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XXV CICLO

Noncommutative Iwasawa theory for Selmer groups of abelian varieties over global function fields

S.S.D. MAT/02 - ALGEBRA

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A.A. 2012 – 2013

To all of you who have a dream and the bravery and the coherence to pursue it.

Introduction

Iwasawa theory dates back to the 1950's with the pioneering work of K. Iwasawa on $\mathbb{Z}_p$-extensions of number fields (see [20], [21] and [22]) and it has led to numerous generalizations, applications and conjectures, almost all in the spirit of the original conjectures or "Main Conjectures". One of the most fruitful application was proposed by Mazur in a series of papers (see [29], [30] and [31]) in the early 1970's. The main goal was to find a new approach to the Birch and Swinnerton-Dyer Conjecture (BSD from now on): its simplest formulation predicts the equality between the rank of the group $\mathcal{E}(F)$, where $\mathcal{E}$ is an elliptic curve defined over a global field F, and the order of zero of the Hasse-Weil L-function $L(\mathcal{E}, s)$ at $s = 1$. Many generalizations, for example to abelian varieties, and refined statements have been provided through the years but the main feature has always been the link between an algebraic object (the group $\mathcal{E}(F)$ and its "algebraic" rank) and an analytic one (the L-function and its order of zero, i.e., the "analytic" rank).

Iwasawa theory, in Mazur's formulation, provides a bridge between this two sides of BSD in the following way. Let K/F be a $\mathbb{Z}_p$-extension of a number field F and A/F an abelian variety. Denote with $\Gamma \simeq \mathbb{Z}_p$ the Galois group of K/F. We can associate two objects to this data. The first one is the p-part of the *Selmer group*, which is denoted by $Sel_A(K)_p$. It is constructed algebraically and is a module over the Iwasawa algebra $\Lambda(\Gamma) := \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$ which controls (i.e., provides a bound for or, sometimes, computes exactly) the algebraic rank. When the Pontrjagin dual of the Selmer group $Sel_A(K)_p^\vee$ is a torsion $\Lambda(\Gamma)$-module, there exist a (uniquely determined) set of non-zero elements $\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_n\} \subset \Lambda(\Gamma)$ such that the sequence of $\Lambda(\Gamma)$-modules

$$\bigoplus_{i=1}^{n} \Lambda(\Gamma)/\Lambda(\Gamma)\boldsymbol{f}_i \hookrightarrow Sel_A(K)_p^\vee \twoheadrightarrow N$$

is exact and N is pseudo-null. We define

$$\boldsymbol{f}_{Sel} := \prod_{i=1}^{n} \boldsymbol{f}_i$$

to be a *characteristic element* for $Sel_A(K)_p^\vee$.
The second object is constructed analytically and is called the *p-adic L-function* $L_p(A,s)$, which is again an element of $\Lambda(\Gamma)$ and interpolates the L-function in its critical value providing some knowledge on the analytic rank (see [30] and [31]).

The *main conjecture* (MC) of Iwasawa theory (roughly speaking, we have a lot of refined statements nowadays) predicts the equality, or a deep relation, between the ideals of $\Lambda(\Gamma)$ generated by $\boldsymbol{f}_{Sel}$ and $L_p(A,s)$.
It is clear that this kind of statement provides a link between the algebraic and the analytic sides of the theory and has major consequences on BSD conjecture. Indeed, the inclusion $L_p(\mathcal{E},s) \subseteq (\boldsymbol{f}_{Sel})$ (proven in some cases for example by Kato, see [25], or Bertolini-Darmon, see [7]) provides, under some mild hypotheses, the inequality

$$\mathrm{rank}_{\mathbb{Z}}\mathcal{E}(\mathrm{F}) \leqslant \mathrm{ord}_{\mathrm{s}=1}\mathrm{L}(\mathcal{E},\mathrm{s}) \ .$$

We mention here also a third object: the *Euler characteristic*

$$\chi(\Gamma, Sel_A(K)_p) := \prod_{i\geqslant 0} |H^i(\Gamma, Sel_A(K)_p)|^{(-1)^i} \ ,$$

assuming that all the quantities on the right are finite. Another classical result in Iwasawa theory, in the same spirit of the MC, relates $\chi(\Gamma, Sel_A(K)_p)$ with the p-adic valuation of $\boldsymbol{f}_{Sel}(0)$, i.e., the image of $\boldsymbol{f}_{Sel}$ under the augmentation map $\Lambda(\Gamma) \to \mathbb{Z}_p$.

The well-known analogy between number fields and function fields has inspired the study of Iwasawa main conjectures in positive characteristic as well. Remarkable works in this direction are, for instance, those of Ochiai and Trihan ([36]) or Ulmer ([49]). More recently the work of Venjakob on modules over noncommutative Iwasawa algebras $\Lambda(G)$, where now G can be any p-adic Lie group of finite dimension and without elements of order p, led to the development of a *noncommutative Iwasawa theory* (see [11]).

This thesis focuses on the algebraic side of the theory: we provide generalizations of classical results on the structure of the (Pontrjagin duals of) Selmer groups over the Iwasawa algebra $\Lambda(G)$. The base field F will be a global function field of characteristic $p > 0$ and $G = \mathrm{Gal}(K/F)$ a noncommutative ℓ-adic Lie group.

In the first chapter we give all basic definitions and provide tools we shall use in later sections. We begin introducing function fields and ℓ-adic Lie groups in Sections 1.1 and 1.2. After recalling the definition of the Iwasawa algebra and its most important properties (Section 1.3), we define the main object of our investigations: the Selmer group $Sel_A(K)_\ell$ (see Section 1.4). Finally, in Sections 1.5 and 1.6 we summarize Venjakob's work on the analogues of *characteristic elements* and *pseudo-nullity* for modules over a noncommutative Iwasawa algebra.

In the second chapter we begin our study of $Sel_A(K)_\ell^\vee$, i.e., the Pontrjagin dual of the Selmer group, via appropriate generalizations of Mazur's Control Theorem ([29]). All results stated there are going to appear in the forthcoming paper [6]. We recall that other generalizations of this theorem to the function field setting and commutative $\mathbb{Z}_\ell^d$-extensions can be found in the works of Bandini and Longhi for elliptic curves ([3], [4]) and Tan ([47]) for abelian varieties.
We begin with the $\ell \neq p$ case. In Section 2.1 we prove the Control Theorem for general ℓ-adic Lie extensions (see Theorem 2.1.2) and, as a special case, for the trivializing extension $F(A[\ell^\infty])/F$ (Theorem 2.1.7). Then, we use these theorems and an easy application of Nakayama's Lemma, to obtain the finitely generated (sometimes even torsion) condition for $Sel_A(K)_\ell^\vee$ as a module over $\Lambda(\mathrm{Gal}(K/F))$. To move a step further we need to assume that it exists $H \lhd_c \mathrm{Gal}(K/F)$ such that $\mathrm{Gal}(K/F)/H \simeq \mathbb{Z}_\ell$. When this is true we provide Theorem 2.1.15, proved with techniques already appearing in the previous Control Theorems, that allows us to describe $Sel_A(K)_\ell^\vee$ as a module over $\Lambda(H)$. The main consequence of these results is the possibility to define a characteristic element in $\Lambda(\mathrm{Gal}(K/F))$ for $Sel_A(K)_\ell^\vee$ (as mentioned in [11], one needs both the $\Lambda(\mathrm{Gal}(K/F))$ and the $\Lambda(H)$-module structure to provide such an element). Finally, in Section 2.2, we treat the same problem for the (arithmetically more interesting) case $\ell = p$: we need a few more sophisticated techniques and some (mild) extra hypotheses to reach similar results (see Theorems 2.2.3 and 2.2.8).

In the last chapter we expand our knowledge of $Sel_A(K)_\ell^\vee$'s structure, for the case $\ell \neq p$ only. Since the characteristic element is independent from the presence of pseudo-null modules (see, for example, the Structure Theorem for modules over a noncommutative Iwasawa algebra in [13]), we want to know when $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule to ensure that no relevant arithmetic information is forgotten. For the number field setting and $K = F(A[\ell^\infty])$, this issue was studied by Ochi and Venjakob ([37, Theorem 5.1]) when A is an elliptic curve, and by Ochi for a general abelian variety in [35]. In Section 3.1 we prove, under suitable hypotheses, that $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule in our setting as well: in Theorem 3.1.10 when $\mathrm{Gal}(K/F)$ has dimension greater than or equal to 3, and in Proposition 3.1.12 when $\mathrm{Gal}(K/F)$ has dimension greater than or equal to 2. At the end, in Section 3.2 we provide some simple calculations on the Euler characteristic of $Sel_A(K)_\ell$ (Theorem 3.2.4).

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Chapter 1

Background

In this chapter we describe the setting in which we shall work and fix notations and conventions. Besides, here we collect all theoretical and technical tools which will be used throughout the thesis.

1.1 Global function fields and extensions

We recall here all basic definitions regarding function fields; however, for a comprehensive study of the subject we refer the reader to [39] and [52].

1.1.1 Function fields

Definition 1.1.1. A *function field in one variable* over a field F, is a field K, containing F and at least one element x, transcendental over F, such that $K/F(x)$ is a finite algebraic extension.
Such a field K has *transcendence degree one* over F.
When F is algebraically closed in K, F is called the *constant field* of K. A *global function field* is a field K of transcendence degree 1 over its *finite* constant field F (this is the only case we will be interested in).

Definition 1.1.2. A *prime* in K is a discrete valuation ring R with maximal ideal v such that $F \subset R$ and the quotient field of R is equal to K.

Example 1.1.3. A classic example of global function field is the rational function field $\mathbb{F}_q(t)$, where q is a power of a fixed prime $p \in \mathbb{Z}$. In this case every non-zero prime ideal corresponds to a unique monic irreducible polynomial $f \in \mathbb{F}_q[t]$. Let v_f be the valuation corresponding to f. Then the valuation ring ϑ_{v_f} is the set of quotients g/h, where $g, h \in \mathbb{F}_q[t]$ and $f \nmid h$, and its maximal ideal is $\mathcal{P}_{v_f}$ which is made up by g/h with $f \mid g$.
If $s = \frac{1}{t}$ then the fields $\mathbb{F}_q(t)$ and $\mathbb{F}_q(s)$ are isomorphic. The prime s is

usually called *the prime at* ∞ (and denoted by ∞ as well): it is associated with the valuation v_∞ and we have

$$\vartheta_\infty = \{f/g \ s.t. \ \deg(f) - \deg(g) \leqslant 0\} \ , \ \mathcal{P}_\infty = \{f/g \ s.t. \ \deg(f) - \deg(g) < 0\} \ .$$

To sum up, the set of valuations over $\mathbb{F}_q(t)$ is exactly

$$\{v_f \ s.t. \ f \in \mathbb{F}_q(t) \text{ monic and irreducible}\} \cup \{v_\infty\}$$

and they are pairwise inequivalent.

As for extensions of function fields we recall the following.
Let K/F be a function field with constant field F and let L be a finite algebraic extension of K. Let E be the algebraic closure of F in L. Then L is a function field with E as its field of constants. If $L = EK$, we say that L is a *constant* (or *arithmetic*) field extension of K. While, if $E = F$, we say that L is a *geometric* extension of K.

Example 1.1.4. Let K be a global function field with the finite field $\mathbb{F}_p$ as its field of constants. An interesting kind of constant extension, from the arithmetical point of view, is $\mathbb{F}_p^{(\ell)}K$, where $\mathbb{F}_p^{(\ell)}$ is the $\mathbb{Z}_\ell$-extension of $\mathbb{F}_p$.

One of the most important property of the constant field extensions is that they are everywhere unramified (for a proof see [39, Chapter 8]).

Finally, write S for a finite and nonempty set of places of K. Define

$$\mathcal{O}_{K,S} = \{a \in K \mid v(a) \geqslant 0 \ \forall \ v \notin S\}$$

to be the *ring of* S*-integers of* K. It can be shown that it is a Dedekind domain and $C\ell_S(K)$, which denotes the S*-ideal class group* (i.e., the class group of $\mathcal{O}_{K,S}$), is finite (see [39, Proposition 14.2] and [39, Theorem 14.5]). The well known analogy between function fields and number fields appears now more clear: $\mathcal{O}_{K,S}$ plays the role of the ring of integers of a number field.

1.1.2 Notations for fields

For any field L we let $G_L := \mathrm{Gal}(L^s/L)$ (L^s a separable closure of L), L^{unr} the maximal unramified extension and $\mathcal{X}_L$ the scheme $Spec(L)$. The last one will essentially appear in Galois and flat cohomology groups so whenever we write a scheme $\mathcal{X}$ we always mean $\mathcal{X}_{fl}$, scheme on flat topology (see Section 1.4.1).
From now on F will be a global function field (of trascendence degree one over its constant field $\mathbb{F}_F = \mathbb{F}_q$, where q is a power of a fixed prime $p \in \mathbb{Z}$). We put $\overline{F}$ for an algebraic closure of F and $F^s \subset \overline{F}$.
For any algebraic extension L/F, let Σ_L be the set of places of L: for any

$v \in \Sigma_L$ we let L_v be the completion of L at v, $\mathcal{O}_v$ its ring of integers with maximal ideal $\mathfrak{m}_v$ and residue field $\mathbb{F}_{L_v}$. Whenever we deal with a local field E (or an algebraic extensions of such field) the above notations will often be replaced by $\mathcal{O}_E$, $\mathfrak{m}_E$ and $\mathbb{F}_E$.
For any place $v \in \Sigma_F$ we choose (and fix) an embedding $\overline{F} \hookrightarrow \overline{F_v}$ (an algebraic closure of F_v), in order to get a restriction map $G_{F_v} := \mathrm{Gal}(\overline{F_v}/F_v) \hookrightarrow G_F$. All algebraic extensions of F (resp. of F_v) will be assumed to be contained in $\overline{F}$ (resp. $\overline{F_v}$).
Let F_S be the maximal Galois extension of F unramified outside S and $G_S(F) := \mathrm{Gal}(F_S/F)$. With $\mathcal{O}_S^\times$ we denote the group of units of $\mathcal{O}_S := \bigcup \mathcal{O}_{L,S}$ where the union is taken over all finite subextensions L of F_S.

1.2 ℓ-adic Lie groups

We recall some facts on profinite groups and ℓ-adic Lie groups, which will be useful later. For further information on these kind of groups the reader is referred to [16] and [55].

Let G be a group, x, y, z elements of G, and A, B subgroups of G. Define $[x, y] = x^{-1}y^{-1}xy$. Then the group

$$[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$$

is the *commutator* subgroup of A and B in G (where $\langle X \rangle$ denotes the subgroup of G generated by a subset X of G).

Let now G be a profinite group and H a subgroup of G. We write $\overline{H}$ for the topological closure of H in G and $|G : H|$ for the index of H in G. For any integer n we put $G^n := \overline{\langle x^n \mid x \in G \rangle}$.

Definition 1.2.1. A pro-ℓ group G is said to be *powerful* if $G/\overline{G^\ell}$ is abelian (when ℓ is odd) or $G/\overline{G^4}$ is abelian (when $\ell = 2$).

Definition 1.2.2. A pro-ℓ group G is said to be *uniform* if

(i) G is (topologically) finitely generated;

(ii) G is powerful;

(iii) for any i, $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$ (where $P_1(G) = G$ and, for $i \geqslant 1$, $P_{i+1}(G) = \overline{P_i(G)^\ell [P_i(G), G]}$).

Example 1.2.3. Fix a positive integer d. Let

$$\Gamma_i = \{\gamma \in GL_d(\mathbb{Z}_\ell) \mid \gamma \equiv 1_d \mod \ell^i\}$$

and put $G = \Gamma_1$ if ℓ is odd, $G = \Gamma_2$ if $\ell = 2$. Then G is a uniform pro-ℓ group (see [16, Theorem 5.2]).

Let $d(G)$ denote the cardinality of a minimal set of topological generators for G. If G is finite, the *rank* is defined to be

$$\mathrm{rk}(G) = \sup\{d(H) \mid H \leqslant G\} .$$

If G is not finite, the rank is one of the following values:

$$\begin{aligned}
r_1 &= \sup\{d(H) \mid H \leqslant_c G\} \\
r_2 &= \sup\{d(H) \mid H \leqslant_c G \text{ and } d(H) \leqslant \infty\} \\
r_3 &= \sup\{d(H) \mid H \leqslant_o G\} \\
r_4 &= \sup\{\mathrm{rk}(G/N) \mid N \lhd_o G\}.
\end{aligned}$$

By [16, Proposition 3.11] $r_1 = r_2 = r_3 = r_4$.

Definition 1.2.4. Let G be a pro-ℓ group of finite rank. The *dimension* of G is

$$\dim(G) = d(H)$$

where H is any open uniform subgroup of G.

Example 1.2.5. Let G be as in Example 1.2.3. Then

$$\dim(G) = \mathrm{rk}(G) = d^2 .$$

Let G be an ℓ-adic Lie group. A very useful characterization of ℓ-adic analytic groups is due to Lazard [27] and states that *a topological group G has the structure of ℓ-adic Lie group if and only if G contains an open subgroup which is a uniform pro-ℓ group* (see also [16, Theorem 8.32]). In this thesis G will always be the Galois group of a field extension, so one must take into account that it is compact.

Theorem 1.2.6. *The following are equivalent for a topological group G.*

(i) G is a compact ℓ-adic Lie group;

(ii) G contains an open, normal, uniform, pro-ℓ subgroup of finite index;

(iii) G is a profinite group containing an open subgroup which is a pro-ℓ group of finite rank.

Proof. See [16, Corollary 8.34]. □

Recall that being profinite implies that every open subgroup is of finite index. Next theorem allows us to define the concept of *dimension* for an ℓ-adic Lie group.

Theorem 1.2.7. *Let G be an ℓ-adic Lie group. Then, there exists a unique non-negative integer n such that every open pro-ℓ subgroup of G has finite rank and dimension n in the sense of Definition 1.2.4.*

Proof. See [16, Theorem 8.37] □

Definition 1.2.8. Let G be an ℓ-adic Lie group. Then the *dimension* $\dim(G)$ of G is the number n specified in Theorem 1.2.7.

Finally, note that if G is an ℓ-adic Lie group without points of order ℓ, then it has finite ℓ-cohomological dimension (denoted by cd_ℓ), which is equal to its dimension as an ℓ-adic Lie group ([41, Corollaire (1) p. 413]).

Example 1.2.9. Let F be a global function field of characteristic $p > 3$ and $\mathcal{E}/F$ a non isotrivial elliptic curve. For any prime $\ell \neq p$ the Galois group $G = \mathrm{Gal}(F(\mathcal{E}[\ell^\infty])/F)$ is open in $GL_2(\mathbb{Z}_\ell)$ and it is an ℓ-adic Lie group of dimension 4.
Moreover, if $\ell > 5$ G has no points of order ℓ. If this is the case, $\mathrm{cd}_\ell(G) = 4$.

1.3 Modules and duals

For any ℓ-adic Lie group G we denote by

$$\Lambda(G) = \mathbb{Z}_\ell[[G]] := \varprojlim_{U} \mathbb{Z}_\ell[G/U]$$

the associated *Iwasawa algebra*, where the limit is taken on the open normal subgroups of G.
From Lazard's work (see [27]), we know that $\Lambda(G)$ is Noetherian and, if G is pro-ℓ and has no elements of order ℓ, then $\Lambda(G)$ is an integral domain. From [16, Theorem 4.5] we also know that, for a finitely generated powerful pro-ℓ group, being torsion free is equivalent to being uniform. Because of this when we need $\Lambda(G)$ to be without zero divisors we will take a torsion free G.

For a $\Lambda(G)$-module M we set the extension groups

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_{\Lambda(G)}(M, \Lambda(G))$$

for any nonnegative integer i and $\mathrm{E}^i(M) = 0$ for $i < 0$ by convention.
Since in our applications G comes from a Galois extension, e.g. $G = \mathrm{Gal}(L/F)$, we denote with G_v the decomposition group of $v \in \Sigma_F$ for some prime $w|v$, $w \in \Sigma_L$, and we use the notation

$$\mathrm{E}^i_v(M) := \mathrm{Ext}^i_{\Lambda(G_v)}(M, \Lambda(G_v)).$$

Note that the definition does not depend on the prime w we choose; indeed, we obtain the other decomposition groups by conjugation, so their Iwasawa algebras are isomorphic.

Let H be a closed subgroup of G. For every $\Lambda(H)$-module N we consider the $\Lambda(G)$-modules

$$\mathrm{Coind}_G^H(N) := \mathrm{Map}_{\Lambda(H)}(\Lambda(G), N) \quad \text{and} \quad \mathrm{Ind}_H^G(N) := \Lambda(G) \otimes_{\Lambda(H)} N\ .^1$$

For a $\Lambda(G)$-module M, we denote by $M^\vee := \mathrm{Hom}_{cont}(M, \mathbb{C}^*)$ its Pontrjagin dual. In the cases considered in this work, M will be a (mostly discrete) topological $\mathbb{Z}_\ell$-module, so that $M^\vee$ can be identified with $\mathrm{Hom}_{cont}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ and it has a natural structure of $\mathbb{Z}_\ell$-module (indeed the category of compact $\Lambda(G)$-modules and the category of discrete $\Lambda(G)$-modules are both abelian and the Pontrjagin duality defines a contravariant equivalence of categories between them).
The reader is reminded that to say that an R-module M (R any ring) is *cofinitely generated* over R means that $M^\vee$ is a finitely generated R-module.

Finally, for every discrete $G_S(F)$-module M which is finitely generated as $\mathbb{Z}$-module and whose torsion has order prime with p we let

$$M' := \mathrm{Hom}(M, \mathcal{O}_S^\times)$$

be the dual $G_S(F)$-module of M. When M is finite the dual module

$$M' = \mathrm{Hom}(M, \mathcal{O}_S^\times) = \mathrm{Hom}(M, \boldsymbol{\mu})$$

is again finite and we have a canonical identification $M'' = M$.

1.4 Selmer groups

Let A be an abelian variety defined over F and of dimension g: we denote by A^t its dual abelian variety. For any positive integer n we let $A[n]$ be the scheme of n-torsion points and, for any prime ℓ, we put $A[\ell^\infty] := \varinjlim_n A[\ell^n]$.

We define Selmer groups via the usual cohomological techniques and, since we deal mostly with the flat scheme of torsion points, we shall use the flat cohomology groups H^i_{fl}.

1.4.1 Flat cohomology

For the basic theory of sites and cohomology on a site we refer the reader to [32, Ch. II and III]. Shortly, for any scheme $\mathcal{X}$ we write $\mathcal{X}_{fl}$ for the subcategory of $\mathbf{Sch}/\mathcal{X}$ (schemes over $\mathcal{X}$) whose structure morphisms are locally of finite type. Furthermore, $\mathcal{X}_{fl}$ is endowed with the *flat topology*: let $\mathcal{Y} \to \mathcal{X}$ be an element of $\mathcal{X}_{fl}$, then a covering of $\mathcal{Y}$ is a family $\{g_i : \mathcal{U}_i \to \mathcal{Y}\}$ such that $\mathcal{Y} = \bigcup g_i(\mathcal{U}_i)$ and each g_i is a flat morphism locally of finite type.

[1] We use the notations of [37], some texts, e.g. [34], switch the definitions of $\mathrm{Ind}_G^H(N)$ and $\mathrm{Coind}_G^H(N)$.

Definition 1.4.1. Let $\mathcal{P}$ be a sheaf on $\mathcal{X}$ and consider the global section functor sending $\mathcal{P}$ to $\mathcal{P}(\mathcal{X})$. The *i-th flat cohomology group* of $\mathcal{X}$ with values in $\mathcal{P}$, denoted by $H^i_{fl}(\mathcal{X},\mathcal{P})$, is the value at $\mathcal{P}$ of the i-th right derived functor of the global section functor.

1.4.2 Selmer groups

Fix a prime $\ell \in \mathbb{Z}$ and consider the exact sequence

$$0 \to A[\ell^n] \to A \xrightarrow{\ell^n} A \to 0 \ .$$

For any finite algebraic extension L/F, take flat cohomology with respect to $\mathcal{X}_L$ to get an injective *Kummer map*

$$A(L)/\ell^n A(L) \hookrightarrow H^1_{fl}(\mathcal{X}_L\,, A[\ell^n]) \ .$$

Taking direct limits one has an injective map

$$\kappa : A(L) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow \varinjlim_n H^1_{fl}(\mathcal{X}_L\,, A[\ell^n]) := H^1_{fl}(\mathcal{X}_L\,, A[\ell^\infty]) \ .$$

Exactly in the same way one can define *local Kummer maps*

$$\kappa_w : A(L_w) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow \varinjlim_n H^1_{fl}(\mathcal{X}_{L_w}\,, A[\ell^n]) := H^1_{fl}(\mathcal{X}_{L_w}\,, A[\ell^\infty])$$

for any place $w \in \Sigma_L$.

Definition 1.4.2. The *ℓ-part of the Selmer group* of A over L is defined to be

$$Sel_A(L)_\ell = Ker\left\{ H^1_{fl}(\mathcal{X}_L, A[\ell^\infty]) \to \prod_{w\in\Sigma_L} H^1_{fl}(\mathcal{X}_{L_w}, A[\ell^\infty])/Im\,\kappa_w \right\}$$

where the map is the product of the natural restrictions between cohomology groups.
For infinite extensions $\mathcal{L}/F$ the Selmer group $Sel_A(\mathcal{L})_\ell$ is defined, as usual, via direct limits.

For an infinite (ℓ-adic Lie) extension K/F, the group $Sel_A(K)_\ell$ admit natural actions by $\mathbb{Z}_\ell$ (because of $A[\ell^\infty]$) and by $G := \mathrm{Gal}(K/F)$. Hence it is a module over the Iwasawa algebra $\Lambda(G)$.
If L/F is a finite extension the group $Sel_A(L)_\ell$ is a cofinitely generated $\mathbb{Z}_\ell$-module (see, e.g. [33, III.8 and III.9]). One can define the *Tate-Shafarevich group* $Ш(A/L)$ as the group that fits into the exact sequence

$$A(L) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow Sel_A(L)_\ell \twoheadrightarrow Ш(A/L)[\ell^\infty] \ .$$

According to the function field version of the Birch and Swinnerton-Dyer conjecture, $\text{Ш}(A/L)$ is finite for any finite extension L of F (some evidences of BSD conjecture in the function field case can be found in [26] and [49]). Taking Pontrjagin duals, it follows that

$$\operatorname{rank}_{\mathbb{Z}_\ell} Sel_A(L)_\ell^\vee = \operatorname{rank}_{\mathbb{Z}} A(L)$$

which provides motivation for the study of (duals of) Selmer groups (recall that the cohomology groups H^i_{fl}, hence the Selmer groups, are endowed with the discrete topology).

Remark 1.4.3. When $\ell \neq p$ the torsion subschemes are Galois modules and we can define Selmer groups via Galois cohomology since, in this case,

$$H^1_{fl}(\mathcal{X}_L, A[\ell^n]) \simeq H^1_{et}(\mathcal{X}_L, A[\ell^n]) \simeq H^1(G_L, A[\ell^n](\overline{F}))$$

(see [32, III.3.9]). In order to lighten notations, each time we work with $\ell \neq p$ we shall use the classical notation $H^i(L, \cdot)$ instead of $H^i(G_L, \cdot)$ and $H^i(L/E, \cdot)$ instead of $H^i(\mathrm{Gal}(L/E), \cdot)$. Moreover we write $A[n]$ for $A[n](\overline{F})$, putting $A[\ell^\infty] := \bigcup A[\ell^n]$.

1.4.3 An equivalent definition for $Sel_A(L)_\ell$

Let S be a finite set of primes of F which contains all primes of bad reduction for the abelian variety A/F. The extension $F(A[\ell^\infty])/F$ is unramified outside S (actually it is ramified exactly at the primes of bad reduction by [42, Corollary 2 (b), p. 497]). So F_S contains all ℓ-torsion points of A and $A[\ell^\infty]$ is an unramified G_{F_v}-module for every $v \notin S$. Thanks to that, for every extension of F contained in F_S, it is possible to give an equivalent definition of the Selmer group (we shall use it in Chapter 3). Before giving it we need the following

Proposition 1.4.4. *In the setting described above, let $\ell \in \mathbb{Z}$ be any prime different from the characteristic of F and L/F a Galois extension such that $L \subseteq F_S$. Let ρ be the map*

$$\rho : H^1(F^s/L, A[\ell^\infty]) \to \prod_{\substack{w|v \\ v \notin S}} H^1(F_v^s/L_w, A[\ell^\infty]) \ .$$

Then

$$Ker(\rho) \simeq H^1(F_S/L, A[\ell^\infty]) \ .$$

Proof. First of all we observe that

$$H^1(F_v^{unr}/L_w, A[\ell^\infty]) = 0 \ \ \forall\ w|v \notin S \ .$$

Indeed, if L_w is local, then it follows from [34, Theorem 7.2.15] and [34, Corollary 7.2.16] (because $A[\ell^\infty]$ is unramified).
If L_w is not local, then we have that $\mathrm{Gal}(F_v^{unr}/L_w)$ has order prime with ℓ. To be more precise, L_w is not local if and only if the degree of the constant field extension $\mathbb{F}_{L_w}/\mathbb{F}_{F_v}$ is infinite. This means that there exists a surjective map

$$\mathrm{Gal}(L_w/F_v) \twoheadrightarrow \mathbb{Z}_\ell \ .$$

Since $\mathrm{Gal}(F_v^{unr}/F_v) \simeq \prod_q \mathbb{Z}_q$, we have that $\mathrm{Gal}(F_v^{unr}/L_w)$ is isomorphic to a subgroup of $\prod_{q\neq\ell} \mathbb{Z}_q$.
For every $v \notin S$ and $w|v$, we consider the map

$$\rho_{S,w} : H^1(F_S/L, A[\ell^\infty]) \to H^1(F_v^s/L_w, A[\ell^\infty]) \ .$$

Let $x \in H^1(F_S/L, A[\ell^\infty])$, then $\rho_{S,w}(x) \in H^1(F_{S,w}/L_w, A[\ell^\infty])$, which, via the inflation map, embeds into $H^1(F_v^{unr}/L_w, A[\ell^\infty]) = 0$. Therefore we have $H^1(F_S/L, A[\ell^\infty]) \subseteq Ker(\rho)$.
Conversely, let $y \in Ker(\rho)$: for every $v \notin S$, $\rho(y)$ is a coboundary, i.e., $\exists\ Q \in A[\ell^\infty]$ such that $\rho(y)(\sigma) = Q^\sigma - Q$ for any $\sigma \in \mathrm{Gal}(F_v^s/L_w)$. We write I_v for the inertia group of v in $\mathrm{Gal}(F_v^s/F_v)$. Since $A[\ell^\infty]$ is unramified at v, $\rho(y)(\sigma) = Q^\sigma - Q = 0$ for any $\sigma \in I_v$. So $\rho(y)_{|I_v} = 0$, $\forall\ v \notin S$. Hence, letting $I_S := \mathrm{Gal}(F^s/F_S)$, we have $\rho(y)_{|I_S} = 0$. From the Inf-Res sequence

$$H^1(F_S/L, A[\ell^\infty]) \hookrightarrow H^1(F^s/L, A[\ell^\infty]) \xrightarrow{Res} H^1(F^s/F_S, A[\ell^\infty]) \ ,$$

it follows that

$$H^1(F_S/L, A[\ell^\infty]) \simeq Ker(Res) \ .$$

Since $y \in Ker(\rho)$ implies $y \in Ker(Res)$, we have $y \in H^1(F_S/L, A[\ell^\infty])$. $\square$

Let us consider the maps:

- $\theta : H^1(F_S/L, A[\ell^\infty]) \to \displaystyle\prod_{\substack{w|v \\ v\in S}} H^1(F_v^s/L_w, A[\ell^\infty])$;

- $\rho : H^1(F^s/L, A[\ell^\infty]) \to \displaystyle\prod_{\substack{w|v \\ v\notin S}} H^1(F_v^s/L_w, A[\ell^\infty])$;

- $\eta : H^1(F^s/L, A[\ell^\infty]) \to \displaystyle\prod_{\substack{w|v \\ v\in S}} H^1(F_v^s/L_w, A[\ell^\infty])$.

In our setting

$$Sel_A(L)_\ell = Ker(\rho) \cap Ker(\eta) \text{ and } H^1(F_S/L, A[\ell^\infty]) \hookrightarrow H^1(F^s/L, A[\ell^\infty]).$$

Besides, $Ker(\theta) \simeq H^1(F_S/L, A[\ell^\infty]) \cap Ker(\eta)$. So we can give the following definition

Definition 1.4.5. If $\ell \neq p$ the *ℓ-part of the Selmer group* of A over L is

$$Sel_A(L)_\ell = Ker\left\{ H^1(F_S/L, A[\ell^\infty]) \to \prod_{\substack{w|v \\ v\in S}} H^1(L_w, A[\ell^\infty]) \right\} .$$

For the applications we have in mind it is useful to note that we can rewrite the above definition as

$$Sel_A(L)_\ell = Ker\left\{ H^1(F_S/L, A[\ell^\infty]) \to \bigoplus_{v\in S} \mathrm{Coind}_G^{G_v} H^1(L_w, A[\ell^\infty]) \right\} .$$

1.5 Characteristic elements

We summarize here the main results of [11] on the appropriate definition of *characteristic elements* for modules over a noncommutative Iwasawa algebra.

Let G be a compact ℓ-adic Lie group and $\Lambda(G)$ its associated Iwasawa algebra. Every $\Lambda(G)$-module M is assumed to be finitely generated. Assume that G contains a closed normal subgroup H such that $G/H \simeq \mathbb{Z}_\ell$.

Definition 1.5.1. A multiplicatively closed subset $\mathscr{T}$ of $\Lambda(G)$ is said to be a left and right *Ore set* if, for each $s \in \mathscr{T}$ and $r \in \Lambda(G)$, there exist $t_1, t_2 \in \mathscr{T}$ and $u_1, u_2 \in \Lambda(G)$ such that

$$su_1 = rt_1 \quad \text{and} \quad u_2 s = t_2 r \ .$$

Definition 1.5.2. Let $\mathscr{S}$ be the set of all $s \in \Lambda(G)$ such that $\Lambda(G)/\Lambda(G)s$ is a finitely generated $\Lambda(H)$-module.

Theorem 1.5.3. *The set $\mathscr{S}$ in Definition 1.5.2 is multiplicatively closed and it is a left and right Ore set in $\Lambda(G)$. The elements of $\mathscr{S}$ are non-zero divisors in $\Lambda(G)$.*

Proof. See [11, Theorem 2.4]. □

Let M be a left or right $\Lambda(G)$-module. Then M is said to be *$\mathscr{S}$-torsion* if, for each $m \in M$, there exists $s \in \mathscr{S}$ such that $sm = 0$ or $ms = 0$ (according to the action). By [11, Proposition 2.3], M is finitely generated over $\Lambda(H)$ if and only if M is $\mathscr{S}$-torsion.
Define the set

$$\mathscr{S}^* := \bigcup_{n\geqslant 0} \ell^n \mathscr{S}$$

which is again a multiplicatively closed left and right Ore set in $\Lambda(G)$ whose elements are non-zero divisors. Let $\Lambda(G)_{\mathscr{S}}$ and $\Lambda(G)_{\mathscr{S}^*}$ be the localizations of $\Lambda(G)$ at $\mathscr{S}$ and $\mathscr{S}^*$, so that

$$\Lambda(G)_{\mathscr{S}^*} = \Lambda(G)_{\mathscr{S}}\left[\frac{1}{\ell}\right] .$$

It is clear that M is $\mathscr{S}^*$-torsion if and only if $M/M(\ell)$ is finitely generated over $\Lambda(H)$ (where $M(\ell)$ denotes the ℓ-primary submodule of M). We define $\mathfrak{M}_H(G)$ to be the category of all finitely generated $\Lambda(G)$-modules, which are $\mathscr{S}^*$-torsion. An application of K-theory (see Appendix A for a quick review of the basic notions needed here) provides a definition of characteristic elements for any object in the category $\mathfrak{M}_H(G)$.
Write $\mathrm{K}_0(\mathfrak{M}_H(G))$ for the Grothendieck group of the category $\mathfrak{M}_H(G)$. There is a connecting homomorphism

$$\delta_G : \mathrm{K}_1(\Lambda(G)_{\mathscr{S}^*}) \to \mathrm{K}_0(\mathfrak{M}_H(G))$$

coming from Theorem A.1.5, which is surjective when G has no element of order ℓ (see [11, Proposition 3.4]). Thanks to that (when G contains a suitable subgroup H and has no elements of finite order ℓ) one can give the following definition.

Definition 1.5.4. For each module M in the category $\mathfrak{M}_H(G)$, a *characteristic element* is any $\xi_M \in \mathrm{K}_1(\Lambda(G)_{\mathscr{S}^*})$ such that

$$\delta_G(\xi_M) = [M] .$$

We now briefly explain why Coates et al. in [11] choose such an element ξ_M as a good candidate to be a characteristic element in Iwasawa theory.
Before seeing this, we recall the following

Definition 1.5.5. Let M be a compact G-module such that

(i) $H_i(G, M)$ is finite for any $i \geqslant 0$;

(ii) $H_i(G, M) = 0$ for all but finitely many i.

Then we define the *Euler characteristic* of M as

$$\chi(G, M) := \prod_{i \geqslant 0} |H_i(G, M)|^{(-1)^i} .$$

Fix any integer $n \geqslant 1$ and consider the continuous homomorphism

$$\rho : G \to GL_n(O)$$

where O is the ring of integers of a finite extension L of $\mathbb{Q}_\ell$.
Put $M_O = M \otimes_{\mathbb{Z}_\ell} O$ and define

$$tw_\rho(M) = M_O \otimes_O O^n .$$

Then, for every $M \in \mathfrak{M}_H(G)$ we have that $tw_\rho(M) \in \mathfrak{M}_H(G)$ (see [11, Lemma 3.2]).
Write $|\cdot|_\ell$ for the valuation of $\overline{\mathbb{Q}}_\ell$ and $m_\rho = [L : \mathbb{Q}_\ell]$, where L is the quotient field of O. Let $\widehat{\rho}$ be the contragradient representation of G, i.e. $\widehat{\rho}(g) = \rho(g^{-1})^{tr}$ for $g \in G$, where $\cdot^{tr}$ denotes the transpose matrix.
The sought relation between a characteristic element of a module M and its Euler characteristic is showed in the following

Theorem 1.5.6. *Let $M \in \mathfrak{M}_H(G)$ and ξ_M a characteristic element for M. Then, for every continuous homomorphism $\rho : G \to GL_n(O)$ such that $\chi(G, tw_{\widehat{\rho}}(M))$ is finite we have $\xi_M(\rho) \neq 0, \infty$*[2] *and*

$$\chi(G, tw_{\widehat{\rho}}(M)) = |\xi_m(\rho)|_\ell^{-m_\rho} \ .$$

Proof. See [11, Theorem 3.6]. □

In Chapter 2 we are able to prove that, under suitable (mild) hypotheses, the module $Sel_A(K)_\ell^\vee$ is in $\mathfrak{M}_H(G)$, so it is possible to associate a characteristic element to it. Hopefully we can find a relation similar to the one of Theorem 1.5.6 for the function field setting as predicted by [11] (for a different approach and some results on this topic see, for example, [56]). In Chapter 3 we provide some calculation on $\chi(G, Sel_A(K)_\ell)$ but the subject is not fully investigated yet.

1.6 Homotopy theory and pseudo-null modules

In this section we recall Venjakob's work on the definition of *pseudo-null* $\Lambda(G)$-module for a noncommutative Iwasawa algebra and Jannsen's homotopy theory leading to diagram (1.3) which will be crucial for the results of Section 3.1.

1.6.1 Auslander regular rings

Let R be a ring. In what follows every R-module M is assumed to be finitely generated.

Definition 1.6.1. (i) Let $M \neq 0$ be a $\Lambda(G)$-module. The *grade* of M is

$$j(M) := \min\{i \,:\, E^i(M) \neq 0\} \ .$$

By convention $j(\{0\}) = \infty$.

[2] For a definition of $\xi_M(\rho)$ see [11, p. 173]

(ii) A Noetherian ring R is called *Auslander-Gorenstein* ring if it has finite injective dimension and the following Auslander condition holds: for any R-module M, any integer m and any submodule N of $\mathrm{E}^m(M)$, the grade of N satisfies $j(N) \geqslant m$.

(iii) A Noetherian ring R is called *Auslander regular* ring if it has finite global homological dimension and the Auslander condition holds.[3]

Let now R be an Auslander regular ring of finite global dimension $\mathfrak{d}$ and M an R-module. There is a canonical filtration

$$T_0(M) \subseteq T_1(M) \subseteq \cdots \subseteq T_{\mathfrak{d}-1}(M) \subseteq T_{\mathfrak{d}}(M) = M \ .$$

By convention $T_i(M) = 0$ for $i < 0$.

Definition 1.6.2. For a non zero module M, the number

$$\delta(M) := \min\{i \, : \, T_i(M) = M\}$$

is called the *dimension* of M. By convention $\delta(\{0\}) = -\infty$.

Definition 1.6.3. We call an R module M *pseudo-null* if

$$\delta(M) \leqslant \mathfrak{d} - 2 \ .$$

Note that when R is a *commutative* local noetherian Gorenstein ring, the dimension $\delta(M)$ is exactly the Krull dimension of $Supp_R(M)$ (see [8, Corollary 3.5.11]).

The Auslander regular rings have some remarkable properties which we collect in the following proposition.

Proposition 1.6.4. *Let R be an Auslander regular ring of finite global dimension $\mathfrak{d}$ and let M be an R-module. Then*

(i) $T_i(M)/T_{i-1}(M) = 0$ *if and only if* $\mathrm{E}^{\mathfrak{d}-i}\mathrm{E}^{\mathfrak{d}-i}(M) = 0$*;*

(ii) $j(M) + \delta(M) = \mathfrak{d}$ *for* $M \neq 0$*;*

(iii) $\delta(T_i(M)) \leqslant i$ *and* $T_i(M)$ *is the maximal submodule of* M *with dimension* δ *less than or equal to* i.

Proof. See [50, Proposition 3.5]. □

The above proposition allows us to provide a more computable version of the definition of pseudo-null module. Indeed, by part *(ii)* we have that M is a pseudo-null module if and only if $E^0(M) = E^1(M) = 0$.

[3]For more details on injective and global homological dimension see [53, Chapter 4].

Moreover, since $\delta(T_i(M)) \leqslant i$ and every $T_i(M)$ is the maximal submodule of M with δ-dimension less than or equal to i (part (iii)), only the submodules $T_0(M),\ldots,T_{\mathfrak{d}-2}(M)$ can be pseudo-null. If $T_0(M) = \cdots = T_{\mathfrak{d}-2}(M) = 0$, M does not have any nonzero pseudo-null submodule. This is the case when $E^iE^i(M) = 0 \;\forall\, i \geqslant 2$ (see (i)).

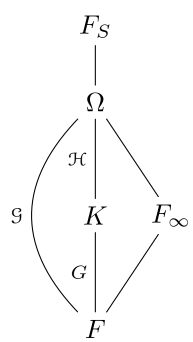
The most important result, which relates the Iwasawa algebra $\Lambda(G)$ to Auslander rings, is the following

Theorem 1.6.5 (O. Venjakob, (2002))**.** *Let G be a compact ℓ-adic Lie group with no points of order ℓ. Then $\Lambda(G)$ is an Auslander regular ring with global dimension $\mathfrak{d} = \mathrm{cd}_\ell(G) + 1$.*

Proof. See [50, Theorem 3.26]. □

1.6.2 Powerful diagram

In this section we will see one of the main tools which will be used in Chapter 3: the *powerful diagram.*
For the missing details of the proofs the reader can consult [34, Chapter V, § 6] or [37, Section 4][4].
We will deal with *ℓ-adic Lie extensions K/F*, i.e., Galois extensions with Galois group an ℓ-adic Lie group. Unless noted otherwise, we always assume that our extensions are unramified outside a nonempty and finite set S of primes of Σ_F which always contains all primes of bad reduction of A/F.
Replacing, if necessary, F by a finite extension we can (and will) assume that K is contained in the maximal pro-ℓ extension of $F_\infty := F(A[\ell^\infty])$ unramified outside S. The prime ℓ is assumed to be different from p.
We have the following picture



where Ω is the maximal pro-ℓ extension of F_∞ contained in F_S. Let us denote $\mathcal{G} = \mathrm{Gal}(\Omega/F)$, $\mathcal{H} = \mathrm{Gal}(\Omega/K)$, $G = \mathrm{Gal}(K/F)$ and $\mathcal{A} := A[\ell^\infty]$. The

[4] Those results hold in our setting as well because we will work with the $\Lambda(G)$-module $A[\ell^\infty]$, where $\ell \neq p$.

extension F_∞/F is called the *trivializing extension.*

Let us denote with $I(\mathcal{G})$ the kernel of the augmentation map $\Lambda(\mathcal{G}) \to \mathbb{Z}_\ell$. Tensoring the natural exact sequence $I(\mathcal{G}) \hookrightarrow \Lambda(\mathcal{G}) \twoheadrightarrow \mathbb{Z}_\ell$ with $\mathcal{A}^\vee \simeq \mathbb{Z}_\ell^{2g}$, one gets

$$I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathcal{A}^\vee \hookrightarrow \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathcal{A}^\vee \twoheadrightarrow \mathcal{A}^\vee \ .$$

Since $\Lambda(\mathcal{G}) \otimes \mathcal{A}^\vee$ is a projective $\Lambda(\mathcal{G})$-module ([37, Lemma 4.2]) the previous sequence yields

$$H_1(\mathcal{H}, \mathcal{A}^\vee) \hookrightarrow (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathcal{A}^\vee)_\mathcal{H} \to (\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathcal{A}^\vee)_\mathcal{H} \twoheadrightarrow (\mathcal{A}^\vee)_\mathcal{H} \ . \tag{1.1}$$

In order to shorten notations we put:

- $X = H_1(\mathcal{H}, \mathcal{A}^\vee)$;
- $Y = (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathcal{A}^\vee)_\mathcal{H}$;
- $J = Ker\{(\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathcal{A}^\vee)_\mathcal{H} \to (\mathcal{A}^\vee)_\mathcal{H}\}$.

So the sequence (1.1) becomes:

$$X \hookrightarrow Y \twoheadrightarrow J \ . \tag{1.2}$$

For our purpose it is useful to think X as $H^1(F_S/K, \mathcal{A})^\vee$. Indeed, $H_1(\mathcal{H}, \mathcal{A}^\vee) \simeq H^1(\Omega/K, \mathcal{A})^\vee \simeq H^1(F_S/K, \mathcal{A})^\vee$.
Let $\mathcal{F}(r)$ denote a free pro-ℓ-group of rank $r = \dim \mathcal{G}$ and denote by $\mathcal{N}$ (resp. $\mathcal{R}$) the kernel of the natural map $\mathcal{F}(r) \to \mathcal{G}$ (resp. $\mathcal{F}(r) \to G$). For any profinite group H, we denote by $H^{ab}(\ell)$ the maximal pro-ℓ- quotient of the maximal abelian quotient of H.

Theorem 1.6.6 (Powerful diagram)**.** *In the above setting, there is the following commutative exact diagram of $\Lambda(G)$-modules*

$$\begin{tikzcd}
H^2(F_S/K,\mathcal{A})^\vee \arrow[r, hook] \arrow[d, equal] & (H^1(\mathcal{N}^{ab}(\ell),\mathcal{A})^{\mathcal{H}})^\vee \arrow[r] \arrow[d, "\simeq"] & H^1(\mathcal{R},\mathcal{A})^\vee \arrow[r, two heads] \arrow[d, hook] & X \arrow[d, hook] \\
H^2(F_S/K,\mathcal{A})^\vee \arrow[r, hook] & (\mathcal{N}^{ab}(\ell) \otimes \mathcal{A}^\vee)_\mathcal{H} \arrow[r] & \Lambda(G)^{2gr} \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads] \\
 & & J \arrow[r, equal] & J
\end{tikzcd} \tag{1.3}$$

Besides, $(\mathcal{N}^{ab}(\ell) \otimes \mathcal{A}^\vee)_\mathcal{H}$ is a projective $\Lambda(G)$-module.

Proof. See [37, Lemma 4.5]. □

From diagram (1.3) it is easy to see that if $H^2(F_S/K,\mathcal{A})=0$ the module Y has projective dimension $\leqslant 1$. Whenever this is true the definition of J provides the isomorphisms:

$$\mathrm{E}^i(X)\simeq \mathrm{E}^{i+1}(J)\quad \forall\, i\geqslant 2\ , \tag{1.4}$$

and

$$E^i(J)\simeq \mathrm{E}^{i+1}((\mathcal{A}^\vee)_{\mathcal{H}})\quad \forall\, i\geqslant 2\ , \tag{1.5}$$

which will be repeatedly used in our computations.
We shall need also a "localized" version of the sequence (1.2). For every $v\in S$ and a $w\in\Sigma_K$ dividing v, we define

$$X_v=H^1(K_w,\mathcal{A})^\vee \qquad \text{and} \qquad Y_v=(I(\mathcal{G}_v)\otimes_{\mathbb{Z}_\ell}\mathcal{A}^\vee)_{\mathcal{H}_v}$$

(with $\mathcal{G}_v$ the decomposition group of v in $\mathcal{G}$ and $\mathcal{H}_v=\mathcal{H}\cap\mathcal{G}_v$). The exact sequence

$$X_v\hookrightarrow Y_v\twoheadrightarrow J_v \tag{1.6}$$

fits into the localized version of diagram (1.3). If K_w is still a local field, then Tate local duality ([34, Theorem 7.2.6]) yields

$$H^2(K_w,\mathcal{A})=H^2(K_w,\varinjlim_n A[\ell^n])\simeq \varprojlim_n H^0(K_w,A^t[\ell^n])^\vee=0\ .$$

If K_w is not local, then ℓ^∞ divides the degree of the extension K_w/F_v and $H^2(K_w,\mathcal{A})=0$ by [34, Theorem 7.1.8 (i)]. Therefore Y_v always has projective dimension $\leqslant 1$ and

$$\mathrm{E}^i(X_v)\simeq \mathrm{E}^{i+1}(J_v)\simeq \mathrm{E}^{i+2}((\mathcal{A}^\vee)_{\mathcal{H}_v})\quad \forall\, i\geqslant 2\ . \tag{1.7}$$

We note that, since $\ell\neq p$, the image of the local Kummer maps is always 0 (see [12, Proposition 4.1 and the subsequent Remark]), hence

$$X_v=H^1(K_w,\mathcal{A})^\vee=(H^1(K_w,\mathcal{A})/Im\,\kappa_w)^\vee\simeq (H^1(K_w,A)[\ell^\infty])^\vee\ .$$

Then Definition 1.4.5 for $L=K$ can be written as

$$Sel_A(K)_\ell=Ker\left\{\psi\,:\,X^\vee\longrightarrow\bigoplus_S \mathrm{Coind}_G^{G_v}X_v^\vee\right\}$$

and, dualizing, we get a map

$$\psi^\vee\,:\,\bigoplus_S \mathrm{Ind}_{G_v}^G X_v\longrightarrow X$$

whose cokernel is exactly $\mathcal{S}:=Sel_A(K)_\ell^\vee$.

Chapter 2

Control Theorems

Let K/F be an ℓ-adic Lie extension unramified outside a finite and nonempty set of primes S. Here bad reduction primes of A are not required to stay in S. As usual, we write G for the Galois group of K/F.
In this chapter we study $Sel_A(K)_\ell^\vee$ and (under some mild hypotheses) prove that it is a finitely generated $\mathbb{Z}_\ell[[G]]$-module via generalizations of Mazur's Control Theorem. If G has no elements of order ℓ and contains a closed normal subgroup H such that $G/H \simeq \mathbb{Z}_\ell$, we are able to give sufficient conditions for $Sel_A(K)_\ell^\vee$ to be finitely generated as $\mathbb{Z}_\ell[[H]]$-module and, consequently, a torsion $\mathbb{Z}_\ell[[G]]$-module. We deal with both cases $\ell \neq p$ and $\ell = p$.
We will consider ℓ-adic Lie extensions K/F' (for any finite extension F'/F) and study the kernels and cokernels of the natural restriction maps

$$Sel_A(F')_\ell \longrightarrow Sel_A(K)_\ell^{\mathrm{Gal}(K/F')}$$

in Theorems 2.1.2 and 2.2.3.
We shall do this via the snake lemma applied to the following diagram

$$\begin{tikzcd}
Sel_A(F')_\ell \arrow[r, hook] \arrow[d, "a_{K/F'}"] & H^1_{fl}(\mathcal{X}_{F'}, A[\ell^\infty]) \arrow[r, two heads] \arrow[d, "b_{K/F'}"] & \mathcal{G}_A(F') \arrow[d, "c_{K/F'}"] \\
Sel_A(K)_\ell^{\mathrm{Gal}(K/F')} \arrow[r, hook] & H^1_{fl}(\mathcal{X}_K, A[\ell^\infty])^{\mathrm{Gal}(K/F')} \arrow[r] & \mathcal{G}_A(K)^{\mathrm{Gal}(K/F')}
\end{tikzcd} \tag{2.1}$$

where, for any field L, we put

$$\mathcal{G}_A(L) = Im \left\{ H^1_{fl}(\mathcal{X}_L, A[\ell^\infty]) \to \prod_{w \in \Sigma_L} H^1_{fl}(\mathcal{X}_{L_w}, A[\ell^\infty])/Im\,\kappa_w \right\} .$$

2.1 Control theorem for $\ell \neq p$.

As pointed out by Coates and Greenberg in [12, Proposition 4.1 and the subsequent Remark], the image of the Kummer map κ_w is trivial for any $w \in \Sigma_L$, because L_w has characteristic $p \neq \ell$. Therefore, in this case, the ℓ-part of the Selmer group is simply

$$Sel_A(L)_\ell = Ker \left\{ H^1(L, A[\ell^\infty]) \to \prod_{w \in \Sigma_L} H^1(L_w, A[\ell^\infty]) \right\} .$$

The following lemma holds for any prime ℓ (including $\ell = p$) and we shall need it in most of our control theorems.

Lemma 2.1.1. *Let G be any compact ℓ-adic Lie group of finite dimension and let M be a discrete $\Lambda(G)$-module which is cofinitely generated over $\mathbb{Z}_\ell$ (resp. finite). Then, for any closed subgroup $\mathcal{V}$ of G, the cohomology groups $H^1(\mathcal{V}, M)$ and $H^2(\mathcal{V}, M)$ are cofinitely generated (resp. finite) $\mathbb{Z}_\ell$-modules as well and their coranks (resp. their orders) are bounded independently of $\mathcal{V}$.*

Proof. Obviously we can restrict our attention to pro-ℓ subgroups of G and, by [16, Ch. 1, Exercise 12], any group of this kind is contained in a pro-ℓ Sylow subgroup P_ℓ. Moreover any pro-ℓ Sylow is open by Theorem 1.2.6. Let $\mathcal{V}$ be any closed pro-ℓ subgroup of some pro-ℓ Sylow P_ℓ and put

$$d_i(\mathcal{V}) := \dim_{\mathbb{Z}/\ell\mathbb{Z}} H^i(\mathcal{V}, \mathbb{Z}/\ell\mathbb{Z}) \qquad i = 1, 2 .$$

Since P_ℓ has finite rank (in the sense of Definition 1.2.4) the cardinalities of a minimal set of topological generators for $\mathcal{V}$, i.e., the $d_1(\mathcal{V})$'s, are all finite and bounded by $d_1(P_\ell)$. Moreover, since P_ℓ contains a uniform open subgroup $\mathcal{U}$ (by Theorem 1.2.6), one has

$$d_1(\mathcal{V}) \leqslant d_1(\mathcal{U}) = d := \text{dimension of } G$$

for all closed subgroups $\mathcal{V}$ of $\mathcal{U}$. For $d_2(\mathcal{V})$ (i.e., the numbers of relations for a minimal set of topological generators of $\mathcal{V}$) the bound is provided by [16, Theorem 4.35] (see also [16, Ch. 4, Exercise 11]) again only in terms of the dimension and rank of G (for example, if $\mathcal{U}$ is as above, then $d_2(\mathcal{U}) = \frac{d(d-1)}{2}$). We put $\tilde{d}_1 = \tilde{d}_1(G)$ (resp. $\tilde{d}_2 = \tilde{d}_2(G)$) as the upper bound for all the $d_1(\mathcal{V})$'s (resp. $d_2(\mathcal{V})$'s) as $\mathcal{V}$ varies among the closed subgroups of any pro-ℓ Sylow of G.
Let M_{div} be the maximal divisible subgroup of M and consider the finite quotient M/M_{div}. By [43, Ch. I, §4, Proposition 20] and the Corollaire right after it, the finite group M/M_{div} admits a $\mathcal{V}$-composition series with

quotients isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. Hence, working as in [18, Proposition 3.1], one immediately finds

$$|H^i(\mathcal{V}, M/M_{div})| \leqslant |M/M_{div}|^{d_i(\mathcal{V})} \leqslant |M/M_{div}|^{\tilde{d}_i}$$

for $i = 1, 2$.
Note that, if M is finite (hence equal to M/M_{div}), we have already completed the proof of the finiteness of the $H^i(\mathcal{V}, M)$.
Moreover, in this case the orders are also bounded independently of $\mathcal{V}$.

To deal with the divisible part note that $M_{div}[\ell]$ is finite (say of order ℓ^λ, where $\lambda = \mathrm{corank}_{\mathbb{Z}_\ell} M$), thus, by what we have just proved,

$$|H^i(\mathcal{V}, M_{div}[\ell])| \leqslant \ell^{\lambda \tilde{d}_i} \quad (i = 1, 2) \ .$$

The cohomology of the exact sequence

$$M_{div}[\ell] \hookrightarrow M_{div} \stackrel{\ell}{\twoheadrightarrow} M_{div}$$

yields surjective morphisms

$$H^i(\mathcal{V}, M_{div}[\ell]) \twoheadrightarrow H^i(\mathcal{V}, M_{div})[\ell] \quad (i = 1, 2) \ ,$$

therefore the groups on the right have finite and bounded orders as well.
Hence the $H^i(\mathcal{V}, M_{div})$ are cofinitely generated $\mathbb{Z}_\ell$-modules with coranks bounded by

$$\tilde{d}_i \lambda = \tilde{d}_i \mathrm{corank}_{\mathbb{Z}_\ell} M$$

(which is a bound for the whole $H^i(\mathcal{V}, M)$ since the $\mathbb{Z}_\ell$-corank of the finite part is 0). □

Theorem 2.1.2. *With the above notations, for every finite extension F' of F contained in K, the kernels and cokernels of the maps*

$$a_{K/F'} : Sel_A(F')_\ell \to Sel_A(K)_\ell^{\mathrm{Gal}(K/F')}$$

are cofinitely generated $\mathbb{Z}_\ell$-modules (i.e., their Pontrjagin duals are finitely generated $\mathbb{Z}_\ell$-modules). If all primes in S and all primes of bad reduction have decomposition groups open in G, then the coranks of kernels and cokernels are bounded independently of F'. Moreover if $A[\ell^\infty](K)$ is finite, then such kernels and cokernels are of finite order.

Proof. We shall use the snake lemma for the diagram (2.1): hence, to prove the theorem, we are going to bound the kernels and cokernels of the maps $b_{K/F'}$ and $c_{K/F'}$.

The maps $b_{K/F'}$ From the Hochschild-Serre spectral sequence we have that

1. $Ker(b_{K/F'}) \simeq H^1(K/F', A[\ell^\infty](K))$
2. $Coker(b_{K/F'}) \subseteq H^2(K/F', A[\ell^\infty](K))$.

We can apply Lemma 2.1.1 with $G = \mathrm{Gal}(K/F)$ and $M = A[\ell^\infty](K)$. Hence $Ker(b_{K/F'})$ and $Coker(b_{K/F'})$ are cofinitely generated $\mathbb{Z}_\ell$-modules whose coranks are bounded by

$$\tilde{d}_1(\mathrm{Gal}(K/F))\mathrm{corank}_{\mathbb{Z}_\ell} A[\ell^\infty](K) \quad \text{and} \quad \tilde{d}_2(\mathrm{Gal}(K/F))\mathrm{corank}_{\mathbb{Z}_\ell} A[\ell^\infty](K)$$

respectively. Moreover, if $A[\ell^\infty](K)$ is finite, we have the bound

$$|H^i(\mathcal{V}, A[\ell^\infty](K))| \leqslant |A[\ell^\infty](K)|^{\tilde{d}_i(\mathrm{Gal}(K/F))}$$

(for $i = 1, 2$).

The maps $c_{K/F'}$ For every prime $v \in \Sigma_F$, let v' be a place of F' lying above v. Observe that

$$Ker(c_{K/F'}) \hookrightarrow \prod_{v' \in \Sigma_{F'}} \bigcap_{\substack{w \in \Sigma_K \\ w | v'}} Ker(d_w)$$

where

$$d_w : H^1(F'_{v'}, A[\ell^\infty]) \to H^1(K_w, A[\ell^\infty])$$

and, by the Inf-Res sequence,

$$Ker(d_w) = H^1(K_w/F'_{v'}, A[\ell^\infty](K_w)) \ .$$

From the Kummer exact sequence one can write the following diagram

$$\begin{array}{ccc} H^1(F'_{v'}, A[\ell^\infty]) & \hookrightarrow & H^1(F'_{v'}, A(F_v^s)) \\ \downarrow{\scriptstyle d_w} & & \downarrow{\scriptstyle f_w} \\ H^1(K_w, A[\ell^\infty]) & \hookrightarrow & H^1(K_w, A(F_v^s)) \end{array}$$

and deduce from it the inclusion

$$Ker(d_w) \hookrightarrow Ker(f_w) \simeq H^1(K_w/F'_{v'}, A(K_w)) \ .$$

Since this last group is obviously trivial if the prime v' splits completely we limit ourselves to the study of this local kernels for primes which are not totally split in K/F' .

If v is a prime of good reduction for A and unramified in K/F', then [33, Ch. I, Proposition 3.8] yields

$$H^1((F'_{v'})^{unr}/F'_{v'}, A((F'_{v'})^{unr})) = 0 \ .$$

Via the inflation map one immediately gets $H^1(K_w/F'_{v'}, A(K_w)) = 0$.
Thus we are left with

$$Ker(c_{K/F'}) \hookrightarrow \prod_{\substack{v' \in \Sigma_{F'} \\ v' | v \in S^*}} \bigcap_{\substack{w \in \Sigma_K \\ w | v'}} Ker(d_w)$$

where S^* is the finite set composed by:

- all primes in S;
- all primes not in S and of bad reduction for A.

To find bounds for these primes we shall use Tate's theorems on (local) duality and (local) Euler-Poincaré characteristic.
By [34, Theorem 7.1.8], the group $H^1(F'_{v'}, A[\ell^\infty])$ is finite and we can bound its order using the Euler characteristic

$$\chi(F'_{v'}, A[\ell^\infty]) := \frac{|H^0(F'_{v'}, A[\ell^\infty])||H^2(F'_{v'}, A[\ell^\infty])|}{|H^1(F'_{v'}, A[\ell^\infty])|} \ .$$

From the pairing on cohomology induced by the Weil pairing (see, for example, [33, Ch. I, Remark 3.5]), one has that the group $H^2(F'_{v'}, A[\ell^\infty])$ is the (Pontrjagin) dual of $H^0(F'_{v'}, A^t[\ell^\infty])$, so all the orders in the formula are finite. Moreover, by [34, Theorem 7.3.1], $\chi(F'_{v'}, A[\ell^\infty]) = 1$ [1], therefore

$$|H^1(F'_{v'}, A[\ell^\infty])| = |A[\ell^\infty](F'_{v'})||A^t[\ell^\infty](F'_{v'})| \ .$$

The inflation map

$$H^1(K_w/F'_{v'}, A[\ell^\infty](K_w)) \hookrightarrow H^1(F'_{v'}, A[\ell^\infty])$$

provides again the finiteness of the kernels (but note that here, in general, the orders are not bounded). $\square$

[1] The statement is for finite modules (i.e., it's true for all the modules $A[\ell^n]$ for any n), but limits are allowed here since the numerator stabilizes, i.e., there is an n such that

$$|A[\ell^m](F'_{v'})||A^t[\ell^m](F'_{v'})| = |A[\ell^n](F'_{v'})||A^t[\ell^n](F'_{v'})|$$

for all $m \geqslant n$.

Remarks 2.1.3.

1. In the previous theorem (and in all control theorems which will follow, i.e., Theorems 2.1.7, 2.1.15, 2.2.3 and 2.2.8) we do not require any assumption on the reduction of A outside S. Hypotheses of that kind were used, for example, in [36, Theorem 1.9].

2. Note that the local kernels are always finite, the additional hypothesis on the finiteness of $A[\ell^\infty](K)$ was only used to bound the orders of $Ker(b_{K/F'})$ and $Coker(b_{K/F'})$.

3. Most of the bounds are independent of F' (in particular the ones for $Ker(a_{K/F'})$), but to get uniform bounds for $Coker(a_{K/F'})$ one also needs to bound the number of nontrivial groups appearing in the product which contains $Ker(c_{K/F'})$. In particular one needs finitely many nontrivial $Ker(d_w)$'s and the only way to get this is assuming that the decomposition groups are open. Note that if there is at least one unramified prime of bad reduction such hypothesis on its decomposition group immediately yields that G contains a subgroup isomorphic to $\mathbb{Z}_\ell$ with finite index.

We can be a bit more precise with the local bounds for the unramified primes of bad reduction thanks to the following

Proposition 2.1.4. *If v is an unramified prime of bad reduction (with nontrivial decomposition group) and $v'|v$, then $H^1(K_w/F'_{v'}, A[\ell^\infty](K_w))$ is finite and its order is bounded independently of F'.*

Proof. The ℓ-part of the Galois group $\mathrm{Gal}(K_w/F'_{v'})$ is a finite cyclic ℓ-group or it is isomorphic to $\mathbb{Z}_\ell$. Moreover

$$A[\ell^\infty](K_w)^{\mathrm{Gal}(K_w/F'_{v'})} = A[\ell^\infty](F'_{v'})$$

is finite, so, in the $\mathbb{Z}_\ell$ case (i.e., when K_w contains $(F'_{v'})^{\ell,unr}$ the unramified $\mathbb{Z}_\ell$-extension of $F'_{v'}$), we can apply [4, Remark 3.5] to get

$$|H^1(K_w/F'_{v'}, A[\ell^\infty](K_w))| \leqslant |A[\ell^\infty](K_w)/A[\ell^\infty](K_w)_{div}| \ .$$

A similar bound (independent from F') holds for the finite case as well. One just uses the inflation map to $H^1(K_w(F'_{v'})^{\ell,unr}/F'_{v'}, A[\ell^\infty](K_w(F'_{v'})^{\ell,unr}))$ which has order bounded by $|A[\ell^\infty](K_w(F_v)^{\ell,unr})/A[\ell^\infty](K_w(F_v)^{\ell,unr})_{div}|$. □

We summarize the given bounds with the following

Corollary 2.1.5. *In the setting of Theorem 2.1.2 assume that all primes in S and all primes of bad reduction have decomposition groups open in G, then one has:*

(i) $\mathrm{corank}_{\mathbb{Z}_\ell} Ker(a_{K/F'}) \leqslant \tilde{d}_1 \mathrm{corank}_{\mathbb{Z}_\ell} A[\ell^\infty](K)$ *and*
$\mathrm{corank}_{\mathbb{Z}_\ell} Coker(a_{K/F'}) \leqslant \tilde{d}_2 \mathrm{corank}_{\mathbb{Z}_\ell} A[\ell^\infty](K)$ *;*

(ii) if $A[\ell^\infty](K)$ is finite, then $|Ker(a_{K/F'})| \leqslant |A[\ell^\infty](K)|^{\tilde{d}_1}$ and

$$|Coker(a_{K/F'})| \leqslant |A[\ell^\infty](K)|^{\tilde{d}_2} \prod_{v'|v \in S^*-S} \alpha_v \prod_{v'|v\in S} \beta_{v'}$$

(where

$$\alpha_v = |A[\ell^\infty](K_w(F_v)^{\ell,unr})/A[\ell^\infty](K_w(F_v)^{\ell,unr})_{div}|$$

and

$$\beta_{v'} = |A[\ell^\infty](F'_{v'})||A^t[\ell^\infty](F'_{v'})|\ ) \ ;$$

(iii) if $A[\ell^\infty](K)$ is finite and, for all primes $v \in S$, $A[\ell^\infty](K_w)$ is finite for any prime $w|v'|v$, then $|Ker(a_{K/F'})| \leqslant |A[\ell^\infty](K)|^{\tilde{d}_1}$ and

$$|Coker(a_{K/F'})| \leqslant |A[\ell^\infty](K)|^{\tilde{d}_2} \prod_{\substack{v'|v \\ v\in S^*\setminus S}} \alpha_v \prod_{\substack{v'|v \\ v\in S}} |A[\ell^\infty](K_w)||A^t[\ell^\infty](K_w)|.$$

The bounds in (i) are independent of F' (while the values appearing in (iii) do not depend on F' but the number of the factors in the product does).

2.1.1 The case $K = F(A[\ell^\infty])$

The finiteness of $A[\ell^\infty]$ is not a necessary condition to get finite kernels and cokernels in the control theorem. An important example is provided by the extension $K = F(A[\ell^\infty])$ (this extensions has been studied in details in [40] in the case $A = \mathcal{E}$, an elliptic curve). Fix a basis of $A[\ell^\infty]$, denote by $T_\ell(A) := \varprojlim_n A[\ell^n]$ the ℓ-adic Tate module of A and consider the continuous Galois representation

$$\rho : \ \mathrm{Gal}(F^s/F) \to \mathrm{Aut}(T_\ell(A)) \simeq GL_{2g}(\mathbb{Z}_\ell)$$

provided by the action of $\mathrm{Gal}(F^s/F)$ on the chosen basis. Since $Ker(\rho)$ is given by the automorphisms which fix $A[\ell^\infty]$, one gets an isomorphism

$$\mathrm{Gal}(K/F) \simeq \rho(\mathrm{Gal}(F^s/F))$$

and, consequently, an embedding

$$\mathrm{Gal}(K/F) \hookrightarrow GL_{2g}(\mathbb{Z}_\ell) \ .$$

Since $\mathrm{Gal}(F^s/F)$ is compact, its image under ρ must be a compact subgroup of $GL_{2g}(\mathbb{Z}_\ell)$. As the latter is Hausdorff, $\rho(\mathrm{Gal}(F^s/F))$ is closed and so it is an ℓ-adic Lie group.

Remark 2.1.6. When A has genus 1 (i.e., it is an elliptic curve), a theorem of Igusa (analogous to Serre's open image theorem) gives a more precise description of $\mathrm{Gal}(K/F)$ (for a precise statement and a proof see [5] and the references there). For a general abelian variety such *open image* statements are not known: some results in this direction for abelian varieties of "Hall type" can be found in [19] and [1] .

Theorem 2.1.7. *Let $K = F(A[\ell^\infty])$, then the kernels and cokernels of the maps*

$$a_{K/F'} : Sel_A(F')_\ell \to Sel_A(K)_\ell^{\mathrm{Gal}(K/F')}$$

are finite.

Proof. Observe that thanks to [44, Corollary 2 (b), p. 497] only primes of bad reduction for A are ramified in the extension $F(A[\ell^\infty])/F$; so, in this case, the set S is obviously finite and there are no unramified primes of bad reduction.
The bounds for $Ker(c_{K/F'})$ are already in Theorem 2.1.2, so we only need to provide bounds for $Ker(b_{K/F'})$ and $Coker(b_{K/F'})$, i.e., for $H^1(K/F', A[\ell^\infty])$ and $H^2(K/F', A[\ell^\infty])$ (note that here $A[\ell^\infty](K) = A[\ell^\infty]$).
Put

$$V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \ ,$$

then one has an exact sequence (of Galois modules)

$$T_\ell(A) \hookrightarrow V_\ell(A) \twoheadrightarrow A[\ell^\infty] \ .$$

By [42, Théorème 2] and the subsequent Corollaire (which hold in our setting as well, as noted in the Remarques following the Corollaire), one has $H^i(K/F', V_\ell(A)) = 0$ for any $i \geqslant 0$, moreover the $H^i(K/F', T_\ell(A))$ are all finite groups. Hence we get isomorphism

$$H^i(K/F', T_\ell(A)) \simeq H^{i-1}(K/F', A[\ell^\infty]) \quad \text{for any } i \geqslant 1$$

which provide the finiteness of $Ker(b_{K/F'})$ and $Coker(b_{K/F'})$. $\square$

2.1.2 Λ-modules for $\ell \neq p$

In this section we assume that our Galois group G (still an ℓ-adic Lie group) has no elements of order ℓ and write $\Lambda(G)$ for the associated Iwasawa algebra. First we describe the structure of $Sel_A(K)_\ell^\vee$ as a $\Lambda(G)$-module, showing that it is a finitely generated (sometimes torsion) $\Lambda(G)$-module. Then, assuming that G contains a subgroup H such that $G/H \simeq \mathbb{Z}_\ell$, we will show that $Sel_A(K)_\ell^\vee$ is finitely generated as a $\Lambda(H)$-module as well (this is equivalent to prove that $Sel_A(K)_\ell^\vee$ is $\mathscr{S}$-torsion, in the language of Section 1.5). For the latter we shall need a slightly modified version of the previous Theorem 2.1.2. As usual the main tool for the proof (along with the control theorem) is the following generalization of Nakayama's Lemma.

Theorem 2.1.8. *Let G be a topologically finitely generated, powerful and pro-ℓ group and I any proper ideal. Let M be a compact $\Lambda(G)$-module, then:*

(i) if M/IM is finitely generated as a $\Lambda(G)/I$-module, then M is finitely generated as a $\Lambda(G)$-module;

(ii) if G is soluble uniform and $M/I_G M$ is finite, then M is a torsion $\Lambda(G)$-module (where $I_G = Ker\{\Lambda(G) \to \mathbb{Z}_\ell\}$ is the augmentation ideal of $\Lambda(G)$).

Proof. See the main results of [2]. □

Let $\mathbb{F}_p^{(\ell)}$ be the unramified $\mathbb{Z}_\ell$-extension of $\mathbb{F}_p$. One of the most (arithmetically) interesting example is provided by extensions K/F containing $\mathbb{F}_p^{(\ell)}F$, where one can take $H = \mathrm{Gal}(K/\mathbb{F}_p^{(\ell)}F)$. This can be considered as a very general setting thanks to the following lemma.

Lemma 2.1.9. *Let K/F be a ℓ-adic Lie extension with $\ell \neq 2$ and p, then there exists a field $K' \supseteq K$ such that*

(i) K' contains $\mathbb{F}_p^{(\ell)}F$;

(ii) K'/F is unramified outside a finite set of places;

(iii) $\mathrm{Gal}(K'/F)$ is a compact ℓ-adic Lie group without elements of order ℓ.

Proof. Since $\ell \neq p$ the proof is the same of [9, Lemma 6.1] (where the statement is for number fields). One basically considers the field $K(\boldsymbol{\mu}_{\ell^\infty})$ (which obviously contains $\mathbb{F}_p^{(\ell)}F$), where $\boldsymbol{\mu}_{\ell^\infty}$ is the set of all ℓ-power roots of unity, and then cuts out elements of order ℓ in $\mathrm{Gal}(K(\boldsymbol{\mu}_{\ell^\infty})/F(\boldsymbol{\mu}_{\ell^\infty}))$ using Kummer theory to describe generators for subextensions of degree ℓ. □

Structure of $Sel_A(K)_\ell^\vee$ as $\Lambda(G)$-module

We are now ready to prove the following

Theorem 2.1.10. *In the setting of Theorem 2.1.2, $Sel_A(K)_\ell^\vee$ is a finitely generated $\Lambda(G)$-module.*

Proof. Consider any open, powerful and pro-ℓ subgroup G' of G. Since $\Lambda(G)$ is finitely generated over $\Lambda(G')$ it is obvious that $Sel_A(K)_\ell^\vee$ is finitely generated over $\Lambda(G)$ if and only if it is finitely generated also over $\Lambda(G')$. So we are going to prove the statement for such G'.
Consider the exact sequence

$$Coker(a_{K/F'})^\vee \hookrightarrow (Sel_A(K)_\ell^{G'})^\vee \to Sel_A(F')_\ell^\vee \twoheadrightarrow Ker(a_{K/F'})^\vee \qquad (2.2)$$

where F' is the fixed field of G'. We know from Theorem 2.1.2 that the modules $Coker(a_{K/F'})^\vee$ and $Ker(a_{K/F'})^\vee$ are finitely generated $\mathbb{Z}_\ell$-modules. Moreover, since F'/F is finite, $Sel_A(F')_\ell^\vee$ is a finitely generated $\mathbb{Z}_\ell$-module ([33, III.8 and III.9]). Hence $(Sel_A(K)_\ell^{G'})^\vee$ is a finitely generated $\mathbb{Z}_\ell$-module thanks to the exactness of the sequence (2.2). Since $(Sel_A(K)_\ell^{G'})^\vee$ is isomorphic to $Sel_A(K)_\ell^\vee/I_{G'}Sel_A(K)_\ell^\vee$, where $I_{G'}$ is the augmentation ideal, our claim follows from Theorem 2.1.8. $\square$

Remark 2.1.11. Note that in the above proof we do not need any hypothesis on the elements of G of order ℓ: it works in general for any compact ℓ-adic Lie group G. That additional hypothesis is necessary only to prove that $Sel_A(K)_\ell^\vee$ is a torsion module, because we need to avoid zero divisors.

Theorem 2.1.12. *Suppose that there exists an open uniform, pro-ℓ and soluble subgroup G' of G, with fixed field F'. Assume that $A[\ell^\infty](K)$ and $Sel_A(F')_\ell^\vee$ are finite. Then $Sel_A(K)_\ell^\vee$ is a torsion $\Lambda(G')$-module.*

Proof. Just use Theorems 2.1.2 and 2.1.8. $\square$

Remark 2.1.13. The hypothesis on the existence of the soluble subgroup G' is necessary. Indeed, when G' is not soluble it is possible to find a non torsion ideal J of $\Lambda(G')$ such that $J/I_{G'}J$ is finite (see [2, p. 228]). However, we observe that when G is finitely generated (not only "topologically" finitely generated) such an open soluble subgroup G' always exists (see [28]).

In the context of non-commutative Iwasawa algebras the right definition of *torsion module* ([50, Definition 2.6]) can be stated in the following way: *a finitely generated $\Lambda(G)$-module M is a $\Lambda(G)$-torsion module if and only if M is a $\Lambda(G')$-torsion module (classical meaning) for some open pro-ℓ subgroup $G' \subseteq G$ such that $\Lambda(G')$ is integral.* So Theorem 2.1.12 immediately yields

Corollary 2.1.14. *Let G be without elements of order ℓ and suppose that there exists an open, uniform, pro-ℓ and soluble subgroup G' of G. If the groups $A[\ell^\infty](K)$ and $Sel_A(F')_\ell^\vee$ (where F' is the fixed field of G') are finite, then $Sel_A(K)_\ell^\vee$ is a torsion $\Lambda(G)$-module.*

Structure of $Sel_A(K)_\ell^\vee$ as $\Lambda(H)$-module

Assume that G contains a closed normal subgroup H such that $G/H = \Gamma \simeq \mathbb{Z}_\ell$. We are going to prove that $Sel_A(K)_\ell^\vee$ is a finitely generated $\Lambda(H)$-module under some mild condictions, as predicted by Conjecture 5.1 of [11]. First note that, letting $\mathbb{F}_p^{(\ell)}$ be the unique $\mathbb{Z}_\ell$-extension of $\mathbb{F}_p$, by [4, Proposition 4.3], one has $K' := K^H = \mathbb{F}_p^{(\ell)}F$. Hence all primes of F are unramified in K' and none of them is totally split.

As mentioned before we need to prove a slightly modified version of the Control Theorem. We will work with the following diagram

$$\begin{tikzcd} Sel_A(K')_\ell \arrow[r, hook] \arrow[d, "a"] & H^1(K', A[\ell^\infty]) \arrow[r, two heads] \arrow[d, "b"] & \mathcal{G}_A(K') \arrow[d, "c"] \\ Sel_A(K)_\ell^H \arrow[r, hook] & H^1(K, A[\ell^\infty])^H \arrow[r] & \mathcal{G}_A(K)^H \end{tikzcd} \tag{2.3}$$

similar to the diagram (2.1) except for the "infinite level" of the upper row, and we will again apply the snake lemma.

Theorem 2.1.15. *With the above notations the kernel and cokernel of the map*

$$a : Sel_A(K')_\ell \to Sel_A(K)_\ell^H$$

are cofinitely generated $\mathbb{Z}_\ell$-modules. Moreover, if $A[\ell^\infty](K)$ is finite and, for any $w|w'|v \in S$, the group $A[\ell^\infty](K_w)$ is finite as well, then $Ker(a)$ and $Coker(a)$ are finite.

Proof. As usual we are going to work on kernels and cokernels of the maps b and c in the diagram (2.3).

The map b From the Hochschild-Serre spectral sequence we have that

$$Ker(b) \simeq H^1(K/K', A[\ell^\infty](K)) \quad \text{and} \quad Coker(b) \subseteq H^2(K/K', A[\ell^\infty](K)) \ .$$

One simply observes that $\mathrm{Gal}(K/K')$ is still an ℓ-adic Lie group and then applies Lemma 2.1.1. Indeed, all arguments are group theoretic and they do not depend on the base field K' being a global field or not. Hence $Ker(b)$ and $Coker(b)$ are cofinitely generated $\mathbb{Z}_\ell$-modules and are finite if $A[\ell^\infty](K)$ is finite.

The map c For every prime $v \in \Sigma_F$ let w' be a place of K' lying above v. As in Section 2.1, from the Inf-Res sequence, one gets

$$Ker(d_w) = H^1(K_w/K'_{w'}, A[\ell^\infty](K_w)) \ .$$

Since $\mathrm{Gal}(K_w/K'_{w'})$ is an ℓ-adic Lie group, every $Ker(d_w)$ is a cofinitely generated $\mathbb{Z}_\ell$-module. The result for $Ker(c)$ will follow from $Ker(d_w) = 0$ for all but finitely many primes. As before the Kummer sequence provides the following diagram

$$\begin{tikzcd} H^1(K'_{w'}, A[\ell^\infty]) \arrow[r, hook] \arrow[d, "d_w"] & H^1(K'_{w'}, A(F_v^s)) \arrow[d, "f_w"] \\ H^1(K_w, A[\ell^\infty]) \arrow[r, hook] & H^1(K_w, A(F_v^s)) \end{tikzcd}$$

and, from it, one has the inclusion

$$Ker(d_w) \hookrightarrow Ker(f_w) \simeq H^1(K_w/K'_{w'}, A(K_w))$$

or, more precisely, the isomorphism

$$Ker(d_w) \simeq H^1(K_w/K'_{w'}, A(K_w))[\ell^\infty] \ .$$

Hence we are not going to consider places which are totally split in K/K', because they obviously provide $Ker(d_w) = 0$.

Unramified primes Let v be unramified in the extension K/F, then, since $\mathrm{Gal}(K'_{w'}/F_v) \simeq \mathbb{Z}_\ell$, $K'_{w'}$ is the maximal unramified pro-ℓ-extension of F_v and the ℓ-part of $\mathrm{Gal}(K_w/K'_{w'})$ is trivial. Therefore the ℓ-part of the (torsion) module $H^1(K_w/K'_{w'}, A(K_w))$ is trivial as well.

Because of the splitting of primes in K' , we are already left with finitely many places. So $Ker(c)$ is a cofinitely generated $\mathbb{Z}_\ell$-module and the first statement on $Ker(a)$ and $Coker(a)$ being cofinitely generated over $\mathbb{Z}_\ell$ is proved.

Ramified primes Let $v \in S$ and recall that $K'_{w'}$ is not a local field anymore. If for any $w|w'|v$ the group $A[\ell^\infty](K_w)$ is finite, then by Lemma 2.1.1 $H^1(K_w/K'_{w'}, A[\ell^\infty](K_w))$ is finite. □

Corollary 2.1.16. *In the setting of Theorem 2.1.15, one has*

(i) if $Sel_A(K')_\ell$ is a cofinitely generated $\mathbb{Z}_\ell$-module, then $Sel_A(K)_\ell^\vee$ is finitely generated over $\Lambda(H)$;

(ii) suppose that there exists an open, soluble, uniform and pro-ℓ subgroup H' of H; if the groups $Sel_A(K^{H'})_\ell$ and $A[\ell^\infty](K)$ are finite and for any $w|w'|v \in S$ also $A[\ell^\infty](K_w)$ is finite, then $Sel_A(K)_\ell^\vee$ is a torsion $\Lambda(H)$-module.

Proof. The arguments are the same we used for the analogous results for $\Lambda(G)$-modules. □

Remark 2.1.17. To get $Sel_A(K)_\ell^\vee$ finitely generated over $\Lambda(G)$ we only need to assume that finitely many primes of F are ramified in the Lie extension K/F. Moreover if $Sel_A(\mathbb{F}_p^{(\ell)}F)_\ell$ is a cofinitely generated $\mathbb{Z}_\ell$-module, then $Sel_A(K)_\ell^\vee$ is finitely generated also over $\Lambda(H)$. Since H has infinite index in G, being finitely generated over $\Lambda(G)$ and $\Lambda(H)$ implies that $Sel_A(K)_\ell^\vee$ is $\Lambda(G)$-torsion (assuming that G does not contain any element of order ℓ). So this is another way to obtain torsion $\Lambda(G)$-modules without assuming the finiteness of $A[\ell^\infty](K)$ (which we needed in Corollary 2.1.14

and is obviously false, for example, for $K = F(A[\ell^\infty])$). Moreover, as mentioned in Section 1.5, proving the finitely generated condition over $\Lambda(G)$ and $\Lambda(H)$ (as done also for all the cases included in Corollary 2.1.16) yields that $Sel_A(K)_\ell^\vee$ is in the category $\mathfrak{M}_H(G)$, i.e., allows us to define a characteristic element for $Sel_A(K)_\ell^\vee$ (without the $\Lambda(H)$-module structure, proving that a module is $\Lambda(G)$-torsion is not enough in the noncommutative case).

Example 2.1.18. Take $K = F(A[\ell^\infty])$ as in Section 2.1.1. This kind of extension realizes naturally most of our assumptions. First of all, from [44], S is just the set of places of bad reduction for A and it is obviously finite (moreover $S = S^*$ in the notations of Section 2.1). As a consequence, by Theorem 2.1.10 we get $Sel_A(K)_\ell^\vee$ always finitely generated over $\Lambda(G)$. Then, since $\mathrm{Gal}(K/F)$ embeds in $GL_{2g}(\mathbb{Z}_\ell)$, it is easy to see that for $\ell > 2g+1$ the Galois group contains no elements of order ℓ so it makes sense to look for torsion modules. By the Weil-pairing we can take H such that K'/F is the unramified $\mathbb{Z}_\ell$-extension of the constant field of F. Because of our choice of H, primes in K' above those in S are finitely many. So, thanks to Theorem 2.1.15 if $Sel_A(K')_\ell^\vee$ is finitely generated over $\mathbb{Z}_\ell$, then $Sel_A(K)_\ell^\vee$ is also finitely generated over $\Lambda(H)$ (hence $\Lambda(G)$-torsion).
One can provide examples of Selmer groups $Sel_A(K')_\ell$ cofinitely generated over $\mathbb{Z}_\ell$ in the work of Pacheco ([38, Proposition 3.6], which generalizes to abelian varieties the analogous statement of Ellenberg in [17, Proposition 2.5] for elliptic curves). For more details on the application of Ellenberg's results to the noncommutative setting of $F(A[\ell^\infty])/F$ (like computations of coranks and Euler characteristic of Selmer groups) see Sechi's (unpublished) PhD thesis [40].

2.2 Control theorem for $\ell = p$

In order to work with the p-torsion part we need to use flat cohomology as mentioned in Section 1.4. We will work with diagram (2.1) but, as we will see, the kernels and cokernels appearing in the snake lemma sequence will still be described in terms of Galois cohomology groups. The main difference is provided by the fact that the images of the local Kummer maps will be nontrivial.
To handle the local kernels we shall need the following

Lemma 2.2.1. *Let E be a local function field and let L/E be an (infinite) p-adic Lie extension. Let A be an abelian variety defined over E with good ordinary reduction. Let I be the inertia group in* $\mathrm{Gal}(L/E)$ *and assume it is nontrivial. Then $H^1(L/E, A(L))$ (Galois cohomology group) is a cofinitely generated $\mathbb{Z}_p$-module. Moreover, if I has finite index (i.e., it is open) in* $\mathrm{Gal}(L/E)$*, then $H^1(L/E, A(L))$ is finite.*

Proof. Let $\widehat{A}$ (resp. $\overline{A}$) be the formal group associated to A (resp. the reduction of A at the prime of E). Because of good reduction the natural map $A(E') \to \overline{A}(\mathbb{F}_{E'})$ is surjective for any extension E'/E. Hence the sequence

$$\widehat{A}(\mathcal{O}_L) \hookrightarrow A(L) \twoheadrightarrow \overline{A}(\mathbb{F}_L)$$

is exact. Taking $\mathrm{Gal}(L/E)$ cohomology (and recalling that $A(E) \to \overline{A}(\mathbb{F}_E)$ is surjective), one gets

$$H^1(L/E, \widehat{A}(\mathcal{O}_L)) \hookrightarrow H^1(L/E, A(L)) \to H^1(L/E, \overline{A}(\mathbb{F}_L))$$

and we will focus on the right and left terms of this sequence from now on. By [47, Theorem 2 (a)], one has an isomorphism

$$H^1(E, \widehat{A}(\mathcal{O}_{\overline{E}})) \simeq Hom(\overline{A^t}[p^\infty](\mathbb{F}_E), \mathbb{Q}_p/\mathbb{Z}_p)$$

(the statement of the theorem requires a $\mathbb{Z}_p^d$-extension but part (a) holds independently of that). Therefore the inflation map provides an inclusion

$$H^1(L/E, \widehat{A}(\mathcal{O}_L)) \hookrightarrow Hom(\overline{A^t}[p^\infty](\mathbb{F}_E), \mathbb{Q}_p/\mathbb{Z}_p)$$

and, since $\overline{A}$ and $\overline{A^t}$ are isogenous, the term on the right is finite of order $|\overline{A^t}[p^\infty](\mathbb{F}_E)| = |\overline{A}[p^\infty](\mathbb{F}_E)|$.

For $H^1(L/E, \overline{A}(\mathbb{F}_L))$ we consider two cases:

Case 1: I is open in $\mathrm{Gal}(L/E)$**.** Just considering p-parts we can assume that I is open in a p-Sylow P_p of $\mathrm{Gal}(L/E)$ and we let L^I be its fixed field. Let $\overline{P_p} := \mathrm{Gal}(L^I/E)$ (a finite group) and note that, since L/L^I is a totally ramified extension, one has that $\mathbb{F}_L = \mathbb{F}_{L^I}$ is still a finite field. The Inf-Res sequence reads as

$$H^1(L^I/E, \overline{A}(\mathbb{F}_{L^I})) \hookrightarrow H^1(L/E, \overline{A}(\mathbb{F}_L)) \to H^1(L/L^I, \overline{A}(\mathbb{F}_L)) \ .$$

The group on the left is obviously finite and for the one on the right we can use [18, Proposition 3.1] as done previously in Lemma 2.1.1 (because I is still a p-adic Lie group) to get

$$|H^1(L/L^I, \overline{A}(\mathbb{F}_L))| \leqslant |\overline{A}(\mathbb{F}_L))|^{\tilde{d}_1}$$

where the exponent $\tilde{d}_1 = \tilde{d}_1(\mathrm{Gal}(L/E))$ only depends on $\mathrm{Gal}(L/E)$.

Case 2: I has infinite index in $\mathrm{Gal}(L/E)$**.** We use the same sequence

$$H^1(L^I/E, \overline{A}(\mathbb{F}_{L^I})) \hookrightarrow H^1(L/E, \overline{A}(\mathbb{F}_L)) \to H^1(L/L^I, \overline{A}(\mathbb{F}_L))$$

but now $\mathbb{F}_L$ is not a finite field anymore: indeed it contains the $\mathbb{Z}_p$-extension of $\mathbb{F}_E$ (because unramified extensions come from extensions of the field of constants). For the group on the right we again work as in Lemma 2.1.1 to prove that $H^1(L/L^I, \overline{A}(\mathbb{F}_L))$ is a cofinitely generated $\mathbb{Z}_p$-module. The only difference with case **1** is that now the p-divisible part of $\overline{A}(\mathbb{F}_L)$ might come into play (moreover note that $H^1(L/L^I, \overline{A}(\mathbb{F}_L))$ is a torsion abelian group, hence its p-primary part is exactly $H^1(L/L^I, \overline{A}(\mathbb{F}_L)[p^\infty])$ and $A(\mathbb{F}_L)[p^\infty]$ has finite $\mathbb{Z}_p$-corank). For $H^1(L^I/E, \overline{A}(\mathbb{F}_{L^I}))$ we observe that the p-part of $\mathrm{Gal}(L^I/E)$ is isomorphic to $\mathbb{Z}_p$ and that the subgroup of $\overline{A}(\mathbb{F}_{L^I})$ fixed by that p-part is finite. Hence, by [4, Lemma 3.4 and Remark 3.5], one has that

$$|H^1(L^I/E, \overline{A}(\mathbb{F}_{L^I}))| \leqslant |\overline{A}(\mathbb{F}_{L^I})/(\overline{A}(\mathbb{F}_{L^I}))_{div}|$$

is finite. □

The lemma provides a bound for the order of $H^1(L/E, A(L))$ which (when I is open in $\mathrm{Gal}(L/E)$) can be written in terms of $\overline{A}[p^\infty](\mathbb{F}_E)$ and $\overline{A}(\mathbb{F}_L)$. If the inertia is infinite (i.e., if I has order divisible by arbitrary high powers of p or, as we will say from now on, *has order divisible by* p^∞) we can prove that the inflation map

$$H^1(L/E, \widehat{A}(\mathcal{O}_L)) \hookrightarrow H^1(E, \widehat{A}(\mathcal{O}_{\overline{E}}))$$

is actually an isomorphism but, since this is not going to improve the bound, we decided to keep this statement out of the lemma and we include it here only for completeness (the proof is just a generalization of the one provided for [47, Theorem 2 (b)]).

Proposition 2.2.2. *In the same setting of the previous lemma, assume that I has order divisible by p^∞ , then*

$$H^1(L/E, \widehat{A}(\mathcal{O}_L)) = H^1(E, \widehat{A}(\mathcal{O}_{\overline{E}}))$$

Proof. The inflation map immediately provides one inclusion so we need to prove the reverse one. By [47, Corollary 2.3.3], $H^1(L/E, A(L))$ is the annihilator of $N_{L/E}(A^t(L))$ with respect to the local Tate pairing (where $N_{L/E}$ is the natural norm map). This provides an isomorphism

$$H^1(L/E, A(L)) \simeq (A^t(E)/N_{L/E}(A^t(L)))^\vee \ . \tag{2.4}$$

Working as in [47, Section 2] (in particular subsections 2.3 and 2.6 in which the results apply to general Galois extensions), one verifies that $H^1(E, \widehat{A}(\mathcal{O}_{\overline{E}}))$ is the annihilator of $\widehat{A^t}(\mathcal{O}_E)$ with respect to the local pairing. Moreover, since $\widehat{A}(\mathcal{O}_{\overline{E}})$ is the kernel of the reduction map, one has

$$\begin{aligned} H^1(E, \widehat{A}(\mathcal{O}_{\overline{E}})) \cap H^1(L/E, A(L)) &\subseteq Ker\Big\{H^1(L/E, A(L)) \to H^1(L/E, \overline{A}(\mathbb{F}_L))\Big\} \\ &= H^1(L/E, \widehat{A}(\mathcal{O}_L)) \ . \end{aligned}$$

Therefore it suffices to show $H^1(E, \widehat{A}(\mathcal{O}_{\overline{E}})) \subseteq H^1(L/E, A(L))$ and, because of the isomorphism (2.4), this is equivalent to proving $N_{L/E}(A^t(L)) \subseteq \widehat{A^t}(\mathcal{O}_E)$.
Take $\alpha \in A^t(L)$ and put $x = N_{L/E}(\alpha)$: we can assume that α belongs to the p-part of $A^t(L)$, so that x is in the p-part of $A^t(E)$. Let E' be an intermediate field containing L^I and such that p^∞ divides $[E' : L^I]$. Consider $z = N_{L/E'}(\alpha) \in A^t(E')$ and let $\overline{z}$ be the image of z in $\overline{A^t}(\mathbb{F}_{E'}) = \overline{A^t}(\mathbb{F}_{L^I})$ (which is a torsion group). Hence $\overline{z}$ has finite order, say p^m, and we can find a field E'' between E' and L^I such that $p^m | [E'' : L^I]$. Put $y = N_{L/E''}(\alpha) = N_{E'/E''}(z) \in A^t(E'')$, so that $x = N_{E''/E}(y)$ and note that $\overline{y}$ (the image of y in $\overline{A^t}(\mathbb{F}_{E''}) = \overline{A^t}(\mathbb{F}_{L^I})$) has order dividing p^m. Since $\mathrm{Gal}(E''/L^I)$ fixes $\overline{y}$, one has that $N_{E''/L^I}(\overline{y})$ is trivial in $\overline{A^t}(\mathbb{F}_{L^I})$, i.e., $N_{E''/L^I}(y) \in \widehat{A^t}(\mathcal{O}_{L^I})$ (which is the kernel of the reduction map). Hence

$$x = N_{E''/E}(y) = N_{L^I/E}(N_{E''/L^I}(y)) \in N_{L^I/E}(\widehat{A^t}(\mathcal{O}_{L^I})) \subseteq \widehat{A^t}(\mathcal{O}_E)\ .$$

□

Now we proceed with our control theorem.

Theorem 2.2.3. *Assume that all ramified primes are of good ordinary or split multiplicative reduction. Then, for any finite extension F'/F contained in K, the kernels and cokernels of the map*

$$a_{K/F'} : Sel_A(F')_p \to Sel_A(K)_p^{\mathrm{Gal}(K/F')}$$

are cofinitely generated $\mathbb{Z}_p$-modules. If all primes in S and all primes of bad reduction have decomposition groups open in G, then the coranks of kernels and cokernels are bounded independently of F'. Moreover if the group $A[p^\infty](K)$ is finite, all places in S are of good reduction and have inertia groups open in their decomposition groups, then the kernels and cokernels are finite (of bounded order if the primes of bad reduction have open decomposition group).

Proof. We work with the usual diagram (2.1), where now we cannot substitute flat cohomology with Galois cohomology and in the groups $\mathcal{G}_A(\cdot)$ the images of the local Kummer maps are nontrivial (in general).
Since the map $\mathcal{X}_K \to \mathcal{X}_{F'}$ is a Galois covering with Galois group $\mathrm{Gal}(K/F')$, the Hochschild-Serre spectral sequence applies and we will study

- $Ker(b_{K/F'}) \simeq H^1(K/F', A[p^\infty](K))$;
- $Coker(b_{K/F'}) \subseteq H^2(K/F', A[p^\infty](K))$;
- $Ker(c_{K/F'})$

noting that the cohomology groups on the right are Galois cohomology groups (see [32, III.2.21 (a), (b) and III.1.17 (d)]).

The map $b_{K/F'}$. Just use Lemma 2.1.1.

The maps $c_{K/F'}$. As before we simply work with the maps

$$d_w : H^1_{fl}(X_{F'_{v'}}, A[p^\infty])/Im\,\kappa_{v'} \to H^1_{fl}(X_{K_w}, A[p^\infty])/Im\,\kappa_w \ .$$

From the Kummer sequence one gets a diagram

$$\begin{array}{ccc} H^1_{fl}(X_{F'_{v'}}, A[p^\infty])/Im\,\kappa_{v'} & \hookrightarrow & H^1_{fl}(X_{F'_{v'}}, A)[p^\infty] \\ \downarrow{\scriptstyle d_w} & & \downarrow{\scriptstyle f_w} \\ H^1_{fl}(X_{K_w}, A[p^\infty])/Im\,\kappa_w & \hookrightarrow & H^1_{fl}(X_{K_w}, A)[p^\infty] \ , \end{array}$$

and (from the Inf-Res sequence)

$$Ker(d_w) \hookrightarrow Ker(f_w) \simeq H^1(K_w/F'_{v'}, A(K_w))[p^\infty] \ .$$

Before moving on observe that $K_w/F'_{v'}$ is a p-adic Lie extension because the decomposition group of any place is closed in $\mathrm{Gal}(K/F')$. Now we distinguish two cases depending on the behaviour of the prime in K/F' (and as usual, we do not consider primes which split completely because they give no contribution to $Ker(c_{K/F'})$).

Unramified primes If v' is unramified, from [33, Proposition I.3.8], we have

$$H^1((F'_{v'})^{unr}/F'_{v'}, A((F'_{v'})^{unr})) = H^1((F'_{v'})^{unr}/F'_{v'}, \pi_0(\boldsymbol{A}(F'_{v'})_0))$$

where $\pi_0(\boldsymbol{A}(F'_{v'})_0)$ is the set of connected components of the closed fiber of the Néron model $\boldsymbol{A}(F'_{v'})$ of A at v'. The latter is a finite module and it is trivial when v' is a place of good reduction. Because of the inflation map

$$H^1(K_w/F'_{v'}, A(K_w)) \hookrightarrow H^1((F'_{v'})^{unr}/F'_{v'}, A((F'_{v'})^{unr}))$$

the same holds for $Ker(d_w)$ as well. Note also that in finite unramified (hence cyclic) extensions, since the group $\pi_0(\boldsymbol{A}(F'_{v'})_0)$ is finite, the order of the H^1 is equal to the order of the H^0 which is uniformly bounded (see for example [47, Lemma 3.3.1]). Thus the order of $H^1((F'_{v'})^{unr}/F'_{v'}, \pi_0(\boldsymbol{A}(F'_{v'})_0))$ is bounded independently of F'.

Ramified primes If v' is of good ordinary reduction, just observe that $\mathrm{Gal}(K_w/F'_{v'})$ is a p-adic Lie group with nontrivial inertia and apply Lemma 2.2.1.
We are left with ramified primes of split multiplicative reduction. Let v'

be such a prime. We have the exact sequence coming from Mumford's uniformization

$$< q'_{A,1}, \dots, q'_{A,g} > \hookrightarrow ((\overline{F_v})^*)^g \twoheadrightarrow A(\overline{F_v})$$

(where we recall that g is the dimension of the variety A, the $q'_{A,i}$ are parameters in $(F'_{v'})^*$ and the morphisms behave well with respect to the Galois action).
Taking cohomology (and using Hilbert's Theorem 90) one finds an injection

$$H^1(K_w/F'_{v'}, A(K_w)) \hookrightarrow H^2(K_w/F'_{v'}, < q'_{A,1}, \dots, q'_{A,g} >) \ .$$

Since $q'_{A,i} \in (F'_{v'})^*$, the action of the Galois group is trivial on them and we have an isomorphism of Galois modules $< q'_{A,1}, \dots, q'_{A,g} > \simeq \mathbb{Z}^g$. Hence

$$H^2(K_w/F'_{v'}, < q'_{A,1}, \dots, q'_{A,g} >) \simeq H^2(K_w/F'_{v'}, \mathbb{Z}^g) \simeq ((\mathrm{Gal}(K_w/F'_{v'})^{ab})^\vee)^g$$

(where the last one is the Pontrjagin dual of the maximal abelian quotient of $\mathrm{Gal}(K_w/F'_{v'})$). The last one is a cofinitely generated $\mathbb{Z}_p$-module, since $\mathrm{Gal}(K_w/F'_{v'})^{ab}$ is virtually[2] a finitely generated $\mathbb{Z}_p$-module. Indeed, $\overline{[\mathrm{Gal}(K_w/F'_{v'}), \mathrm{Gal}(K_w/F'_{v'})]}$ is a closed normal subgroup of $\mathrm{Gal}(K_w/F'_{v'})$ and their quotient is still a p-adic Lie group (see [16, Theorem 9.6 (ii)] and, for the $\mathbb{Z}_p$-module structure, [16, Theorem 4.9] and [16, Theorem 4.17]). $\square$

Remarks 2.2.4.

1. The hypotheses on the reduction of A at ramified primes are necessary. Indeed, if v is a ramified prime of good supersingular reduction and $\mathrm{Gal}(K/F) \simeq \mathbb{Z}_p^d$ (a deeply ramified extension in the sense of [12, Section 2]), then K.-S. Tan has shown that $H^1(K_w/F'_{v'}, A(K_w))[p^\infty]$ has infinite $\mathbb{Z}_p$-corank (see [48, Theorem 3.6.1]).

2. When $A = \mathcal{E}$ is an elliptic curve it is easy to see that the number of torsion points of p-power order in a separable extension of F is finite (see, for example, [3, Lemma 4.3]). For a general abelian variety A, in [47, Lemma 2.5.1] K.-S. Tan shows that $A[p^\infty](K) = A[p^\infty](K \cap F^{unr})$. Hence, for a p-adic Lie extension, we are interested only in the number of p-power torsion points in $\mathbb{F}_p^{(p)}F$ (or $\mathbb{F}_p^{(p)}F'$ for some finite unramified extension F' of F).

2.2.1 Λ-modules for $\ell = p$

The Selmer groups are again $\Lambda(G)$-modules and we will investigate their structure: as in the case $l \neq p$, we assume that G does not contain any element of order p.

[2] A profinite group G is said to have a property P *virtually* if G has an open normal subgroup H such that H verifies P.

Structure of $Sel_A(K)_p^\vee$ as $\Lambda(G)$-module

Theorem 2.2.5. *With the above notations, if all places in S are of good ordinary or split multiplicative reduction, then $Sel_A(K)_p^\vee$ is a finitely generated $\Lambda(G)$-module.*

Proof. This is the same proof of Theorem 2.1.10, using Theorems 2.2.3 and 2.1.8. □

Theorem 2.2.6. *Assume that there exists an open soluble, uniform and pro-p subgroup G' of G. Suppose that $A[p^\infty](K)$ and $Sel_A(F')_p^\vee$, where F' is the fixed field of G', are finite. If all ramified places of F are of good ordinary reduction for A and have inertia groups open in their decomposition groups, then $Sel_A(K)_p^\vee$ is a torsion $\Lambda(G)$-module.*

Proof. It is sufficient to show that $Sel_A(K)_p^\vee$ is a torsion $\Lambda(G')$-module (classical meaning). In order to do this just use Theorems 2.2.3 and 2.1.8. □

Structure of $Sel_A(K)_p^\vee$ as $\Lambda(H)$-module

Assume that there exists a closed normal subgroup H in G such that $G/H = \Gamma \simeq \mathbb{Z}_p$ and let K' be its fixed field.
We need the following

Lemma 2.2.7. *With notations as above, let $K' := K^H$ be the fixed field of H. If $v \in \Sigma_F$ is unramified in K/F, then*

(i) v splits completely in K'/F or

(ii) the decomposition group, in $H = \mathrm{Gal}(K/K')$, of any prime w' of K' dividing v is finite and has order prime to p.

Proof. Consider the diagram

$$\begin{tikzcd}
K \arrow[r, dash] \arrow[d, dash, "H"] & K_w \arrow[d, dash] \\
K' \arrow[r, dash] \arrow[d, dash, "\Gamma"] & K'_{w'} \arrow[d, dash] \\
F \arrow[r, dash] \arrow[uu, dash, bend left, "G"] & F_v
\end{tikzcd}$$

where $w|w'|v$.
Assume v is unramified, then, since $\mathrm{Gal}(K_w/F_v)$ is still a p-adic Lie group and the p-part of an unramified p-adic Lie extension of local fields is at most a $\mathbb{Z}_p$-extension, we have to deal with two cases.

1. $\mathrm{Gal}(K_w/F_v)$ *is finite.* Then $\mathrm{Gal}(K'_{w'}/F_v)$ is a finite subgroup of $\Gamma \simeq \mathbb{Z}_p$, hence it is trivial and v splits completely if K'/F (i.e., *(i)* holds).

2. $\mathrm{Gal}(K_w/F_v) \simeq \Gamma$. Then $\mathrm{Gal}(K'_{w'}/F_v)$ can be trivial or isomorphic to Γ. If it is trivial we are back to case *(i)*. If it is Γ, then $K'_{w'}$ is the unramified $\mathbb{Z}_p$-extension of F_v . Hence the extension $K_w/K'_{w'}$ is unramified of (finite) order prime to p (i.e., *(ii)* holds).

□

As in the $\ell \neq p$ case, we will show a slightly modified version of Theorem 2.2.3.

Theorem 2.2.8. *Assume that*

(i) all places in S are of split multiplicative reduction for A;

(ii) all places in S and all places of bad reduction for A split in finitely many primes in K'/F.

Then the map

$$a : Sel_A(K')_p \to Sel_A(K)_p^H$$

has cofinitely generated kernel and cokernel (viewed as $\mathbb{Z}_p$-modules).

Proof. We go directly to the local kernels for places which do not split completely

$$Ker(d_w) \simeq H^1(K_w/K'_{w'}, A(K_w))[p^\infty] \ .$$

Unramified primes of good reduction Let v be an unramified prime of good reduction for A. From Lemma 2.2.7 we know that $K'_{w'} = F_v$ is a local field or $\mathrm{Gal}(K_w/K'_{w'})$ has finite order prime with p. In the first case [33, Ch. I, Proposition 3.8] shows that $H^1(K_w/K'_{w'}, A(K_w)) = 0$ and we get our claim. In the second case the p-part of the torsion module $H^1(K_w/K'_{w'}, A(K_w))$ is obviously trivial.

Because of our hypothesis on the splitting of primes we are already left with finitely many local kernels, now we check the behaviour of the remaining ones.

Unramified primes of bad reduction As above if $\mathrm{Gal}(K'_{w'}/F_v) \simeq \Gamma$ (i.e., $K'_{w'}$ is not a local field), then the p-part of $H^1(K_w/K'_{w'}, A(K_w))$ is trivial. The other case (i.e., $K'_{w'} = F_v$) cannot happen because we are assuming that bad reduction primes do not split completely in K' .

Ramified primes For the ramified ones we use Mumford's parametrization as in the proof of Theorem 2.2.3. First we get an exact sequence

$$< q_{A,1}, \dots, q_{A,g} > \hookrightarrow (K_w^*)^g \twoheadrightarrow A(K_w)$$

(where we can always assume that the periods are in the base field F_v). Then $\mathrm{Gal}(K_w/K'_{w'})$-cohomology provides an injection

$$H^1(K_w/K'_{w'}, A(K_w)) \hookrightarrow H^2(K_w/K'_{w'}, \mathbb{Z}^g) \simeq ((\mathrm{Gal}(K_w/K'_{w'})^{ab})^\vee)^g \ .$$

$\square$

As a consequence we have the following.

Corollary 2.2.9. *In the setting of Theorem 2.2.8, assume that $Sel_A(K')_p^\vee$ is a finitely generated $\mathbb{Z}_p$-module. Then $Sel_A(K)_p^\vee$ is finitely generated over $\Lambda(H)$ (hence torsion over $\Lambda(G)$).*

Final summary Take the function fields K and K' such that the hypothesis on the splitting of primes in Theorem 2.2.3 is verified.
For $Sel_A(K)_p^\vee$ to be finitely generated as $\Lambda(G)$-module we need just to assume that all primes in S are of good ordinary or split multiplicative reduction for A. To move a step further and find $\Lambda(G)$-torsion modules we can assume

1. all ramified primes are of good ordinary reduction and have open inertia groups;

2. $A[p^\infty]$ and $Sel_A(F')_p^\vee$ are finite (where F' is the fixed field of an open soluble, uniform and pro-p subgroup G' of G, if such a subgroup exists);

and use Theorem 2.2.6.
Another way to find $\Lambda(G)$-torsion modules is provided by Corollary 2.2.9. It somehow complements the previous one because it requires a different type of reduction for the ramified places. The assumptions for this case are

3. all ramified primes are of split multiplicative reduction;

4. $Sel_A(K')_p$ is cofinitely generated over $\mathbb{Z}_p$.

One example for this second case is given by the usual arithmetic extension $K' = \mathbb{F}_p^{(p)}F$ for which the hypothesis **1** above obviously does not hold and the hypothesis **3** is proved (in some cases) in [36, Theorem 1.8].

Remark 2.2.10. We found conditions to get $\Lambda(G)$-torsion modules for different types of reduction for the ramified primes, but we remind that, to get a characteristic element for a $\Lambda(G)$-torsion module for a noncommutative group G, one needs to examine the $\Lambda(H)$-module structure as well. Hence our results provide characteristic elements for $Sel_A(K)_p^\vee$ only when the ramified primes are of split multiplicative reduction.

Chapter 3

Pseudo-null submodules and Euler characteristic

In this chapter we will look closely at the Selmer group. In particular, in Section 3.1 we shall prove that, under certain conditions, $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule; while in Section 3.2 we will provide some computations for the Euler characteristic of $Sel_A(K)_\ell$.

The setting and the notations are as explained in Section 1.6.2.

3.1 Pseudo-null submodules

Before stating and proving the main Theorem (see Theorem 3.1.10) we need some intermediate results that will be fundamental in our calculations.

Theorem 3.1.1 (U. Jannsen)**.** *Let G be an ℓ-adic Lie group without elements of order ℓ and of dimension d. Let M be a $\Lambda(G)$-module which is finitely generated as $\mathbb{Z}_\ell$-module. Then $\mathrm{E}^i(M)$ is a finitely generated $\mathbb{Z}_\ell$-module and, in particular,*

(i) if M is $\mathbb{Z}_\ell$-free, then $E^i(M) = 0$ for any $i \neq d$ and $E^d(M)$ is free;

(ii) if M is finite, then $E^i(M) = 0$ for any $i \neq d+1$ and $E^{d+1}(M)$ is finite.

Proof. See [23, Corollary 2.6]. □

Corollary 3.1.2. *With notations as above:*

(i) if $H^2(F_S/K, A[\ell^\infty]) = 0$*, then, for* $i \geqslant 2$*,*

$$\mathrm{E}^i(X) \text{ is } \begin{cases} finite & \text{if } i = d-1 \\ free & \text{if } i = d-2 \\ 0 & \text{otherwise} \end{cases} \quad ;$$

(ii) $\mathrm{E}_v^i \mathrm{E}_v^{i-1}(X_v) = 0$ *for* $i \geqslant 3$.

Proof. (i) The hypothesis yields the isomorphism

$$\mathrm{E}^i(X) \simeq E^{i+2}((A[\ell^\infty]^\vee)_{\mathcal{H}}) \ .$$

Since

$$(A[\ell^\infty]^\vee)_{\mathcal{H}} \simeq (A[\ell^\infty]^{\mathcal{H}})^\vee = A[\ell^\infty](K)^\vee \simeq \mathbb{Z}_\ell^r \oplus \Delta$$

(with $0 \leqslant r \leqslant 2g$ and Δ a finite group) and $\mathrm{E}^i(\mathbb{Z}_\ell^r \oplus \Delta) = \mathrm{E}^i(\mathbb{Z}_\ell^r) \oplus \mathrm{E}^i(\Delta)$, the claim follows from Theorem 3.1.1.
(ii) Use Theorem 3.1.1 and the isomorphism in (1.7). ☐

Lemma 3.1.3. *If* $H^2(F_S/K, A[\ell^\infty]) = 0$*, then there is the following commutative diagram*

$$\begin{tikzcd}
\mathrm{E}^1(Y) \arrow[r, "g_1"] \arrow[d] & \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^1(Y_v) \arrow[r, two heads] \arrow[d] & Coker(g_1) \arrow[d] \\
\mathrm{E}^1(X) \arrow[r, "h_1"] \arrow[d, two heads] & \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^1(X_v) \arrow[r, two heads] \arrow[d, two heads] & Coker(h_1) \arrow[d, two heads, "f"] \\
\mathrm{E}^2(J) \arrow[r, "\bar{g}_1"] & \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^2(J_v) \arrow[r, two heads] & Coker(\bar{g}_1) \, .
\end{tikzcd}$$

Proof. The inclusions $\mathcal{G}_v \subseteq \mathcal{G}$ and $\mathcal{H}_v \subseteq \mathcal{H}$ induce the maps

$$(I(\mathcal{G}_v) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v} \to (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v} \to (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \ .$$

We have a homomorphism of $\Lambda(G)$-modules $g : \bigoplus_S \mathrm{Ind}_{G_v}^G Y_v \to Y$ which, restricted to the X_v's, provides the map $h : \bigoplus_S \mathrm{Ind}_{G_v}^G X_v \to X$. So we have the following situation

$$\begin{tikzcd}
X \arrow[d, hook] & \bigoplus_S \mathrm{Ind}_{G_v}^G X_v \arrow[l, "h"'] \arrow[d, hook] \\
Y \arrow[d, two heads] & \bigoplus_S \mathrm{Ind}_{G_v}^G Y_v \arrow[l, "g"'] \arrow[d, two heads] \\
J & \bigoplus_S \mathrm{Ind}_{G_v}^G J_v \arrow[l, dashed, "\bar{g}"']
\end{tikzcd} \tag{3.1}$$

where $\bar{g}$ is induced by g and the diagram is obviously commutative.
Since Y and the Y_v's have projective dimension $\leqslant 1$ (i.e., $\mathrm{E}^2(Y) = \mathrm{E}^2(Y_v) = 0$), the lemma follows by taking Ext in diagram (3.1) and recalling that, for any $i \geqslant 0$, $\mathrm{E}^i_v(\mathrm{Ind}^G_{G_v}(X_v)) = \mathrm{Ind}^G_{G_v}\mathrm{E}^i_v(X_v)$ (see [37, Lemma 5.5]). □

In the next paragraph we are going to describe the structure of $Coker(g_1)$.

Homotopy theory and $Coker(g_1)$

For every finitely generated $\Lambda(G)$-module M choose a presentation $P_1 \to P_0 \to M \to 0$ of M by projectives and define the *transpose* functor DM by the exactness of the sequence

$$0 \to \mathrm{E}^0(M) \to \mathrm{E}^0(P_0) \to \mathrm{E}^0(P_1) \to DM \to 0.$$

Then it can be shown that the functor D is well-defined and one has $D^2 = Id$ (see [23]). The most important use of the transpose functor is explained by the next lemma.

Lemma 3.1.4. *Assume $\Lambda(G)$ is an integral domain. Let M be a finitely generated $\Lambda(G)$-module. Then the torsion elements of M form a submodule of M, and it is isomorphic to $\mathrm{E}^1(DM)$.*

Proof. See [37, Lemma 3.1]. □

Definition 3.1.5. Let L be an extension of F contained in F_S. Then we define

$$Z(L) := H^0(F_S/L, \varinjlim_{m} D_2(A[\ell^m]))^\vee$$

where

$$D_2(A[\ell^m]) = \varinjlim_{F\subset E\subset F_S} (H^2(F_S/E, A[\ell^m]))^\vee$$

and the limit in $\varinjlim_{m} D_2(A[\ell^m])$ is taken with respect to the ℓ-power map $A[\ell^{m+1}] \xrightarrow{\ell} A[\ell^m]$.
In the same way we define $Z(L)$ for any Galois extension L of F_v.

An alternative description of the module Z is provided by the following

Lemma 3.1.6. *Let K be a fixed extension of F contained in F_S and K_w its completion for some $w|v \in S$. Then*

(i) $Z(K) \simeq \varprojlim_{F\subseteq L\subseteq K} H^2(F_S/L, T_\ell(A))$;

(ii) $Z(K_w) \simeq \varprojlim_{F_v\subseteq L\subseteq K_w} H^2(L, T_\ell(A))$.

Proof. **Global case.** For any global field L, let

$$Ш^i(F_S/L, A[\ell^\infty]) := Ker\left\{ H^i(F_S/L, A[\ell^\infty]) \to \bigoplus_{w|v\in S} H^i(L_w, A[\ell^\infty]) \right\} .$$

We have already seen that $H^2(L_w, A[\ell^\infty]) = 0$, hence $H^2(F_S/L, A[\ell^\infty]) \simeq Ш^2(F_S/L, A[\ell^\infty])$. Using the pairing of [33, Ch. I, Proposition 6.9], we get

$$\begin{aligned}
Z(K) &= H^0(F_S/K, \varinjlim_m \varinjlim_{F\subseteq L\subseteq F_S} Ш^2(F_S/L, A[\ell^m])^\vee)^\vee \\
&= H^0(F_S/K, \varinjlim_m \varinjlim_{F\subseteq L\subseteq F_S} Ш^0(F_S/L, A^t[\ell^m]))^\vee \\
&= (\varinjlim_m \varinjlim_{F\subseteq L\subseteq F_S} Ш^0(F_S/L, A^t[\ell^m])^{\mathrm{Gal}(F_S/K)})^\vee \\
&= (\varinjlim_m \varinjlim_{F\subseteq L\subseteq K} Ш^0(F_S/L, A^t[\ell^m]))^\vee \\
&= \varprojlim_m \varprojlim_{F\subseteq L\subseteq K} (H^2(F_S/L, A[\ell^m])^\vee)^\vee \\
&= \varprojlim_{F\subseteq L\subseteq K} H^2(F_S/L, T_\ell(A)) .
\end{aligned}$$

Local case. The proof is similar, just use Tate local duality [34, Theorem 7.2.6]. Indeed

$$\begin{aligned}
Z(K_w) &= H^0(K_w, \varinjlim_m \varinjlim_{F_v\subseteq L\subseteq F_v^s} H^2(L, A[\ell^m])^\vee)^\vee \\
&= H^0(K_w, \varinjlim_m \varinjlim_{F_v\subseteq L\subseteq F_v^s} H^0(L, A[\ell^m]'))^\vee \\
&= (\varinjlim_m \varinjlim_{F_v\subseteq L\subseteq F_v^s} H^0(L, A[\ell^m]')^{\mathrm{Gal}(F_v^s/K_w)})^\vee \\
&= (\varinjlim_m \varinjlim_{F_v\subseteq L\subseteq K_w} H^0(L, A[\ell^m]'))^\vee \\
&= \varprojlim_m \varprojlim_{F_v\subseteq L\subseteq K_w} (H^2(L, A[\ell^m])^\vee)^\vee \\
&= \varprojlim_{F_v\subseteq L\subseteq K_w} H^2(L, T_\ell(A)) .
\end{aligned}$$

$\square$

We recall that our group G has no elements of order ℓ, hence $\Lambda(G)$ is an integral domain. Moreover for any open subgroup U of G we have that (see [23, Lemma 2.3])

$$\mathrm{E}^i(U) \simeq \mathrm{E}^i(G) \quad \forall\, i \in \mathbb{Z}$$

is an isomorphism of $\Lambda(U)$-modules. An ℓ-adic Lie group G always contains an open pro-ℓ subgroup (see Theorem 1.2.6), so, in order to use properly the usual definitions of "torsion submodule" and "rank" for a finitely generated $\Lambda(G)$-module, with no loss of generality, we will assume that G is pro-ℓ.

Proposition 3.1.7. *Let M be a finitely generated $\Lambda(G)$-module. Then for any $i \geqslant 1$, $\mathrm{E}^i(M)$ is a finitely generated torsion $\Lambda(G)$-module.*

Proof. Take a finite presentation $P_1 \to P_0 \to M \to 0$ with finitely generated and projective $\Lambda(G)$-modules P_1 and P_0, and the consequent exact sequence

$$0 \to R_1 \to P_0 \to M \to 0 \tag{3.2}$$

for a suitable submodule R_1 of P_1. Since M and $\mathrm{Hom}_{\Lambda(G)}(M, \Lambda(G))$ have the same $\Lambda(G)$-rank, computing ranks in the sequence coming from (3.2)

$$\begin{aligned}&\mathrm{Hom}_{\Lambda(G)}(M, \Lambda(G)) \hookrightarrow \mathrm{Hom}_{\Lambda(G)}(P_0, \Lambda(G)) \to \mathrm{Hom}_{\Lambda(G)}(R_1, \Lambda(G)) \to \mathrm{E}^1(M) \to \\ &\to 0 \to \mathrm{E}^1(R_1) \to \mathrm{E}^2(M) \to 0 \to \cdots \to 0 \to \mathrm{E}^{i-1}(R_1) \to \mathrm{E}^i(M) \to 0 \to \cdots\end{aligned}$$

one finds $\mathrm{rank}_{\Lambda(G)}(\mathrm{E}^1(M)) = 0$ for any finitely generated $\Lambda(G)$-module M. Therefore $\mathrm{E}^1(R_1)$ is torsion, which yields $\mathrm{E}^2(M) \simeq \mathrm{E}^1(R_1)$ is torsion. Iterating $\mathrm{E}^i(M) \simeq \mathrm{E}^{i-1}(R_1)$ is $\Lambda(G)$-torsion $\forall\ i \geqslant 2$. ☐

Lemma 3.1.8. *Let F_n be subfields of K such that $G = \varprojlim_n \mathrm{Gal}(F_n/F)$. Then*

$$H^2_{Iw}(K_w, T_\ell(A)) := \varprojlim_{n,m} H^2(F_{v_n}, A[\ell^m])$$

is a torsion $\Lambda(G_v)$-module. If $H^2(F_S/K, A[\ell^\infty]) = 0$, then

$$H^2_{Iw}(K, T_\ell(A)) := \varprojlim_{n,m} H^2(F_S/F_n, A[\ell^m])$$

is a $\Lambda(G)$-torsion as well.

Proof. The proofs are identical so we only show the second statement. From the spectral sequence

$$\mathrm{E}_2^{p,q} = E^p(H^q(F_S/K, A[\ell^\infty])^\vee) \implies H^{p+q}_{Iw}(K, T_\ell(A))$$

due to Jannsen (see [24]), we have a filtration for $H^2_{Iw}(K, T_\ell(A))$

$$0 = H^2_3 \subseteq H^2_2 \subseteq H^2_1 \subseteq H^2_0 = H^2_{Iw}(K, T_\ell(A))\ , \tag{3.3}$$

which provides the following sequences:

$$\begin{aligned}&\mathrm{E}^0(H^1(F_S/K, A[\ell^\infty])^\vee) \to \mathrm{E}^2(H^0(F_S/K, A[\ell^\infty])^\vee) \to H^2_1 \\ &\qquad\to \mathrm{E}^1(H^1(F_S/K, A[\ell^\infty])^\vee) \to \mathrm{E}^3(H^0(F_S/K, A[\ell^\infty])^\vee)\end{aligned}$$

and

$$H_1^2 \hookrightarrow H^2_{Iw}(K, T_\ell(A)) \twoheadrightarrow \mathrm{E}_\infty^{0,2} \ .$$

By hypothesis $\mathrm{E}_\infty^{0,2} \simeq \mathrm{E}_2^{0,2} = 0$, so $H_1^2 \simeq H^2_{Iw}(K, T_\ell(A))$.
Since $H^i(F_S/K, A[\ell^\infty])^\vee$ is finitely generated (as a module over $\Lambda(G)$, for $i = 0, 1$), Proposition 3.1.7 yields that the groups $\mathrm{E}^2(H^0(F_S/K, A[\ell^\infty])^\vee)$ and $\mathrm{E}^1(H^1(F_S/K, A[\ell^\infty])^\vee)$ are $\Lambda(G)$-torsion. Hence H_1^2 is torsion as well. □

Lemma 3.1.9. *With notations and hypotheses as in Lemma 3.1.3 we have that $Coker(g_1)$ is a finitely generated $\mathbb{Z}_\ell$-module.*

Proof. Lemma 3.1.6 yields $Z(K) = H^2_{Iw}(K, T_\ell(A))$ so, using [37, Proposition 4.10], one has $DH^2_{Iw}(K, T_\ell(A)) \simeq Y$. Therefore $\mathrm{E}^1(DH^2_{Iw}(K, T_\ell(A))) \simeq \mathrm{E}^1(Y)$. Since $H^2_{Iw}(K, T_\ell(A))$ is a $\Lambda(G)$-torsion module, Lemma 3.1.4 implies $\mathrm{E}^1(DH^2_{Iw}(K, T_\ell(A)) \simeq H^2_{Iw}(K, T_\ell(A))$, i.e.,

$$H^2_{Iw}(K, T_\ell(A)) \simeq \mathrm{E}^1(Y)$$

(the same holds for the "local" modules). The map g_1 of Lemma 3.1.3 then reads as

$$g_1 : \varprojlim_n H^2(F_S/F_n, T_\ell(A)) \to \bigoplus_S \mathrm{Ind}^G_{G_v} \varprojlim_n H^2(F_{v_n}, T_\ell(A)) \ .$$

The claim follows from the Poitou-Tate sequence (see [34, 8.6.10 p. 488]), since

$$Coker(g_1) \simeq \varprojlim_{n,m} H^0(F_S/F_n, (A[\ell^m])') \ .$$

□

3.1.1 The main Theorem

We are now ready to prove the following

Theorem 3.1.10. *Let $G = \mathrm{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order ℓ and of positive dimension $d \geqslant 3$ (as ℓ-adic Lie group). If $H^2(F_S/K, A[\ell^\infty]) = 0$ and the map ψ in the sequence*

$$Sel_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \mathrm{Coind}^{G_v}_G H^1(K_w, A)[\ell^\infty] \qquad (3.4)$$

is surjective, then $\mathcal{S} := Sel_A(K)_\ell^\vee$ has no non trivial pseudo-null submodules.

Proof. We need to prove that

$$\mathrm{E}^i\mathrm{E}^i(\mathcal{S}) = 0 \ \ \forall\, i \geqslant 2 \ ,$$

and we consider two cases.

<u>Case $i = 2$.</u> Let $\mathcal{D} := \bar{g}_1(\mathrm{E}^2(J))$. Then

$$Coker(\bar{g}_1) = \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D} \ .$$

Observe that $\mathcal{D} \simeq \bar{g}_1(\mathrm{E}^3(A[\ell^\infty]_{\mathcal{H}}^\vee))$ is a finitely generated $\mathbb{Z}_\ell$-module (it is zero if $d \neq 3$ and a free $\mathbb{Z}_\ell$-module if $d = 3$), so $\mathrm{E}^1(\mathcal{D}) = 0$. Even if the theorem is limited to $d \geqslant 3$ we remark here that, for $d = 2$, $\mathcal{D}$ is finite and, for $d = 1$, $\mathcal{D} = 0$: hence $\mathrm{E}^1(\mathcal{D}) = 0$ in any case.
Moreover

$$\begin{aligned}\mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)) &= \mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^3(A[\ell^\infty]_{\mathcal{H}_v}^\vee)) \\ &= \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2\mathrm{E}^3(A[\ell^\infty]_{\mathcal{H}_v}^\vee) = 0 \ ,\end{aligned}$$

so, taking Ext in the sequence,

$$\mathcal{D} \hookrightarrow \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v) \twoheadrightarrow \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D} \ , \tag{3.5}$$

one finds

$$\mathrm{E}^1(\mathcal{D}) \to \mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D}) \to \mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)) \ .$$

Therefore

$$\mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D}) = 0 \ . \tag{3.6}$$

Recall the sequences

$$\bigoplus_S \mathrm{Ind}_{G_v}^G X_v \hookrightarrow X \twoheadrightarrow \mathcal{S} \tag{3.7}$$

$$Ker(f) \hookrightarrow Coker(h_1) \twoheadrightarrow Coker(\bar{g}_1) \tag{3.8}$$

provided (respectively) by the hypothesis on ψ and by Lemma 3.1.3. Take Ext on (3.7) to get

$$\mathrm{E}^1(X) \xrightarrow{h_1} \mathrm{E}^1(\bigoplus_S \mathrm{Ind}_{G_v}^G X_v) \to \mathrm{E}^2(\mathcal{S}) \to \mathrm{E}^2(X) \ .$$

If $d \geqslant 5$, then $\mathrm{E}^2(X) = \mathrm{E}^3(J) = \mathrm{E}^4(A[\ell^\infty]_{\mathcal{H}}^\vee) = 0$. When this is the case $Coker(h_1) \simeq \mathrm{E}^2(\mathcal{S})$ and sequence (3.8) becomes

$$Ker(f) \hookrightarrow \mathrm{E}^2(\mathcal{S}) \twoheadrightarrow \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D} \ .$$

By Lemma 3.1.9, $Ker(f)$ is a finitely generated $\mathbb{Z}_\ell$-module. Taking Ext, one has

$$\mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D}) \to \mathrm{E}^2\mathrm{E}^2(\mathcal{S}) \to \mathrm{E}^2(Ker(f)) \ ,$$

where the first and third term are trivial, so $\mathrm{E}^2\mathrm{E}^2(\mathcal{S}) = 0$ as well.
We are left with $d = 3, 4$. We know that $\mathrm{E}^4(A[\ell^\infty]^\vee_{\mathcal{H}}) = \mathrm{E}^2(X)$ is free over $\mathbb{Z}_\ell$ if $d = 4$ or finite if $d = 3$ (again we remark it is 0 if $d = 1, 2$). Anyway $\mathrm{E}^2\mathrm{E}^2(X) = 0$ in all cases. From the sequence

$$Coker(h_1) \hookrightarrow \mathrm{E}^2(\mathcal{S}) \xrightarrow{\eta} \mathrm{E}^2(X)$$

one writes

$$Coker(h_1) \hookrightarrow \mathrm{E}^2(\mathcal{S}) \twoheadrightarrow Im(\eta) \tag{3.9}$$

where $Im(\eta)$ is free over $\mathbb{Z}_\ell$ if $d = 4$ or finite if $d = 3$.
Taking Ext in (3.8) one has

$$\mathrm{E}^2(Coker(\bar{g}_1)) \to \mathrm{E}^2(Coker(h_1)) \to \mathrm{E}^2(Ker(f))$$

with the first (see equation (3.6)) and third term equal to zero. As a result $\mathrm{E}^2(Coker(h_1)) = 0$. This fact in sequence (3.9) implies

$$0 = \mathrm{E}^2(Im(\eta)) \to \mathrm{E}^2\mathrm{E}^2(\mathcal{S}) \to \mathrm{E}^2(Coker(h_1)) = 0 \ ,$$

so $\mathrm{E}^2\mathrm{E}^2(\mathcal{S}) = 0$.
Case $i \geqslant 3$. From sequence (3.7) we get the following

$$\begin{aligned} \mathrm{E}^{i+1}(A[\ell^\infty]^\vee_{\mathcal{H}}) \simeq \mathrm{E}^{i-1}(X) \to \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^{i-1}(X_v) \to \\ \to \mathrm{E}^i(\mathcal{S}) \to \mathrm{E}^i(X) \simeq \mathrm{E}^{i+2}(A[\ell^\infty]^\vee_{\mathcal{H}}) \ . \end{aligned} \tag{3.10}$$

We have four cases, depending on whether $\mathrm{E}^{i-1}(X)$ and $\mathrm{E}^i(X)$ are trivial or not.
Case 1. Assume $\mathrm{E}^{i-1}(X) = \mathrm{E}^i(X) = 0$.
From (3.10) we obtain the isomorphism

$$\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^{i-1}(X_v) \simeq \mathrm{E}^i(\mathcal{S}) \ ,$$

so

$$\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^i\mathrm{E}_v^{i-1}(X_v) \simeq \mathrm{E}^i\mathrm{E}^i(\mathcal{S}) = 0$$

thanks to Corollary 3.1.2 part *(ii)*. We remark that this is the only case to consider when $d = 1, 2$.

Case 2. Assume $\mathrm{E}^{i-1}(X) = 0$ and $\mathrm{E}^i(X) \neq 0$.
This happens when $i = d-2$ or $i = d-1$ and $A[\ell^\infty]^\vee_{\mathcal{H}}$ is finite. From (3.10) we have

$$\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-3}_v \hookrightarrow \mathrm{E}^{d-2}(\mathcal{S}) \twoheadrightarrow N$$

$$(\text{ resp. } \quad \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-2}_v \hookrightarrow \mathrm{E}^{d-1}(\mathcal{S}) \twoheadrightarrow N \ )$$

where N is a submodule of the free module $\mathrm{E}^{d-2}(X)$ (resp. of the finite module $\mathrm{E}^{d-1}(X)$). Therefore $\mathrm{E}^{d-2}(N) = 0$ (resp. $\mathrm{E}^{d-1}(N) = 0$) and, moreover, $\mathrm{E}^{d-2}_v\mathrm{E}^{d-3}_v(X_v) = 0$ (resp. $\mathrm{E}^{d-1}_v\mathrm{E}^{d-2}_v(X_v) = 0$) by Corollary 3.1.2 part *(ii)*. Hence $\mathrm{E}^{d-2}\mathrm{E}^{d-2}(\mathcal{S}) = 0$ (resp. $\mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S}) = 0$).
Case 3. Assume $\mathrm{E}^{i-1}(X) \neq 0$ and $\mathrm{E}^i(X) = 0$.
This happens when $i = d$ or $i = d-1$ and $A[\ell^\infty]^\vee_{\mathcal{H}}$ is free. The sequence (3.10) gives

$$N \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-1}_v(X_v) \twoheadrightarrow \mathrm{E}^d(\mathcal{S})$$

$$(\text{ resp. } \quad N \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-2}_v(X_v) \twoheadrightarrow \mathrm{E}^{d-1}(\mathcal{S}) \ )$$

where now N is a quotient of the finite module $\mathrm{E}^{d-1}(X)$ (resp. of the free module $\mathrm{E}^{d-2}(X)$). Then $\mathrm{E}^d(N) = 0$ (resp. $\mathrm{E}^{d-1}(N) = 0$) and

$$\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^d_v\mathrm{E}^{d-1}_v(X_v) \simeq \mathrm{E}^d\mathrm{E}^d(\mathcal{S}) = 0$$

$$(\text{ resp. } \quad \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-1}_v\mathrm{E}^{d-2}_v(X_v) \simeq \mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S}) = 0 \ ) \ .$$

Case 4. Assume $\mathrm{E}^{i-1}(X) \neq 0$ and $\mathrm{E}^i(X) \neq 0$.
This happens when $i = d-1$ and $A[\ell^\infty]^\vee_{\mathcal{H}}$ has nontrivial rank and torsion. From sequence (3.10) we have

$$\mathrm{E}^{d-2}(X) \to \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-2}_v(X_v) \to \mathrm{E}^{d-1}(\mathcal{S}) \to \mathrm{E}^{d-1}(X) \ .$$

Let N_1, N_2 and N_3 be modules such that:

- N_1 is a quotient of $\mathrm{E}^{d-2}(X)$ (which is torsion free so that $\mathrm{E}^{d-2}(N_1) = 0$);
- N_2 is a submodule of $\mathrm{E}^{d-1}(X)$ (which is finite so that $\mathrm{E}^{d-1}(N_2) = 0$);
- N_3 is a module such that the sequences

  $$N_1 \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^{d-2}_v(X_v) \twoheadrightarrow N_3 \quad \text{and} \quad N_3 \hookrightarrow \mathrm{E}^{d-1}(\mathcal{S}) \twoheadrightarrow N_2$$

  are exact.

Applying the functor Ext we find

$$\mathrm{E}^{d-2}(N_1) \to \mathrm{E}^{d-1}(N_3) \to \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^{d-1}\mathrm{E}_v^{d-2}(X_v)$$

(which yields $\mathrm{E}^{d-1}(N_3) = 0$), and

$$\mathrm{E}^{d-1}(N_2) \to \mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S}) \to \mathrm{E}^{d-1}(N_3)$$

which proves $\mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S}) = 0$. □

Remark 3.1.11. As pointed out in various steps of the previous proof, most of the statements still hold for $d = 1, 2$. The only missing part is $\mathrm{E}^2(Ker(f)) = 0$ for $i = 2$, in that case only our calculations to get $\mathrm{E}^2\mathrm{E}^2(\mathcal{S}) = 0$ fail. In particular the same proof shows that $\mathrm{E}^2\mathrm{E}^2(\mathcal{S}) = 0$ when $Ker(f)$ is free and $d = 1$ or when $Ker(f)$ is finite and $d = 2$ or, obviously, for any d if f is injective.

We can extend the previous result to the $d \geqslant 2$ case with some extra assumptions.

Proposition 3.1.12. *Let $G = \mathrm{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order ℓ and of dimension $d \geqslant 2$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and $\mathrm{cd}_\ell(G_v) = 2$ for any $v \in S$, then $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.*

Proof. Since $\mathrm{cd}_\ell(F_v) = 2$ (by [34, Theorem 7.1.8]), our hypothesis implies that $\mathrm{Gal}(\overline{F_v}/K_w)$ has no element of order ℓ (see also [34, Theorem 7.5.3]). Hence $H^1(K_w, A[\ell^\infty])^\vee = 0$ and $Sel_A(K)_\ell^\vee \simeq X$ embeds in Y. Now $H^2(F_S/K, A[\ell^\infty]) = 0$ yields Y has projective dimension $\leqslant 1$, so Y has no nontrivial pseudo-null submodule (by [37, Proposition 2.5]). □

Remark 3.1.13. When $d = 2$, $\mathrm{cd}_\ell(G_v) = 2$ for any $v \in S$ holds if and only if every $v \in S$ splits in finitely many primes in K.

The hypotheses on $H^2(F_S/K, A[\ell^\infty])$ and ψ

Let F_m be extensions of F such that $\mathrm{Gal}(K/F) \simeq \varprojlim_m \mathrm{Gal}(F_m/F)$. To provide some cases in which the main hypotheses hold we consider the Poitou-

Tate sequence for the module $A[\ell^n]$, from which one can extract the sequence

$$\begin{tikzcd} Ker(\psi_{m,n}) \arrow[r, hook] & H^1(F_S/F_m, A[\ell^n]) \arrow[r, "\psi_{m,n}"] & \prod\limits_{\substack{v_m|v \\ v\in S}} H^1(F_{v_m}, A[\ell^n]) \arrow[d, "\phi_{m,n}"] \\ \prod\limits_{\substack{v_m|v \\ v\in S}} H^2(F_{v_m}, A[\ell^n]) \arrow[d] & H^2(F_S/F_m, A[\ell^n]) \arrow[l] & Ker(\psi^t_{m,n}))^\vee \arrow[l] \\ H^0(F_S/F_m, A^t[\ell^n])^\vee \arrow[r] & 0 & \end{tikzcd} \tag{3.11}$$

(where $\psi^t_{m,n}$ is the analogue of $\psi_{m,n}$ for the dual abelian variety A^t , i.e., their kernels represent the Selmer groups over F_m for the modules $A^t[\ell^n]$ and $A[\ell^n]$ respectively). Taking direct limits on n and recalling that $H^2(F_{v_m}, A[\ell^\infty]) = 0$, the sequence (3.11) becomes

$$\begin{tikzcd} Sel_A(F_m)_\ell \arrow[r, hook] & H^1(F_S/F_m, A[\ell^\infty]) \arrow[r, "\psi_m"] & \prod_{\substack{v_m|v \\ v\in S}} H^1(F_{v_m}, A[\ell^\infty]) \arrow[d, "\phi_m"] \\ 0 & H^2(F_S/F_m, A[\ell^\infty]) \arrow[l] & (\varprojlim_n Ker(\psi^t_{m,n}))^\vee \arrow[l] \end{tikzcd} \tag{3.12}$$

(for more details one can consult [15, Chapter 1]). One way to prove that $H^2(F_S/K, A[\ell^\infty]) = 0$ and ψ is a surjective map is to show that $(\varprojlim_n Ker(\psi^t_{m,n}))^\vee = 0$ for any m. We mention here two cases in which the hypothesis on the vanishing of $H^2(F_S/K, A[\ell^\infty])$ is verified. The following is basically [15, Proposition 1.9].

Proposition 3.1.14. *Let F_m be as above and assume that $Sel_{A^t}(F_m)_\ell$ is finite for any m, then*

$$H^2(F_S/K, A[\ell^\infty]) = 0 \ .$$

Proof. From [33, Chapter I Remark 3.6] we have the isomorphism

$$A^t(F_{v_m})^* \simeq H^1(F_{v_m}, A[\ell^\infty])^\vee \ ,$$

where $A^t(F_{v_m})^* \simeq \varprojlim_n A^t(F_{v_m})/\ell^n A^t(F_{v_m})$.

Taking inverse limits on n in the exact sequence

$$A^t(F_m)/\ell^n A^t(F_m) \hookrightarrow Ker(\psi^t_{m,n}) \twoheadrightarrow Ш(A^t/F_m)[\ell^n] \ ,$$

and noting that $|Ш(A^t/F_m)[\ell^\infty]| < \infty$ yields $T_\ell(Ш(A^t/F_m)) = 0$, we find

$$A^t(F_m)^* \simeq \varprojlim_n Ker(\psi^t_{m,n}) \ .$$

Therefore (3.12) becomes

$$\begin{array}{ccccc} Sel_A(F_m)_\ell \hookrightarrow & H^1(F_S/F_m, A[\ell^\infty]) & \xrightarrow{\psi} & \prod_{\substack{v_m|v \\ v\in S}} (A^t(F_{v_m})^*)^\vee & \\ & & & \downarrow \widetilde{\phi} & \\ 0 \longleftarrow & H^2(F_S/F_m, A[\ell^\infty]) & \longleftarrow & (A^t(F_m)^*)^\vee & \end{array} \tag{3.13}$$

By hypothesis $A^t(F_m)^*$ is finite, therefore $H^2(F_S/F_m, A[\ell^\infty])$ is finite as well. From the cohomology of the sequence

$$A[\ell] \hookrightarrow A[\ell^\infty] \xrightarrow{\ell} \!\!\!\!\rightarrow A[\ell^\infty]$$

(and the fact that $H^3(F_S/F_m, A[\ell]) = 0$, because $cd_\ell(\mathrm{Gal}(F_S/F_m)) = 2$), one finds

$$H^2(F_S/F_m, A[\ell^\infty]) \xrightarrow{\ell} \!\!\!\!\rightarrow H^2(F_S/F_m, A[\ell^\infty]) \ ,$$

i.e., $H^2(F_S/F_m, A[\ell^\infty])$ is divisible. Being divisible and finite implies that $H^2(F_S/F_m, A[\ell^\infty])$ must be 0 for any m and the claim follows. □

We can also prove the vanishing of $H^2(F_S/K, A[\ell^\infty])$ for the extension $K = F(A[\ell^\infty])$.

Proposition 3.1.15. *If $K = F(A[\ell^\infty])$, then $H^2(F_S/K, A[\ell^\infty]) = 0$.*

Proof. $\mathrm{Gal}(F_S/K)$ has trivial action on $A[\ell^\infty]$ and (by the Weil pairing) on $\boldsymbol{\mu}_{\ell^\infty}$, so

$$H^2(F_S/K, A[\ell^\infty]) \simeq H^2(F_S/K, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}) \simeq H^2(F_S/K, (\boldsymbol{\mu}_{\ell^\infty})^{2g}) \ .$$

Let $F_n = F(A[\ell^n])$, using the notations of Lemma 3.1.6, Poitou-Tate duality ([34, Theorem 8.6.7]) and the isomorphism $Ш^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z}) \simeq \mathrm{Hom}(C\ell_S(F_n), \mathbb{Z}/\ell^m\mathbb{Z})$ ([34, Lemma 8.6.3]), one has

$$\begin{aligned} H^2(F_S/K, \boldsymbol{\mu}_{\ell^\infty}) &\simeq Ш^2(F_S/K, \boldsymbol{\mu}_{\ell^\infty}) \simeq \varinjlim_{n,m} Ш^2(F_S/F_n, \boldsymbol{\mu}_{\ell^m}) \\ &\simeq \varinjlim_{n,m} Ш^1(F_S/F_n, \boldsymbol{\mu}'_{\ell^m})^\vee \simeq \varinjlim_{n,m} Ш^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z})^\vee \\ &\simeq \varinjlim_{n,m} \mathrm{Hom}(C\ell_S(F_n), \mathbb{Z}/\ell^m\mathbb{Z})^\vee \simeq \varinjlim_{n,m} C\ell_S(F_n)/\ell^m \\ &\simeq \varinjlim_{n} C\ell_S(F_n) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0 \end{aligned}$$

since $C\ell_S(F_n)$ is finite. □

Remark 3.1.16. The above proposition works in the same way for any ℓ-adic Lie extension, unramified outside S, which contains the trivializing extension.

Example 3.1.17. Let A be an abelian variety: by Proposition 3.1.15, the extension $K = F(A[\ell^\infty])$ realizes the hypotheses of Proposition 3.1.12 when every bad reduction prime is of split multiplicative reduction (in order to have $\mathrm{cd}_\ell(G_v) = 2$) and $\ell > 2g+1$ (by [44] and the embedding $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$). Therefore, under these hypotheses $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.
When $A = \mathcal{E}$ is a non isotrivial elliptic curve and $p > 3$ (using Igusa's theorem, see, e.g., [5]) one can prove that $\dim \mathrm{Gal}(K/F) = 4$ and also the surjectivity of the map ψ (which, in this case, is not needed to prove the absence of pseudo-null submodules): more details can be found in [40].
The same problem over number fields cannot (in general) be addressed in the same way and one needs the surjectivity of the map ψ. The topic is treated (for example) in [10, Section 4.2].

3.2 Euler characteristic

The main result of this section is Theorem 3.2.4 where we provide an Euler characteristic formula for $Sel_A(K)_\ell$.
Since the homology theory for profinite groups is dual to the cohomological one (see [34, Theorem 2.2.9]) we will do our calculations using cohomological groups.

3.2.1 Preliminary results

First we list some useful intermediate results.

Proposition 3.2.1. *Let F_m be extensions of F such that $\mathrm{Gal}(K/F) \simeq \varprojlim_m \mathrm{Gal}(F_m/F)$. If $|Sel_{A^t}(F_m)_\ell| < \infty$ for any m, then*

$$H^2(F_S/F_m, A[\ell^\infty]) = 0 \quad \text{for any } m\,.$$

Proof. See Proposition 3.1.14. □

Lemma 3.2.2. *If $Sel_{A^t}(F)_\ell$ is finite and $H^2(F_S/K, A[\ell^\infty]) = 0$ we have*

(i) $H^i(K/F, H^1(F_S/K, A[\ell^\infty])) \simeq H^{i+2}(K/F, A[\ell^\infty](K)) \ \forall\, i \geqslant 1$;

(ii) $H^i(K/F, H^1(F_S/K, A[\ell^\infty])) = 0 \ \forall\, i \geqslant 2$.

Moreover, let w any prime of K such that $w|v \in S$. Then

(iii) $H^i(K_w/F_v, H^1(K_w, A[\ell^\infty])) \simeq H^{i+2}(K_w/F_v, A[\ell^\infty](K_w)) \ \forall\, i \geqslant 1$.

Proof. *(i)* By [34, Corollary 10.1.3 (iii) and Proposition 3.3.5] the ℓ-cohomological dimension of $\mathrm{Gal}(F_S/K)$ is less than or equal to 2. Therefore our hypothesis on $H^2(F_S/K, A[\ell^\infty])$ yields

$$H^i(F_S/K, A[\ell^\infty]) = 0 \quad \forall\, i \geqslant 2\ .$$

From the Hochschild-Serre spectral sequence, we have:

$$
\begin{array}{l}
0 \to H^1(K/F, A[\ell^\infty](K)) \to H^1(F_S/F, A[\ell^\infty]) \longrightarrow H^1(F_S/K, A[\ell^\infty])^{\mathrm{Gal}(K/F)} \to \\
\to H^2(K/F, A[\ell^\infty](K)) \to H^2(F_S/F, A[\ell^\infty]) \longrightarrow H^1(K/F, H^1(F_S/K, A[\ell^\infty])) \to \\
\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\to H^i(K/F, A[\ell^\infty](K)) \to H^i(F_S/F, A[\ell^\infty]) \to H^{i-1}(K/F, H^1(F_S/K, A[\ell^\infty])) \to
\end{array}
$$

(see [34, Lemma 2.1.3]). Since $\mathrm{cd}_\ell(G_S(F)) = 2$ and (by Proposition 3.2.1) $H^2(F_S/F, A[\ell^\infty]) = 0$, the claim follows.

(ii) By [34, Lemma 2.1.4] we have the following isomorphisms:

$$H^i(K/F, H^1(F_S/K, A[\ell^\infty])) = E_2^{i,1} \simeq E^{i+1} = H^{i+1}(F_S/F, A[\ell^\infty]) \quad \forall\, i \geqslant 1\ .$$

Then

$$H^i(K/F, H^1(F_S/K, A[\ell^\infty])) = 0 \quad \forall\, i \geqslant 2\ .$$

(iii) The argument is the same of part *(i)*; just note that $\mathrm{Gal}(K_w/F_v)$ has at most ℓ-cohomological dimension 2. ☐

Lemma 3.2.3. *If $Sel_A(F)_\ell$ is finite, then $H^1(F_S/F, A[\ell^\infty])$ is finite too.*

Proof. Observe that

$$H^2(F_S/F, A^t[\ell^\infty]) \simeq Ш^2(F_S/F, A^t[\ell^\infty])$$

is finite thanks to [34, Theorem 8.6.8 (Poitou-Tate Duality)]. By [33, Chapter 1, Theorem 4.20]

$$Ш^2(F_S/F, A^t[\ell^\infty]) \simeq Ш^1(F_S/F, A[\ell^\infty])^\vee.$$

Since our hypothesis implies $H^2(F_S/F, A^t[\ell^\infty]) = 0$ (see Proposition 3.2.1), we have that $Ш^1(F_S/F, A[\ell^\infty]) = 0$ as well. Then,

$$H^1(F_S/F, A[\ell^\infty]) \hookrightarrow \prod_{v\in S} H^1(F_v, A[\ell^\infty])$$

and the statement follows from [34, Theorem 7.1.8]. ☐

3.2.2 Calculations for Euler Characteristic

Now we can prove the main theorem of this section.

Theorem 3.2.4. *Assume that $Sel_A(F)_\ell$ is finite, $H^2(F_S/K, A[\ell^\infty]) = 0$, $\chi(G, A[\ell^\infty])$ and $\chi(G_v, A[\ell^\infty])$ $\forall\ v \in S$ are well defined and that the map ψ in the sequence*

$$Sel_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \mathrm{Coind}_G^{G_v} H^1(K_w, A[\ell^\infty]) \tag{3.14}$$

is surjective. Then the Euler characteristic of $Sel_A(K)_\ell$ is well defined and

$$\chi(G, Sel_A(K)_\ell) = \prod_{v\in S} \frac{1}{c_v} \cdot \frac{|H^1(F_S/K, A[\ell^\infty])^G|}{|H^3(G, A[\ell^\infty](K))|}$$

where $c_v = |H^1(K_w, A[\ell^\infty])^{G_v}|$ for every $v \in S$.

Proof. Let us consider the sequence (B.2) and take its cohomology with respect to G (recall $\mathrm{cd}_\ell(G) = d = \dim(G)$ and, to shorten notations, $\mathcal{A} := A[\ell^\infty]$) to get

$$\begin{array}{ccccccc}
H^0(G, Sel_A(K)_\ell) & \hookrightarrow & H^0(G, H^1(F_S/K, \mathcal{A})) & \to & \bigoplus_S H^0(G, \mathrm{Coind}_G^{G_v} H^1(K_w, \mathcal{A})) & \to \\
\cdots & & \cdots & & \cdots & \\
H^i(G, Sel_A(K)_\ell) & \to & H^i(G, H^1(F_S/K, \mathcal{A})) & \to & \bigoplus_S H^i(G, \mathrm{Coind}_G^{G_v} H^1(K_w, \mathcal{A})) & \to \\
\cdots & & \cdots & & \cdots & \\
H^d(G, Sel_A(K)_\ell) & \to & H^d(G, H^1(F_S/K, \mathcal{A})) & \twoheadrightarrow & \bigoplus_S H^d(G, \mathrm{Coind}_G^{G_v} H^1(K_w, \mathcal{A})) & .
\end{array}$$

Using Shapiro's Lemma ([34, Proposition 1.6.3]) and the isomorphisms of Lemma 3.2.2 the above sequence becomes

$$\begin{tikzcd}
Sel_A(K)_\ell^G \arrow[r, hook] & H^1(F_S/K, A[\ell^\infty])^G \arrow[r, "\widetilde{\psi}"] & \bigoplus_S H^1(K_w, A[\ell^\infty])^{G_v} \arrow[d] \\
& H^3(G, A[\ell^\infty](K)) & H^1(G, Sel_A(K)_\ell) . \arrow[l, two heads]
\end{tikzcd} \tag{3.15}$$

By hypotheses on $\chi(G, A[\ell^\infty])$ and $\chi(G_v, A[\ell^\infty])$ and Lemma 3.2.3 we have that all terms in (3.15) are finite. So, we find that

$$\chi(G, Sel_A(K)_\ell) = \prod_{v\in S} \frac{1}{c_v} \cdot \frac{|H^1(F_S/K, A[\ell^\infty])^G|}{|H^3(G, A[\ell^\infty](K))|} .$$

$\square$

We can provide a few cases in which some hypotheses of Theorem 3.2.4 are satisfied.

Lemma 3.2.5. *If $A[\ell^\infty](K)$ (resp. $A[\ell^\infty](K_w)$ for any $w|v \in S$) is finite, then $\chi(G, A[\ell^\infty])$ (resp. $\chi(G_v, A[\ell^\infty])$) is well defined.*

Proof. With no loss of generality, we can assume that G is pro-ℓ. Thanks to [27, Théorème 2.5.8], we know that G is a Poincaré group of dimension d. Since the G-module $A[\ell^\infty](K)$ is finite, by [43, Exercice (a) § 4.1] the claim follows.
The argument for the local part is the same. □

For the local case we can provide also the following

Lemma 3.2.6. *If $\mathrm{cd}_\ell(G_v) = 2$ then*

$$\chi(G_v, A[\ell^\infty]) = \frac{H^0(G_v, A[\ell^\infty](K_w))}{H^1(G_v, A[\ell^\infty](K_w))}$$

is well defined.

Proof. Our hypotheses implies that $H^i(K_w, A[\ell^\infty]) = 0$ for any $i \geqslant 1$ and any $w|v$. The Hochschild-Serre spectral sequence provides isomorphisms

$$H^n(G_v, A[\ell^\infty](K_w)) \simeq H^n(F_v, A[\ell^\infty]) \ \ \forall\, n \geqslant 0 \ .$$

The claim follows thanks to [34, Theorem 7.1.8] and the fact that Tate local duality ([34, Theorem 7.2.6]) yields

$$H^2(F_v, A[\ell^\infty]) = H^2(F_v, \varinjlim_n A[\ell^n]) \simeq \varprojlim_n H^0(F_v, A^t[\ell^n])^\vee = 0 \ .$$

□

Example 3.2.7. Take $K = F(A[\ell^\infty])$ and $\ell > 2g+1$. By Proposition 3.1.15 we know that $H^2(F_S/K, A[\ell^\infty]) = 0$. Moreover, we observe that $\chi(G, A[\ell^\infty])$ is well defined. Indeed, by [42, Théorème 2] and the subsequent Corollaire one has $|H^i(K/F, A[\ell^\infty])| < \infty$ for any $i \geqslant 0$. Besides, if all primes in S are of split multiplicative reduction (so $\mathrm{cd}_\ell(G_v) = 2\ \forall\, v \in S$) and $Sel_A(F)_\ell$ is finite

$$\chi(G, Sel_A(K)_\ell) = \frac{|H^1(F_S/K, A[\ell^\infty])^G|}{|H^3(G, A[\ell^\infty](K))|}$$

and it is well defined.

Example 3.2.8. Suppose that $G \simeq \mathbb{Z}_\ell^d$ and take $H \lhd_c G$ such that $G/H \simeq \mathbb{Z}_\ell$. If $A = \mathcal{E}$ is an elliptic curve, by [5, Theorem 4.2] we know that $\mathcal{E}[\ell^\infty](K)$ is a finite group, then $|H^i(H, \mathcal{E}[\ell^\infty](K))| < \infty\ \forall\ i \geqslant 0$. This implies (see [14]) that

$$\chi(G, \mathcal{E}[\ell^\infty]) = 1.$$

Final remark It would be interesting to provide an Euler characteristic formula when $\ell = p$. Unfortunately, the cohomological dimension of $\mathcal{X}_F$ is not finite unless F is a perfect field (see [45, Theorem 1]), so a similar calculation seems to make no sense.
Dealing with $\ell = p$ shows us the different nature of function fields with respect to number fields. As at the beginning the analogies between them led us to write this thesis, now this differences seem a good starting point for new researches: a new approach (maybe more geometric then the one we used) to Iwasawa theory is necessary in order to provide a path between a characteristic element of $Sel_A(K)_p^\vee$ and an interpolated L-function.

Appendix A

K-Theory

Our intent here is just to provide a little "dictionary" for the basic objects of K-theory appearing in Section 1.5 in order to simplify the reading for the reader.
For more details the reader is referred to [46].

A.1 The basic theory

Definition A.1.1. Let $\mathcal{A}$ be a small abelian category. The *Grothendieck group* $\mathrm{K}_0(\mathcal{A})$ of $\mathcal{A}$ is the abelian group generated by isomorphism classes $[A]$, where A runs over the objects of $\mathcal{A}$, and relations $[A] = [A'] + [A'']$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$.

Example A.1.2. Let $\mathcal{A}$ be the category of finite dimensional vector spaces V over the field k. If V has dimension n, then $[V] = n[k]$. Consider the function

$$\dim : \text{ objects}(\mathcal{A}) \to \mathbb{Z}\,.$$

If $0 \to V' \to V \to V'' \to 0$ is an exact sequence of vector spaces, then $\dim(V) = \dim(V') + \dim(V'')$. Hence $\mathrm{K}_0(\mathcal{A})$ is isomorphic to $\mathbb{Z}$.

Let $\mathcal{B}$ be a full subcategory of an abelian category which contains 0, is closed under $\oplus$, and is equivalent to a small category. Recall that $\mathcal{B}[x, x^{-1}]$ is the category whose objects are pairs (B, f) where $B \in \mathcal{B}$ and $f : B \to B$ is an isomorphism, and whose morphisms $(B, f) \to (B', f')$ are represented by a map $h : B \to B'$ such that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "h"] \\ B' \arrow[r, "f'"] & B' \end{tikzcd}$$

commutes. A sequence $0 \to (B', f') \to (B, f) \to (B'', f'') \to 0$ is exact in $\mathcal{B}[x, x^{-1}]$ if and only if $0 \to B' \to B \to B'' \to 0$ is exact in $\mathcal{B}$.

Definition A.1.3. With the above notation, $\mathrm{K}_1(\mathcal{B})$ is the abelian group whose generators are isomorphism classes $[(B, f)]$ of objects in $(\mathcal{B}[x, x^{-1}])$ with relations

- if $0 \to (B', f') \to (B, f) \to (B'', f'') \to 0$ is exact in $\mathcal{B}[x, x^{-1}]$, then $[(B, f)] = [(B', f')] + [(B'', f'')]$;
- $[(B, fg)] = [(B, f)] + [(B, g)]$.

Definition A.1.4. Let R and R' be rings and $f : R \to R'$ a homomorphism. Write $\mathcal{F}_f$ for the category whose objects are triples (A, g, B), where A and B are finitely generated projective R-modules and $g : R' \otimes_R A \to R' \otimes_R B$ is an isomorphism of R'-modules. A map

$$(h, m) : (A, g, B) \to (A', g', B')$$

is a pair of maps $h : A \to A'$ and $m : B \to B'$ such that the square

$$\begin{tikzcd} R' \otimes_R A \arrow[r, "1 \otimes h"] \arrow[d, "g"'] & R' \otimes_R A' \arrow[d, "g'"] \\ R' \otimes_R B \arrow[r, "1 \otimes m"] & R' \otimes_R B' \end{tikzcd}$$

commutes. A sequence

$$0 \to (A', g', B') \to (A, g, B) \to (A'', g'', B'') \to 0$$

in $\mathcal{F}_f$ is exact if $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ are exact sequence of R-modules.
Then, $\mathrm{K}_0(R, f)$ is the abelian group with generators the objects of $\mathcal{F}_f$ and relations

- $[(A, g, B)] = [(A', g', B')] + [(A'', g'', B'')]$ if
  $0 \to (A', g', B') \to (A, g, B) \to (A'', g'', B'') \to 0$ is exact in $\mathcal{F}_f$;

- $[(A, gh, B)] = [(A, h, C)] + [(C, g, B)]$.

Theorem A.1.5. *Let $f : R \to R'$ be an homomorphism of rings. Then there exists an exact sequence*

$$\mathrm{K}_1(R) \to \mathrm{K}_1(R') \to \mathrm{K}_0(R, f) \to \mathrm{K}_0(R) \to \mathrm{K}_0(R')\,.$$

Proof. See [46, Theorem 15.5]. $\square$

To link the above sequence to that described in Section 1.5 see [51, § 3].

Appendix B

Spectral sequences

The purpose of this appendix is to show how we can obtain information about some low-degree terms of a given spectral sequence. We shall provide calculations only for a specific case, but the method we describe can be used in greater generality (we used this method in the proof of Lemma 3.1.8). For more details on spectral sequences one can consult [34, Chapter II] or [54].

B.1 Basic definitions

We start with formal definitions and notations.

Definition B.1.1. An a^{th}*-stage (first quadrant cohomological) spectral sequence* ia a collection of abelian groups $E_r^{p,q}$ for all $p, q \geqslant 0$ and for all $r \geqslant a$ for some positive integer a, together with maps

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that

$$d_r^{p+r,q-r+1} d_r^{p,q} = 0$$

and

$$E_{r+1}^{p,q} \simeq Ker(d_r^{p,q})/Im(d_r^{p-r,q+r-1})$$

for all p, q and r as above.

By convention (we deal with *first* quadrant cohomological spectral sequence) we have $E_r^{p,q} = 0$ if either p or q is negative.

In any spectral sequence the (p, q) place eventually stabilizes; we denote this stable value of $E_r^{p,q}$ by $E_\infty^{p,q}$. Next definition will tell us how to relate these values to the spectral sequence.

Definition B.1.2. An a^{th}-stage spectral sequence $E_r^{p,q}$ is said to *converge* to groups H^n, written

$$E_r^{p,q} \Rightarrow H^{p+q},$$

if there is a filtration

$$0 = H_{n+1}^n \subseteq H_n^n \subseteq H_{n-1}^n \subseteq \cdots \subseteq H_1^n \subseteq H_0^n = H^n$$

such that

$$E_\infty^{p,n-p} \simeq H_p^n / H_{p+1}^n \tag{B.1}$$

for all p.

B.2 A special case of Hochschild-Serre spectral sequence

Let G be a profinite group, H a closed normal subgroup and A a G-module. Consider the *Hochschild-Serre spectral sequence*

$$E_2^{p,q} = H^p(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A)$$

and suppose that $\mathrm{cd}(G/H) = 1$.

Step one is drawing a picture of the initial sheet with entries $E_2^{p,q}$:

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
E_2^{0,2} & E_2^{1,2} & E_2^{2,2} & E_2^{3,2} & \cdots \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & \cdots \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & \cdots
\end{array}$$

For the Hochschild-Serre spectral sequence with $\mathrm{cd}(G/H) = 1$ the above sheet is:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H^0(G/H, H^n(H, A)) & H^1(G/H, H^n(H, A)) & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
H^0(G/H, H^1(H, A)) & H^1(G/H, H^1(H, A)) & 0 & \cdots \\
H^0(G/H, H^0(H, A)) & H^1(G/H, H^0(H, A)) & 0 & \cdots
\end{array}$$

Step two is calculating $E_3^{p,q}$ for every $p, q \geqslant 0$. We use the maps given in Definition B.1.1. In particular,

$$E_3^{p,q} \simeq Ker(d_2^{p,q})/Im(d_2^{p-2,q+1}) \quad \forall\, p, q \geqslant 0\,.$$

Observe that

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1} = 0$$

and

$$d_2^{p-2,q+1} : 0 = E_2^{p-2,q+1} \to E_2^{p,q}\,.$$

So, $Ker(d_2^{p,q}) \simeq E_2^{p,q}$ and $Im(d_2^{p-2,q+1}) = 0$. Thus, in the first sheet we have already all stabilized terms: $E_2^{p,q} \simeq E_\infty^{p,q}$ $\forall\, p, q \geqslant 0$.

Now, we have to match this information with the notion of convergence.

Let us start with a filtration for $H^1(G, A)$. It is

$$0 = H_2^1 \subseteq H_1^1 \subseteq H_0^1 = H^1(G, A).$$

By (B.1) we have:

$$H_1^1 \simeq E_\infty^{1,0} \quad \text{and} \quad H_0^1/H_1^1 \simeq E_\infty^{0,1}\,.$$

Thus, we have the short exact sequence

$$E_\infty^{1,0} \hookrightarrow H^1(G, A) \twoheadrightarrow E_\infty^{0,1}$$

which actually is

$$H^1(G/H, H^0(H, A)) \hookrightarrow H^1(G, A) \twoheadrightarrow H^0(G/H, H^1(H, A))\,.$$

Now we go directly to the general case and calculate a filtration for $H^n(G, A)$, $n \geqslant 2$:

$$0 = H_{n+1}^n \subseteq H_n^n \subseteq \cdots \subseteq H_1^n \subseteq H_0^n = H^n.$$

We have that $H_n^n \simeq E_\infty^{n,0} = 0$, $H_{n-1}^n \simeq E_\infty^{n-1,1} = 0$ (if $n \geqslant 3$), $H_{n-2}^n \simeq E_\infty^{n-2,2} = 0$ (if $n \geqslant 4$) and so on till

$$H_1^n \simeq E_\infty^{1,n-1} \quad \text{and} \quad H^n(G, A)/H_1^n \simeq E_\infty^{0,n}\,.$$

Thus, we obtain the short exact sequence

$$E_\infty^{1,n-1} \hookrightarrow H^n(G, A) \twoheadrightarrow E_\infty^{0,n}\,.$$

To sum up, the Hochschild-Serre spectral sequence when $\mathrm{cd}(G/H) = 1$ provides the sequences

$$H^1(G/H, H^{n-1}(H, A)) \hookrightarrow H^n(G, A) \twoheadrightarrow H^0(G/H, H^n(H, A)) \quad \forall\, n \geqslant 1\,.$$

Bibliography

[1] S. Arias-de-Reyna, W. Gajda, S. Petersen *Big monodromy theorem for abelian varieties over finitely generated fields*, arXiv:1201.2335v1 [math.AG] (2011).

[2] P.N. Balister - S. Howson *Note on Nakayama's lemma for compact Λ-modules*, Asian J. Math. **1**, no. 2 (1997), 224–229.

[3] A. Bandini, I. Longhi *Control Theorems for elliptic curves over function fields*, Int. J. Number Theory, Vol. **5**, no. 2 (2009), 229–256.

[4] A. Bandini, I. Longhi *Selmer groups for elliptic curves in $\mathbb{Z}_\ell^d$-extensions of function fields of characteristic p*, Ann. Inst. Fourier **59**, no. 6 (2009), 2301–2327.

[5] A. Bandini, I. Longhi, S. Vigni *Torsion points on elliptic curves over function fields and a theorem of Igusa*, Expo. Math. **27**, no. 3 (2009), 175–209.

[6] A. Bandini, M. Valentino *Control Theorems for ℓ-adic Lie extensions of global function fields*, accepted for publication in *Ann. Sc. Norm. Super. Pisa Cl. Sci.*.

[7] M. Bertolini, H. Darmon *Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions*, Ann. of Math. (2) **162** (2005), no. 1, 1–64.

[8] W. Bruns, J. Herzog *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics, **39** Cambridge Univ. Press 1993.

[9] D. Burns, O. Venjakob *On descent theory and main conjectures in non-commutative Iwasawa theory*, J. Inst. Math. Jussieu **10** (2011), 59–118.

[10] J. Coates *Fragments of the GL_2 Iwasawa theory of elliptic curves without complex multiplication*, in *Arithmetic Theory of Elliptic Curves, (Cetraro, 1997)*, Ed. C. Viola, Lecture Notes in Mathematics 1716, Springer (1999), 1–50.

[11] J. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob *The GL_2 main conjecture for elliptic curves without complex multiplication* Inst. Hautes Etud. Sci. Publ. Math., **101** (2005), 163–208.

[12] J. Coates, R. Greenberg *Kummer theory for abelian varieties over local fields*, Invent. Math. **124** (1996), 129–174.

[13] J. Coates, P. Schneider, R. Sujatha *Modules over Iwasawa algebras*, J. Inst. Math. Jussieu **2** (2003), no. 1, 73–108.

[14] J.Coates, R. Sujatha *Euler-Poincaré characteristic of abelian varieties*, C.R. Acad. Sci. Paris, Série I (1999), 309–313.

[15] J. Coates, R. Sujatha *Galois cohomology of elliptic curves. 2nd Edition* Narosa (2010).

[16] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal *Analytic pro-p groups. 2nd Edition*, Cambridge Studies in Advanced Mathematics **61**, Cambridge Univ. Press (1999).

[17] J.S. Ellenberg *Selmer groups and Mordell-Weil groups of elliptic curves over towers of function fields*, Compos. Math. **142**, no. 5 (2006), 1215–1230.

[18] R. Greenberg *Galois Theory for the Selmer Group of an Abelian Variety*, Comp. Math **136** (2003), 255–297.

[19] C. Hall *An open image theorem for a general class of abelian varieties*, arXiv:0803.1682v1 [math.NT] (2011).

[20] K. Iwasawa *On Γ-extensions of algebraic number fields*, Bull. Amer. Math. Soc. **65** (1959), 183–226.

[21] K. Iwasawa *On the theory of cyclotomic fields*, Ann. of Math. (2) **70** (1959), 530–561.

[22] K. Iwasawa *On $\mathbb{Z}_\ell$-extensions of algebraic number fields*, Ann. of Math. (2), **98** (1973), 246–326.

[23] U. Jannsen *Iwasawa modules up to isomorphism*, Advanced Studies in Pure Mathematics 17 (1989), Algebraic Number Theory - in honor of K. Iwasawa, 171-207.

[24] U. Jannsen *A spectral sequence for Iwasawa adjoints*, arXiv:1307.6547v1 [math.NT] (2013).

[25] K. Kato *p-adic Hodge theory and values of zeta functions of modular forms. Cohomologies p-adiques et applications arithmtiques. III.* Astrisque No. 295 (2004), ix, 117–290.

[26] K. Kato, F. Trihan *On the conjectures of Birch and Swinnerton-Dyer in characteristic* $p > 0$, Invent. Math. **153** (2003), 537–592.

[27] M. Lazard *Groupes analytiques p-adiques*, Publ. Math. I.H.E.S. **26** (1965), 389–603.

[28] A. Lubotzky, A. Mann, D. Segal *Finitely generated groups of polynomial subgroup growth*, Israel J. Math. **82**, no. 1-3 (1993), 363–371.

[29] B. Mazur *Rational points of abelian varieties with values in towers of number fields*, Invent. Math. **18** (1972), 183–266.

[30] B. Mazur, P. Swinnerton-Dyer *Arithmetic of Weil curves*, Invent. Math. **25** (1974), 1–61.

[31] B. Mazur, J. Tate, J. Teitelbaum *On p -adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), no. 1, 1–48.

[32] J.S. Milne *Étale Cohomology*, Princeton Mathematical Series, Vol. 33 Princeton University Press, Princeton, N. J., 1980.

[33] J.S. Milne *Arithmetic Duality Theorems*, BookSurge, LLC, Second edition, 2006.

[34] J. Neuchirch, A. Schmidt, K. Wingberg *Cohomology of number fields - Second edition*, GTM 323, Springer-Verlag, (2008).

[35] Y. Ochi *A note on Selmer groups of abelian varieties over the trivializing extensions*, Proc. Amer. Math. Soc. **134** (2006), no. 1, 31–37 .

[36] T. Ochiai - F. Trihan *On the Selmer groups of abelian varieties over function fields of characteristic* $p > 0$ Math. Proc. Cambridge Philos. Soc. **146**, no. 1 (2009), 23–43.

[37] Y. Ochi, O. Venjakob *On the structure of Selmer groups over p -adic Lie extensions* J. Algebraic Geom. 11 (2002), no. **3**, 547-580.

[38] A. Pacheco *Selmer groups of abelian varieties in extensions of function fields*, Math. Z. **261**, no. 4 (2009), 787–804

[39] M. Rosen *Number theory in function fields*, GTM 210, Springer, New York, 2002.

[40] G. Sechi GL_2 *Iwasawa Theory of Elliptic Curves over Global Function Fields*, PhD thesis, University of Cambridge, (2006).

[41] J.P. Serre *Sur la dimension cohomologique des groupes profinis*, Topology **3** (1965), 413–420.

[42] J.P. Serre *Sur les groupes de congruence des variétés abéliennes II* Izv. Akad. Nauk SSSR, Ser. Mat. **35** (1971), 731–737.

[43] J.P. Serre *Cohomologie Galoiesienne. 5ème éd., rév. et complété*, Lecture Notes in Mathematics **5**, Springer-Verlag, 1994.

[44] J.P. Serre, J. Tate *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517.

[45] S.S. Shatz *The Cohomological Dimension of Certain Grothendieck Topologies*, Ann. of Math (2), **83** No. 3 (1966), 572–595.

[46] R. Swan *Algebraic K-theory*, LNM **76**, Springer, 1968.

[47] K.S. Tan *A generalized Mazur's theorem and its applications*, Trans. Amer. Math. Soc. **362** (2010), 4433–4450.

[48] K.S. Tan *Selmer groups over $\mathbb{Z}_p^d$-extensions*, arXiv:1205.3907v1 [math.NT] (2012).

[49] D. Ulmer *L-functions with large analytic rank and abelian varieties with large algebraic rank over function fields*, Invent. Math. **167** (2007), 379–408.

[50] O. Venjakob *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc. (JEMS) **4**, no. 3 (2002), 271–311.

[51] O. Venjakob *Characteristic elements in noncommutative Iwasawa theory*, J. reine angew. Math. **583** (2005), 193–236.

[52] G.D. Villa Salvador *Topics in the theory of algebraic function fields*, Birkhäuser Boston, 2006.

[53] C.A. Weibel *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge Univ. Press (1995).

[54] T. Weston *The inflation-restriction sequence: an introduction to spectral sequences*
http://www.math.mcgill.ca/goren/SeminarOnCohomology/infres.pdf

[55] J.S. Wilson *Profinite groups* London Mathematical Society Monographs. New Series, 19. The Clarendon Press, Oxford University Press, New York, 1998.

[56] M. Witte *On a noncommutative Iwasawa main conjecture for varieties over finite fields*, to appear in J. Eur. Math. Soc. (JEMS), arXiv:1004.2481v2 [math.NT] (2012).

Riassunto

Siano F un campo di funzioni globale di caratteristica $p > 0$, A/F una varietà abeliana e K/F un'estensione di campi tale che $G = \mathrm{Gal}(K/F)$ sia un gruppo di Lie ℓ-adico. Questo lavoro di tesi è concentrato sull'aspetto algebrico della Teoria di Iwasawa non commutativa. In particolare, vengono generalizzati alcuni risultati sulla struttura dei duali di Pontrjagin dei gruppi di Selmer, visti come moduli sull'algebra di Iwasawa $\Lambda(G)$.

Nel primo capitolo vengono introdotte le nozioni di base di cui si avrà bisogno nel prosieguo del lavoro. Si comincia con i campi di funzioni e i gruppi di Lie ℓ-adici nelle Sezioni 1.1 e 1.2. Dopo aver richiamato la definizione di algebra di Iwasawa e le sue proprietà più importanti (Sezione 1.3), viene data la definizione del principale oggetto di indagine di questo lavoro: il gruppo di Selmer (vd. paragrafo 1.4). Infine, nelle Sezioni 1.5 e 1.6 si riassume il lavoro di Venjakob sull'appropriata definizione di *elemento caratteristico* e *modulo pseudo-nullo* per moduli definiti su un algebra di Iwasawa non commutativa.

Nel secondo capitolo comincia lo studio di $Sel_A(K)_\ell^\vee$, cioè il duale di Pontrjagin del gruppo di Selmer, attraverso alcune generalizzazioni del Teorema del Controllo di Mazur.
Per il caso $\ell \neq p$, nella Sezione 2.1 si dimostra il seguente:

Teorema (Theorem 2.1.2)
Sia K/F estensione non ramificata fuori da un insieme finito S di primi di F. Per ogni estensione finita F'/F contenuta in K, i nuclei e i conuclei delle mappe

$$a_{K/F'} \,:\, Sel_A(F')_\ell \longrightarrow Sel_A(K)_\ell^{\mathrm{Gal}(K/F')}$$

sono $\mathbb{Z}_\ell$-moduli cofinitamente generati. Se tutti i primi di S e quelli di cattiva riduzione hanno gruppi di decomposizione aperti in G, i coranghi dei nuclei e dei conuclei sono limitati indipendentemente da F'. Inoltre, se $A[\ell^\infty](K)$ è finito, allora tali nuclei e conuclei hanno ordine finito.

Applicando il Lemma di Nakayama, si trovano poi le condizioni affinché $Sel_A(K)_\ell^\vee$ sia finitamente generato (sotto ulteriori ipotesi anche di torsione) su $\Lambda(G)$. Per ottenere maggiori informazioni sulla struttura del Selmer c'è bisogno inoltre di assumere che $\exists\ H \lhd_c \mathrm{Gal}(K/F)$ tale che $\mathrm{Gal}(K/F)/H \simeq \mathbb{Z}_\ell$ e studiarlo come modulo su $\Lambda(H)$. Per farlo si dimostra un teorema analogo al precedente, questa volta per la mappa

$$a_{K/K^H} \,:\, Sel_A(K^H)_\ell \longrightarrow Sel_A(K)_\ell^H \,.$$

Teorema (Theorem 2.1.15)
Sia K/F una estensione di Lie ℓ-adica non ramificata fuori da un insieme

finito S di primi di F. Se esiste $H \lhd_c G := \mathrm{Gal}(K/F)$ tale che $G/H \simeq \mathbb{Z}_\ell$, allora il nucleo e il conucleo della mappa

$$a : Sel_A(K^H)_\ell \to Sel_A(K)_\ell^H$$

sono $\mathbb{Z}_\ell$-moduli cofinitamente generati. Inoltre, se $A[\ell^\infty](K)$ è finito, e per ogni $w|w'|v \in S$ il gruppo $A[\ell^\infty](K_w)$ è anch'esso finito, allora $Ker(a)$ e $Coker(a)$ sono finiti.

La principale conseguenza dei risultati summenzionati riguarda l'esistenza di elementi caratteristici in $\Lambda(G)$ per $Sel_A(K)_\ell^\vee$.
Nella Sezione 2.2, viene trattato lo stesso problema nel caso $\ell = p$, ottenendo il seguente

Teorema (Theorem 2.2.3)
Sia K/F una estensione di Lie p-adica non ramificata fuori da un insieme finito S di primi di F. Se tutti i primi ramificati hanno riduzione buona o moltiplicativa spezzata, allora, per ogni estensione finita F'/F contenuta in K, i nuclei e i conuclei delle mappe

$$a_{K/F'} \ : Sel_A(F')_p \longrightarrow Sel_A(K)_p^{\mathrm{Gal}(K/F')}$$

sono $\mathbb{Z}_p$-moduli cofinitamente generati. Se il gruppo $A[p^\infty](K)$ è finito, tutti i primi in S sono di buona riduzione e hanno gruppi di inerzia aperti nei loro gruppi di decomposizione, allora i nuclei e i conuclei sono finiti (di ordine limitato se i primi di cattiva riduzione hanno gruppo di decomposizione aperto).

Nell'ultimo capitolo si dà un quadro più completo della struttura del Selmer, ma solo per il caso $\ell \neq p$. In particolare, si studiano le condizioni affinché $Sel_A(K)_l^\vee$ non abbia sottomoduli non banali pseudo-nulli. Il risultato principale è il seguente:

Teorema (Theorem 3.1.10)
Sia $G = \mathrm{Gal}(K/F)$ un gruppo di Lie ℓ-adico senza elementi di ordine ℓ e di dimensione $d \geqslant 3$ (come gruppo di Lie ℓ-adico). Se $H^2(F_S/K, A[\ell^\infty]) = 0$ e la mappa ψ nella successione

$$Sel_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \mathrm{Coind}_G^{G_v} H^1(K_w, A)[\ell^\infty]$$

è suriettiva, allora $\mathcal{S} := Sel_A(K)_\ell^\vee$ non ha sottomoduli non banali pseudo-nulli.

Il capitolo si conclude con il calcolo della caratteristica di Eulero di $Sel_A(K)_\ell$:

Teorema (Theorem 3.2.4)
Se $Sel_A(F)_\ell$ *è finito,* $H^2(F_S/K, A[\ell^\infty]) = 0$*,* $\chi(G, A[\ell^\infty])$ *e* $\chi(G_v, A[\ell^\infty])$ *(*$v \in S$*) sono ben definiti e la mappa* ψ *nella successione*

$$Sel_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \mathrm{Coind}_G^{G_v} H^1(K_w, A[\ell^\infty])$$

è suriettiva, allora la caratteristica di Eulero di $Sel_A(K)_\ell$ *è ben definita e*

$$\chi(G, Sel_A(K)_\ell) = \prod_{v \in S} \frac{1}{c_v} \cdot \frac{|H^1(F_S/K, A[\ell^\infty])^G|}{|H^3(G, A[\ell^\infty](K))|}$$

dove $c_v = |H^1(K_w, A[\ell^\infty])^{G_v}|$ *per ogni* $v \in S$*.*

Acknowledgements

At the end of my Ph.D. journey let me thank all those people who made this thesis possible.

Foremost, I would like to thank my advisor, Dr. Andrea Bandini. It would sound trite thanking you only for your astonishing patience, fatherly advices and exhausting “full immersions” in mathematics during our meetings in Parma. More than any other reason, I want to thank you for helping me to build my own wings. I do not know how far I can fly. Anyway, wherever I will go, thank you for the mathematician I am.

Besides my advisor, I wish to thank Prof. Dr. Otmar Venjakob. Your support during some critical stages of my Ph.D. experience is invaluable.

I would also to thank Prof. John van Bon. You became my point of reference in the department when Andrea moved to Parma.

I gratefully acknowledge all colleagues and friends who shared with me this last years. I will always cherish memories of our laughs, confidences, tears, chatter, coffee breaks and walks around the Europe.

My sincere thank also goes to my parents and my brother. Thank you for lying close to me as always.

Last, but not the least, I thank my husband Giacomo. Handling me is really something. Thank you for loving me.