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On the enhancement of the approximation order of Shepard operators

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On the enhancement of the approximation order of Shepard operators

Ph.D. Thesis Filomena Di Tommaso

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To all those who have believed

and still believe in me...

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Introduzione

Nel 1968 Donald Shepard [65] introdusse un metodo globale delle distanze inverse pesate per l'interpolazione di dati scattered bivariati e applicò tale schema nel contesto nell'approssimazione di dati geografici e demografici [45]. La superficie interpolante era definita come una combinazione lineare di funzioni peso con coefficienti che sono le valutazioni funzionali nei punti di interpolazione. Il termine "scattered" si riferisce al fatto che i punti sono irregolarmente distribuiti, cioè non soddisfano alcuna particolare condizione geometrica. Nell'operatore di Shepard, ogni funzione peso è la normalizzazione della distanza inversa dai dati e quindi, anche se essa diminuisce all'aumentare della distanza dal punto, è influenzata da punti che sono lontani dal punto di approssimazione [67]. Il metodo di Shepard, essendo sensibile a tutti i punti, appartiene alla classe dei metodi globali. La natura globale del metodo di Shepard non è da preferire per grandi set di dati. Per evitare questo problema Franke e Nielson [43] svilupparono una modifica del metodo in cui ogni funzione peso è localizzata in modo tale da avere un supporto compatto [40].

Un ulteriore svantaggio del metodo di Shepard è che esso riproduce in maniera esatta solo polinomi costanti. A partire da Shepard, diversi autori hanno studiato come migliorare la qualità di riproduzione dell'operatore di Shepard, in presenza di diversi tipi di dati funzionali e di dati derivati, nel caso multivariato così come in quello univariato. Shepard [65] sostituì le valutazioni funzionali in ogni punto con il polinomio di Taylor del primo ordine; in questo modo, egli ottenne un operatore di interpolazione che riproduce in maniera esatta polinomi fino al grado uno e interpola le valutazioni funzionali, così come le derivate del primo ordine, nei punti di interpolazione. Farwig [40] considerò il caso più generale del polinomio di Taylor di qualsiasi ordine e investigò la velocità di convergenza della formula di interpolazione di Shepard globale. Franke and Nielson [43] sostituirono il polinomio di Taylor del secondo ordine nei punti di interpolazione con la funzione quadratica che approssima i valori dati in un set di punti vicini nel senso dei minimi quadrati pesati. Seguendo l'idea di Franke, Renka e Brown [62] considerarono come funzioni nodali la cubica e la serie di coseni a 10 parametri al posto della funzione quadratica e svilupparono tre subroutine in Fortran 77 per testare l'accuratezza dei vari algoritmi. Successivamente, Thacker et al. [67] svilupparono un pacchetto contenente la traduzione in Fortran 95 degli algoritmi di Renka e Brown e la variante lineare dell'algoritmo originale di Shepard da essere usati quando è noto che i dati contengono valori anomali. Questi pacchetti trovano applicazione nella meteorologia, nelle stazioni di osservazione meteorologiche e geografiche, quando sono disponibili solo le valutazioni funzionali.

Per quanto riguarda il caso univariato, Coman and Trîmbiţaş [20] studiarono la velocità di convergenza dell'operatore di Shepard combinato con gli operatori di interpolazione polinomiale più conosciuti , come Taylor, Lagrange, Hermite, Birkhoff. Cătinaş [30] introdusse la combinazione dell'operatore di Shepard con il polinomio di interpolazione di Lidstone [52], mentre Caira e Dell'Accio [15] si occuparono della combinazione dell'operatore di Shepard con il polinomio di Taylor generalizzato, proposto da Costabile [21]. Caira e Dell'Accio dimostrarono l'accuratezza di approssimazione dell'operatore univariato di Shepard combinato nell'interpolazione della soluzione discreta di problemi al valore iniziale, per equazioni differenziali ordinarie.

Nel 2003 Han Xuli [71] sollevò il problema relativo all'aumento della qualità di riproduzione di un operatore lineare univariato nel caso in cui siano noti valori di derivate di ordine più alto. Allo scopo di costruire uno sviluppo di ordine superiore di una funzione, Han Xuli propose di sostituire il classico polinomio di Taylor con una versione modificata (o ridotta), sebbene quest'ultima interpoli le valutazioni funzionali ma perda le proprietà di interpolazione delle derivate di ordine superiore. Nel 2005 Carl de Boor [34] diede una dimostrazione alternativa del risultato di Han Xuli, basata sulle differenze divise, e mostrò che il ruolo dell'operatore lineare di Xuli può essere preso da un qualsiasi operatore lineare dello spazio delle funzioni continue su un intervallo reale. L'estensione a dati arbitrari nel caso multivariato è meno naturale da definire, rispetto al caso univariato. Nel 2009

Guessab, Nouisser and Schmeisser [46] studiarono un caso multivariato analogo al risultato di Xuli, sostituendo l'intervallo reale con un dominio compatto convesso e le derivate della funzione con le derivate direzionali, fino a un certo ordine fissato $p, p \in \mathbb{N}$. Esempi notevoli di funzioni peso per l'operatore lineare sono i polinomi di Bernstein, i polinomi di Lagrange e le funzioni peso di Shepard.

Nel caso particolare dell'operatore di Shepard, che riproduce solo polinomi costanti, l'utilizzo del polinomio di Taylor ridotto piuttosto che quello di Taylor, da un lato non aumenta la qualità di riproduzione della combinazione e dall'altro comporta una sostanziale perdita di informazioni, dovuta alla mancata interpolazione dei dati derivati supplementari utilizzati. Pertanto è naturale chiedersi se sia possibile aumentare la qualità di riproduzione dell'operatore di Shepard, oltre che la sua accuratezza nell'approssimazione, in presenza di dati derivati supplementari e mantenendo allo stesso tempo le proprietà di interpolazione. Un primo tentativo di soluzione è da attribuire a Cătinaş [31, 32], la quale sostituì le valutazioni funzionali con l'estensione tensoriale del polinomio di Bernoulli [22] e del polinomio di Lidstone [6]. Dato che l'estensione tensoriale è basata sui vertici di un rettangolo, le combinazioni risultanti necessitano di dati strutturati in maniera particolare e quando vengono applicati al caso dell'interpolazione di dati scattered è necessario aggiungere alcuni punti fittizi allo scopo di creare i rettangoli.

Una nuova procedura, introdotta da Dell'Accio e Di Tommaso nel 2010 [36], permette di costruire, sotto le stesse condizioni, operatori di Shepard combinati che interpolano tutti i dati forniti nei punti di interpolazione in molteplici situazioni, che includono anche i casi di dati lacunari difficili da gestire. Nel caso bivariato, questa procedura è basata sull'associazione, a ogni punto di interpolazione, di un triangolo con un vertice in esso e gli altri due vertici nel suo intorno. L'associazione è realizzata in modo da ridurre l'errore di un opportuno polinomio di interpolazione locale tre punti, basato sui vertici del triangolo. Il polinomio tre punti è usato al posto delle valutazioni funzionali nel metodo di Shepard, analogamente al polinomio di interpolazione di Taylor nella combinazione di Farwig [40] o del polinomio di Taylor ridotto nell'approccio di Xuli [71]. In particolare, a seconda dei dati disponibili in ogni punto di interpolazione, definiamo la combinazione dell'operatore di Shepard con

1. il polinomio di interpolazione di Hermite sul triangolo [26];
2. il polinomio di interpolazione di Lidstone tre punti [16];
3. il polinomio di interpolazione Complementary Lidstone tre punti [27].

Assumendo che l'insieme dei dati contenga, insieme alle valutazioni funzionali, tutte le derivate parziali fino a un certo ordine fissato p, l'operatore bivariato di Shepard-Hermite [26], presentato nel Capitolo 2, interpola tutti i dati utilizzati e riproduce polinomi di grado minore o uguale a p, ed è quindi migliore dell'operatore di Shepard-Taylor che utilizza gli stessi dati. L'operatore di Shepard-Hermite è definito considerando alcuni casi particolari di una classe generale di polinomi di interpolazione di Hermite sul triangolo, introdotta da Chui e Lai nel 1990 [18]. Questi particolari polinomi usano e interpolano le valutazioni funzionali e tutte le derivate in un vertice fissato del triangolo (il punto di interpolazione in considerazione) e alcuni dei dati disponibili nei rimanenti vertici. Di conseguenza, hanno un grado totale m maggiore del massimo ordine p di dati derivati utilizzati e sono univocamente definiti nello spazio $\mathcal{P}_{\boldsymbol{x}}^m$ dei polinomi di grado totale minore o uguale a $m, m \in \mathbb{N}$.

I dati di tipo Lidstone consistono nelle valutazioni funzionali e in tutte le derivate di ordine pari in ogni punto di interpolazione, fino a un certo ordine $2m-2$. Il nome è in onore di G. J. Lidstone, il quale nel 1929 [52] fornì un'espressione esplicita di un polinomio che approssima una data funzione nell'intorno di due punti piuttosto che uno, generalizzando in questo modo il polinomio di Taylor. Dal punto di vista pratico tale sviluppo è molto utile [6] dato che è la base per la soluzione di problemi al valore al bordo di tipo Lidstone (guardare i lavori pionieristici [2, 4, 5, 7]). Nel 2005 Costabile e Dell'Accio [24], risposero a un problema posto da Agarwal e Wong in [2] “sulla scarsità di letteratura sull'estensione di alcuni risultati sull'approssimazione di funzioni univariate, in termini di polinomi di Lidstone per funzioni di due variabili indipendenti su un dominio non rettangolare”. Essi estesero l'espansione univariata di Lidstone a un'espansione bivariata che interpola nei vertici del triangolo standard di $\mathbb{R}^2$, e la cui restrizione a ogni lato del triangolo è l'espansione di Lidstone univariata. Nel Capitolo 3 ridefiniamo l'operatore di Shepard-Lidstone utilizzando la generalizzazione di questa espansione polinomiale a un generico triangolo di $\mathbb{R}^2$ evitando, in tale modo, lo svantaggio dell'aggiunta dei punti che si presenta nel caso dell'estensione di Cătinaş [31]. In particolare, l'operatore di Shepard-Lidstone usa, interpolandole, le valutazioni funzionali e tutte le derivate di ordine pari fino all'ordine $2m-2$ e riproduce i polinomi fino al grado $2m-1$.

I dati di tipo Complementary Lidstone consistono in valutazioni funzionali in qualche punto di interpolazione e in tutte le derivate di ordine pari, fino a un certo ordine $2m-1$, in ogni punto di interpolazione. L'interpolazione Complementary Lidstone complementa in maniera naturale l'interpolazione di Lidstone e fu introdotta da Costabile, Dell'Accio e Luceri nel 2005 [29]. Successivamente, viste le sue applicazioni, essa fu sviluppata ulteriormente da Agarwal, Pinelas e Wong [1] e da Agarwal e Wong [8, 3]. Nel Capitolo 4 estendiamo il polinomio di interpolazione Complementary Lidstone a un polinomio bivariato tre punti sui vertici di un generico triangolo di $\mathbb{R}^2$ e, combinandolo con l'operatore di Shepard, forniamo una soluzione al problema dell'interpolazione di tipo Complementary Lidstone su un insieme di dati scattered [27]. In particolare l'operatore di Shepard Complementary Lidstone utilizza derivate fino all'ordine $2m-1$ e le interpola riproducendo polinomi fino al grado $2m$.

Per ognuno degli operatori elencati, viene fornita un'espressione esplicita del termine residuale nel caso di funzioni sufficientemente differenziabili. Tale espressione è ottenuta sommando, al classico resto di Taylor, un termine polinomiale che è la differenza tra ogni espansione tre punti e l'espansione di Taylor dello stesso grado. Limitazioni per il resto sono ottenute, da cui deduciamo un criterio per l'associazione del triangolo ad ogni punto di interpolazione. I risultati numerici sulle funzioni test generalmente usate per l'approssimazione su dati scattered, mostrano che questi operatori hanno una buona accuratezza di approssimazione, comparabile con quella degli operatori di Shepard-Taylor che riproducono polinomi dello stesso grado, anche nei casi di dati lacunari.

Quando sono note solo le valutazioni funzionali, i polinomi tre punti, che utilizzano le valutazioni funzionali in ogni punto di interpolazione, si riducono al polinomio di interpolazione lineare tre punti. Un metodo di F. Little, chiamato Shepard triangolare e introdotto nel 1982 [53], combina funzioni di base simili a quelle di Shepard con polinomi di interpolazione lineare tre punti. Esso è una combinazione convessa di interpolanti lineari su un insieme di triangoli, in cui ogni funzione peso è definita come il prodotto della distanza inversa dai vertici del corrispondente triangolo. Il metodo riproduce polinomi lineari senza utilizzare alcun dato derivato e Little notò che esso supera largamente il metodo di Shepard dal punto di vista estetico. Durante il mio periodo presso l'USI-Università della Svizzera Italiana, lavorando con il Prof. K. Hormann, abbiamo ricavato lo stesso metodo in maniera indipendente come un modo naturale per rompere l'asimmetria che si ha combinando l'operatore di Shepard con polinomi di interpolazione lineare. Nel Capitolo 5 studiamo attentamente le proprietà di questo operatore e, come novità, dimostriamo la sua velocità di convergenza quadratica per configurazioni generali di triangoli, rendendo il metodo indipendente da qualsiasi mesh che connetta i punti di interpolazione, quindi meshless o meshfree. Un lavoro scientifico su questo argomento è in fase di elaborazione [37].

Introduction

In 1968 Donald Shepard [65] introduced a global inverse distance weighted method for interpolating scattered bivariate data points and applied this scheme in the context of geographic and demographic data fitting [45]. The interpolating surface was defined as a linear combination of weight functions with coefficients that are the function evaluations at the interpolation points. The term "scattered" refers to the fact that the points are irregularly spaced, i.e. they are not assumed to satisfy any particular geometric condition. In the Shepard operator, each weight function is a normalization of the inverse distance from the data points and therefore, even if it diminishes as the distance from the interpolation point increases, it accords too much influence to points that are far away from the point of approximation [67]. The Shepard method, being sensitive to all data points, belongs to the class of global methods. The global nature of the Shepard method is not desirable for large sets of data. To avoid this problem Franke and Nielson [43] developed a modification of the Shepard method in which each weight function is localized such that it has small compact support [40].

A further drawback of the Shepard's method is that it reproduces exactly only constant polynomials. Starting from Shepard, many authors have studied how to improve the reproduction quality of the Shepard operator, in presence of different types of functional and derivative data, in the multivariate as well as in the univariate case. Shepard [65] substituted the function evaluation at each interpolation point with the first order Taylor polynomial; in this way, he obtained an interpolation operator that reproduces exactly polynomials up to the degree one and interpolates function evaluations, as well as first order derivative data, at the interpolation points. Farwig [40] considered the more general case of the Taylor polynomial of any order and investigated the rate of convergence of Shepard's global interpolation formula. Franke and Nielson [43] replaced the second order Taylor polynomial at the interpolation points by quadratic functions that fit the data values on a set of nearby nodes in a weighted least square sense. Following Franke's idea, Renka and Brown [62] considered the cubic and the 10-parameter cosine series nodal functions in place of the quadratics and developed three Fortran 77 subroutines to test the accuracy of various algorithms. Afterwards, Thacker et al. [67] developed a package containing the Fortran 95 translations of Renka and Brown's algorithms and linear variations of the original Shepard algorithm to be used when the data is known to contain outliers. These packages find application to meteorology, weather observation stations and geography, when only function evaluations are available.

Regarding the univariate case, Coman and Trîmbiţaş [20] studied the rate of convergence of the Shepard operator combined with most known polynomial interpolation operators, such as Taylor, Lagrange, Hermite, Birkhoff. Cătinaş [30] introduced the combination of the Shepard operator with the Lidstone interpolating polynomial [52], while Caira and Dell'Accio [15] dealt with the combination of the Shepard operator with the generalized Taylor polynomial, proposed by Costabile [21]. Caira and Dell'Accio demonstrated the accuracy of approximation of the univariate combined Shepard operators to interpolate the discrete solution of initial value problem for ordinary differential equation.

In 2003 Han Xuli [71] raised the problem related to the enhancement of the reproduction quality of a univariate linear operator if higher order derivatives of a function are given. With the aim of constructing an higher order expansion of a function, Han Xuli proposed to substitute the classical Taylor polynomial with a modified (or reduced) version, despite the latter interpolates function evaluations but loses the property of interpolation of all successive derivatives. In 2005 Carl de Boor [34] gave an alternative proof of Han Xuli's result, based on divided differences, and showed that the role of Xuli's linear operator can be taken by any bounded linear operator on the space of continuous function on a real interval. Extension to arbitrary data in multiple dimensions is less natural to define, with respect to the univariate case. In 2009 Guessab, Nouisser and Schmeisser [46] established a multivariate analogue of Han Xuli's result, by substituting the real interval with a compact convex domain and the derivatives of the function with the directional derivatives up to a fixed order $p, p \in \mathbb{N}$. Remarkable examples of the weight functions for the linear operator are Bernstein polynomials, Lagrange polynomials and Shepard weight functions.

In the particular case of the Shepard operator, that reproduces only constant polynomials, the use of the reduced Taylor polynomial rather than the Taylor one, on the one hand does not increase the reproduction quality of the combination, on the other it involves a substantially loss of information due to the lack of interpolation of used supplementary derivative data. Therefore, it is natural to ask whether it is possible to enhance the reproduction quality of the Shepard operator, as well as its accuracy of approximation, in presence of supplementary derivative data, maintaining at the same time the interpolation properties. A first attempt of solution must be ascribed to Cătinaş [31, 32], who substituted the function evaluations with the tensorial extensions of the Bernoulli [22] and Lidstone polynomials [6]. Since the tensorial extension is based on the vertices of a rectangle, the resulting combinations need specially structured data and, therefore, when they are applied in the case of scattered data interpolation it is necessary to add some other fictive points in order to create the rectangles.

A new procedure, introduced in 2010 by Dell'Accio and Di Tommaso [36], allows to construct, under the same conditions, combined Shepard operators that interpolate all given data at each interpolation point in several situations, that includes even the hard to handle case of lacunary data. In the bivariate case, this procedure is based on the association, to each interpolation point, of a triangle with a vertex in it and the other two vertices in its neighborhood. The association is realized to reduce the error of a suitable three point local interpolating polynomial, based on the vertices of the triangle. The three point polynomial is used instead of the function evaluations in the Shepard method, similarly to the Taylor polynomial in the Farwig's one [40] or the reduced Taylor polynomial in the Xuli's approach [71]. In particular, according to the data available at each interpolation point, we define the combination of the Shepard operator with:

1. the Hermite interpolating polynomial on the triangle [26];

2. the three point Lidstone interpolating polynomial [16];

3. the three point Complementary Lidstone interpolating polynomial [27].

By assuming that the set of data contains, together with function evaluations, all partial derivatives up to a fixed order p, the bivariate Shepard-Hermite operator [26], presented in Chapter 2, interpolates all given data and reproduces polynomials of degree greater than p, that is better than the Shepard-Taylor operators which use the same set of data. The Shepard-Hermite operator is defined by considering some particular cases of a general class of Hermite interpolation polynomials on the triangle, introduced by Chui and Lai in 1990 [18]. These particular polynomials use and interpolate function evaluations and all derivative data at a fixed vertex of the triangle (the interpolation point under consideration) and some of the given data at the remaining vertices. As a consequence, they have a total degree m greater than the maximum order p of used derivative data and are uniquely defined in the space $\mathcal{P}_{\boldsymbol{x}}^{m}$ of polynomials of total degree less than or equal to $m, m \in \mathbb{N}$.

Lidstone type data consist of function evaluations and all even order derivatives at each interpolation point, up to some order $2m-2$. The name is in honor of G. J. Lidstone, who in 1929 [52] provided an explicit expression of a polynomial which approximates a given function in the neighborhood of two points instead of one, generalizing in such a way the Taylor polynomial. From the practical point of view such development is very useful [6] since it is the basis for the solution of the Lidstone boundary value problems (see the pioneering papers [2, 4, 5, 7]). In 2005 Costabile and Dell'Accio [24], answering to a question "on the lack of literature on the extension of some results on the approximation of univariate functions by means of Lidstone polynomials to functions of two independent variables over non-rectangular domains", posed by Agarwal and Wong in [2], extended the univariate Lidstone expansion to a bivariate expansion interpolating on the vertices of the standard simplex of $\mathbb{R}^2$, which is the univariate Lidstone expansion when restricted to each side of the simplex. In Chapter 3 we redefine the bivariate Shepard-Lidstone operator by using the generalization of this polynomial expansion on a generic triangle of $\mathbb{R}^2$ avoiding, in this way, the drawback of the addition of points that occurs in the Cătinaş' extension [31]. In particular, the bivariate Shepard-Lidstone operator uses function evaluations and all even derivatives data up to the order $2m-2$, interpolating them and reproduces polynomials up to the degree $2m-1$.

Complementary Lidstone type data consist of function evaluations in some interpolation point and all odd order derivatives, up to some order $2m-1$, at each interpolation point. Complementary Lidstone interpolation naturally complements Lidstone interpolation and was introduced by Costabile, Dell'Accio and Luceri in 2005 [29]. Afterwards it was drawn on by Agarwal, Pinelas and Wong [1] and Agarwal and Wong [8, 3] for its applications. In Chapter 4 we extend the Complementary Lidstone polynomial to a three point bivariate polynomial on the vertices of a generic triangle of $\mathbb{R}^2$ and, by combining it with the Shepard operator, we provide a solution to the Complementary Lidstone interpolation problem on scattered data set [27]. In particular the Shepard Complementary Lidstone operator uses derivatives data up to the order $2m-1$ interpolating them and reproduces polynomial up to the degree $2m$.

For each operator listed above, explicit expression for the remainder is given in the case of sufficiently differentiable functions. This expression results from the addition, to the classical Taylor remainder, of a polynomial term which is the difference between each three point expansion and the Taylor expansion of the same degree. Bound for the remainder are obtained, from which we deduce a criteria for the association of the triangle to each interpolation point. Numerical results, on generally used test functions for scattered data approximation [62], show that these operators have a good accuracy of approximation, comparable with the one of the Shepard-Taylor operators which reproduce polynomials of the same degree, even in the lacunary data cases.

When only function evaluations are given, the three point polynomials, which use function evaluations at each interpolation point, reduce to the three point linear interpolation polynomial. A method by F. Little, called triangular Shepard and introduced in 1982 [53], combines Shepard-like basis functions and three point linear interpolation polynomials as well. It is a convex combination of the linear interpolants of a set of triangles, in which each weight function is defined as the product of the inverse distance from the vertices of the corresponding triangle. The method reproduces linear polynomials without using any derivative data and Little noticed that it surpasses Shepard's method greatly in esthetic behavior. During my period at the USI-Università della Svizzera Italiana, working with Prof. K. Hormann, we discovered the same method independently as a natural way to break the asymmetry that occurs in combining the Shepard operator with three point linear interpolation polynomials. In Chapter 5 we deeply study the properties of this operator and, as a novelty, we demonstrate its quadratic rate of convergence for general configuration of triangles, making the method not dependent from any mesh which connects the interpolation points, i.e. a meshless or meshfree method. A paper on this topic is in preparation [37].

Chapter 1

Preliminary results and posing of the problem

1.1 Enhancement of the algebraic precision of a linear operator

Let $\Omega \subset \mathbb{R}^s, s \in \mathbb{N}$, be a compact convex domain whose interior is non-empty. Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\} \subset \Omega, n \in \mathbb{N}$, be a set of n distinct points (called *nodes* or *sample points*) with associated function evaluations $f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_n)$, $f \in C^p(\Omega)$, $p \geq 0$, and let $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\} \subset C^p(\Omega)$ be a set of functions which depend only on the node set X. We suppose that functions in $\mathcal{F}$ are cardinal, i.e.

$$\phi_i(\boldsymbol{x}_k) = \delta_{ik} \quad i, k = 1, \ldots, n \tag{1.1}$$

and form a partition of unity, i.e.

$$\sum_{i=1}^{n} \phi_i(\boldsymbol{x}) = 1. \tag{1.2}$$

Han Xuli [71] considers, for $s = 1$, the linear operator

$$L[f](\boldsymbol{x}) = \sum_{i=1}^{n} \phi_i(\boldsymbol{x}) f(\boldsymbol{x}_i). \tag{1.3}$$

Since (1.1) and (1.2) hold, $L[f]$ interpolates f at each sample point $\boldsymbol{x}_i$ and reproduces exactly constant functions. An important characteristic of an interpolation operator is the polynomial reproduction property. By denoting with $\mathcal{P}_{\boldsymbol{x}}^m$ the space of all polynomials in $\boldsymbol{x}$ of degree at most m, we say that the interpolation operator L reproduces exactly polynomials up to the degree m if $L\left[f\right] = f$ for all $f \in \mathcal{P}_{\boldsymbol{x}}^m$. In addition, if there exists $g \in \mathcal{P}_{\boldsymbol{x}}^{m+1}$ such that $L\left[g\right] \neq g$, then we say that L has polynomial reproduction degree equal to m. This number is also known as algebraic degree of precision or algebraic degree of exactness. We choose this last denomination, according to the notation in [20], and denote it by "dex $(\cdot)$". The algebraic degree of exactness m of L depends on $\mathcal{F}$. Possible choices for $\mathcal{F}$ are: Bernstein basis functions (for which $m = 1$), Lagrange basis functions ($m = n - 1$) and Shepard basis functions ($m = 0$).

Let us suppose that function evaluations and all the derivatives up to a fixed order $p \geq 1$ are given at each sample point. Under this assumption, it is possible to prove [71] that, by replacing each function evaluations $f(\boldsymbol{x}_i)$ in (1.3) with the Taylor polynomial of order p for f centered at $\boldsymbol{x}_i$, $i = 1, \ldots, n$

$$T_p[f, \boldsymbol{x}_i](\boldsymbol{x}) = \sum_{j=0}^{p} \frac{f^{(j)}\left(\boldsymbol{x}_i\right)}{j!} \left(\boldsymbol{x} - \boldsymbol{x}_i\right)^j \tag{1.4}$$

the combined operator

$$L_{T_p}\left[f\right]\left(\boldsymbol{x}\right) = \sum_{i=1}^{n} \phi_i\left(\boldsymbol{x}\right) T_p[f, \boldsymbol{x}_i](\boldsymbol{x}) \tag{1.5}$$

has algebraic degree of exactness $\max\{m, p\}$. In order to further enhance the algebraic precision of the operator (1.3), the idea of Xuli in the univariate case [71], in embryonic stage also in the Ph.D. thesis of Kraaijpoel [51] and subsequently drawn on by many authors in the multivariate case [46, 47, 51, 73], consists on the replacement of the Taylor polynomial T_p in (1.5) with a modified version $\tilde{T}_p$ defined as

$$\widetilde{T}_p[f, \boldsymbol{x}_i](\boldsymbol{x}) = \sum_{j=0}^{p} a_j \frac{f^{(j)}\left(\boldsymbol{x}_i\right)}{j!} \left(\boldsymbol{x} - \boldsymbol{x}_i\right)^j \tag{1.6}$$

where

$$a_j := \frac{p!(m + p - j)!}{(m + p)!(p - j)!}, \quad j = 0, 1, \ldots, p. \tag{1.7}$$

The coefficients a_j in (1.7) are all less than or equal to 1. For this reason the modified Taylor polynomial is also called the *reduced Taylor polynomial* [51] and in the following we use this last denomination. After the replacement, we obtain the operator

$$L_{\tilde{T}_p}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \phi_i(\boldsymbol{x})\, \widetilde{T}_p[f, \boldsymbol{x}_i](\boldsymbol{x}) \tag{1.8}$$

which has algebraic degree of exactness $m+p$ [71], interpolates f at each sample point $\boldsymbol{x}_i$, like the operators L and L_{T_p}, but loses the property of interpolation of derivatives of f, if L_{T_p} has this property. This is the case, for example, of Shepard basis functions. Moreover, in this particular case in which $m = 0$, the substitution of the Taylor polynomial with the reduced Taylor one does not cause the enhancement of the algebraic degree of exactness of the combination since $\max\{m, p\} = p = m + p$. From now on, we focus on the case of Shepard basis functions and we start by recalling some basic facts and results on the theory of Shepard operators.

1.2 The Shepard operator and its modifications

The *classical Shepard operator* is the original global inverse distance weighted interpolation method due to Shepard [65]. It is defined by taking $\phi_i(\boldsymbol{x}) = A_{\mu,i}(\boldsymbol{x})$ with

$$A_{\mu,i}(\boldsymbol{x}) = \frac{||\boldsymbol{x} - \boldsymbol{x}_i||_2^{-\mu}}{\sum\limits_{k=1}^{n} ||\boldsymbol{x} - \boldsymbol{x}_k||_2^{-\mu}} \tag{1.9}$$

where $||\cdot||_2$ denotes the Euclidean norm and $\mu > 0$ is a real parameter that controls the range of influence of the data values. The resulting operator,

$$S_\mu[f](\boldsymbol{x}) = \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x})\, f(\boldsymbol{x}_i), \tag{1.10}$$

is a weighted average of the data values, where closer points have greater influence. This operator is stable, in the sense that

$$\min_i f(\boldsymbol{x}_i) \le S_\mu[f](\boldsymbol{x}) \le \max_i f(\boldsymbol{x}_i), \quad \boldsymbol{x} \in \mathbb{R}^2.$$

Each weight function (1.9) can be rewritten in the form [20]

$$A_{\mu,i}(\boldsymbol{x}) = \frac{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} ||\boldsymbol{x} - \boldsymbol{x}_i||_2^{\mu}}{\sum\limits_{k=1}^{n} \prod\limits_{\substack{j=1\\ j\neq k}}^{n} ||\boldsymbol{x} - \boldsymbol{x}_k||_2^{\mu}}, \tag{1.11}$$

the denominator of (1.11) never vanishes and therefore each weight function is a globally defined continuous function. In addition, if μ is an even integer, then each $A_{\mu,i}(\boldsymbol{x})$ is the quotient of polynomials with denominators that do not vanish and so S_μ is in C^∞ [11].

The exponent μ strongly influences the surface to be reconstructed. More precisely, for $0 < \mu < 1$, the Shepard operator (1.10) has cusps at each sample point, and, for $\mu = 1$, it has corners. For $\mu > 1$, the tangent plane at each sample point is parallel to the xy-plane, which produces flat spots at $\boldsymbol{x}_i$. The effect of μ on the surface is illustrated in [45, 57]. The continuity class of Shepard's operator depends on μ and, for $\mu > 0$, is as follows

(i) if μ is an even integer, then $S_\mu[f] \in C^\infty$,

(ii) if μ is an odd integer, then $S_\mu[f] \in C^{\mu-1}$,

(iii) if μ is not an integer, then $S_\mu[f] \in C^{\lfloor\mu\rfloor}$, where $\lfloor\mu\rfloor$ is the larger integer less than or equal to μ.

The cusps and corners are unsatisfactory and the flat spots have discouraged the use of this method, since there is no reason to believe that all functions f have stationary points at all nodes. In order to avoid the drawback of having stationary points and to increase the algebraic degree of exactness as well, many authors [11, 15, 20, 30, 31, 32, 36, 43, 60, 61, 62, 65, 67] suggest to replace the constant values $f(\boldsymbol{x}_i)$ in (1.10) with the values of interpolation polynomials $P_i[\cdot](\boldsymbol{x})$ at $\boldsymbol{x}_i$ applied to f, to obtain the *combined Shepard operator [20]*

$$S_P(\boldsymbol{x}) := \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x}) P_i[f](\boldsymbol{x}). \tag{1.12}$$

The combined Shepard operator enhances the algebraic degree of exactness of the original Shepard method to $\operatorname{dex}(S_P) = \min_{i=1,\dots,n} \operatorname{dex}(P_i[\cdot])$; moreover, it has the property of interpolation of derivative data, i.e.

$$\begin{gathered}\frac{\partial^{r+s}}{\partial x^r \partial y^s} S_P(\boldsymbol{x}_i) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} P_i[f](\boldsymbol{x}_i) \\ i = 1, \dots, n, \quad r, s \in \mathbb{N},\ 1 \le r+s < \mu,\end{gathered} \tag{1.13}$$

since [11]

$$\begin{gathered}\frac{\partial^{r+s}}{\partial x^r \partial y^s} A_{\mu,i}(\boldsymbol{x}_k) = 0, \\ i, k = 1, ..., n, \quad r, s \in \mathbb{N},\ 1 \le r+s < \mu.\end{gathered}$$

Shepard himself proposed to substitute the function evaluation $f(\boldsymbol{x}_i)$ in (1.10) with the first order Taylor polynomial for f centered at $\boldsymbol{x}_i$, obtaining an operator that interpolates function evaluation, as well as first order derivative data at each sample point and has algebraic degree of exactness 1. In [40] Farwig investigated the rate of convergence of the Shepard-Taylor operator

$$S_{T_p}[f](\boldsymbol{x}) = \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x})\, T_p[f, \boldsymbol{x}_i](\boldsymbol{x}) \tag{1.14}$$

as a function of μ in the s-dimensional case. Let us introduce some useful notations and report the Farwig's result on the rate of convergence of the Shepard-Taylor operator in the case $s = 2$.

A function f is said to be of class $C^{k,1}(\Omega)$, $k \geq 0$, if and only if f is of class $C^k(\Omega)$ and all its partial derivatives of order k are Lipschitz continuous in Ω. We associate with this space the seminorm $|\cdot|_{k,1}$ [40]

$$|f|_{k,1} = \sup\left\{ \frac{\left|\frac{\partial^k f}{\partial x^{k-i}\partial y^i}(\boldsymbol{u}) - \frac{\partial^k f}{\partial x^{k-i}\partial y^i}(\boldsymbol{v})\right|}{|\boldsymbol{u}-\boldsymbol{v}|} ; \boldsymbol{u}, \boldsymbol{v} \in \Omega, \boldsymbol{u} \neq \boldsymbol{v}, i = 0, \ldots, k \right\}. \tag{1.15}$$

We denote with $||\cdot||$ the maximum norm (i.e. $||\boldsymbol{x}||$ is the maximum of the moduli of the components of $\boldsymbol{x}$), with $B_\rho(\boldsymbol{x})$ the closed ball, in the maximum norm, centered at $\boldsymbol{x}$ and of radius ρ (i.e. the closed square with side length 2ρ) and set

$$r = \inf\left\{\rho > 0 : \text{for every } \boldsymbol{x} \in \Omega, B_\rho(\boldsymbol{x}) \text{ contains at least one element of } X\right\}.$$

If Ω is a compact convex domain, Farwig's main theorem on the rate of convergence of the Shepard-Taylor operator reduce to the following

Theorem 1 *Let Ω be a compact convex domain. If $f \in C^{p,1}(\Omega)$ then the supremum norm $\left|\left|S_{T_p}[f] - f\right|\right|_\Omega$ of $S_{T_p}[f] - f$ in Ω as the following estimate*

$$\left|\left|S_{T_p}[f] - f\right|\right|_\Omega \leq CM\, |f|_{p,1}\, \varepsilon_\mu^p(r),$$

where

$$\varepsilon_\mu^p(r) = \begin{cases} |\log r|^{-1}, & \mu = 2, \\ r^{\mu-2}, & \mu - 2 < p+1, \mu > 2, \\ r^{\mu-2}|\log r|, & \mu - 2 = p+1, \\ r^{p+1}, & \mu - 2 > p+1. \end{cases}$$

The basis function $A_{\mu,i}$ is global, i.e. it gives too much influence to data points that are far away from the point of approximation and this may be not so good in some cases. Franke and Nielson [43] developed a modification of the original Shepard operator in which the value at a point $\boldsymbol{x} \in \mathbb{R}^2$ depends only on a set $\mathcal{N}_{\boldsymbol{x}}$ of nodes nearest to $\boldsymbol{x}$, and this significantly improves the quality of interpolation, making the Shepard method local and very useful to the interpolation of scattered data. The local Shepard method is obtained by multiplying the Euclidean distances $||\boldsymbol{x} - \boldsymbol{x}_j||_2, j = 1, \ldots, n$, in the expression of $A_{\mu,i}$ (1.9), by Franke-Little weights [10]

$$\left(1 - \frac{||\boldsymbol{x} - \boldsymbol{x}_j||_2}{R_{w_j}}\right)_+^{\mu}, \quad R_{w_j} > 0 \text{ for each } j = 1, \ldots, n,$$

where $(\cdot)_+$ is the positive part function, i.e.

$$(t)_+ := \begin{cases} t, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

The local Shepard basis functions are then defined by

$$\widetilde{W}_{\mu,i}(\boldsymbol{x}) = \frac{W_{\mu,i}(\boldsymbol{x})}{\sum\limits_{k=1}^{n} W_{\mu,k}(\boldsymbol{x})}, \quad \mu > 0, \tag{1.16}$$

where

$$W_{\mu,i}(\boldsymbol{x}) = \left(\frac{1}{||\boldsymbol{x} - \boldsymbol{x}_i||_2} - \frac{1}{R_{w_i}}\right)_+^{\mu}$$

and R_{w_i} is the radius of influence about node $\boldsymbol{x}_i$. In practice R_{w_i} is chosen just large enough to include N_w points in the open ball $B(\boldsymbol{x}_i, R_{w_i}) = \{\boldsymbol{x} : ||\boldsymbol{x} - \boldsymbol{x}_i||_2 < R_{w_i}\}$ [59]. As a consequence, the value at a point $\boldsymbol{x} \in \Omega$ of the *local Shepard operator*

$$\widetilde{S}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) f(\boldsymbol{x}_i) \tag{1.17}$$

depends only on the data $\mathcal{N}_{\boldsymbol{x}} = \{\boldsymbol{x}_i \in X : ||\boldsymbol{x} - \boldsymbol{x}_i||_2 < R_{w_i}\}$.

Like the Shepard operator (1.10), the local Shepard operator $\widetilde{S}[\cdot]$ is stable, interpolates function evaluations at each node but reproduces only constant functions. For $\mu \in \mathbb{N}$, $\mu \geq 1$, the basis functions $\widetilde{W}_{\mu,i}(\boldsymbol{x})$ are of class $C^{\mu-1}(\Omega)$ and

$$\frac{\partial^{r+s}}{\partial x^r \partial y^s} \widetilde{W}_{\mu,i}(\boldsymbol{x}_k) = 0, \quad i, k = 1, \ldots, n; \quad r, s \in \mathbb{N}, \quad 1 \leq r + s < \mu. \tag{1.18}$$

Therefore the *local Shepard-Taylor operator* [40]

$$\widetilde{S}_{T_p}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \sum_{j=0}^{p} \frac{f^{(j)}(\boldsymbol{x}_i)}{j!} (\boldsymbol{x}-\boldsymbol{x}_i)^j \tag{1.19}$$

has algebraic degree of exactness p and interpolates all the data required for its definition, provided that $\mu \geq p+1$.

The Shepard-reduced Taylor operator

$$\widetilde{S}_{\widetilde{T}_p}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \sum_{j=0}^{p} a_j \frac{f^{(j)}(\boldsymbol{x}_i)}{j!} (\boldsymbol{x}-\boldsymbol{x}_i)^j, \tag{1.20}$$

has been recently studied in [73]. As pointed out above in the case of a general linear operator L, it does not increase the algebraic degree of exactness of the local Shepard-Taylor operator and loses the property of interpolation of derivative data.

1.3 Statement of the main results

Recently, in [36], we introduced a new technique for combining local bivariate Shepard operators with three point interpolation polynomials. This technique is based on the association, to each sample point, of a triangle with a vertex in it and the other two vertices in its neighborhood, so as to locally reduce the error of the three point polynomial interpolant. The combination inherits both the degree of exactness and the interpolation conditions of the polynomial interpolant at each sample point.

In this thesis, we want to consider some more kind of combinations of the Shepard operator with three point interpolation polynomial to increase, as much as possible, the algebraic precision of the combined operator, depending on the information we have at each sample point. More precisely, we discuss the following situations.

Case 1 *At each node are given function evaluations and derivative data, up to a fixed order p. We enhance the algebraic degree of exactness of the Shepard operator to $m = p + q$, with $q > 0$, by considering the combination of the Shepard operator with some particular cases of a general class of Hermite interpolation polynomials on the triangle, introduced by Chui and Lai in 1990 [18]. The resulting operators maintain the interpolation properties of the*

Shepard-Taylor operator $S_{T_p}[f]$ *and have the accuracy of approximation of the operator* $S_{T_m}[f]$.

Case 2 *At the nodes* $\boldsymbol{x}_i$, $i = 1, \ldots, n$ *are given some particular kind of lacunary data, such as function evaluations and even order derivative data at each sample point or function evaluations at some nodes and odd order derivative data at all nodes. We enhance the algebraic degree of exactness of the Shepard operator by considering the three point Lidstone and Complementary Lidstone polynomial, respectively. The resulting operators interpolates on all used data and have the accuracy of approximation of the Shepard-Taylor operators which have the same degree of exactness.*

Case 3 *At each node are given only function evaluations. The three point interpolating polynomial, which uses function evaluation at each node, is the three point linear interpolating polynomial. In combining the Shepard operator with that three point polynomial, a kind of asymmetry occurs as we combine "point-based" Shepard basis function with "triangle-based" polynomials. To solve this asymmetry we consider the triangular Shepard operator [53]. We study its properties and provide that this operator has a quadratic approximation, rather than linear.*

The three point interpolation polynomials relate to the vertices of triangles, therefore it is convenient to express them by using barycentric coordinates. To this aim, in the next section, we recall the definition of the barycentric coordinates of a point with respect to a triangle in $\mathbb{R}^2$, introduce notations for the derivatives along the directed sides of the triangle and state some related results which are useful in the study of the remainder term of all the proposed three point interpolation polynomials.

1.4 Barycentric coordinates, directional derivatives and related bounds

Let $\mathbb{Z}_+^2$ denote the set of all pairs with non-negative integer components in the euclidean space $\mathbb{R}^2$. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, we use the notations $|\beta| = \beta_1 + \beta_2$, $\beta! = \beta_1!\beta_2!$ and

$\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for all $i = 1, 2$. Let $V_k = \left(V_k^1, V_k^2\right), k = 0, 1, 2$ be non-collinear points in an anticlockwise ordering and let $\Delta_2\left(V_0, V_1, V_2\right)$ denote the two-dimensional simplex with vertices V_0, V_1, V_2. The *barycentric coordinates* $\left(\lambda_0\left(\boldsymbol{x}\right), \lambda_1\left(\boldsymbol{x}\right), \lambda_2\left(\boldsymbol{x}\right)\right)$, of a generic point $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$, relative to the simplex $\Delta_2\left(V_0, V_1, V_2\right)$, are defined by

$$\lambda_0\left(\boldsymbol{x}\right) = \frac{A\left(\boldsymbol{x}, V_1, V_2\right)}{A\left(V_0, V_1, V_2\right)}, \quad \lambda_1\left(\boldsymbol{x}\right) = \frac{A\left(V_0, \boldsymbol{x}, V_2\right)}{A\left(V_0, V_1, V_2\right)}, \quad \lambda_2\left(\boldsymbol{x}\right) = \frac{A\left(V_0, V_1, \boldsymbol{x}\right)}{A\left(V_0, V_1, V_2\right)} \tag{1.21}$$

where $A\left(V_0, V_1, V_2\right)$ is the signed area of the simplex of vertices V_0, V_1, V_2

$$A\left(V_0, V_1, V_2\right) = \begin{vmatrix} 1 & 1 & 1 \\ V_0^1 & V_1^1 & V_2^1 \\ V_0^2 & V_1^2 & V_2^2 \end{vmatrix}.$$

If f is a differentiable function, and V_i and V_j are two distinct vertices of the simplex $\Delta_2\left(V_0, V_1, V_2\right)$, the derivative of f along the directed line segment from V_i to V_j (side of the simplex) at $\boldsymbol{x}$ is denoted by

$$D_{ij} f\left(\boldsymbol{x}\right) := \left(V_i - V_j\right) \cdot \nabla f\left(\boldsymbol{x}\right), \quad i, j = 0, 1, 2, i \neq j, \tag{1.22}$$

where $\cdot$ is the dot product and $\nabla f\left(\boldsymbol{x}\right) = \left(\frac{\partial f}{\partial x}\left(\boldsymbol{x}\right), \frac{\partial f}{\partial y}\left(\boldsymbol{x}\right)\right)$. The composition of derivatives along the directed sides of the simplex (1.22) are denoted by

$$D_0^\beta = D_{10}^{\beta_1} D_{20}^{\beta_2}, \quad D_1^\beta = D_{01}^{\beta_1} D_{21}^{\beta_2}, \quad D_2^\beta = D_{02}^{\beta_1} D_{12}^{\beta_2}. \tag{1.23}$$

By setting

$$r = \max\left\{\left\|V_0 - V_1\right\|_2, \left\|V_1 - V_2\right\|_2, \left\|V_2 - V_0\right\|_2\right\} \tag{1.24}$$

and

$$S = \frac{1}{A\left(V_0, V_1, V_2\right)}, \tag{1.25}$$

the barycentric coordinates $\lambda_1\left(\boldsymbol{x}\right)$ and $\lambda_2\left(\boldsymbol{x}\right)$ are bounded by

$$\left|\lambda_i\left(\boldsymbol{x}\right)\right| \leq \left(rS\right) \left\|\boldsymbol{x} - V_0\right\|_2, \quad i = 1, 2. \tag{1.26}$$

In fact,

$$\lambda_i\left(\boldsymbol{x}\right) = (-1)^i \frac{\left(V_i^2 - V_0^2\right)\left(x - V_0^1\right) - \left(V_i^1 - V_0^1\right)\left(y - V_0^2\right)}{A\left(V_0, V_1, V_2\right)}, \quad i = 1, 2$$

and, by the Cauchy-Schwartz inequality [9]

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right), \quad a_i, b_i \in \mathbb{R},$$

(1.24) and (1.25), we get (1.26). Moreover, since

$$\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) = \frac{\left(V_2^2 - V_1^2\right)\left(x - V_0^1\right) - \left(V_2^1 - V_1^1\right)\left(y - V_0^2\right)}{A\left(V_0, V_1, V_2\right)}$$

we have

$$|\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})| \leq (rS)\,||\boldsymbol{x} - V_0||_2\,. \tag{1.27}$$

Regarding the bounds for the composition of the derivatives of f along the directed sides of the simplex $\Delta_2(V_0, V_1, V_2)$, we have

Corollary 2 *Let $f \in C^{k,1}(\Omega)$. The derivatives of f along the directed sides of the simplex $\Delta_2(V_0, V_1, V_2)$ satisfy*

$$\left|D_j^{\beta} f(\boldsymbol{x})\right| \leq 2^{|\beta|} r^{|\beta|} |f|_{|\beta|-1,1}\,, \quad j = 0, 1, 2, \quad |\beta| = k+1, \quad \boldsymbol{x} \in \Omega. \tag{1.28}$$

Proof. Let us consider the case $j = 0$. By (1.23) and (1.22) we have

$$\begin{aligned} D_0^{\beta} f(\boldsymbol{x}) = &\sum_{i=0}^{\beta_1} \tbinom{\beta_1}{i} (V_1^1 - V_0^1)^{\beta_1 - i} \left(V_1^2 - V_0^2\right)^i \\ &\sum_{j=0}^{\beta_2} \tbinom{\beta_2}{j} (V_2^1 - V_0^1)^{\beta_2 - j} \left(V_2^2 - V_0^2\right)^j \tfrac{\partial^{|\beta|} f(\boldsymbol{x})}{\partial x^{|\beta|-i-j} \partial y^{i+j}}. \end{aligned}$$

Since $|\beta| \geq 1$, at least one between $|\beta| - i - j$ and $i + j$ is greater than or equal to 1, $i = 0, \ldots, \beta_1$, $j = 0, \ldots, \beta_2$. If $|\beta| - i - j \geq 1$

$$\left|\frac{\partial^{|\beta|} f(\boldsymbol{x})}{\partial x^{|\beta|-i-j} \partial y^{i+j}}\right| = \left|\lim_{h \to 0} \frac{\frac{\partial^{|\beta|-1} f(x+h,y)}{\partial x^{|\beta|-i-j-1} \partial y^{i+j}} - \frac{\partial^{|\beta|-1} f(x,y)}{\partial x^{|\beta|-i-j-1} \partial y^{i+j}}}{h}\right| \leq |f|_{|\beta|-1,1}$$

otherwise

$$\left|\frac{\partial^{|\beta|} f(\boldsymbol{x})}{\partial x^{|\beta|-i-j} \partial y^{i+j}}\right| = \left|\lim_{h \to 0} \frac{\frac{\partial^{|\beta|-1} f(x,y+h)}{\partial x^{|\beta|-i-j} \partial y^{i+j-1}} - \frac{\partial^{|\beta|-1} f(x,y)}{\partial x^{|\beta|-i-j} \partial y^{i+j-1}}}{h}\right| \leq |f|_{|\beta|-1,1}$$

$i = 0, \ldots, \beta_1$, $j = 0, \ldots, \beta_2$. Therefore we have

$$\begin{aligned}
\left| D_0^\beta f(\boldsymbol{x}) \right| = & \left| \sum_{i=0}^{\beta_1} \binom{\beta_1}{i} (V_1^1 - V_0^1)^{\beta_1 - i} \left(V_1^2 - V_0^2\right)^i \right. \\
& \left. \sum_{j=0}^{\beta_2} \binom{\beta_2}{j} (V_2^1 - V_0^1)^{\beta_2 - j} \left(V_2^2 - V_0^2\right)^j \frac{\partial^{|\beta|} f(\boldsymbol{x})}{\partial x^{|\beta|-i-j} \partial y^{i+j}} \right| \\
& \leq \sum_{i=0}^{\beta_1} \binom{\beta_1}{i} \left|\left|V_1^1 - V_0^1\right|\right|_2^{\beta_1 - i} \left|\left|V_1^2 - V_0^2\right|\right|_2^i \\
& \sum_{j=0}^{\beta_2} \binom{\beta_2}{j} \left|\left|V_2^1 - V_0^1\right|\right|_2^{\beta_2 - j} \left|\left|V_2^2 - V_0^2\right|\right|_2^j \left| \frac{\partial^{|\beta|} f(\boldsymbol{x})}{\partial x^{|\beta|-i-j} \partial y^{i+j}} \right| \\
& \leq 2^{|\beta|} \left|\left|V_1 - V_0\right|\right|_2^{\beta_1} \left|\left|V_2 - V_0\right|\right|_2^{\beta_2} \left|f\right|_{|\beta|-1,1} \\
& \leq 2^{|\beta|} r^{|\beta|} \left|f\right|_{|\beta|-1,1} .
\end{aligned}$$

The cases $j = 1, 2$ are analogous. ■

1.5 On the remainder term of three point interpolation polynomials

Let $\Delta_2(V_0) := \Delta_2(V_0, V_1, V_2; V_0)$ denote the simplex $\Delta_2(V_0, V_1, V_2)$ with fixed vertex V_0 and let $P_k^{\Delta_2(V_0)}[\cdot], k \in \mathbb{N}_0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, be a three point polynomial operator based on the vertices of $\Delta_2(V_0)$. For a function $f \in C^p(\Omega)$ we suppose that the polynomial $P_k^{\Delta_2(V_0)}[f]$ is expressed in terms of the functional or derivative data of a set $\mathcal{I}_f \subset \left\{ D_i^\beta f(V_i), i = 0, 1, 2, \beta \in \mathbb{N}_0^2, |\beta| \leq p \right\}$ of cardinality k, i.e.

$$P_k^{\Delta_2(V_0)}[f](\boldsymbol{x}) = \sum_{\substack{i,\beta \\ D_i^\beta f(V_i) \in \mathcal{I}_f}} B_{i,\beta}(\boldsymbol{x}) D_i^\beta f(V_i), \tag{1.29}$$

where $B_{i,\beta}(\boldsymbol{x}) \in \mathcal{P}_{\boldsymbol{x}}^k$. We suppose also that $\mathrm{dex}\left(P_k^{\Delta_2(V_0)}[\cdot]\right) = k$ or, equivalently, that the interpolation problem

$$D_i^\beta \left(P_k^{\Delta_2(V_0)}[f](V_i) \right) = f_i^\beta, \quad i, \beta \text{ such that } D_i^\beta f(V_i) \in \mathcal{I}_f$$

has a unique solution in $\mathcal{P}_{\boldsymbol{x}}^k$ for any choice of the numbers $f_i^\beta \in \mathbb{R}$, i.e. that

$$\left\{ B_{i,\beta}(\boldsymbol{x}), i, \beta \text{ such that } D_i^\beta f(V_i) \in \mathcal{I}_f \right\} \tag{1.30}$$

is a basis of $\mathcal{P}_{\boldsymbol{x}}^{k}$ [33]. If $f \in C^{k,1}(\Omega)$ we provide an explicit expression for the remainder term

$$R_k^{P,\Delta_2(V_0)}[f] = f - P_k^{\Delta_2(V_0)}[f] \tag{1.31}$$

through the $(k-|\beta|)$-th order truncated Taylor expansions $T_{k-|\beta|}\left[D_i^{\beta} f, V_0\right](V_i)$ of the data $D_i^{\beta} f(V_i)$ centered at V_0 with integral remainder along the segment $[V_0, V_i]$ (note that for $i=0$ we have $T_{k-|\beta|}\left[D_0^{\beta} f, V_0\right](V_0) = D_0^{\beta} f(V_0)$). In fact, for each i, β such that $D_i^{\beta} f(V_i) \in \mathcal{I}_f$, we have

$$\begin{aligned} D_i^{\beta} f(V_i) &= T_{k-|\beta|}\left[D_i^{\beta} f, V_0\right](V_i) \\ &+ \int_0^1 \frac{D_{i0}^{k-|\beta|+1} D_i^{\beta} f(V_0 + t(V_i - V_0))}{(k-|\beta|)!}(1-t)^{k-|\beta|}\, dt \end{aligned} \tag{1.32}$$

where

$$T_{k-|\beta|}\left[D_i^{\beta} f, V_0\right](V_i) = \sum_{0 \le \gamma_i \le k-|\beta|} \frac{1}{\gamma_i!} D_{i0}^{\gamma_i} D_i^{\beta} f(V_0)$$

and the Lebesgue integral on the right hand side of (1.32) is well defined and finite. In fact $f \in C^{k,1}(\Omega)$ implies that the k-th order derivatives of f are univariate absolutely continuous functions when restricted to the segments $[V_0, V_i], i = 1, 2$ [63, pag.157]. A Lebesgue Theorem [50, pag.340] ensures therefore that the $(k+1)$-th order derivatives of f exist almost everywhere, are Lebesgue integrable and (1.32) holds after a repeated integration by parts. By substituting (1.32) in (1.29), after some rearrangement we get

$$P_k^{\Delta_2(V_0)}[f](\boldsymbol{x}) = \widetilde{T}_k[f, V_0](\boldsymbol{x}) + \delta_k^{P,\Delta_2(V_0)}(\boldsymbol{x}) \tag{1.33}$$

where

$$\widetilde{T}_k[f, V_0](\boldsymbol{x}) = \sum_{\substack{i,\beta \\ D_i^{\beta} f(V_i) \in \mathcal{I}_f}} B_{i,\beta}(\boldsymbol{x})\, T_{k-|\beta|}\left[D_i^{\beta} f, V_0\right](V_i)$$

is a polynomial in $\boldsymbol{x}$ which is expressed only in terms of directional derivatives of f up to the order k at V_0 and

$$\delta_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x}) = \sum_{\substack{i,\beta \\ D_i^{\beta} f(V_i) \in \mathcal{I}_f}} B_{i,\beta}(\boldsymbol{x}) \int_0^1 \frac{D_{i0}^{k-|\beta|+1} D_i^{\beta} f(V_0 + t(V_i - V_0))}{(k-|\beta|)!}(1-t)^{k-|\beta|}\, dt \tag{1.34}$$

is a polynomial in $\boldsymbol{x}$ which is expressed only in terms of the integral of $(k+1)$-th order directional derivatives of f. Since $\delta_k^{P,\Delta_2(V_0)}[f] \equiv 0$ for each $f \in \mathcal{P}_{\boldsymbol{x}}^{k}$ and $\operatorname{dex}\left(P_k^{\Delta_2(V_0)}[\cdot]\right) =$

k, the polynomial operator $\widetilde{T}_k[\cdot, V_0]$ reproduces exactly polynomials up to the degree k. For this reason we can affirm that $\widetilde{T}_k[f, V_0]$ is the k-th order Taylor polynomial $T_k[f, V_0]$ for f centered at V_0

$$T_k[f, V_0](\boldsymbol{x}) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(V_0)(\boldsymbol{x} - V_0)^\alpha \tag{1.35}$$

where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \tag{1.36}$$

and

$$(\boldsymbol{x} - V_0)^\alpha = \left(x - V_0^1\right)^{\alpha_1} \left(y - V_0^2\right)^{\alpha_2}.$$

In fact, $\widetilde{T}_k[f, V_0]$ can be expressed, after some computation and rearrangement, in terms of some partial derivatives of f up to the order k at a point $V_0 \in \Omega$

$$\widetilde{T}_k[f, V_0](\boldsymbol{x}) = \sum_{\substack{|\alpha| \leq k \\ \alpha \in \mathbb{N}_0^2}} D^\alpha f(V_0)\, p_\alpha(\boldsymbol{x}),$$

where $p_\alpha(\boldsymbol{x})$ are polynomials of degree at most k. Since $\widetilde{T}_k[f, V_0] = f$ for each $f \in \mathcal{P}_{\boldsymbol{x}}^k$ then $\{p_\alpha(\boldsymbol{x}) : |\alpha| \leq k\}$ generates $\mathcal{P}_{\boldsymbol{x}}^k$, and therefore $p_\alpha(\boldsymbol{x}) \neq 0$ for each α. Since $\widetilde{T}_k[\cdot, V_0]$ reproduces exactly all polynomials in $\mathcal{P}_{\boldsymbol{x}}^k$ it follows that

$$\widetilde{T}_k[T_k[f, V_0], V_0](\boldsymbol{x}) = T_k[f, V_0](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega; \tag{1.37}$$

on the other hand

$$\widetilde{T}_k[T_k[f, V_0], V_0](\boldsymbol{x}) = \widetilde{T}_k[f, V_0](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.38}$$

since for each α, such that $|\alpha| \leq k$

$$D^\alpha T_k[f, V_0](V_0) = D^\alpha f(V_0).$$

Therefore, by equaling the right hand side terms of (1.37) and (1.38) we get

$$\widetilde{T}_k[f, V_0](\boldsymbol{x}) = T_k[f, V_0](\boldsymbol{x}).$$

From (1.34), the polynomial $\delta_k^{P, \Delta_2(V_0)}[f](\boldsymbol{x})$ depends only on the integral of derivatives $D_{i0}^{k-|\beta|+1} D_i^\beta f(V_0 + t(V_i - V_0))$ of f of order $k+1$ restricted to the segments $[V_0, V_i]$, $i =$

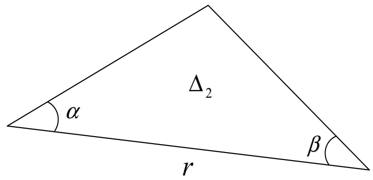


Figure 1.1: The angles adjacent to the longest edge of triangle Δ_2 with length r are denoted by α and β.

$1, 2$ and on the polynomials $B_{i,\beta}(\boldsymbol{x})$ which can be expressed in terms of the barycentric coordinates $(\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x}))$. By using bounds (1.26), (1.27) and (1.28) we can get a bound for the polynomial $\delta_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x})$ in terms of the quantities r, $(r^2 S)$ and $||\boldsymbol{x} - V_0||_2$ (in this work this is done, case by case, for the expansions of Chapters 2, 3, 4 and 5). These bounds have the following general form

$$\left|\delta_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 2^k |f|_{k,1} \left(\sum_{\alpha=0}^{k} C_\alpha r^{k+1-\alpha} \left(r^2 S\right)^\alpha ||\boldsymbol{x} - V_0||_2^\alpha\right), \quad \forall \boldsymbol{x} \in \Omega, \tag{1.39}$$

where C_α are non-negative constants which depend on $P_k^{\Delta_2(V_0)}$. By settings (1.24) and (1.25) the quantity $r^2 S$ depends only on the shape of the fixed triangle $\Delta_2(V_0)$. In fact, if α, β denote the adjacent angles to the side of length r, then $r^2 S = \frac{\sin(\alpha+\beta)}{\sin\alpha \sin\beta} = \cot(\alpha) + \cot(\beta) \geq \frac{2}{\sqrt{3}}$ (See Figure 1.1).

Since each term in (1.39) has the factor $r^{k+1-\alpha}$ with $k+1-\alpha \geq 1$, from previous considerations it follows that

$$\lim_{\substack{r \to 0 \\ r^2 S = const}} \left\|\delta_k^{P,\Delta_2(V_0)}[f]\right\|_\Omega = 0. \tag{1.40}$$

In other words, the limit in (1.40) means that when V_1 and V_2 tends to V_0 so as the initial shape of $\Delta_2(V_0)$ does not change, the polynomial $P_k^{\Delta_2(V_0)}[f](\boldsymbol{x})$ tends, in the uniform norm, to the k-th order Taylor polynomial for f centered at V_0. This property is also implied by the results of a paper by Ciarlet and Raviart [19].

For the remainder (1.31) we easily get

$$\begin{aligned} &R_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x}) = f(\boldsymbol{x}) - P_k^{\Delta_2(V_0)}[f](\boldsymbol{x}) \\ &= f(\boldsymbol{x}) - T_k[f, V_0](\boldsymbol{x}) + T_k[f, V_0](\boldsymbol{x}) - P_k^{\Delta_2(V_0)}[f](\boldsymbol{x}) \\ &= R_k^T[f](\boldsymbol{x}) + \delta_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x}), \qquad\qquad \boldsymbol{x} \in \Omega \end{aligned}$$

where $R_k^T[f](\boldsymbol{x})$ is the remainder in the Taylor expansion. Hence we have

$$\begin{aligned}
&\left|R_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \le \left|R_k^T[f](\boldsymbol{x})\right| + \left|\delta_k^{P,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \\
&\le 2^k |f|_{k,1}\left(\frac{||\boldsymbol{x}-V_0||_2^{k+1}}{(k-1)!} + \sum_{\alpha=0}^{k} C_\alpha r^{k+1-\alpha}\left(r^2 S\right)^\alpha ||\boldsymbol{x}-V_0||_2^\alpha\right), \quad \boldsymbol{x}\in\Omega
\end{aligned} \tag{1.41}$$

1.6 Shepard operator combined with three point interpolation polynomials

Here we use the results of previous Section to define a new class of combined Shepard operators. The technique, introduced in [36], is based on the association, to each sample point $\boldsymbol{x}_i \in X$, of a triangle $\Delta_2(i)$ with a vertex in $\boldsymbol{x}_i$ (the fixed vertex) and the other two vertices in a neighborhood $B(\boldsymbol{x}_i, R_{w_i})$ of $\boldsymbol{x}_i$. Among all eligible triangles in $B(\boldsymbol{x}_i, R_{w_i})$, $\Delta_2(i)$ is chosen so as to locally reduce $\delta_k^{P,\Delta_2(i)}[f]$ through the bound (1.39). More precisely, for each set of indices $\{j,k\}$ such that $\Delta_2(\boldsymbol{x}_i,\boldsymbol{x}_j,\boldsymbol{x}_k;\boldsymbol{x}_i) \subset B(\boldsymbol{x}_i, R_{w_i})$ we set $r_{i,j,k} = \max\left\{||\boldsymbol{x}_i-\boldsymbol{x}_j||_2, ||\boldsymbol{x}_i-\boldsymbol{x}_k||_2, ||\boldsymbol{x}_k-\boldsymbol{x}_j||_2\right\}$ and $S_{i,j,k} = \frac{1}{A(\boldsymbol{x}_i,\boldsymbol{x}_j,\boldsymbol{x}_k)}$; therefore in the ball $B(\boldsymbol{x}_i, r_{i,j,k})$ we get

$$\begin{aligned}
\left|\delta_k^{P,\Delta_2(\boldsymbol{x}_i,\boldsymbol{x}_j,\boldsymbol{x}_k;\boldsymbol{x}_i)}[f](\boldsymbol{x})\right| &\le 2^k |f|_{k,1}\left(\sum_{\alpha=0}^{k} C_\alpha r_{i,j,k}^{k+1}\left(r_{i,j,k}^2 S_{i,j,k}\right)^\alpha\right) \\
&\le 2^k |f|_{k,1}\, r_{i,j,k}^{k+1}\left(r_{i,j,k}^2 S_{i,j,k}\right)^k\left(\sum_{\alpha=0}^{k} C_\alpha\right).
\end{aligned}$$

We set

$$\Delta_2(i) = \Delta_2\left(\boldsymbol{x}_i, \boldsymbol{x}_{\overline{j}}, \boldsymbol{x}_{\overline{k}}; \boldsymbol{x}_i\right)$$

where the indices $\overline{j},\overline{k}$ are such that the quantity $r_{i,\overline{j},\overline{k}}^{k+1}\left(r_{i,\overline{j},\overline{k}}^2 S_{i,\overline{j},\overline{k}}\right)^k$ is as small as possible. This kind of choice is a compromise between the choice of a small triangle (which, however, could have very small angles and therefore a large value of r^2S) and a regular triangles (for which r^2S is near to the minimum value but which could have a big area).

We combine the local Shepard operator (1.17) with three point interpolation polynomials as follows

$$S_P[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x})\, P_k^{\Delta_2(i)}[f](\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega. \tag{1.42}$$

As usual, the combination (1.42) inherits the degree of exactness of the operators $P_k^{\Delta_2(i)}[\cdot]$ and, for each $\mu > p+1$, the interpolation conditions satisfied by each local polynomial interpolant $P_k^{\Delta_2(i)}[\cdot]$ at the vertex $\boldsymbol{x}_i$, since the basis functions $\widetilde{W}_{\mu,i}$ form a partition of unity and satisfy the cardinality property (1.1) and (1.18).

To give a bound for the remainder term

$$\begin{aligned} R_S[f](\boldsymbol{x}) &= f(\boldsymbol{x}) - S_P[f](\boldsymbol{x}) \\ &= \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \left(f(\boldsymbol{x}) - P_k^{\Delta_2(i)}[f](\boldsymbol{x}) \right), \end{aligned} \tag{1.43}$$

in line with [40, 72], we set

1. $\mathcal{I}_{\boldsymbol{x}} = \{i \in \{1,\ldots,n\} : ||\boldsymbol{x} - \boldsymbol{x}_i||_2 < R_{w_i}\}$, $\boldsymbol{x} \in \Omega$;

2. $M = \sup\limits_{\boldsymbol{x}\in\Omega} \{\#(\mathcal{I}_{\boldsymbol{x}})\}$, where # is the cardinality operator.

By easy calculation we get

$$\begin{aligned} &|R_S[f](\boldsymbol{x})| = |f(\boldsymbol{x}) - S_P[f](\boldsymbol{x})| \\ &\le \sum_{i\in\mathcal{I}_{\boldsymbol{x}}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \left| f(\boldsymbol{x}) - P_k^{\Delta_2(i)}[f](\boldsymbol{x}) \right| \\ &\le 2^k |f|_{k,1} \sum_{i\in\mathcal{I}_{\boldsymbol{x}}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \left(\frac{||\boldsymbol{x} - V_{i_0}||_2^{k+1}}{(k-1)!} + \sum_{\alpha=0}^{k} C_{\alpha,i} r_i^{k+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} ||\boldsymbol{x} - V_{i_0}||_2^{\alpha} \right) \\ &\le 2^k |f|_{k,1} \sum_{i\in\mathcal{I}_{\boldsymbol{x}}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \left(\frac{R_{w_i}^{k+1}}{(k-1)!} + \sum_{\alpha=0}^{k} C_{\alpha,i} r_i^{k+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R_{w_i}^{\alpha} \right) \\ &\le 2^k |f|_{k,1} M \max_{i\in\mathcal{I}_{\boldsymbol{x}}} \left\{ \frac{R_{w_i}^{k+1}}{(k-1)!} + \sum_{\alpha=0}^{k} C_{\alpha,i} r_i^{k+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R_{w_i}^{\alpha} \right\}. \end{aligned} \tag{1.44}$$

We see that the error estimate (1.44) is essentially that of Zuppa's paper [72] for the modified local Shepard's interpolation formula apart for the term $\sum\limits_{\alpha=0}^{k} C_{\alpha,i} r_i^{k+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R_{w_i}^{\alpha}$. In the practice R_{w_i} is chosen just large enough to include N_w nodes in the ball $B(\boldsymbol{x}_i, R_{w_i})$; for a set of n uniformly distributed nodes and $N_w \ll n$ we can assume $r_i \approx R_{w_i}$ and therefore

$$|R_S[f](\boldsymbol{x})| \lessapprox 2^k |f|_{k,1} N_w \left(\frac{1}{(k-1)!} + \max_{i\in\mathcal{I}_{\boldsymbol{x}}} \left\{ \sum_{\alpha=0}^{k} C_{\alpha,i} \left(r_i^2 S_i\right)^{\alpha} \right\} \right) \max_{i\in\mathcal{I}_{\boldsymbol{x}}} \left\{ R_{w_i}^{k+1} \right\}.$$

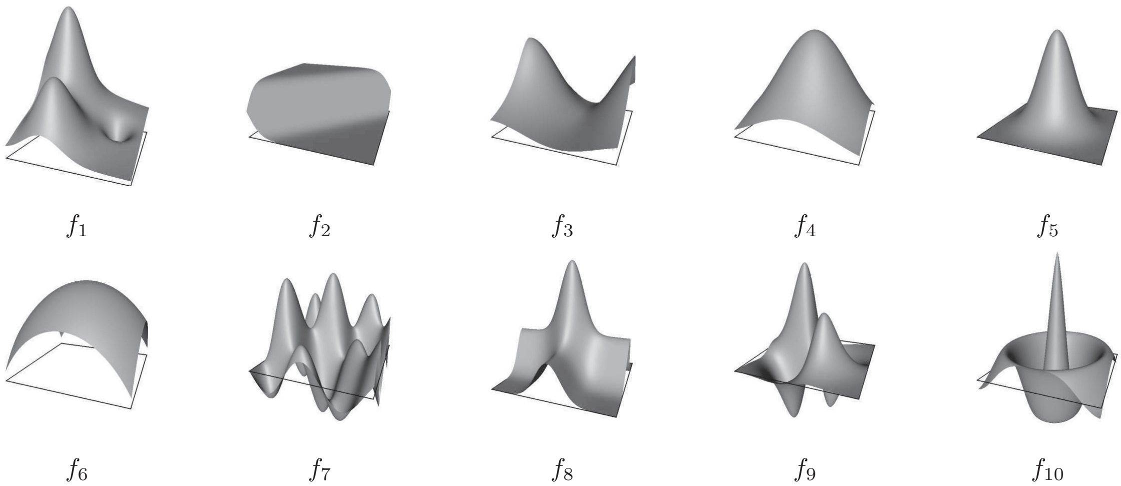
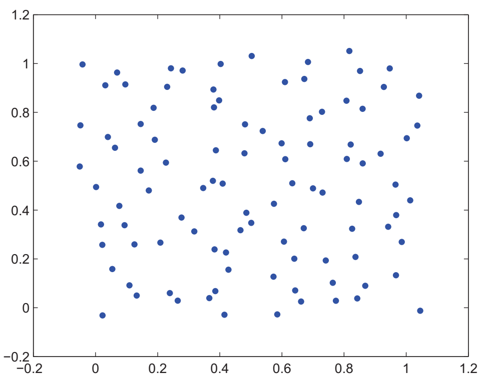
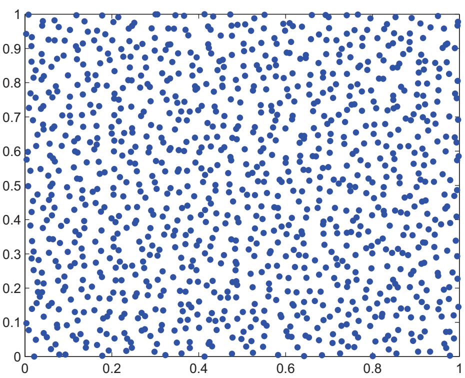


Figure 1.2: Test functions used in our numerical experiments. The definitions can be found in [62].



Franke dataset



Sparse set of $n = 1089$ uniformly distributed interpolation nodes

Figure 1.3: Set of nodes

1.7 Settings for the numerical experiments

To test the accuracy of approximation of the combined Shepard operators (1.42) in the bivariate interpolation of large sets of scattered data, we carry out various numerical experiments with different sets of nodes (See Figure 1.3) and 10 test functions (see Figure 1.2), including those introduced by Franke [41, 42] and Renka and Brown [62].

$$\begin{aligned}\text{Exponential:}\quad F_1 = {} & 0.75\exp\Big(-\frac{(9x-2)^2+(9y-2)^2}{4}\Big)\\ & +0.50\exp\Big(-\frac{(9x-7)^2+(9y-3)^2}{4}\Big)\\ & +0.75\exp\Big(-\frac{(9x+1)^2}{49}-\frac{(9y+1)^2}{10}\Big)\\ & -0.20\exp\Big(-(9x-4)^2-(9y-7)^2\Big),\end{aligned}$$

$$\text{Gentle:}\qquad F_2 = \frac{\exp\Big(-\frac{81}{16}\left((x-0.5)^2+(y-0.5)^2\right)\Big)}{3},$$

$$\text{Sphere:}\qquad F_3 = \frac{\sqrt{64-81((x-0.5)^2+(y-0.5)^2)}}{9} - 0.5,$$

$$\text{Saddle:}\qquad F_4 = \frac{1.25+\cos(5.4y)}{6+6(3x-1)^2},$$

$$\text{Cliff:}\qquad F_5 = \frac{\tanh(9y-9x)+1}{9},$$

$$\text{Steep:}\qquad F_6 = \frac{\exp\Big(-\frac{81}{4}((x-0.5)^2+(y-0.5)^2)\Big)}{3},$$

$$F_7 = 2\cos(10x)\sin(10y)+\sin(10xy),$$

$$F_8 = \exp\Big(-\frac{(5-10x)^2}{2}\Big) + 0.75\exp\Big(-\frac{(5-10y)^2}{2}\Big)$$
$$+0.75\exp\Big(-\frac{(5-10x)^2}{2}\Big)\exp\Big(-\frac{(5-10y)^2}{2}\Big),$$

$$F_9 = \left(\left(\tfrac{20}{3}\right)^3\exp\Big(\frac{10-20x}{3}\Big)\exp\Big(\frac{10-20y}{3}\Big)\right)^2$$
$$\times\left(\left(\frac{1}{1+\exp\Big(\frac{10-20x}{3}\Big)}\right)\left(\frac{1}{1+\exp\Big(\frac{10-20y}{3}\Big)}\right)\right)^5$$
$$\times\left(\exp\Big(\frac{10-20x}{3}\Big) - \frac{2}{1+\exp\Big(\frac{10-20x}{3}\Big)}\right),$$

$$F_{10} = \exp\Big(-0.04\sqrt{(80x-40)^2+(90y-45)^2}\Big)$$
$$\times\cos\Big(0.15\sqrt{(80x-40)^2+(90y-45)^2}\Big).$$

We compare numerical results obtained by applying the Shepard-operators (1.42) with those obtained by applying the Shepard-Taylor operators with the same algebraic degree of exactness. We remark that the second operator uses function and derivatives that cannot be available in some particular problems.

We compute numerical approximations by using the Franke dataset [49] or a sparse set of 1089 uniformly distributed interpolation nodes in the unit square R (See Figure 1.3), introduced in [36]. The resulting approximations are computed on a regular grid of $n_e = 101 \times 101$ points of R. In achieving the numerical comparisons we use three error metrics [67]. These are the maximum absolute error $e_{\max}$, the mean absolute error e_{mean} and mean square error e_{MS}. The absolute approximation error is defined by

$$e_i = \|S_P[F_j](z_i) - F_j(z_i)\|_2, \quad z_i \in R, j = 1, \ldots, 10$$

where S_P is one the combined Shepard operator (1.42), F_j is one of the test functions and z_i are the points of the regular grid of evaluation points of R. Therefore, the maximum absolute error is

$$e_{\max} = \max_{1\le i\le n_e} e_i, \tag{1.45}$$

the mean absolute error is

$$e_{mean} = \frac{1}{n_e} \sum_{i=1}^{n_e} e_i, \tag{1.46}$$

and the mean square error is defined as

$$e_{MS} = \sqrt{\frac{\sum_{i=1}^{n_e} e_i^2}{n_e}}. \tag{1.47}$$

Chapter 2

The Shepard-Hermite operators

Over all this chapter we assume that, together with function evaluations, all partial derivatives up to a certain order $p \in \mathbb{N}$ are given at each sample point. Under this assumption, we combine the Shepard operator with Hermite interpolation polynomials on the simplex in order to enhance, as much as possible, the algebraic degree of exactness of the combined operator. For one variable, the Hermite interpolation problem consists in finding a polynomial that interpolates, on a set of distinct nodes, values of a function and its successive derivatives up to a certain order, which may depend on the node of interpolation. As well known, for each set of n distinct nodes $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ and associated set of n corresponding non negative integers $\{p_1, \ldots, p_n\}$, there exists only one polynomial of degree $m = n + \sum\limits_{i=1}^{n} p_i - 1$ that interpolates at $\boldsymbol{x}_i$, $i = 1, \ldots, n$ function evaluation and successive derivatives up to the order p_i of a given function f.

In the case of several variables the definition of Hermite interpolation problem is less clear [64]; in the literature it is often defined as interpolation of all partial derivatives up to a certain order [54]. Since we consider the case of three not collinear nodes, in line with [18], we find more convenient to state the problem in terms of directional derivatives along the side of the two dimensional simplex, instead of using classical partial derivatives. To do this we need to take care not only of the maximum order p of derivative data given at each of the three vertices, but also of the dimension of the polynomial space we work with. In fact, for a fixed p, different subsets of derivative data at each vertex of the simplex can be fixed such that the associated three point interpolation problem is unisolvent in a polynomial space of opportunely total degree. Since we want that the Shepard-Hermite operator interpolates all given data, we require that these subsets contain all available data at the fixed vertex V_0 of the simplex $\Delta_2(V_0, V_1, V_2; V_0)$ and some of the available data at the remaining vertices V_1, V_2 for a total of $\dim(\mathcal{P}^m_{\boldsymbol{x}}) = \binom{m+2}{2}$ interpolation conditions. In this way the resulting Shepard-Hermite operator has algebraic degree of exactness $m > p$ that is better than the Shepard-Taylor operators which use the same set of data.

2.1 Some remarks on the Hermite polynomial on the simplex

The formulation of the Hermite interpolation polynomials on the vertices of a simplex $\Delta_s(V_0, V_1, \ldots, V_s) \subset \mathbb{R}^s$, given by Chui and Lai in a famous paper of 1991 [18] in connection with the notion of super vertex splines, is here readapted to the case $\Delta_2(V_0, V_1, V_2) \subset \mathbb{R}^2$. Moreover, let $c_i, i = 0, 1, 2$ be the projection from $\mathbb{Z}^3_+$ to $\mathbb{Z}^2_+$ which associates to each $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3_+$ the pair obtained from α deprived of the component i, i.e.

$$c_0\alpha = (\alpha_1, \alpha_2), \quad c_1\alpha = (\alpha_0, \alpha_2), \quad c_2\alpha = (\alpha_0, \alpha_1).$$

The following Theorem is proven in [18, Theorem 3.1.1] in the general case of $\Delta_s(V_0, V_1, \ldots, V_s) \subset \mathbb{R}^s$.

Theorem 3 *In Bézier representation with respect to $\Delta_2(V_0, V_1, V_2)$, the Taylor polynomial of order m of a sufficiently smooth function f centered at the vertex V_0 is given by*

$$T_m[f, V_0](\boldsymbol{x}) = \sum_{\substack{|\alpha|=m \\ \alpha\in\mathbb{Z}^3_+}} \sum_{\substack{\beta\leq c_0\alpha \\ \beta\in\mathbb{Z}^2_+}} \binom{c_0\alpha}{\beta} \frac{(m-|\beta|)!}{m!} D_0^\beta f(V_0)\, \phi^m_\alpha(\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) \tag{2.1}$$

where

$$\phi^m_\alpha(\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) = \frac{m!}{\alpha!} \lambda_0(\boldsymbol{x})^{\alpha_0} \lambda_1(\boldsymbol{x})^{\alpha_1} \lambda_2(\boldsymbol{x})^{\alpha_2}.$$

Now we specify certain Hermite-type interpolation conditions on the vertices of a two–dimensional simplex which ensure uniqueness of interpolation in bivariate polynomial spaces $\mathcal{P}^m_{\boldsymbol{x}}$. To this purpose, we need some additional definitions.

Definition 4 *A subset* M^2 *of* $\mathbb{Z}_+^2$ *is called a lower set if for each* $\beta, \gamma \in \mathbb{Z}_+^2$, $\beta \in M^2$ *and* $0 \leq \gamma \leq \beta$ *it results* $\gamma \in M^2$.

Let $\Gamma_p^2 := \left\{\beta \in \mathbb{Z}_+^2 : |\beta| \leq p\right\}$, $\Lambda_p^3 := \left\{\alpha \in \mathbb{Z}_+^3 : |\alpha| = p\right\}$ and A_i^p the raising map from Γ_p^2 to Λ_p^3 defined by

$$A_0^p\beta = (p - |\beta|, \beta_1, \beta_2), \quad A_1^p\beta = (\beta_1, p - |\beta|, \beta_2), \quad A_2^p\beta = (\beta_1, \beta_2, p - |\beta|), \quad \beta \in \mathbb{Z}_+^2.$$

Definition 5 *A collection of subsets* M_0^2, M_1^2, M_2^2 *of* Γ_p^2 *is said to form a partition of* Λ_p^3 *if*

1. $A_i^p M_i^2 \cap A_j^p M_j^2 = \emptyset$ *for* $i \neq j$, *and*
2. $\bigcup_{i=0}^{2} A_i^p M_i^2 = \Lambda_p^3$.

The following Theorem is proven in [18, Theorem 3.1.4] in the general case of $\Delta_s(V_0, V_1, \ldots, V_s) \subset \mathbb{R}^s$ and lower sets $M_0^s, M_1^s, ..., M_n^s$ forming a partition of Λ_p^{s+1}.

Theorem 6 *Let* $M_0^2 = \left\{\beta \in \mathbb{Z}_+^2 : |\beta| \leq p\right\}$ *and* M_1^2, M_2^2 *lower sets forming a partition of* Λ_p^3. *Then for any given set of data* $\left\{f_{i,\beta} \in \mathbb{R} : \beta \in M_i^2, i = 0, 1, 2\right\}$ *there exists a unique polynomial* $H_{m,p}^{\Delta_2(V_0)}$ *of total degree* $m = p + q$ *satisfying*

$$D_i^\beta H_{m,p}^{\Delta_2(V_0)}(V_i) = f_{i,\beta}, \quad \beta \in M_i^2, i = 0, 1, 2.$$

Moreover, $H_{m,p}^{\Delta_2(V_0)}(\boldsymbol{x})$ *may be formulated in the Bézier representation of total degree* m *with respect to the simplex* $\Delta_2(V_0)$ *as follows*

$$H_{m,p}^{\Delta_2(V_0)}(\boldsymbol{x}) = \sum_{i=0}^{2} \sum_{\beta \in M_i^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(m - |\gamma|)!}{m!} f_{i,\gamma} \right\} \phi_{A_i^m\beta}^m (\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) \qquad (2.2)$$

where

$$\phi_{A_i^m\beta}^m (\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) = \frac{m!}{(A_i^m\beta)!} \lambda_0(\boldsymbol{x})^{\left(A_i^m\beta\right)_0} \lambda_1(\boldsymbol{x})^{\left(A_i^m\beta\right)_1} \lambda_2(\boldsymbol{x})^{\left(A_i^m\beta\right)_2}.$$

Let f be a function of class $C^m(\Omega)$. Following above notations we set $f_{i,\beta} = D_i^\beta f(V_i)$, $\beta \in M_i^2, i = 0, 1, 2$; for each $\boldsymbol{x} \in \Omega$, we have

$$f(\boldsymbol{x}) = H_{m,p}^{\Delta_2(V_0)}[f](\boldsymbol{x}) + R_m^{H,\Delta_2(V_0)}[f](\boldsymbol{x}) \qquad (2.3)$$

where $R_m^{H,\Delta_2(V_0)}[f](\boldsymbol{x})$ is the remainder term. We emphasize that $H_{m,p}^{\Delta_2(V_0)}[f]$ interpolates function evaluations at the vertices of $\Delta_2(V_0)$, *all partial derivatives* of f up to the order p at V_0 and *some directional derivatives* at the remaining vertices for a total of $\dim(\mathcal{P}_{\boldsymbol{x}}^m) = \binom{m+2}{2}$ interpolation conditions. As a result of the uniqueness of the interpolation polynomial, $H_{m,p}^{\Delta_2(V_0)}[\cdot]$ leaves invariant all polynomials of total degree not greater than m. In order to give a more compact expression for the bound of the error, let us consider the disjoint union of M_1^2 and M_2^2, i.e.

$$M_1^2 \cup^* M_2^2 = \left(M_1^2 \times \{0\}\right) \cup \left(M_2^2 \times \{1\}\right); \tag{2.4}$$

$\theta \in M_1^2 \cup^* M_2^2$ is a pair of type $(\beta, 0)$ or $(\beta, 1)$. In the following, with an abuse of notation, we write $\beta \in M_1^2 \cup^* M_2^2$ by intending the component β of θ. The following Theorem is a particular case of the more general result stated in Section 1.5.

Theorem 7 *Let Ω be a compact convex domain containing $\Delta_2(V_0)$ and $f \in C^{m,1}(\Omega)$. Then, for each $\boldsymbol{x} \in \Omega$, we have*

$$H_{m,p}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = T_m[f, V_0](\boldsymbol{x}) + \delta_m^{H,\Delta_2(V_0)}(\boldsymbol{x}) \tag{2.5}$$

where $T_m[f, V_0](\boldsymbol{x})$ is the Taylor polynomial of order m for f centered at V_0 (1.4) and

$$\left|\delta_m^{H,\Delta_2(V_0)}(\boldsymbol{x})\right| \leq 2^{m+1} |f|_{m,1} \sum_{\beta \in M_1^2 \cup^* M_2^2} \sum_{j=0}^{\beta_1} C_\beta r^{j+1} \left(r^2 S\right)^{m-j} \|\boldsymbol{x} - V_0\|_2^{m-j} \tag{2.6}$$

with

$$C_\beta = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{\binom{\beta_1}{\beta_1 - k}(m - |\gamma|)!}{(m - |\gamma| + 1)!(m - |\beta|)!\beta!} 2^{\beta_1 - k}. \tag{2.7}$$

Proof. Let us rewrite (2.2) by expanding the sum of index i, i.e.

$$\begin{aligned} H_{m,p}^{\Delta_2(V_0)}(\boldsymbol{x}) = & \sum_{\beta \in M_0^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(m - |\gamma|)!}{m!} D_0^\gamma f(V_0) \right\} \phi_{A_0^m \beta}^m (\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) \\ & + \sum_{\beta \in M_1^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(m - |\gamma|)!}{m!} D_1^\gamma f(V_1) \right\} \phi_{A_1^m \beta}^m (\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) \\ & + \sum_{\beta \in M_2^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(m - |\gamma|)!}{m!} D_2^\gamma f(V_2) \right\} \phi_{A_2^m \beta}^m (\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})). \end{aligned} \tag{2.8}$$

By considering the Taylor expansion of order m of $D_i^{\gamma} f(V_i), i = 1, 2$ centered at V_0 with integral remainder of we get

$$\begin{aligned} &\delta_m^{H,\Delta_2(V_0)}(\boldsymbol{x}) = \sum_{\beta \in M_1^2} \sum_{\gamma \le \beta} \tbinom{\beta}{\gamma} \tfrac{(m-|\gamma|)!}{m!} \int_0^1 \tfrac{D_{10}^{m-|\gamma|+1} D_1^{\gamma} f(V_0+t(V_1-V_0))}{(m-|\gamma|+1)!} (1-t)^{m-|\gamma|+1}\, dt \\ &\times \tfrac{m!}{(m-|\beta|)!\beta_1!\beta_2!} \lambda_0(\boldsymbol{x})^{\beta_1} \lambda_1(\boldsymbol{x})^{m-|\beta|} \lambda_2(\boldsymbol{x})^{\beta_2} \\ &+ \sum_{\beta \in M_2^2} \sum_{\gamma \le \beta} \tbinom{\beta}{\gamma} \tfrac{(m-|\gamma|)!}{m!} \int_0^1 \tfrac{D_{20}^{m-|\gamma|+1} D_2^{\gamma} f(V_0+t(V_2-V_0))}{(m-|\gamma|+1)!} (1-t)^{m-|\gamma|+1}\, dt \\ &\times \tfrac{m!}{(m-|\beta|)!\beta_1!\beta_2!} \lambda_0(\boldsymbol{x})^{\beta_1} \lambda_1(\boldsymbol{x})^{\beta_2} \lambda_2(\boldsymbol{x})^{m-|\beta|}, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \widetilde{T}_m[f, V_0](\boldsymbol{x}) &= \sum_{\beta \in M_0^2} \sum_{\gamma \le \beta} \tbinom{\beta}{\gamma} \tfrac{(m-|\gamma|)!}{m!} D_0^{\gamma} f(V_0)\, \phi^m_{A_0^m \beta}(\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) \\ &+ \sum_{\beta \in M_1^2} \sum_{\gamma \le \beta} \tbinom{\beta}{\gamma} \tfrac{(m-|\gamma|)!}{m!} \sum_{\substack{\delta_1 \le m-|\gamma| \\ \delta \in \mathbb{Z}_+^2}} \tfrac{D_{10}^{\delta_1} D_1^{\gamma} f(V_0)}{\delta_1!} \phi^m_{A_1^m \beta}(\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})) \\ &+ \sum_{\beta \in M_2^2} \sum_{\gamma \le \beta} \tbinom{\beta}{\gamma} \tfrac{(m-|\gamma|)!}{m!} \sum_{\substack{\delta_2 \le m-|\gamma| \\ \delta \in \mathbb{Z}_+^2}} \tfrac{D_{20}^{\delta_2} D_2^{\gamma} f(V_0)}{\delta_2!} \phi^m_{A_2^m \beta}(\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})). \end{aligned}$$

Let us recall the equality $\lambda_0(\boldsymbol{x}) = 1 - \lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})$. By expanding $\lambda_0(\boldsymbol{x})^{\beta_1}$ in (2.9) by the binomial theorem, by bounds (1.26) and by considering the disjoint union of M_1^2 and M_2^2 (2.4), we have the following bound for $\delta_{m,p}^{H,\Delta_2(V_0)}(\boldsymbol{x})$

$$\begin{aligned} &\left|\delta_{m,p}^{H,\Delta_2(V_0)}(\boldsymbol{x})\right| \le 2^{m+1} |f|_{m,1} \sum_{\beta \in M_1^2 \cup^* M_2^2} \sum_{j=0}^{\beta_1} \sum_{\gamma \le \beta} \tbinom{\beta}{\gamma} \tfrac{\binom{\beta_1}{\beta_1-k}(m-|\gamma|)!}{(m-|\gamma|+1)!(m-|\beta|)!\beta!} 2^{\beta_1-k} \\ &\times r^{j+1} \left(r^2 S\right)^{m-j} ||\boldsymbol{x} - V_0||_2^{m-j} \end{aligned}$$

and $\widetilde{T}_m[f, V_0](\boldsymbol{x})$ is the Taylor polynomial for f of order m centered at V_0. ■

Corollary 8 *In the hypothesis of Theorem 7 for all $\boldsymbol{x} \in \Omega$ we have*

$$R_m^{H,\Delta_2(V_0)}[f](\boldsymbol{x}) = R_m^T[f, V_0](\boldsymbol{x}) - \delta_m^{H,\Delta_2(V_0)}(\boldsymbol{x}) \tag{2.10}$$

where $R_m^T[f, V_0](\boldsymbol{x})$ is the remainder term in Taylor expansion of order m for f centered at V_0; moreover

$$\begin{aligned} &\left|R_m^{H,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \le 2^{m+1} |f|_{m,1} \\ &\times \left(\sum_{\beta \in M_1^2 \cup^* M_2^2} \sum_{j=0}^{\beta_1} C_\beta r^{j+1} \left(r^2 S\right)^{m-j} ||\boldsymbol{x} - V_0||_2^{m-j} + \frac{||\boldsymbol{x} - V_0||_2^{m+1}}{(m-1)!} \right). \end{aligned} \tag{2.11}$$

2.2 Particular cases

In the following we test the accuracy of approximation of the Shepard-Hermite operator in the case that, at each node, are available function evaluations and first order derivative, i.e. $p = 1$, or function evaluations, first and second order derivative, i.e. $p = 2$. In order to get these expansions, we graphically represent the corresponding lower sets by Bézier nets as in [18, Theorem 3.1.4]. In particular, we denote function evaluations by circles on the vertices of the simplex and derivatives along the directed sides of the simplex by circles on the corresponding side; mixed derivatives are then denoted by circles in the interior of the simplex. As in Theorem 6, we fix M_0^2 such that we interpolate f and all its partial derivatives up to the order p at the vertex V_0 and lower sets M_1^2, M_2^2 so as to maintain a certain symmetry in the distribution of the remaining interpolation conditions at the vertices V_1, V_2.

Case $p = 1$

Let us suppose that function evaluations and first order derivatives of a function f are given at the vertices of $\Delta_2(V_0)$, that is $p = 1$. Let us fix lower sets

$$\begin{aligned} M_0^2 &= \{(0,0), (1,0), (0,1)\}, \\ M_1^2 &= \{(0,0), (0,1)\}, \\ M_2^2 &= \{(0,0)\}. \end{aligned} \tag{2.12}$$

In Figure 2.1 we graphically represent the Bézier net according to the corresponding M_i^2, $i = 0, 1, 2$ in (2.12). The corresponding interpolation polynomial in Bézier representation is

$$\begin{aligned} &H_{2,1}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = f(V_0)\lambda_0(\boldsymbol{x})(\lambda_0(\boldsymbol{x}) + 2\lambda_1(\boldsymbol{x}) + 2\lambda_2(\boldsymbol{x})) \\ &+f(V_1)\lambda_1(\boldsymbol{x})(\lambda_1(\boldsymbol{x}) + 2\lambda_2(\boldsymbol{x})) + f(V_2)\lambda_2^2(\boldsymbol{x}) + D_{10}f(V_0)\lambda_0(\boldsymbol{x})\lambda_1(\boldsymbol{x}) \\ &+D_{20}f(V_0)\lambda_0(\boldsymbol{x})\lambda_2(\boldsymbol{x}) + D_{21}f(V_1)\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}). \end{aligned} \tag{2.13}$$

The polynomial $H_{2,1}^{\Delta_2(V_0)}[f](\boldsymbol{x})$ has algebraic degree of exactness 2. Therefore in the case $p = 1$, $H_{2,1}^{\Delta_2(V_0)}$ has degree of precision increased by 1 with respect to the Taylor polynomial

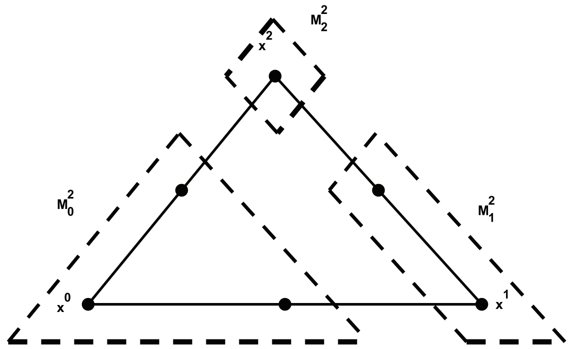


Figure 2.1: Lower sets for $H_{2,1}^{\Delta_2(V_0)}[f](\boldsymbol{x})$.

$T_1[f, V_0](\boldsymbol{x})$ that uses the same data at V_0. The remainder term is bounded by

$$\left|R_2^{H,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 8\,|f|_{2,1}\left(\frac{5}{3}r\left(r^2 S\right)^2 ||\boldsymbol{x}-V_0||_2^2 + ||\boldsymbol{x}-V_0||_2^3\right).$$

Case $p = 2$

Let us suppose that function evaluations, first and second order derivatives of a function f are given at the vertices of $\Delta_2(V_0)$, that is $p = 2$. Let us fix lower sets

$$\begin{aligned} M_0^2 &= \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}, \\ M_1^2 &= \{(0,0),(0,1)\}, \\ M_2^2 &= \{(0,0),(0,1)\}. \end{aligned} \tag{2.14}$$

In Figure 2.2 we graphically represent the Bézier net according to the corresponding M_i^2, $i = 0,1,2$ in (2.14). In this case we fix lower sets M_1^2, M_2^2 to obtain an interpolant with degree of exactness 3, that is, we fix 10 interpolation conditions. The corresponding

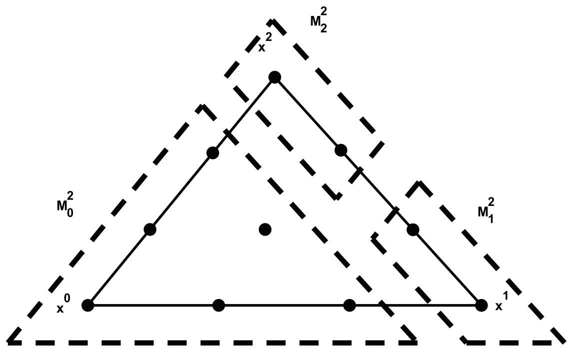


Figure 2.2: Lower sets for $H_{3,2}^{\Delta_2(V_0)}[f](\boldsymbol{x})$.

interpolation polynomial is

$$\begin{aligned}
&H_{3,2}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = f(V_0)\,\lambda_0(\boldsymbol{x})\,(\lambda_0(\boldsymbol{x})\,(\lambda_0(\boldsymbol{x}) + 3\lambda_1(\boldsymbol{x}) + 3\lambda_2(\boldsymbol{x}))\\
&+3\,(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))^2\Big) + f(V_1)\,\lambda_1^2(\boldsymbol{x})\,(\lambda_1(\boldsymbol{x}) + 3\lambda_2(\boldsymbol{x}))\\
&+f(V_2)\,\lambda_2^2(\boldsymbol{x})\,(3\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\\
&+D_{10}f(V_0)\,\lambda_0(\boldsymbol{x})\,\lambda_1(\boldsymbol{x})\,(\lambda_0(\boldsymbol{x}) + 2\lambda_1(\boldsymbol{x}) + 2\lambda_2(\boldsymbol{x}))\\
&+D_{20}f(V_0)\,\lambda_0(\boldsymbol{x})\,\lambda_2(\boldsymbol{x})\,(\lambda_0(\boldsymbol{x}) + 2\lambda_1(\boldsymbol{x}) + 2\lambda_2(\boldsymbol{x}))\\
&+D_{21}f(V_1)\,\lambda_1^2(\boldsymbol{x})\,\lambda_2(\boldsymbol{x}) + D_{12}f(V_2)\,\lambda_1(\boldsymbol{x})\,\lambda_2^2(\boldsymbol{x})\\
&+\tfrac{1}{2}D_{10}^2f(V_0)\,\lambda_0(\boldsymbol{x})\,\lambda_1^2(\boldsymbol{x}) + \tfrac{1}{2}D_{20}^2f(V_0)\,\lambda_0(\boldsymbol{x})\,\lambda_2^2(\boldsymbol{x})\\
&+D_{10}D_{20}f(V_0)\,\lambda_0(\boldsymbol{x})\,\lambda_1(\boldsymbol{x})\,\lambda_2(\boldsymbol{x})
\end{aligned} \tag{2.15}$$

The polynomial $H_{3,2}^{\Delta_2(V_0)}[f](\boldsymbol{x})$ has algebraic degree of exactness 3. Therefore, also in the case $p = 2$, $H_{3,2}^{\Delta_2(V_0)}$ has degree of exactness increased by 1 with respect to the Taylor polynomial $T_2[f, V_0](\boldsymbol{x})$ that uses the same data at V_0. The remainder term is bounded by

$$\left|R_3^{H,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 16\,|f|_{3,1}\left(r\left(r^2S\right)^3\|\boldsymbol{x} - V_0\|_2^3 + \frac{\|\boldsymbol{x} - V_0\|_2^4}{2}\right).$$

Nevertheless, in case $p = 2$ it is possible to fix others lower sets, as for example those depicted in figure 2.3.

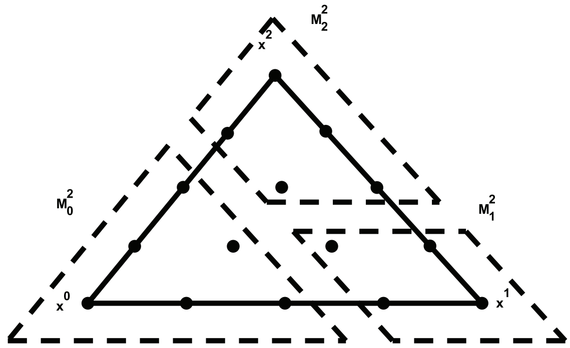


Figure 2.3: Lower sets for $H_{4,2}^{\Delta_2(V_0)}[f](\boldsymbol{x})$

In this case we obtain polynomial

$$\begin{aligned}
&H_{4,2}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = f(V_0)\lambda_0^2(\boldsymbol{x})\left(\lambda_0^2(\boldsymbol{x}) + 2(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))(2\lambda_0(\boldsymbol{x}) + 3(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})))\right)\\
&+f(V_1)\lambda_1^2(\boldsymbol{x})(4\lambda_0(\boldsymbol{x})(\lambda_1(\boldsymbol{x}) + 3\lambda_2(\boldsymbol{x})) + \lambda_1(\boldsymbol{x})(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})))\\
&f(V_0)\lambda_2^2(\boldsymbol{x})((3\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))(4\lambda_0(\boldsymbol{x}) + 2\lambda_1(\boldsymbol{x})) + \lambda_2(\boldsymbol{x})(2\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})))\\
&+D_{10}f(V_0)\lambda_0^2(\boldsymbol{x})\lambda_1(\boldsymbol{x})(\lambda_0(\boldsymbol{x}) + 3\lambda_1(\boldsymbol{x}) + 3\lambda_2(\boldsymbol{x}))\\
&+D_{20}f(V_0)\lambda_0^2(\boldsymbol{x})\lambda_2(\boldsymbol{x})(\lambda_0(\boldsymbol{x}) + 3\lambda_1(\boldsymbol{x}) + 3\lambda_2(\boldsymbol{x}))\\
&+D_{01}f(V_1)\lambda_0(\boldsymbol{x})\lambda_1^2(\boldsymbol{x})(\lambda_1(\boldsymbol{x}) + 3\lambda_2(\boldsymbol{x}))\\
&+D_{21}f(V_1)\lambda_1^2(\boldsymbol{x})\lambda_2(\boldsymbol{x})(3\lambda_0(\boldsymbol{x}) + \lambda_1(\boldsymbol{x}))\\
&+D_{02}f(V_2)\lambda_0(\boldsymbol{x})\lambda_2^2(\boldsymbol{x})(3\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\\
&+D_{12}f(V_2)\lambda_1(\boldsymbol{x})\lambda_2^2(\boldsymbol{x})(3\lambda_0(\boldsymbol{x}) + 3\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\\
&+\tfrac{1}{2}D_{10}^2(V_0)\lambda_0^2(\boldsymbol{x})\lambda_1^2(\boldsymbol{x}) + \tfrac{1}{2}D_{20}f(V_0)\lambda_0^2(\boldsymbol{x})\lambda_2^2(\boldsymbol{x})\\
&+D_{10}D_{20}f(V_0)\lambda_0^2(\boldsymbol{x})\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}) + D_{01}D_{21}f(V_1)\lambda_0(\boldsymbol{x})\lambda_1^2(\boldsymbol{x})\lambda_2(\boldsymbol{x})\\
&+\tfrac{1}{2}D_{12}^2f(V_2)\lambda_1^2(\boldsymbol{x})\lambda_2(\boldsymbol{x})^2 + D_{02}D_{12}f(V_2)\lambda_0(\boldsymbol{x})\lambda_1(\boldsymbol{x})\lambda_2^2(\boldsymbol{x}).
\end{aligned} \tag{2.16}$$

The algebraic degree of exactness of $H_{4,2}^{\Delta_2(V_0)}$ is 4, hence it increases by 2 the algebraic degree of exactness of the Taylor polynomial $T_2[f, V_0](\boldsymbol{x})$ that uses the same data at V_0. The remainder term is bounded by

$$\left|R_4^{H,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \le 32\,|f|_{4,1}\left(\frac{848}{15}r^2\left(r^2S\right)^3||\boldsymbol{x} - V_0||_2^3 + \frac{1352}{15}r\left(r^2S\right)^4||\boldsymbol{x} - V_0||_2^4 + \frac{||\boldsymbol{x} - V_0||_2^5}{6}\right).$$

2.3 The bivariate Shepard-Hermite operator

For each $i = 1, ..., n$ we associate to $\boldsymbol{x}_i$ the simplex $\Delta_2(i)$ with fixed vertex $\boldsymbol{x}_i$ and vertices in $B\left(\boldsymbol{x}_i, R_{w_i}\right)$ which reduces the quantity $r_i^{m+1}\left(r_i^2 S_i\right)^m$, $i = 1, \ldots, n$ according to Section 1.6.

Definition 9 *For each $\mu > 0$ and $m = 1, 2, \ldots$ the bivariate Shepard-Hermite operator is defined by*

$$S_{H_{m,p}}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x})\, H_{m,p}^{\Delta_2(i)}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{2.17}$$

where $H_{m,p}^{\Delta_2(i)}[f](\boldsymbol{x})$ is the Hermite interpolating polynomial (2.2) on the simplex $\Delta_2(i)$, $i = 1, \ldots, n$. The remainder term is

$$R_{H_m}[f](\boldsymbol{x}) = f(\boldsymbol{x}) - S_{H_{m,p}}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{2.18}$$

Convergence results can be obtained by following the well known approaches [68, §15.4], [36, 73].

Theorem 10 *If $f \in C^{m-1,1}(\Omega)$ we have*

$$\begin{aligned} &\left|R_{H_m}[f](\boldsymbol{x})\right| \leq 2^{m+1}\, |f|_{m,1}\, M \\ &\times \max_{i \in \mathcal{I}_{\boldsymbol{x}}} \left\{ \sum_{\beta \in \left(M_1^2 \cup^* M_2^2\right)_i} \sum_{j=0}^{\beta_1} \left(C_\beta\right)_i r_i^{j+1} \left(r_i^2 S_i\right)^{m-j} R_{w_i}^{m-j} + \frac{R_{w_i}^{m+1}}{(m-1)!} \right\} \end{aligned}$$

where, for each i, $\left(C_\beta\right)_i$ is the constant defined in (2.7).

The following statements can be easily checked.

Theorem 11 *The operator $S_{H_{m,p}}[\cdot]$ interpolates on all data required for its definition, provided that $\mu > p + 1$.*

Theorem 12 *The algebraic degree of exactness of the operator $S_{H_{m,p}}[\cdot]$ is m.*

2.4 Numerical results

To test the accuracy of approximation of the bivariate Shepard-Hermite operators in the multivariate interpolation of large sets of scattered data, we test the local Shepard operator combined with Hermite interpolation polynomials (2.13), (2.15) and (2.16) on the unit square $R = [0, 1] \times [0, 1]$. For each function of the set (See Figure 1.2) we compare numerical results obtained by applying the Shepard-Hemite operators $S_{H_{m,p}}[f]$, $m = 2, 3, 4$, $p = 1, 2$ having degree of exactness 2, 3 and 4 with those obtained by applying the local Shepard-Taylor operators $\widetilde{S}_{T_m}[f]$, $m = 2, 3, 4$ (1.19) having the same degree of exactness of $S_{H_{m,p}}[f]$ respectively. In the following we set $m = \mu = 2, 3, 4$ and $p = 1, 2$. We emphasize that operators $\widetilde{S}_{T_m}[f]\,, m = 2, 3, 4$ make use, at each sample point, of function evaluations and partial derivative data up to the orders 2, 3 and 4 respectively, while operators $S_{H_{m,p}}[f]$ make use, at each sample point, of function evaluation and first order derivatives data for $m = 2$, and function evaluation and first and second order derivatives data for $m = 3, 4$. We compute numerical approximations by using the Franke dataset and the set of 1089 sparse uniformly distributed interpolation nodes (See Figure 1.2) in the unit square R. We compute the resulting approximations at the points of a regular grid of 101×101 points of Ω. We show in table 2.1 and Table 2.2 maximum (1.45), mean (1.46) and mean square (1.47) interpolation errors, computed for the parameter value $N_w = 30$ for the operators $\widetilde{S}_{T_m}$ and $N_w = 13$ for the operators $S_{H_{m,p}}$. The local Shepard-Hermite operators defined in this Chapter give a positive answer to the problem of the enhancement of the degree of exactness of the Shepard operators by using supplementary derivative data. These operators are realized as combination of the local version of the Shepard operators with Hermite interpolation polynomials on the simplex by using the procedure described in Section 1.6 [36, 16, 27]. The local Shepard-Hermite operators $S_{H_{m,p}}[f]$ allow us not only to raise the algebraic precision of the Shepard-Taylor operators $\widetilde{S}_{T_p}[f]$ that use the same data, maintaining at the same time their interpolation properties, but also to achieve the accuracy of approximation of the Shepard-Taylor operators $\widetilde{S}_{T_m}[f]$ with the same algebraic precision. Numerical results confirm the theoretical ones and show that the proposed Shepard-Hermite operators apply well to the scattered data interpolation problem.

Table 2.1: Comparison of among the operators $\widetilde{S}_{T_p}, p = 2, 3, 4$ and the operators $S_{H_{m,p}}$, $m = 2, 3, 4,\ p = 1, 2$ applied to the 10 test functions in Figure 1.2 using the Franke data set in Figure 1.3.

		$\tilde{S}_{T_2}$	$S_{H_{2,1}}$	$\tilde{S}_{T3}$	$S_{H_{3,2}}$	$\tilde{S}_{T_4}$	$S_{H_{4.2}}$
f_1	$e_{\max}$	7.88e-02	1.28e-01	5.15e-02	4.03e-02	9.91e-02	1.53e-02
	e_{mean}	5.78e-03	9.32e-03	1.48e-03	1.31e-03	1.33e-03	4.55e-04
	e_{MS}	1.37e-04	3.27e-04	1.88e-05	1.67e-05	2.91e-05	1.92e-06
f_2	$e_{\max}$	4.50e-03	5.36e-03	6.73e-04	8.39e-04	3.53e-04	1.35e-04
	e_{mean}	3.54e-04	6.30e-04	4.13e-05	4.02e-05	1.13e-05	3.03e-06
	e_{MS}	3.43e-07	9.92e-07	6.27e-09	6.60e-09	8.08e-10	8.19e-11
f_3	$e_{\max}$	3.73e-03	3.38e-03	5.10e-04	1.20e-03	1.92e-04	4.57e-04
	e_{mean}	1.84e-04	6.02e-04	2.07e-05	9.93e-05	5.53e-06	1.39e-05
	e_{MS}	1.44e-07	6.38e-07	2.11e-09	2.98e-08	2.14e-10	1.42e-09
f_4	$e_{\max}$	7.27e-03	1.11e-02	1.81e-03	1.27e-03	1.47e-03	6.60e-04
	e_{mean}	5.78e-04	1.16e-03	8.20e-05	7.95e-05	3.99e-05	1.96e-05
	e_{MS}	1.08e-06	3.35e-06	3.71e-08	2.94e-08	1.52e-08	3.56e-09
f_5	$e_{\max}$	3.08e-02	2.38e-02	1.46e-02	1.96e-02	2.37e-02	4.72e-03
	e_{mean}	2.18e-03	2.70e-03	7.16e-04	8.08e-04	6.69e-04	2.41e-04
	e_{MS}	1.91e-05	2.06e-05	3.01e-06	3.71e-06	4.00e-06	3.72e-07
f_6	$e_{\max}$	4.16e-02	4.69e-02	6.64e-03	2.33e-03	9.15e-03	4.34e-03
	e_{mean}	2.10e-03	2.58e-03	2.68e-04	1.38e-04	1.78e-04	6.73e-05
	e_{MS}	2.04e-05	2.93e-05	3.97e-07	7.83e-08	3.35e-07	6.60e-08
f_7	$e_{\max}$	1.03	1.05	2.11e-01	1.81e-01	2.84e-01	1.53e-01
	e_{mean}	9.86e-02	1.44e-01	1.67e-02	1.01e-02	1.35e-02	5.61e-03
	e_{MS}	2.65e-02	4.67e-02	9.21e-04	4.96e-04	1.04e-03	2.18e-04
f_8	$e_{\max}$	4.88e-01	5.73e-01	1.95e-01	1.63e-01	1.13e-01	8.59e-02
	e_{mean}	3.09e-02	3.92e-02	9.44e-03	6.83e-03	4.16e-03	2.03e-03
	e_{MS}	3.90e-03	5.31e-03	4.80e-04	2.45e-04	1.06e-04	3.87e-05
f_9	$e_{\max}$	2.85e+01	2.64e+01	8.59	4.45	8.19	4.21
	e_{mean}	1.70	1.99	3.69e-01	1.93e-01	2.70e-01	8.43e-02
	e_{MS}	1.29e+01	1.75e+01	8.69e-01	1.94e-01	5.50e-01	7.85e-02
f_{10}	$e_{\max}$	4.75e-01	5.36e-01	2.73e-01	2.45e-01	7.73e-02	3.45e-02
	e_{mean}	2.27e-02	3.18e-02	6.51e-03	3.83e-03	2.96e-03	1.12e-03
	e_{MS}	2.39e-03	3.68e-03	3.64e-04	2.49e-04	6.48e-05	7.85e-06

Table 2.2: Comparison of among the operators $\widetilde{S}_{T_p}, p = 2, 3, 4$ and the operators $S_{H_{m,p}}, m = 2, 3, 4,\ p = 1, 2$ applied to the 10 test functions in Figure 1.2 using the sparse set of 1089 nodes in Figure 1.3.

		$\tilde{S}_{T_2}$	$S_{H_{2,1}}$	$\tilde{S}_{T3}$	$S_{H_{3,2}}$	$\tilde{S}_{T_4}$	$S_{H_{4.2}}$
f_1	$e_{\max}$	1.82e-03	2.81e-03	3.79e-04	3.50e-04	1.25e-04	4.99e-05
	e_{mean}	9.24e-05	1.16e-04	8.38e-06	9.38e-06	1.42e-06	4.51e-07
	e_{MS}	3.27e-08	5.22e-08	4.36e-10	5.04e-10	2.63e-11	3.88e-12
f_2	$e_{\max}$	8.22e-05	6.03e-05	7.72e-06	6.80e-06	3.23e-07	8.16e-08
	e_{mean}	5.96e-06	4.34e-06	2.87e-07	3.58e-07	1.31e-08	2.64e-09
	e_{MS}	9.13e-11	5.02e-11	3.20e-13	4.18e-13	7.43e-16	3.60e-17
f_3	$e_{\max}$	1.98e-04	2.00e-04	1.09e-05	1.40e-05	1.85e-06	1.04e-06
	e_{mean}	3.22e-06	4.62e-06	1.60e-07	3.29e-07	1.12e-08	5.84e-09
	e_{MS}	8.41e-11	1.14e-10	2.48e-13	6.34e-13	3.66e-15	9.60e-16
f_4	$e_{\max}$	1.63e-04	1.99e-04	3.88e-05	3.14e-05	4.46e-06	8.62e-07
	e_{mean}	1.03e-05	8.39e-06	5.84e-07	7.14e-07	4.99e-08	1.37e-08
	e_{MS}	3.33e-10	2.70e-10	2.35e-12	2.86e-12	2.75e-14	2.07e-15
f_5	$e_{\max}$	2.03e-03	1.91e-03	3.22e-04	4.22e-04	3.80e-04	9.86e-05
	e_{mean}	5.36e-05	8.69e-05	5.96e-06	6.46e-06	2.34e-06	1.09e-06
	e_{MS}	1.96e-08	3.96e-08	3.96e-10	3.17e-10	1.31e-10	2.05e-11
f_6	$e_{\max}$	4.34e-04	4.95e-04	5.92e-05	7.25e-05	8.10e-06	2.11e-06
	e_{mean}	2.21e-05	2.18e-05	1.60e-06	1.85e-06	1.76e-07	3.99e-08
	e_{MS}	1.85e-09	1.91e-09	1.74e-11	2.09e-11	2.39e-13	1.47e-14
f_7	$e_{\max}$	2.55e-02	2.26e-02	3.67e-03	2.93e-03	5.67e-04	2.08e-04
	e_{mean}	1.67e-03	1.96e-03	1.27e-04	1.66e-04	1.45e-05	4.03e-06
	e_{MS}	7.51e-06	8.81e-06	6.15e-08	8.42e-08	1.11e-09	9.94e-11
f_8	$e_{\max}$	9.06e-03	1.11e-02	1.34e-03	1.50e-03	5.42e-04	1.37e-04
	e_{mean}	4.21e-04	5.87e-04	4.07e-05	5.00e-05	7.50e-06	2.55e-06
	e_{MS}	6.44e-07	1.04e-06	9.20e-09	1.04e-08	4.75e-10	4.91e-11
f_9	$e_{\max}$	7.16e-01	7.00e-01	1.04e-01	1.23e-01	3.24e-02	1.30e-02
	e_{mean}	2.56e-02	2.96e-02	2.24e-03	2.49e-03	3.55e-04	9.86e-05
	e_{MS}	3.26e-03	4.39e-03	4.34e-05	4.97e-05	1.43e-06	1.61e-07
f_{10}	$e_{\max}$	3.74e-02	4.97e-02	3.51e-02	3.68e-02	2.63e-02	4.05e-02
	e_{mean}	3.69e-04	4.50e-04	4.54e-05	5.11e-05	2.10e-05	1.23e-05
	e_{MS}	1.36e-06	2.18e-06	2.35e-07	2.54e-07	1.45e-07	1.89e-07

Chapter 3

The bivariate Shepard-Lidstone operator

Over all this chapter we assume that, together with function evaluations, all even order partial derivatives up to a fixed order $2m-2, m \in \mathbb{N}$, are given at each sample point. We call such kind of data *Lidstone type data*, in honor of G. J. Lidstone, who, in 1929 [52], provided an explicit expression of a polynomial which approximates a given function in the neighborhood of two points instead of one, generalizing in such a way the Taylor polynomial. This polynomial, known as Lidstone interpolating polynomial [2], uses function evaluations and all even order derivatives up to the order $2m-2$, interpolating them. The interest for this kind of expansion lies in the fact that it finds application to several problems of numerical analysis such as approximation of solutions of some boundary value problems, polynomial approximation, construction of splines with application to finite elements, etc. [2, 4, 5, 6, 7]. In a remark made in [2, p.37] reference was made to the lack of literature on the extension of some results on the approximation of univariate functions by means of Lidstone polynomials to functions of two independent variables over non-rectangular domains. Costabile and Dell'Accio [24] answered to this question by providing a new polynomial approximation formula, which uses function evaluations and even order derivatives at the vertices of the simplex and is the univariate Lidstone expansion when restricted to each side.

In this Chapter we combine the Lidstone approximation formula on the triangle with the local Shepard operators to provide an interpolation operator which satisfies bivariate Lidstone interpolation conditions. This operator uses function evaluation and all even order derivatives data up to the order $2m-2$ interpolating them, reproduces polynomials up to the degree $2m-1$ and has the same accuracy of approximation of the Shepard Taylor operator which has the same degree of exactness, despite the Shepard-Lidstone operator uses lacunary data.

3.1 Some remarks on the univariate Lidstone interpolation polynomial

Lidstone [52] in 1929 and independently Poritsky [58] in 1932 introduced a generalization of the Taylor series that approximates a given function in the neighborhood of two points instead of one

$$f(x) = L_{2m-1}[f,0,1](x) + R^L_{2m-1}[f,0,1](x), \quad x \in [0,1]. \tag{3.1}$$

$L_{2m-1}[f,0,1](x)$ is the polynomial of degree not greater than $2m-1$ defined by

$$L_{2m-1}[f,0,1](x) = \sum_{k=0}^{m-1} \left[\Lambda_k(1-x)f^{(2k)}(0) + \Lambda_k(x)f^{(2k)}(1)\right] \tag{3.2}$$

where the polynomials $\Lambda_k(x),\ k=0,1,...,$ are Lidstone polynomials [6] defined recursively by

$$\begin{cases} \Lambda_0(x) = x, & \\ \Lambda_k''(x) = \Lambda_{k-1}(x), & k \geq 1, \\ \Lambda_k(0) = \Lambda_k(1) = 0, & k \geq 1. \end{cases} \tag{3.3}$$

The remainder term $R^L_{2m-1}[f,0,1](x)$ in the expansion formula (3.1) has the following Cauchy's representation [6]

$$R^L_{2m-1}[f,0,1](x) = E_{2m}(x)f^{(2m)}(\xi), \quad \xi \in (0,1),$$

where

$$E_{2m}(x) = \Lambda_m(x) + \Lambda_m(1-x) \tag{3.4}$$

are the Euler polynomials [6]. The algebraic degree of exactness of the operator $L_{2m-1}[\cdot,0,1](x)$ is therefore $2m-1$.

For $f \in C^{2m-2}([a,b])$, $m = 1, 2,...$ and $a, b \in \mathbb{R}$, $a < b$, the Lidstone interpolation problem

$$\begin{cases} L^{(2i)}_{2m-1}[f,a,b](a) = f^{(2i)}(a), & i = 0, \ldots, m-1, \\ L^{(2i)}_{2m-1}[f,a,b](b) = f^{(2i)}(b), & i = 0, \ldots, m-1, \end{cases}$$

has a unique solution in the space of polynomials of degree not greater than $2m-1$ that is

$$L_{2m-1}[f,a,b](x) = \sum_{k=0}^{m-1} h^{2k} \left[\Lambda_k \left(\frac{b-x}{h} \right) f^{(2k)}(a) + \Lambda_k \left(\frac{x-a}{h} \right) f^{(2k)}(b) \right], \tag{3.5}$$

where we set $h = b - a$. As remarked in [25], by the continuity of the Birkhoff interpolation [39]

$$\lim_{h \to 0} L_{2m-1}[f,a,b](x) = T_{2m-1}[f,a](x), \quad \text{for each } f \in C^{2m-1}([a,b]) \tag{3.6}$$

where $T_{2m-1}[f,a](x)$ is the Taylor polynomial of order $2m-1$ for f centered at a, i.e.

$$T_{2m-1}[f,a](x) = \sum_{k=0}^{2m-1} \frac{f^{(k)}(a)}{k!} (x-a)^k. \tag{3.7}$$

A first consequence of equation (3.6) is that

$$L_{2m-1}[f,a,b](x) = T_{2m-1}[f,a](x) + \delta_{2m-1}[f,a,b](x), \quad x \in [a,b]$$

where

$$\lim_{h \to 0} \delta_{2m-1}[f,a,b](x) = 0.$$

We can get some expressions of the difference $\delta_{2m-1}[f,a,b](x)$ in terms of Lidstone polynomials (3.3) and higher order derivatives of f. Each of these expressions provides an associated representation of the error in Lidstone interpolation by the equality

$$R^L_{2m-1}[f,a,b](x) = R^T_{2m-1}[f,a](x) - \delta_{2m-1}[f,a,b](x)$$

where

$$R^L_{2m-1}[f,a,b](x) = f(x) - L_{2m-1}[f,a,b](x),$$
$$R^T_{2m-1}[f,a](x) = f(x) - T_{2m-1}[f,a](x).$$

If $f \in C^{2m-1,1}([0,1])$ the integral form of the remainder in Taylor expansion holds [9, § 7.5]

$$R_{2m-1}^{T}[f,a](x) = \int_a^x \frac{f^{(2m)}(t)}{(2m-1)!}(x-t)^{2m-1}\,dt \tag{3.8}$$

provided that the integral involved is understood as Lebesgue integral and we get the following

Theorem 13 *If $f \in C^{2m-1,1}([a,b])$ we have*

$$L_{2m-1}[f,a,b](x) = T_{2m-1}[f,a](x) + \delta_{2m-1}[f,a,b](x)$$

where $L_{2m-1}[f,a,b](x)$ is the Lidstone interpolating polynomial (3.5), $T_{2m-1}[f,a](x)$ is the Taylor polynomial (3.7) and

$$\delta_{2m-1}[f,a,b](x) = h^{2m}\sum_{k=0}^{m-1} h^{2k}\Lambda_k\left(\frac{x-a}{h}\right)\left(\int_0^1 \frac{f^{(2m)}(a+th)(1-t)^{2m-2k-1}}{(2m-2k-1)!}dt\right).$$

Proof. By applying the Taylor theorem with integral remainder to $f^{(2k)}(b)$ we get

$$f^{(2k)}(b) = \sum_{i=0}^{2m-1}\frac{h^i}{i!}f^{(2k+i)}(a) + h^{2m}\int_0^1 \frac{f^{(2m)}(a+th)(1-t)^{2m-2k-1}}{(2m-2k-1)!}dt. \tag{3.9}$$

By substituting relation (3.9) in (3.5) we have

$$L_{2m-1}[f,a,b](x) = T_{2m-1}[f,a](x) + \delta_{2m-1}[f,a,b](x)$$

where

$$\begin{aligned} T_{2m-1}[f,a](x) = {} & \sum_{k=0}^{m-1} h^{2k}\left(\Lambda_k\left(\tfrac{b-x}{h}\right)f^{(2k)}(a)\right. \\ & \left. + \Lambda_k\left(\tfrac{x-a}{h}\right)\sum_{i=0}^{2m-1}\tfrac{h^i}{i!}f^{(2k+i)}(a)\right) \end{aligned}$$

and

$$\delta_{2m-1}[f,a,b](x) = h^{2m}\sum_{k=0}^{m-1} h^{2k}\Lambda_k\left(\frac{x-a}{h}\right)\int_0^1 \frac{f^{(2m)}(a+th)(1-t)^{2m-2k-1}}{(2m-2k-1)!}dt.$$

■

Corollary 14 *If $f \in C^{2m-1,1}([a,b])$ we have*

$$\begin{aligned} R_{2m-1}^{L}[f,a,b](x) = {} & \int_a^x \frac{f^{(2m)}(t)(x-t)^{2m-1}}{(2m-1)!}dt \\ & - h^{2m}\sum_{k=0}^{m-1} h^{2k}\Lambda_k\left(\frac{x-a}{h}\right)\left(\int_0^1 \frac{f^{(2m)}(a+th)(1-t)^{2m-2k-1}}{(2m-2k-1)!}dt\right) \end{aligned}$$

and

$$\left|R_{2m-1}^{L}\left[f,a,b\right](x)\right| \leq |f|_{2m-1,1}\left(\frac{(x-a)^{2m}}{(2m)!}+h^{2m}\sum_{k=0}^{m-1}h^{2k}\left(-1\right)^{k}\frac{\Lambda_{k}\left(\dfrac{x-a}{h}\right)}{\left(2\left(m-k\right)\right)!}\right)$$

where $|f|_{2m-1,1}$ *is the seminorm (1.15) on* $[a,b]$*.*

Proof. We use [6, Remark 1.2.1]

$$\left|\Lambda_{k}\left(x\right)\right| = \left(-1\right)^{k}\Lambda_{k}\left(x\right), \quad x\in\left[a,b\right].$$

■

3.2 The three point Lidstone interpolation polynomial

In [24] the univariate expansion (3.1) has been extended to a bivariate polynomial expansion

$$f\left(\boldsymbol{x}\right) = L_{2m-1}^{\Delta_2}\left[f\right]\left(\boldsymbol{x}\right) + R_{2m-1}^{L,\Delta_2}\left[f\right]\left(\boldsymbol{x}\right) \tag{3.10}$$

for functions of class $C^{2m-2}\left(\Omega\right)$ on the standard simplex $\Delta_2 = \left\{(x,y)\in\mathbb{R}^2 : x\geq 0, y\geq 0, x+y\leq 1\right\}$. In the following we consider the generalization of the expansion given in [24] by setting

$$\begin{aligned}
L_{2m-1}^{\Delta_2}[f](\boldsymbol{x}) = &\\
&\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1}\left(\frac{\partial^{2j+2k}f}{\partial x^{2j}\partial\nu^{2k}}\left(0,0\right)\right)2^k\left(x+y\right)^{2k}\Lambda_j(1-x-y)\Lambda_k\left(\tfrac{x}{x+y}\right)\\
&+\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1}\left(\frac{\partial^{2j+2k}f}{\partial x^{2j}\partial\nu^{2k}}\left(1,0\right)\right)2^k\left(x+y\right)^{2k}\Lambda_j(x+y)\Lambda_k\left(\tfrac{x}{x+y}\right)\\
&+\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1}\left(\frac{\partial^{2j+2k}f}{\partial y^{2j}\partial\nu^{2k}}\left(0,0\right)\right)2^k\left(x+y\right)^{2k}\Lambda_j(1-x-y)\Lambda_k\left(\tfrac{y}{x+y}\right)\\
&+\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1}\left(\frac{\partial^{2j+2k}f}{\partial y^{2j}\partial\nu^{2k}}\left(0,1\right)\right)2^k\left(x+y\right)^{2k}\Lambda_j(x+y)\Lambda_k\left(\tfrac{y}{x+y}\right)
\end{aligned} \tag{3.11}$$

where $\frac{\partial}{\partial\nu} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right)$ is the derivative in the direction of the slanted side of the simplex. In the space $\mathcal{P}_{\boldsymbol{x}}^{2m-1}$ the polynomial $L_{2m-1}^{\Delta_2}[f](\boldsymbol{x})$ is the unique solution of the interpolation problem

$$\frac{\partial^{2i}L_{2m-1}^{\Delta_2}\left[f\right]}{\partial x^{2i-j}\partial y^{j}}\left(0,0\right) = \frac{\partial^{2i}f}{\partial x^{2i-j}\partial y^{j}}\left(0,0\right) \tag{3.12}$$

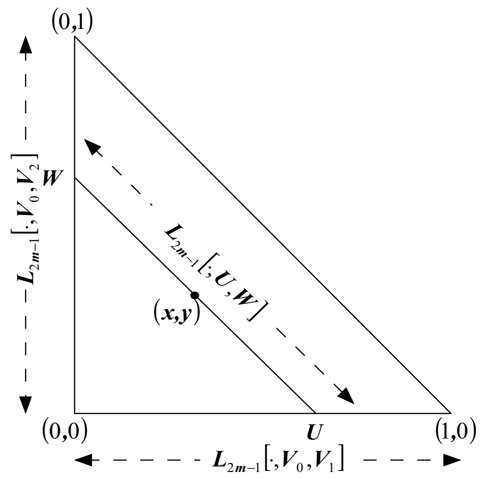


Figure 3.1: Lidstone interpolation extension technique.

for each $i = 0, 1, ..., m-1,\ j = 0, 1, ..., 2i$ and

$$\frac{\partial^{2i} L_{2m-1}^{\Delta_2}[f]}{\partial x^{2i-j} \partial \nu^j}(1,0) = \frac{\partial^{2i} f}{\partial x^{2i-j} \partial \nu^j}(1,0)$$

$$\frac{\partial^{2i} L_{2m-1}^{\Delta_2}[f]}{\partial y^{2i-j} \partial \nu^j}(0,1) = \frac{\partial^{2i} f}{\partial y^{2i-j} \partial \nu^j}(0,1)$$

for each $i = 0, 1, ..., m-1,\ j = 0, 2, 4, ..., 2i$. As a consequence we have $L_{2m-1}^{\Delta_2}[f] = f$ for each $f \in \mathcal{P}_{\boldsymbol{x}}^{2m-1}$ and the approximation formula (3.10) is the univariate Lidstone expansion when restricted on each side of the boundary $\partial\Delta_2$. We emphasize the asymmetry of the interpolation conditions: we count $1 + 3 + ... + (2m-1) = m^2$ interpolation conditions at vertex $(0,0)$ but only $1 + 2 + ... + m = \frac{m(m+1)}{2}$ interpolation conditions at each of the remaining vertices; this asymmetry is a consequence of the extension technique introduced in [23]. More precisely, fixed a point $\boldsymbol{x} = (x, y) \in \Delta_2$ and the straight line through (x, y) parallel to the slanted edge of the simplex, we assign (x, y) to the segment of end points $U = (x+y, 0)$ and $W = (0, x+y)$ and we expand f along this segment by univariate Lidstone expansion, the vertices of the simplex are then reached by expanding f and its directional derivatives along the horizontal and vertical sides of the simplex by using the univariate Lidstone polynomial (See Figure 3.1).

Nevertheless, the symmetry of univariate Lidstone interpolating polynomial (3.5) with respect to the midpoint of the interval $[a,b]$ causes a symmetry of the bivariate polynomial (3.11) with respect to the line $y=x$. We note that the polynomial $L_{2m-1}^{\Delta_2}[f]$ requires only even order derivatives at the vertices for its definition, up to the order $2m-2$.

Let $V_i \in \mathbb{R}^2, i=0,1,2$ be not collinear points, to write the generalization of the univariate expansion (3.1) to the triangle $\Delta_2(V_0,V_1,V_2)$ with vertices V_0,V_1,V_2 we apply the same technique used in [24] to the case of a generic simplex of $\mathbb{R}^2$ and we write the resulting expansion in barycentric coordinates. In analogy with [24, Theorem 2] we state the following

Theorem 15 *Let f be a function of class C^{2m-2} in a convex domain $\Omega \supset \Delta_2$. Then for each $\boldsymbol{x} \in \Omega$ we have*

$$f(\boldsymbol{x}) = L_{2m-1}^{\Delta_2(V_0)}[f](\boldsymbol{x}) + R_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x})$$

where

$$\begin{aligned}
&L_{2m-1}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = \\
&\sum_{k=0}^{m-1}\left(\sum_{j=0}^{m-1-k}\left(D_2^{(2j,2k)}f(V_0)\Lambda_j(1-\lambda_1-\lambda_2)+\right.\right.\\
&\left.+D_2^{(2j,2k)}f(V_2)\Lambda_j(\lambda_1+\lambda_2)\right)\Lambda_k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)+\\
&+\sum_{j=0}^{m-1-k}\left(D_1^{(2j,2k)}f(V_0)\Lambda_j(1-\lambda_1-\lambda_2)+\right.\\
&\left.\left.+D_1^{(2j,2k)}f(V_1)\Lambda_j(\lambda_1+\lambda_2)\right)\Lambda_k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\right)(\lambda_1+\lambda_2)^{2k}.
\end{aligned} \tag{3.13}$$

Proof. Let $V=\boldsymbol{x}\in\Delta_2(V_0)$ be an interior point. We set $U(\boldsymbol{x})=(x+\lambda_2 h_1, y+\lambda_2 k_1)$, $W(\boldsymbol{x})=(x-\lambda_1 h_1, y-\lambda_1 k_1)$, where $(h_1,k_1)=V_3-V_2$ and $(h_2,k_2)=V_3-V_1$. We assign point V to the line segment $S(\boldsymbol{x})=[U,W]$ parameterized by

$$\begin{cases} x(\lambda)=x+\lambda_2 h_1-\lambda(\lambda_1+\lambda_2)h_1 \\ y(\lambda)=y+\lambda_2 k_1-\lambda(\lambda_1+\lambda_2)k_1 \end{cases}, \quad \lambda\in[0,1].$$

The restriction of f to S is the univariate function $f(x(\lambda),y(\lambda))$ in $[0,1]$ and we expand it by univariate Lidstone polynomial (3.2). In this expansion we replace the parameter λ by the value $\frac{\lambda_2}{\lambda_1+\lambda_2}$ which corresponds to the point V. This results in an expansion in terms of the values of f at U and its even order derivative values in the direction of side of vertices V_1, V_2 at U,W. We reach the vertices of the simplex $\Delta_2(V_0)$ by assigning points

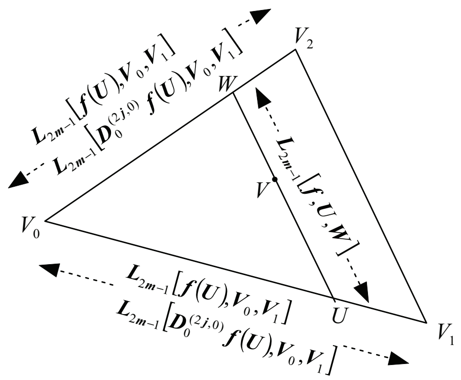


Figure 3.2: Construction scheme of the Lidstone bivariate polynomial (3.13).

U, W to the segments with endpoints V_0, V_1 and V_0, V_2 respectively and by expanding $f(U)$ by the univariate Lidstone polynomial (3.2) with $a = 0, b = 1$ and even order directional derivatives of f at U, W by the univariate Lidstone polynomial (3.2). The above procedure, summarized in Figure 3.2, allows us to obtain expansion (3.13). ■

For the remainder term we have

Theorem 16 *Let $\Omega \supset \Delta_2$ and let $f \in C^{2m-1,1}(\Omega)$. Then for each $\boldsymbol{x} \in \Omega$ we have*

$$L_{2m-1}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = T_{2m-1}[f, V_0](\boldsymbol{x}) + \delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x}) \tag{3.14}$$

where $T_{2m-1}[f, V_0](\boldsymbol{x})$ is the $(2m-1)$-th order Taylor polynomial for f centered at V_0 (1.35) and

$$\left|\delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 2^{2m-1}|f|_{2m-1,1} \sum_{\gamma=0}^{2m-1} C_{\gamma,m} r^{2m-\gamma}\left(r^2 S\right)^{\gamma} ||\boldsymbol{x} - V_0||_2^{\gamma} \tag{3.15}$$

with

$$C_{\gamma,m} = 4 \sum_{k=0}^{\lceil\frac{\gamma+1}{2}\rceil-1} \sum_{\substack{\alpha=\gamma-2k \\ \alpha \text{ odd}}}^{2(m-1-k)+1} \sum_{l=0}^{2k+1} \frac{\binom{\alpha}{\alpha-\gamma+2k}\binom{2k+1}{l}\left|B_{\alpha-\gamma+2k}\left(\frac{1}{2}\right)\right|\left|B_l\left(\frac{1}{2}\right)\right| 2^{\alpha-\gamma+2k+l}}{(2m-2k-\alpha+1)!\alpha!(2k+1)!}. \tag{3.16}$$

for each $\gamma = 0, \ldots, 2m-1$.

Proof. For each $k = 0, \ldots, m-1$, $j = 0, \ldots, m-1-k$ let us consider the Taylor expansion of $D_2^{(2j,2k)} f(V_2)$ and $D_1^{(2j,2k)} f(V_1)$, centered at V_0 with integral remainder. By substituting these expansions in (3.13) we obtain

$$L_{2m-1}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = T_{2m-1}[f, V_0](\boldsymbol{x}) + \delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x})$$

where $T_{2m-1}[f, V_0](\boldsymbol{x})$ is the Taylor polynomial of order $2m-1$ for f at V_0 and

$$\begin{aligned}
&\delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x}) = \\
&\quad \sum_{k=0}^{m-1}\left(\sum_{j=0}^{m-1-k}\left(\int_0^1 \tfrac{D_2^{(2(m-k),2k)} f(V_0+t(V_2-V_0))}{(2(m-k-j)-1)!}(1-t)^{2(m-k-j)-1}\,dt\right)\Lambda_j(\lambda_2+\lambda_3)\Lambda_k\left(\frac{\lambda_3}{\lambda_2+\lambda_3}\right)\right. \\
&\quad \left. + \sum_{j=0}^{m-1-k}\left(\int_0^1 \tfrac{D_1^{(2(m-k),2k)} f(V_0+t(V_1-V_0))}{(2(m-k-j)-1)!}(1-t)^{2(m-k-j)-1}\,dt\right)\Lambda_j(\lambda_2+\lambda_3)\Lambda_k\left(\frac{\lambda_2}{\lambda_2+\lambda_3}\right)\right) \\
&\quad \times(\lambda_2+\lambda_3)^{2k}.
\end{aligned} \tag{3.17}$$

By the Whittaker identities [69], using relations given in [48] we get

$$\begin{aligned}
\Lambda_m(x) &= \tfrac{2^{2m+1}}{(2m+1)!}B_{2m+1}\left(\frac{1+x}{2}\right) \\
&= \tfrac{2^{2m+1}}{(2m+1)!}\sum_{i=0}^{2m+1}\tbinom{2m+1}{i}B_i\left(\tfrac{1}{2}\right)\left(\tfrac{x}{2}\right)^{2m+1-i}, \\
m &= 0,1,2,...
\end{aligned} \tag{3.18}$$

and therefore

$$\Lambda_j(\lambda_1+\lambda_2) = \tfrac{2^{2j+1}}{(2j+1)!}\sum_{i=0}^{2j+1}\tbinom{2j+1}{i}B_i\left(\tfrac{1}{2}\right)\left(\tfrac{\lambda_1+\lambda_2}{2}\right)^{2j+1-i} \tag{3.19}$$

$$\begin{aligned}
\Lambda_k\left(\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right) &= \tfrac{2^{2k+1}}{(2k+1)!}\sum_{l=0}^{2k+1}\tbinom{2k+1}{l}B_l\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{2k+1-l} \\
\Lambda_k\left(\tfrac{\lambda_1}{\lambda_1+\lambda_2}\right) &= \tfrac{2^{2k+1}}{(2k+1)!}\sum_{l=0}^{2k+1}\tbinom{2k+1}{l}B_l\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\tfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{2k+1-l}.
\end{aligned}$$

Then we get

$$\begin{aligned}
&\Lambda_j(\lambda_1+\lambda_2)\Lambda_k\left(\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right)(\lambda_1+\lambda_2)^{2k} \\
&= \frac{\sum_{i=0}^{2j+1}\sum_{l=0}^{2k+1}\binom{2j+1}{i}B_i\left(\frac{1}{2}\right)\binom{2k+1}{l}2^{i+l}B_l\left(\frac{1}{2}\right)\lambda_2^{2k+1-l}(\lambda_1+\lambda_2)^{2j-i+l}}{(2j+1)!(2k+1)!}
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\Lambda_j(\lambda_1+\lambda_2)\Lambda_k\left(\tfrac{\lambda_1}{\lambda_1+\lambda_2}\right)(\lambda_1+\lambda_2)^{2k} \\
&= \frac{\sum_{i=0}^{2j+1}\sum_{l=0}^{2k+1}\binom{2j+1}{i}B_i\left(\frac{1}{2}\right)\binom{2k+1}{l}B_l\left(\frac{1}{2}\right)2^{i+l}\lambda_1^{2k+1-l}(\lambda_1+\lambda_2)^{2j-i+l}}{(2j+1)!(2k+1)!}.
\end{aligned} \tag{3.21}$$

By substituting (3.20) and (3.21) in (3.17) we have

$$\begin{aligned}
&\delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x}) = \\
&\sum_{k=0}^{m-1}\left(\sum_{j=0}^{m-1-k}\left(\int_0^1 \frac{D_2^{(2(m-k),2k)}f(V_0+t(V_2-V_0))}{(2(m-k-j)-1)!}(1-t)^{2(m-k-j)-1}\,dt\right)\right. \\
&\frac{\sum_{i=0}^{2j+1}\sum_{l=0}^{2k+1} 2^{i+l}\binom{2j+1}{i}B_i\left(\frac{1}{2}\right)\binom{2k+1}{l}B_l\left(\frac{1}{2}\right)}{(2j+1)!(2k+1)!}\lambda_2^{2k+1-l}(\lambda_1+\lambda_2)^{2j-i+l} \\
&+\sum_{j=0}^{m-1-k}\left(\int_0^1 \frac{D_1^{(2(m-k),2k)}f(V_0+t(V_1-V_0))}{(2(m-k-j)-1)!}(1-t)^{2(m-k-j)-1}\,dt\right) \\
&\left.\frac{\sum_{i=0}^{2j+1}\sum_{l=0}^{2k+1} 2^{i+l}\binom{2j+1}{i}B_i\left(\frac{1}{2}\right)\binom{2k+1}{l}B_l\left(\frac{1}{2}\right)}{(2j+1)!(2k+1)!}\lambda_1^{2k+1-l}(\lambda_1+\lambda_2)^{2j-i+l}\right).
\end{aligned}$$

By taking the modulus of both sides and by relations (1.26), (1.27) and (1.28) we have

$$\begin{aligned}
&\left|\delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \le 2^{2m+1}|f|_{2m-1,1} \\
&\sum_{k=0}^{m-1}\sum_{j=0}^{m-1-k}\frac{\sum_{i=0}^{2j+1}\sum_{l=0}^{2k+1}\binom{2j+1}{i}\binom{2k+1}{l}\left|B_i\left(\frac{1}{2}\right)\right|\left|B_l\left(\frac{1}{2}\right)\right|2^{i+l}}{(2(m-k-j))!(2j+1)!(2k+1)!} \\
&\times r^{2m-(2k+2j-i+1)}\left(r^2S\right)^{2k+2j-i+1}\|\boldsymbol{x}-V_0\|_2^{2k+2j-i+1}
\end{aligned} \tag{3.22}$$

After some changes of dummy indices and by rearranging the order of summation we get the following bound for $\delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x})$

$$\begin{aligned}
&2^{2m+1}|f|_{2m-1,1} \\
&\times\sum_{k=0}^{m-1}\sum_{\gamma=2k}^{2m-1}\sum_{\substack{\alpha=\gamma-2k \\ \alpha \text{ odd}}}^{2(m-1-k)+1}\sum_{l=0}^{2k+1}\frac{\binom{\alpha}{\alpha-\gamma+2k}\binom{2k+1}{l}\left|B_{\alpha-\gamma+2k}\left(\frac{1}{2}\right)\right|\left|B_l\left(\frac{1}{2}\right)\right|2^{\alpha-\gamma+2k+l}}{(2m-2k-\alpha+1)!\alpha!(2k+1)!} \\
&\times r^{2m-\gamma}\left(r^2S\right)^{\gamma}\|\boldsymbol{x}-V_0\|_2^{\gamma} \\
&= 2^{2m+1}|f|_{2m-1,1} \\
&\times\sum_{\gamma=0}^{2m-1}\sum_{k=0}^{\left\lceil\frac{\gamma+1}{2}\right\rceil-1}\sum_{\substack{\alpha=\gamma-2k \\ \alpha \text{ odd}}}^{2(m-1-k)+1}\sum_{l=0}^{2k+1}\frac{\binom{\alpha}{\alpha-\gamma+2k}\binom{2k+1}{l}\left|B_{\alpha-\gamma+2k}\left(\frac{1}{2}\right)\right|\left|B_l\left(\frac{1}{2}\right)\right|2^{\alpha-\gamma+2k+l}}{(2m-2k-\alpha+1)!\alpha!(2k+1)!} \\
&\times r^{2m-\gamma}\left(r^2S\right)^{\gamma}\|\boldsymbol{x}-V_0\|_2^{\gamma}.
\end{aligned}$$

(3.15) follows by setting $C_{\gamma,m}$ as in (3.16). ■

Corollary 17 *In the hypothesis of Theorem 16, for all $\boldsymbol{x}\in\Omega$, we have*

$$R_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x}) = R_{2m-1}^{T}[f,V_0](\boldsymbol{x}) - \delta_{2m-1}^{L,\Delta_2(V_0)}[f](\boldsymbol{x}) \tag{3.23}$$

where $R^{T}_{2m-1}[f,V_0](\boldsymbol{x})$ *is the remainder term in Taylor expansion; moreover*

$$\left|R^{L,\Delta_2(V_0)}_{2m-1}[f](\boldsymbol{x})\right| \le 2^{2m-1}|f|_{2m-1,1} \left(\sum_{\gamma=0}^{2m-1} C_{\gamma,m} r^{2m-\gamma}\left(r^2S\right)^{\gamma}||\boldsymbol{x}-V_0||_2^{\gamma} + \frac{||\boldsymbol{x}-V_0||_2^{2m}}{(2m-2)!}\right) \tag{3.24}$$

with $C_{\gamma,m}$ defined in (3.16).

In the case $m=2$, which is of interest in the numerical experiments, we get

$$\left|R^{L,\Delta_2(V_0)}_{3}[f](\boldsymbol{x})\right| \le 2|f|_{3,1}\left(r^3\left(r^2S\right)||\boldsymbol{x}-V_0||_2 + 2r\left(r^2S\right)^3||\boldsymbol{x}-V_0||_2^3 + 2||\boldsymbol{x}-V_0||_2^4\right).$$

3.3 The bivariate Shepard-Lidstone operator

In defining the bivariate Shepard-Lidstone operator we associate with each sample point $\boldsymbol{x}_i$ the triangle $\Delta_2(i)$ in $B(\boldsymbol{x}_i, R_{w_i})$ which minimizes the quantity $r_i^{2m}\left(r_i^2S_i\right)^{2m-1}, i = 1,\dots,n$.

Definition 18 *For each fixed $\mu > 0$ and $m = 1,2,\dots$ the bivariate Shepard-Lidstone operator is defined by*

$$S_{L_{2m-1}}[f](\boldsymbol{x}) = \sum_{i=1}^{n}\widetilde{W}_{\mu,i}(\boldsymbol{x})\,L^{\Delta_2(i)}_{2m-1}[f](\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega \tag{3.25}$$

where $L^{\Delta_2(i)}_{2m-1}[f](\boldsymbol{x}),\ i=1,\dots,n$ *is the three point Lidstone interpolation polynomial (3.13) over the whole domain* Ω *and* $\widetilde{W}_{\mu,i}(\boldsymbol{x})$ *is the local Shepard basis function (1.16). The remainder term is*

$$R_{L_{2m-1}}[f](\boldsymbol{x}) = f(\boldsymbol{x}) - S_{L_{2m-1}}[f](\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega.$$

Theorem 19 *Let $\boldsymbol{x}\in\Omega$. The following bound holds*

$$\left|R_{L_{2m-1}}[f](\boldsymbol{x})\right| \le 2^{2m-1}|f|_{2m-1,1}M \times \max_{i\in\mathcal{I}_{\boldsymbol{x}}}\left\{\sum_{\gamma=0}^{2m-1}(C_{\gamma,m})_i r_i^{2m-\gamma}\left(r_i^2S_i\right)^{\gamma}R_{w_i}^{\gamma} + \frac{R_{w_i}^{2m}}{(2m-2)!}\right\}$$

where, for each i, $C_{\gamma,m}$ is the constant defined in (3.16).

The following statements can be easily checked.

Theorem 20 *The operator* $S_{L_{2m-1}}[\cdot]$ *is an interpolation operator in* V_i, $i = 1, ..., n$.

Theorem 21 *The algebraic degree of exactness of the operator* $S_{L_{2m-1}}[\cdot]$ *is* $2m-1$, *i.e.* $S_{L_{2m-1}}[f] = f$ *for each bivariate polynomial* $f \in \mathcal{P}_{\boldsymbol{x}}^{2m-1}$.

As discussed in Section 1.2 the continuity class of the Shepard operators depend on μ, by which we can deduce the continuity class of the Shepard-Lidstone operators.

Theorem 22 *If* $\mu \in \mathbb{N}$, $\mu > 0$ *the continuity class of the operator (3.25) is* $\mu - 1$.

Theorem 23 *For each* $r, s \in \mathbb{N}$ *with* $1 \leq r + s < \mu + 1$ *we have*

$$\frac{\partial^{r+s}}{\partial x^r \partial y^s} S_{L_{2m-1}}[f](\boldsymbol{x})\bigg|_{\boldsymbol{x}=\boldsymbol{x}_i} = \frac{\partial^{r+s}}{\partial x^r \partial y^s} L_{2m-1}^{\Delta_2(i)}[f](\boldsymbol{x})\bigg|_{\boldsymbol{x}=\boldsymbol{x}_i}, \quad i = 1, \ldots, n.$$

3.4 Numerical results

We test the bivariate Shepard-Lidstone approximation operator in the multivariate interpolation of large sets of scattered data. For each function of the set in Figure 1.2 we compare the numerical results obtained by applying the approximation operator $S_{L_{2m-1}}[f]$ with those obtained by applying the local Shepard-Taylor operator (1.19) $\widetilde{S}_{T_{2m-1}}[f]$ (with the same algebraic degree of exactness $2m-1$ as $S_{L_{2m-1}}$). In the following we set $m = 2$ and $\mu = 3$ then, according to Theorem 21 both operators $\widetilde{S}_{T_3}[f]$ and $S_{L_3}[f]$ have algebraic degree of exactness 3 and class of differentiability 2 in virtue of Theorem 22. In achieving the numerical comparisons between the two operators we compute maximum (1.45), mean (1.46) and mean square (1.47) interpolation errors for the parameter value $N_w = 30$ [62] for the Shepard-Taylor operators $\widetilde{S}_{T_{2m-1}}$ and $N_w = 13$ for the operators $S_{L_{2m-1}}$. We compute numerical approximations by using the Franke dataset and the sparse set of 1089 uniformly distributed interpolation nodes in the unit square R (See Figure 1.3). We compute the resulting approximations at the points of a regular grid of 101×101 points of R. The numerical results are given in Table 3.1. We note that the approximations are comparable even if S_{L_3} uses lacunary data (functional evaluation and only second order derivatives at each node).

Table 3.1: Comparison among the operators $\widetilde{S}_{T_3}$ and S_{L_3} applied to the 10 test functions in Figure 1.2 using the sets of interpolation nodes in Figure 1.3.

		Franke data set		1089 scattered data	
		S_{L_3}	$\widetilde{S}_{T_3}$	S_{L_3}	$\widetilde{S}_{T_3}$
f_1	e_{max}	8.54e-02	4.03e-02	6.49e-04	3.50e-04
	e_{mean}	3.00e-03	1.31e-03	1.76e-05	9.38e-06
	e_{MS}	5.24e-05	1.67e-05	1.48e-09	5.04e-10
f_2	e_{max}	9.45e-04	8.39e-04	1.62e-05	6.80e-06
	e_{mean}	9.24e-05	4.02e-05	5.77e-07	3.58e-07
	e_{MS}	2.01e-08	6.60e-09	9.31e-13	4.18e-13
f_3	e_{max}	3.72e-03	1.20e-03	1.31e-05	1.40e-05
	e_{mean}	5.51e-05	9.93e-05	3.17e-07	3.29e-07
	e_{MS}	3.86e-08	2.98e-08	7.44e-13	6.34e-13
f_4	e_{max}	1.31e-02	1.27e-03	3.05e-05	3.14e-05
	e_{mean}	2.33e-04	7.95e-05	1.22e-06	7.14e-07
	e_{MS}	6.41e-07	2.94e-08	6.21e-12	2.86e-12
f_5	e_{max}	3.97e-02	1.96e-02	7.56e-04	4.22e-04
	e_{mean}	1.45e-03	8.08e-04	1.17e-05	6.46e-06
	e_{MS}	1.51e-05	3.71e-06	1.17e-09	3.17e-10
f_6	e_{max}	7.93e-03	2.33e-03	6.73e-05	7.25e-05
	e_{mean}	4.73e-04	1.38e-04	3.31e-06	1.85e-06
	e_{MS}	8.38e-07	7.83e-08	4.33e-11	2.09e-11
f_7	e_{max}	4.25e-01	1.81e-01	2.27e-02	2.93e-03
	e_{mean}	4.30e-02	1.01e-02	2.79e-04	1.66e-04
	e_{MS}	4.62e-03	4.96e-04	4.43e-07	8.42e-08
f_8	e_{max}	1.93e-01	1.63e-01	2.18e-03	1.50e-03
	e_{mean}	1.21e-02	6.83e-03	8.53e-05	5.00e-05
	e_{MS}	5.38e-04	2.45e-04	3.14e-08	1.04e-08
f_9	e_{max}	1.18e+01	4.45	1.62e-01	1.23e-01
	e_{mean}	6.67e-01	1.93e-01	4.86e-03	2.49e-03
	e_{MS}	1.97	1.94e-01	1.26e-04	4.97e-05
f_{10}	e_{max}	2.29e-01	2.45e-01	3.75e-02	3.68e-02
	e_{mean}	8.89e-03	3.83e-03	9.42e-05	5.11e-05
	e_{MS}	3.86e-04	2.49e-04	3.78e-07	2.54e-07

Chapter 4

Complementary Lidstone Interpolation on Scattered Data Sets

Despite classical Lidstone interpolation (LI) has a long history [13, 12, 14, 44, 52, 69, 70] Complementary Lidstone Interpolation (CLI) has been only recently introduced by Costabile, Dell'Accio and Luceri in [29] and drawn on by Agarwal, Pinelas and Wong in two successive papers [1, 8]. CLI naturally complements LI: both interpolation polynomials are based on two points (say them a and b) and interpolate all data required for their definition, but while the LI polynomial requires the use of odd order Bernoulli polynomials, function evaluations and even order derivative data in both points, the CLI polynomial requires even order Bernoulli polynomials, function evaluation at a (or b) and odd order derivative data in both boundary points. To generalize this kind of interpolation in the context of bivariate scattered data we need firstly to extend the CLI polynomial to a three point bivariate polynomial. The target polynomial will be realized by the well known extension technique [23, 24, 28] opportunely modified, taking in major account the interpolation conditions satisfied by the three point polynomial, rather than its property to be univariate CLI polynomial when restricted to the axes as in [28]. Therefore we introduce new interpolation polynomials on the standard simplex with the following properties: i) they reproduce polynomials of total degree not greater than $2m$; ii) they interpolate the function evaluation in a vertex, all odd order derivatives up to order $2m-1$ at the first vertex and some odd order derivatives at the remaining vertices for a total of $2m^2+3m+1 = \dim \mathcal{P}_{\boldsymbol{x}}^{2m}$ interpolation conditions. The last feature allows us to extend the notion of CLI to scattered data sets by combination with Shepard operators: the local combined Shepard-CLI operator has algebraic degree of exactness $2m$ and uses function evaluations in some points and odd order derivative data in all points up to the order $2m-1$, interpolating them.

4.1 Univariate Complementary-Lidstone interpolation

The CLI polynomial was recently introduced by Costabile, Dell'Accio and Luceri [29]. For a function f of class C^{2m-1} in the closed interval with end-points a, b the CLI problem

$$\begin{cases} CL_{2m}[f,a,b](a) = f(a), \\ CL_{2m}^{(2j-1)}[f,a,b](a) = f^{(2j-1)}(a), \quad j = 1, \ldots, m, \\ CL_{2m}^{(2j-1)}[f,a,b](b) = f^{(2j-1)}(b), \quad j = 1, \ldots, m, \end{cases}$$

has a unique solution in $\mathcal{P}_x^{2m}$ that is

$$\begin{aligned} CL_{2m}[f,a,b](x) = f\left(a\right) + \sum_{j=1}^{m} h^{2j-1} \Bigg(f^{(2j-1)}\left(b\right) \left(v_j \left(\frac{x-a}{h} \right) - v_j\left(0\right) \right) \\ - f^{(2j-1)}\left(a\right) \left(v_j \left(\frac{b-x}{h} \right) - v_j\left(1\right) \right) \Bigg), \end{aligned} \tag{4.1}$$

where $v_j(x)$, $j = 1, ..., m$ are even order degree polynomials, called Lidstone second type polynomials [29], defined recursively by

$$\begin{cases} v_0\left(x\right) = 1, \\ v_j'\left(x\right) = \int_0^x v_{j-1}\left(t\right) dt, \quad j \geq 1, \\ \int_0^1 v_j\left(x\right) dx = 0, \qquad\qquad j \geq 1. \end{cases} \tag{4.2}$$

We note that, while $L_{2m-1}\left[f,a,b\right] = L_{2m-1}\left[f,b,a\right]$, CLI polynomial does not satisfy this property since $CL_{2m}[f,a,b] \neq CL_{2m}[f,b,a]$. Lidstone (3.3) and Lidstone second type polynomials (4.2) are related to well known Bernoulli polynomials by equations [6, 29]

$$\Lambda_j\left(x\right) = \tfrac{2^{2j+1}}{(2j+1)!} B_{2j+1}\left(\tfrac{1+x}{2}\right), \quad v_j\left(x\right) = \tfrac{2^{2j}}{(2j)!} B_{2j}\left(\tfrac{1+x}{2}\right), \quad j = 0, 1, \ldots. \tag{4.3}$$

Agarwal, Pinelas and Wong drawn on researches on Complementary Lidstone interpolation in [29] and in two successive papers [1, 8]; in particular they used the terms Complementary Lidstone polynomials and CLI polynomials for polynomials $v_j(x)$ and $CL_{2m}[f,a,b](x)$ respectively; we find this terminology more appropriate and we shall adopt it in the following.

4.2 Three point Complementary Lidstone polynomials in $\mathbb{R}^2$

In [8] the univariate CLI polynomial (4.1) has been extended to a bivariate polynomial based on the vertices of a rectangle of $\mathbb{R}^2$ by using classical tensor extension technique [17]. The purpose of the paper was to develop piecewise CLI in one and two variables and to establish explicit error bounds for the derivatives in L_∞ and L_2 norms. In this section we extend univariate CLI polynomial to bivariate polynomials based on the vertices of a simplex of $\mathbb{R}^2$ and, in line with the idea presented in [28], we use univariate Lidstone and Complementary Lidstone polynomials in combination. However, while in [28], in analogy with papers [23, 24], the main goal was to get an intrinsic bivariate polynomial in the standard simplex Δ_2, which is the univariate CLI polynomial when restricted to the axes x and y, here, in the extension process, we pay attention mainly on the data that these polynomials use and on the interpolation conditions that they satisfy as well. Moreover, since we use these polynomials in combination with Shepard operators for interpolation of scattered data, we develop they in a more general context than the standard simplex as we did for Lidstone polynomials.

The following expansions hold:

Theorem 24 *Let $f \in C^{2m-1,1}(\Omega)$. Then for each $\boldsymbol{x} \in \Omega$ we have*

$$f(\boldsymbol{x}) = CL_{i,2m}^{\Delta_2(V_0)}[f](\boldsymbol{x}) + R_{i,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x}), \quad i = 0,1,2 \tag{4.4}$$

where $CL_{i,2m}^{\Delta_2(V_0)}[f] \in \mathcal{P}_{\boldsymbol{x}}^{2m-1}$ *are defined by*

$$\begin{aligned}
CL_{0,2m}^{\Delta_2(V_0)}[f](\boldsymbol{x}) &= f(V_0) + \sum_{j=1}^{m}\Big(D_0^{(2j-1,0)}f(V_1)\left(v_j(\lambda_1+\lambda_2)-v_j(0)\right) \\
&-D_0^{(2j-1,0)}f(V_0)\left(v_j(1-\lambda_1-\lambda_2)-v_j(1)\right)\Big) \\
&-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\Big(D_2^{(2k,2j-1)}f(V_0)\Lambda_k(1-\lambda_1-\lambda_2)+D_2^{(2k,2j-1)}f(V_2)\Lambda_k(\lambda_1+\lambda_2)\Big) \\
&\times(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)\right) \\
&-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\Big(D_1^{(2k,2j-1)}f(V_0)\Lambda_k(1-\lambda_1-\lambda_2)+D_1^{(2k,2j-1)}f(V_1)\Lambda_k(\lambda_1+\lambda_2)\Big) \\
&\times(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(1)\right),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
CL_{1,2m}^{\Delta_2(V_0)}[f](\boldsymbol{x}) &= f(V_1) + \sum_{j=1}^{m}\Big(D_0^{(2j-1,0)}f(V_1)\left(v_j(\lambda_1+\lambda_2)-v_j(1)\right) \\
&-D_0^{(2j-1,0)}f(V_0)\left(v_j(1-\lambda_1-\lambda_2)-v_j(0)\right)\Big) \\
&-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\Big(D_2^{(2k,2j-1)}f(V_0)\Lambda_k(1-\lambda_1-\lambda_2)+D_2^{(2k,2j-1)}f(V_2)\Lambda_k(\lambda_1+\lambda_2)\Big) \\
&\times(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)\right) \\
&-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\Big(D_1^{(2k,2j-1)}f(V_0)\Lambda_k(1-\lambda_1-\lambda_2)+D_1^{(2k,2j-1)}f(V_1)\Lambda_k(\lambda_1+\lambda_2)\Big) \\
&\times(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(1)\right),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
CL_{2,2m}^{\Delta_2(V_0)}[f](\boldsymbol{x}) &= f(V_2) + \sum_{j=1}^{m}\Big(D_0^{(0,2j-1)}f(V_2)\left(v_j(\lambda_1+\lambda_2)-v_j(1)\right) \\
&-D_0^{(0,2j-1)}f(V_0)\left(v_j(1-\lambda_1-\lambda_2)-v_j(0)\right)\Big) \\
&+\sum_{j=1}^{m}\sum_{k=0}^{m-j}\Big(D_1^{(2k,2j-1)}f(V_0)\Lambda_k(1-\lambda_1-\lambda_2)+D_1^{(2k,2j-1)}f(V_1)\Lambda_k(\lambda_1+\lambda_2)\Big) \\
&(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)\right) \\
&+\sum_{j=1}^{m}\sum_{k=0}^{m-j}\Big(D_2^{(2k,2j-1)}f(V_0)\Lambda_k(1-\lambda_1-\lambda_2)+D_2^{(2k,2j-1)}f(V_2)\Lambda_k(\lambda_1+\lambda_2)\Big) \\
&(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(1)\right)
\end{aligned} \tag{4.7}$$

and $R_{i,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x}),\, i=0,1,2$ *are the remainder terms.*

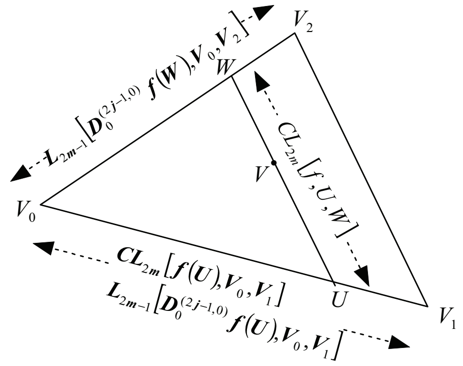


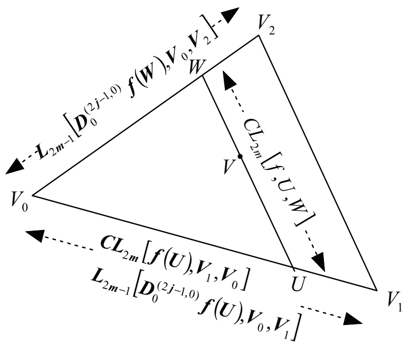
Figure 4.1: Construction scheme of the CLI bivariate polynomial (4.5).

Proof. Let $V = \boldsymbol{x} \in \Delta_2(V_0)$ be an interior point. We set $U(\boldsymbol{x}) = (x + \lambda_2 h_1, y + \lambda_2 k_1)$, $W(\boldsymbol{x}) = (x - \lambda_1 h_1, y - \lambda_1 k_1)$, where $(h_1, k_1) = V_3 - V_2$ and $(h_2, k_2) = V_3 - V_1$. We assign the point V to the line segment $S(\boldsymbol{x}) = [U, W]$ parameterized by

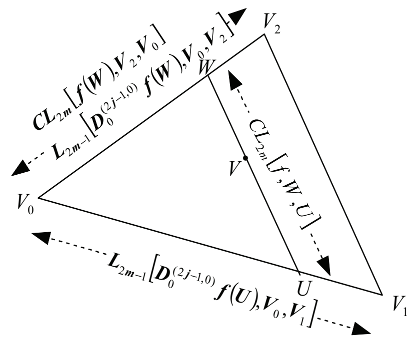
$$\begin{cases} x(\lambda) = x + \lambda_2 h_1 - \lambda(\lambda_1 + \lambda_2) h_1 \\ y(\lambda) = y + \lambda_2 k_1 - \lambda(\lambda_1 + \lambda_2) k_1 \end{cases}, \quad \lambda \in [0, 1].$$

The restriction of f to S is the univariate function $f(x(\lambda), y(\lambda))$ in $[0, 1]$ and we expand it by univariate CLI polynomial (4.1) with $a = 0, b = 1$. In this expansion we replace the parameter λ by the value $\frac{\lambda_2}{\lambda_1 + \lambda_2}$ which corresponds to the point V. This results in an expansion in terms of the values of f at U and its odd order derivative values in the direction of the side of vertices V_1, V_2 at U, W. We reach the vertices of the simplex $\Delta_2(V_0)$ by assigning points U, W to the segments with endpoints V_0, V_1 and V_0, V_2 respectively and by expanding $f(U)$ by the univariate CLI polynomial (4.1) with $a = 0, b = 1$ and odd order directional derivatives of f at U, W by the univariate LI polynomial (3.2). The above procedure, summarized in fig. 4.1, allows us to obtain expansion (4.5). Expansions (4.6), (4.7) are obtained by analogy, according to the schemes in Figure 4.2(a), 4.2(b) respectively. ∎

Expressions for remainder terms $R_{i,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x}), i = 0, 1, 2$ can be obtained by using remainder representations in univariate LI and CLI [6, 29] in combination, as in [28]. These expressions are in terms of the derivatives of order $2m$ of f and therefore expansions



(a)



(b)

Figure 4.2: Construction schemes of polynomials $CL_{1,2m}^{\Delta_2(V_0)}[f]$ (a) and $CL_{2,2m}^{\Delta_2(V_0)}[f]$ (b).

(4.4) reproduce exactly polynomials of total degree not greater than $2m-1$. Then the following Theorem holds.

Theorem 25 *Let $f \in C^{2m,1}(\Omega)$. Then for each $\boldsymbol{x} \in \Omega$, we have*

$$CL_{i,2m}^{\Delta_2(V_0)}[f](\boldsymbol{x}) = T_{2m}[f,V_0](\boldsymbol{x}) + \delta_{i,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x}), \quad i=0,1,2 \tag{4.8}$$

where $T_{2m}[f,V_0](\boldsymbol{x})$ is the Taylor polynomial of order $2m$ for f at V_0 and

$$\begin{aligned}\left|\delta_{0,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq{}& 2^{2m+1}|f|_{2m,1} \\ & \sum_{\alpha=1}^{2m}\left(C_{0,\alpha,m}^1 + C_{0,\alpha,m}^2\right) r^{2m+1-\alpha}\left(r^2 S\right)^{\alpha}||\boldsymbol{x}-V_0||_2^{\alpha}\end{aligned} \tag{4.9}$$

$$\begin{aligned}\left|\delta_{1,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq{}& 2^{2m+1}|f|_{2m,1} \\ & \left(\frac{r^{2m+1}}{(2m+1)!} + \sum_{\alpha=0}^{2m}\left(C_{1,\alpha,m}^1 + C_{1,\alpha,m}^2\right) r^{2m+1-\alpha}\left(r^2 S\right)^{\alpha}||\boldsymbol{x}-V_0||_2^{\alpha}\right)\end{aligned} \tag{4.10}$$

where

$$\begin{aligned} C_{0,\alpha,m}^1 &= \sum_{\substack{\gamma=1+\left\lfloor\frac{\alpha-1}{4}\right\rfloor \\ 2\gamma-\alpha\geq 0}}^{m} \frac{\binom{2\gamma-1}{2\gamma-\alpha}\left|B_{2\gamma-\alpha}\left(\frac{1}{2}\right)\right|2^{2\gamma-\alpha}}{(2\gamma-1)!} \\ C_{0,\alpha,m}^2 &= \sum_{\gamma=1}^{\left\lfloor\frac{\alpha+1}{2}\right\rfloor} \frac{\displaystyle\sum_{\substack{\beta=\left\lfloor\frac{\alpha-1}{2\gamma}\right\rfloor \\ 2\gamma-\alpha+2\beta\geq 0}}^{m-\gamma} \sum_{l=0}^{2\gamma-1} \binom{2\beta+1}{2\gamma-\alpha+2\beta}\binom{2\gamma-1}{l}\left|B_l\left(\frac{1}{2}\right)\right|\left|B_{2\gamma-\alpha+2\beta}\left(\frac{1}{2}\right)\right|2^{2\gamma-\alpha+2\beta+l+1}}{(2m-2\gamma-2\beta+2)!(2\beta+1)!(2\gamma-1)!}. \end{aligned} \tag{4.11}$$

$$
\begin{aligned}
C^1_{1,\alpha,m} &= \sum_{j=\lceil\frac{\alpha+1}{2}\rceil}^{m} \sum_{i=0}^{2j-1-\alpha} \tfrac{\binom{2j-1}{i}\binom{2j-1-i}{\alpha}\left|B_i\left(\frac{1}{2}\right)\right|2^i}{(2m-2j+2)!(2j-1)!} \\
C^2_{1,\alpha,m} &= 2\sum_{\beta=1}^{\lceil\frac{\alpha+1}{2}\rceil} \sum_{\substack{\gamma=\lceil\frac{\alpha+1}{4}\rceil \\ 2\gamma-\alpha\geq 0}}^{m} \sum_{l=0}^{2\beta-1} \tfrac{\binom{2\gamma-2\beta+1}{2\gamma-\alpha}\binom{2\beta-1}{l}\left|B_{2\gamma-\alpha}\left(\frac{1}{2}\right)\right|\left|B_l\left(\frac{1}{2}\right)\right|2^{2\gamma-\alpha+l}}{(2m-2\gamma+2)!(2\gamma-2\beta+1)!(2\beta-1)!}
\end{aligned} \tag{4.12}
$$

and the bound for $\delta^{CL,\Delta_2(V_0)}_{2,2m}[f](\boldsymbol{x})$ *is analogous to (4.10).*

Proof. For each $j=1,\ldots,m$, $k=0,\ldots,m-j$, let us consider the Taylor expansion of order $2m$ of $D_0^{(2j-1,0)}f(V_1)$, $D_2^{(2k,2j-1)}f(V_2)$ and $D_1^{(2k,2j-1)}f(V_1)$ centered at V_0 with integral remainder. By substituting these expansions in (4.5) we have

$$CL^{\Delta_2(V_0)}_{0,2m}[f](\boldsymbol{x}) = T_{2m}[f,V_0](\boldsymbol{x}) + \delta^{CL,\Delta_2(V_0)}_{0,2m}[f](\boldsymbol{x})$$

where $T_{2m}[f,V_0](\boldsymbol{x})$ is the Taylor polynomial of order $2m$ centered at V_0 and

$$
\begin{aligned}
\delta^{CL,\Delta_2(V_0)}_{0,2m}[f](\boldsymbol{x}) &= \sum_{j=1}^{m}\left(\int_0^1 \tfrac{D_0^{(2m+1,0)}f(V_0+t(V_1-V_0))(1-t)^{2m-2j+1}}{(2m-2j+1)!}dt\right)\left(v_j(\lambda_1+\lambda_2)-v_j(0)\right) \\
&-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\left(\int_0^1 \tfrac{D_2^{(2m-2j+2,2j-1)}f(V_0+t(V_2-V_0))(1-t)^{2m-2j-2k+1}}{(2m-2k-2j+1)!}dt\right)\Lambda_k(\lambda_1+\lambda_2) \\
&\times(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)\right) \\
&+\sum_{j=1}^{m}\sum_{k=0}^{m-j}\left(\int_0^1 \tfrac{D_1^{(2m-2j+2,2j-1)}f(V_0+t(V_1-V_0))(1-t)^{2m-2j-2k+1}}{(2m-2k-2j+1)!}dt\right)\Lambda_k(\lambda_1+\lambda_2) \\
&\times(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(1)\right).
\end{aligned} \tag{4.13}
$$

By the Whittaker identities [69] we get (3.18) and

$$v_k(x) = \frac{2^{2k}}{(2k)!}\sum_{l=0}^{2k}\binom{2k}{l}B_l\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^{2k-l}. \tag{4.14}$$

To get the bound (4.9) we use repetitively the Mean-Value Theorem and we get

$$
\begin{aligned}
&v_j(\lambda_1+\lambda_2)-v_j(0) = v_j'(\xi_1)(\lambda_1+\lambda_2) = \Lambda_{j-1}(\xi_1)(\lambda_1+\lambda_2), \\
&\xi_1\in\left(\min\{0,\lambda_1+\lambda_2\},\max\{0,\lambda_1+\lambda_2\}\right),
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
&v_j\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0) = v_j'(\xi_2)\frac{\lambda_2}{\lambda_1+\lambda_2} = \Lambda_{j-1}(\xi_2)\frac{\lambda_2}{\lambda_1+\lambda_2} \\
&\xi_2\in\left(\min\left\{0,\frac{\lambda_2}{\lambda_1+\lambda_2}\right\},\max\left\{0,\frac{\lambda_2}{\lambda_1+\lambda_2}\right\}\right),
\end{aligned} \tag{4.16}
$$

$$v_j\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)-v_j\left(1\right)=v_j'\left(\xi_3\right)\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)=\Lambda_{j-1}\left(\xi_3\right)\frac{\lambda_2}{\lambda_1+\lambda_2}$$
$$\xi_3\in\left(\min\left\{1,\frac{\lambda_1}{\lambda_1+\lambda_2}\right\},\max\left\{1,\frac{\lambda_1}{\lambda_1+\lambda_2}\right\}\right) \tag{4.17}$$

for each $j=1,\dots,m$, $k=0,\dots,m-j$. By (3.18) and (4.14) we have

$$\begin{aligned}v_j\left(\lambda_1+\lambda_2\right)-v_j\left(0\right)&=\Lambda_{j-1}\left(\xi_1\right)\left(\lambda_1+\lambda_2\right)\\&=\sum_{i=0}^{2j-1}\frac{\binom{2j-1}{i}B_i\left(\frac{1}{2}\right)2^i\xi_1^{2j-1-i}\left(\lambda_1+\lambda_2\right)}{(2j-1)!},\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\Lambda_k\left(\lambda_1+\lambda_2\right)\left(\lambda_1+\lambda_2\right)^{2j-1}\left(v_j\left(\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j\left(0\right)\right)\\&=\sum_{i=0}^{2k+1}\sum_{l=0}^{2j-1}\frac{\binom{2k+1}{i}\binom{2j-1}{l}B_l\left(\frac{1}{2}\right)B_i\left(\frac{1}{2}\right)2^{i+l}\left(\lambda_1+\lambda_2\right)^{2k-i+2j-1}\xi_2^{2j-1-l}\lambda_2}{(2k+1)!\,(2j-1)!}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&\Lambda_k\left(\lambda_1+\lambda_2\right)\left(\lambda_1+\lambda_2\right)^{2j-1}\left(v_j\left(1-\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j\left(1\right)\right)\\&=\sum_{i=0}^{2k+1}\sum_{l=0}^{2j-1}\frac{\binom{2k+1}{i}\binom{2j-1}{l}B_i\left(\frac{1}{2}\right)B_l\left(\frac{1}{2}\right)2^{i+l}\left(\lambda_1+\lambda_2\right)^{2k+2j-i-1}\xi_3^{2j-1-l}\lambda_2}{(2k+1)!\,(2j-1)!};\end{aligned} \tag{4.20}$$

therefore by using relations (4.18)-(4.20) in (4.13) we have

$$\begin{aligned}\delta_{0,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})&=\sum_{j=1}^{m}\left(\int_0^1\frac{D_0^{(2m+1,0)}f(V_0+t(V_1-V_0))(1-t)^{2m-2j+1}}{(2m-2j+1)!}dt\right)\\&\times\frac{\sum_{i=0}^{2j-1}\binom{2j-1}{i}B_i\left(\frac{1}{2}\right)2^i\xi_1^{2j-1-i}(\lambda_1+\lambda_2)}{(2j-1)!}\\&-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\left(\int_0^1\frac{D_2^{(2m-2j+2,2j-1)}f(V_0+t(V_2-V_0))(1-t)^{2m-2j-2k+1}}{(2m-2k-2j+1)!}dt\right)\\&\times\frac{\sum_{i=0}^{2k+1}\sum_{l=0}^{2j-1}\binom{2k+1}{i}\binom{2j-1}{l}B_l\left(\frac{1}{2}\right)B_i\left(\frac{1}{2}\right)2^{i+l}(\lambda_1+\lambda_2)^{2k-i+2j-1}\xi_2^{2j-1-l}\lambda_2}{(2k+1)!(2j-1)!}\\&+\sum_{j=1}^{m}\sum_{k=0}^{m-j}\left(\int_0^1\frac{D_1^{(2m-2j+2,2j-1)}f(V_0+t(V_1-V_0))(1-t)^{2m-2j-2k+1}}{(2m-2k-2j+1)!}dt\right)\\&\times\frac{\sum_{i=0}^{2k+1}\sum_{l=0}^{2j-1}\binom{2k+1}{i}\binom{2j-1}{l}B_i\left(\frac{1}{2}\right)B_l\left(\frac{1}{2}\right)2^{i+l}(\lambda_1+\lambda_2)^{2k+2j-i-1}\xi_3^{2j-1-l}\lambda_2}{(2k+1)!(2j-1)!}.\end{aligned}$$

By taking the modulus of both sides, by bounding ξ_1,ξ_2,ξ_3 with the length of the segment they belong to and by relations (1.26), (1.27) and (1.28) we have that, after some rearrangement and changes of dummy indices, $\delta_{0,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})$ is bounded by

$$\left|\delta_{0,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 2^{2m+1}|f|_{2m,1}\sum_{\alpha=1}^{2m}\left(\sum_{\substack{\gamma=1+\left\lfloor\frac{\alpha-1}{4}\right\rfloor\\2\gamma-\alpha\geq 0}}^{m}\frac{\binom{2\gamma-1}{2\gamma-\alpha}\left|B_{2\gamma-\alpha}\left(\frac{1}{2}\right)\right|2^{2\gamma-\alpha}}{(2\gamma-1)!}\right.$$

$$\left.+\sum_{\gamma=1}^{\left\lfloor\frac{\alpha+1}{2}\right\rfloor}\frac{\sum\limits_{\substack{\beta=\left\lfloor\frac{\alpha-1}{2\gamma}\right\rfloor\\2\gamma-\alpha+2\beta\geq 0}}^{m-\gamma}\sum\limits_{l=0}^{2\gamma-1}\binom{2\beta+1}{2\gamma-\alpha+2\beta}\binom{2\gamma-1}{l}\left|B_l\left(\frac{1}{2}\right)\right|\left|B_{2\gamma-\alpha+2\beta}\left(\frac{1}{2}\right)\right|2^{2\gamma-\alpha+2\beta+l+1}}{(2m-2\gamma-2\beta+2)!(2\beta+1)!(2\gamma-1)!}\right)$$

$$\times r^{2m+1-\alpha}\left(r^2S\right)^{\alpha}\|\boldsymbol{x}-V_0\|_2^{\alpha}.$$

(4.9) follows by setting $C_{0,\alpha,m}^1$ and $C_{0,\alpha,m}^2$ as in (4.11).

Let us now prove the bound (4.10). We proceed, as in the previous case, by considering the $(2m)$-th Taylor expansion of $f(V_1)$, $D_0^{(2j-1,0)}f(V_1)$, $D_2^{(2k,2j-1)}f(V_2)$ and $D_1^{(2k,2j-1)}f(V_1)$ centered at V_0 with integral remainder. Moreover, we need to take care of

$$\begin{gathered}v_j(\lambda_1+\lambda_2)-v_j(1)=v_j'(\xi_1)(1-\lambda_1-\lambda_2)=\Lambda_{j-1}(\xi_1)(1-\lambda_1-\lambda_2)\\ \xi_1\in(\min\{1,\lambda_1+\lambda_2\},\max\{1,\lambda_1+\lambda_2\}),\end{gathered}\tag{4.21}$$

$$\begin{gathered}v_j\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)=v_j'(\xi_3)\frac{\lambda_1}{\lambda_1+\lambda_2}=\Lambda_{j-1}(\xi_3)\frac{\lambda_1}{\lambda_1+\lambda_2}\\ \xi_3\in\left(\min\left\{0,\frac{\lambda_1}{\lambda_1+\lambda_2}\right\},\max\left\{0,\frac{\lambda_1}{\lambda_1+\lambda_2}\right\}\right).\end{gathered}\tag{4.22}$$

Therefore, from (4.16), (4.21) and (4.22) we get

$$v_j(\lambda_1+\lambda_2)-v_j(1)=\frac{\sum\limits_{i=0}^{2j-1}\binom{2j-1}{i}B_i\left(\frac{1}{2}\right)2^i\xi_1^{2j-1-i}(1-\lambda_1-\lambda_2)}{(2j-1)!},$$

$$\begin{gathered}\Lambda_k(\lambda_1+\lambda_2)(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)\right)\\ =\sum_{i=0}^{2k+1}\sum_{l=0}^{2j-1}\frac{\binom{2k+1}{i}\binom{2j-1}{l}B_i\left(\frac{1}{2}\right)B_l\left(\frac{1}{2}\right)2^{i+l}(\lambda_1+\lambda_2)^{2k+2j-i-1}\xi_2^{2j-1-l}\lambda_2}{(2k+1)!(2j-1)!},\end{gathered}$$

$$\begin{gathered}\Lambda_k(\lambda_1+\lambda_2)(\lambda_1+\lambda_2)^{2j-1}\left(v_j\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)-v_j(0)\right)\\ =\sum_{i=0}^{2k+1}\sum_{l=0}^{2j-1}\frac{\binom{2k+1}{i}\binom{2j-1}{l}B_i\left(\frac{1}{2}\right)B_l\left(\frac{1}{2}\right)2^{i+l}(\lambda_1+\lambda_2)^{2k+2j-i-1}\xi_3^{2j-1-l}\lambda_1}{(2k+1)!(2j-1)!}.\end{gathered}$$

Then we have

$$\delta_{1,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x}) = \int_0^1 \frac{D_0^{(2m+1,0)} f(V_0+t(V_1-V_0))(1-t)^{2m}}{(2m)!}dt$$
$$+\sum_{j=1}^{m}\left(\int_0^1 \frac{D_0^{(2m+1,0)} f(V_0+t(V_1-V_0))(1-t)^{2m-2j+1}}{(2m-2j+1)!}dt\right)$$
$$\times\frac{\sum\limits_{i=0}^{2j-1}\binom{2j-1}{i}B_i\left(\frac{1}{2}\right)2^i\xi_1^{2j-1-i}(1-\lambda_1-\lambda_2)}{(2j-1)!}$$
$$-\sum_{j=1}^{m}\sum_{k=0}^{m-j}\left(\int_0^1 \frac{D_2^{(2m-2j+2,2j-1)} f(V_0+t(V_2-V_1))(1-t)^{2m-2j-2k+1}}{(2m-2k-2j+1)!}dt\right)$$
$$\times\frac{\sum\limits_{i=0}^{2k+1}\sum\limits_{l=0}^{2j-1}\binom{2k+1}{i}\binom{2j-1}{l}B_i\left(\frac{1}{2}\right)B_l\left(\frac{1}{2}\right)2^{i+l}(\lambda_1+\lambda_2)^{2k+2j-i-1}\xi_2^{2j-1-l}\lambda_2}{(2k+1)!(2j-1)!}$$
$$+\sum_{j=1}^{m}\sum_{k=0}^{m-j}\left(\int_0^1 \frac{D_1^{(2m-2j+2,2j-1)} f(V_0+t(V_1-V_0))(1-t)^{2m-2j-2k+1}}{(2m-2k-2j+1)!}dt\right)$$
$$\times\frac{\sum\limits_{i=0}^{2k+1}\sum\limits_{l=0}^{2j-1}\binom{2k+1}{i}\binom{2j-1}{l}B_i\left(\frac{1}{2}\right)B_l\left(\frac{1}{2}\right)2^{i+l}(\lambda_1+\lambda_2)^{2k+2j-i-1}\xi_3^{2j-1-l}\lambda_1}{(2k+1)!(2j-1)!};$$

by taking the modulus of both sides, by bounding ξ_1, ξ_2, ξ_3 with the length of the segment they belong to and by relations (1.26), (1.27) and (1.28) we have that, after some rearrangement and changes of dummy indices, $\delta_{1,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})$ is bounded by

$$\left|\delta_{1,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 2^{2m+1}\left|f\right|_{2m,1} r^{2m+1}\frac{1}{(2m+1)!}$$
$$+2^{2m+1}\left|f\right|_{2m,1}\sum_{\alpha=0}^{2m}\left(\sum_{j=\left\lceil\frac{\alpha+1}{2}\right\rceil}^{m}\sum_{i=0}^{2j-1-\alpha}\frac{\binom{2j-1}{i}\binom{2j-1-i}{\alpha}\left|B_i\left(\frac{1}{2}\right)\right|2^i}{(2m-2j+2)!(2j-1)!}\right.$$
$$\left.+2\sum_{\beta=1}^{\left\lceil\frac{\alpha+1}{2}\right\rceil}\sum_{\substack{\gamma=\left\lceil\frac{\alpha+1}{4}\right\rceil\\ 2\gamma-\alpha\geq 0}}^{m}\sum_{l=0}^{2\beta-1}\frac{\binom{2\gamma-2\beta+1}{2\gamma-\alpha}\binom{2\beta-1}{l}\left|B_{2\gamma-\alpha}\left(\frac{1}{2}\right)\right|\left|B_l\left(\frac{1}{2}\right)\right|2^{2\gamma-\alpha+l}}{(2m-2\gamma+2)!(2\gamma-2\beta+1)!(2\beta-1)!}\right)$$
$$\times r^{2m+1-\alpha}\left(r^2S\right)^{\alpha}\left|\left|\boldsymbol{x}-V_0\right|\right|_2^{\alpha}.$$

The bound (4.10) follows by setting $C_{1,\alpha,m}^1$ and $C_{1,\alpha,m}^2$ as in (4.12). ■

Corollary 26 *In the hypothesis of Theorem 25 for all $\boldsymbol{x} \in \Omega$ we have*

$$\left|R_{0,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 2^{2m}\left|f\right|_{2m,1}$$
$$\left(2\sum_{\alpha=1}^{2m}\left(C_{0,\alpha,m}^1+C_{0,\alpha,m}^2\right)r^{2m+1-\alpha}\left(r^2S\right)^{\alpha}\left|\left|\boldsymbol{x}-V_0\right|\right|_2^{\alpha}+\frac{\left|\left|\boldsymbol{x}-V_0\right|\right|_2^{2m+1}}{(2m-1)!}\right) \tag{4.23}$$

and

$$\left|R_{1,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 2^{2m}|f|_{2m,1}$$
$$\left(2\tfrac{r^{2m+1}}{(2m+1)!}+4\sum_{\alpha=0}^{2m}\left(C_{1,\alpha,m}^{1}+C_{1,\alpha,m}^{2}\right)r^{2m+1-\alpha}\left(r^{2}S\right)^{\alpha}||\boldsymbol{x}-V_0||_2^{\alpha}+\tfrac{||\boldsymbol{x}-V_0||_2^{2m+1}}{(2m-1)!}\right) \tag{4.24}$$

where $C_{0,\alpha,m}^{k}, C_{1,\alpha,m}^{k}, k=1,2$ *are defined in (4.11) and (4.12). The bound for* $R_{2,2m}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})$ *is the same of the bound (4.24).*

In the case $m=1$, which we use in the numerical results, we get

$$\left|R_{0,2}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 4|f|_{2,1}\left(3r\left(r^2S\right)^2||\boldsymbol{x}-V_0||_2^2+||\boldsymbol{x}-V_0||_2^3\right)$$
$$\left|R_{k,2}^{CL,\Delta_2(V_0)}[f](\boldsymbol{x})\right| \leq 4|f|_{2,1}\left(\frac{4}{3}r^3+r^2\left(r^2S\right)||\boldsymbol{x}-V_0||_2+2r\left(r^2S\right)^2||\boldsymbol{x}-V_0||_2^2+||\boldsymbol{x}-V_0||_2^3\right)$$

$k=1,2$.

4.3 The bivariate Shepard-Complementary Lidstone operator

In defining the bivariate Shepard-CLI operator we suppose that function data of one of the following type are associated to each node $\boldsymbol{x}_i$,

$$\left\{f(\boldsymbol{x}_i), \frac{\partial^{2k-1}f}{\partial x^{2k-1-j}\partial y^{j}}(\boldsymbol{x}_i)\right\}_{\substack{k=1,\ldots,m\\ j=0,\ldots,2k-1}} \tag{4.25}$$

or

$$\left\{\frac{\partial^{2k-1}f}{\partial x^{2k-1-j}\partial y^{j}}(\boldsymbol{x}_i)\right\}_{\substack{k=1,\ldots,m\\ j=0,\ldots,2k-1}}. \tag{4.26}$$

Such data, clearly, generalize classic Complementary Lidstone univariate data to the bivariate case. Now we introduce an explicit interpolant of these data, by using local Shepard operators in combination with three point CLI polynomials (4.5), (4.6) and (4.7). For the nature of the three point polynomial interpolants we need to associate to each sample point $\boldsymbol{x}_i, i=1,\ldots,n$ a triangle with a vertex in $\boldsymbol{x}_i$ and other two vertices in its neighborhood; we propose to do the choice by the following priority criteria: let R_{w_i} be the smallest positive real number such that the ball $B(\boldsymbol{x}_i, R_{w_i})$ includes at least N_w nodes and at least a node with associated data (4.25);

1. if the function data in $\boldsymbol{x}_i$ are of type (4.25) then we associate to $\boldsymbol{x}_i$ the triangle in $B(\boldsymbol{x}_i, R_{w_i})$ which minimizes the quantity $r_i^{2m+1}\left(r_i^2 S_i\right)^{2m}$,

2. if the function data in $\boldsymbol{x}_i$ are of type (4.26) and $\boldsymbol{x}_j$ is a vertex with associated data of type (4.25) then we associate to $\boldsymbol{x}_i$ the triangle in $B(\boldsymbol{x}_i, R_{w_i})$ with vertices in $\boldsymbol{x}_i$, $\boldsymbol{x}_j$ which minimizes the quantity $r_i^{2m+1}\left(r_i^2 S_i\right)^{2m}$.

As before, we denote by $\Delta_2(i)$ the triangle associated to $\boldsymbol{x}_i$ according to the above procedure and we assume that its vertices are listed counterclockwise. To define the Shepard-CLI operator at $\boldsymbol{x} \in \Omega$, according to (1.12), we substitute the value $f(\boldsymbol{x}_i)$ in (1.10) with:

a. the value at $\boldsymbol{x}$ of the interpolation polynomial (4.5) in case (1);

b. the value at $\boldsymbol{x}$ of the interpolation polynomial (4.6) in case (2) if $\boldsymbol{x}_j$ is a vertex with associated data of type (4.25);

c. the value at $\boldsymbol{x}$ of the interpolation polynomial (4.7) in case (2) if $\boldsymbol{x}_j$ is a vertex with associated data of type (4.26).

Definition 27 *For each fixed $\mu > 0$ and $m = 1, 2, \ldots$ the bivariate Shepard-CLI operators are defined by*

$$S_{CL_{2m}}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x})\, CL_{2m}^{\Delta_2(i)}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{4.27}$$

where $CL_{2m}^{\Delta_2(i)}[f](\boldsymbol{x})$, $i = 1, \ldots, n$ is one of the CLI polynomials (4.5), (4.6) or (4.7) according to the above rules. The remainder term is

$$R_{CL_{2m}}[f](\boldsymbol{x}) = f(\boldsymbol{x}) - S_{CL_{2m}}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{4.28}$$

Convergence results can be obtained by following the known approaches [68, § 15.4], [36, 73]. Let us denote by $\mathcal{I}_e$ the set of indices of nodes with functional data of type (4.25) and by $\mathcal{I}_{ne}$ the set of indices of nodes with functional data of type (4.26) we have the following bound for the remainder term (4.28)

Theorem 28 *Let $\boldsymbol{x} \in \Omega$. The following bound holds:*

$$\begin{aligned}
&|R_{CL_{2m}}[f](\boldsymbol{x})| \leq 2^{2m} |f|_{2m,1} \sup_{\boldsymbol{x}\in\Omega} \#(\mathcal{I}_{\boldsymbol{x}} \cap \mathcal{I}_e)\, 2 \max_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_e} \left\{ \sum_{\alpha=1}^{2m} \left(\left(C^1_{0,\alpha,m}\right)_i + \left(C^2_{0,\alpha,m}\right)_i \right) \right. \\
&\left. \times\, r_i^{2m+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R^{\alpha}_{w_i} + \frac{R^{2m+1}_{w_i}}{(2m-1)!} \right\} \\
&+ 2^{2m} |f|_{2m,1} \sup_{\boldsymbol{x}\in\Omega} \#(\mathcal{I}_{\boldsymbol{x}} \cap \mathcal{I}_{ne}) \max_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_{ne}} \left\{ 2\frac{r_i^{2m+1}}{(2m+1)!} + 4\sum_{\alpha=0}^{2m} \left(\left(C^1_{1,\alpha,m}\right)_i + \left(C^2_{1,\alpha,m}\right)_i \right) \right. \\
&\left. \times\, r_i^{2m+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R^{\alpha}_{w_i} + \frac{R^{2m+1}_{w_i}}{(2m-1)!} \right\}.
\end{aligned} \tag{4.29}$$

where $C^k_{0,\alpha,m}, C^k_{1,\alpha,m}$, $k = 1, 2$ are defined in (4.11) and (4.12).

Proof. From (4.23), (4.24) we have

$$\begin{aligned}
&|R_{CL_{2m}}[f](\boldsymbol{x})| \leq 2^{2m} |f|_{2m,1} \sum_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_e} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \\
&\times \left(2 \sum_{\alpha=1}^{2m} \left(\left(C^1_{0,\alpha,m}\right)_i + \left(C^2_{0,\alpha,m}\right)_i \right) r_i^{2m+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} ||\boldsymbol{x} - V_{i_0}||_2^{\alpha} + \frac{||\boldsymbol{x}-V_{i_0}||_2^{2m+1}}{(2m-1)!} \right) \\
&+ 2^{2m} |f|_{2m,1} \sum_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_{ne}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \\
&\times \left(2\frac{r_i^{2m+1}}{(2m+1)!} + 4\sum_{\alpha=0}^{2m} \left(\left(C^1_{1,\alpha,m}\right)_i + \left(C^2_{1,\alpha,m}\right)_i \right) r_i^{2m+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} ||\boldsymbol{x} - V_{i_0}||_2^{\alpha} + \frac{||\boldsymbol{x}-V_{i_0}||_2^{2m+1}}{(2m-1)!} \right) \\
&\leq 2^{2m} |f|_{2m,1} \sup_{\boldsymbol{x}\in\Omega} \#(\mathcal{I}_{\boldsymbol{x}} \cap \mathcal{I}_e) \max_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_e} \left\{ 2 \sum_{\alpha=1}^{2m} \left(\left(C^1_{0,\alpha,m}\right)_i + \left(C^2_{0,\alpha,m}\right)_i \right) \right. \\
&\left. \times\, r_i^{2m+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R^{\alpha}_{w_i} + \frac{R^{2m+1}_{w_i}}{(2m-1)!} \right\} \\
&+ 2^{2m} |f|_{2m,1} \sup_{\boldsymbol{x}\in\Omega} \#(\mathcal{I}_{\boldsymbol{x}} \cap \mathcal{I}_{ne}) \max_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_{ne}} \left(2\frac{\left(r_i^{2m+1}\right)}{(2m+1)!} + 4\sum_{\alpha=0}^{2m} \max_{i\in\mathcal{I}_{\boldsymbol{x}}\cap\mathcal{I}_{ne}} \left(\left(C^1_{1,\alpha,m}\right)_i + \left(C^2_{1,\alpha,m}\right)_i \right) \right. \\
&\left. \times r_i^{2m+1-\alpha} \left(r_i^2 S_i\right)^{\alpha} R^{\alpha}_{w_i} + \frac{R^{2m+1}_{w_i}}{(2m-1)!} \right\}.
\end{aligned}$$

■

The following statements can be easily checked.

Theorem 29 *The operator $\widetilde{S}_{CL_{2m}}[\cdot]$ interpolates at V_i, $i = 1, ..., n$ if the functional data in V_i are of type (4.25).*

Theorem 30 *The degree of exactness of the operator $\widetilde{S}_{CL_{2m}}[\cdot]$ is $2m$, i.e. $\widetilde{S}_{CL_{2m}}[f] = f$ for each bivariate polynomial $f \in \mathcal{P}^{2m}_{\boldsymbol{x}}$.*

In [10] is given the continuity class of the Shepard operators, and consequently we can deduce the continuity class of the Shepard-CLI operators.

Theorem 31 *If $\mu \in \mathbb{N}$, $\mu > 0$ the continuity class of the operator (4.27) is $\mu - 1$.*

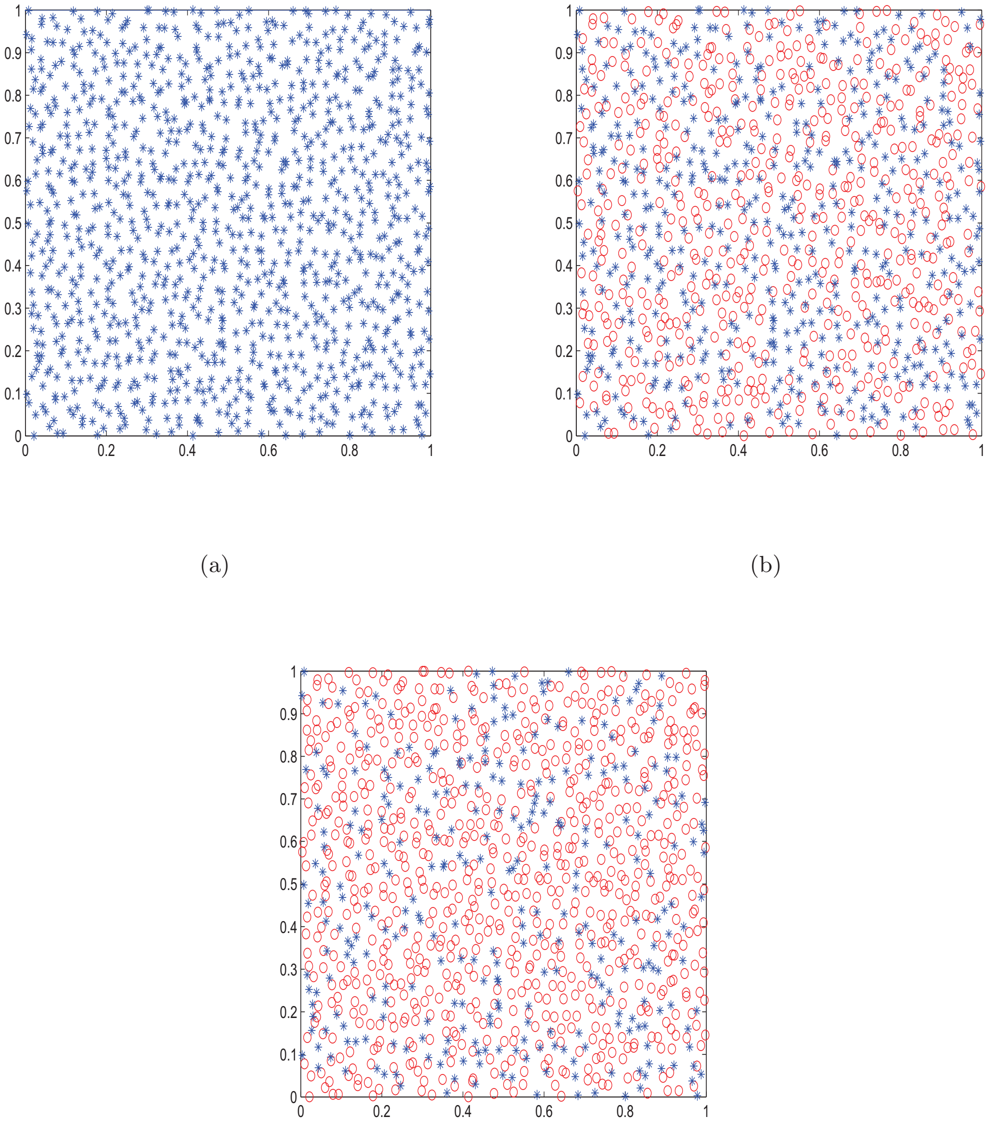
Theorem 32 *For each $k, j \in \mathbb{N}$ with $1 \leq 2k-1 < \mu+1, 0 \leq j \leq 2k-1$ we have*

$$\left.\frac{\partial^{2k-1}}{\partial x^{2k-1-j}\partial y^{j}} S_{CL_{2m}}[f](\boldsymbol{x})\right|_{\boldsymbol{x}=\boldsymbol{x}_i} = \left.\frac{\partial^{2k-1} f}{\partial x^{2k-1-j}\partial y^{j}}(\boldsymbol{x})\right|_{\boldsymbol{x}=\boldsymbol{x}_i}, \quad i = 1, \ldots, n.$$

4.4 Numerical results

In the following we set $\mu = 2$. According to Theorem 30 both operators have degree of exactness 2 and class of differentiability 1 in virtue of Theorem 31. We note that the operator $\widetilde{S}_{T_2}$ uses function evaluations and all partial derivatives up to the order 2 at each sample point, while the operator $\widetilde{S}_{CL_2}$ is a lacunary interpolation operator which uses function evaluations and partial derivatives of order not greater than 1. In achieving the numerical comparisons between the two operators we compute maximum (1.45), mean (1.46), mean square (1.47) interpolation errors for the parameter values $N_w = 19$ for the operator $\widetilde{S}_{T_2}$ [62] and $N_w = 7$ for operator $\widetilde{S}_{CL_2}$. We compute numerical approximations by using a 1089 sparse set of uniformly distributed interpolation nodes in the unit square R. To better show the accuracy of operator $\widetilde{S}_{CL_2}$ by varying the number of lacunas, we compute approximations in cases of: (a) 100% of data of type (4.25) coupled with 0% of data of type (4.26); (b) 66% of data of type (4.25) coupled with 34% of data of type (4.26); (c) 33% of data of type (4.25) coupled with 67% of data of type (4.26).

We compute the resulting approximations at the points of a regular grid of 101×101 points of R. The numerical results are given in Table 4.1. We note that the obtained approximations are comparable even if $\widetilde{S}_{CL_2}$ uses lacunary data.



(a) (b) (c)

Figure 4.3: Sparse set of $n = 1089$ uniformly distributed interpolation points in R: (a) 100% of data of type (4.25) coupled with 0% of data of type (4.26); (b) 66% of data of type (4.25) coupled with 34% of data of type (4.26); (c) 33% of data of type (4.25) coupled with 67% of data of type (4.26)).

Table 4.1: Comparison between the operator $\widetilde{S}_{T_2}$ and the operator S_{CL_2} applied to the 10 test functions in Figure 1.2 using the sparse set of 1089 interpolation nodes in Figure 1.3 with the different type of data in Figure 4.3(a), Figure 4.3(b) and Figure 4.3(c).

Type of data			Figure 4.3(a)	Figure 4.3(b)	Figure 4.3(c)
		$\tilde{S}_{T_2}$	S_{CL_2}	S_{CL_2}	S_{CL_2}
f_1	$e_{\max}$	1.82e-03	1.98e-03	3.87e-03	5.26e-03
	e_{mean}	6.58e-05	9.83e-05	1.51e-04	1.96e-04
	e_{MS}	1.91e-08	3.76e-08	8.67e-08	1.53e-07
f_2	$e_{\max}$	4.80e-05	9.44e-05	1.30e-04	4.35e-04
	e_{mean}	2.88e-06	7.04e-06	1.11e-05	1.73e-05
	e_{MS}	2.20e-11	1.24e-10	2.92e-10	1.01e-09
f_3	$e_{\max}$	1.29e-04	1.71e-04	2.21e-04	7.92e-04
	e_{mean}	2.75e-06	3.39e-06	5.32e-06	9.02e-06
	e_{MS}	5.23e-11	8.30e-11	1.73e-10	1.17e-09
f_4	$e_{\max}$	1.30e-04	1.96e-04	3.36e-04	8.33e-04
	e_{mean}	5.18e-06	1.11e-05	1.80e-05	2.72e-05
	e_{MS}	1.01e-10	4.00e-10	1.01e-09	3.35e-09
f_5	$e_{\max}$	1.61e-03	2.24e-03	2.23e-03	2.13e-02
	e_{mean}	5.07e-05	4.98e-05	6.46e-05	1.77e-04
	e_{MS}	1.65e-08	1.85e-08	2.47e-08	9.73e-07
f_6	$e_{\max}$	2.85e-04	5.85e-04	7.39e-04	1.71e-03
	e_{mean}	1.23e-05	2.38e-05	3.45e-05	5.27e-05
	e_{MS}	6.44e-10	2.33e-09	4.49e-09	1.54e-08
f_7	$e_{\max}$	1.79e-02	2.67e-02	5.39e-02	1.79e-01
	e_{mean}	1.11e-03	1.69e-03	2.56e-03	4.39e-03
	e_{MS}	3.17e-06	7.75e-06	1.84e-05	1.15e-04
f_8	$e_{\max}$	6.41e-03	1.43e-02	1.26e-02	2.63e-02
	e_{mean}	3.26e-04	4.19e-04	6.04e-04	8.58e-04
	e_{MS}	3.57e-07	7.38e-07	1.31e-06	3.42e-06
f_9	$e_{\max}$	5.42e-01	8.37e-01	9.48e-01	1.71
	e_{mean}	1.71e-02	2.61e-02	3.76e-02	5.34e-02
	e_{MS}	1.58e-03	3.64e-03	6.35e-03	1.47e-02
f_{10}	$e_{\max}$	4.79e-02	3.66e-02	3.65e-02	3.34e-02
	e_{mean}	2.66e-04	3.93e-04	5.86e-04	8.91e-04
	e_{MS}	1.35e-06	1.31e-06	1.93e-06	4.51e-06

Chapter 5

Triangular Shepard method

When only function evaluations are given, the three point polynomials of Chapters 2 and 3, reduce to the three point linear interpolation polynomial. A method by F. Little, called *triangular Shepard* and introduced in 1982 [53], combines Shepard-like basis functions and three point linear interpolation polynomials as well. It is a convex combination of the linear interpolants of a set of triangles, in which each weight function is defined as the product of the inverse distance from the vertices of the corresponding triangle. The method reproduces linear polynomials without using any derivative data and Little noticed that it surpasses Shepard's method greatly in esthetic behavior.

During my period at the USI-Università della Svizzera Italiana, working with Prof. K. Hormann, we rediscovered the same method independently as a natural way to break the asymmetry that occurs in combining the Shepard operator with three point linear interpolation polynomials. In this Chapter we deeply study the properties of the new weight functions and of the triangular Shepard operator. Some results, here presented, are totally new, in particular: equations (5.4) and (5.7) which can be useful for making the triangular Shepard method able to interpolate derivative and Proposition 34 and Theorem 36 which demonstrate the quadratic rate of convergence for general configuration of triangles. The validity of the triangular Shepard method is confirmed by a series of experiments, based on different Delaunay triangulations with increasing resolution and completely arbitrary triangulations as well. In fact Proposition 34 and Theorem 36, give information on how to

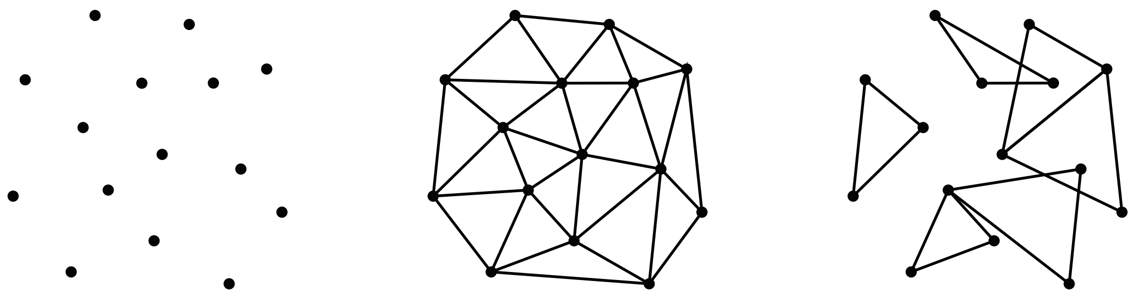


Figure 5.1: Example of a node set X (left), its Delaunay triangulation (center), and an arbitrary triangulation T of X (right).

choose the triangulation in order to get good accuracy of approximation, i.e. they make the method not dependent from any mesh which connects the interpolation points, i.e. a meshless or meshfree method.

5.1 Triangle-based basis functions

To extend the point-based basis functions in (1.9) to triangle-based basis functions, let us consider a triangulation $T = \{t_1, t_2, \ldots, t_m\}$ of the nodes X. That is, each $t_j = [\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3}]$ is a triangle with vertices in X and each node $\boldsymbol{x}_i \in X$ is the vertex of at least one triangle, hence

$$\bigcup_{j=1}^{m} \{j_1, j_2, j_3\} = \{1, 2, \ldots, n\}. \tag{5.1}$$

For example, T can be the *Delaunay triangulation* [35] of X, but we also allow for general triangulations with overlapping or disjoint triangles (see Figure 5.1).

The *triangle-based basis functions* with respect to the triangulation T are then defined by

$$B_{\mu,j}(\boldsymbol{x}) = \frac{\displaystyle\prod_{\ell=1}^{3} \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|_2^{\mu}}}{\displaystyle\sum_{k=1}^{m} \prod_{\ell=1}^{3} \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_{k_\ell}\|_2^{\mu}}}, \qquad j = 1, \ldots, m, \tag{5.2}$$

where $\mu > 0$ is again a control parameter. Like Shepard's basis functions, the triangle-based basis functions (5.2) are non-negative and form a partition of unity, but instead of being cardinal they satisfy the following properties.

Proposition 33 *The triangle-based basis function in* (5.2) *and its gradient (that exists for* $\mu > 1$*) vanish at all nodes* $\boldsymbol{x}_i \in X$ *that are not a vertex of the corresponding triangle* t_j*. That is,*

$$B_{\mu,j}(\boldsymbol{x}_i) = 0, \tag{5.3}$$

$$\nabla B_{\mu,j}(\boldsymbol{x}_i) = 0, \quad \mu > 1, \tag{5.4}$$

for any $j = 1, \ldots, m$ *and* $i \notin \{j_1, j_2, j_3\}$*.*

Proof. If we multiply both the numerator and the denominator of (5.2) with $\|\boldsymbol{x} - \boldsymbol{x}_i\|_2^{\mu}$, then

$$B_{\mu,j}(\boldsymbol{x}) = \frac{C_j(\boldsymbol{x})}{\sum\limits_{k=1}^{m} C_k(\boldsymbol{x})},$$

where

$$C_k(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{x}_i\|_2^{\mu} \prod_{\ell=1}^{3} \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_{k_\ell}\|_2^{\mu}}, \qquad k = 1, \ldots, m.$$

Let us denote by J_i the set of indices of all triangles which have $\boldsymbol{x}_i$ as a vertex,

$$J_i = \big\{k \in \{1, \ldots, m\} : i \in \{k_1, k_2, k_3\}\big\}, \tag{5.5}$$

and note that $J_i \neq \emptyset$ by (5.1). Then, $C_k(\boldsymbol{x}_i) = 0$ if and only if $k \notin J_i$, and (5.3) follows because $j \notin J_i$. Moreover, if $\mu > 1$, then $C_k(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}_i$, and (5.4) follows because

$$\nabla B_{\mu,j}(\boldsymbol{x}) = \frac{\nabla C_j(\boldsymbol{x}) \sum_{k=1}^{m} C_k(\boldsymbol{x}) + C_j(\boldsymbol{x}) \sum_{k=1}^{m} \nabla C_k(\boldsymbol{x})}{\left(\sum_{k=1}^{m} C_k(\boldsymbol{x})\right)^2}$$

and $\nabla C_j(\boldsymbol{x}_i) = 0$ for $i \notin \{j_1, j_2, j_3\}$. ■

As an immediate consequence of Proposition 33 and the partition of unity property we have, for each $i = 1, \ldots, n$,

$$\sum_{j \in J_i} B_{\mu,j}(\boldsymbol{x}_i) = 1, \tag{5.6}$$

$$\sum_{j \in J_i} \nabla B_{\mu,j}(\boldsymbol{x}_i) = 0, \quad \mu > 1 \tag{5.7}$$

where J_i is the index set from (5.5).

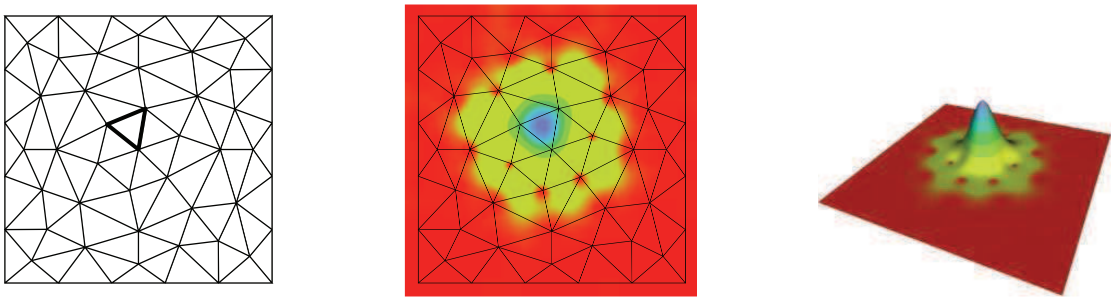


Figure 5.2: Triangle-based basis function $B_{2,j}(\boldsymbol{x})$ for the indicated triangle t_j with respect to a Delaunay triangulation T.

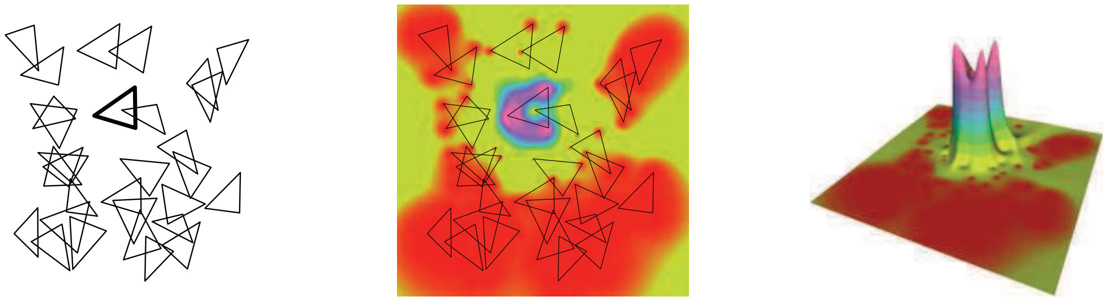


Figure 5.3: Triangle-based basis function $B_{2,j}(\boldsymbol{x})$ for the indicated triangle t_j with respect to an arbitrary triangulation T.

Figures 5.2 and 5.3 show examples of triangle-based basis functions for $\mu = 2$. Function values close to 0 are marked by red and the ten colours from green via blue to pink correspond to the ten uniform intervals of function values between 0 and 1. As predicted by Proposition 33, each basis function has local minima with function value 0 at all nodes, except at the vertices $\boldsymbol{x}_i$, $i \in \{j_1, j_2, j_3\}$ of the corresponding triangle t_j. According to (5.6), if $J_i = \{j\}$, so that t_j is the only triangle adjacent to x_i, then the basis function has a local maximum with function value 1 at $\boldsymbol{x}_i$ (see Figure 5.3).

5.2 Local linear interpolants

The second ingredient for Little's extension of the Shepard operator [53] are the linear polynomials that locally interpolate the given data at the vertices of each of the triangles. For $t_j \in T$, this polynomial $L_j \colon \mathbb{R}^2 \to \mathbb{R}$ can be written as

$$L_j(\boldsymbol{x}) = \sum_{\ell=1}^{3} \lambda_{j,j_\ell}(\boldsymbol{x}) f_{j_\ell}, \tag{5.8}$$

where $\lambda_{j,j_\ell}(\boldsymbol{x})$, $\ell = 1, 2, 3$ are the *barycentric coordinates* [56] of $\boldsymbol{x}$ with respect to the triangle $t_j = [\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3}]$, that is,

$$\lambda_{j,j_1}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3})}{A(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3})}, \quad \lambda_{j,j_2}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}_{j_1}, \boldsymbol{x}, \boldsymbol{x}_{j_3})}{A(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3})}, \quad \lambda_{j,j_3}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x})}{A(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3})},$$

with $A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ denoting the signed area of the triangle $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$. In general, $\lambda_{j,i}(\boldsymbol{x})$ with $j \in J_i$ and $i \in \{j_1, j_2, j_3\}$ is the unique linear polynomial with $\lambda_{j,i}(\boldsymbol{x}_i) = 1$ and $\lambda_{j,i}(\boldsymbol{x}) = 0$ for $\boldsymbol{x}$ on the line defined by the edge opposite $\boldsymbol{x}_i$ in the triangle t_j.

To study the approximation order of the interpolating polynomial (5.8), we let $\Omega \subset \mathbb{R}^2$ be a non-empty, compact, and convex domain with $X \subset \Omega$ (e.g. the convex hull of X). Moreover, we denote the edge vectors of the triangle t_j by $e_{j_1} = \boldsymbol{x}_{j_3} - \boldsymbol{x}_{j_2}$, $e_{j_2} = \boldsymbol{x}_{j_3} - \boldsymbol{x}_{j_1}$, and $e_{j_3} = \boldsymbol{x}_{j_1} - \boldsymbol{x}_{j_2}$.

Proposition 34 *Let Ω be a compact convex domain which contains X and $f \in C^{1,1}(\Omega)$. Then,*

$$|f(\boldsymbol{x}) - L_j(\boldsymbol{x})| \le |f|_{1,1} \left(2\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2^2 + 4h_j C_j \|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2 \right) \tag{5.9}$$

for any $\boldsymbol{x} \in \Omega$, with $h_j = \max\{\|e_{j_1}\|_2, \|e_{j_2}\|_2, \|e_{j_3}\|_2\}$ and C_j a constant which depends only on the shape of t_j.

Proof. Let us consider the first order Taylor expansion of $f(\boldsymbol{x}_{j_2})$ and $f(\boldsymbol{x}_{j_3})$ at $\boldsymbol{x}_{j_1}$ with integral remainder [9],

$$\begin{aligned} f(\boldsymbol{x}_{j_2}) &= f(\boldsymbol{x}_{j_1}) + e_{j_3} \cdot \nabla f(\boldsymbol{x}_{j_1}) + \|e_{j_3}\|_2^2 \int_0^1 \frac{\partial^2 f(\boldsymbol{x}_{j_1} + t e_{j_3})}{\partial \nu_{j_3}^2} (1-t)\, dt, \\ f(\boldsymbol{x}_{j_3}) &= f(\boldsymbol{x}_{j_1}) + e_{j_2} \cdot \nabla f(\boldsymbol{x}_{j_1}) + \|e_{j_2}\|_2^2 \int_0^1 \frac{\partial^2 f(\boldsymbol{x}_{j_1} + t e_{j_2})}{\partial \nu_{j_2}^2} (1-t)\, dt, \end{aligned} \tag{5.10}$$

where $\frac{\partial^2}{\partial \nu_{j_k}^2}$ are the second order directional derivatives along the unit vectors $\nu_{j_k} = \frac{e_{j_k}}{\|e_{j_k}\|_2}$. Substituting (5.10) in (5.8) we get

$$L_j(\boldsymbol{x}) = T_1[f, \boldsymbol{x}_{j_1}](x) + \delta_j(\boldsymbol{x}),$$

where $T_1[f, \boldsymbol{x}_{j_1}]$ is the first order Taylor polynomial of f at $\boldsymbol{x}_{j_1}$ and

$$\begin{aligned}\delta_j(\boldsymbol{x}) = \lambda_{j,j_2}(\boldsymbol{x})\,||e_{j_3}||_2^2 \int_0^1 \frac{\partial^2 f(\boldsymbol{x}_{j_1} + te_{j_3})}{\partial \nu_{j_3}^2}(1-t)dt \\ + \lambda_{j,j_3}(\boldsymbol{x})\,||e_{j_2}||_2^2 \int_0^1 \frac{\partial^2 f(\boldsymbol{x}_{j_1} + te_{j_2})}{\partial \nu_{j_2}^2}(1-t)dt.\end{aligned}$$

By the triangle inequality,

$$|f(\boldsymbol{x}) - L_j(\boldsymbol{x})| \leq |f(\boldsymbol{x}) - T_1[f, \boldsymbol{x}_{j_1}](\boldsymbol{x})| + |T_1[f, \boldsymbol{x}_{j_1}](\boldsymbol{x}) - L_j(\boldsymbol{x})|. \tag{5.11}$$

While the first term in (5.11) is bounded as usual [40] by

$$|f(\boldsymbol{x}) - T_1[f, \boldsymbol{x}_{j_1}](\boldsymbol{x})| \leq 2|f|_{1,1}\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2^2,$$

the second term is bounded by

$$|T_1[f, \boldsymbol{x}_{j_1}](\boldsymbol{x}) - L_j(\boldsymbol{x})| = |\delta_j(\boldsymbol{x})| \leq 4|f|_{1,1} r_j C_j \|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2.$$

In fact,

$$|\lambda_{j,j_k}(\boldsymbol{x})| \leq \frac{h_j}{|A(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3})|}\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2, \qquad x \in \Omega$$

for $k = 2, 3$, and by writing $\frac{\partial^2 f}{\partial \nu_{j_k}^2}$ as the limit of difference quotients of the functions $\frac{\partial f}{\partial \nu_{j_k}}$ as well as (1.15), we get

$$\frac{\left|\frac{\partial f}{\partial \nu_{j_k}}(\boldsymbol{u}) - \frac{\partial f}{\partial \nu_{j_k}}(\boldsymbol{v})\right|}{|\boldsymbol{u} - \boldsymbol{v}|} \leq 2\|f\|_{1,1}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \Omega, \quad \boldsymbol{u} \neq \boldsymbol{v}.$$

Finally, (5.9) follows by setting

$$C_j = \frac{r_j^2}{|A(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3})|} = \frac{\sin(\alpha_j + \beta_j)}{\sin \alpha_j \sin \beta_j} = \cot(\alpha_j) + \cot(\beta_j),$$

where α_j and β_j are the angles adjacent to the longest edge of t_j (see Figure 1.1). Note that C_j is large for small angles α_j and β_j, hence it is advantageous to use as T the Delaunay triangulation of X, because it maximizes the smallest angle. Moreover, the bound in (5.9) also holds if we replace the reference vertex $\boldsymbol{x}_{j_1}$ by any of the other two vertices $\boldsymbol{x}_{j_2}$ and $\boldsymbol{x}_{j_3}$ of t_j. ■

5.3 Triangle-based Shepard operator

For any $\mu > 0$ the triangular Shepard operator is defined by

$$K_\mu[f](\boldsymbol{x}) = \sum_{j=1}^{m} B_{\mu,j}(\boldsymbol{x}) L_j(\boldsymbol{x}), \tag{5.12}$$

where $L_j(\boldsymbol{x})$ is the linear interpolating polynomial (5.8) over triangle t_j and $B_{\mu,j}(\boldsymbol{x})$ is the corresponding triangle-based basis function from (5.2). For the special case $\mu = 2$, this operator was proposed by [53] and he noticed the following properties.

Proposition 35 *The operator K_μ is an interpolation operator, that is,*

$$K_\mu[f](\boldsymbol{x}_i) = f_i, \qquad i = 1, \ldots, n,$$

and $\operatorname{dex}(K_\mu) = 1$.

Proof. If $\boldsymbol{x}_i$ is a vertex of triangle t_j (i.e., $i \in \{j_1, j_2, j_3\}$), then $L_j(\boldsymbol{x}_i) = f_i$ by (5.8), otherwise $B_{\mu,j}(\boldsymbol{x}_i) = 0$ by Proposition 33. Using (5.6) we then have

$$K_\mu[f](\boldsymbol{x}_i) = \sum_{j=1}^{m} B_{\mu,j}(\boldsymbol{x}_i) L_j(\boldsymbol{x}_i) = \sum_{j \in J_i} B_{\mu,j}(\boldsymbol{x}_i) f_i = f_i.$$

Moreover, K_μ reproduces polynomials up to degree 1, because $\operatorname{dex}(L_j) = 1$ for $j = 1, \ldots, m$ by construction and because the basis functions $B_{\mu,j}$ are a partition of unity. ■

Let us now turn to the approximation order of the operator K_μ, which was not studied by [53]. To this purpose, we follow [40] and let $\|\cdot\|$ be the maximum norm and $R_\rho(\boldsymbol{x}) = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x} - \boldsymbol{y}\| \le \rho\}$ be the axis-aligned closed square with centre $\boldsymbol{y}$ and edge length 2ρ. With $V(t)$ denoting the set of vertices of a triangle $t \in T$, we then define

$$h' = \inf\{\rho > 0 : \forall \boldsymbol{x} \in \Omega \; \exists t \in T : R_\rho(\boldsymbol{x}) \cap V(t) \neq \emptyset\} \tag{5.13}$$

and

$$h'' = \inf\{\rho > 0 : \forall t \in T \; \exists \; \boldsymbol{x} \in \Omega : t \subset R_\rho(\boldsymbol{x})\},$$

and finally

$$h = \max\{h', h''\}. \tag{5.14}$$

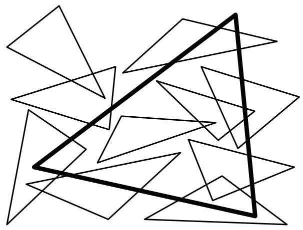


Figure 5.4: The value h' is small for this triangulation, but h'' is large because of the indicated large triangle.

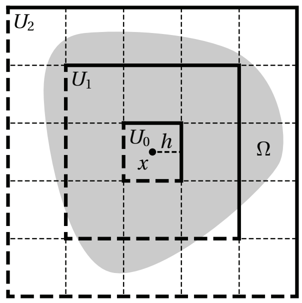


Figure 5.5: Example of a covering of Ω by annuli U_j.

A small value of h' corresponds to a rather uniform triangle distribution, but does not exclude the presence of large triangles (see Figure 5.4). The latter cannot occur if h'' and then also h are small, because each triangle is contained in a square with edge length $2h$. Note that in the maximum norm, the length of each triangle edge does not exceed $2h$. We further let

$$M = \sup_{\boldsymbol{x}\in\Omega} \#\{t \in T : R_h(\boldsymbol{x}) \cap V(t) \neq \emptyset\}, \tag{5.15}$$

where $\#$ is the cardinality operator, be the maximum number of triangles with at least one vertex in some square with edge length $2h$. Small values of M imply that there are no clusters of triangles.

Theorem 36 *Let Ω be a compact convex domain which contains X, $f \in C^{1,1}(\Omega)$, and $\mu > 4/3$. Then,*

$$|f(\boldsymbol{x}) - K_\mu[f](\boldsymbol{x})| \leq CM\, |f|_{1,1}\, h^2$$

for any $\boldsymbol{x} \in \Omega$, with C a positive constant which depends on T and μ.

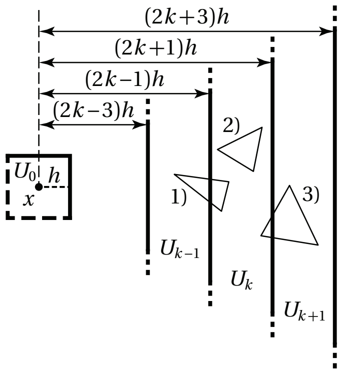


Figure 5.6: Possible cases of a triangle with at least one vertex in U_k.

Proof. For $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$ let

$$Q_\rho(\boldsymbol{y}) = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : y_k - \rho < x_k \leq y_k + \rho, k = 1, 2\}$$

be the axis-aligned half-open square with centre $\boldsymbol{y}$ and edge length 2ρ. Now let $\boldsymbol{x} \in \Omega$ be fixed and consider the following disjoint covering of Ω. For $k \in \mathbb{N}_0$ we define by

$$U_k = \bigcup_{\nu \in \mathbb{Z}^2,\ \|\nu\| = k} Q_h(\boldsymbol{x} + 2h\nu)$$

the half-open annulus with centre $\boldsymbol{x}$, radius $2kh$, and width h. For example, $U_0 = Q_h(\boldsymbol{x})$ and U_1 is the union of the 8 congruent half-open squares surrounding U_0 (see Figure 5.5). Since Ω is compact, there exists some $N \in \mathbb{N}$, independent of $\boldsymbol{x}$ and of order $O(1/h)$, such that

$$\Omega \subset \bigcup_{k=0}^{N} U_k.$$

Noticing that U_k is composed of $8k$ congruent copies of $Q_h(\boldsymbol{x})$, the number of triangles with at least one vertex in U_k is bounded,

$$\#\{t \in T : U_k \cap V(t) \neq \emptyset\} \leq 8kM, \qquad k = 1, \ldots, N. \tag{5.16}$$

For any triangle t with at least one vertex in U_k one of the following cases (see Figure 5.6) holds:

$$\begin{aligned}
&1) \quad V(t) \cap U_{k-1} \neq \emptyset \quad &&\Rightarrow \quad (2k-3)h \leq \|\boldsymbol{x} - \boldsymbol{v}\| \leq (2k+1)h \quad \forall v \in V(t),\\
&2) \quad V(t) \subset U_k &&\Rightarrow \quad (2k-1)h \leq \|\boldsymbol{x} - \boldsymbol{v}\| \leq (2k+1)h \quad \forall v \in V(t),\\
&3) \quad V(t) \cap U_{k+1} \neq \emptyset &&\Rightarrow \quad (2k-1)h \leq \|\boldsymbol{x} - \boldsymbol{v}\| \leq (2k+3)h \quad \forall v \in V(t).
\end{aligned} \tag{5.17}$$

Let us now denote by T_0 the set of all triangles with at least one vertex in U_0. By the definitions of h' in (5.13) and M in (5.15), this set contains at least one and at most M triangles and for each triangle $t_j = [\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3}] \in T_0$ we have

$$\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|\|\boldsymbol{x} - \boldsymbol{x}_{j_2}\|\|\boldsymbol{x} - \boldsymbol{x}_{j_3}\| \leq h \cdot (3h) \cdot (3h) = 9h^3, \tag{5.18}$$

because one vertex of t_j is inside U_0 and the other two are in $U_0 \cup U_1$. For $k = 1, \ldots, N$ we further denote by T_k the set of all triangles with at least one vertex in U_k and no vertex in U_{k-1}. By (5.16), this set contains at most $8kM$ triangles and by case 3 in (5.17) we have

$$\big((2k-1)h\big)^3 \leq \|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|\|\boldsymbol{x} - \boldsymbol{x}_{j_2}\|\|\boldsymbol{x} - \boldsymbol{x}_{j_3}\| \leq \big((2k+3)h\big)^3 \tag{5.19}$$

for each triangle $t_j = [\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3}] \in T_k$. Further note that by construction,

$$\bigcup_{k=0}^{N} T_k = T \qquad \text{and} \qquad \bigcap_{k=0}^{N} T_k = \emptyset.$$

Let us now turn to the approximation error

$$e(\boldsymbol{x}) = |f(\boldsymbol{x}) - K_\mu[f](\boldsymbol{x})|$$

of the triangular Shepard interpolant at $\boldsymbol{x}$. By (5.12) and the fact that the basis function $B_{\mu,j}$ are non-negative and form a partition of unity,

$$e(\boldsymbol{x}) = \left|\sum_{j=1}^{m} B_{\mu,j}(\boldsymbol{x}) f(\boldsymbol{x}) - \sum_{j=1}^{m} B_{\mu,j}(\boldsymbol{x} L_j(\boldsymbol{x})\right| \leq \sum_{j=1}^{m} |f(\boldsymbol{x}) - L_j(\boldsymbol{x})| B_{\mu,j}(x).$$

Using Proposition 34 and (5.2) we then get

$$\begin{aligned}
e(\boldsymbol{x}) &\leq |f|_{1,1} \sum_{j=1}^{m} \big(2\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2^2 + 4r_j C_j \|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|_2\big) \frac{\prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|_2^{-\mu}}{\sum_{k=1}^{m} \prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{k_\ell}\|_2^{-\mu}},\\
&\leq C' \, |f|_{1,1} \sum_{j=1}^{m} \big(2\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|^2 + 4r_j C_j \|\boldsymbol{x} - \boldsymbol{x}_{j_1}\|\big) \frac{\prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|^{-\mu}}{\sum_{k=1}^{m} \prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{k_\ell}\|^{-\mu}},
\end{aligned}$$

where $C' = \sqrt{2}^{3m\mu}$ is a constant which depends on the fact that we bound the Euclidean norm with the maximum norm.

Now let $t_i \in T$ be a triangle such that

$$\prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{i_\ell}\| = \min_{j=1,\dots,m} \prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|.$$

Since at least one triangle of T belongs to T_0, we know from (5.18) that

$$\prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{i_\ell}\| \le 9h^3.$$

For each $t_j \in T_0$ we then have

$$\prod_{\ell=1}^{3} \frac{\|\boldsymbol{x} - \boldsymbol{x}_{i_\ell}\|}{\|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|} \le 1$$

and for each $t_j \in T_k$, $k = 1, \dots, N$, using (5.19),

$$\prod_{\ell=1}^{3} \frac{\|\boldsymbol{x} - \boldsymbol{x}_{i_\ell}\|}{\|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|} \le \frac{9h^3}{((2k-1)h)^3} = \frac{9}{(2k-1)^3}.$$

Therefore,

$$\frac{\prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|^{-\mu}}{\sum_{k=1}^{m} \prod_{\ell=1}^{3} \|\boldsymbol{x} - \boldsymbol{x}_{k_\ell}\|^{-\mu}} \le \prod_{\ell=1}^{3} \frac{\|\boldsymbol{x} - \boldsymbol{x}_{j_\ell}\|^{-\mu}}{\|\boldsymbol{x} - \boldsymbol{x}_{i_\ell}\|^{-\mu}} \le \begin{cases} 1, & \text{if } t_j \in T_0, \\ 9^\mu/(2k-1)^{3\mu}, & \text{if } t_j \in T_k. \end{cases}$$

Further assuming without loss of generality that $\boldsymbol{x}_{j_1} \in U_k$ for $t_j \in T_k$, so that $\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\| \le h$ for each $t_j \in T_0$ and $\|\boldsymbol{x} - \boldsymbol{x}_{j_1}\| \le (2k+1)h$ for each $t_j \in T_k$, $k = 1, \dots, N$, and taking into account that $r_j \le \sqrt{8}h \le 3h$, we get

$$\begin{aligned} e(\boldsymbol{x}) &\le C\,|f|_{1,1} \left(\sum_{t_j \in T_0} \big(2h^2 + 4h_jC_jh\big) + \sum_{k=1}^{N} \sum_{t_j \in T_k} \big(2(2k+1)^2h^2 + 4h_jC_j(2k+1)h\big) \frac{9^\mu}{(2k-1)^{3\mu}} \right) \\ &\le C'\,|f|_{1,1} \left(\sum_{t_j \in T_0} (2 + 12C''^\mu \sum_{k=1}^{N} \sum_{t_j \in T_k} \frac{2(2k+1)^2 + 12(2k+1)C''}{(2k-1)^{3\mu}} \right) h^2, \end{aligned}$$

where $C'' = \max\{C_1, C_2, \dots, C_m\}$. Using (5.16) we then have

$$\begin{aligned} e(\boldsymbol{x}) &\le C'M\,|f|_{1,1} \left((2 + 12C''^\mu \sum_{k=1}^{N} 8k \frac{2(2k+1)^2 + 12(2k+1)C''}{(2k-1)^{3\mu}} \right) h^2 \\ &= C'M\,|f|_{1,1} \left(2 + 12C''^\mu \sum_{k=1}^{N} \frac{k(2k+1)^2}{(2k-1)^{3\mu}} + 96 \cdot 9^\mu C'' \sum_{k=1}^{N} \frac{k(2k+1)}{(2k-1)^{3\mu}} \right) h^2. \end{aligned}$$

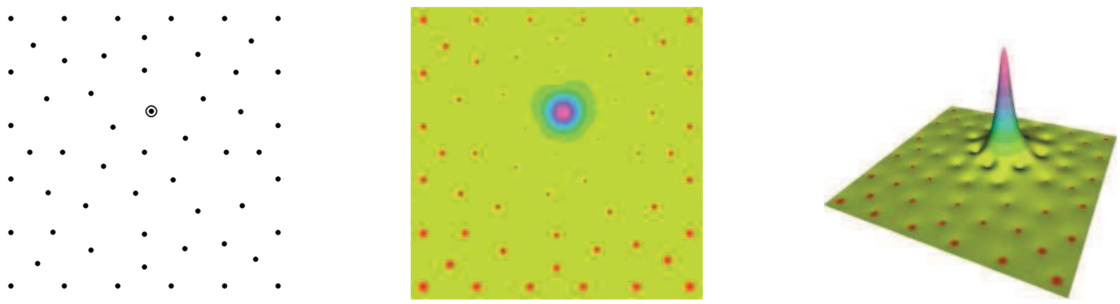


Figure 5.7: Point-based Shepard basis function $A_{2,i}(\boldsymbol{x})$ for the indicated node $\boldsymbol{x}_i \in X$ with respect to a set of scattered points X.

■

As the series $\sum_{k=1}^{\infty} \frac{k(2k+1)}{(2k-1)^{3\mu}}$ and $\sum_{k=1}^{\infty} \frac{k(2k+1)^2}{(2k-1)^{3\mu}}$ converge for $\mu > 4/3$, we conclude that the approximation order of K_μ is $O(h^2)$.

5.4 Cardinal basis functions

Applying K_μ to the *unit data* δ_i with $f_i = 1$ and $f_k = 0$ for $k \neq i$, gives the *cardinal basis functions*

$$\hat{B}_{\mu,i}(\boldsymbol{x}) = K_\mu[\delta_i](\boldsymbol{x}) = \sum_{j \in J_i} B_{\mu,j}(\boldsymbol{x}) \lambda_{j,i}(\boldsymbol{x}), \qquad i = 1, \ldots, n. \tag{5.20}$$

As K_μ is linear, they allow us to rewrite the triangular Shepard operator as

$$K_\mu[f](\boldsymbol{x}) = \sum_{i=1}^{n} \hat{B}_{\mu,i}(\boldsymbol{x}) f_i,$$

and as the constant function $f(\boldsymbol{x}) \equiv 1$ is in the precision set of K_μ, it follows that the cardinal basis functions $\hat{B}_{\mu,i}$ form a partition of unity. However, in contrast to their classical counterparts $A_{\mu,i}$, they are not necessarily positive away from the nodes $\boldsymbol{x}_i$.

Figures 5.8 and 5.10 show examples of these cardinal basis functions for $\mu = 2$. Negative function values are marked by grey colour, while the small brown regions near $\boldsymbol{x}_i$ indicate function values greater than 1. Comparing the basis functions to their classical counterparts shown in Figures 5.7 and 5.9, we observe that $\hat{B}_{2,i}$ is less "spiky" than $A_{2,i}$ and tends to zero faster with increasing distance from $\boldsymbol{x}_i$. Another notable property is the

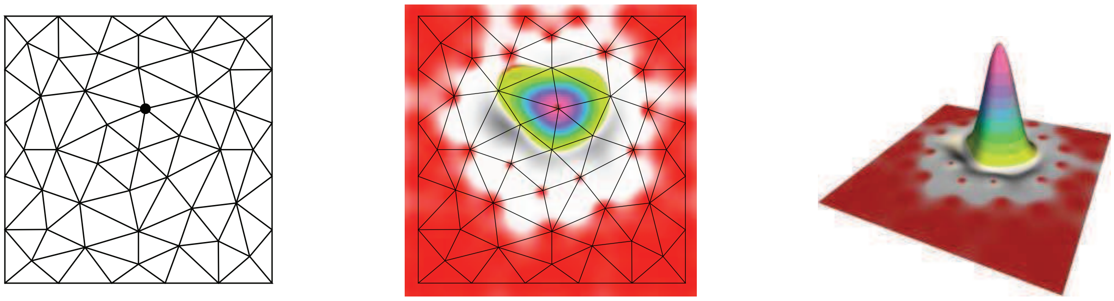


Figure 5.8: New basis function $\hat{B}_{2,i}(\boldsymbol{x})$ for the indicated node $\boldsymbol{x}_i \in X$ with respect to the Delaunay triangulation T of the points X in Figure 5.7.

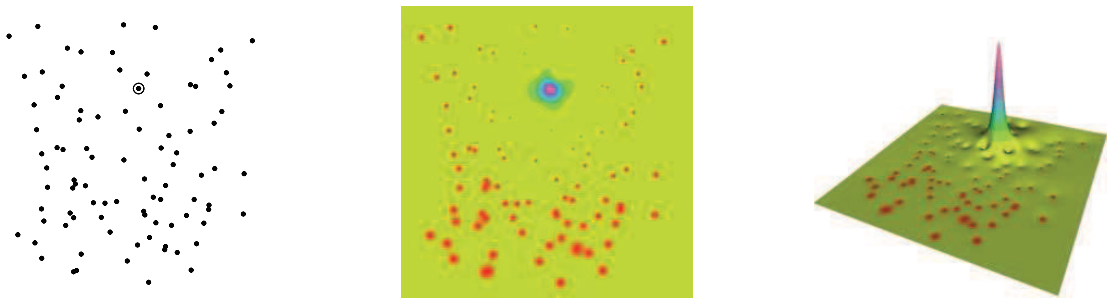


Figure 5.9: Point-based Shepard basis function $A_{2,i}(\boldsymbol{x})$ for the indicated node $\boldsymbol{x}_i \in X$ with respect to a set of scattered points X.

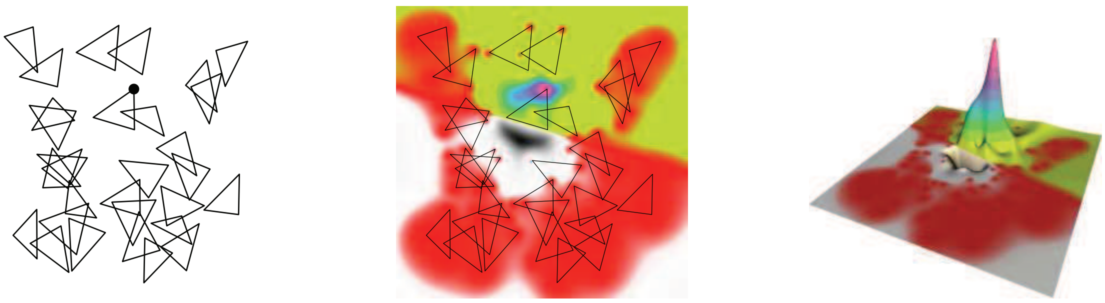


Figure 5.10: New basis function $\hat{B}_{2,i}(\boldsymbol{x})$ for the indicated node $\boldsymbol{x}_i \in X$ with respect to an arbitrary triangulation T of the points X in Figure 5.9.

behaviour of the gradient of $\hat{B}_{\mu,i}$ at the nodes. To this end let

$$I_i = \bigcup_{j \in J_i} \{j_1, j_2, j_3\}$$

be the set of indices of all nodes that share a triangle with $\boldsymbol{x}_i$, that is, which are neighbours of $\boldsymbol{x}_i$ in T, including the index i itself.

Proposition 37 *For any $i = 1, \ldots, n$ and $\mu > 1$, the cardinal basis function in* (5.20) *has local extrema at almost all nodes,*

$$\nabla \hat{B}_{\mu,i}(\boldsymbol{x}_k) = 0, \quad k \notin I_i, \tag{5.21}$$

and

$$\nabla \hat{B}_{\mu,i}(\boldsymbol{x}_k) = \textstyle\sum_{j \in J_i} B_{\mu,j}(\boldsymbol{x}_k) c_{j,i}, \quad k \in I_i, \tag{5.22}$$

where $c_{j,i}$ are some constant vectors which depend only on t_j and $\boldsymbol{x}_i$.

Proof. First note that

$$\nabla \hat{B}_{\mu,i}(\boldsymbol{x}) = \sum_{j \in J_i} \left(\nabla B_{\mu,j}(\boldsymbol{x}) \lambda_{j,i}(x) + B_{\mu,j}(\boldsymbol{x}) \nabla \lambda_{j,i}(\boldsymbol{x}) \right).$$

If $k \notin I_i$, then it follows from (5.3) and (5.4) that both $B_{\mu,j}(\boldsymbol{x}_k)$ and $\nabla B_{\mu,j}(\boldsymbol{x}_k)$ vanish for any $j \in J_i$, which is sufficient to establish (5.21). To get (5.22), we distinguish two cases. On the one hand, if $k = i$, then $\lambda_{j,i}(\boldsymbol{x}_i) = 1$ for any $j \in J_i$ and $\sum_{j \in J_i} \nabla B_{\mu,j}(\boldsymbol{x}_i) = 0$, as mentioned in (5.7). On the other hand, if $k \in I_i \setminus \{i\}$, then for any $j \in J_i$ we have either $k \in \{j_1, j_2, j_3\}$, implying $\lambda_{j,i}(\boldsymbol{x}_k) = 0$, or $k \notin \{j_1, j_2, j_3\}$, so that $\nabla B_{\mu,j}(\boldsymbol{x}_k) = 0$. Overall, this gives (5.22) with $c_{j,i} = \nabla \lambda_{j,i}(\boldsymbol{x})$, which is a constant vector because $\lambda_{j,i}$ is linear. ■

It follows from Proposition 37 that the interpolant $K_\mu[f]$ does not necessarily have flat spots at the nodes.

5.5 Numerical results

To verify the quadratic approximation order of the triangular Shepard operator K_μ in (5.12) predicted by Theorem 36, we carried out various numerical experiments with different sets of nodes and 12 test functions (see Figure 1.2), including those introduced

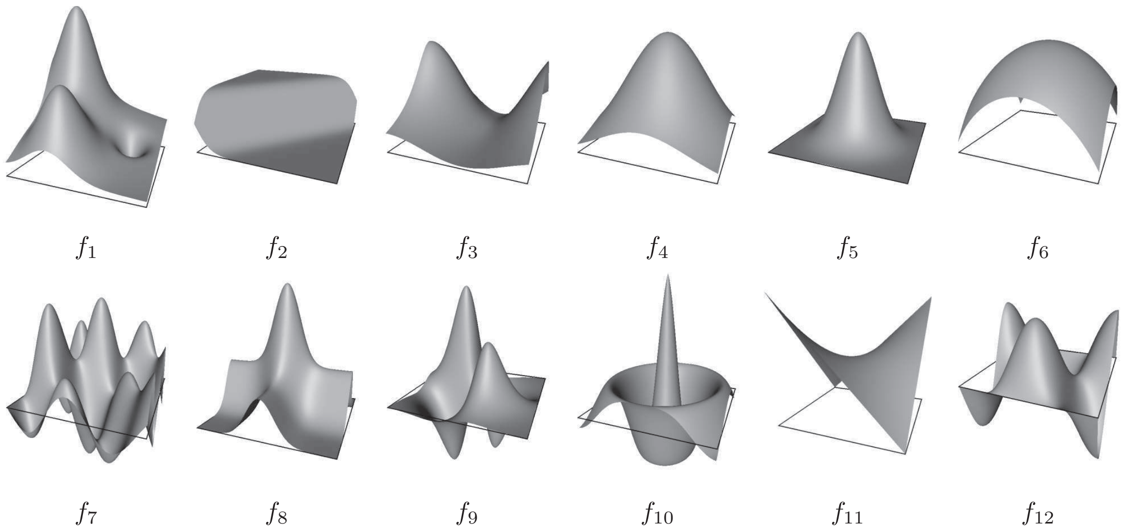


Figure 5.11: Test functions used in our numerical experiments. The definitions of the first 10 functions can be found in [62], the last two functions are $f_{11}(x, y) = ((2x-1)(1-2y)+1)/2$ and $f_{12}(x, y) = \sin(2\pi x)\cos(2\pi y/2$.

by [55], [41, 42], and [62]. We report the results of some of these experiments for different kinds of triangulations in Sections 5.5.1 and 5.5.2. In Section 5.5.3 we further present a comparison of the approximation accuracy of K_2 and some alternative state-of-the-art interpolation methods.

5.5.1 Approximation order for Delaunay triangulations

Our first series of experiments is based on six different Delaunay triangulations with increasing resolution (see Figure 5.12). These triangulations were generated by prescribing N uniformly distributed nodes along the boundary of the unit square $R = [0, 1] \times [0, 1]$ and then using Shewchuk's TRIANGLE program [66] to create a *conforming Delaunay triangulation* of R with no angle smaller than 20 degrees and no triangle area greater than $a_{\max} = 4\sqrt{3}/N^2$, by inserting Steiner points. Note that $a_{\max}$ is the area of an equilateral triangle with edge length $4/N$, the spacing of the prescribed boundary nodes. Table 5.1 lists the number of vertices and triangles, as well as the maximum edge length $h_T = \max\{h_1, \ldots, h_m\}$ for the six triangulations. Note that for Delaunay triangulations h_T is of the same order as h in Theorem 36.

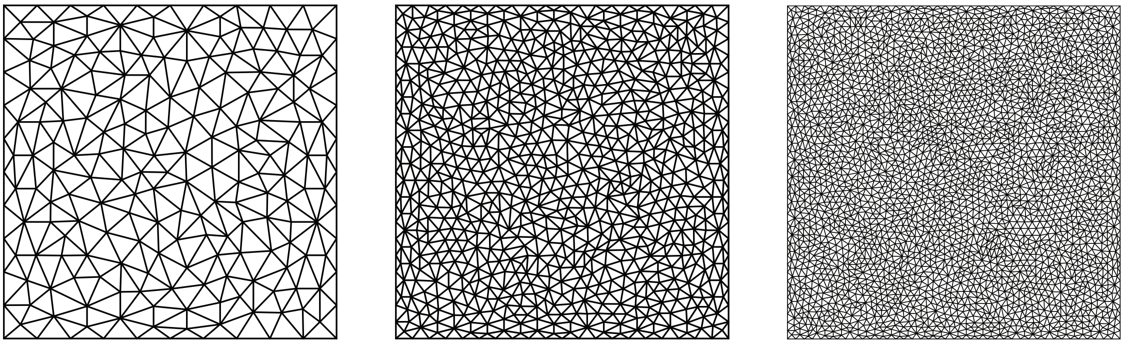


Figure 5.12: Three of the six Delaunay triangulations used in our numerical experiments with $N = 40$ (left), $N = 80$ (middle), and $N = 160$ (right) boundary nodes (compare Table 5.1).

Table 5.1: Starting from N uniformly distributed nodes on the boundary of the unit square, we generated Delaunay triangulations with n vertices, m triangles, and maximum edge length h_T (compare Figure 5.12).

N	n	m	h_T
40	202	362	0.122196
80	777	1472	0.068061
160	2991	5820	0.034871
320	11638	22954	0.018046
640	46176	91710	0.008993
1280	195323	389364	0.004494

For each of the 12 test functions f_k we constructed the triangular Shepard interpolant $K_2[f_k]$ and determined the maximum approximation error $e_{\max}$ by evaluating $|f_k(\boldsymbol{x}) - K_2[f_k](\boldsymbol{x})|$ at $100,000$ random points $x \in R$ and recording the maximum value. The results are shown in Figure 5.13 and clearly demonstrate the quadratic approximation order of the operator K_2. For other values of $\mu > 4/3$ we obtained similar results.

Figure 5.14 shows a comparison between the reconstruction of test function f_1 using the classical Shepard interpolant $S_2[f_1]$ and the triangular Shepard interpolant $K_2[f_1]$, both based on samples taken at the nodes of the first four of our six Delaunay triangulations. While it is well-known that the Shepard interpolant behaves rather poorly, which has led to various improvements (see [67] and references therein), we include this comparison because the construction of the triangular Shepard operator is as simple as that of the classical Shepard operator, and we believe that it can be further improved significantly by following

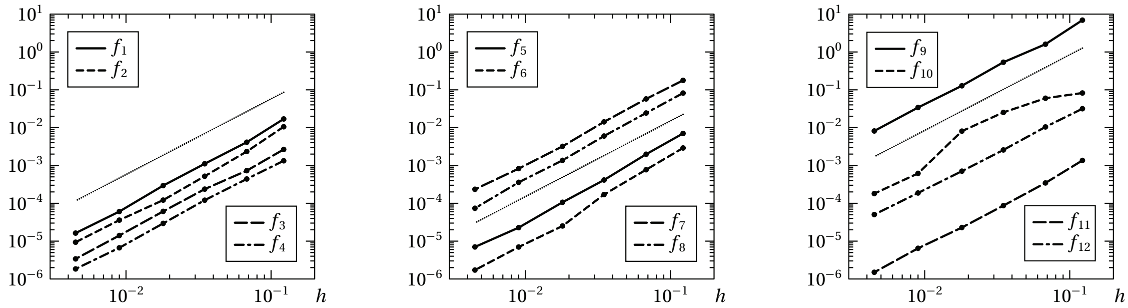


Figure 5.13: Log-log-plot of the approximation error $e_{\max}$ over the maximum edge length h_T for the 12 test functions in Figure 1.2 and the Delaunay triangulations in Figure 5.12 and Table 5.1. As reference, the dotted line indicates a perfect quadratic trend.

Table 5.2: Starting from the set of nodes given by the six Delaunay triangulations in Figure 5.12, we generated compact triangulations with n vertices, m triangles and maximum edge length h_T (compare Figure 5.15).

n	m	h_T
202	133	0.130966
777	490	0.065158
2991	1860	0.034552
11638	7151	0.016804
46176	28582	0.008769
195323	120350	0.004651

ideas similar to the ones used for extending the Shepard interpolant. Note that the superior aesthetic behaviour of the triangular Shepard interpolants K_2, compared to the Shepard interpolants, had also been observed by [53].

5.5.2 Approximation order for general triangulations

In a second series of experiments, we tested the approximation order of K_2 when used with respect to general triangulations, constructed in the following way. For each node $\boldsymbol{x}_i \in X$ we choose among the 15 triangles that connect $\boldsymbol{x}_i$ with 2 of its 6 nearest neighbours in X the one which locally reduces the error bound of the associated linear interpolant in (5.9). After omitting duplicate triangles we get a triangulation T of the nodes with $m \leq n$ triangles, where some of the triangles may overlap each other. As the number of triangles is only about 1/3 the number of triangles in the Delaunay triangulation of X, we call T a *compact triangulation* of X. For our numerical experiments we created compact

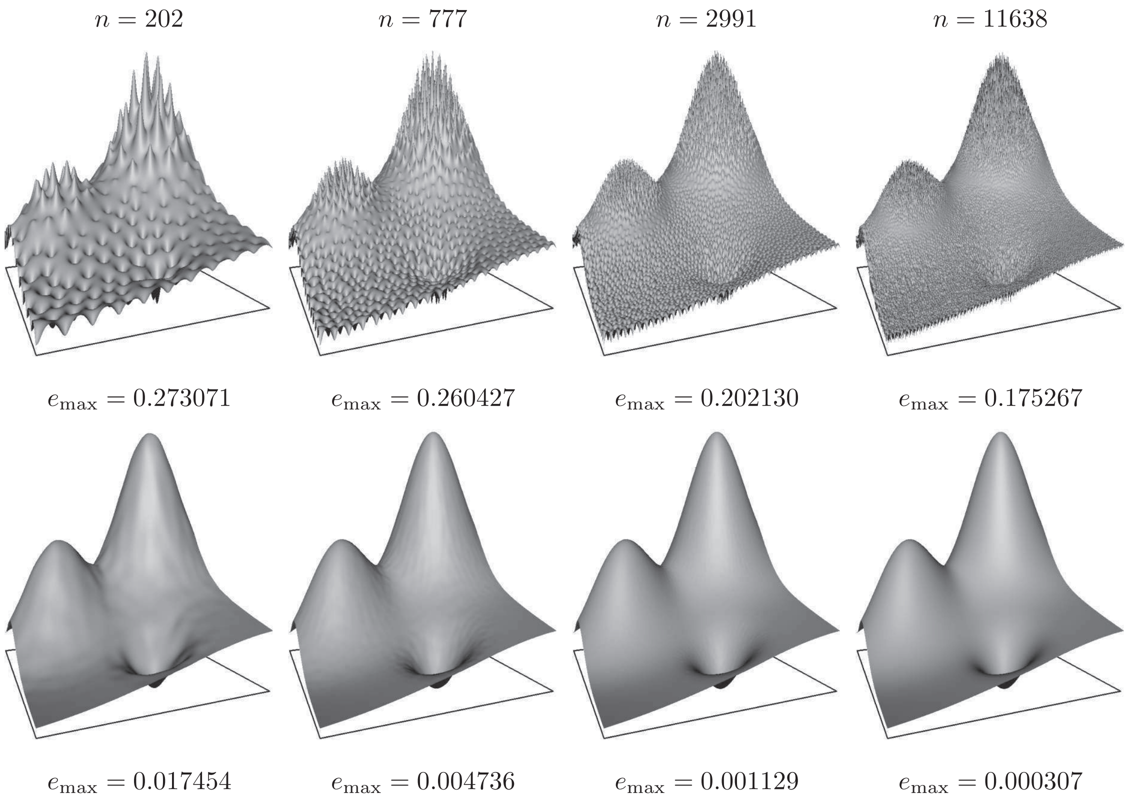


Figure 5.14: Reconstruction of Franke's principal test function f_1 from n samples using classical Shepard interpolation (top) and triangular Shepard interpolation (bottom).

triangulations of the sets of nodes given by the six Delaunay triangulations in Figure 5.12 (see Figure 5.15 and Table 5.2) and determined the approximation errors for the triangular Shepard interpolant as described in Section 5.5.1. Figure 5.16 summarizes the results and confirms that the approximation order is again quadratic.

5.5.3 Approximation accuracy

We finally carried out a series of experiments to compare the approximation accuracies of the triangular Shepard operator K_2 and

1. the global version of the classical Shepard operator S_2 given in (1.10);
2. the global version of the first order Shepard–Taylor operator S_{T_1} (1.14), which is obtained by substituting in (1.10) functional evaluation at $\boldsymbol{x}_i$ with the first order

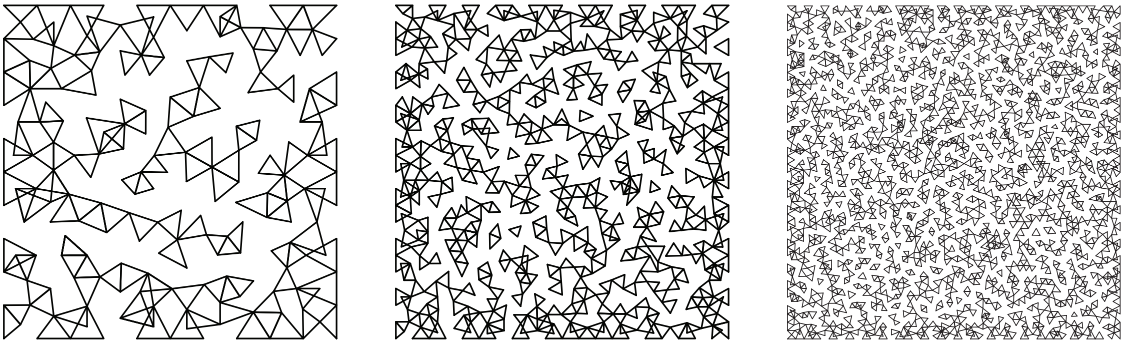


Figure 5.15: Three of the six compact triangulations used in our numerical experiments with $m = 133$ (left), $m = 490$ (middle), and $m = 1860$ (right) triangles (compare Table 5.2).

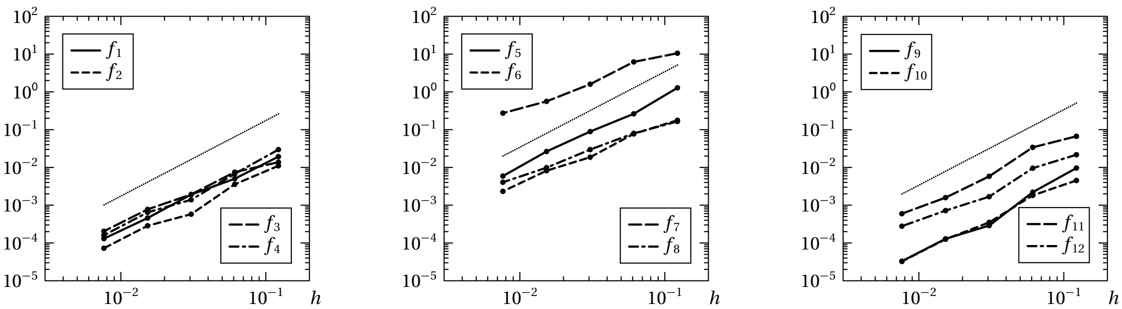


Figure 5.16: Log-log-plot of the approximation error $e_{\max}$ over the maximum edge length h_T for the 12 test functions in Figure 1.2 and the triangulations in Figure 5.15 and Table 5.2. As reference, the dotted line indicates a perfect quadratic trend.

Taylor polynomial of f at $\boldsymbol{x}_i$

3. the local version of the classical Shepard operator $\tilde{S}_2$, where the Shepard basis functions (1.9) are multiplied by Franke–Little weights [10] in order to make them compactly supported;

4. the local version of the first order Shepard–Taylor operator $\tilde{S}_{T_1}$;

5. the linear Shepard operator LSHEP [67], which substitutes in $\tilde{S}_{T_1}$ the first order Taylor polynomial of f at $\boldsymbol{x}_i$ with the linear polynomial that interpolates f at $\boldsymbol{x}_i$ and fits the data at the 4 nodes closest to $\boldsymbol{x}_i$ best in the least squares sense.

Note that the global and the local version of the classical Shepard operator, S_2 and S_{T_1}, as well as the LSHEP operator use the same data as K_2, that is, they rely on

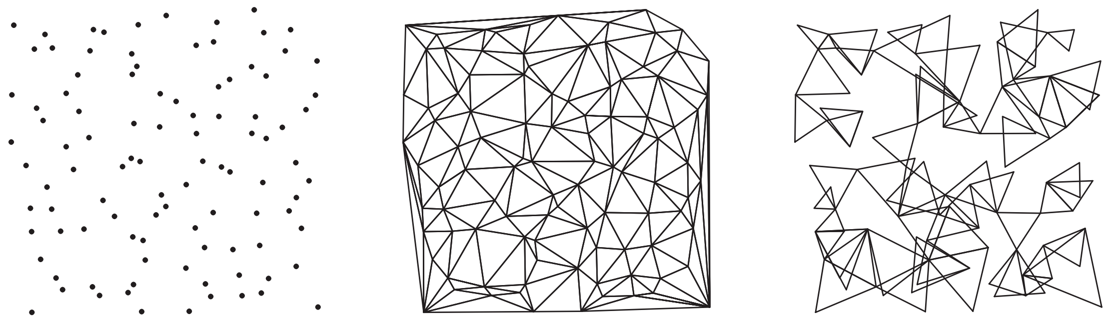


Figure 5.17: The 100 nodes of the Franke data set (left), the corresponding Delaunay triangulation with $m = 188$ triangles (middle), and the $m = 59$ triangles of the compact triangulation (right) used in our comparison of six interpolation operators (see Table 5.3).

function values only. Instead, the global and local version of the first order Shepard–Taylor operator, $\tilde{S}_2$ and $\tilde{S}_{T_1}$, require the first order derivatives at each sample point in addition. Moreover, the first order operators (S_{T_1}, $\tilde{S}_{T_1}$, and LSHEP) have the same degree of exactness as K_2, namely degree 1, while the other two operators (S_2 and $\tilde{S}_2$) reproduce only constant functions.

The results show that the *global* triangular Shepard operator K_2 is comparable to the *local* Shepard interpolation methods. This encourages us to develop and analyse a local version of K_2 in future work and to study combined operators, based on K_2, with higher degrees of exactness.

Table 5.3: Comparison of six interpolation operators applied to the 12 test functions in Figure 5.11 using the interpolation nodes of the Franke data set (see Figure 5.17). The smallest error in each row is marked in boldface.

		S_2	S_{T_1}	K_2 Delaunay	K_2 compact	$\tilde{S}_2$	$\tilde{S}_{T_1}$	LSHEP
f_1	e_{max}	4.34e-1	1.06e-1	**5.16e-2**	1.00e-1	2.15e-1	1.19e-1	2.71e-1
	e_{mean}	5.93e-2	2.31e-2	**7.79e-3**	1.47e-2	2.55e-2	1.14e-2	3.32e-2
	e_{MS}	8.24e-3	9.40e-4	**1.46e-4**	5.10e-4	1.84e-3	3.76e-4	3.02e-3
f_2	e_{max}	6.17e-2	2.65e-2	2.66e-2	4.20e-2	6.79e-2	**2.28e-2**	9.43e-2
	e_{mean}	2.27e-2	7.15e-3	**2.64e-3**	4.53e-3	5.95e-3	3.03e-3	1.21e-2
	e_{MS}	6.88e-4	7.71e-5	**2.27e-5**	6.21e-5	1.29e-4	2.54e-5	3.60e-4
f_3	e_{max}	9.48e-2	4.16e-2	9.46e-2	**2.31e-2**	4.45e-2	3.31e-2	6.44e-2
	e_{mean}	2.12e-2	7.16e-3	**2.71e-3**	3.39e-3	7.44e-3	3.35e-3	1.01e-2
	e_{MS}	7.71e-4	1.05e-4	6.46e-5	**2.64e-5**	1.07e-4	2.83e-5	2.04e-4
f_4	e_{max}	9.24e-2	6.32e-2	3.07e-2	**2.43e-2**	3.32e-2	2.49e-2	5.37e-2
	e_{mean}	2.06e-2	3.12e-2	**1.47e-3**	2.73e-3	8.28e-3	2.40e-3	1.07e-2
	e_{MS}	7.13e-4	1.15e-3	**9.38e-6**	1.55e-5	1.10e-4	1.72e-5	1.95e-4
f_5	e_{max}	1.86e-1	4.28e-2	**1.97e-2**	3.53e-2	9.36e-2	5.96e-2	1.07e-1
	e_{mean}	2.01e-2	1.66e-2	**1.81e-3**	3.86e-3	7.41e-3	3.82e-3	1.08e-2
	e_{MS}	1.08e-3	3.61e-4	**1.03e-5**	3.74e-5	2.27e-4	5.40e-5	3.38e-4
f_6	e_{max}	1.30e-1	1.25e-1	8.57e-2	2.18e-2	7.29e-2	**2.07e-2**	8.30e-2
	e_{mean}	1.87e-2	4.41e-2	**2.51e-3**	2.96e-3	9.64e-3	5.32e-3	1.08e-2
	e_{MS}	5.97e-4	2.28e-3	5.76e-5	**1.85e-5**	1.62e-4	4.12e-5	2.13e-4
f_7	e_{max}	1.77	1.00	1.08	1.35	1.33	**6.14e-1**	1.51
	e_{mean}	4.77e-1	2.11e-1	1.47e-1	2.46e-1	3.14e-1	**1.33e-1**	3.35e-1
	e_{MS}	3.72e-1	8.42e-2	4.03e-2	1.03e-1	1.65e-1	**3.24e-2**	1.93e-1
f_8	e_{max}	1.45	5.19e-1	4.99e-1	4.54e-1	8.91e-1	**3.04e-1**	9.33e-1
	e_{mean}	1.74e-1	9.77e-2	3.69e-2	6.29e-2	9.18e-2	**3.61e-2**	1.03e-1
	e_{MS}	6.50e-2	1.59e-2	4.36e-3	1.04e-2	2.33e-2	**2.87e-3**	2.68e-2
f_9	e_{max}	9.30e+1	3.08e+1	**1.93e+1**	2.67e+1	5.01e+1	2.84e+1	6.55e+1
	e_{mean}	1.13e+1	7.35	**1.78**	3.36	5.04	2.51	6.20
	e_{MS}	3.14e+2	8.37e+1	**9.81**	3.35e+1	8.13e+1	2.05e+1	1.10e+2
f_{10}	e_{max}	1.05	7.12e-1	5.01e-1	8.10e-1	9.41e-1	**1.95e-1**	9.70e-1
	e_{mean}	1.09e-1	4.86e-2	2.80e-2	6.13e-2	6.98e-2	**2.75e-2**	7.12e-2
	e_{MS}	2.49e-2	6.44e-3	3.19e-3	1.10e-2	1.42e-2	**1.44e-3**	1.49e-2
f_{11}	e_{max}	2.15e-1	9.66e-2	1.32e-2	1.63e-2	9.02e-2	**5.83e-3**	1.32e-1
	e_{mean}	2.78e-2	1.17e-2	1.33e-3	3.00e-3	1.15e-2	**9.73e-4**	1.87e-2
	e_{MS}	1.86e-3	3.12e-4	3.69e-6	1.58e-5	2.40e-4	**1.82e-6**	6.43e-4
f_{12}	e_{max}	1.97e-1	1.40e-1	1.11e-1	1.68e-1	1.63e-1	**9.76e-2**	1.99e-1
	e_{mean}	7.32e-2	3.26e-2	**1.64e-2**	3.30e-2	4.11e-2	2.58e-2	4.91e-2
	e_{MS}	7.50e-3	1.71e-3	**5.15e-4**	1.89e-3	2.55e-3	1.02e-3	3.72e-3

Conclusions and Future Perspective

The scope of this thesis is to enhance the approximation order of the Shepard operator by using supplementary derivative data. The idea is to consider the combination of the Shepard operator with different three point interpolation polynomials, depending on the data available at each interpolation node. The combined operators are realized by a procedure, introduced by Dell'Accio and Di Tommaso in [36], which is based on the association, to each interpolation point, of a triangle with a vertex in it and the other two vertices in its neighborhood. After some preliminary results on the Shepard operator and the posing of the problem in Chapter 1, we start to describe our proposals of combination in Chapter 2, where we discuss the problem of interpolation of *Hermite type data* (function evaluations and all partial derivative data up to a certain order at each node) on scattered data set. The Shepard-Hermite operator is defined through some particular cases of a general class of Hermite interpolation polynomials on the triangle, introduced by Chui and Lai in 1990 [18]. The resulting operators allow to enhance the approximation order of the Shepard-Taylor operators which use the same data maintaining, at the same time, the interpolation conditions and the accuracy of approximation of the Shepard-Taylor operators with the same degree of exactness. In Chapter 3 and in Chapter 4 we consider the lacunary data cases of *Lidstone type data* (function evaluations and all even order derivative data up to a certain order at each node) and *Complementary Lidstone type data* (function evaluations in some nodes and all odd order derivative data up to a certain order at each node). The Shepard-Lidstone operator and the Shepard-Complementary Lidstone operator, besides enhancing the approximation order of the Shepard operator, give a solution to lacunary data interpolation problem in these special cases. Numerical results show that these operators have a good accuracy of approximation comparable with the one of the Shepard-Taylor operators which reproduce polynomials of the same degree. These results are carried out by numerical experiments on generally used test functions for scattered data interpolation. In Chapter 5 we try to resolve the asymmetry that occurs in combining the point-based Shepard basis functions with three point interpolation polynomials in the case in which only function evaluations are given. A natural way to break this asymmetry is to act on the basis functions and, during my period at the USI-Università della Svizzera Italiana, in collaboration with Prof. K. Hormann, we studied the possibility to use triangle-based basis functions rather than point-based basis functions. Few months ago, thanks to a private communication with Prof. O. Davydov, we realized that this idea was already considered by F. Little in an hard to find paper of 1982 [53]. In the last Chapter we deeply study the properties of this triangular Shepard operator and, as a novelty, we demonstrate its quadratic rate of convergence for general configuration of triangles, making the method not dependent from any mesh which connects the interpolation points, i.e. a meshless or meshfree method.

Future works will develop in two directions. The first one will concern the more general case of the interpolation of *Birkhoff type data* (lacunary data of "general" type) on scattered data set by the point-triangle association technique [38]. The second one will focus on the study of localized versions of the triangular Shepard operator, as well as on the definition of new combinations to increase its approximation order.

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