \sum_{i=0}^{n-1}(-1)^{i+1}\, 2\frac{\left[A_{2(n-i)-1}(1)-A_{2(n-i)-1}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n-1)!}{(2(n-i)-1)!}, \quad k=1,2,... \tag{2.58}$$

Proof. For the expansion of the coefficients we proceed in the same way of even degree. After further integrations by part, we have

$$a_0 = 2\int_0^1 A_{2n-1}(x)dx = \frac{2}{2n}\left[A_{2n}(1) - A_{2n}(0)\right],$$

$$\frac{1}{2}a_k = \int_0^1 A_{2n-1}(x)\cos 2\pi kx dx = \sum_{i=0}^{n-1}(-1)^i \frac{\left[A_{2(n-1-i)}(1) - A_{2(n-1-i)}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n-1)!}{(2(n-1-i))!},$$

$$\frac{1}{2}b_k = \int_0^1 A_{2n-1}(x)\sin 2\pi kx dx = \sum_{i=0}^{n-1}(-1)^{i+1} \frac{\left[A_{2(n-i)-1}(1) - A_{2(n-i)-1}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n-1)!}{(2(n-i)-1)!}.$$

■

2.5 Examples

In this section we consider classic examples, in particular Bernoulli, Euler, Hermite and Laguerre polynomials and their possible generalizations not studied in the literature so far.

2.5.1 Bernoulli polynomials

Placing

$$\begin{cases} \beta_0 = 1, \\ \beta_i = \frac{1}{i+1}, \quad i = 1, ..., n, \end{cases} \tag{2.59}$$

in (2.29) and (2.30), the resulting Appell polynomial is known as Bernoulli polynomial [5]. The determinantal form of this polynomial has been considered in [12] and the fundamental properties have been also obtained by employing basic tools of linear algebra.

Moreover the following identity can be derived.

Theorem 60 *For Bernoulli polynomials $B_n(x)$ we have*

$$m^{n-1}\sum_{i=0}^{m-1} B_n\left(x+\frac{i}{m}\right)=\sum_{i=0}^{n}\binom{n}{i}B_i(x)(m-1)^{n-i}x^{n-i}, \qquad \begin{array}{l} n=0,1,...,\\ \\ m=1,2,... \end{array} \tag{2.60}$$

Proof. It is known [1] that

$$B_n(mx)=m^{n-1}\sum_{i=0}^{m-1} B_n\left(x+\frac{i}{m}\right), \qquad \begin{array}{l} n=0,1,...,\\ \\ m=1,2,... \end{array} \tag{2.61}$$

and hence from (2.40) and (2.61) the proof is concluded. ■

Fourier series of Bernoulli polynomials

In the case of Bernoulli polynomials, for which we have

$$B_{2i-1}(0)=\begin{cases} -\frac{1}{2} & \text{for } i=1 \\ \\ 0 & \text{for } i=2,3,... \end{cases}$$

$$B_{2i-1}(1)=-B_{2i-1}(0)$$

$$B_{2i}(1)=B_{2i}(0)$$

the Fourier expansion is the following

$$B_{2n}(x)=\sum_{k=1}^{\infty}(-1)^{n-1}2\frac{[B_1(1)-B_1(0)]}{(2\pi k)^{2n}}(2n)!\cos 2\pi kx= \tag{2.62}$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{n-1}2(2n)!}{(2\pi k)^{2n}}\cos 2\pi kx \tag{2.63}$$

$$B_{2n-1}(x)=\sum_{k=1}^{\infty}(-1)^{n}2\frac{[B_1(1)-B_1(0)]}{(2\pi k)^{2n-1}}(2n-1)!\sin 2\pi kx= \tag{2.64}$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{n}2(2n-1)!}{(2\pi k)^{2n}}\sin 2\pi kx \tag{2.65}$$

2.5.2 Generalized Bernoulli polynomials

By direct inspection of (2.59) we deduce

$$\beta_i = \int_0^1 x^i dx, \quad i = 0, .., n. \tag{2.66}$$

Analogously, we can consider the weighed coefficients

$$\beta_i^w = \int_0^1 w(x)\, x^i dx, \quad i = 0, .., n, \tag{2.67}$$

where $w(x)$ is a general weight function.

In particular by taking the classical Jacobi weight, $w(x) = (1-x)^{\alpha} x^{\beta},\ \alpha, \beta > -1,$ we obtain

$$\beta_i^w = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+2)}, \quad i = 0, ..., n \tag{2.68}$$

The relative Appell polynomials, called now Bernoulli-Jacobi, are not considered in the literature to our knowledge, except for the case $\alpha = \beta = 0$ for which we find again the Bernoulli polynomials. For the case $\alpha = \beta = -\frac{1}{2}$ it is useful to normalize by setting

$$\beta_i^{\overline{w}} = \frac{1}{\pi}\frac{\Gamma(\alpha+1)\,\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+2)}, \quad i = 0, ..., n. \tag{2.69}$$

2.5.3 Hermite normalized polynomials

Assuming

$$\begin{cases} \beta_0 = 1, \\ \beta_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} x^i dx = \begin{cases} 0 & \text{for } i \text{ odd} \\ \frac{(i-1)(i-3)\cdot\dots\cdot 3\cdot 1}{2^{\frac{i}{2}}} & \text{for } i \text{ even} \end{cases}, \ i = 1, ..., n, \end{cases} \tag{2.70}$$

in (2.29) and (2.30), the related Appell polynomials coincide with the well-known Hermite normalized polynomials [6].

It's known [49] that Hermite normalized polynomials are the only ones which are, at the same time, orthogonal and Appell polynomials.

The Hessemberg determinantal form does not seem to be known in literature.

2.5.4 Generalized Hermite polynomials

Assuming

$$\beta_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-|x|^{\alpha}} x^i dx = \begin{cases} 0 & \text{for } i \text{ odd} \\ \frac{2}{\alpha\sqrt{\pi}} \Gamma\left(\frac{i+1}{\alpha}\right) & \text{for } i \text{ even} \end{cases}, \quad \begin{matrix} i = 0, ..., n, \\ \alpha > 0 \end{matrix} \tag{2.71}$$

in (2.29) and (2.30), we obtain a wider class of Appell polynomials.

2.5.5 Generalized Laguerre polynomials

Placing

$$\beta_i = \int_{0}^{+\infty} e^{-sx} x^i dx = \frac{1}{s^{i+1}} \Gamma\left(i+1\right) = \frac{i!}{s^{i+1}}, \quad s > 0, \quad i = 0, ..., n, \tag{2.72}$$

in (2.29) and (2.30), we obtain a new class of Appell polynomials, called now Appell-Laguerre, that does not seem to be known in literature, except for the case $s = 1$ [6].

2.5.6 Euler polynomials

Placing

$$\begin{cases} \beta_0 = 1, \\ \beta_i = \frac{1}{2}, \quad i = 1, ..., n, \end{cases} \tag{2.73}$$

in (2.29) and (2.30), the resulting Appell polynomials are known as Euler polynomials [23]. The determinantal form seems new. In fact we have

$$E_0(x) = 1, \tag{2.74}$$

$$E_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \cdots & \frac{1}{2}\binom{n-1}{1} & \frac{1}{2}\binom{n}{1} \\ 0 & 0 & 1 & \cdots & \cdots & \frac{1}{2}\binom{n-1}{2} & \frac{1}{2}\binom{n}{2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \frac{1}{2}\binom{n}{n-1} \end{vmatrix}, \quad n = 1, 2, ... \tag{2.75}$$

Concerning Euler polynomials, all the properties proved in general for Appell polynomials hold. In particular we have the following result.

Theorem 61 *For Euler polynomials $E_n(x)$ we have*

$$E_n(x) = x^n - \frac{1}{2^n} \sum_{k=0}^{n-1} \binom{n}{k} E_k(x), \quad n = 1, 2, ... \tag{2.76}$$

Proof. The claimed thesis follows from (2.31). ∎

Theorem 62 *For Euler polynomials $E_n(x)$ we have*

$$\sum_{i=0}^{n}\binom{n}{i}E_i(x)(m-1)^{n-i}x^{n-i}=$$
$$=\begin{cases} m^n\sum\limits_{i=0}^{m-1}(-1)^i E_n\left(x+\frac{i}{m}\right), & \begin{array}{l} n=0,1,..., \\ m=1,3,..., \end{array} \\ -\frac{2}{n+1}m^n\sum\limits_{i=0}^{m-1}(-1)^i B_{n+1}\left(x+\frac{i}{m}\right), & \begin{array}{l} n=0,1,..., \\ m=2,4,... \end{array} \end{cases} \tag{2.77}$$

Proof. In literature [1] it is known that

$$E_n(mx)=\begin{cases} m^n\sum\limits_{i=0}^{m-1}(-1)^i E_n\left(x+\frac{i}{m}\right), & \begin{array}{l} n=0,1,..., \\ m=1,3,..., \end{array} \\ -\frac{2}{n+1}m^n\sum\limits_{i=0}^{m-1}(-1)^i B_{n+1}\left(x+\frac{i}{m}\right), & \begin{array}{l} n=0,1,..., \\ m=2,4,... \end{array} \end{cases} \tag{2.78}$$

and therefore, from (2.40) and (2.78), we desired result follows. ■

Fourier series of Euler polynomials

Concerning Euler polynomials, for which we have

$$E_{2i}(0)=E_{2i}(1)=\begin{cases} 1 & \text{for } i=0 \\ 0 & \text{for } i=1,2,... \end{cases}$$

$$E_{2i-1}(1)=-E_{2i-1}(0)$$

the Fourier expansion is the following

$$E_{2n-1}(x) = \sum_{k=1}^{\infty}\sum_{i=0}^{n-1}(-1)^{i+1}\,2\frac{\left[E_{2(n-i)-1}(1)-E_{2(n-i)-1}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n-1)!}{(2(n-i)-1)!}\sin 2\pi kx =$$
$$= \sum_{k=1}^{\infty}\sum_{i=0}^{n-1}(-1)^{i}\,4\frac{\left[E_{2(n-i)-1}(0)\right]}{(2\pi k)^{2i+1}}\frac{(2n-1)!}{(2(n-i)-1)!}\sin 2\pi kx$$

$$E_{2n}(x) = \frac{1}{2n+1}\left[E_{2n+1}(1)-E_{2n+1}(0)\right]+$$
$$+\sum_{k=1}^{\infty}\sum_{i=0}^{n-1}(-1)^{i}\,2\frac{\left[E_{2(n-i)-1}(1)-E_{2(n-i)-1}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n)!}{(2(n-i)-1)!}\cos 2\pi kx$$
$$= -\frac{2}{2n+1}E_{2n+1}(0)+\sum_{k=1}^{\infty}\sum_{i=0}^{n-1}(-1)^{i+1}\,4\frac{\left[E_{2(n-i)-1}(0)\right]}{(2\pi k)^{2(i+1)}}\frac{(2n)!}{(2(n-i)-1)!}\cos 2\pi kx.$$

2.5.7 Generalized Euler polynomials

From (2.73) we can write

$$\beta_i = Mx^i, \quad i = 0, ..., n, \tag{2.79}$$

where $Mf = \frac{f(1)+f(0)}{2}$. In a similar way, we can consider the weighed coefficients

$$\beta_i^w = M^w x^i, \quad i = 0, ..., n, \tag{2.80}$$

where $M^w f = \frac{w_1 f(1)+w_2 f(0)}{w_1+w_2}, \quad w_1, w_2 > 0$, i.e:

$$\begin{cases} \beta_0^w = 1, \\ \beta_i^w = \frac{w_1}{w_1+w_2}, \quad i = 1, ..., n. \end{cases} \tag{2.81}$$

2.6 Numerical examples

In this section we provide explicitly same classes of Appell polynomials, by using an ad hoc Mathematica code based on the new definition.

By the choice of the coefficients β_i in definition (2.30) we can compute the relative Appell polynomial

$$A_n(x) = c_0 + c_1 x + ... + c_n x^n. \tag{2.82}$$

2.6.1 Bernoulli-Jacobi/Tchebichev polynomials

Placing in (2.69) $\alpha = \beta = -\frac{1}{2}$ we have

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$n=0$	1								
$n=1$	$-\frac{1}{2}$	1							
$n=2$	$\frac{1}{8}$	-1	1						
$n=3$	$\frac{1}{16}$	$\frac{3}{8}$	$-\frac{3}{2}$	1					
$n=4$	$-\frac{7}{128}$	$\frac{1}{4}$	$\frac{3}{4}$	-2	1				
$n=5$	$-\frac{13}{256}$	$-\frac{35}{128}$	$\frac{5}{8}$	$\frac{5}{4}$	$-\frac{5}{2}$	1			
$n=6$	$\frac{71}{1024}$	$-\frac{39}{128}$	$-\frac{105}{128}$	$\frac{5}{4}$	$\frac{15}{8}$	-3	1		
$n=7$	$\frac{187}{2048}$	$\frac{497}{1024}$	$-\frac{273}{256}$	$-\frac{245}{128}$	$\frac{35}{16}$	$\frac{21}{8}$	$-\frac{7}{2}$	1	
$n=8$	$-\frac{5479}{32768}$	$\frac{187}{256}$	$\frac{497}{256}$	$-\frac{91}{32}$	$-\frac{245}{64}$	$\frac{7}{2}$	$\frac{7}{2}$	-4	1

2.6.2 Generalized Laguerre polynomials

Setting in (2.72) $s = 1$ we have the normalized Appel-Laguerre polynomials [6]

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
$n = 0$	1										
$n = 1$	-1	1									
$n = 2$	0	-2	1								
$n = 3$	0	0	-3	1							
$n = 4$	0	0	0	-4	1						
$n = 5$	0	0	0	0	-5	1					
$n = 6$	0	0	0	0	0	-6	1				
$n = 7$	0	0	0	0	0	0	-7	1			
$n = 8$	0	0	0	0	0	0	0	-8	1		
$n = 9$	0	0	0	0	0	0	0	0	-9	1	
$n = 10$	0	0	0	0	0	0	0	0	0	-10	1

2.6.3 Generalized Euler polynomials

Placing in (2.81) $w_1 = \frac{1}{2},\ w_2 = \frac{1}{3}$ we find

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$n=0$	1								
$n=1$	$-\frac{3}{5}$	1							
$n=2$	$\frac{3}{25}$	$-\frac{6}{5}$	1						
$n=3$	$\frac{33}{125}$	$\frac{9}{25}$	$-\frac{9}{5}$	1					
$n=4$	$-\frac{141}{625}$	$\frac{132}{125}$	$\frac{18}{25}$	$-\frac{12}{5}$	1				
$n=5$	$-\frac{267}{625}$	$-\frac{141}{125}$	$\frac{66}{25}$	$\frac{6}{5}$	-3	1			
$n=6$	$\frac{2751}{3125}$	$-\frac{1602}{625}$	$-\frac{423}{125}$	$\frac{132}{25}$	$\frac{9}{5}$	$-\frac{18}{5}$	1		
$n=7$	$\frac{20109}{15625}$	$\frac{19257}{3125}$	$-\frac{5607}{625}$	$-\frac{987}{125}$	$\frac{231}{25}$	$\frac{63}{25}$	$-\frac{21}{5}$	1	
$n=8$	$-\frac{448761}{78125}$	$\frac{160872}{15625}$	$\frac{77028}{3125}$	$-\frac{14952}{625}$	$-\frac{1974}{125}$	$\frac{1848}{125}$	$\frac{84}{25}$	$-\frac{24}{5}$	1

Chapter 3

Appell interpolation problem

The algebraic approach proposed in chapter 2 allows the solution, expressed by a determinantal form, using a basis of Appell polynomials, of a remarkable general linear interpolation problem.

3.1 The Appell interpolation problem

Let us consider the linear space $X = C^n[a,b]$ and let P_n be the space of polynomials of degree $\leq n$. Let L_0 be a linear functional defined on P_n, such that $L_0(1) \neq 0$.

Reminding the determinantal definition of Appell polynomials (2.29, 2.30) we provides now the following

Definition 63 *The sequence of polynomials defined by*

$$A_0^{L_0}(x) = \frac{1}{\beta_0},$$

$$A_n^{L_0}(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, ... \tag{3.1}$$

with $\beta_i = L_0(x^i)$, $i = 0, ..., n$, is called sequence of Appell polynomials related to the functional L_0.

Problem 64 *Let L_0 be a given linear functional defined on $\mathcal{P}_n$ such that $L_0(1) \neq 0$. For a given set of values $\omega_0, \omega_1, ..., \omega_n \in \mathbb{R}$ there exists a unique polynomial $P_n(x)$ of degree $\leq n$ such that*

$$\frac{L_0\left(P_n^{(i)}\right)}{i!} = \omega_i, \quad i = 0, 1, ...n? \tag{3.2}$$

The answer is provided by the following

Theorem 65 *For every choice of $\omega_0, \omega_1, ..., \omega_n \in \mathbb{R}$ the polynomial of degree $\leq n$*

$$P_n(x) = \sum_{i=0}^{n} \omega_i A_i^{L_0}(x),$$

where the $A_i^{L_0}(x)$ are the Appell polynomials related to the functional L_0 defined by (3.1), is the unique solution of the interpolation problem (3.2).

Proof. Let us define n linear functionals L_i, $i=1,...n$ on $\mathcal{P}_n$ such that

$$L_i\left(x^j\right)=\frac{L_0\left(\left(x^j\right)^{(i)}\right)}{i!}\quad i=1,...n,\quad j=0,...,n. \tag{3.3}$$

By Lemma 1 and Theorem 2, the problem (3.2) possesses a unique solution if and only if

$$G=\left|L_i(x^j)\right|\neq 0,\quad i,j=0,1,...,n. \tag{3.4}$$

Since,

$$L_i\left(x^j\right)=\begin{cases}\frac{L_0\left(\left(x^j\right)^{(i)}\right)}{i!}=\binom{j}{i}L_0\left(x^{j-i}\right) & j=i,i+1,...\\ \frac{L_0\left(\left(x^j\right)^{(i)}\right)}{i!}=L_0\left(0\right)=0 & j=0,...,i-1\end{cases},\ i=1,2,...,$$

setting

$$L_0\left(1\right)=\beta_0\neq 0,$$

$$L_0\left(x^j\right)=\beta_j,\quad j=1,2,...\ \beta_j\in\mathbb{R},$$

we have

$$G=\begin{vmatrix}\beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n\\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1}\\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2}\\ \vdots & & & \ddots & & \vdots & \vdots\\ \vdots & & & & \ddots & \vdots & \vdots\\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1\end{vmatrix}=\left(\beta_0\right)^{n+1}. \tag{3.5}$$

By the representation theorem 21 we can calculate the solution of the interpolation problem

(3.2)

$$P_n(x) = -\frac{1}{G}\begin{vmatrix} 0 & x^0 & x^1 & x^2 & ... & ... & x^n \\ \omega_0 & L_0(x^0) & L_0(x^1) & L_0(x^2) & ... & ... & L_0(x^n) \\ \omega_1 & L_1(x^0) & L_1(x^1) & L_1(x^2) & ... & ... & L_1(x^n) \\ \vdots & & & & & & \vdots \\ \omega_n & L_n(x^0) & L_n(x^1) & L_n(x^2) & ... & ... & L_n(x^n) \end{vmatrix} = \tag{3.6}$$

$$= -\frac{1}{(\beta_0)^{n+1}}\begin{vmatrix} 0 & 1 & x^1 & x^2 & \cdots & x^{n-1} & x^n \\ \omega_0 & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ \omega_1 & 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ \omega_2 & 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ \omega_{n-1} & 0 & \cdots & \cdots & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \\ \omega_n & 0 & \cdots & \cdots & \cdots & 0 & \beta_0 \end{vmatrix}. \tag{3.7}$$

We expand the determinant in (3.7) by the minors of its first row

$$P_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{(\beta_0)^{n+1}} \omega_i \begin{vmatrix} 1 & x^1 & \ldots & x^{i-1} & x^i & x^{i+1} & \ldots & \ldots & x^n \\ \beta_0 & \beta_1 & \ldots & \beta_{i-1} & \beta_i & \beta_{i+1} & \ldots & \ldots & \beta_n \\ 0 & \beta_0 & \ldots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} & \binom{i+1}{1}\beta_i & \ldots & \ldots & \binom{n}{1}\beta_{n-1} \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & \ldots & \ldots & \beta_0 & \binom{i}{i-1}\beta_1 & \binom{i+1}{i-1}\beta_2 & \ldots & \ldots & \binom{n}{i-1}\beta_{n-i+1} \\ 0 & \ldots & \ldots & 0 & 0 & \beta_0 & \ldots & \ldots & \binom{n}{i+1}\beta_{n-i-1} \\ \vdots & & & & & & \ddots & & \vdots \\ \vdots & & & & & & & \ddots & \binom{n}{n-1}\beta_1 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \beta_0 \end{vmatrix},$$

We keep on expanding the determinants in the last one by the minors of last row for $n-i$ times. Finally we obtain

$$P_n(x) = \sum_{i=0}^{n} \omega_i \frac{(-1)^i}{(\beta_0)^{i+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} =$$

$$= \sum_{i=0}^{n} \omega_i A_i^{L_0}(x).$$

■

Theorem 66 *Let L_0 be a linear functional defined on X such that $L_0(1) \neq 0$. The polyno-*

mial

$$P_n(f,x) = \sum_{i=0}^{n} \frac{L_0\left(f^{(i)}\right)}{i!} A_i^{L_0}(x) \tag{3.8}$$

is the unique element of $\mathcal{P}_n$ that satisfy the conditions

$$L_0\left((P_n(f,x))^{(i)}\right) = L_0\left(f^{(i)}\right), \quad i = 0, 1, ..., n \tag{3.9}$$

Proof. It follows by previous theorem setting

$$\omega_i = \frac{L_0\left(f^{(i)}\right)}{i!}, \quad i = 0, 1, ..., n.$$

■

We can, now, provides the following

Definition 67 *Let L_0 be a linear functional defined on X such that $L_0(1) \neq 0$. The polynomial*

$$P_n(L_0 f, x) = \sum_{i=0}^{n} \frac{L_0\left(f^{(i)}\right)}{i!} A_i^{L_0}(x) \tag{3.10}$$

is called Appell interpolation polynomial related to the function f and to the functional L_0.

3.2 The remainder

In order to estimate the difference between the function $f(x)$ and the interpolation polynomial $P_n(L_0 f, x)$ we introduce the remainder

$$R_n(f,x) = f(x) - P_n(L_0 f, x), \quad x \in [a,b] \tag{3.11}$$

Observe that if $f \in \mathcal{P}_n$ the remainder identically vanishes. In fact, the following theorem hold.

Theorem 68 *The Appell interpolation problem has degree of exactness n, i.e.:*

$$P_n(x^j,x)=x^j \quad j=0,1,....,n \tag{3.12}$$

Proof. By definition (67) we have

$$\begin{aligned}
P_n(x^j,x) &= \sum_{i=0}^{n}\frac{L_0\left(\left(x^j\right)^{(i)}\right)}{i!}A_i^{L_0}(x)= \\
&= \sum_{i=0}^{j-1}L_i\left(x^j\right)A_i^{L_0}(x)+L_j\left(x^j\right)A_j^{L_0}(x)+\sum_{i=j+1}^{n}L_i\left(x^j\right)A_i^{L_0}(x)= \\
&= \sum_{i=0}^{j-1}\binom{j}{i}\beta_{j-i}A_i^{L_0}(x)+\beta_0A_j^{L_0}(x),
\end{aligned}$$

and by the recurrence formula (2.31),

$$P_n(x^j,x)=\sum_{i=0}^{j-1}\binom{j}{i}\beta_{j-i}A_i^{L_0}(x)+\beta_0\frac{1}{\beta_0}\left(x^j-\sum_{i=0}^{j-1}\binom{j}{i}\beta_{j-i}A_i^{L_0}(x)\right)=x^j$$

■

For a fixed x we may consider the remainder $R_n(f,x)$ as a linear functional which operates on f and which annihilates all elements of $\mathcal{P}_n$. By Peano's Theorem ([21, p. 69]), if a linear functional has this property, then it must also have a simple representation in terms of $f^{(n+1)}$.

Theorem 69 *Let $f\in C^{n+1}[a,b]$, and $P_n(L_0f,x)$ be the related Appell interpolation polynomial, then*

$$\forall x\in[a,b],\ R_n(f,x)=f(x)-P_n(L_0f,x) \tag{3.13}$$

with

$$R_n(f,x)=\frac{1}{n!}\int_a^x(x-t)^n f^{(n+1)}(t)\,dt-\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}A_i^{L_0}(x)\int_a^b L_0\left((x-t)_+^{n-i}\right)f^{(n+1)}(t)\,dt. \tag{3.14}$$

Proof. The remainder

$$R_n(f,x) = f(x) - P_n(L_0 f, x)$$

is a linear functional that, by the (3.12), vanishes for polynomials degree $\leq n$. By applying the Peano's Theorem we have

$$K_n\left(x,t\right) = \frac{1}{n!} R_n\left[(x-t)_+^n\right] = \frac{1}{n!}\left((x-t)_+^n - \sum_{i=0}^{n}\binom{n}{i} L_0\left((x-t)_+^{n-i}\right) A_i^{L_0}(x)\right),$$

and, hence,

$$\begin{aligned}
R_n(f,x) &= \int_a^b K_n\left(x,t\right) f^{(n+1)}\left(t\right) dt = \\
&= \frac{1}{n!}\int_a^b (x-t)_+^n f^{(n+1)}\left(t\right) dt - \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i} A_i^{L_0}(x) \int_a^b L_0\left((x-t)_+^{n-i}\right) f^{(n+1)}\left(t\right) dt = \\
&= \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}\left(t\right) dt - \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i} A_i^{L_0}(x) \int_a^b L_0\left((x-t)_+^{n-i}\right) f^{(n+1)}\left(t\right) dt.
\end{aligned}$$

■

3.3 Examples

In the following we provides some examples of Appell interpolation polynomials related to a given functional.

3.3.1 The Taylor interpolation polynomial

Let $f \in C^n[a,b]$ and $L_0(f) = f(x_0)$, $x_0 \in [a,b]$. Assigned $\beta_j = L_0\left(x^j\right) = (x_0)^j$, $j = 0, 1, \ldots$ the sequence of Appell polynomials related to the functional L_0 is given by

$$A_0^{L_0}(x) = 1,$$

$$A_n^{L_0}(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ 1 & x_0 & (x_0)^2 & \cdots & \cdots & (x_0)^{n-1} & (x_0)^n \\ 0 & 1 & \binom{2}{1} x_0 & \cdots & \cdots & \binom{n-1}{1}(x_0)^{n-2} & \binom{n}{1}(x_0)^{n-1} \\ 0 & 0 & 1 & \cdots & \cdots & \binom{n-1}{2}(x_0)^{n-3} & \binom{n}{2}(x_0)^{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \binom{n}{n-1} x_0 \end{vmatrix} = (x - x_0)^n,$$

$$n = 1, 2, \ldots \quad (3.15)$$

where in the last one, because of

$$\frac{d}{dx^i}\left(A_n^{L_0}(x)\right) = (-1)^n \begin{vmatrix} 0 & 0 & \cdots & i! & \cdots & i!\binom{n-1}{i} x^{n-1-i} & i!\binom{n}{i} x^{n-i} \\ 1 & x_0 & \cdots & (x_0)^i & \cdots & (x_0)^{n-1} & (x_0)^n \\ 0 & 1 & \cdots & \binom{i}{1}(x_0)^{i-1} & \cdots & \binom{n-1}{1}(x_0)^{n-2} & \binom{n}{1}(x_0)^{n-1} \\ \vdots & & \ddots & & & \vdots & \vdots \\ \vdots & & & 1 & \cdots & \binom{n-1}{i}(x_0)^{n-1-i} & \binom{n}{i}(x_0)^{n-i} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & & 0 & 1 & \binom{n}{n-1} x_0 \end{vmatrix},$$

$$i = 1, \ldots, n-1, \quad (3.16)$$

it is clear that $A_n^{L_0}(x)$ and its derivatives to order $n-1$ vanish in x_0. Therefore, x_0 is a zero of multiplicity n for the polynomial $A_n^{L_0}(x)$ of degree n. Moreover, expanding the determinant by the minors of its first row we have that the leading coefficient is precisely 1 and, finally, $A_n^{L_0}(x) = (x-x_0)^n$.

Reminding that $L_0\left(f^{(i)}\right) = f^{(i)}(x_0)\,, \quad i = 1, ...n$, the Appell interpolation polynomial related to the function f is

$$T_n(f,x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i, \tag{3.17}$$

known as Taylor polynomial centered at x_0.

The remainder

Theorem 70 *Let $f \in C^{n+1}[a,b]$, then $\forall x \in [a,b]$*

$$R_n(f,x) = f(x) - T_n(f,x) = -\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(x-x_0)^i \int_x^{x_0} (x_0-t)^{n-i} f^{(n+1)}(t)dt \tag{3.18}$$

Proof. By Peano's Theorem,

$$R_n(f,x) = \int_a^b K_n(x,t)\, f^{(n+1)}(t)\, dt \tag{3.19}$$

where

$$K_n(x,t) = \frac{1}{n!} R_n\left[(x-t)_+^n\right], \tag{3.20}$$

because of

$$((x-t)_+^n)^{(i)} = \frac{n!(x-t)_+^{n-i}}{(n-i)!} \tag{3.21}$$

we have

$$K_n(x,t) = \frac{1}{n!}\left((x-t)_+^n - \sum_{i=0}^{n}\binom{n}{i}(x_0-t)_+^{n-i}(x-x_0)^i\right).$$

To fix ideas, set $x < x_0$,

$$R_n(f,x) = \int_a^b K_n(x,t)\, f^{(n+1)}(t)\, dt = \tag{3.22}$$

$$= \frac{1}{n!}\int_a^b (x-t)_+^n\, f^{(n+1)}(t)\, dt - \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(x-x_0)^i \int_a^b (x_0-t)_+^{n-i} f^{(n+1)}(t)\, dt =$$

$$\tag{3.23}$$

$$= \frac{1}{n!}\int_a^x (x-t)^n\, f^{(n+1)}(t)\, dt - \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(x-x_0)^i \int_a^x (x_0-t)^{n-i} f^{(n+1)}(t)\, dt -$$

$$\tag{3.24}$$

$$- \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(x-x_0)^i \int_x^{x_0} (x_0-t)^{n-i} f^{(n+1)}(t)\, dt \tag{3.25}$$

and from the exactness, we have

$$(x-t)^n = \sum_{i=0}^{n}\binom{n}{i}(x_0-t)^{n-i}(x-x_0)^i$$

from which the thesis follows. ∎

Corollary 71 ([1]) *Let $f \in C^{n+1}[a,b]$, then $\forall x \in [a,b]$*

$$R_n(f,x) = f(x) - T_n(f,x) = \frac{f^{(n+1)}(\xi)\,(x-x_0)^{n+1}}{(n+1)!}, \quad \xi \in [x, x_0]$$

Proof. Because of $(x_0 - t) > 0$ in the interval (x, x_0) we have

$$\begin{aligned} R_n(f,x) &= -\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(x-x_0)^i f^{(n+1)}(\xi)\int_x^{x_0}(x_0-t)^{n-i}dt = \\ &= -\frac{1}{n!}f^{(n+1)}(\xi)\sum_{i=0}^{n}\binom{n}{i}(x-x_0)^i\frac{(-1)^{n-i+1}\,(x-x_0)^{n-i+1}}{n-i+1} = \\ &= \frac{1}{n!}f^{(n+1)}(\xi)(x-x_0)^{n+1}\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{n-i}}{n-i+1} \end{aligned}$$

From

$$\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{n-i}}{n-i+1} = \frac{1}{n+1}$$

finally,

$$R_n(f,x)=\frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!},\quad \xi\in[x,x_0]$$

■

3.3.2 The Bernoulli interpolation polynomial

Let $f\in C^n[0,1]$ and $L_0(f)=\int_0^1 f(x)\,dx$. Setting $\beta_j=L_0\left(x^j\right)=\frac{1}{j+1}$, $j=0,1,...$ the Appell polynomials related to the functional L_0 coincide with the Bernoulli polynomials $B_i(x)$. From $L_0\left(f^{(i)}\right)=f^{(i-1)}(1)-f^{(i-1)}(0)$, $i=1,...n$, the Bernoulli interpolation polynomial related to the function f is

$$B_n(f,x)=\int_0^1 f(x)\,dx+\sum_{i=1}^{n}\frac{f^{(i-1)}(1)-f^{(i-1)}(0)}{i!}B_i(x) \tag{3.26}$$

The remainder

Theorem 72 *Let $f\in C^{n+1}[0,1]$, then $\forall x\in[0,1]$*

$$f(x)-B_n(f,x)=\frac{1}{n!}\sum_{i=0}^{n}\frac{B_i(x)}{n-i+1}\binom{n}{i}\int_0^1 G_i(x,t)f^{(n+1)}(t)dt \tag{3.27}$$

where

$$G_i(x,t)=\begin{cases}(-1)^{n-i}t^{n-i+1} & 0\le t\le x\\ -(1-t)^{n-i+1} & x\le t\le 1\end{cases} \tag{3.28}$$

Proof. By Peano's Theorem,

$$R_n(f,x)=\int_0^1 f^{(n+1)}(t)K(t)dt \tag{3.29}$$

with

$$K(t)=\frac{1}{n!}R_{n_x}[(x-t)_+^n]. \tag{3.30}$$

Because of

$$\frac{((x-t)_+^n)^{(i-1)}}{i!} = \frac{\binom{n}{i}}{n-i+1}(x-t)_+^{n-i+1} \tag{3.31}$$

we have

$$K(t) = \frac{1}{n!}R_{n_x}[(x-t)_+^n] = \tag{3.32}$$

$$= \frac{1}{n!}[(x-t)_+^n - \int_0^1 (x-t)_+^n dx + \sum_{i=1}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(1-t)_+^{n-i+1} - (-t)_+^{n-i+1}]] =$$

$$= \frac{1}{n!}[(x-t)_+^n - \frac{(1-t)_+^{n+1}}{n+1} - \sum_{i=1}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(1-t)_+^{n-i+1} - (-t)_+^{n-i+1}]] \tag{3.33}$$

that is

$$K(t) = \begin{cases} \frac{1}{n!}\left[(x-t)^n - \frac{(1-t)^{n+1}}{n+1} - \sum_{i=1}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}(1-t)^{n-i+1}\right] & \text{se} \quad 0 \le t \le x \\ -\frac{1}{n!}\left[-\frac{(1-t)^{n+1}}{n+1} - \sum_{i=1}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}(1-t)^{n-i+1}\right] & \text{se} \quad x \le t \le 1 \end{cases} \tag{3.34}$$

Since, moreover,

$$(x-t)^n = \int_0^1 (x-t)^n dx + \sum_{i=1}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(1-t)^{n-i+1} - (-t)^{n-i+1}] =$$

$$= (1-t)^{n-i+1} - (-t)^{n-i+1} + \sum_{i=1}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(1-t)^{n-i+1} - (-t)^{n-i+1}] =$$

$$= \sum_{i=0}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(1-t)^{n-i+1} - (-t)^{n-i+1}]$$

and, replacing the last one in the (3.34), we have

$$K(t) = \begin{cases} \frac{1}{n!}\sum_{i=0}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(-1)^{n-i}t^{n-i+1}] & \text{se} \quad 0 \le t \le x \\ \frac{1}{n!}\sum_{i=0}^{n} \frac{B_i(x)}{n-i+1}\binom{n}{i}[(1-t)^{n-i+1}] & \text{se} \quad x \le t \le 1 \end{cases} \tag{3.35}$$

from which the thesis follows. ■

3.3.3 The Bernoulli generalized interpolation polynomial

Let $f \in C^n[0,1]$, $w(x)$ be a general weight function and $L_0^w(f) = \int_0^1 w(x)f(x)\,dx$. Setting $\beta_j^w = L_0^w\left(x^j\right)$, $j = 0,1,...$ the Appell polynomials related to the functional L_0^w coincide with the Bernoulli generalized polynomials $B_i^w(x)$. Since $L_0^w\left(f^{(i)}\right) = \int_0^1 w(x)f^{(i)}(x)\,dx$, $i = 1,...n$, the Bernoulli generalized interpolation polynomial related to the function f is

$$B_n^w(f,x) = \sum_{i=0}^{n} \frac{1}{i!} B_i^w(x) \int_0^1 w(x) f^{(i)}(x)\,dx = \tag{3.36}$$

$$= \int_0^1 w(x) f(x)\,dx + \sum_{i=0}^{n} \frac{1}{i!} B_i^w(x) \int_0^1 w(x) f^{(i)}(x)\,dx \tag{3.37}$$

The remainder

Theorem 73 *Let* $f \in C^{n+1}[0,1]$, *then* $\forall x \in [0,1]$

$$R_n(f,x) = f(x) - B_n^w(f,x) = \int_0^1 f^{(n+1)}(t)K(t)dt =$$

$$= \frac{1}{n!}\left(\int_0^1 (x-t)_+^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \frac{1}{i!} B_i^w(x) \int_0^1 \left(\int_0^1 w(x)\left((x-t)_+^n\right)^{(i)} dx\right) f^{(n+1)}(t)dt\right) =$$

$$= \frac{1}{n!}\left(\int_0^x (x-t)^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \frac{1}{i!} B_i^w(x) \int_0^1 \left(\int_0^1 w(x) \frac{n!(x-t)_+^{n-i}}{(n-i)!} dx\right) f^{(n+1)}(t)dt\right) =$$

$$= \frac{1}{n!}\left(\int_0^x (x-t)^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \binom{n}{i} B_i^w(x) \int_0^1 \left(\int_0^1 w(x)(x-t)_+^{n-i} dx\right) f^{(n+1)}(t)dt\right).$$

Consider, for example, $w(x) = \dfrac{1}{\sqrt{x(1-x)}}$. We have

$$B_n^w(f,x) = \sum_{i=0}^{n} \frac{B_i^w(x)}{i!} \int_0^1 \frac{f^{(i)}(x)}{\sqrt{x(1-x)}} dx = \tag{3.38}$$

$$= \int_0^1 \frac{f(x)}{\sqrt{x(1-x)}} dx + \sum_{i=1}^{n} \frac{B_i^w(x)}{i!} \int_0^1 \frac{f^{(i)}(x)}{\sqrt{x(1-x)}} dx \tag{3.39}$$

and, hence,

$$R_n(f,x) = \frac{1}{n!}\left(\int_0^x (x-t)^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \binom{n}{i} B_i^w(x) \int_0^1 \left(\int_0^1 \frac{(x-t)_+^n - i}{\sqrt{x(1-x)}} dx\right) f^{(n+1)}(t)dt\right).$$

3.3.4 The Euler interpolation polynomial

Let $f \in C^n[0,1]$ and $L_0(f) = \dfrac{f(0)+f(1)}{2}$. Setting $\beta_0 = L_0(1) = 1$, $\beta_j = L_0\left(x^j\right) = \frac{1}{2}$, $j = 0, 1, \ldots$ the Appell polynomials related to the functional L_0 coincide with the Euler polynomials $E_i(x)$. Reminding that $L_0\left(f^{(i)}\right) = \dfrac{f^{(i)}(0)+f^{(i)}(1)}{2}$, $i = 1, \ldots n$, the Euler interpolation polynomial related to the function f is

$$E_n(f,x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)+f^{(i)}(1)}{2i!} E_i(x) = \tag{3.40}$$

$$= \frac{f(0)+f(1)}{2} + \sum_{i=0}^{n} \frac{f^{(i)}(0)+f^{(i)}(1)}{2i!} E_i(x) \tag{3.41}$$

The remainder

Theorem 74 *Let $f \in C^{n+1}[0,1]$, then $\forall x \in [0,1]$*

$$f(x) - E_n(f,x) = \frac{1}{n!} \sum_{i=0}^{n} \frac{E_i(x)}{2} \binom{n}{i} \int_0^1 G_i(x,t) f^{(n+1)}(t)dt \tag{3.42}$$

where

$$G_i(x,t) = \begin{cases} (-1)^{n-i} t^{n-i} & 0 \le t \le x \\ -(1-t)^{n-i} & x \le t \le 1 \end{cases} \tag{3.43}$$

Proof. By Peano's Theorem,

$$R_n(f,x) = f(x) - E_n(f,x) = \int_0^1 f^{(n+1)}(t) K(t) dt$$

with

$$K(t) = \frac{1}{n!} R_{n_x}[(x-t)_+^n]. \tag{3.44}$$

Because of

$$((x-t)_+^n)^{(i)} = \frac{n!(x-t)_+^{n-i}}{(n-i)!} \tag{3.45}$$

we have

$$
\begin{aligned}
K(t) &= \frac{1}{n!}R_{n_x}[(x-t)_+^n] = \\
&= \frac{1}{n!}[(x-t)_+^n - \sum_{i=0}^{n}\frac{E_i(x)}{2i!}[\frac{n!(1-t)_+^{n-i}}{(n-i)!} + \frac{n!(-t)_+^{n-i}}{(n-i)!}]] = \\
&= \frac{1}{n!}[(x-t)_+^n - \sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}[(1-t)_+^{n-i} + (-t)_+^{n-i}]]
\end{aligned}
\tag{3.46}
$$

that is

$$
K(t) = \begin{cases} \dfrac{1}{n!}\left[(x-t)^n - \displaystyle\sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}(1-t)^{n-i}\right] & \text{se} \quad 0 \le t \le x \\ -\dfrac{1}{n!}\displaystyle\sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}(1-t)^{n-i} & \text{se} \quad x \le t \le 1 \end{cases}
\tag{3.47}
$$

Reminding that

$$L_i((x-t)^n) = \frac{1}{2}\binom{n}{i}[(1-t)^{n-i} + (-t)^{n-i}]$$

and, moreover,

$$\sum_{i=0}^{n}E_i(x)L_i((x-t)^n) = E_n((x-t)^n, x) = (x-t)^n$$

we have

$$
\begin{aligned}
(x-t)^n &= \sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}[(1-t)^{n-i} + (-t)^{n-i}] = \\
&= \sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}(1-t)^{n-i} + \sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}(-t)^{n-i}.
\end{aligned}
$$

Replacing the last one in the (3.47) we obtain

$$
K(t) = \begin{cases} \dfrac{1}{n!}\displaystyle\sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}(-1)^{n-i}t^{n-i} & \text{se} \quad 0 \le t \le x \\ -\dfrac{1}{n!}\displaystyle\sum_{i=0}^{n}E_i(x)\frac{1}{2}\binom{n}{i}(1-t)^{n-i} & \text{se} \quad x \le t \le 1 \end{cases}
\tag{3.48}
$$

from which the thesis follows. ■

3.3.5 The Euler generalized interpolation polynomial

Let $f \in C^n[0,1]$ and $L_0^{\overline{w}}(f) = \dfrac{\sum\limits_{k=0}^{n} w_k f\left(\frac{k}{n}\right)}{\sum\limits_{k=0}^{n} w_k}$, $w_k \in \mathbb{R}$, $w_k \geq 0$. Setting $\beta_0^{\overline{w}} = L_0^w(1) = 1$, $\beta_j^{\overline{w}} = L_0^w\left(x^j\right) = \dfrac{\sum\limits_{k=0}^{n} w_k \left(\frac{k}{n}\right)^j}{\sum\limits_{k=0}^{n} w_k}$, $j = 0, 1, ...$ the Appell polynomials related to the functional $L_0^{\overline{w}}$ coincide with the Euler generalized polynomials $E_i^{\overline{w}}(x)$. Reminding that $L_0^{\overline{w}}\left(f^{(i)}\right) = \dfrac{\sum\limits_{k=0}^{n} w_k f^{(i)}\left(\frac{k}{n}\right)}{\sum\limits_{k=0}^{n} w_k}$, $i = 1, ...n$, the Euler generalized interpolation polynomial related to the function f is

$$E_n^{\overline{w}}(f,x) = \sum_{i=0}^{n} \frac{\sum\limits_{k=0}^{n} w_k f^{(i)}\left(\frac{k}{n}\right)}{i! \sum\limits_{k=0}^{n} w_k} E_i^{\overline{w}}(x) \tag{3.49}$$

In particular, for $n = 1$ we have $L_0^w(f) = \dfrac{w_1 f(1) + w_2 f(0)}{w_1 + w_2}$ and $\beta_j^w = L_0^w\left(x^j\right) = \dfrac{w_1}{w_1 + w_2}$, $j = 0, 1, ...$ The Appell polynomials related to the functional L_0^w coincide with the Euler generalized polynomials $E_i^w(x)$. Reminding that $L_0^w\left(f^{(i)}\right) = \dfrac{w_1 f^{(i)}(1) + w_2 f^{(i)}(0)}{w_1 + w_2}$, $i = 1, ...n$, the Euler generalized interpolation polynomial related to the function f is

$$E_n^w(f,x) = \sum_{i=0}^{n} \frac{w_1 f^{(i)}(1) + w_2 f^{(i)}(0)}{(w_1 + w_2)\, i!} E_i^w(x) \tag{3.50}$$

The remainder

Theorem 75 *Let $f \in C^{n+1}[0,1]$, then $\forall x \in [0,1]$,*

$$f(x) - E_n^w(f,x) = \frac{1}{n!} \sum_{i=0}^{n} E_i^w(x) \frac{w_1}{(w_1 + w_2)} \binom{n}{i} \int_0^1 G_i^w(x,t) f^{(n+1)}(t)\, dt \tag{3.51}$$

where

$$G_i^w(x,t) = \begin{cases} (-t)^{n-i} & 0 \le t \le x \\ -(1-t)^{n-i} & x \le t \le 1 \end{cases} \tag{3.52}$$

Proof. By Peano's Theorem,

$$R_n(f,x) = f(x) - E_n^w(f,x) = \int_0^1 f^{(n+1)}(t)K(t)dt \tag{3.53}$$

with

$$K(t) = \frac{1}{n!}R_{n_x}[(x-t)_+^n]. \tag{3.54}$$

Because of

$$((x-t)_+^n)^{(i)} = \frac{n!(x-t)_+^{n-i}}{(n-i)!} \tag{3.55}$$

we have

$$\begin{aligned} K(t) &= \frac{1}{n!}R_{n_x}[(x-t)_+^n] = \\ &= \frac{1}{n!}[(x-t)_+^n - \sum_{i=0}^{n} \frac{E_i^w(x)}{(w_1+w_2)\, i!}[\frac{w_1 n!(1-t)_+^{n-i}}{(n-i)!} + \frac{w_2 n!(-t)_+^{n-i}}{(n-i)!}]] = \\ &= \frac{1}{n!}[(x-t)_+^n - \sum_{i=0}^{n} E_i^w(x)\frac{1}{(w_1+w_2)}\binom{n}{i}[w_1(1-t)_+^{n-i} + w_2(-t)_+^{n-i}]] \end{aligned} \tag{3.56}$$

that is

$$K(t) = \begin{cases} \frac{1}{n!}\left[(x-t)^n - \sum_{i=0}^{n} E_i^w(x)\frac{w_1}{(w_1+w_2)}\binom{n}{i}(1-t)^{n-i}\right] & \text{se} \quad 0 \le t \le x \\ -\frac{1}{n!}\sum_{i=0}^{n} E_i^w(x)\frac{w_1}{(w_1+w_2)}\binom{n}{i}(1-t)^{n-i} & \text{se} \quad x \le t \le 1 \end{cases} \tag{3.57}$$

Reminding that

$$L_i^w((x-t)^n) = \frac{w_1}{(w_1+w_2)}\binom{n}{i}[(1-t)^{n-i} + (-t)^{n-i}]$$

and, moreover,

$$\sum_{i=0}^{n} E_i^w(x) L_i^w((x-t)^n) = E_n^w((x-t)^n, x) = (x-t)^n$$

we have

$$\begin{aligned}(x-t)^n &= \sum_{i=0}^{n} E_i^w(x) \frac{w_1}{(w_1+w_2)} \binom{n}{i} [(1-t)^{n-i} + (-t)^{n-i}] = \\ &= \sum_{i=0}^{n} E_i^w(x) \frac{w_1}{(w_1+w_2)} \binom{n}{i} (1-t)^{n-i} + \sum_{i=0}^{n} E_i^w(x) \frac{w_1}{(w_1+w_2)} \binom{n}{i} (-t)^{n-i}.\end{aligned}$$

Replacing the last one in the (3.57) we obtain

$$K(t) = \begin{cases} \frac{1}{n!} \sum_{i=0}^{n} E_i^w(x) \frac{w_1}{(w_1+w_2)} \binom{n}{i} (-t)^{n-i} & \text{se} \quad 0 \le t \le x \\ -\frac{1}{n!} \sum_{i=0}^{n} E_i^w(x) \frac{w_1}{(w_1+w_2)} \binom{n}{i} (1-t)^{n-i} & \text{se} \quad x \le t \le 1 \end{cases} \tag{3.58}$$

from which the thesis follows. ■

3.3.6 The Hermite normalized interpolation polynomial

Let $f \in C^n(\mathbb{R})$ and $L_0(f) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f(x)\, dx$. Setting $\beta_j = L_0(x^j)$, $j = 0, 1, ...$ the Appell polynomials related to the functional L_0 coincide with the Hermite normalized polynomials $H_i(x)$. Since $L_0(f^{(i)}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f^{(i)}(x)\, dx, \quad i = 1, ...n,$ the Hermite normalized interpolation polynomial related to the function f is

$$H_n(f, x) = \frac{1}{\sqrt{\pi}} \sum_{i=0}^{n} \frac{1}{i!} H_i(x) \int_{-\infty}^{+\infty} e^{-x^2} f^{(i)}(x)\, dx \tag{3.59}$$

The remainder

Theorem 76 *Let $f \in C^{n+1}(\mathbb{R})$, then $\forall x \in \mathbb{R}$,*

$R_n(f,x) = f(x) - B_n^w(f,x) = \int_{-\infty}^{+\infty} f^{(n+1)}(t)K(t)dt =$

$= \frac{1}{n!}\left(\int_{-\infty}^{+\infty}(x-t)_+^n f^{(n+1)}(t)dt - \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\frac{1}{i!}H_i(x)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-x^2}\left((x-t)_+^n\right)^{(i)}dx\right)f^{(n+1)}(t)dt\right) =$

$= \frac{1}{n!}\left(\int_{-\infty}^{+\infty}(x-t)_+^n f^{(n+1)}(t)dt - \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\frac{1}{i!}H_i(x)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-x^2}\frac{n!(x-t)_+^{n-i}}{(n-i)!}dx\right)f^{(n+1)}(t)dt\right) =$

$= \frac{1}{n!}\left(\int_{-\infty}^{x}(x-t)^n f^{(n+1)}(t)dt - \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\binom{n}{i}H_i(x)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-x^2}(x-t)_+^{n-i}dx\right)f^{(n+1)}(t)dt\right).$

3.3.7 The Hermite generalized interpolation polynomial

Let $f \in C^n(\mathbb{R})$, $\alpha > 0$ and $L_0^w(f) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-|x|^\alpha}f(x)\,dx$. Setting $\beta_j^w = L_0^w(x^j)$, $j = 0, 1, ...$ the Appell polynomials related to the functional L_0^w coincide with the Hermite generalized polynomials $H_i^w(x)$. Since $L_0^w\left(f^{(i)}\right) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-|x|^\alpha}f^{(i)}(x)\,dx$, $i = 1, ...n$, the Hermite generalized interpolation polynomial related to the function f is

$$H_n^w(f,x) = \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\frac{1}{i!}H_i^w(x)\int_{-\infty}^{+\infty}e^{-|x|^\alpha}f^{(i)}(x)\,dx \tag{3.60}$$

The remainder

Theorem 77 *Let $f \in C^{n+1}(\mathbb{R})$, then $\forall x \in \mathbb{R}$,*

$R_n(f,x) = f(x) - B_n^w(f,x) = \int_{-\infty}^{+\infty} f^{(n+1)}(t)K(t)dt =$

$= \frac{1}{n!}\left(\int_{-\infty}^{+\infty}(x-t)_+^n f^{(n+1)}(t)dt - \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\frac{1}{i!}H_i^w(x)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-|x|^\alpha}\left((x-t)_+^n\right)^{(i)}dx\right)f^{(n+1)}(t)dt\right) =$

$= \frac{1}{n!}\left(\int_{-\infty}^{+\infty}(x-t)_+^n f^{(n+1)}(t)dt - \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\frac{1}{i!}H_i^w(x)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-|x|^\alpha}\frac{n!(x-t)_+^{n-i}}{(n-i)!}dx\right)f^{(n+1)}(t)dt\right) =$

$= \frac{1}{n!}\left(\int_{-\infty}^{x}(x-t)^n f^{(n+1)}(t)dt - \frac{1}{\sqrt{\pi}}\sum_{i=0}^{n}\binom{n}{i}H_i^w(x)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-|x|^\alpha}(x-t)_+^{n-i}dx\right)f^{(n+1)}(t)dt\right).$

3.3.8 The Laguerre generalized interpolation polynomial

Let $f \in C^n(\mathbb{R}^+)$, $s > 0$ and $L_0^w(f) = \int_0^{+\infty} e^{-sx} f(x)\,dx$. Setting $\beta_j^w = L_0^w\left(x^j\right)$, $j = 0, 1, ...$ the Appell polynomials related to the functional L_0^w coincide with the Laguerre generalized polynomials $La_i^w(x)$. Since $L_0^w\left(f^{(i)}\right) = \int_0^{+\infty} e^{-sx} f^{(i)}(x)\,dx$, $i = 1, ...n$, the Laguerre generalized interpolation polynomial related to the function f is

$$La_n^w(f,x) = \sum_{i=0}^{n} \frac{1}{i!} La_i^w(x) \int_0^{+\infty} e^{-sx} f^{(i)}(x)\,dx \tag{3.61}$$

The remainder

Theorem 78 *Let $f \in C^{n+1}(\mathbb{R}^+)$, then $\forall x \in [0, +\infty)$,*

$$R_n(f,x) = f(x) - B_n^w(f,x) = \int_0^{+\infty} f^{(n+1)}(t) K(t)dt =$$

$$= \frac{1}{n!}\left(\int_0^{+\infty}(x-t)_+^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \frac{1}{i!} La_i^w(x) \int_0^{+\infty}\left(\int_0^{+\infty} e^{-sx}\left((x-t)_+^n\right)^{(i)} dx\right) f^{(n+1)}(t)dt\right) =$$

$$= \frac{1}{n!}\left(\int_0^{+\infty}(x-t)_+^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \frac{1}{i!} La_i^w(x) \int_0^{+\infty}\left(\int_0^{+\infty} e^{-sx}\frac{n!(x-t)_+^{n-i}}{(n-i)!} dx\right) f^{(n+1)}(t)dt\right) =$$

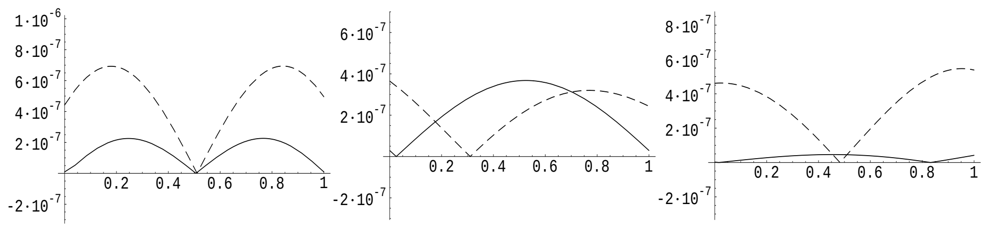
$$= \frac{1}{n!}\left(\int_0^{x}(x-t)^n f^{(n+1)}(t)dt - \sum_{i=0}^{n} \binom{n}{i} La_i^w(x) \int_0^{+\infty}\left(\int_0^{+\infty} e^{-sx}(x-t)_+^{n-i} dx\right) f^{(n+1)}(t)dt\right).$$

3.4 Numerical examples

Now we consider certain interpolation test problems and report the numerical results obtained by using an ad hoc "Mathematica" code. In particular, we compare the error committed approximating a given function with the interpolation Appell polynomials studied in the previous section.

Example 79 *Let us consider the function*

$$f(x) = \sqrt{e^{\frac{x}{2}}}, \quad x \in [0, 1]. \tag{3.62}$$



(a) Bernoulli interpolation (b) Euler interpolation (c) Hermite interpolation

Figure 3.1: Interpolation error of function (3.62).

The error in the interpolation of the function (3.62) is plotted in Figure 3.1.

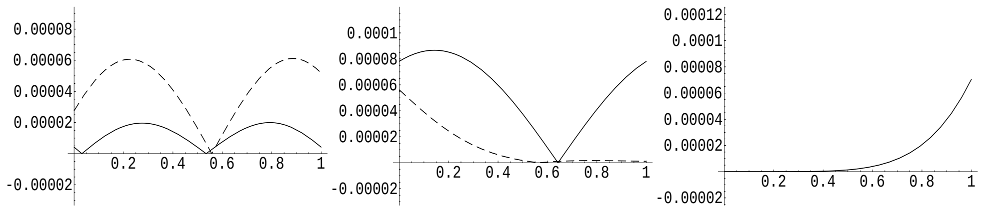
In the case of Bernoulli interpolation (solid line) and Bernoulli generalized interpolation with $\alpha = \beta = -\frac{1}{2}$ *(dashed line) we obtain an error less than or equal to the requested tolerance* $\epsilon = 10^{-6}$ *for* $n = 4$*. To obtain an error of the same order with Euler interpolation (solid line) and Euler generalized interpolation with* $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{3}$ *(dashed line) we need an interpolation polynomial of degree* $n = 5$*. In the case of Hermite interpolation (solid line) and Hermite generalized interpolation with* $\alpha = 3$ *(dashed line) we obtain an error less than or equal to the same requested tolerance, respectively, for* $n = 6$ *and* $n = 5$*.*

Example 80 *Let be*

$$f(x) = \ln(x^2 + 10), \quad x \in [0, 1]. \tag{3.63}$$

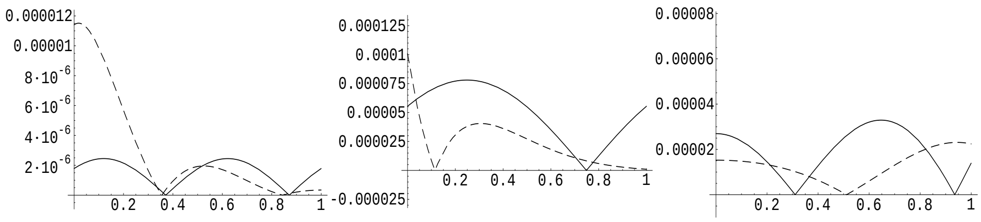
The error in the interpolation of the function (3.63) is plotted in Figure 3.2.

In the case of Bernoulli interpolation (solid line) and Bernoulli generalized interpolation with $\alpha = \beta = -\frac{1}{2}$ *(dashed line) we obtain an error less than or equal to the requested tolerance* $\epsilon = 10^{-4}$ *for* $n = 4$*. To obtain an error of the same order with Euler interpolation (solid line) and Euler generalized interpolation with* $w_1 = 10$, $w_2 = \frac{1}{5}$ *(dashed*



(a) Bernoulli interpolation (b) Euler interpolation (c) Laguerre interpolation

Figure 3.2: Interpolation error of function (3.63).



(a) Bernoulli interpolation (b) Euler interpolation (c) Hermite interpolation

Figure 3.3: Interpolation error of function (3.64).

line) we need an interpolation polynomial, respectively, of degree $n = 7$ *and* $n = 5$*. In the case of Laguerre generalized interpolation with* $s = 3$ *(solid line) we obtain an error less than or equal to the same requested tolerance for* $n = 6$*.*

Example 81 *Let be*

$$f(x) = 10\cos(x) + \frac{\sin^2(x)}{10}, \quad x \in [0,1]. \tag{3.64}$$

The error in the interpolation of the function (3.64) is plotted in Figure 3.3.

In the case of Bernoulli interpolation (solid line) and Bernoulli generalized interpolation with $\alpha = -\frac{1}{2}, \beta = 1$ *(dashed line) we obtain an error less than or equal to the requested tolerance* $\epsilon = 10^{-4}$ *for* $n = 6$*. To obtain an error of the same order with Euler* *interpolation (solid line) and Euler generalized interpolation with* $w_1 = 1$, $w_2 = \frac{1}{100}$ *(dashed line) we need an interpolation polynomial, respectively, of degree* $n = 13$ *and* $n = 6$*. In the case of Hermite interpolation (solid line) and Hermite generalized interpolation with* $\alpha = 4$ *(dashed line) we obtain an error less than or equal to the same requested tolerance, respectively, for* $n = 10$ *and* $n = 6$.

Chapter 4

Bernoulli, Lidstone and Fourier series for real entire functions of exponential type

In this chapter we consider the series expansion of the interpolation Bernoulli polynomial in order to investigate its convergence. In this analysis we also take into account the Lidstone ([6]) and Lidstone of second type ([13]) series expansions since they have interesting analogies with Bernoulli series.

4.1 Introduction

A problem of fundamental interest in classical analysis is to study the representability of analytic function $f(z)$ as a series $\sum c_n p_n(z)$ where $\{p_n\}$ is a prescribed sequence of functions, and the connections between the function f and the coefficients $\{c_n\}$. There is a wide literature on the subject: see [6], [11], [25], [26], [27], and references therein. Now we will prove that the Bernoulli, Lidstone ([6]) and Lidstone of second type ([13]) series coincide with the Fourier series, for real entire functions of exponential type.

4.2 Preliminary and known results

The Bernoulli polynomials $\{B_n(x)\}$ can be expanded in the Fourier series in $[0,1]$ (2.63, 2.65):

$$B_{2n-1}(x) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{k^{2n-1}}, \quad n > 1, \tag{4.1}$$

$$B_{2n}(x) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{k^{2n}}, \quad n > 1. \tag{4.2}$$

As we shall always deal with functions defined in $[0,1]$, in all the sequel we will simply say "uniformly" instead of "uniformly in $[0,1]$".

The two following theorems can be found in ([6, p. 29]) and in [6, p. 14] respectively. These theorems involve entire functions f in $\mathbb{C}$ of exponential type τ, i. e.:

$$|f(z)| \leq Ce^{\tau|z|}, \quad \text{for each } z \in \mathbb{C} \tag{4.3}$$

where $C > 0$ and $\tau > 0$ are two constants.

Theorem 82 (Bernoulli series) *Any entire function of exponential type less than 2π has the convergent expansion*

$$f(x) = \int_0^1 f(x)dx + \sum_{n=1}^{\infty} \left[f^{(n-1)}(1) - f^{(n-1)}(0) \right] \frac{B_n(x)}{n!}, \quad \forall x \in [0,1]. \tag{4.4}$$

Parallel there are the Lidstone polynomials introduced in ([38]) by

$$\begin{cases} \Lambda_0(x) = x, \\ \Lambda_n(x) = \int_0^1 G_n(x,t)\, t dt, \quad n \geq 1, \end{cases} \tag{4.5}$$

where

$$G_1(x,t) = \begin{cases} (x-1)t, & 0 \leq t < x \leq 1, \\ (t-1)x, & 0 \leq x < t \leq 1, \end{cases} \tag{4.6}$$

$$G_n(x,t) = \int_0^1 G_1(x,t)\, G_{n-1}(y,t)\, dy, \quad n = 2, ... \tag{4.7}$$

Other definitions are possible ([19]) but here we prefer the expansion ([52], [51])

$$\frac{\sinh xt}{\sinh x} = \sum_{n=1}^{\infty} \Lambda_n(x) \frac{t^{2n}}{(2n)!}, \tag{4.8}$$

where the series converges uniformly. There is also the Fourier expansion ([52], [51])

$$\Lambda_n(x) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi x, \quad n \geq 1, \tag{4.9}$$

uniformly.

Theorem 83 (Lidstone first type series) *Any entire function $f(x)$ of exponential type less than π has a convergent Lidstone representation*

$$f(x) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-x) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(x), \quad \forall x \in [0,1]. \tag{4.10}$$

Recently a new class of polynomials, called Lidstone polynomials of second type, was introduced in ([13]) by means of the recursion formulae:

$$\begin{cases} \upsilon_0(x) = 1, \\ \upsilon_n''(x) = \upsilon_{n-1}(x), \quad n \geq 1, \\ \upsilon_n'(1) = \upsilon_n'(0) = 0, \quad n \geq 1. \end{cases} \tag{4.11}$$

For these polynomials it has been proved in ([13]) that

$$\upsilon_n(x) = \frac{2^{2n}}{(2n)!} B_{2n}\left(\frac{1+x}{2}\right), \tag{4.12}$$

then applying (4.2) we get the Fourier expansion

$$\upsilon_n(x) = (-1)^{n-1} \frac{2}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} \cos \pi k x. \tag{4.13}$$

Using the same method as in ([13]) it is possible to prove the following theorem:

Theorem 84 (Lidstone second type series) *Any real entire function $f(x)$ of exponential type less than π has the absolutely and uniformly convergent expansion*

$$f(x) = \int_0^1 f(x)dx + \sum_{n=1}^{\infty} \left[f^{(2n-1)}(1)\upsilon_n(x) - f^{(2n-1)}(0)\upsilon_n(1-x) \right]. \tag{4.14}$$

4.3 Equivalence of Bernoulli and Lidstone series with Fourier series

In order to make our results independent of the theory of entire functions of exponential type we may introduce the class of functions denoted by $M(h)$, $h > 0$, as follows:

Definition 85 *A real entire function $f(x)$ belongs to the class $M(h)$, $h > 0$ if there exists a positive number $p < h$ such that*

$$f^{(n)}(0) = O(p^n), \quad n \to \infty. \tag{4.15}$$

Concerning this class of functions we have the following result:

Lemma 86 *([51]) If $f(x)$ belongs to $M(h)$ then there exists a positive number $p < h$ such that*

$$f^{(n)}(x) = O(p^n), \quad \forall x \in [0, 1]. \tag{4.16}$$

We note explicitly that if $f(x)$ belongs to the class $M(h)$ then it is of exponential type less than h.

Then we have the following theorem:

Theorem 87 *If $f(x)$ belongs to $M(2\pi)$ we have*

$$f(x)=\int_0^1 f(x)dx+\sum_{k=1}^{\infty}a_k\cos 2\pi kx+\sum_{k=1}^{\infty}b_k\sin 2\pi kx,\quad \forall x\in[0,1], \tag{4.17}$$

where

$$a_k=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2k\pi)^{2n}}\left[f^{(2n-1)}(1)-f^{(2n-1)}(0)\right], \tag{4.18}$$

$$b_k=2\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2k\pi)^{2n-1}}\left[f^{(2n-2)}(1)-f^{(2n-2)}(0)\right]. \tag{4.19}$$

Proof. We set

$$\Delta f^{(k)}=f^{(k)}(1)-f^{(k)}(0)\,,\quad k=0,1,...$$

If $f(x)$ belongs to $M(2\pi)$ by theorem 1 we have

$$\begin{aligned} f(x) &= \int_0^1 f(x)dx+\sum_{n=1}^{\infty}B_n(x)\frac{\Delta f^{(n-1)}}{n!}= \\ &= \int_0^1 f(x)dx+\sum_{n=1}^{\infty}B_{2n}(x)\frac{\Delta f^{(2n-1)}}{(2n)!}+\sum_{n=1}^{\infty}B_{2n-1}(x)\frac{\Delta f^{(2n-2)}}{(2n-1)!},\quad \forall x\in[0,1]\,. \end{aligned}$$

Then, replacing into the last one the Bernoulli polynomials with their Fourier expansion (4.1 and 4.2), after a few calculations and exchanging the order of summation we obtain (4.17), (4.18), and (4.19). ■

It is known that the Fourier series of $f(x)$ in $[0,1]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos 2\pi kx+\sum_{k=1}^{\infty}b_k\sin 2\pi kx \tag{4.20}$$

where

$$a_0 = 2\int_0^1 f(x)dx, \tag{4.21}$$

$$\begin{cases} a_k = 2\int_0^1 f(x)\cos 2\pi kx dx, \\ b_k = 2\int_0^1 f(x)\sin 2\pi kx dx, \end{cases} \quad k \geq 1. \tag{4.22}$$

It follows, from the uniqueness of representation by trigonometric series ([53, p. 272]), that (4.17) must be the Fourier series of $f(x)$; therefore we have, also, the equalities

$$\begin{array}{l} \int_0^1 f(x)\cos 2\pi kx dx = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2k\pi)^{2n}} \left[f^{(2n-1)}(1) - f^{(2n-1)}(0)\right], \\ \int_0^1 f(x)\sin 2\pi kx dx = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n}}{(2k\pi)^{2n-1}} \left[f^{(2n-2)}(1) - f^{(2n-2)}(0)\right], \end{array} \quad k \geq 1, \tag{4.23}$$

from which we obtain the equivalence between the Fourier series and Bernoulli series for function of the class $M(2\pi)$.

For Lidstone series we have, instead, the following result:

Theorem 88 *If $f(x)$ belongs to $M(\pi)$ and $f(1) = 0 = f(0)$ then*

$$f(x) = \sum_{k=1}^{\infty} b_k \sin k\pi x, \quad \forall x \in [0,1] \tag{4.24}$$

with

$$b_k = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(k\pi)^{2n+1}} \left[f^{(2n)}(0) + (-1)^{k+1} f^{(2n)}(1)\right], \quad k = 1, 2, ... \tag{4.25}$$

Proof. If $f(x)$ belongs to M(π) it is known (4.10) that

$$f(x) = \sum_{n=0}^{\infty} \left[f^{(2n)}(0)\Lambda_n(1-x) + f^{(2n)}(1)\Lambda_n(x)\right]$$

and that the series converges uniformly. After some calculations and applying (4.9), we have (4.24) with (4.25). ■

From the uniqueness of representation by trigonometric series, (4.24) must be the Fourier series in only sin of $f(x)$ and therefore we have also

$$\int_0^1 f(x)\sin \pi kx dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(k\pi)^{2n+1}} \left[f^{(2n)}(0) + (-1)^{k+1} f^{(2n)}(1) \right]. \tag{4.26}$$

Similarly we have

Theorem 89 *If $f(x)$ belongs to $M(\pi)$ then*

$$f(x) = \sum_{k=0}^{\infty} a_k \cos k\pi x, \quad \forall x \in [0,1] \tag{4.27}$$

with

$$a_0 = \int_0^1 f(x)dx, \tag{4.28}$$

$$a_k = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(k\pi)^{2n}} \left[f^{(2n-1)}(1)\,(-1)^k - f^{(2n-1)}(0) \right], \quad k = 1, 2, ... \tag{4.29}$$

Proof. If $f(x)$ belongs to $M(\pi)$ by theorem 3 we have

$$f(x) = \int_0^1 f(x)dx + \sum_{n=1}^{\infty} \left[f^{(2n-1)}(1)\upsilon_n(x) - f^{(2n-1)}(0)\upsilon_n(1-x) \right], \tag{4.30}$$

uniformly. Then, replacing into the last one the Lidstone polynomials of second type with their Fourier expansion (4.13), after a few calculations and exchanging the order of summation we obtain (4.27) with (4.28) and (4.29). ∎

It follows, from the uniqueness of representation by trigonometric series, that (4.24) must be the Fourier series of $f(x)$ in only *cos* and therefore

$$\int_0^1 f(x)\cos k\pi x dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(k\pi)^{2n}} \left[f^{(2n-1)}(1)\,(-1)^k - f^{(2n-1)}(0) \right]. \tag{4.31}$$

Remark 90 *We note explicitly that we can prove the theorems* 1, 2, 3 *by integration by part of the coefficients of the Fourier series and applying the Fourier series of Bernoulli, Lidstone and Lidstone of second type polynomials. That gives a proof by elementary tools of convergence of Bernoulli and Lidstone series and, to the authors' knowledge, a more elementary proof is not available in the literature. In this form all this is presented in [17].*

Chapter 5

A Birkhoff interpolation problem

In this chapter we introduce a new interpolation problem of Birkhoff kind and we provide the explicit solution of it with estimation error. Moreover we study the convergence of the method. This problem is the basis for a new general collocation method for a BVP that we will expose in chapter 6. Furthermore, it allows us to define a new class of quadrature formulae, called of Birkhoff and Gauss-Birkhoff type.

5.1 Introduction

Interpolation theory is concerned with reconstructing functions on the basis of certain functional information assumed known. Now if f is a real function defined in the interval $[-1,1]$ and twice differentiable on the distinct and fixed points $x_i \in (-1,1)$ $i = 1,...,n-1,\ n>1$ we consider the functional information

$$f(-1),\ f''(x_i)\ i=1,...,n-1,\ f(1) \tag{5.1}$$

and we will reconstruct the function f.

The interpolatory problem with data (5.1) is a Birkhoff interpolation problem, in fact, if

$$-1 = x_0 < x_1 < ... < x_{n-1} < x_n = 1, \tag{5.2}$$

we search the polynomial of degree $\leq n$, $P_n(x)$, such that

$$P_n(-1) = f(-1),\ P_n''(x_i) = f''(x_i)\ i = 1, ..., n-1,\ P_n(1) = f(1) \tag{5.3}$$

and this is the Birkhoff interpolation problem with incidence matrix ([40]) $E = \{e_{i,j}\}$ $i, j = 0, ..., n$ defined by

$$e_{0,0} = e_{n,0} = 1 = e_{i,2}, \quad , i = 1, ..., n-1 \tag{5.4}$$

and $e_{i,j} = 0$ for all the other indexes i, j.

It is possible show that the incidence matrix E defined by (5.4) is regular and therefore the interpolatory problem (5.3) has a solution for all values of $f(\pm 1)$, $f''(x_i)$ $i = 1, ..., n-1$, but in the following we give a constructive and direct study for (5.3).

5.2 The solution

The interpolation problem (5.3) can be generalized, in fact:

let $\omega_i \in \mathbb{R}$ $\ i = 0, ..., n$ and

$$-1 = x_0 < x_1 < ... < x_{n-1} < x_n = 1, \tag{5.5}$$

we search a polynomial $P_n(x)$ of degree $\leq n$ such that

$$P_n(x_0) = \omega_0,\ P_n''(x_i) = \omega_i \quad i = 1, ..., n-1,\ P_n(x_n) = \omega_n, \tag{5.6}$$

then we have the following

Theorem 91 ([14]) *Let $l_i(x)$ be the fundamental polynomials of Lagrange calculated on the $n-1$ points x_i $i = 1, ..., n-1$ and*

$$p_{n,i}(x) = \int_{-1}^{1} G(x,t) l_i(t) dt \quad i = 1, ..., n-1 \tag{5.7}$$

where

$$G(x,t) = \begin{cases} \frac{(t+1)(x-1)}{2} & t \leq x \\ \frac{(x+1)(t-1)}{2} & x \leq t \end{cases} \tag{5.8}$$

then the polynomial

$$P_n(x) = \frac{\omega_n + \omega_0}{2} + \frac{\omega_n - \omega_0}{2} x + \sum_{i=1}^{n-1} p_{n,i}(x) \omega_i \tag{5.9}$$

is the unique polynomial of degree $\leq n$ which satisfies the interpolation problem (5.6).

Proof. Observing that $p_{n,i}(x)$ is the solution of the boundary value problem

$$\begin{cases} p''_{n,i}(x) = l_i(x) \\ p_{n,i}(\pm 1) = 0 \end{cases} \quad i = 1, ..., n-1, \tag{5.10}$$

the polynomial (5.9) satisfies the interpolatory conditions (5.6). For the proof of the uniqueness we suppose that there is a polynomial $\overline{P}_n$ of degree $\leq n$ such that $\overline{P}_n \neq P_n$ and satisfies the interpolation conditions i. e.:

$$\overline{P}_n(x_0) = \omega_0, \ \overline{P}''_n(x_i) = \omega_i \quad i = 1, ..., n-1, \ \overline{P}_n(x_n) = \omega_n. \tag{5.11}$$

Let consider the function

$$\phi_n(x) = \overline{P}_n(x) - P_n(x).$$

We know that

$$\phi''_n(x_i) = 0, \qquad i = 1, .., n-1;$$

$\phi_n''(x)$ is a polynomial of degree $\leq n-2$ and it vanishes at $n-1$ distinct points, therefore

$$\phi_n''(x) \equiv 0 \tag{5.12}$$

from which we have

$$\phi_n'(x) \equiv c, \quad c \in \mathbb{R}. \tag{5.13}$$

Now, we can observe that

$$\phi_n(\pm 1) = 0$$

and by the Rolle's theorem there is a point $\xi \in (-1, 1)$ for which

$$\phi_n'(\xi) = 0.$$

From the (5.13) and the last one we obtain

$$\phi_n^{'}(x) \equiv 0.$$

Therefore we have

$$\phi_n(x) = c,$$

and from $\phi_n(\pm 1) = 0$, we obtain

$$\phi_n(x) \equiv 0.$$

That is in contrast with the hypothesis $\overline{P}_n \neq P_n$. ■

Remark 92 *Let $Q_{n-2}(x)$ be the interpolation polynomial*

$$Q_{n-2}(x) = \sum_{i=1}^{n-1} l_i(x)\omega_i, \tag{5.14}$$

the polynomial (5.9) can be written in the form

$$P_n(x) = \frac{\omega_n + \omega_0}{2} + \frac{\omega_n - \omega_0}{2}x + \int_{-1}^{1} G(x,t)Q_{n-2}(t)dt. \tag{5.15}$$

We can give an alternative representation of polynomials basis $p_{n,i}(x)$, in fact the following theorem holds:

Theorem 93 *With the notation already introduced we have*

$$p_{n,i}(x) = \frac{(-1)^i}{G}\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 1 & -1 & 1 & -1 & \cdots & (-1)^n \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 2 & 6x_1 & \cdots & n(n-1)(x_1)^{n-2} \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 2 & 6x_{i-1} & \cdots & n(n-1)\left(x_{i-1}\right)^{n-2} \\ 0 & 0 & 2 & 6x_{i+1} & \cdots & n(n-1)\left(x_{i+1}\right)^{n-2} \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 2 & 6x_{n-1} & \cdots & n(n-1)\left(x_{n-1}\right)^{n-2} \end{vmatrix}, \quad i = 1, ..., n-1 \tag{5.16}$$

with

$$G = 2 * n!(n-1)! \prod_{\substack{i=1 \\ i>j}}^{n-1} (x_i - x_j). \tag{5.17}$$

Proof. Let be $L_0, L_1, ..., L_n$, $n+1$ linear functionals on the space P_n of polynomials of degree $\leq n$, defined as follows:

$$L_0(p) = p(x_0), \quad L_i(p) = p''(x_i) \;\; i = 1, ..., n-1, \quad L_n(p) = p(x_n). \tag{5.18}$$

The functionals $L_0, L_1, ..., L_n$ are linear independent in P_n^* iff $G = \left|L_j(x^i)\right| \neq 0,\ i, j = 0, 1, ..., n$ [21, p. 26].

From (5.18) we have

$$G = \begin{vmatrix} 1 & -1 & 1 & -1 & ... & (-1)^{n-1} & (-1)^n \\ 1 & 1 & 1 & 1 & ... & 1 & 1 \\ 0 & 0 & 2 & 6x_1 & ... & (n-1)(n-2)\,(x_1)^{n-3} & n(n-1)\,(x_1)^{n-2} \\ \vdots & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ 0 & 0 & 2 & 6x_{n-2} & & (n-1)(n-2)\,(x_{n-2})^{n-3} & n(n-1)\,(x_{n-2})^{n-2} \\ 0 & 0 & 2 & 6x_{n-1} & ... & (n-1)(n-2)\,(x_{n-1})^{n-3} & n(n-1)\,(x_{n-1})^{n-2} \end{vmatrix} =$$

$$= 2 * n!(n-1)! \prod_{\substack{i=1 \\ i>j}}^{n-1} (x_i - x_j) \neq 0.$$

Let be $\omega_0, \omega_1, ..., \omega_{n-1}, \omega_n \in \mathbb{R}$, from a known theorem of representation [21, p. 27], with the hypothesis $G \neq 0$, the polynomial

$$P_n(x) = \tfrac{\omega_n+\omega_0}{2} + \tfrac{\omega_n-\omega_0}{2}x + \sum_{i=1}^{n-1} \omega_i \tfrac{(-1)^i}{G} \begin{vmatrix} 1 & x^1 & x^2 & x^3 & ... & x^n \\ 1 & -1 & 1 & -1 & ... & (-1)^n \\ 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 2 & 6x_1 & ... & n(n-1)\,(x_1)^{n-2} \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 2 & 6x_{i-1} & ... & n(n-1)\,(x_{i-1})^{n-2} \\ 0 & 0 & 2 & 6x_{i+1} & ... & n(n-1)\,(x_{i+1})^{n-2} \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 2 & 6x_{n-1} & ... & n(n-1)\,(x_{n-1})^{n-2} \end{vmatrix} \tag{5.19}$$

is the unique element of P_n that satisfies the interpolation conditions

$$L_0(P_n) = \omega_0,\ L_i(P_n) = \omega_i \quad i = 1, ..., n-1,\ L_n(P_n) = \omega_n. \tag{5.20}$$

From the uniqueness of the solution of the problem (5.20) that is, from the definition (5.18), equivalent to the problem (5.6), we have the result. ■

5.3 The remainder

We suppose, now, that

$$\omega_0 = f(-1),\ \omega_i = f''(x_i) \quad i = 1, ..., n-1,\ \omega_n = f(1) \tag{5.21}$$

for a real function f defined and twice differentiable in $[1, 1]$. Then the polynomial

$$P_n[f, x] = \frac{f(1) + f(-1)}{2} + \frac{f(1) - f(-1)}{2} x + \sum_{i=1}^{n-1} p_{n,i}(x) f''(x_i) \tag{5.22}$$

is the unique polynomial of degree $\leq n$ that satisfies the interpolation conditions

$$P_n[f, -1] = f(-1),\ P_n''[f, x_i] = f''(x_i) \quad i = 1, ..., n-1,\ P_n[f, 1] = f(1). \tag{5.23}$$

Therefore for $z \in (-1, 1)$ we are interested to the remainder

$$R_n[f, z] = f(z) - P_n[f, z]. \tag{5.24}$$

We have the following

Theorem 94 *Let $Q_{n-2}[f'', x]$ be the interpolation polynomial of $f''(x)$ in the fixed points $x_i \ \ i = 1, ..., n-1$ and $T_n[f'', z]$ the remainder in a fixed point z, then we have*

$$R_n[f, z] = \int_{-1}^{1} G(z, t) T_n[f'', t] dt \tag{5.25}$$

where $G(x, t)$ is defined in (5.8).

Proof. The following identity holds:

$$f(x) = \frac{f(1) + f(-1)}{2} + \frac{f(1) - f(-1)}{2}x + \int_{-1}^{1} G(x,t)f''(t)dt \tag{5.26}$$

moreover, from definition we have that

$$Q_{n-2}[f'', x] = \sum_{i=1}^{n-1} l_i(x)f''(x_i) \tag{5.27}$$

and

$$f''(x) = Q_{n-2}[f'', x] + T_n[f'', x]. \tag{5.28}$$

Putting that in (5.26) we have the result. ■

Remark 95 *Let $f(x) \in C^2[-1,1]$ and suppose that $f^{(n+1)}(x)$ exists at each point of $(-1,1)$. In the hypothesis (5.2) we have*

$$R_n[f, z] = \int_{-1}^{1} G(z,t)\frac{(t - x_1)...(t - x_{n-1})}{(n-1)!} f^{(n+1)}(\xi_t)\, dt. \tag{5.29}$$

Proof. It is known [21, p. 6] that

$$T_n[f'', z] = \frac{(z - x_1)...(z - x_{n-1})}{(n-1)!} f^{(n+1)}(\xi_z) \tag{5.30}$$

where $min(z, x_1, ..., x_{n-1}) < \xi_z < \max(z, x_1, ..., x_{n-1})$, i.e. the point ξ_z depends upon $z, x_1, ..., x_{n-1}$ and f''. The result follows from (5.25) and (5.30). ■

Remark 96 (Exactness) *The interpolation polynomial (5.22) has degree of exactness n, i.e:*

$$R_n[x^k, x] = 0 \quad k = 0, ..., n. \tag{5.31}$$

Proof. It follows from (5.29). ■

Remark 97 *With the notation already introduced the following identity holds:*

$$R_n[f,z] = \frac{1}{(n-2)!}\int_{-1}^{1} G(z,t)\left[\sum_{i=1}^{n-1} l_i(t)\int_{x_i}^{t}(x_i-s)^{n-2}f^{(n+1)}(s)ds\right]dt. \tag{5.32}$$

Proof. It is known [21, p. 72] that

$$T_n[f'',z] = \frac{1}{(n-2)!}\sum_{i=1}^{n-1} l_i(z)\int_{x_i}^{z}(x_i-s)^{n-2}f^{(n+1)}(s)ds. \tag{5.33}$$

The result follows from (5.25) and (5.33). ■

Theorem 98 (Bounds) *Let* $\left|f^{(n+1)}(x)\right| \leq M_n$*, then the following bound for the remainder holds:*

$$|R_n[f,x]| \leq \frac{M_n}{2(n-1)!}\int_{-1}^{1}|(t-x_1)...(t-x_{n-1})|\,dt \leq \frac{2^{n-1}M_n}{(n-1)!}. \tag{5.34}$$

Proof. From (5.29) we have

$$\begin{aligned}
|R_n[f,x]| &= \left|\int_{-1}^{1} G(x,t)\frac{(t-x_1)...(t-x_{n-1})}{(n-1)!}f^{(n+1)}(\xi_t)\,dt\right| \leq \\
&\leq \int_{-1}^{1}\left|G(x,t)\frac{(t-x_1)...(t-x_{n-1})}{(n-1)!}f^{(n+1)}(\xi_t)\right|dt = \\
&= \int_{-1}^{1}|G(x,t)|\left|\frac{(t-x_1)...(t-x_{n-1})}{(n-1)!}\right|\left|f^{(n+1)}(\xi_t)\right|dt \leq \\
&\leq \frac{M_n}{2(n-1)!}\int_{-1}^{1}|(t-x_1)...(t-x_{n-1})|\,dt \leq \frac{2^{n-1}M_n}{(n-1)!}.
\end{aligned}$$

■

Remark 99 *For the convergence of* $P_n[f,x]$ *to* $f(x)$*, with standard techniques, analogous results to that ones obtained for Lagrange interpolation can be proven. In particular, the convergence in mean holds.*

If the interpolation nodes are zeros of orthogonal polynomials with respect to the weight function $w(t)$ on the interval (a, b) we have that the relation

$$\lim_{n\to\infty} \int_a^b w(t) \{P_n[f,t] - f(t)\}^2 \, dt \to 0$$

holds for every function $f \in C^2[a, b]$.

5.4 Examples

In order to calculate the polynomial (5.9) or (5.22) we can use the determinantal form (5.16) of the polynomials $p_{n,i}(x)$ and generate them by an ad hoc "Mathematica" code.

Remark 100 *In addition to the definition (5.7) and (5.16) we can express the polynomials $p_{n,i}(x)$ in the form*

$$p_{n,i}(x) = \sum_{j=0}^{n} c_j^{n,i} x^{n-j} \tag{5.35}$$

The coefficients $c_j^{n,i}$ can be obtained from (1.6) by double integration of the $l_i(x)$ and the coefficients $a_j^{n,i}$ of the polynomials $l_i(x)$ can be calculated with recursion using the known values of the sums $S_k^{n,i} = x_1^k + ... + x_{i-1}^k + x_{i+1}^k + ... + x_{n-1}^k$, $k = 1, ..., n-2$. The relation between the coefficients $a_j^{n,i}$ and the sums $S_k^{n,i}$ are known as Newton's Identities ([50]). With the notation until now introduced we have

$$c_j^{n,i} = A^{n,i} \frac{a_j^{n,i}}{(n-j)(n-1-j)}, \qquad j = 0, ..., n-2, \tag{5.36}$$

$$c_{n-1}^{n,i} = -\frac{1}{2} \sum_{j=0}^{n-2} c_j^{n,i} [1 - (-1)^{n-j}], \tag{5.37}$$

$$c_n^{n,i} = -\frac{1}{2} \sum_{j=0}^{n-2} c_j^{n,i} [1 + (-1)^{n-j}]. \tag{5.38}$$

where

$$A^{n,i} = \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n-1} (x_i - x_j)} \tag{5.39}$$

and

$$a_0^{n,i} = 1, \tag{5.40}$$

$$a_j^{n,i} = -\frac{S_j + \sum\limits_{k=1}^{j-1} S_{j-k} a_k^{n,i}}{j}, \quad j = 1, ..., n-2. \tag{5.41}$$

This algorithm can be implemented in a symbolic language. For example, a "Mathematica" code is reported in Appendix A.

Now, we consider equidistant nodes and Chebyshev of second kind nodes.

5.4.1 Equidistant nodes

Let us consider $n-1$ equidistant nodes in the interval $[-1, 1]$ defined as follows

$$x_i = -1 + \frac{2i}{n},\ i = 1, .., n-1. \tag{5.42}$$

The polynomials $p_{n,i}(x)$, $i = 1, ..., n-1$, for $n = 2, ..., 8$ on the set of nodes (5.42) can be calculated using the "Mathematica" code reported in Appendix A.

For this set of nodes the following bound for the remainder holds:

Theorem 101 *Let* $\left|f^{(n+1)}(x)\right| \leq M_n$, *then*

$$|R_n[f, x]| \leq M_n \left(\frac{2}{n}\right)^{n-1}. \tag{5.43}$$

Proof. Let define

$$U_{n-1}(x) = (x - x_1)...(x - x_{n-1}), \tag{5.44}$$

it's known ([44]) that

$$|U_{n-1}(x)| \le \left(\frac{2}{n}\right)^{n-1} (n-1)!, \quad \text{for} \quad -1 \le x \le 1. \tag{5.45}$$

Therefore, the result follows from the last one and theorem 98. ■

5.4.2 Chebyshev of second kind nodes

Let us consider the zeros of the Chebyshev polynomials of the second kind of degree $n-1$ in the interval $[-1,1]$ defined as follows

$$x_i = \cos(\frac{i}{n}\pi), \quad i = 1,..,n-1 \tag{5.46}$$

For this set of nodes the polynomials $p_{n,i}(x)$ defined in (5.7) or equivalently in (5.16) can be written in the form ([18])

$$p_{n,i}(x) = \frac{1}{n}\sin\frac{\pi i}{n}\left[\sum_{k=2}^{n-1}\frac{G_k(x)}{k}\sin\frac{k\pi i}{n} + (x^2-1)\sin\frac{\pi i}{n}\right], \tag{5.47}$$

where

$$G_k(x) = \frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1} + \begin{cases} \frac{2x}{k^2-1} & \text{even } k \\ \frac{2}{k^2-1} & \text{odd } k \end{cases} \tag{5.48}$$

and $T_k(x)$ are the Chebyshev polynomials of first kind of degree k.

In fact, as proven in [18], it results that

$$p''_{n,i}(x_j) = \delta_{i,j}, \quad i,j = 1,...,n-1, \tag{5.49}$$

and then

$$p''_{n,i}(x) = l_i(x), \quad i = 1,...,n-1, \tag{5.50}$$

where $l_i(x)$ is the elementary polynomials of Lagrange on the nodes $x_1,...,x_{n-1}$.

For this set of nodes the following bound for the remainder holds:

Theorem 102 *Let* $\left|f^{(n+1)}(x)\right| \leq M_n$*, then*

$$|R_n[f,x]| \leq \frac{M_n}{2^{n-1}(n-1)!}. \tag{5.51}$$

Proof. Let define

$$\hat{U}_{n-1}(x) = (x-x_1)...(x-x_{n-1}), \tag{5.52}$$

it's known ([41, page 17]) that

$$\int_{-1}^{1} \left|\hat{U}_{n-1}(t)\right| dt = \frac{1}{2^{n-2}}. \tag{5.53}$$

Therefore, the thesis follows from the last one and theorem 98. ■

5.5 Quadrature formulae

With the notations until now introduced we consider the identity

$$f(x) = P_n(f,x) + R_n[f,x] \tag{5.54}$$

and, from an integration between -1 and 1, we obtain the class of quadrature formulae of interpolatory type, which are exact for polynomials of degree $\leq n$

$$\int_{-1}^{1} f(x)dx = f(-1) + \sum_{i=1}^{n-1} A_{n,i} f''(x_i) + f(1) + \int_{-1}^{1} R_n[f,x] \tag{5.55}$$

where

$$A_{n,i} = \int_{-1}^{1} p_{n,i}(x)dx =$$

$$= \begin{cases} \frac{(-1)^i}{G} \begin{vmatrix} 2 & 0 & \frac{2}{3} & 0 & ... & \frac{2}{n+1} \\ 1 & -1 & 1 & -1 & ... & 1 \\ 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 2 & 6x_1 & ... & n(n-1)\left(x_1\right)^{n-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & 2 & 6x_{i-1} & ... & n(n-1)\left(x_{i-1}\right)^{n-2} \\ 0 & 0 & 2 & 6x_{i+1} & ... & n(n-1)\left(x_{i+1}\right)^{n-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & 2 & 6x_{n-1} & ... & n(n-1)\left(x_{n-1}\right)^{n-2} \end{vmatrix}, & \text{even } n, \\ \\ \frac{(-1)^i}{G} \begin{vmatrix} 2 & 0 & \frac{2}{3} & 0 & ... & 0 \\ 1 & -1 & 1 & -1 & ... & -1 \\ 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 2 & 6x_1 & ... & n(n-1)\left(x_1\right)^{n-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & 2 & 6x_{i-1} & ... & n(n-1)\left(x_{i-1}\right)^{n-2} \\ 0 & 0 & 2 & 6x_{i+1} & ... & n(n-1)\left(x_{i+1}\right)^{n-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & 2 & 6x_{n-1} & ... & n(n-1)\left(x_{n-1}\right)^{n-2} \end{vmatrix}, & \text{odd } n, \\ & i = 1, .., n-1. \end{cases} \tag{5.56}$$

If x_i $i = 1, ..., n-1$ are the zeros of the Jacobi polynomial $P_{n-1}^{(1,1)}$ then $A_{n,i}$ are given by

$$A_{n,i} = -\frac{w_i}{2} \tag{5.57}$$

where ([41, page 325])

$$w_i = \frac{8n}{n+1} . \frac{1}{(1-x_i^2)\left[\frac{d}{dx}P_{n-1}^{(1,1)}(x_i)\right]^2}, \quad i = 1, ..., n-1. \tag{5.58}$$

In fact, if we consider the identity

$$\int_{-1}^{1} f(x)dx = f(1) + f(-1) - \frac{1}{2}\int_{-1}^{1}(1-t^2)f''(t)dt, \tag{5.59}$$

applying the formula of Gauss-Jacobi with $\alpha = \beta = 1$ to the integral at second member, we obtain

$$\int_{-1}^{1}(1-t^2)f''(t)dt = \sum_{i=1}^{n-1} w_i f''(x_i) + T_n(f). \tag{5.60}$$

where $T_n(f)$ is the remainder term in the Gauss-Jacobi formula.

If it is the case, the quadrature formula (5.55) can be named Gauss-Birkhoff quadrature, which is exact for all polynomials of degree at most $2n-1$. This result can be extended to Gauss-Birkhoff quadratures with respect to the weight w on $(-1, 1)$, for example taking the Jacobi weight $w(t) = (1-t)^\alpha(1+t)^\beta$, with parameters $\alpha, \beta > -1$. In fact, starting from Eq. (5.26), i.e.:

$$f(x) = \frac{f(1)+f(-1)}{2} + \frac{f(1)-f(-1)}{2}x + \int_{-1}^{1} G(x,t)f''(t)dt \tag{5.61}$$

with Green's function

$$G(x,t) = \begin{cases} \frac{(t+1)(x-1)}{2} & t \le x \\ \frac{(x+1)(t-1)}{2} & x \le t \end{cases} \tag{5.62}$$

multiply it by the weight function w, and integrating we get

$$\int_{-1}^{1} f(x)w(x)dx = \frac{f(1)+f(-1)}{2}\mu_0 + \frac{f(1)-f(-1)}{2}\mu_1 + \int_{-1}^{1} W(t)f''(t)dt, \tag{5.63}$$

where $\mu_k = \int_{-1}^{1} x^k w(x)dx$, $k = 0, 1$ and the new weight function is given by

$$W(t) = \int_{-1}^{1} w(x)G(x,t)dx.$$

Now, applying Gaussian quadrature with respect to weight $W(t)$, i.e:

$$\int_{-1}^{1} W(t)g(t)dt = \sum_{i=1}^{n-1} W_i g(x_i) + T_n(g)$$

where x_i $i = 1, ..., n-1$ are the zeros of orthogonal polynomials with respect to the weight W on $(-1, 1)$ and the weights are given by

$$W_i = \int_{-1}^{1} W(t)l_i(t)dt$$

in which the $l_i(t)$ are the Lagrange fundamental polynomials on the nodes x_i $i = 1, ..., n-1$, we obtain the extended Gauss-Birkhoff quadrature with respect to the weight function w on $(-1, 1)$:

$$\int_{-1}^{1} f(x)w(x)dx = \tfrac{f(1)+f(-1)}{2}\mu_0 + \tfrac{f(1)-f(-1)}{2}\mu_1 + \sum_{i=1}^{n-1} A_{n,i} f''(x_i) + T_n(f) \tag{5.64}$$

where $A_{n,i} = W_i$, $i = 1, ..., n-1$.

5.5.1 Examples

For equidistant nodes and $w(x) \equiv 1$, with $n = 2$ we have

$$\int_{-1}^{1} f(x)dx = f(-1) - \frac{2f''(0)}{3} + f(1) + \int_{-1}^{1} R_2[f, x]; \tag{5.65}$$

for $n = 3, ..., 8$ the coefficients $A_{n,i}$ are reported in appendix B.

For the Chebyshev of second kind nodes and $w(x) \equiv 1$ we have the class of quadrature formulae

$$\int_{-1}^{1} f(x)dx = f(-1) + \sum_{i=1}^{n-1} A_{n,i} f''(x_i) + f(1) + \int_{-1}^{1} R_n[f,x] \tag{5.66}$$

with

$$A_{n,i} = \frac{1}{n} \sin \frac{\pi i}{n} \left[\sum_{k=1}^{[\frac{n-1}{2}]} \frac{4}{8k^3 + 12k^2 + 4k} \sin \frac{(2k+1)\pi i}{n} - \frac{4}{3} \sin \frac{\pi i}{n} \right], \tag{5.67}$$

in fact,

$$\int_{-1}^{1} G_k(x)dx = \begin{cases} 0 & \text{even } k \\ \frac{4}{k^2-4} & \text{odd } k. \end{cases} \tag{5.68}$$

For $n = 2$ we obtain the formula (5.65); for $n = 3, ..., 8$ the coefficients $A_{n,i}$ are reported in appendix C.

Chapter 6

A new collocation method for a BVP

In this chapter, as application of the interpolation problem before exposed, we propose a new collocation method for a BVP.

6.1 The method

Let us consider the boundary value problem

$$\begin{cases} y''(x) = f(x, y(x), y'(x)) \quad x \in [-1, 1]\,, \quad -\infty < y, y' < \infty \\ y(-1) = y_a \\ y(1) = y_b \end{cases} \tag{6.1}$$

We assume that $f(x, z_1, z_2)$ is a real function defined and continuous on the strip $S = [-1, 1] \times R^2$ where it has continuous derivatives which satisfy, for some positive constant

M,

$$\frac{\partial f}{\partial z_1} > 0, \qquad \left| \frac{\partial f}{\partial z_2} \right| \leq M.$$

Moreover, we assume that f satisfies a uniform Lipschitz condition in z_1 e z_2, which means that there exist two nonnegative constants L and K such that, whenever (x, y_1, y_2) and $(x, \overline{y}_1, \overline{y}_2)$ are in the domain of f, the inequality

$$|f(x, y_1, y_2) - f(x, \overline{y}_1, \overline{y}_2)| \leq L\,|y_1 - \overline{y}_1| + K\,|y_2 - \overline{y}_2|\,.$$

holds. Under this hypotheses the BVP has a unique solution $y(x)$.

Let be $x_1, ..., x_{n-1}$, $n-1$ distinct points in $]-1,1[$. Let us consider the polynomials $p_{n,i}(x)$ defined in (5.7) as

$$p_{n,i}(x) = \int_{-1}^{1} G(x,t) l_i(t) dt \quad i = 1, ..., n-1 \tag{6.2}$$

where

$$G(x,t) = \begin{cases} \frac{(t+1)(x-1)}{2} & t \leq x \\ \frac{(x+1)(t-1)}{2} & x \leq t \end{cases} \tag{6.3}$$

and $l_i(x)$ are the fundamental polynomials of Lagrange calculated on the points x_i.

Theorem 103 ([15]) *For $n > 1$, the polynomial of degree n, implicitly defined by*

$$y_n(x) = \frac{y_b + y_a}{2} + \frac{y_b - y_a}{2} x + \sum_{i=1}^{n-1} p_{n,i}(x) f(x_i, y_n(x_i), y_n'(x_i)) \tag{6.4}$$

satisfies the relations

$$\begin{cases} y_n''(x_i) = f(x_i, y_n(x_i), y_n'(x_i)), \quad i = 1, ..., n-1, \\ y_n(-1) = y_a, \\ y_n(1) = y_b, \end{cases} \tag{6.5}$$

i.e., it is a collocation polynomial for the problem (6.1) on the set of nodes $x_1, ..., x_{n-1}$.

Proof. Follows from theorem 91 setting

$$\begin{cases} \omega_i = f(x_i, y_n(x_i), y_n'(x_i)), \quad i = 1, ..., n-1, \\ \omega_0 = y_a, \\ \omega_n = y_b. \end{cases}$$

■

6.2 Global error

For the estimation of the global error

$$y^{(s)}(x) - y_n^{(s)}(x), \quad s = 0, 1, 2,$$

by putting

$$\Lambda_n = \max_{-1 \le x \le 1} \sum_{i=1}^{n-1} |l_i(x)| \tag{6.6}$$

and

$$R = \max_{-1 \le x, \xi_x \le 1} \left| \frac{y^{(n+1)}(\xi_x)(x - x_1) \cdots (x - x_{n-1})}{(n-1)!} \right| \tag{6.7}$$

we have the following theorem:

Theorem 104 ([18]) *If* $y \in C^{n+3}([-1, 1])$, L *and* K *are the Lipschitz constants of function* f *such that* $1 - 2\Lambda_n K > 0$ *and* $1 - \Lambda_n(2K + L) > 0$, *then*

$$\left\| y^{(s)}(x) - y_n^{(s)}(x) \right\|_\infty \le \frac{cR}{1 - \Lambda_n(2K + L)}, \quad s = 0, 1, 2, \tag{6.8}$$

where

$$c = \begin{cases} 1 & \text{if } s = 0, \ 2 \\ 2 & \text{if } s = 1. \end{cases} \tag{6.9}$$

Remark 105 ([18]) *If f does not depend on $y'(x)$, then*

$$\left\| y^{(s)}(x) - y_n^{(s)}(x) \right\|_\infty \leq \frac{cR}{1 - \Lambda_n L}. \tag{6.10}$$

6.3 The numerical algorithm and numerical comparison

In order to calculate the approximate solution of problem (6.1) by (6.4) at $x \in [-1, 1]$, we need the values $y_i = y_n(x_i), \quad i = 1, ..., n-1$. To do this, we can solve the following system:

$$\begin{cases} y_i = \dfrac{y_b + y_a}{2} + \dfrac{y_b - y_a}{2} x_i + \sum\limits_{k=1}^{n-1} p_{n,k}(x_i) f(x_k, y_k, y_k') & i = 1, ..., n-1 \\ y_i' = \dfrac{y_b - y_a}{2} + \sum\limits_{k=1}^{n-1} p_{n,k}'(x_i) f(x_k, y_k, y_k') & i = 1, ..., n-1 \end{cases} \tag{6.11}$$

Let us set

$$A = \begin{pmatrix} p_{n,1}(x_1) & \cdots & p_{n,n-1}(x_1) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n,1}(x_{n-1}) & \cdots & p_{n,n-1}(x_{n-1}) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p_{n,1}'(x_1) & \cdots & p_{n,n-1}'(x_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & p_{n,1}'(x_{n-1}) & \cdots & p_{n,n-1}'(x_{n-1}) \end{pmatrix},$$

$$f_k = f(x_k, y_k, y_k'), \qquad F(Y_n) = (f_1, ..., f_{n-1}, f_1, ..., f_{n-1})^T,$$

$$Y_n = (y_1, ..., y_{n-1}, y_1', ..., y_{n-1}')^T, \qquad a_i = \frac{y_a - y_b}{2} x_i - \frac{y_a + y_b}{2},$$

$$b_i = \frac{y_a - y_b}{2}, \qquad C_n = (a_1, ..., a_{n-1}, b_1, ..., b_{n-1})^T,$$

the previous system can be written in the form

$$AF(Y_n) - Y_n = C_n,$$

or equivalently

$$Y_n = G(Y_n)$$

where

$$G(Z) = AF(Z) - C_n.$$

For the existence and uniqueness of the solution we provide the following theorem:

Theorem 106 ([18]) *If $\|A\| L < 1$, then the previous system has a unique solution that can be calculated with the iterative method*

$$\begin{aligned} &Y_n^{(0)} \text{ arbitrary,} \\ &Y_n^{(\nu+1)} = G(Y_n^{(\nu)}), \quad \nu = 1, ... \end{aligned}$$

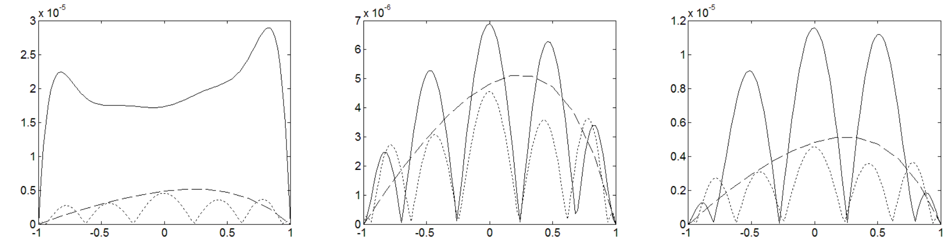
Now we report certain numerical results for classical test problems and compare the error committed applying our method, with $n = 7$, on different sets of nodes, in particular between Chebyshev nodes of the second kind (already examined in Ref. [18]) with equidistant, Chebyshev of the first kind and Legendre nodes. We also compare these results with the ones obtained by applying the Matlab ODE solver bvp4c.

Problem 107 *Let us consider the problem*

$$\begin{cases} y'' = y + 2e^x \\ y(-1) = 0 \\ y(1) = 2e \end{cases} \quad \text{with the solution } \; y(x) = (x+1)e^x.$$

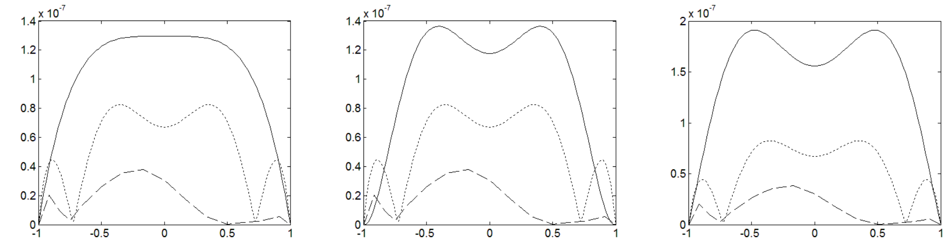
The error in the approximation of the solution is plotted in Figure 6.1.

In the case of bvp4c (dashed line) 100 function evaluations are needed for this problem on a mesh of 13 points; our method applied on Chebyshev second kind nodes (dotted



(a) Equidistant nodes (b) Legendre nodes (c) Cheb I kind nodes

Figure 6.1: Error functions of problem 107.



(a) Equidistant nodes (b) Legendre nodes (c) Cheb I kind nodes

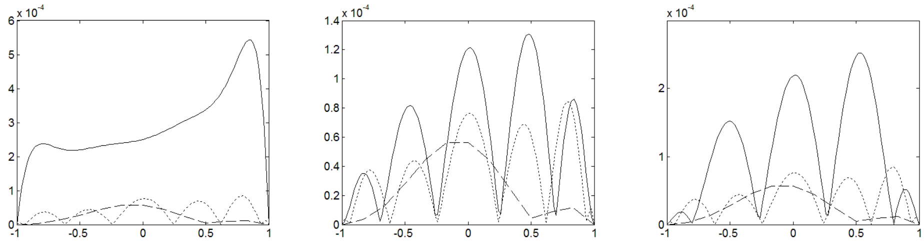
Figure 6.2: Error functions of problem 108.

line) or on one of the other three set of nodes considered (solid line) requires only 36 function evaluations to obtain an error less than or equal to the requested tolerance $\epsilon = 10^{-5}$.

Problem 108 *Let be*

$$\begin{cases} y'' = yy'^2 - 2(2x^4 - 1) \\ y(-1) = 1 \\ y(1) = 1 \end{cases} \qquad \textit{with the solution} \;\; y(x) = x^2.$$

For this problem our method applied on Chebyshev second kind nodes (dotted line) or on one of the other three set of nodes considered (solid line) requires only 102 function evaluations to obtain an error less than or equal to the requested tolerance $\epsilon = 10^{-7}$.



(a) Equidistant nodes (b) Legendre nodes (c) Cheb I kind nodes

Figure 6.3: Error functions of problem 109.

To obtain an error of the same order (10^{-7}) bvp4c (dashed line) requires 337 function evaluations on a mesh of 19 points.

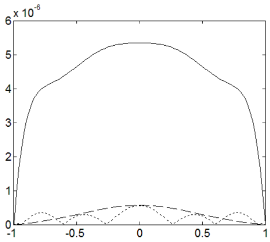
Problem 109 *Let us consider the problem*

$$\begin{cases} y'' = 2yy' \\ y(-1) = \frac{1}{3} \\ y(1) = 1 \end{cases} \quad \textit{with the solution} \quad y(x) = \frac{1}{2-x}.$$

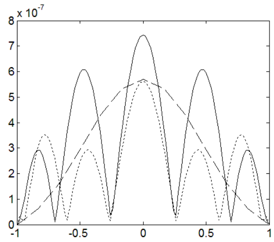
In the case of bvp4c (dashed line) 201 function evaluations are needed for this problem on a mesh of 13 points; our method applied on Chebyshev second kind nodes (dotted line) or on one of the other three set of nodes considered (solid line) requires only 132 function evaluations to obtain an error less than or equal to the requested tolerance $\epsilon = 10^{-4}$.

Problem 110 *Let us consider the problem*

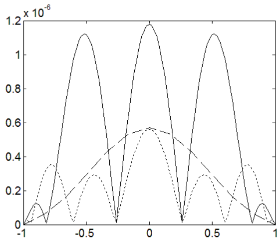
$$\begin{cases} y'' = -(1 + 0.01y^2)y + 0.01\cos^3 x \\ y(-1) = \cos(-1) \\ y(1) = \cos(1) \end{cases} \quad \textit{with the solution} \quad y(x) = \cos(x).$$



(a) Equidistant nodes



(b) Legendre nodes



(c) Cheb I kind nodes

Figure 6.4: Error functions of problem 110.

In this case bvp4c (dashed line) requires 60 function evaluations on a mesh of 13 points; our method applied on Chebyshev second kind nodes (dotted line) or on one of the other three set of nodes considered (solid line) requires only 7 function evaluations to obtain an error less than or equal to the requested tolerance $\epsilon = 10^{-6}$.

Appendix A

Mathematica code for computation of polynomials $p_{n,i}(x)$

We report below an ad hoc "Mathematica" code for the calculation of the polynomials $p_{n,i}(x)$ defined in (5.7).

$n = \text{Input}[\text{"Insert degree of polynomial } p_{n,i}(\text{x})\text{"}];$

$X = \{\};$

$\text{OPT} = \text{Input}[\text{"Press 1 for equidistant nodes;}$

$\quad \text{Press 2 for Chebyshev of second kinds nodes;}$

$\quad \text{Press 3 for a generic set of nodes"}];$

$\text{For}[i = 1, i \leq n-1, i{+}{+},$

$\{\text{node} = \text{Which}[\text{OPT} == 1, -1 + \frac{2*i}{n}, \text{OPT} == 2, \text{Cos}[\frac{i*\pi}{n}],$

$\text{OPT} == 3, \text{Input}[\text{"Insert node"}]]; X = \text{Append}[X, \text{node}]\}];$

$A[\text{i\_}]{:=}(\prod_{j=1}^{i-1} \frac{1}{X[[i]]-X[[j]]}) * (\prod_{j=i+1}^{n-1} \frac{1}{X[[i]]-X[[j]]});$

$$S[\text{k\_}, \text{i\_}] := \sum_{j=1}^{i-1} (X[[j]])^k + \sum_{j=i+1}^{n-1} (X[[j]])^k;$$

$$a[\text{j\_}, \text{i\_}] := \text{Which}[j == 0, 1, 0 < j \leq n-2,$$
$$-\tfrac{1}{j}(S[j,i] + \sum_{k=1}^{j-1} (a[k,i] * S[j-k,i]))];$$

$$c[\text{j\_}, \text{i\_}] := \text{Which}[0 \leq j < n-1, A[i] * \tfrac{a[j,i]}{(n-j)(n-1-j)},$$
$$j == n-1, -\tfrac{1}{2} \sum_{k=0}^{n-2} (c[k,i] * (1 - (-1)^{n-k})),$$
$$j == n, -\tfrac{1}{2} \sum_{k=0}^{n-2} (c[k,i] * (1 + (-1)^{n-k}))];$$

$$\text{Poly}[\text{i\_}, \text{x\_}] := \text{If}[0 < i < n, \text{Expand}[\sum_{j=0}^{n} c[j,i] * x^{n-j}]];$$

Appendix B

Computation of coefficients $A_{n,i}$ for equidistant nodes

The coefficients $A_{n,i} = \int_{-1}^{1} p_{n,i}(x)dx$, for $n = 3, ..., 8$, for equidistant nodes are reported below:

$$
\begin{aligned}
&A_{3,1} = -\tfrac{1}{3}, \\
&A_{4,1} = -\tfrac{4}{15}, \quad A_{4,2} = -\tfrac{2}{15}, \\
&A_{5,1} = -\tfrac{1}{6}, \quad A_{5,2} = -\tfrac{1}{6}, \\
&A_{6,1} = -\tfrac{1}{7}, \quad A_{6,2} = -\tfrac{1}{35}, \quad A_{6,3} = -\tfrac{34}{105}, \\
&A_{7,1} = -\tfrac{73}{720}, \quad A_{7,2} = -\tfrac{1}{16}, \quad A_{7,3} = -\tfrac{61}{360}, \\
&A_{8,1} = -\tfrac{184}{2025}, \quad A_{8,2} = \tfrac{116}{4725}, \quad A_{8,3} = -\tfrac{328}{945}, \quad A_{8,4} = \tfrac{454}{2835}.
\end{aligned}
$$

We observe that, for reason of symmetry, $A_{n,i} = A_{n,n-i}$, $i = 1,...,n-1$.

Appendix C

Computation of coefficients $A_{n,i}$ for Chebyshev of second kind nodes

The coefficients $A_{n,i} = \int_{-1}^{1} p_{n,i}(x)dx$, for $n = 3, ..., 8$, for Chebyshev of second kind nodes are reported below:

$$A_{3,1} = -\tfrac{1}{3},$$

$$A_{4,1} = -\tfrac{2}{15}, \qquad A_{4,2} = -\tfrac{2}{5},$$

$$A_{5,1} = -\tfrac{25-7\sqrt{5}}{150}, \qquad A_{5,2} = -\tfrac{2+7\sqrt{5}}{150},$$

$$A_{6,1} = -\tfrac{2}{63}, \qquad A_{6,2} = -\tfrac{6}{35}, \qquad A_{6,3} = -\tfrac{82}{315},$$

$$A_{8,1} = -\tfrac{26-16\sqrt{2}}{315}, \quad A_{8,2} = -\tfrac{22}{315}, \quad A_{8,3} = -\tfrac{26+16\sqrt{2}}{315}, \quad A_{8,4} = -\tfrac{62}{315}.$$

Explicit expressions for $A_{7,i}$ are complicated but their numerical values are:

$$A_{7,1} = -0.01789750707562690, \qquad A_{7,2} = -0.1076559342057399,$$

$$A_{7,3} = -0.2077798920519665.$$

Also for this set of nodes it results that $A_{n,i} = A_{n,n-i}$, $i = 1, ..., n-1$.

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